KU LEUVEN

FACULTEIT INGENIEURSWETENSCHAPPEN

B-KUL-H04X3A: Control Theory

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Assignment 3: State feedback and state estimation

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Declaration of Originality

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1 State estimator and state feedback controller design using pole placement

The goal of this assignment is to control the position of the cart along a straight line. The state is the cart position $x(t)$ [m] along the x-axis, with the wall at the origin $x = 0$ m. The cart is positioned in front of the wall, so $x < 0$. The infrared sensor measures the positive distance to the wall, i.e. it measures $-x$.

The system parameters are:

- Wheel radius: $r = 0.033$ m
- Sampling time: $T_s = 0.01$ s (100 Hz control loop)

The input $u(t) = \omega(t)$ [rad/s] is the common wheel angular velocity setpoint, applied equally to both motors A and B via the inner velocity controllers designed in Assignment 2. The velocity control loop is assumed ideal, meaning the actual wheel velocity tracks the setpoint perfectly.

1.1 Discrete-time state equation (1a)

Continuous-time state equation: The velocity control loop from Assignment 2 is assumed ideal, so the actual cart velocity equals the commanded velocity. The continuous-time kinematic relation is:

$$\dot{x}(t) = v(t) = r \cdot \omega(t) = r \cdot u(t), \tag{1}$$

where $v(t)$ [m/s] is the linear cart velocity and $u(t) = \omega(t)$ [rad/s] is the wheel angular velocity input.

In state-space form [1] with scalar state $x(t)$ [m] and input $u(t)$ [rad/s]:

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad A_c = [0], \quad B_c = [r] = [0.033 \text{ m}]. \tag{2}$$

Forward Euler discretization: The forward Euler method approximates the derivative as:

$$\dot{x}[k] \approx \frac{x[k+1] - x[k]}{T_s}. \tag{3}$$

Substituting into the continuous-time equation:

$$\frac{x[k+1] - x[k]}{T_s} = A_c x[k] + B_c u[k] = 0 \cdot x[k] + r \cdot u[k]. \tag{4}$$

Solving for $x[k+1]$:

$$x[k+1] = x[k] + T_s \cdot r \cdot u[k]. \tag{5}$$

The discrete-time state-space model is:

$$x[k+1] = A_d x[k] + B_d u[k], \tag{6}$$

with the discrete-time system matrices:

$$\boxed{A_d = [1], \qquad B_d = [T_s \cdot r] = 0.01 \times 0.033 \ = 3.3 \times 10^{-4} \text{ m/(rad/s)}.} \tag{7}$$

1.2 Measurement equation (1b)

The front infrared (IR) sensor measures the distance from the cart to the wall. Since the cart is at position $x < 0$ (in front of the wall at $x = 0$ m), the measured distance is a positive quantity equal to $|x| = -x$. The measurement equation (ignoring noise) is:

$$y[k] = -x[k]. \tag{8}$$

In state-space form:

$$y[k] = Cx[k] + Du[k], \tag{9}$$

with the output matrices:

$$\boxed{C = [-1], \qquad D = [0].} \tag{10}$$

1.3 Design of state feedback controller gain K using pole placement (1c)

The position controller outputs the desired wheel angular velocity based on the position error. Assuming full state feedback (no estimator), the control law is:

$$u[k] = K \cdot (x_{\mathrm{ref}}[k] - x[k]), \tag{11}$$

where $x_{\mathrm{ref}}[k]$ [m] is the reference position, $x[k]$ [m] the measured position, K [rad/(s·m)] the state feedback gain, and $u[k]$ [rad/s] the commanded wheel angular velocity.

Derivation of closed-loop pole as function of K: Substituting the control law into the discrete-time state equation:

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] && (12)\\ &= x[k] + T_s r \cdot K(x_{\mathrm{ref}}[k] - x[k]) && (13)\\ &= (1 - T_s r K)x[k] + T_s r K \cdot x_{\mathrm{ref}}[k]. && (14)\end{aligned}$$

The closed-loop system has the form:

$$x[k+1] = A_{\mathrm{cl}} x[k] + B_{\mathrm{cl}} x_{\mathrm{ref}}[k], \tag{15}$$

where the closed-loop system matrix is:

$$\boxed{A_{\mathrm{cl}} = [1 - T_s r K].} \tag{16}$$

For this first-order system, the closed-loop pole equals the system matrix:

$$\boxed{z_{\mathrm{cl}}(K) = 1 - T_s r K = 1 - 0.01 \times 0.033 \times K.} \tag{17}$$

Pole behavior as function of K: At $K = 0$, $z_{\mathrm{cl}} = 1$ (marginally stable integrator). As K increases, the pole moves left along the real axis, reaching $z_{\mathrm{cl}} = 0$ ("deadbeat") at $K \approx 3030$ rad/(s·m) and the stability boundary $z_{\mathrm{cl}} = -1$ at $K \approx 6060$ rad/(s·m).

