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Monte-Carlo and Gauss-Legendre Numerical Integration Methods for 1- to 3-D Functions

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Gauss-Legendre Integration Test:

Function #1 Gauss-Legendre

```
% Define test function for 1D, 2D & 3D case
fun1 = @(x) (2*pi).^(-1/2)* exp(0.5*(-x.^2));
fun2 = @(x, y) 1 / (2 * pi) * exp(-0.5 * (x.^2 + y.^2));
fun3 = @(x,y,z) (2*pi).^(-3/2)*exp(0.5*(-x.^2-y.^2-z.^2));

% Define the limits of integration for each dimension
a_x = -3; b_x = 3; % Limits for x
a_y = -3; b_y = 3; % Limits for y
a_z = -3; b_z = 3; % Limits for y

% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 0.9973
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)              % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 0.9973
```

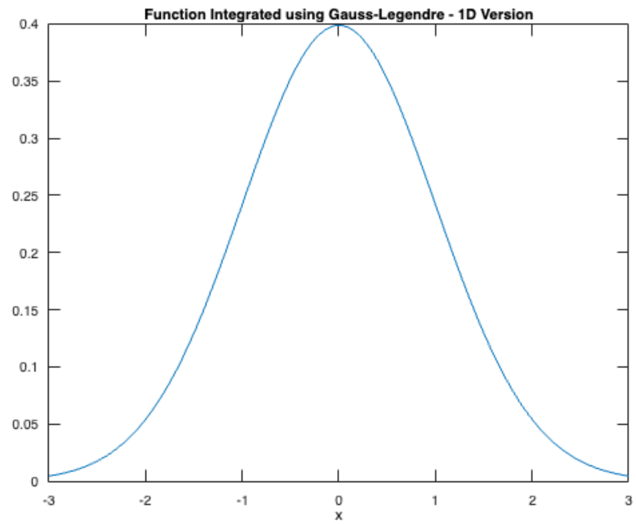
```
toc                                                    % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001584 seconds.
```

```
absoluteError1D = abs ((value_GL1-value1) / value1 )     % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 3.1923e-05
```

```
plot_1D( fun1, a_x,b_x)                                  % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = 0.9946
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)    % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = 0.9945
```

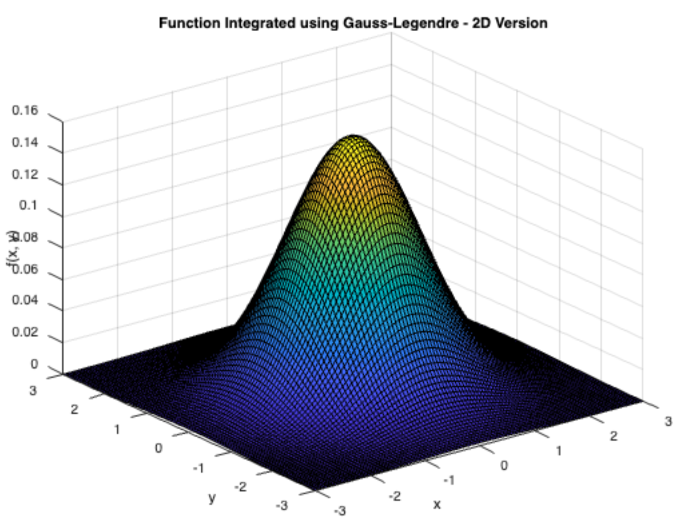
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.006080 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)       % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 6.3845e-05
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = 0.9919
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = 0.9918
```

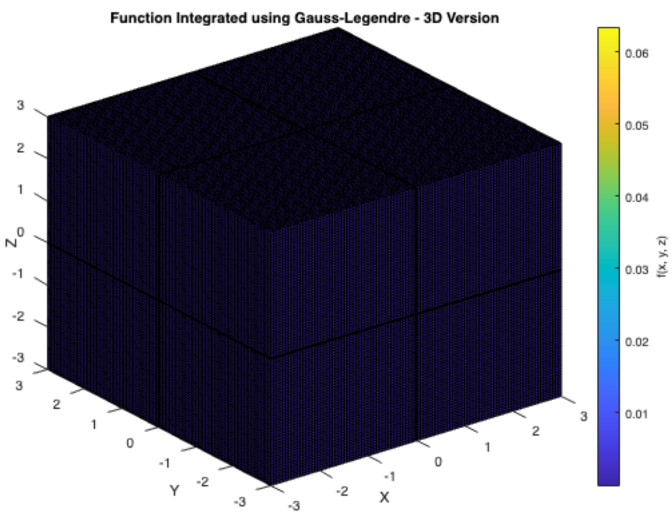
```
toc                                                                  % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.010011 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                 % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 9.5765e-05
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                         % Plot func1 - 3D Case
```



Function #2 Gauss-Legendre

```
% Define test function for 1D,2D 3D case
fun1 = @(x) abs(4*x-2);
fun2 = @(x, y) abs(4*x-2).*abs(4*y-2);
fun3 = @(x, y,z) abs(4*x-2).*abs(4*y-2).*abs(4*z-2);

% Define the limits of integration for each dimension
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y



% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 1
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)              % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 1.0115
```

```
toc                                                    % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001610 seconds.
```

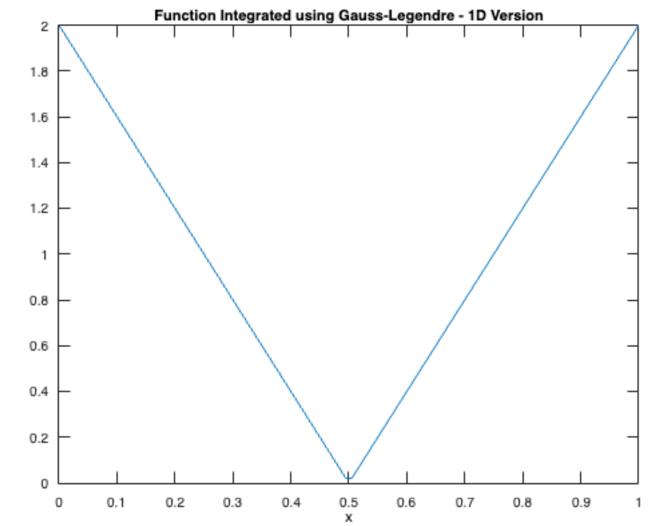
```

absoluteError1D = abs ((value_GL1-value1) / value1 )   % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 0.0115
```

```

plot_1D( fun1, a_x,b_x)                                % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = 1.0000
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)    % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = 1.0232
```

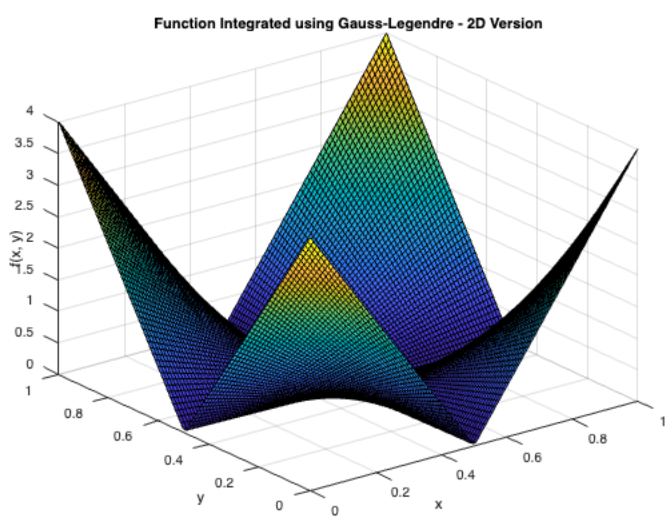
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.003053 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)       % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 0.0232
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = 1.0000
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = 1.0350
```

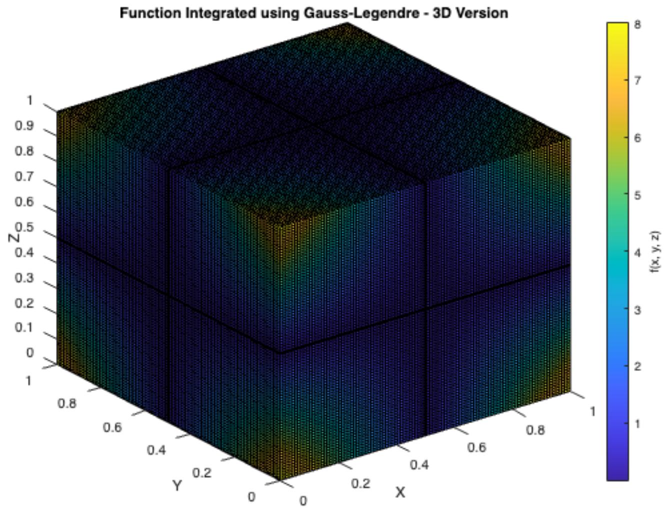
```
toc                                                                  % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.010868 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                   % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 0.0350
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                           % Plot func1 - 3D Case
```



Function #3 Gauss-Legendre

```
% Define the constants used in functions
a = 5;
u = 0.5;

% Define test function for 1D, 2D & 3D case
fun1 = @(x) 1 ./ (a.^(-2) + (x - u).^2);
fun2 = @(x, y) 1 ./ (a.^(-2) + (x - u).^2) .* 1 ./ (a.^(-2) + (y - u).^2);
fun3 = @(x, y, z) 1 ./ (a.^(-2) + (x - u).^2) .* 1 ./ (a.^(-2) + (y - u).^2) .* 1 ./ (a.^(-2) + (z - u).^2);



% Define the limits of integration for each dimension
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y



% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 11.9029
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)              % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 11.8620
```

```
toc                                                    % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001037 seconds.
```

