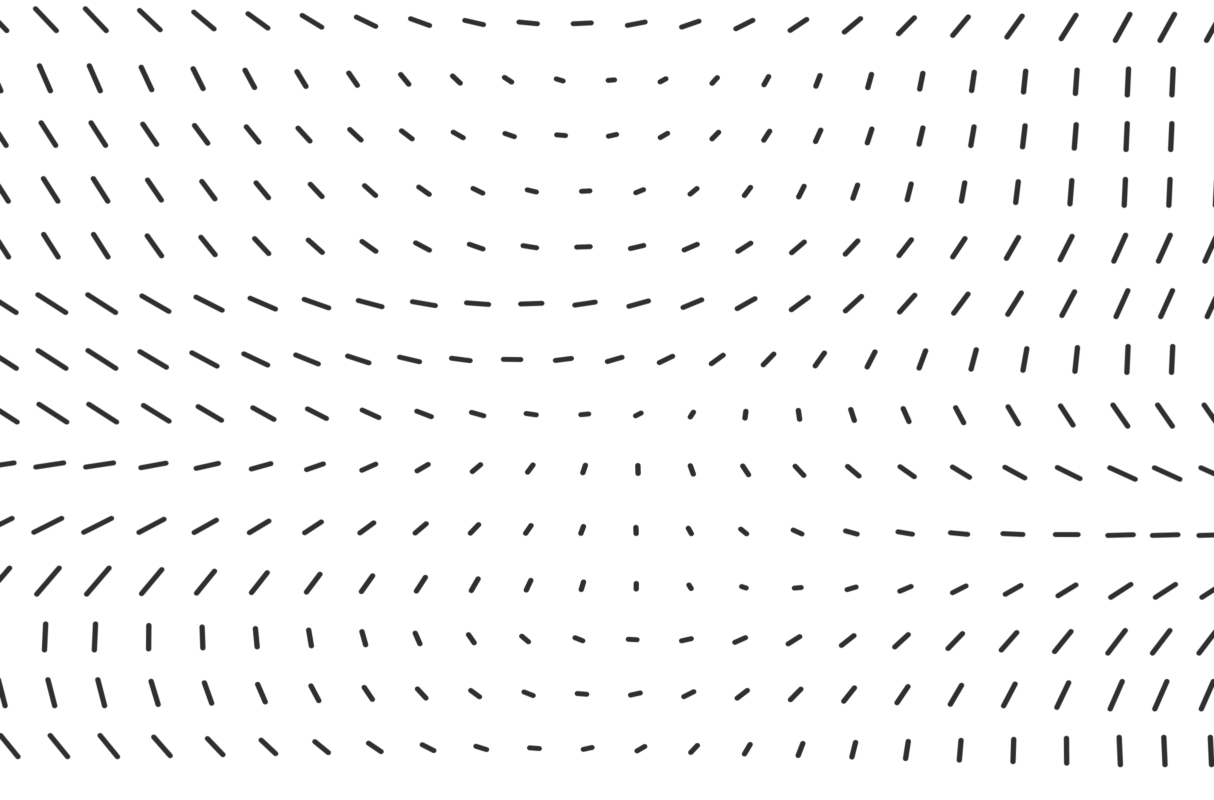


1.3 Encuentre la función de densidad espectral (transformada de Fourier) para las siguientes señales (sin aplicar propiedades): a) $e^{-a|t|}$, $a\in\mathbb{R}^+$; b) $\cos(w_c t)$, $w_c\in\mathbb{R}$; c) $\sin(w_s t)$, $w_s\in\mathbb{R}$; d) $f(t)\cos(w_c t)$, $w_c\in\mathbb{R}$, $f(t)\in\mathbb{R},\mathbb{C}$; e) $e^{-a|t|^2}$, $a\in\mathbb{R}^+$. f) $A\mathrm{rect}_d(t)$, $A, d\in\mathbb{R}$.

$$X(\omega)=F\{x(t)\}=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}\,\partial t$$

$$;\ F\{e^{j\omega_0 t}\}=2\pi\delta(\omega-\omega_0)$$

a) $x(t)=e^{-a|t|}$, $a\in\mathbb{R}^+$ Para $|t|$ se puede expresar como $-t\in(-\infty,0)$ y $t\in(0,\infty)$

$$F\{e^{-a|t|}\}=\int_{-\infty}^{\infty}e^{-a|t|}e^{-j\omega t}\partial t=\int_{-\infty}^{0}e^{-a\cdot -t}e^{-j\omega t}\partial t+\int_{0}^{\infty}e^{-a\cdot t}e^{-j\omega t}\partial t=\int_{-\infty}^{0}e^{(a-j\omega)t}\partial t+\int_{0}^{\infty}e^{-(a+j\omega)t}\partial t=\left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0}+\left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$=\frac{1}{a-j\omega}(e^{0}-e^{-\infty})-\frac{1}{(a+j\omega)}(e^{-\infty}-e^{0})=\frac{1}{a-j\omega}\Big(1-\underbrace{\cancel{\tfrac{1}{e^{\infty}}}}_{0}\Big)-\frac{1}{(a+j\omega)}\Big(\underbrace{\cancel{\tfrac{1}{e^{\infty}}}}_{0}-1\Big)=\frac{1}{a-j\omega}+\frac{1}{a+j\omega}=\frac{a+\cancel{j\omega}+a-\cancel{j\omega}}{a^2-\underbrace{j^2}_{-1}\omega^2}=\frac{2a}{a^2+\omega^2}$$

$$X(\omega)=\frac{2a}{a^2+\omega^2}$$

b) $x(t)=\cos(\omega_c t)$; $\omega_c\in\mathbb{R}$ $\quad$ Si $\cos(\omega_c t)=\dfrac{e^{j\omega_c t}+e^{-j\omega_c t}}{2}$