Stability analysis: For discrete-time stability, the pole must lie inside the unit circle:

$$|z_{\mathrm{cl}}(K)| = |1 - T_s r K| < 1. \tag{18}$$

This yields the stability condition:

$$\boxed{0 < K < \frac{2}{T_s r} = \frac{2}{0.01 \cdot 0.033} \approx 6060 \text{ rad/(s·m)}.} \tag{19}$$

This means the system can become unstable if $K > 6060$ rad/(s·m), causing the pole to exit the unit circle through $z = -1$.

Pole-zero map: Figure 1 shows the closed-loop pole location for varying K. The pole moves along the real axis from $z = 1$ (at $K = 0$) toward $z = -1$ (at $K = K_{\max} \approx 6060$ rad/(s·m)).

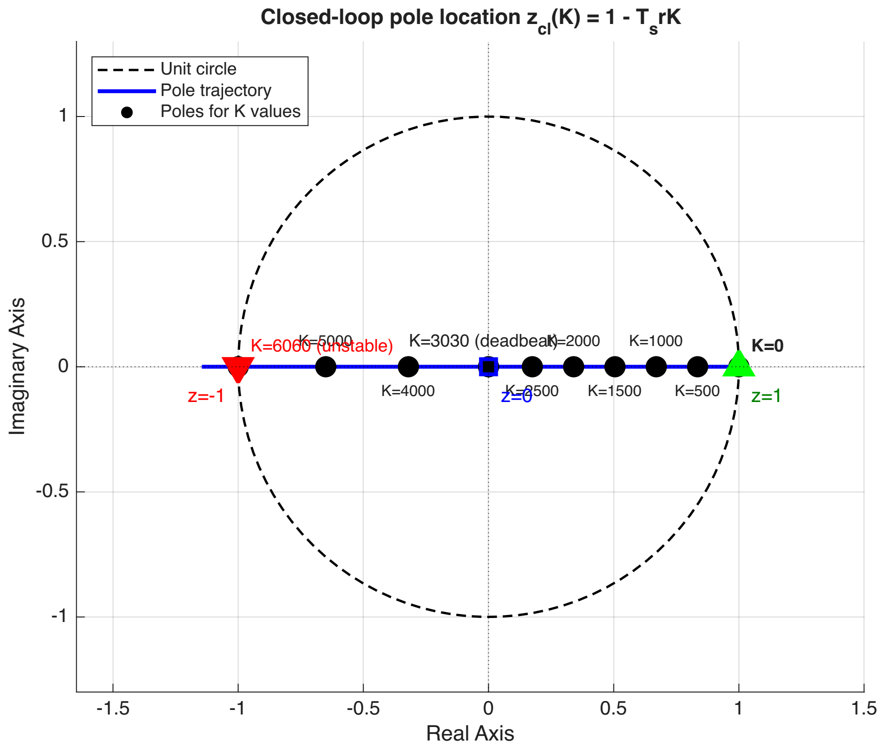


Figure 1: Closed-loop pole location $z_{\mathrm{cl}}(K) = 1 - T_s r K$ for varying K. Unit circle is shown for stability reference.

Simulated step responses: The discrete-time step response is simulated for $K \in \{20, 40, 80\}$ rad/(s·m):

$$x[k+1] = (1 - T_s r K)x[k] + T_s r K \cdot x_{\mathrm{ref}}, \quad x[0] = x_0. \tag{20}$$

K [rad/(s·m)]	z_{cl}	Pole location	Expected behavior
20	0.9934	close to 1	slow convergence, long settling time
40	0.9868	moderate	faster response, good compromise
80	0.9736	closer to 0	fast response, higher control effort

Table 1: Closed-loop poles for different K values.