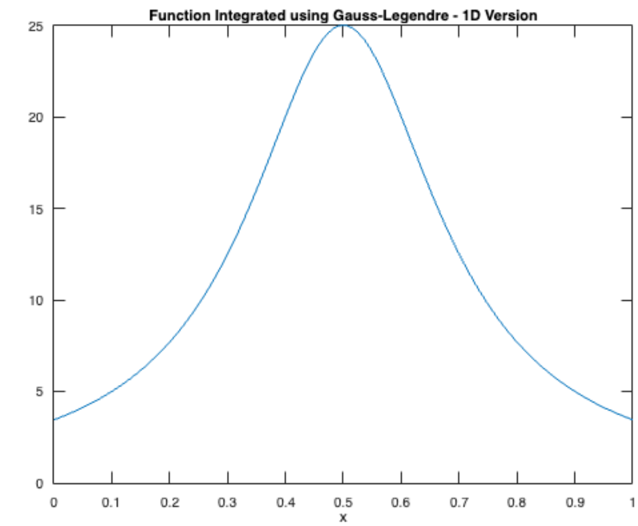
```

absoluteError1D = abs ((value_GL1-value1) / value1 )    % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 0.0034
```

```

plot_1D( fun1, a_x,b_x)                                % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = 141.6790
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)    % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = 140.7059
```

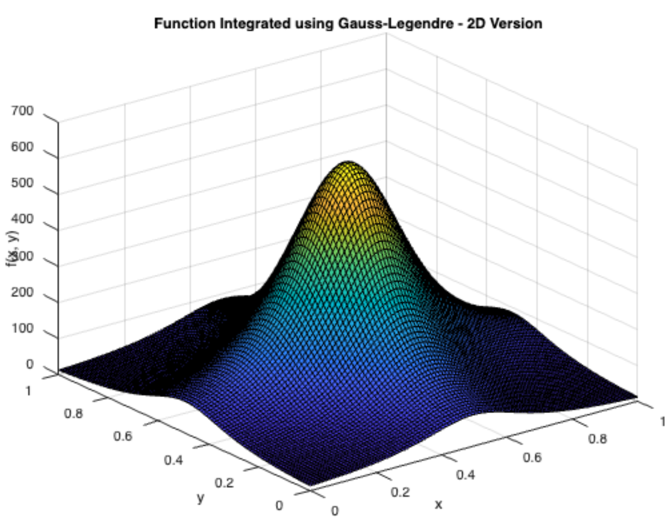
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.003121 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)       % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 0.0069
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = 1.6864e+03
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = 1.6690e+03
```

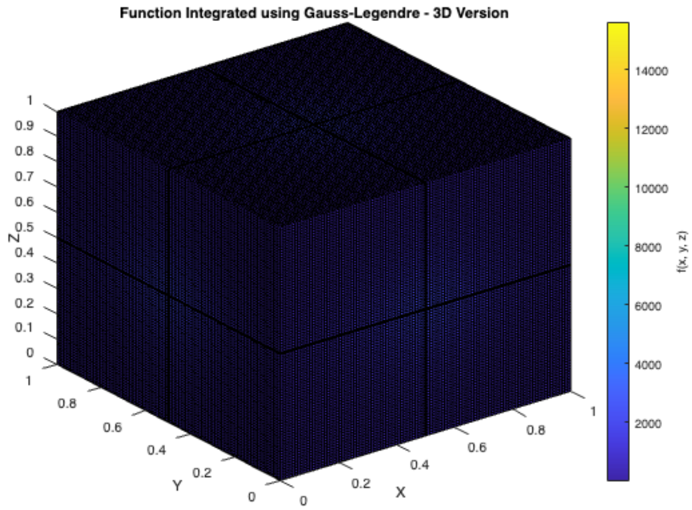
```
toc                                                                  % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.009754 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 0.0103
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                        % Plot func1 - 3D Case
```



Function #4 Gauss-Legendre

```
% Define the limits of integration for each dimension
a = 5;
u = 0.5;
% Define test function for 1D, 2D & 3D case
fun1 = @(x) cos(2.*pi.*u + a.*x);
fun2 = @(x, y) cos(2.*pi.*u + a.*x+a.*y);
fun3 = @(x, y, z) cos(2.*pi.*u + a.*x+a.*y+a.*z);



% Define the limits of integration for each dimension
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y


% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 0.1918
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)             % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 0.1918
```

```
toc                                                   % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001088 seconds.
```

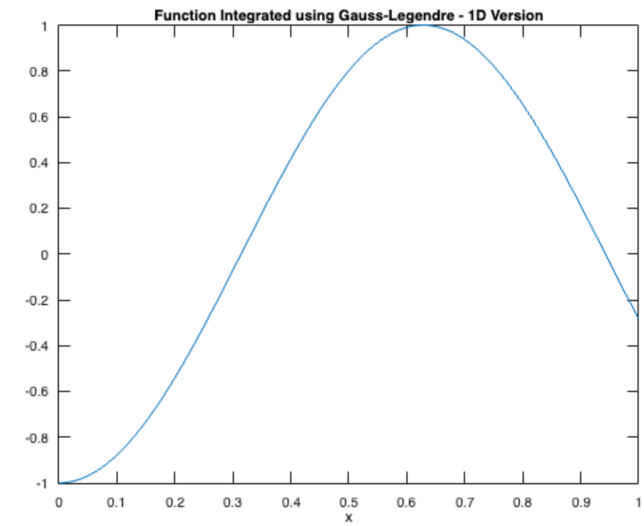
```

absoluteError1D = abs ((value_GL1-value1) / value1 )    % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 9.9181e-12
```

```

plot_1D( fun1, a_x,b_x)                                 % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = -0.0163
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)    % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = -0.0163
```

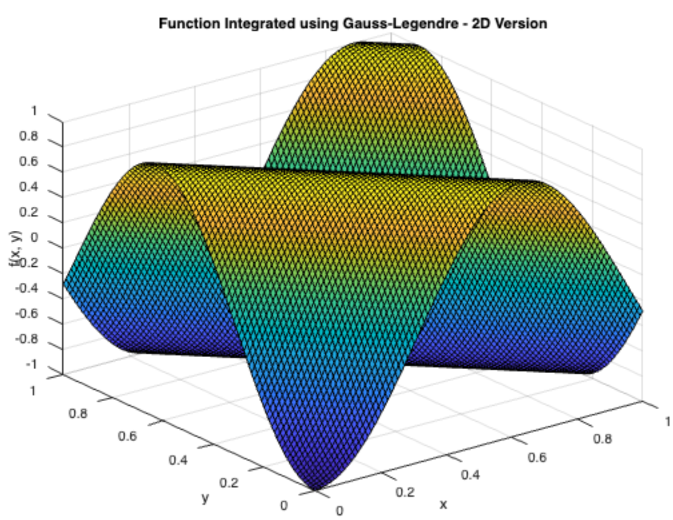
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.003888 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)       % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 2.6996e-09
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = -0.0048
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = -0.0048
```

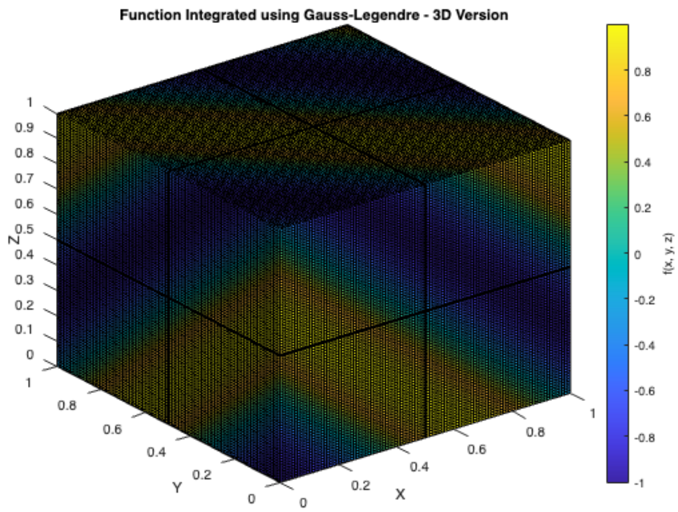
```
toc                                                                  % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.009326 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 8.4939e-10
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                      % Plot func1 - 3D Case
```



Function #5 Gauss-Legendre

```
% Define the constant used in functions
a = 5;

% Define test function for 1D, 2D & 3D case
fun1 = @(x) (abs(4.*x-2)+a)./(1+a);
fun2 = @(x,y) (abs(4.*x-2)+a)./(1+a) .* (abs(4.*y-2)+a)./(1+a);
fun3 = @(x,y,z) (abs(4.*x-2)+a)./(1+a) .* (abs(4.*y-2)+a)./(1+a).* (abs(4.*z-2)+a)./(1+a);



% Define the limits of integration for each dimension
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y


% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 1.0000
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)               % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 1.0019
```

```
toc                                                     % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001280 seconds.
```

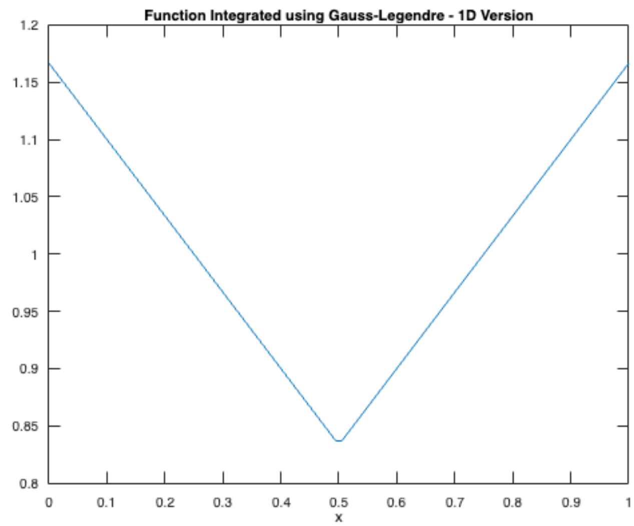
```

absoluteError1D = abs ((value_GL1-value1) / value1 )    % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 0.0019
```

```

plot_1D( fun1, a_x,b_x)                                 % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = 1.0000
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)   % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = 1.0038
```