$$F\{\cos(\omega_c t)\}=\int_{-\infty}^{\infty}\cos(\omega_c t)e^{-j\omega t}\partial t=\int_{-\infty}^{\infty}\frac{e^{j\omega_c t}+e^{-j\omega_c t}}{2}e^{-j\omega t}\partial t=\frac{1}{2}\int_{-\infty}^{\infty}e^{j\omega_c t}e^{-j\omega t}\partial t+\frac{1}{2}\int_{-\infty}^{\infty}e^{-j\omega_c t}e^{-j\omega t}\partial t=\frac{1}{2}\int_{-\infty}^{\infty}e^{-(\omega-\omega_c)jt}\partial t+\frac{1}{2}\int_{-\infty}^{\infty}e^{-(\omega+\omega_c)jt}\partial t$$

$$=\frac{1}{\cancel{2}}\cdot\cancel{2}\pi\delta(\omega-\omega_c)+\frac{1}{\cancel{2}}\cdot\cancel{2}\pi\delta(\omega+\omega_c)=\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$$

$$\int_{-T}^{T}e^{-\overbrace{(\omega-\omega_0)}^{\alpha}jt}\partial t=2\pi\delta(\omega-\omega_0)$$

$$\int_{-T}^{T}e^{-\alpha jt}\partial t=\frac{e^{-j\alpha T}-e^{j\alpha T}}{-j\alpha}=\frac{(e^{j\alpha T}-e^{-j\alpha T})\cdot 2}{j\alpha\cdot 2}$$

$$=\frac{2\,\mathrm{Sen}(\alpha T)}{\alpha}$$

$$\text{Si } \lim_{T\to\infty}\frac{\mathrm{Sen}(\alpha T)}{\pi\alpha}=\delta(\alpha)\to\lim_{T\to\infty}\frac{2\,\mathrm{Sen}(\alpha T)}{\alpha}=2\pi\delta(\alpha)$$

$$\text{Si } \alpha=\omega-\omega_c\to 2\pi\delta(\omega-\omega_c)$$

$$\text{Si } \alpha=\omega+\omega_c\to 2\pi\delta(\omega+\omega_c)$$

$$X(\omega)=\pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]$$

c) $x(t)=\mathrm{sen}(\omega_s t)$; $\omega_s\in\mathbb{R}$ $\quad$ Si $\mathrm{sen}(\omega_s t)=\dfrac{e^{j\omega_s t}-e^{-j\omega_s t}}{j2}$

$$\int_{-\infty}^{\infty}e^{-(\omega-\omega_0)jt}\partial t=2\pi\delta(\omega-\omega_0)$$

$$F\{\mathrm{sen}(\omega_s t)\}=\int_{-\infty}^{\infty}\mathrm{sen}(\omega_s t)e^{-j\omega t}\partial t=\int_{-\infty}^{\infty}\frac{e^{j\omega_s t}-e^{-j\omega_s t}}{j2}e^{-j\omega t}\partial t=\frac{1}{j2}\int_{-\infty}^{\infty}e^{j\omega_s t}e^{-j\omega t}\partial t-\frac{1}{j2}\int_{-\infty}^{\infty}e^{-j\omega_s t}e^{-j\omega t}\partial t=\frac{1}{j2}\int_{-\infty}^{\infty}e^{-(\omega-\omega_s)jt}\partial t-\frac{1}{j2}\int_{-\infty}^{\infty}e^{-(\omega+\omega_s)jt}\partial t$$

$$=\frac{1}{j\cancel{2}}\cdot\cancel{2}\pi\delta(\omega-\omega_s)-\frac{1}{j\cancel{2}}\cdot\cancel{2}\pi\delta(\omega+\omega_s)=\frac{\pi}{j}[\delta(\omega-\omega_s)-\delta(\omega+\omega_s)]=\frac{j}{j}\cdot\frac{\pi}{j}[\delta(\omega-\omega_s)-\delta(\omega+\omega_s)]=\frac{j\pi}{\underbrace{j^2}_{-1}}[\delta(\omega-\omega_s)-\delta(\omega+\omega_s)]$$

$$=-j\pi[\delta(\omega-\omega_s)-\delta(\omega+\omega_s)]$$

$$=j\pi[\delta(\omega+\omega_s)-\delta(\omega-\omega_s)]$$

$$X(\omega)=j\pi[\delta(\omega+\omega_s)-\delta(\omega-\omega_s)]$$

d) $X(t)=f(t)\cos(w_ct);\ w_c\in\mathbb{R},\ f(t)\in\mathbb{R},\mathbb{C}$

$$F\{f(t)\cos(w_ct)\}=\int_{-\infty}^{\infty}f(t)\cos(w_ct)e^{-jwt}\partial t=\int_{-\infty}^{\infty}f(t)\frac{e^{jw_ct}+e^{-jw_ct}}{2}e^{-jwt}\partial t=\frac{1}{2}\int_{-\infty}^{\infty}f(t)(e^{jw_ct}+e^{-jw_ct})e^{-jwt}\partial t=\frac{1}{2}\int_{-\infty}^{\infty}f(t)\,e^{jw_ct}e^{-jwt}\partial t+\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-jw_ct}e^{-jwt}\partial t$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-(w-w_c)jt}\partial t+\frac{1}{2}\int_{-\infty}^{\infty}f(t)e^{-(w+w_c)jt}\partial t$$

Si $\int_{-\infty}^{\infty}f(t)e^{-j(w-w_c)t}\partial t=F(w-w_c)$ y $\int_{-\infty}^{\infty}f(t)e^{-j(w+w_c)t}\partial t=F(w+w_c)$

Es la transformada de t desplazada.