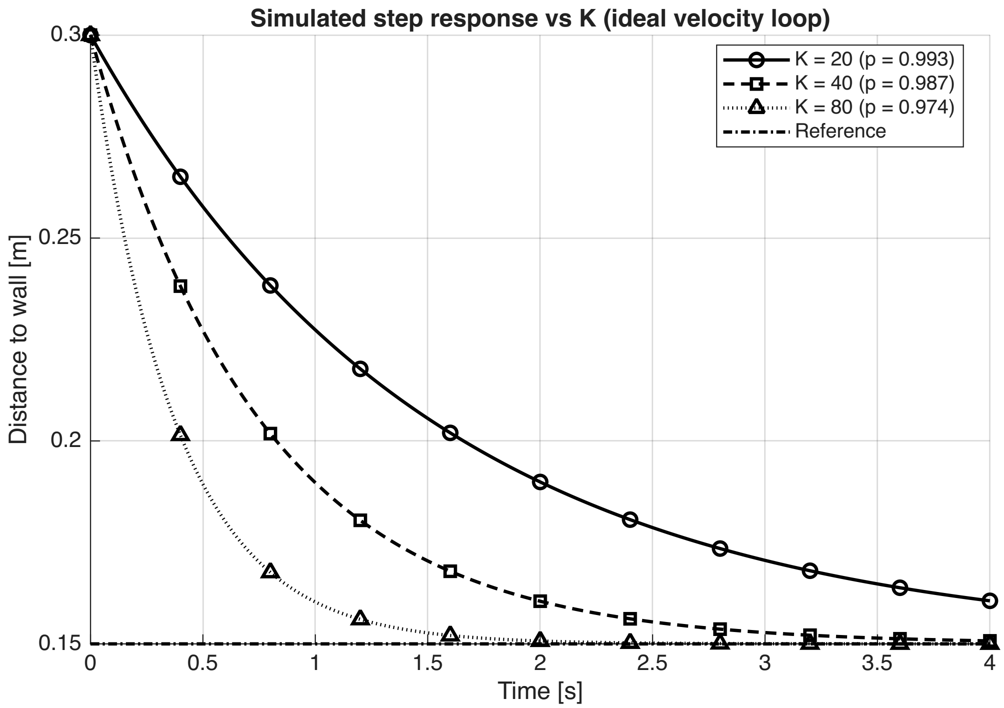


Figure 2: Simulated closed-loop step responses. Larger K yields faster convergence but higher peak velocity commands.

Table 1 and Figure 2 confirm that larger K moves z_{cl} toward zero, yielding faster settling but higher control effort. For $K > K_{\max}$, the pole exits the unit circle and the system becomes unstable.

Choice of K: Beyond the theoretical limit $K_{\max} \approx 6060$ rad/(s·m), practical constraints—actuator saturation (≈11–12 V), sensor noise amplification, and IR range (5–30 cm)—are more restrictive. The selected value

$$\boxed{K_{nom} = 40 \text{ rad/(s·m)}} \tag{21}$$

yields $z_{\mathrm{cl}} = 0.9868$, providing $t_s \approx 3$ s with control effort within actuator limits.

1.4 Design of state estimator gain L using pole placement (1d)

A discrete-time Luenberger observer is used to estimate the cart position from the IR sensor measurement:

$$\hat{x}[k+1] = A_d \hat{x}[k] + B_d u[k] + L \cdot \nu[k], \tag{22}$$

where $\nu[k] = y[k] - C\hat{x}[k]$ is the innovation (measurement prediction error).
With $A_d = [1]$, $B_d = [T_s r]$, and $C = [-1]$:

$$\hat{x}[k+1] = \hat{x}[k] + T_s r \cdot u[k] + L \cdot (y[k] + \hat{x}[k]). \tag{23}$$

Derivation of estimator pole as function of L: Define the estimation error:

$$e[k] = x[k] - \hat{x}[k]. \tag{24}$$

The true state evolves as $x[k+1] = x[k] + T_s r \cdot u[k]$. Subtracting the observer equation:

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] && (25)\\ &= (x[k] + T_s r \cdot u[k]) - (\hat{x}[k] + T_s r \cdot u[k] + L(y[k] - C\hat{x}[k])) && (26)\\ &= (x[k] - \hat{x}[k]) - L \cdot C \cdot e[k] && (27)\\ &= e[k] - L \cdot C \cdot e[k] && (28)\\ &= (1 - LC) \cdot e[k]. && (29) \end{aligned}$$