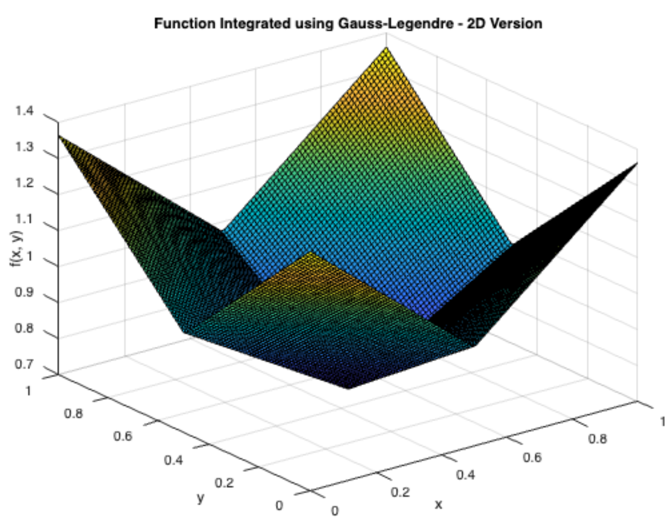
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.003385 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)      % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 0.0038
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = 1.0000
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = 1.0058
```

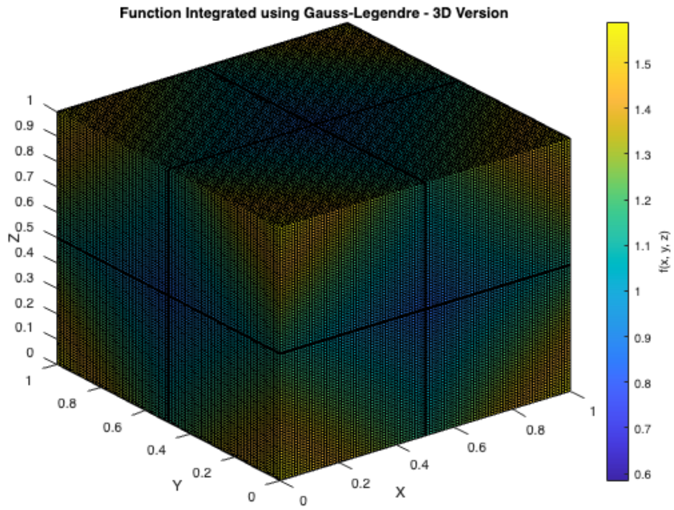
```
toc                                                                  % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.008982 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                    % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 0.0058
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                            % Plot func1 - 3D Case
```



Function #6 Gauss-Legendre

```
% Define the constant used in functions
a = 5;

% Define test function for 1D, 2D & 3D case
fun1 = @(x) (1+a*x).^(-2);
fun2 = @(x,y)  (1+a*x+a*y).^(-3);
fun3 = @(x,y,z) (1+a*x+a*y+a*z).^(-4);


% Define the limits of integration for each dimension
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y


% 1D Integral
value1 = integral(fun1,a_x,b_x)
```

```
value1 = 0.1667
```

```
tic
value_GL1 = integral_GL1(fun1,8, a_x,b_x)               % Calculate 1D integral using Gauss-Legendre Quadrature
```

```
value_GL1 = 0.1667
```

```
toc                                                     % Time taken for compute 1D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.001219 seconds.
```

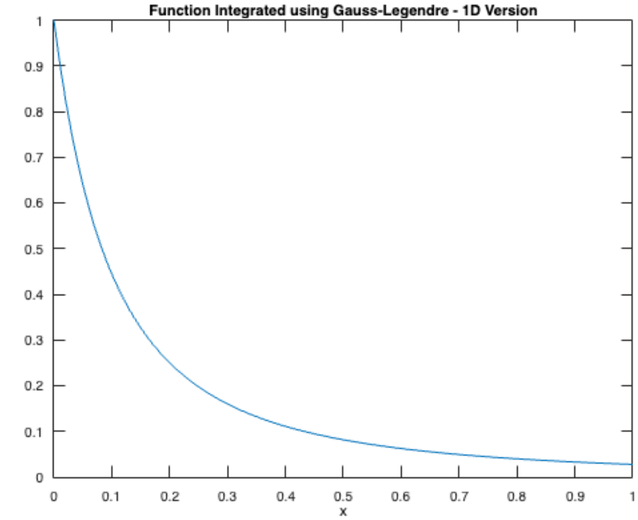
```

absoluteError1D = abs ((value_GL1-value1) / value1 )    % Calculate Absolute Error of the 1D integral using Gauss-Legendre Qua
```

```
absoluteError1D = 1.9877e-05
```

```

plot_1D( fun1, a_x,b_x)                                 % Plot func1 - 1D Case
```




```
% 2D Integral
value2 = integral2(fun2,a_x,b_x,a_y,b_y)
```

```
value2 = 0.0152
```

```
tic
value_GL2 = integral_GL2(fun2,[8,8], a_x,b_x,a_y,b_y)    % Calculate 2D integral using Gauss-Legendre Quadrature
```

```
value_GL2 = 0.0152
```

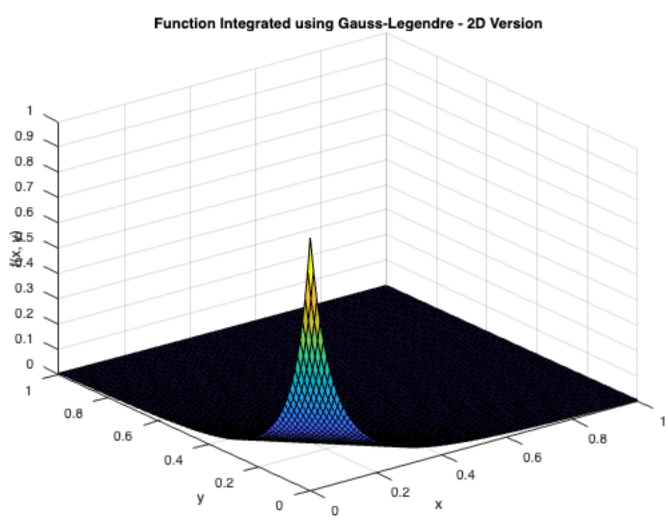
```
toc                                                      % Time taken for compute 2D integral using Gauss-Legendre Quadrature
```

```
Elapsed time is 0.003098 seconds.
```

```
absolutError2D = abs ((value_GL2-value2) / value2)       % Calculate Absolute Error of the 2D integral using Gauss-Legendre Qua
```

```
absolutError2D = 4.3427e-05
```

```
plot_2D(fun2, a_x, b_x, a_y, b_y)                        % Plot func1 - 2D Case
```




```
% 3D Integral
value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z)
```

```
value3 = 9.4697e-04
```

```
tic
value_GL3 = integral_GL3(fun3,[8,8,8], a_x, b_x, a_y, b_y, a_z, b_z) % Calculate 3D integral using Gauss-Legendre Quadrature
```

```
value_GL3 = 9.4690e-04
```

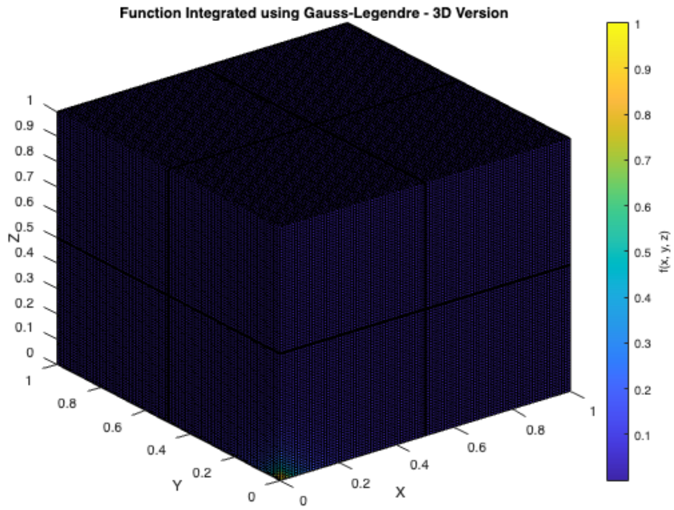
```
toc                                                                   % Time taken for compute 3D integral using Gauss-Legendre
```

```
Elapsed time is 0.008616 seconds.
```

```
absolutError3D = abs (( value_GL3-value3 ) / value3)                % Calculate Absolute Error of the 3D integral using Gauss
```