$$=\frac{1}{2}F(w-w_c)+\frac{1}{2}F(w+w_c)$$

$$=\frac{1}{2}\big(F(w-w_c)+F(w+w_c)\big)$$

$$X(w)=\frac{1}{2}\big(F(w-w_c)+F(w+w_c)\big)$$

e) $X(t)=e^{-a|t|^2};\ a\in\mathbb{R}^+$ Para $a>0$

$$F\{e^{-at^2}\}=\int_{-\infty}^{\infty}e^{-at^2}e^{-jwt}\partial t$$

$$-at^2-jwt=-a\left(t^2+\frac{jw}{a}t\right)\qquad t^2+\frac{jw}{a}t=\left(t+\frac{jw}{2a}\right)^2-\left(\frac{jw}{2a}\right)^2$$

$$=-a\left(\left(t+\frac{jw}{2a}\right)^2-\left(\frac{jw}{2a}\right)^2\right)$$

$$=-a\left(\left(t+\frac{jw}{2a}\right)^2-\frac{j^2w^2}{2^2a^2}\right)=-a\left(\left(t+\frac{jw}{2a}\right)^2-\frac{-1\cdot w^2}{4a^2}\right)=-a\left(t+\frac{jw}{2a}\right)^2+\cancel{a}\frac{-1\cdot w^2}{4a^{\cancel{2}}}=-a\left(t+\frac{jw}{2a}\right)^2-\frac{w^2}{4a}$$

$$=\int_{-\infty}^{\infty}e^{-at^2-jwt}\partial t$$

$$=\int_{-\infty}^{\infty}e^{-\frac{w^2}{4a}}e^{-a\left(t+\frac{jw}{2a}\right)^2}\partial t$$

$$=e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty}e^{-a\left(t+\frac{jw}{2a}\right)^2}\partial t$$

Si $u=t+\frac{jw}{2a}$ $\quad\partial u=\partial t$

$$=e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty}e^{-au^2}\partial u$$

$$I=\int_{-\infty}^{\infty}e^{-au^2}\partial u$$

En coordenadas polares $x=r\cos\theta,\ y=r\,\text{sen}\,\theta,\ \partial x\partial y=r\partial r\partial\theta$
$r\in[0,\infty),\ \theta\in[0,2\pi)$

$$I^2=\left(\int_{-\infty}^{\infty}e^{-ax^2}\partial x\right)\left(\int_{-\infty}^{\infty}e^{-ay^2}\partial y\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-a(x^2+y^2)}\partial x\partial y$$

$$I^2=\int_0^{2\pi}\int_0^{\infty}e^{-ar^2}r\partial r\partial\theta=2\pi\int_0^{\infty}re^{-ar^2}\partial r$$

$s=ar^2\quad \partial s=2ar\partial r\rightarrow r\partial r=\frac{\partial s}{2a}$

Cuando $r:0\to\infty$
$s:0\to\infty$

$$I^2=2\pi\int_0^{\infty}e^{-s}\frac{\partial s}{2a}=\frac{2\pi}{2a}\int_0^{\infty}e^{-s}\partial s=\frac{\pi}{a}\cdot\left(-\cancelto{0}{\frac{1}{e^{\infty}}}-(-1)\right)=\frac{\pi}{a}\longrightarrow I=\sqrt{\frac{\pi}{a}}$$

$$=e^{-\frac{w^2}{4a}}\sqrt{\frac{\pi}{a}}$$

$$X(w)=e^{-\frac{w^2}{4a}}\sqrt{\frac{\pi}{a}}$$

f) $X(t)=A\,\text{rect}_d(t);\ A,d\in\mathbb{R}$ Con $\text{rect}_d(t)=\begin{cases}1 & |t|\leq d/2\\ 0 & |t|>d/2\end{cases}$

Si $\text{rect}_d(t)=1$ Solo en $[-d/2,d/2]$

$$F\{A\,\text{rect}_d(t)\}=\int_{-\infty}^{\infty}A\,\text{rect}_d(t)e^{-jwt}\partial t$$

$$=\int_{-d/2}^{d/2}A\cdot1e^{-jwt}\partial t=A\cdot\left(\frac{e^{-jwt}}{-jw}\right)\Bigg|_{-d/2}^{d/2}=-\frac{A}{jw}\left(e^{-jw\cdot d/2}-e^{-jw\cdot-d/2}\right)=\frac{A}{w}\frac{2\cdot\left(e^{-jw\cdot-d/2}-e^{-jw\cdot d/2}\right)}{2\cdot j}=\frac{2A}{w}\,\text{Sen}(wd/2)$$

$$X(w)=\frac{2A}{w}\cdot\text{Sen}(wd/2)$$

1.4 Aplique las propiedades de la transformada de Fourier para resolver: a) $\mathscr{F}\{e^{-jw_1 t}\cos(w_c t)\}$, $w_1, w_c \in \mathbb{R}$; b) $\mathscr{F}\{u(t)\cos^2(w_c t)\}$, $w_c \in \mathbb{R}$; c) $\mathscr{F}^{-1}\{\frac{7}{w^2+6w+45} * \frac{10}{(8+jw/3)^2}\}$, d) $\mathscr{F}\{3t^3\}$, e) $\frac{B}{T}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{a^2+(w-n\omega_o)^2}+\frac{1}{a+j(w-n\omega_o)}\right)$, donde $n \in \{0, \pm 1, \pm 2, \dots\}$, $\omega_o = 2\pi/T$ y $B, T \in \mathbb{R}^+$. Ver Tablas de propiedades y Tablas transformada de Fourier.

$$F\{X(t)\} = X(w) = \int_{-\infty}^{\infty} X(t) e^{-jwt} \partial t$$

$$F^{-1}\{X(w)\} = X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w) e^{jwt} \partial w$$

a) $F[e^{-jw_1 t}\cos(w_c t)]$; $w_1, w_c \in \mathbb{R}$ Si $\cos(w_c t) = \frac{e^{jw_c t}+e^{-jw_c t}}{2}$

$$e^{-jw_1 t}\cos(w_c t) = \frac{e^{jw_c t}+e^{-jw_c t}}{2}\cdot e^{-jw_1 t}\partial t = \frac{1}{2}\left(e^{jw_c t}e^{-jw_1 t}+e^{-jw_c t}e^{-jw_1 t}\right) = \frac{1}{2}\left(e^{-jt(w_1-w_c)}+e^{-jt(w_1+w_c)}\right)$$

