Exercises

Chapter 4

1. Show that the firing-rate distribution that maximizes the entropy when the firing rate is constrained to lie in the range $0 \le r \le r_{\max}$ is given by equation 4.22, and that its entropy for a fixed resolution Δr is given by equation 4.23. Use a Lagrange multiplier (see the Mathematical Appendix) to constrain the integral of $p[r]$ to one.

2. Show that the firing-rate distribution that maximizes the entropy when the mean of the firing rate is held fixed is an exponential, and compute its entropy for a fixed resolution Δr. Assume that the firing rate can fall anywhere in the range from 0 to ∞. Use Lagrange multipliers (see the Mathematical Appendix) to constrain the integral of $p[r]$ to 1 and the integral of $p[r]r$ to the fixed average firing rate.

3. Show that the distribution that maximizes the entropy when the mean and variance of the firing rate are held fixed is a Gaussian, and compute its entropy for a fixed resolution Δr. To simplify the mathematics, allow the firing rate to take any value between $-\infty$ and $+\infty$. Use Lagrange multipliers (see the Mathematical Appendix) to constrain the integral of $p[r]$ to 1, the integral of $p[r]r$ to the fixed average firing rate $\langle r \rangle$, and the integral of $p[r](r - \langle r \rangle)^2$ to the fixed variance.

4. Using Fourier transforms, solve equation 4.37, using equation 4.36, to obtain the result of equation 4.42.

5. Suppose the filter $L_s(\vec{a})$ has a correlation function that satisfies equation 4.37. Consider a new filter constructed in terms of this old one by writing

$$L'_s(\vec{a}) = \int d\vec{c}\, U(\vec{a}, \vec{c}) L_s(\vec{c}) \,. \tag{1}$$

   Show that if $U(\vec{a}, \vec{c})$ satisfies the condition of an orthogonal transformation,

$$\int d\vec{c}\, U(\vec{a}, \vec{c}) U(\vec{b}, \vec{c}) = \delta(\vec{a} - \vec{b}) \,, \tag{2}$$

   the correlation function for this new filter also satisfies equation 4.37.

6. Consider a stimulus $s_r = s_s + \eta$ that is given by the sum of a true stimulus s_s and a noise term η. Values of the true stimulus s_s are drawn from a Gaussian distribution with mean 0 and variance Q_{ss}. Values of the noise term η are also obtained from a Gaussian distribution, with mean 0 and variance $Q_{\eta\eta}$. The two terms η and s_s are independent of each other. Using the formula for the continuous entropy of a Gaussian random variable calculated in problem 3, calculate the mutual information between s_r and s_s.

7. Consider a multivariate signal $\mathbf{s}_{\mathrm{s}}$ drawn from a Gaussian distribution with mean $\mathbf{0}$ and covariance matrix $\mathbf{Q}_{\mathrm{ss}}$. Compute the continuous entropy of $\mathbf{s}$ in terms of the eigenvalues of $\mathbf{Q}_{\mathrm{ss}}$, up to the usual resolution term for a continuous entropy.

8. Suppose that a stimulus at one point on the retina, and at a given time, $s_{\mathrm{r}} = s_{\mathrm{s}} + \eta$, is the sum of a true stimulus s_{s} and a noise term η, as in exercise 6. Model the retinal processing at this particular location as producing a signal at the thalamus

$$s_{\mathrm{l}} = D_{\mathrm{s}} s_{\mathrm{r}} + \eta_{\mathrm{l}} \, ,$$

where D_{s} is a parameter called the transfer constant, and η_{l} represents an additional, independent source of noise that can be modeled as being drawn from a Gaussian distribution with mean 0 and variance $Q_{\eta_{\mathrm{l}}\eta_{\mathrm{l}}}$. Calculate the mutual information I_{l} between s_{l} and s_{s} as a function of D_{s}. The power of the signal produced by the retina is defined as $P_{\mathrm{r}} = \langle (D_{\mathrm{s}} s_{\mathrm{r}})^2 \rangle$. By maximizing

$$I_{\mathrm{l}} - k P_{\mathrm{r}}$$

as a function of D_{s}, find the transfer constant that maximizes the mutual information for a given value of k (with $k > 0$), a parameter that controls the trade-off between information and power. What happens when Q_{ss}, describing the visual signal, gets much smaller than $Q_{\eta\eta}$? (Based on a problem from Dawei Dong.)

9. Consider two independent inputs s and s' drawn from Gaussian distributions with means 0 and with different variances Q_{ss} and $Q_{\mathrm{s's'}}$. These generate two thalamic signals, as in exercise 8.

$$s_{\mathrm{l}} = D_{\mathrm{s}} s + \eta \quad \mathrm{a}nd \quad s'_{\mathrm{l}} = D_{\mathrm{s'}} s' + \eta' \, ,$$

defined by two separate transfer constants, D_{s} and $D_{\mathrm{s'}}$, and two independent noise terms with variances $Q_{\eta\eta}$ and $Q_{\eta'\eta'}$. Find the transfer constants that maximize the total mutual information $I_{\mathrm{l}} + I'_{\mathrm{l}}$ for a fixed total power $P_{\mathrm{r}} + P'_{\mathrm{r}}$, where the non-primes and primes denote the information and power for s_{l} and s'_{l}, respectively.