```
absolutError3D = 6.9120e-05
```

```
plot_3D( fun3, a_x, b_x, a_y, b_y, a_z, b_z)                        % Plot func1 - 3D Case
```

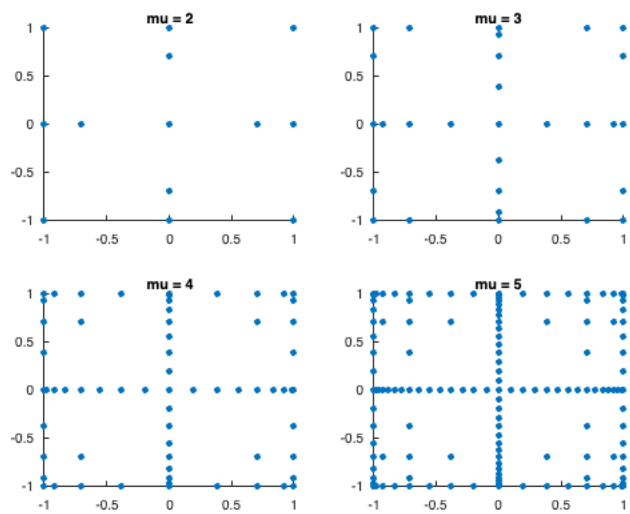


Smolyak Sparse Grid Points

This algorithm was implemented to demonstrate the generation of a subset of the nodes. The Smolyak Sparse Grid is generated with Chebyshev polynomial instead of Legendre and only a select number of points based on the Smolyak conditions.

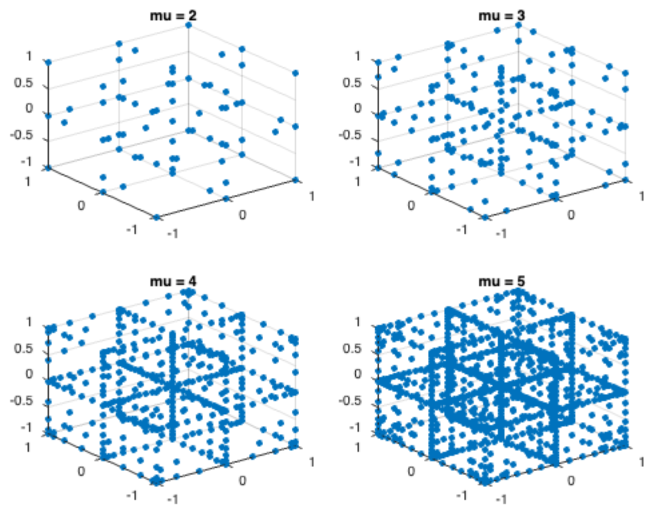
Smolyak Sparse Grid Points 2D

```
figure;
plotSmolyak2D();
```



Smolyak Sparse Grid Points 3D

```
figure;
plotSmolyak3D();
```



Monte-Carlo Integration Test:

Function #1 Monte-Carlo

```
% Define the limits of integration for each dimension
fun1 = @(x) (2*pi).^(-1/2)* exp(0.5*(-x.^2));
fun2 = @(x, y) 1 / (2 * pi) * exp(-0.5 * (x.^2 + y.^2));
fun3 = @(x,y,z) (2*pi).^(-3/2)*exp(0.5*(-x.^2-y.^2-z.^2));

% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = -3;
b_u = 3;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:0.9973
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

```
Elapsed time is 0.240421 seconds.
```

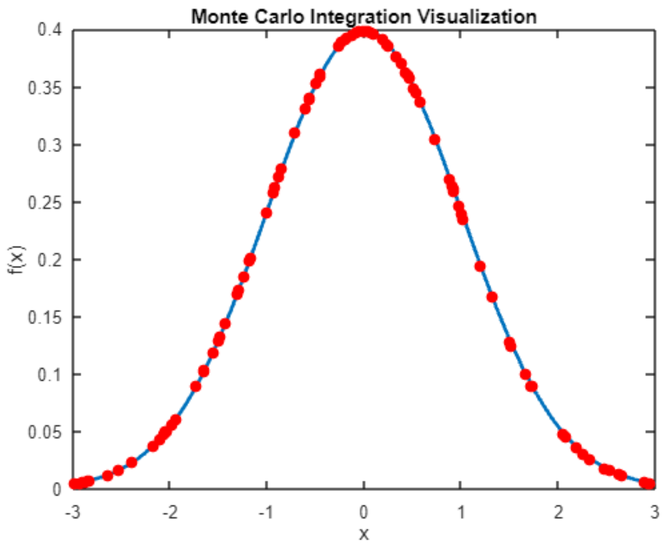
```
disp("1D Monte Carlo Integral:"+monte_value);
```

```
1D Monte Carlo Integral:0.99873
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.1433
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:0.99461
```

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

```
Elapsed time is 0.111504 seconds.
```

```
disp("2D Monte Carlo Integral:"+monte_value);
```

```
2D Monte Carlo Integral:0.99514
```

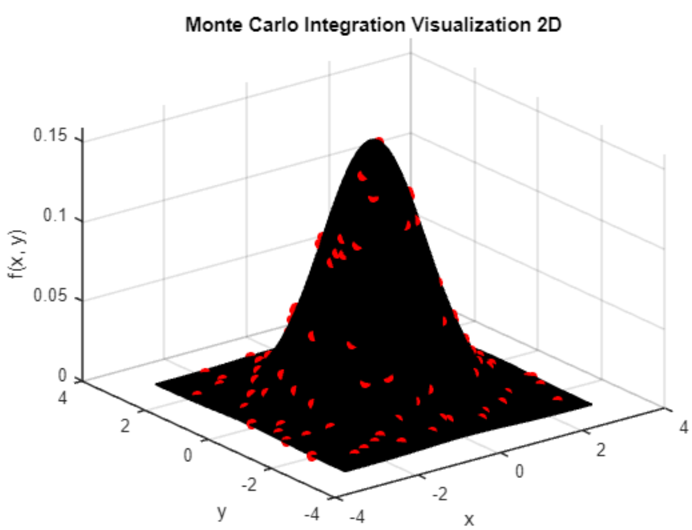
```

%Calculate the absolute value of the relative error

Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.053953
```

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```




```

%Determine the number of samples
```

```
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:0.99192
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.272592 seconds.
```

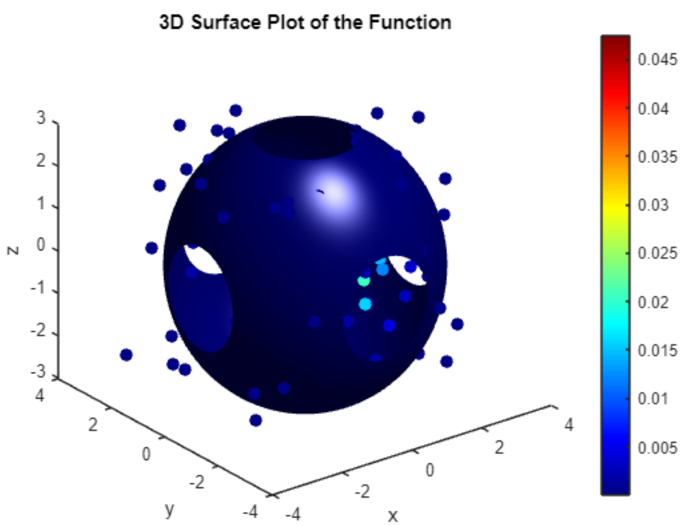
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:0.9902
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.17414
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Function #2 Monte-Carlo

```
% Define the limits of integration for each dimension
fun1 = @(x) abs(4*x-2);
fun2 = @(x, y) abs(4*x-2).*abs(4*y-2);
fun3 = @(x, y,z) abs(4*x-2).*abs(4*y-2).*abs(4*z-2);

% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = 0;
b_u = 1;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:1
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

Elapsed time is 0.064190 seconds.

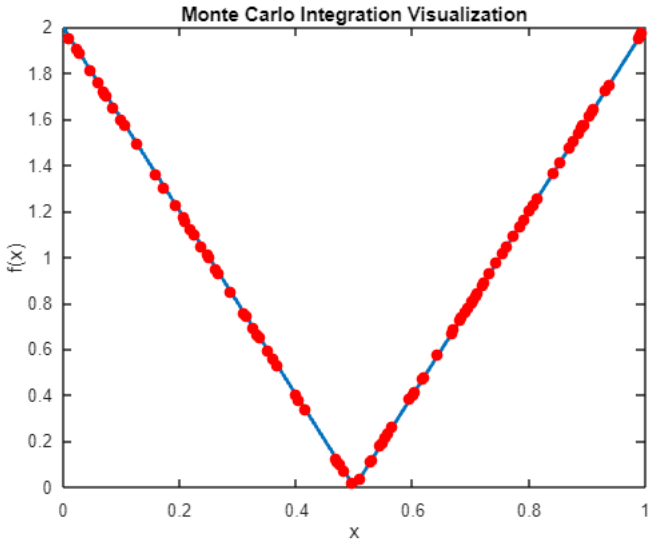
```
disp("1D Monte Carlo Integral:"+monte_value);
```

1D Monte Carlo Integral:1

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

Abserror:0.0019363

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

matlab:1

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

Elapsed time is 0.140716 seconds.

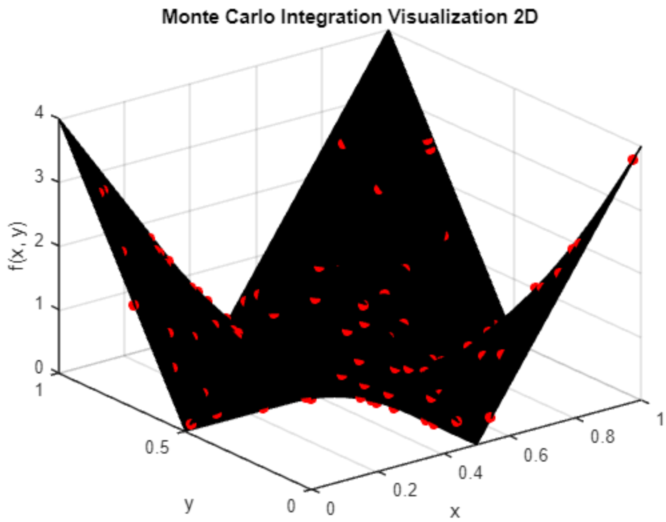
```
disp("2D Monte Carlo Integral:"+monte_value);
```

2D Monte Carlo Integral:0.99946

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

Abserror:0.054199

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:1
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.132036 seconds.
```