Si $F[e^{jw_0 t}] = 2\pi\delta(w-w_0)$

$$F\left\{\frac{1}{2}\left(e^{-jt(w_1-w_c)}+e^{-jt(w_1+w_c)}\right)\right\} = \frac{1}{2}\left(F\left[e^{-jt(w_1-w_c)}\right]+F\left[e^{-jt(w_1+w_c)}\right]\right) = \frac{1}{2}\left(2\pi\delta(w-(-w_1+w_c))+2\pi\delta(w-(-w_1-w_c))\right)$$

$$= \frac{2\pi}{2}\left(\delta(w+w_1-w_c)+\delta(w+w_1+w_c)\right) = \pi\left(\delta(w+w_1-w_c)+\delta(w+w_1+w_c)\right)$$

$$X(w) = \pi\left(\delta(w+w_1-w_c)+\delta(w+w_1+w_c)\right)$$

b) $F\{u(t)\cos^2(w_c t)\}$ $w_c \in \mathbb{R}^+$ Si $u(t) = 1$ de $[0,\infty)$

$$\cos^2(w_c t) = \frac{1+\cos(w_c t)}{2} \; ; \; u(t)\cos^2(w_c t) = \frac{1}{2}u(t) + \frac{1}{2}u(t)\cos(2w_c t)$$

Si $F[u(t)] = F[sgn(t)] + F[1/2] = \frac{1}{jw} + \pi\delta(w)$

$F[u(t)\cos(w_0 t)] = \frac{\pi}{2}(\delta(w-w_0)+\delta(w+w_0)) + \frac{jw}{w_0^2-w^2}$

$$F[u(t)\cos^2(w_c t)] = \frac{1}{2}F[u(t)] + \frac{1}{2}F[u(t)\cos(2w_c t)]$$

$$= \frac{1}{2}\left(F[u(t)] + F[u(t)\cos(2w_c t)]\right)$$

$$= \frac{1}{2}\left(\frac{1}{jw} + \pi\delta(w) + \frac{\pi}{2}(\delta(w-2w_c)+\delta(w+2w_c)) + \frac{jw}{(2w_c)^2-w^2}\right)$$

$$X(w) = \frac{1}{2}\left(\frac{1}{jw} + \pi\delta(w) + \frac{\pi}{2}(\delta(w-2w_c)+\delta(w+2w_c)) + \frac{jw}{(2w_c)^2-w^2}\right)$$

c) $F^{-1}\left\{\frac{7}{w^2+6w+45} * \frac{10}{(8+jw/3)^2}\right\}$

Producto en frecuencia ⟷ convolucion en t

$F^{-1}\{G(w)H(w)\}(t) = \frac{1}{2\pi}(g*h)(t)$

$g(t) = F^{-1}\{G(w)\}$ y $h(t) = F^{-1}\{H(w)\}$

$G(w) = \frac{7}{w^2+6w+45}$ → $w^2+6w+45 = (w+3)^2+36 = (w+3)^2+6^2$

$G(w) = \frac{7}{(w+3)^2+6^2}$ → Si $e^{-a|t|} \leftrightarrow \frac{2a}{a^2+w^2}$; $\frac{1}{(w-w_0)^2+a^2} \leftrightarrow \frac{1}{2a}e^{-a|t|}e^{jw_0 t}$

→ $(w+3)^2+6^2 = (w-(-3))^2+6^2$

$$G(w) = \frac{7}{(w-(-3))^2+6^2}$$

$$g(t) = F^{-1}\left\{\frac{7}{w-(-3))^2+6^2}\right\} = \frac{7}{2\cdot 6}e^{-6|t|}e^{-j3t} = \frac{7}{12}e^{-6|t|}e^{-j3t}$$

$$H(w)=\frac{10}{(8+jw/3)^2} \longrightarrow 8+\frac{jw}{3}=\frac{1}{3}(24+jw)$$
$$\left(8+\frac{jw}{3}\right)^2=\frac{1}{9}(24+jw)^2$$

$$H(w)=\frac{10}{\frac{1}{9}(24+jw)^2}$$

$$H(w)=90\cdot\frac{1}{(24+jw)^2} \longrightarrow \text{Si } F\{te^{-at}u(t)\}=\frac{1}{(a+jw)^2} \quad \text{Para } a>0$$

$$h(t)=F^{-1}\left\{90\cdot\frac{1}{(24+jw)^2}\right\}=90te^{-24t}u(t)$$

$$x(t)=F^{-1}\{G\cdot H\}=\frac{1}{2\pi}(g*h)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\tau)h(t-\tau)\partial\tau$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{7}{12}e^{-6|\tau|}e^{-j3\tau}\cdot 90(t-\tau)e^{-24(t-\tau)}u(t-\tau)\partial\tau$$

$$x(t)=\frac{1}{2\pi}\cdot\frac{7\cdot 90}{12}\int_{-\infty}^{\infty}e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}u(t-\tau)\partial\tau$$

$$x(t)=\frac{105}{4\pi}\int_{-\infty}^{\infty}e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}u(t-\tau)\partial\tau \longrightarrow \text{Si } u(t-\tau)=1 \text{ en } (-\infty,\tau]$$

$$x(t)=\frac{105}{4\pi}\int_{-\infty}^{t}e^{-6|\tau|}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\partial\tau$$

$$x(t)=\frac{105}{4\pi}\left(\int_{-\infty}^{0}e^{6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\partial\tau+\int_{0}^{t}e^{-6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\partial\tau\right)$$