Substituting $C = [-1]$:

$$e[k+1] = (1 - L \cdot (-1)) \cdot e[k] = (1 + L) \cdot e[k]. \tag{30}$$

The estimator error dynamics have the closed-loop pole:

$$\boxed{z_{\text{est}}(L) = 1 + L.} \tag{31}$$

Pole behavior as function of L:

- at $L = 0$: $z_{\text{est}} = 1$ (no correction, estimator ignores measurements)
- as L becomes more negative: pole moves left along the real axis
- at $L = -1$: $z_{\text{est}} = 0$ (deadbeat estimator, instant convergence)
- at $L = -2$: $z_{\text{est}} = -1$ (stability boundary)

Stability analysis: For the estimator to be stable:

$$|z_{\text{est}}(L)| = |1 + L| < 1. \tag{32}$$

This yields:

$$\boxed{-2 < L < 0.} \tag{33}$$

This means the estimator can become unstable:

- if $L > 0$: pole $z_{\text{est}} > 1$, estimation error grows exponentially
- if $L < -2$: pole $z_{\text{est}} < -1$, estimation error diverges with oscillations

Trade-offs: Faster estimators (larger $|L|$, pole closer to 0) converge quickly but amplify measurement noise; slower estimators (pole closer to 1) reject noise but respond sluggishly to initialization errors. The estimator pole is typically placed 2–6 times faster than the controller pole. For Section 2.3, the specification requires setting it 10 times *slower*.

Chosen gain: A nominal value:

$$\boxed{L_{nom} = -0.18} \tag{34}$$

gives the closed-loop pole

$$z_{\text{est}} = 1 + L_{nom} = 0.82 \tag{35}$$

which is fast but noise-aware. Alternative gains $L \in \{-0.05, -0.18, -0.35\}$ are swept in the experiments below.

2 Implementation and testing of state estimator and state feedback controller

2.1 Estimator only: wrong initial estimate, different L (2a)

The controller is disabled; the estimator starts from a wrong initial position $\hat{x}[0]$.

Configuration:

- estimator active, controller off; the cart is moved manually, or driven with a known, simple input signal
- the state estimator is initialized with a wrong position estimate $\hat{x}[0]$ (e.g. 10 cm closer to the wall than reality).
- different values of L are tested (e.g. $L \in \{-0.05, -0.18, -0.35\}$).

For each L, the measured distance $y[k]$ (positive, measured by IR sensor) and the estimated state $\hat{x}[k]$ are logged. Figure 3 overlays the measured distance $y = -x$ and the estimated distance $-\hat{x}$ for $L \in \{-0.05, -0.18, -0.35\}$. Larger $|L|$ yields faster convergence but more noise on $\hat{x}$ and ν.