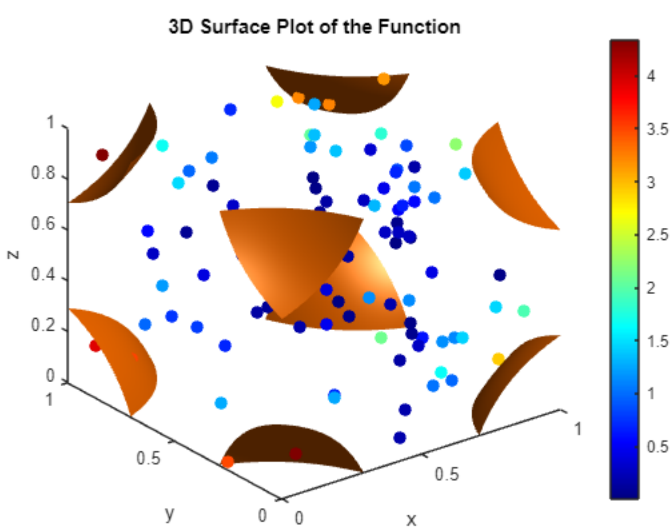
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:0.9984
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.16026
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Function #3 Monte-Carlo

```
% Define the limits of integration for each dimension
a = 5;
u = 0.5;

fun1 = @(x) 1 ./ (a.^(-2) + (x - u).^2);
fun2 = @(x, y) 1 ./ (a.^(-2) + (x - u).^2) .* 1 ./ (a.^(-2) + (y - u).^2);
fun3 = @(x, y, z) 1 ./ (a.^(-2) + (x - u).^2) .* 1 ./ (a.^(-2) + (y - u).^2) .* 1 ./ (a.^(-2) + (z - u).^2);


% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = 0;
b_u = 1;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:11.9029
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

```
Elapsed time is 0.216759 seconds.
```

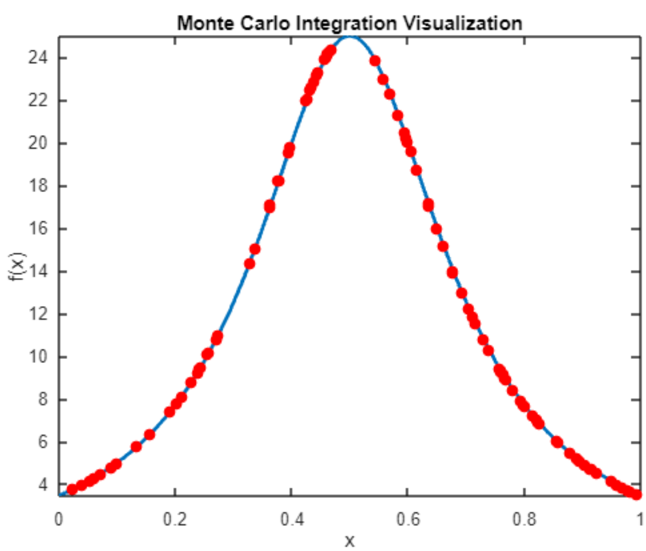
```
disp("1D Monte Carlo Integral:"+monte_value);
```

```
1D Monte Carlo Integral:11.9048
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.015572
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:141.679
```

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

```
Elapsed time is 0.224052 seconds.
```

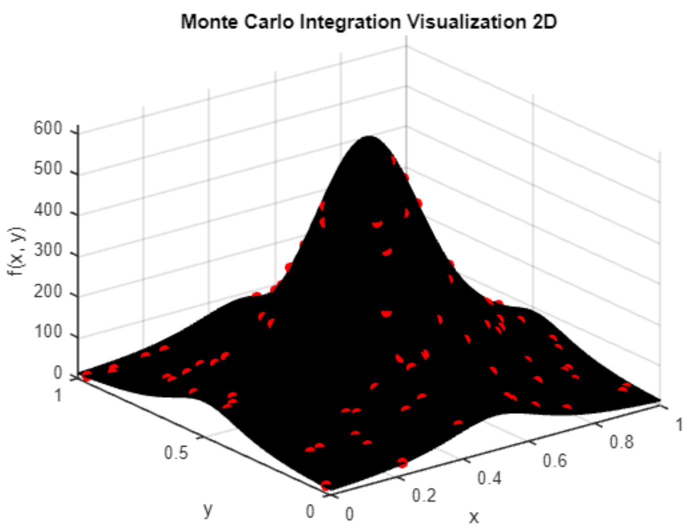
```
disp("2D Monte Carlo Integral:"+monte_value);
```

```
2D Monte Carlo Integral:141.6106
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.048283
```

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:1686.3909
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.300367 seconds.
```

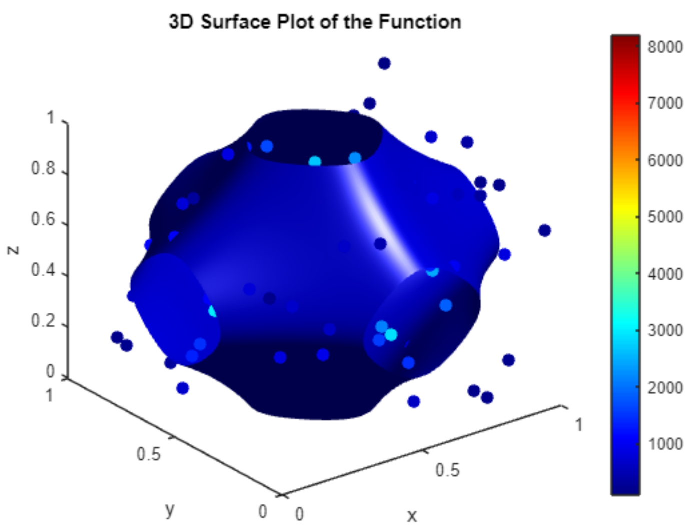
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:1683.7876
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.15437
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Function #4 Monte-Carlo

```
% Define the limits of integration for each dimension
a = 5;
u = 0.5;

fun1 = @(x) cos(2.*pi.*u + a.*x);
fun2 = @(x, y) cos(2.*pi.*u + a.*x+a.*y);
fun3 = @(x, y, z) cos(2.*pi.*u + a.*x+a.*y+a.*z);



% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = 0;
b_u = 1;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:0.19178
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

```
Elapsed time is 0.102554 seconds.
```

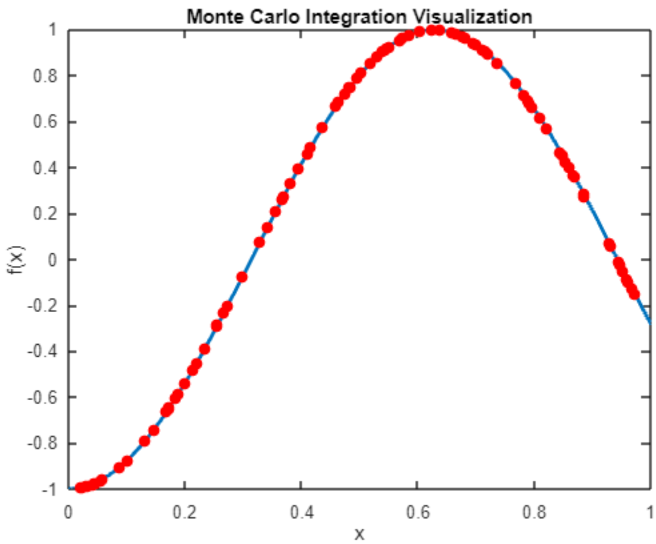
```
disp("1D Monte Carlo Integral:"+monte_value);
```

```
1D Monte Carlo Integral:0.19217
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.20239
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:-0.016256
```

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

```
Elapsed time is 0.105107 seconds.
```

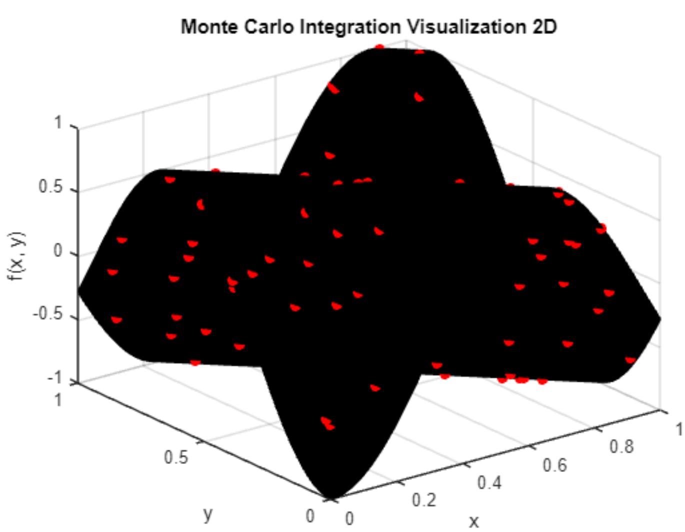
```
disp("2D Monte Carlo Integral:"+monte_value);
```

```
2D Monte Carlo Integral:-0.015548
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:4.3543
```

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```



```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:-0.0047554
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.142891 seconds.
```

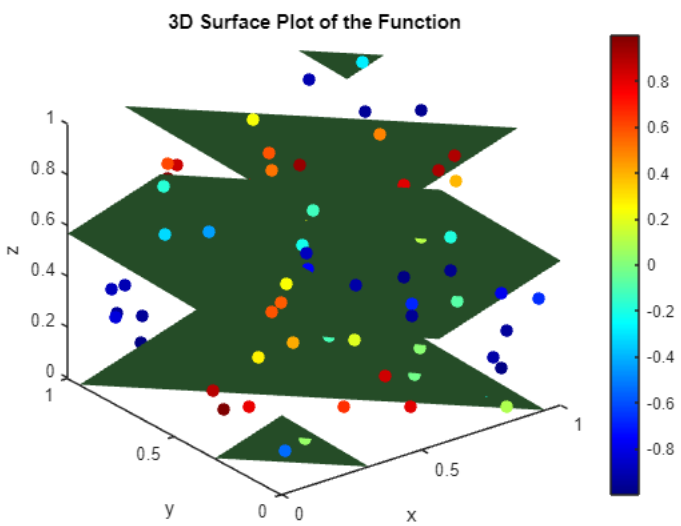
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:-0.00425
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:10.6265
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Function #5 Monte-Carlo