$$x(t)=\frac{105}{4\pi}\left(\int_{-\infty}^{0}e^{6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\partial\tau+\int_{0}^{t}e^{-6\tau}e^{-j3\tau}(t-\tau)e^{-24(t-\tau)}\partial\tau\right)$$

$$x(t)=\frac{105}{4\pi}e^{-24t}\left(\int_{-\infty}^{0}e^{6\tau}e^{-j3\tau}(t-\tau)e^{24\tau}\partial\tau+\int_{0}^{t}e^{-6\tau}e^{-j3\tau}(t-\tau)e^{24\tau}\partial\tau\right)$$

$$x(t)=\frac{105}{4\pi}e^{-24t}\left(\int_{-\infty}^{0}(t-\tau)e^{\tau(30-j3)}\partial\tau+\int_{0}^{t}(t-\tau)e^{\tau(-30-j3)}\partial\tau\right)$$

$$\int_{-\infty}^{0}(t-\tau)e^{(30-j3)\tau}\partial\tau=\int_{-\infty}^{0}te^{(30-j3)\tau}\partial\tau-\int_{-\infty}^{0}\tau e^{(30-j3)\tau}\partial\tau$$

$$=t\cdot\frac{e^{(30-j3)\tau}}{30-j3}\Bigg|_{-\infty}^{0}-\left(\frac{\tau e^{(30-j3)\tau}}{30-j3}-\frac{e^{(30-j3)\tau}}{(30-j3)^2}\right)\Bigg|_{-\infty}^{0}$$

$$=\frac{t}{30-j3}\left(\cancel{e^{0}}^{\,1}-\cancel{\frac{1}{e^{\infty}}}^{\,0}\right)-\left(\cancel{\frac{0\cdot e^{0}}{30-j3}}^{\,0}-\frac{\cancel{e^{0}}^{\,1}}{(30-j3)^2}-\left(\cancel{\frac{1}{(30-j3)\cdot e^{\infty}}}^{\,0}-\cancel{\frac{1}{(30-j3)^2e^{\infty}}}^{\,0}\right)\right)$$

$$=\frac{t}{30-j3}+\frac{1}{(30-j3)^2}$$

$$\int_{0}^{t}(t-\tau)e^{(-30-j3)\tau}\partial\tau=\int_{0}^{t}te^{(-30-j3)\tau}\partial\tau-\int_{0}^{t}\tau e^{(-30-j3)\tau}\partial\tau$$

$$=t\cdot\frac{e^{(-30-j3)\tau}}{-30-j3}\Bigg|_{0}^{t}-\left(\frac{\tau e^{(-30-j3)\tau}}{-30-j3}-\frac{e^{(-30-j3)\tau}}{(-30-j3)^2}\right)\Bigg|_{0}^{t}$$

$$=\frac{t}{-30-j3}\left(e^{(-30-j3)t}-\cancel{e^{0}}^{\,1}\right)-\left(\frac{te^{(-30-j3)t}}{-30-j3}-\frac{e^{(-30-j3)t}}{(-30-j3)^2}-\left(\cancel{\frac{0\cdot e^{0}}{-30-j3}}^{\,0}-\frac{\cancel{e^{0}}^{\,1}}{(-30-j3)^2}\right)\right)$$

$$=\frac{t}{-30-j3}\left(e^{(-30-j3)t}-1\right)-\left(\frac{te^{(-30-j3)t}}{-30-j3}-\frac{e^{(-30-j3)t}}{(-30-j3)^2}+\frac{1}{(-30-j3)^2}\right)$$

$$=\cancel{\frac{te^{(-30-j3)t}}{-30-j3}}-\frac{t}{-30-j3}-\cancel{\frac{te^{(-30-j3)t}}{-30-j3}}+\frac{e^{(-30-j3)t}}{(-30-j3)^2}-\frac{1}{(-30-j3)^2}$$

$$=\frac{e^{(-30-j3)t}-1}{(30+j3)^2}+\frac{t}{30+j3}$$

$$X(t)=\frac{105}{4\pi}e^{-24t}\left(\frac{t}{30-j3}+\frac{1}{(30-j3)^2}+\frac{e^{(-30-j3)t}-1}{(30+j3)^2}+\frac{t}{30+j3}\right)$$

$$X(t)=\frac{105}{4\pi}e^{-24t}\left(t\left(\frac{1}{30-j3}+\frac{1}{30+j3}\right)+\left(\frac{1}{(30-j3)^2}-\frac{1}{(30+j3)^2}\right)+\frac{e^{(-30-j3)t}}{(30+j3)^2}\right)$$

$$X(t)=\frac{105}{4\pi}e^{-24t}\left(t\frac{60}{30^2+3^2}+\frac{360j}{909^2}+\frac{e^{(-30-j3)t}}{(30+j3)^2}\right)$$

d) $F\{3t^3\}$ → Si $F\{t^n X(t)\}=j^n\frac{\partial^n}{\partial w^n}X(w)$; $X(t)=1\rightarrow F\{1\}=2\pi\delta(w)$

$$F\{3t^3\}=3j^3\frac{\partial^3}{\partial w^3}(2\pi\delta(w))=j^2 j\cdot 6\pi\delta^{(3)}(w)=-j6\pi\delta^{(3)}(w)$$

→ tercera derivada.

e) $\frac{B}{T}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{a^2+(w-nw_0)^2}+\frac{1}{a+j(w-nw_0)}\right)$ donde $n\in\{0,\pm1,\pm2,\dots\}$, $w_0=2\pi/T$ y $B,T\in\mathbb{R}^+$

$$X(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{B}{T}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{a^2+(w-nw_0)^2}+\frac{1}{a+j(w-nw_0)}\right)e^{jwt}\partial w=\frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{\infty}\left(\frac{1}{a^2+(w-nw_0)^2}+\frac{1}{a+j(w-nw_0)}\right)e^{jwt}\partial w$$