```
% Define the limits of integration for each dimension
a = 5;


fun1 = @(x) (abs(4.*x-2)+a)./(1+a);
fun2 = @(x,y) (abs(4.*x-2)+a)./(1+a) .* (abs(4.*y-2)+a)./(1+a);
fun3 = @(x,y,z) (abs(4.*x-2)+a)./(1+a) .* (abs(4.*y-2)+a)./(1+a).* (abs(4.*z-2)+a)./(1+a);



% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = 0;
b_u = 1;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:1
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

```
Elapsed time is 0.121458 seconds.
```

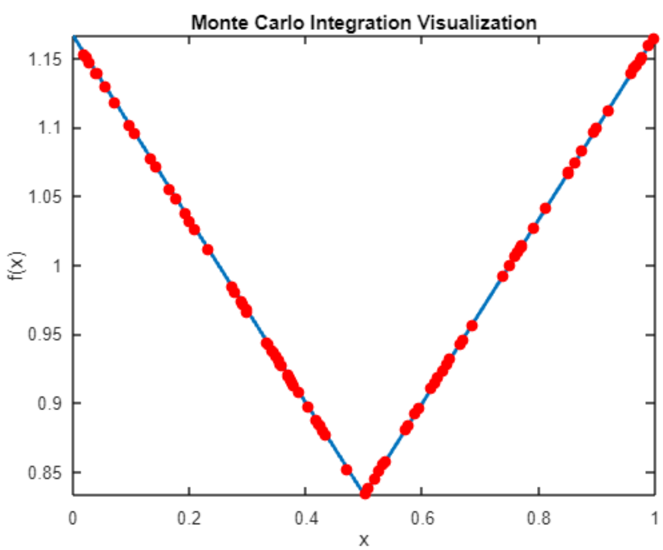
```
disp("1D Monte Carlo Integral:"+monte_value);
```

```
1D Monte Carlo Integral:1.0001
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.014149
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:1
```

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

```
Elapsed time is 0.088652 seconds.
```

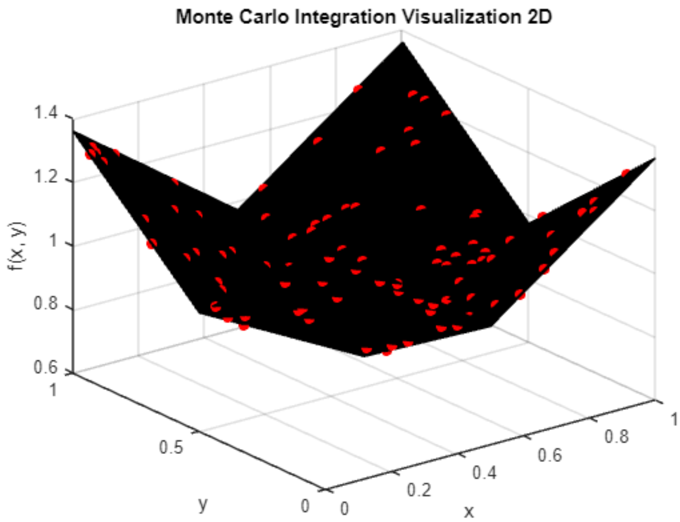
```
disp("2D Monte Carlo Integral:"+monte_value);
```

```
2D Monte Carlo Integral:0.9999
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.0095259
```

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:1
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.127645 seconds.
```

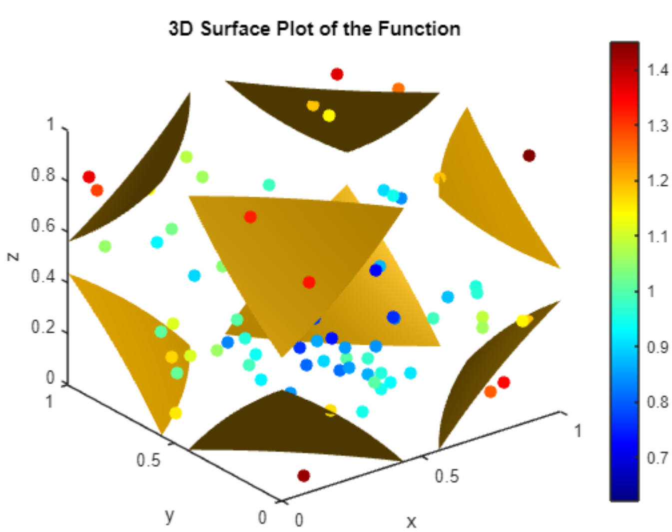
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:1
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.0033476
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Function #6 Monte-Carlo

```
% Define the limits of integration for each dimension
a = 5;


fun1 = @(x) (1+a*x).^(-2);
fun2 = @(x,y)  (1+a*x+a*y).^(-3);
fun3 = @(x,y,z) (1+a*x+a*y+a*z).^(-4);


% Define the limits of integration for each dimension,since for all 3
% variables are the same, we can only define the lower bound and upper
% bound.
a_l = 0;
b_u = 1;

%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral(fun1, a_l, b_u);
disp("matlab:"+matlab);
```

```
matlab:0.16667
```

```
% 1D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun1,a_l,b_u,N);
toc
```

```
Elapsed time is 0.257893 seconds.
```

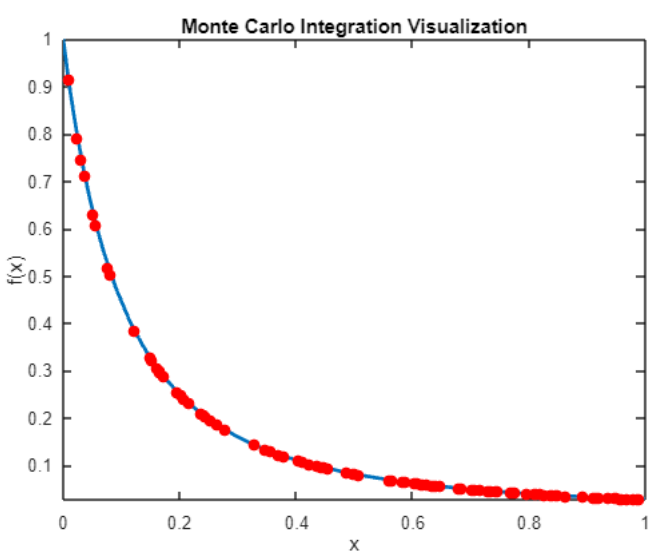
```
disp("1D Monte Carlo Integral:"+monte_value);
```

```
1D Monte Carlo Integral:0.16669
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.016293
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_1D(fun1, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral2(fun2, a_l, b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:0.015152
```

```
% 2D Monte-Carlo Integral
tic
monte_value = monte_carlo_integration(fun2,a_l,b_u,N);
toc
```

```
Elapsed time is 0.239887 seconds.
```

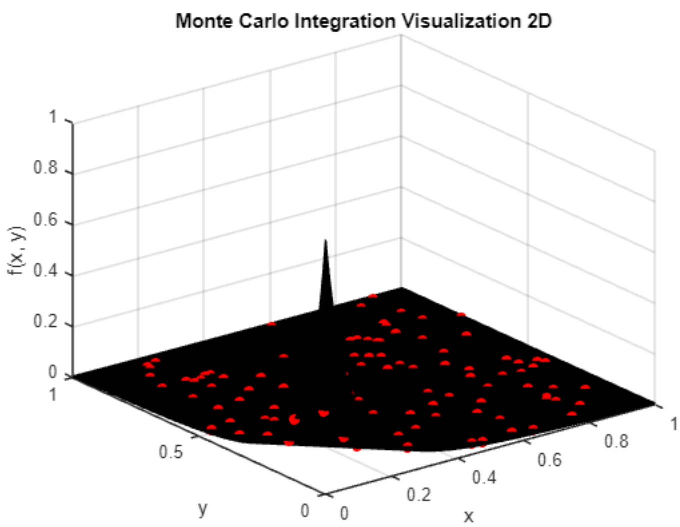
```
disp("2D Monte Carlo Integral:"+monte_value);
```

```
2D Monte Carlo Integral:0.015182
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.1997
```

```
%In order to show the relationsip between each sample and the function
N=100;
monte_carlo_integration_visualization_2D(fun2, a_l, b_u, N);
```




```
%Determine the number of samples
N=1000000;

%Use build-in function to calculate the integral
matlab = integral3(fun3, a_l, b_u,a_l,b_u,a_l,b_u);
disp("matlab:"+matlab);
```

```
matlab:0.00094697
```

```
% 3D Monte-Carlo Integral
tic
monte_value=monte_carlo_integration(fun3,a_l,b_u,N);
toc
```

```
Elapsed time is 0.266365 seconds.
```

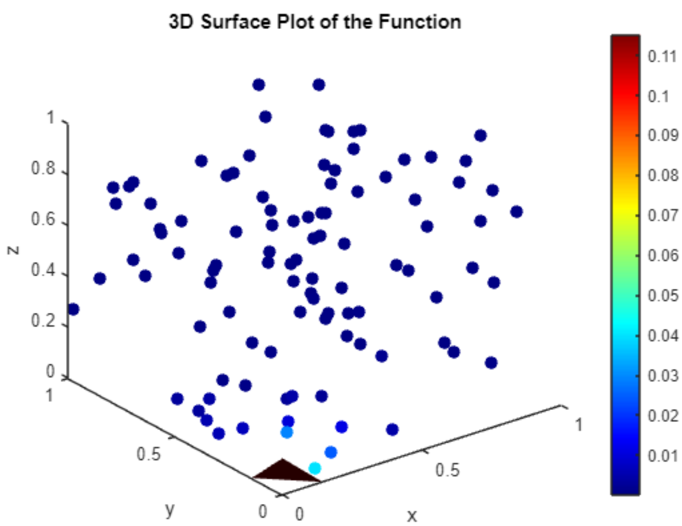
```
disp("3D Monte Carlo Integral:"+monte_value);
```

```
3D Monte Carlo Integral:0.00095435
```

```
%Calculate the absolute value of the relative error
Abserror = abs(monte_value - matlab) / abs(matlab) * 100;
disp("Abserror:"+Abserror);
```

```
Abserror:0.77923
```

```
%In order to show the relationsip between each sample and the function
N = 100;
monte_carlo_integration_visualization_3D(fun3, a_l, b_u, N);
```



Numerical Integration Algorithm Comparison

Function 4

The number of nodes used for Gauss-Legendre is naturally cubed. The array `N_gl` holds the number of nodes in each dimension for Gauss-Legendre, and gives the equivalent number of nodes for the Monte-Carlo algorithm. Like with the previous sections, the time is recorded and the error is compared to the value found with MatLab's `integral3`.

```
N_gl = [5 10 50 100 250 300 600]; % Unidimensional number of nodes
N_mc = N_gl.^3; % Number of nodes for Monte-Carlo
u = 0.5;
a = 5;
fun3 = @(x, y, z) cos(2.*pi.*u + a.*x+a.*y+a.*z); % Function 4 is used here
a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y
a_l = 0; b_u = 1; % Limits for all 3 (M-C)

value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z); % MatLab calculated value for Error

GLError = zeros([1,length(N_gl)]);
MCError = zeros([1,length(N_gl)]);
GLTime = zeros([1,length(N_gl)]);
MCTime = zeros([1,length(N_gl)]);

for i = 1:length(N_gl) % evaluate for each node value
    mcStart = tic;
    value_mc = monte_carlo_integration(fun3, a_l, b_u, N_mc(i));
    MCTime(i) = toc(mcStart);
    MCError(i) = abs(value_mc - value3) / abs(value3) * 100;
    glStart = tic;
    value_GL3 = integral_GL3(fun3,[N_gl(i), N_gl(i), N_gl(i)], a_x, b_x, a_y, b_y, a_z, b_z);
    GLError(i) = abs(value_GL3 - value3) / abs(value3) * 100;
    GLTime(i) = toc(glStart);

end
disp(GLError);
```

```
    0.0042    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000
```

```
disp(MCError);
```

```
  874.6657   21.9151   15.5572   16.3587    1.3946    0.2569    0.7461
```

```
disp(GLTime);
```

```
    0.0067    0.0052    0.0179    0.0919    1.4272    2.5558   20.8358
```


```
disp(GLTime);
```

```
    0.0067    0.0052    0.0179    0.0919    1.4272    2.5558   20.8358
```


Function 3

The same is done for Function 3.

```
N_gl = [5 10 50 100 250 300 400 600];
N_mc = N_gl.^3;
```

```
% Function 3 is used here
fun3 = @(x, y, z) 1 ./ (a.^(-2) + (x - u).^2) .* 1 ./ (a.^(-2) + (y - u).^2) .* 1 ./ (a.^(-2) + (z - u).^2);

a_x = 0; b_x = 1; % Limits for x
a_y = 0; b_y = 1; % Limits for y
a_z = 0; b_z = 1; % Limits for y
a_l = 0; b_u = 1; % Limits for all 3 (M-C)

value3 = integral3(fun3, a_x, b_x, a_y, b_y, a_z, b_z); % MatLab calculated value for error comparison

GLError = zeros([1,length(N_gl)]);
MCError = zeros([1,length(N_gl)]);
GLTime = zeros([1,length(N_gl)]);
MCTime = zeros([1,length(N_gl)]);

for i = 1:length(N_gl)
    mcStart = tic;
    value_mc = monte_carlo_integration(fun3, a_l, b_u, N_mc(i));
    MCTime(i) = toc(mcStart);
    MCError(i) = abs(value_mc - value3) / abs(value3) * 100;
    glStart = tic;
    value_GL3 = integral_GL3(fun3,[N_gl(i), N_gl(i), N_gl(i)], a_x, b_x, a_y, b_y, a_z, b_z);
    GLError(i) = abs(value_GL3 - value3) / abs(value3) * 100;
    GLTime(i) = toc(glStart);

end
disp(GLError);
```

```
   11.2088    0.2173    0.0000    0.0000    0.0000    0.0000    0.0000    0.0000
```

```
disp(MCError);
```

```
    2.8651    2.1959    0.4937    0.0413    0.0542    0.0137    0.0034    0.0010
```

```
disp(GLTime);
```

```
    0.0104    0.0009    0.0675    0.2445    3.8041    6.7257   15.9323   51.9239
```

```
disp(GLTime);
```

```
    0.0104    0.0009    0.0675    0.2445    3.8041    6.7257   15.9323   51.9239
```

Appendix

Gauss-Legendre Integration

Gauss-Legendre Polynomials - Nodes and Weights

```
function [x, w] = gausslegendre(n, a, b)
% Compute the Gauss-Legendre quadrature nodes and weights on the interval [a, b]
% n: Number of quadrature points
% a, b: Lower and upper limits of integration
% From Gaussian Quadrature and the Eigenvalue Problem - John A. Gubner

    sqrtBeta = sqrt(1./(4-1./ [1:n-1].^2));                 % Beta coefficients 1...n-1 which go in the super and subdiagona
    % Note: alpha coefficent are all 0, hence they are not explicitly calculated.
    jacobiMatrix = diag(sqrtBeta,1) + diag(sqrtBeta,-1);      % Jacobi Matrix containing the sqrtBeta in super and subdiagonal
    [V, Lambda] = eig(jacobiMatrix);                        % Perform Eigen decomposition of jacobiMatrix.
    [x, i] = sort(diag(Lambda));                            % Genetrate Nodes and Index | Corresponding to the eigen values
    w = 2 * V(1,i).^2;                                      % Calculate the weights by the Golub-Welsh Algorithm.
    x = (b - a) / 2 * x + (a + b) / 2;                      % Rescales the Nodes for domain [a,b].
    w = (b - a) / 2 * w;                                    % Rescales the Weigths for domain [a,b].
end
```

Gauss-Legendre 1D Integration

```
function integral_value = integral_GL1(func, num_points, a_x, b_x)
% Compute the 1D integral using Gauss-Legendre quadrature on the interval [a_x, b_x]
% [num_points]: Number of quadrature points
% a_x, b_x:     Lower and upper limits of integration

    num_points_x = num_points(1);                            % Extract number of points from array.

    [x, w_x] = gausslegendre(num_points_x, a_x, b_x);        % Generate x -> nodes and w_x -> weights for 1D


    integral_value = 0;                                      % Initilize integral sum
    for i = 1:num_points_x                                   % Loop from 1 to number of nodes

        integral_value = integral_value + func(x(i)) * w_x(i); % update integral sum <- sum + f(x) * w
    end
end
```

Gauss-Legendre 2D Integration

```
function integral_value = integral_GL2(func, num_points, a_x, b_x, a_y, b_y)
% Compute the 2D integral using Gauss-Legendre quadrature on the interval [a_x, b_x] X [a_y, b_y]
% [num_points] -> [num_points_x,num_points_y] : Number of quadrature points
% a_x, b_x:     Lower and upper limits of integration for x-axis
% a_y, b_y:     Lower and upper limits of integration for y-axis


    num_points_x = num_points(1);                                               % Extract number of points in X from arr
    num_points_y = num_points(2);                                               % Extract number of points in Y from arr

    [x, w_x] = gausslegendre(num_points_x, a_x, b_x);                           % Generate x -> nodes in X and w_x -> we
    [y, w_y] = gausslegendre(num_points_y, a_y, b_y);                           % Generate y -> nodes in Y and w_y -> we


    integral_value = 0;                                                         % Initilize integral sum
    % Iterations = num_points_x * num_points_y
    for i = 1:num_points_x                                                      % Loop1 from 1 to number of nodes
        for j = 1:num_points_y                                                  % Loop2 from 1 to number of nodes
            integral_value = integral_value + func(x(i), y(j)) * w_x(i) * w_y(j); % update integral sum <- sum + f(x,y) *
        end
    end
end
```

Gauss-Legendre 3D Integration

```
function integral_value = integral_GL3(func, num_points, a_x, b_x, a_y, b_y, a_z, b_z)
% Compute the 3D integral using Gauss-Legendre quadrature on the interval [a_x, b_x] X [a_y, b_y] X [a_z, b_z]
% [num_points] -> [num_points_x,num_points_y,num_points_z] : Number of quadrature points
% a_x, b_x:     Lower and upper limits of integration for x-axis
% a_y, b_y:     Lower and upper limits of integration for y-axis
% a_z, b_z:     Lower and upper limits of integration for z-axis
    num_points_x = num_points(1);                                                            % Extract number of poi
    num_points_y = num_points(2);                                                            % Extract number of poi
    num_points_z = num_points(3);                                                            % Extract number of poi

    [x, w_x] = gausslegendre(num_points_x, a_x, b_x);                                        % Generate x -> nodes i
    [y, w_y] = gausslegendre(num_points_y, a_y, b_y);                                        % Generate y -> nodes i
    [z, w_z] = gausslegendre(num_points_z, a_z, b_z);                                        % Generate z -> nodes i


    integral_value = 0;                                                                      % Initilize integral su
    % Iterations = num_points_x * num_points_y * num_points_z
    for i = 1:num_points_x                                                                   % Loop1 from 1 to numbe
        for j = 1:num_points_y                                                               % Loop2 from 1 to numbe
            for k = 1:num_points_z                                                           % Loop3 from 1 to numbe
                integral_value = integral_value + func(x(i), y(j), z(k)) * w_x(i) * w_y(j) * w_z(k);   % update integral sum <
            end
        end
    end
end
```

Gauss-Legendre Plots

Gauss-Legendre 1D Plot

```

function plot_1D( f, lower_limit_x, upper_limit_x)
    x_values = linspace(lower_limit_x, upper_limit_x, 100);
    y_values = f(x_values);