→ $v=w-nw_0$ $\quad \partial w=\partial v$

$w=v+nw_0$

$$=\frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}\left(\int_{-\infty}^{\infty}\frac{1}{a^2+v^2}e^{j(v+nw_0)t}\partial v+\int_{-\infty}^{\infty}\frac{1}{a+jv}e^{j(v+nw_0)t}\partial v\right)$$

$$=\frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}\left(e^{jnw_0t}\int_{-\infty}^{\infty}\frac{1}{a^2+v^2}e^{jvt}\partial v+e^{jnw_0t}\int_{-\infty}^{\infty}\frac{1}{a+jv}e^{jvt}\partial v\right)$$

$$=\frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}e^{jnw_0t}\left(\int_{-\infty}^{\infty}\frac{1}{a^2+v^2}e^{jvt}\partial v+\int_{-\infty}^{\infty}\frac{1}{a+jv}e^{jvt}\partial v\right)$$

→ Si $A(v)=\frac{1}{a^2+v^2}\rightarrow a(t)=F^{-1}\{A(v)\}=\frac{1}{2a}e^{-a|t|}$ Para $a>0$

$B(v)=\frac{1}{a+jv}$ Para $a>0\rightarrow b(t)=F^{-1}\{B(v)\}=e^{-at}u(t)$

$$=\frac{B}{T2\pi}\sum_{n=-\infty}^{+\infty}e^{jnw_0t}\left(2\pi\cdot\frac{1}{2a}e^{-a|t|}+2\pi e^{-at}u(t)\right)$$

$$=\frac{B\cancel{2\pi}}{T\cancel{2\pi}}\left(\frac{1}{2a}e^{-a|t|}+e^{-at}u(t)\right)\sum_{n=-\infty}^{+\infty}e^{jnw_0t}$$

$$=\frac{B}{T}\left(\frac{1}{2a}e^{-a|t|}+e^{-at}u(t)\right)\sum_{n=-\infty}^{+\infty}e^{jnw_0t}$$

→ $\sum_{n=-\infty}^{\infty}e^{jnw_0t}=2\pi\sum_{k=-\infty}^{\infty}\delta(w_0t-2\pi k)=2\pi\sum_{k=-\infty}^{\infty}\frac{1}{w_0}\delta\left(t-\frac{2\pi k}{w_0}\right)$; $w_0=\frac{2\pi}{T}\rightarrow T=\frac{2\pi}{w_0}$

$=\frac{2\pi}{w_0}\sum_{k=-\infty}^{\infty}\delta(t-Tk)=T\sum_{k=-\infty}^{\infty}\delta(t-Tk)$

$$=\frac{B}{\cancel{T}}\left(\frac{1}{2a}e^{-a|t|}+e^{-at}u(t)\right)\cancel{T}\sum_{k=-\infty}^{\infty}\delta(t-Tk)$$

$$=B\left(\frac{1}{2a}e^{-a|t|}+e^{-at}u(t)\right)\sum_{k=-\infty}^{\infty}\delta(t-Tk)$$

2.3 Demuestre si los siguientes sistemas de la forma $y = \mathcal{H}\{x\}$, son sistemas lineales e invariantes en el tiempo (SLIT) (simule los sistemas en Python):

- $y[n] = x[n]/3 + 2x[n-1] - y[n-1]$.
- $y[n] = \sum_{k=-\infty}^{n} x^2[k]$.
- $y[n] = \text{median}(x[n])$; donde median es la función mediana sobre una ventana de tamaño 3.
- $y(t) = Ax(t) + B$; $A, B \in \mathbb{R}$.

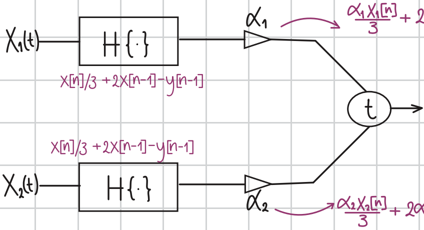
1. $y[n] = x[n]/3 + 2x[n-1] - y[n-1]$

Siempre se asume linealidad a la entrada $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$\mathcal{H}\{x[n]\} = y[n] = x[n]/3 + 2x[n-1] - y[n-1] = \frac{\alpha_1 x_1[n] + \alpha_2 x_2[n]}{3} + 2(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) - y[n-1]$$

$y[n-1] = \alpha_1 y_1[n-1] + \alpha_2 y_2[n-1]$

$$= \frac{\alpha_1 x_1[n]}{3} + \frac{\alpha_2 x_2[n]}{3} + 2\alpha_1 x_1[n-1] + 2\alpha_2 x_2[n-1] - \alpha_1 y_1[n-1] - \alpha_2 y_2[n-1]$$



$\frac{\alpha_1 x_1[n]}{3} + 2\alpha_1 x_1[n-1] - \alpha_1 y[n-1]$

$\tilde{y} = \frac{\alpha_1 x_1[n]}{3} + 2\alpha_1 x_1[n-1] - \alpha_1 y[n-1] + \frac{\alpha_2 x_2[n]}{3} + 2\alpha_2 x_2[n-1] - \alpha_2 y[n-1]$

$\frac{\alpha_2 x_2[n]}{3} + 2\alpha_2 x_2[n-1] - \alpha_2 y[n-1]$

$y[n] = \tilde{y}[n]$ El sistema es lineal si consideramos la dependencia de $y[n-1]$ y condiciones iniciales nulas

Si $y[n] = \mathcal{H}\{x[n]\}$, $x_d[n] = x[n-n_0]$, $y_d[n] = \mathcal{H}\{x_d[n]\}$, Si $y_d[n] = y[n-n_0]$

$$y_d[n] = \frac{x[n-n_0]}{3} + 2x[n-n_0-1] - y_d[n-1]$$
$$y[n-n_0] = \frac{x[n-n_0]}{3} + 2x[n-n_0-1] - y[n-n_0-1]$$

Si $y[n]=0$; $n<0 \rightarrow y_d[n] = y[n-n_0]$ para todo n.