    % Plot the surface
    figure;
    plot(x_values, y_values);
    hold on

    % Labeling and title
    xlabel('x');

    zlabel('f(x)');
    title('Function Integrated using Gauss-Legendre - 1D Version');
    hold off
end
```

Gauss-Legendre 2D Plot

```
function plot_2D( f, lower_limit_x, upper_limit_x, lower_limit_y, upper_limit_y)
    [x_values, y_values] = meshgrid(linspace(lower_limit_x, upper_limit_x, 100), linspace(lower_limit_y, upper_limit_y, 100));
    z_values = f(x_values, y_values);
```

```
    % Plot the surface
    figure;
    surf(x_values, y_values, z_values);
    hold on

    % Labeling and title
    xlabel('x');
    ylabel('y');
    zlabel('f(x, y)');
    title('Function Integrated using Gauss-Legendre - 2D Version');
    hold off
end
```

Gauss-Legendre 3D Plot

```
function plot_3D(f, lower_limit_x, upper_limit_x, lower_limit_y, upper_limit_y, lower_limit_z, upper_limit_z)
    x = linspace(lower_limit_x, upper_limit_x, 100);
    y = linspace(lower_limit_y, upper_limit_y, 100);
    z = linspace(lower_limit_z, upper_limit_z, 100);

    [X, Y, Z] = meshgrid(x, y, z);

    % Generate slicing planes
    xslices = linspace(lower_limit_x, upper_limit_x, 3);
    yslices = linspace(lower_limit_y, upper_limit_y, 3);
    zslices = linspace(lower_limit_z, upper_limit_z, 3);

    % Plot the slices
    figure;
    for i = 1:length(xslices)
        for j = 1:length(yslices)
            for k = 1:length(zslices)
                slice(X, Y, Z, f(X, Y, Z), xslices(i), yslices(j), zslices(k));
                hold on; % To overlay multiple slices
            end
        end
    end
    xlabel('X');
    ylabel('Y');
    zlabel('Z');
    title('Function Integrated using Gauss-Legendre - 3D Version');
    colorbar;
    clim([min(f(X, Y, Z), [], 'all'), max(f(X, Y, Z), [], 'all')]);
    cbar = colorbar;
    cbar.Label.String = 'f(x, y, z)';
    hold off;
end
```

Monte-Carlo Based Integration

```
function value = monte_carlo_integration(f,a,b,N)
    %f is the function to be integrated
    %a is lower bound
    %b is upper bound
    %N is the number of samples

    sum = 0; % Initialize the sum
    num_vars = nargin(f); %Determine the dimension of the function
    switch num_vars

        case 1

            X=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound

            %Update the value of the sum
            for i = 1:N
                k = f(X(i));
                sum = sum + k;
            end

            value = (sum*(b-a))/N; %Use the formula (volume/Total number)*Summation




        case 2
            X=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound
            Y=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound
```

```
            %Update the value of the sum
            for i = 1:N
                k = f(X(i),Y(i));
                sum = sum + k;
            end

            value = (sum*(b-a)*(b-a))/N; %Use the formula (volume/Total number)*Summation




        case 3
            X=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound
            Y=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound
            Z=rand(1,N)*(b-a)+a; %Randomly generate the points from lower bound to upper bound

             %Update the value of the sum
            for i = 1:N
                k = f(X(i),Y(i),Z(i));
                sum = sum + k;
            end

            value = (sum*(b-a)*(b-a)*(b-a))/N; %Use the formula (volume/Total number)*Summation




        otherwise
            disp("Unsupported number of variables!");
    end
end
```

Monte-Carlo Plots

Monte-Carlo 1D Plot

```
function monte_carlo_integration_visualization_1D(f, a, b, N)
    % Generate the function graph and the random sampling points within the interval [a, b] for each variable
    X = rand(1, N) * (b - a) + a;

    % Calculate function values at the sampling points
    Y = f(X);

    % Plot the function
    figure;
    fplot(f, [a, b], 'LineWidth', 2); % Plot the function
    hold on;

    % Plot the sampled points on the function curve
    plot(X, Y, 'ro', 'MarkerFaceColor', 'r'); % Plot the samples

    % Title and labels
    title('Monte Carlo Integration Visualization');
    xlabel('x');
    ylabel('f(x)');

    % Hold off to release the plot
    hold off;
end
```

Monte-Carlo 2D Plot

```
function monte_carlo_integration_visualization_2D(f, a, b, N)
    % Generate the function graph and the random sampling points within the interval [a, b] for each variable
    X_rand = rand(1, N) * (b - a) + a;
    Y_rand = rand(1, N) * (b - a) + a;

    % Calculate function values at the sampling points
    Z_rand = f(X_rand, Y_rand);

    % Plot the function surface
    figure;
    x = linspace(a, b, 1000);
    y = linspace(a, b, 1000);
    [X, Y] = meshgrid(x, y);
    Z = f(X, Y);
    surf(X, Y, Z);
    title('3D Surface Plot of the Function');
    xlabel('x');
    ylabel('y');
```

```
    zlabel('f(x, y)');
    hold on;

    % Plot the sampled points on the function surface
    scatter3(X_rand, Y_rand, Z_rand, 'filled', 'MarkerFaceColor', 'r'); % Plot the samples

    % Set title and labels
    title('Monte Carlo Integration Visualization 2D');
    xlabel('x');
    ylabel('y');
    zlabel('f(x, y)');

    hold off;
end
```

Monte-Carlo 3D plot

```
function monte_carlo_integration_visualization_3D(f, a, b, N)
    % Generate the function graph and the random sampling points within the interval [a, b] for each variable
    X_rand = rand(1, N) * (b - a) + a;
    Y_rand = rand(1, N) * (b - a) + a;
    Z_rand = rand(1, N) * (b - a) + a;
    W_rand = f(X_rand, Y_rand, Z_rand);

    % Plot the function surface
    figure;
    x = linspace(a, b, 100);
    y = linspace(a, b, 100);
    z = linspace(a, b, 100);
    [X, Y, Z] = meshgrid(x, y, z);
    W = f(X, Y, Z);
    isosurface(X, Y, Z, W);
    title('3D Surface Plot of the Function');
    xlabel('x');
    ylabel('y');
    zlabel('z');
    hold on;

    % Plot sampled points with color representing the fourth dimension (W)
    scatter3(X_rand, Y_rand, Z_rand, 50, W_rand, 'filled');
    colormap(jet);
    colorbar;

    hold off;
end
```

Smolyak Sparse Grids (Extra)

Smolyak Sparse Grids

```
function A = genSets(mu, d)
%   generate sets of CGL points unidimensional
%   max number of pts would be d + mu: lowest i combo is 1 + i = d + mu
    i = d + mu - 1;
    A = cell(1, i); % creates a i x i
    A{1} = cglnodes(1);
    A{2} = [-1;1];

    for idx = 3:i
        prevSet = cglnodes(2^(idx-2) + 1);
        curSet = cglnodes(2^(idx-1) + 1);
        A{idx} = setdiff(curSet, prevSet);
    end

end

function [sgPoints] = tensorCombo(A, d)
%   A is the cell with the sets you want to combine, in order of dimensions
%   A{1} = x sets, A{2} = y sets, A{3} = z sets
    n = length(A{1});
    m = length(A{2});
    if (d == 2)
        l = 1;
    else
        l = length(A{3});
    end

    sgPoints = cell(n*m*l,1);
    idx = 1;
```

```matlab
    for i = 1:l
        for j = 1:m
            for k = 1:n
                if (d == 2)
                    sgPoints{idx} = [A{1}(k) A{2}(j)];
                else
                    sgPoints{idx} = [A{1}(k) A{2}(j) A{3}(i)];
                end
                idx = idx + 1;
            end
        end
    end
    sgPoints = cell2mat(sgPoints);
end


function S = chooseSets(A, I)
%   A is the cell with the unidimensional point sets
%   I is a vector with the i_1, i_2, ... i_n indices
    d = length(I);
    S = {};
    for i = 1:d
        S = [S; A{I(i)}];
    end

end


function x = cglnodes(N)
% From Appendix A of Judd. K.-L. et al. (2014)
    if (N == 1)
        x = 0;
        return;
    end
    x = -cos(pi*(0:N-1)/(N-1))';
end

function I = findIs(mu, d)
%   the largest value any can take for I is mu + 1
%   to satisfy d ≤ sum(i_n) ≤ d + mu
    if (d == 2)
        l = 1;
    else
        l = mu + 1;
    end

    I = [];

    for i = 1:l
        for j = 1:mu + d
            for k = 1:mu + d
                iSums = i + j + k;
                if (iSums >= d) && (iSums <= mu + d)
                    % satisfies condition
                    I = [I; k j i];
                end
            end
        end
    end

    if (d == 2)
        I = I(:, 1:2);
    end
end

function S = smolyakPoints(mu, dim)
    mu = mu + 1;
    A = genSets(mu,dim);
    I = findIs(mu,dim);

    for i = 1:size(I, 1)
        curSets = chooseSets(A, I(i, :));
        StoAppend = tensorCombo(curSets, dim);
        if (i == 1)
        S = StoAppend;
        else
        S = [S; StoAppend];
        end


    end
```

```
end
```

Smolyak Sparse Grids - Graphs

```
function plotSmolyak2D()

muRange = 2:5;
for i = 1:4
    subplot(2,2,i);
    s = smolyakPoints(muRange(i), 2);

    scatter(s(:,1), s(:,2), 25, 'filled');
    title(sprintf('mu = %d', muRange(i)));
end

end

function plotSmolyak3D()

muRange = 2:5;
for i = 1:4
    subplot(2,2,i);
    s = smolyakPoints(muRange(i), 3);

    scatter3(s(:,1), s(:,2), s(:,3), 25, 'filled');
    title(sprintf('mu = %d', muRange(i)));
end

end
```