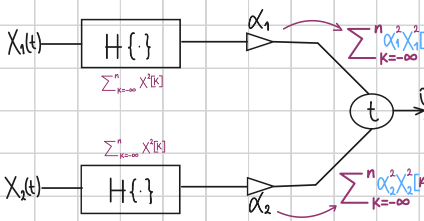
El sistema es invariante en t

En conclusión el sistema es SLIT para condiciones iniciales $= 0$

2. $y[n] = \sum_{k=-\infty}^{n} x^2[k]$

Siempre se asume linealidad a la entrada $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$\mathcal{H}\{x[n]\} = y[n] = \sum_{k=-\infty}^{n} x^2[k] = \sum_{k=-\infty}^{n} (\alpha_1 x_1[k] + \alpha_2 x_2[k])^2$$
$$= \sum_{k=-\infty}^{n} \alpha_1^2 x_1^2[k] + 2\alpha_1 x_1[k]\alpha_2 x_2[k] + \alpha_2^2 x_2^2[k]$$



$\sum_{k=-\infty}^{n} \alpha_1^2 x_1^2[k]$

$\tilde{y} = \sum_{k=-\infty}^{n} \alpha_1^2 x_1^2[k] + \sum_{k=-\infty}^{n} \alpha_2^2 x_2^2[k] = \sum_{k=-\infty}^{n} \alpha_1^2 x_1^2[k] + \alpha_2^2 x_2^2[k]$

$\sum_{k=-\infty}^{n} \alpha_2^2 x_2^2[k]$

$y[n] \neq \tilde{y}[n]$

Si $y[n]=\mathcal{H}\{x[n]\}$, $x_d[k]=x[k-n_0]$, $y_d[n]=\mathcal{H}\{x_d[k]\}$, Si $y_d[n]=y[n-n_0]$

$$y_d[n]=\sum_{k=-\infty}^{n}(x[k-n_0])^2$$

$$y[n-n_0]=\sum_{m=-\infty}^{n-n_0}(x[m])^2 \quad \text{Si } m=k-n_0 \rightarrow k=m+n_0\,;\, k\rightarrow-\infty \rightarrow n: m=k-n_0 \rightarrow n-n_0$$

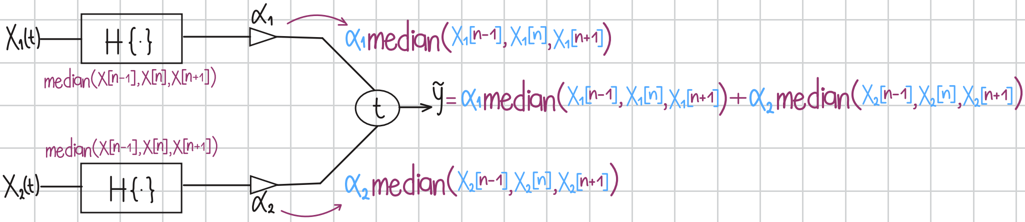
$$y[n-n_0]=\sum_{k=-\infty}^{n}(x[k-n_0])^2 \rightarrow y_d[n]=y[n-n_0]$$

En conclusión el sistema no es SLIT porque aunque es invariante en el tiempo, no es lineal.

3. $y[n]=\text{median}(x[n])$; donde median es la función mediana sobre una ventana de tamaño 3

Siempre se asume linealidad a la entrada $x[n]=\alpha_1 x_1[n]+\alpha_2 x_2[n]$

$$\mathcal{H}\{x[n]\}=y[n]=\text{median}(x[n-1],x[n],x[n+1])=\text{median}(\alpha_1 x_1[n-1]+\alpha_2 x_2[n-1],\alpha_1 x_1[n]+\alpha_2 x_2[n],\alpha_1 x_1[n+1]+\alpha_2 x_2[n+1])$$



$y[n]\neq\tilde{y}[n]$

Si $y[n]=\mathcal{H}\{x[n]\}$, $x_d[n]=x[n-n_0]$, $y_d[n]=\mathcal{H}\{x_d[n]\}$, Si $y_d[n]=y[n-n_0]$

$$y_d[n]=\text{median}(x_d[n-n_0-1],x_d[n-n_0],x_d[n-n_0+1])$$

$$y[n-n_0]=\text{median}(x_d[n-n_0-1],x_d[n-n_0],x_d[n-n_0+1])$$

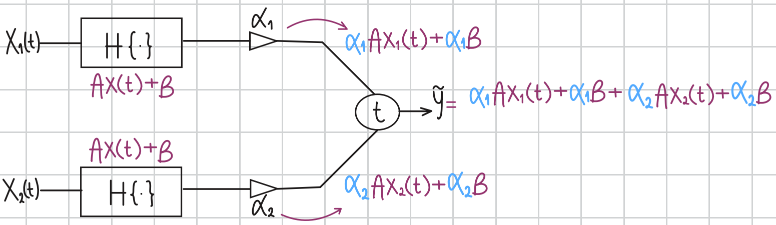
$$y_d[n]=y[n-n_0]$$

En conclusión el sistema no es SLIT porque aunque es invariante en el tiempo, no es lineal.

4. $y(t)=Ax(t)+B$; $A,B\in\mathbb{R}$

Siempre se asume linealidad a la entrada $x(t)=\alpha_1 x_1(t)+\alpha_2 x_2(t)$

$$\mathcal{H}\{x[n]\}=y[n]=Ax(t)+B=A(\alpha_1 x_1(t)+\alpha_2 x_2(t))+B=A\alpha_1 x_1(t)+A\alpha_2 x_2(t)+B$$



$y[n]\neq\tilde{y}[n]\rightarrow$ Para $B\neq 0$

$y[n]=\tilde{y}[n]\rightarrow$ Solo es lineal si $B=0$ o $\alpha_1=\alpha_2=\frac{1}{2}$

Si $y(t)=\mathcal{H}\{x(t)\}$, $x_d(t)=x(t-t_0)$, $y_d(t)=\mathcal{H}\{x_d(t)\}$, Si $y_d(t)=y(t-t_0)$

$$y_d(t)= A(x(t-t_0))+B$$
$$y(t-t_0)=A(x(t-t_0))+B$$
$$y_d(t)=y(t-t_0)$$

En conclusión el sistema es invariante en el tiempo y es SLIT solo cuando $B=0$ o $\alpha_1=\alpha_2=\frac{1}{2}$ porque de otra forma no es lineal.

2.5 Sea la señal Gaussiana $x(t) = e^{-at^2}$ con $a \in \mathbb{R}^+$, el sistema A con relación entrada-salida $y_A(t) = x^2(t)$, y el sistema lineal e invariante con el tiempo B con respuesta al impulso $h_B(t) = Be^{-bt^2}$: a) Encuentre la salida del sistema en serie $x(t) \to h_B(t) \to y_A(t) \to y(t)$ b) Encuentre la salida del sistema en serie $x(t) \to y_A(t) \to h_B(t) \to y(t)$.

$$X(t)*\delta(t)=\int_{-\infty}^{\infty}X(\tau)\delta(t-\tau)\partial\tau$$

respuesta al impulso desplazado

$$y(t)=\int_{-\infty}^{\infty}X(\tau)h(t-\tau)\partial\tau \qquad \boxed{y(t)=X(t)*h(t)}$$

$h_B=Be^{-bt^2}$

$X(t)=e^{-at^2}$

$y_A(t)=X^2(t)$

a) $X(t)\xrightarrow{h_B}v(t)\xrightarrow{A}y(t)$

$$v(t)=X(t)*h_B=\int_{-\infty}^{\infty}e^{-a\tau^2}Be^{-b(t-\tau)^2}\partial\tau=B\int_{-\infty}^{\infty}e^{-(a\tau^2+b(t-\tau)^2)}\partial\tau$$

$$a\tau^2+b(t-\tau)^2=a\tau^2+b(t^2-2t\tau+\tau^2)$$
$$=a\tau^2+bt^2-2bt\tau+b\tau^2$$
$$=(a+b)\tau^2-2bt\tau+bt^2$$

$$(a+b)\left(\tau^2-\frac{2bt}{a+b}\tau\right)=(a+b)\left[\left(\tau-\frac{bt}{a+b}\right)^2-\left(\frac{bt}{a+b}\right)^2\right]$$

$$a\tau^2+b(t-\tau)^2=(a+b)\left(\tau-\frac{bt}{a+b}\right)^2bt^2-(a+b)\left(\frac{bt}{a+b}\right)^2$$

$$bt^2-(a+b)\frac{bt^2}{(a+b)^2}=bt^2-\frac{b^2t^2}{a+b}=\frac{b(a+b)-b^2}{a+b}t^2=\frac{ab}{a+b}t^2$$

$$a\tau^2+b(t-\tau)^2=(a+b)\left(\tau-\frac{bt}{a+b}\right)^2+\frac{ab}{a+b}t^2$$

$$=B\int_{-\infty}^{\infty}e^{-\left((a+b)\left(\tau-\frac{bt}{a+b}\right)^2+\frac{ab}{a+b}t^2\right)}\partial\tau=Be^{-\frac{ab}{a+b}t^2}\int_{-\infty}^{\infty}e^{-\left((a+b)\left(\tau-\frac{bt}{a+b}\right)^2\right)}\partial\tau$$

Integral gaussiana:

$$=Be^{-\frac{ab}{a+b}t^2}\sqrt{\frac{\pi}{a+b}}$$

$$y(t)=X^2(t)=\left(Be^{-\frac{ab}{a+b}t^2}\sqrt{\frac{\pi}{a+b}}\right)^2=\frac{\pi}{a+b}B^2e^{-\frac{2ab}{a+b}t^2}$$

$$a)\to y(t)=\frac{B^2\pi}{a+b}e^{-\frac{2ab}{a+b}t^2}$$

b) $X(t)\xrightarrow{A}u(t)=y_A(t)\xrightarrow{h_B}y(t)$

$$u(t)=X^2(t)=e^{-2at^2}$$

$$y(t)=u(t)*h_B(t)=\int_{-\infty}^{\infty}e^{-2a\tau^2}Be^{-b(t-\tau)^2}\partial\tau=B\int_{-\infty}^{\infty}e^{-(2a\tau^2+b(t-\tau)^2)}\partial\tau$$

$$2a\tau^2+b(t-\tau)^2=(2a+b)\tau^2-2bt\tau+bt^2$$

$$(2a+b)\left(\tau-\frac{bt}{2a+b}\right)^2bt^2-(2a+b)\left(\frac{bt}{2a+b}\right)^2$$

$$bt^2-\frac{b^2t^2}{2a+b}=\frac{b(2a+b)-b^2}{2a+b}t^2=\frac{2ab}{2a+b}t^2$$

$$(2a+b)\left(\tau-\frac{bt}{2a+b}\right)^2+\frac{2ab}{2a+b}t^2$$

$$=B\int_{-\infty}^{\infty}e^{-\left((2a+b)\left(\tau-\frac{bt}{2a+b}\right)^2+\frac{2ab}{2a+b}t^2\right)}\partial\tau=Be^{-\frac{2ab}{2a+b}t^2}\int_{-\infty}^{\infty}e^{-(2a+b)\left(\tau-\frac{bt}{2a+b}\right)^2}\partial\tau$$

$$=Be^{-\frac{2ab}{2a+b}t^2}\sqrt{\frac{\pi}{2a+b}}$$

$$b)\to y(t)=B\sqrt{\frac{\pi}{2a+b}}e^{-\frac{2ab}{2a+b}t^2}$$