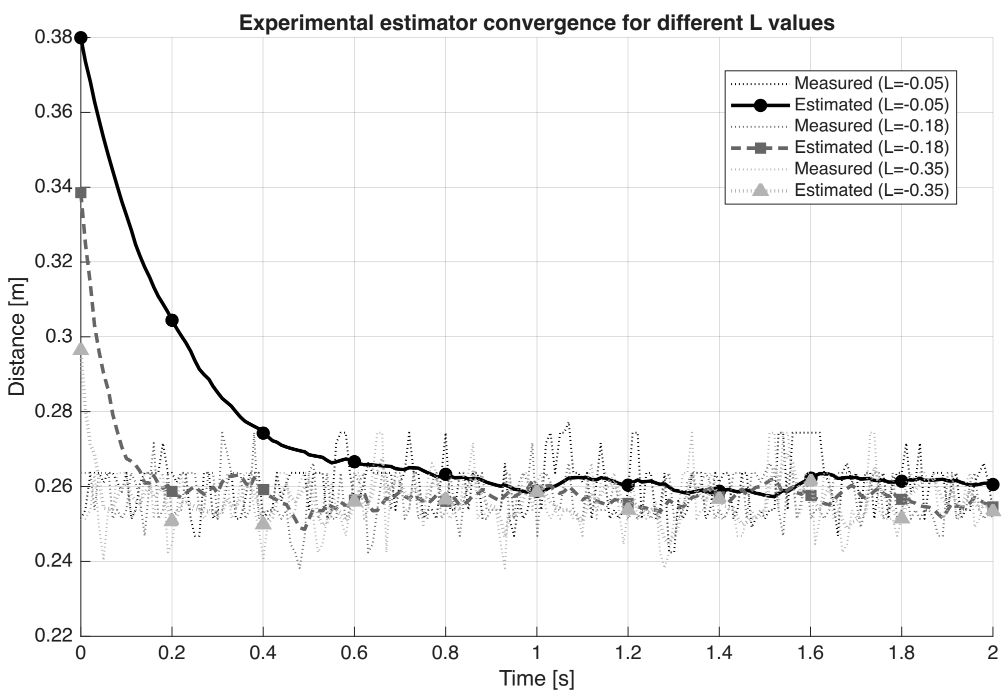


Figure 3: Measured vs. estimated distance for different L (wrong initial estimate).

Interpretation: The experimental results confirm the theoretical predictions from Section 1.4: larger $|L|$ (pole closer to 0) yields faster convergence but amplifies sensor noise, while smaller $|L|$ (pole closer to 1) produces smoother but slower estimates. The chosen $L_{\text{nom}} = -0.18$ ($z_{\text{est}} = 0.82$) balances settling time and noise rejection.

2.2 Controller only: proportional feedback for different K (2b)

The estimator is disabled, feedback uses the raw distance $y[k] = -x[k]$. For different values of $K \in \{K_{slow}, K_{nom}, K_{fast}\} = \{20, 40, 80\}$ rad/(s $\cdot m$), a 0.25 m step reference is applied.

Configuration:

- controller active, estimator off; the controller directly uses the IR measurement (converted to position) for feedback
- the control law becomes: $u[k] = K(r[k] - x[k])$, with $x[k]$ obtained from the IR sensor relation $x[k] = -y[k]$
- a 0.25 m step in position reference is applied

This experiment is repeated for several values of K (for $K \in \{20, 40, 80\}$ rad/(s $\cdot m$)).
Figure 4 shows the step reference (dashed line) and position responses for different values of K. Figure 5 shows the corresponding motor voltages, for the same three values of K.

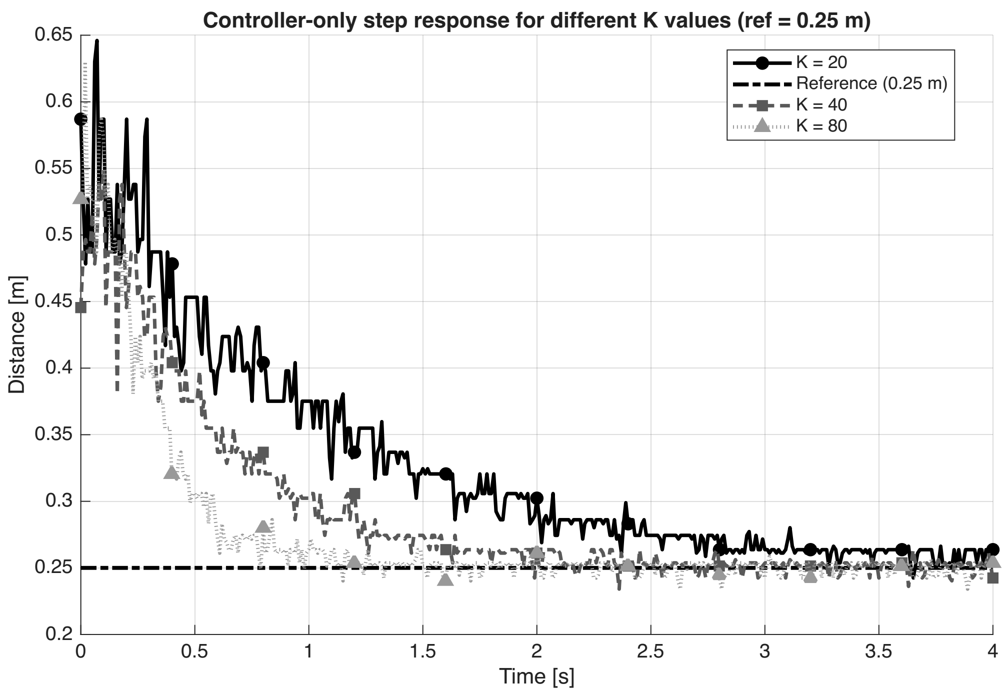


Figure 4: Measured position step responses for different K (estimator disabled).

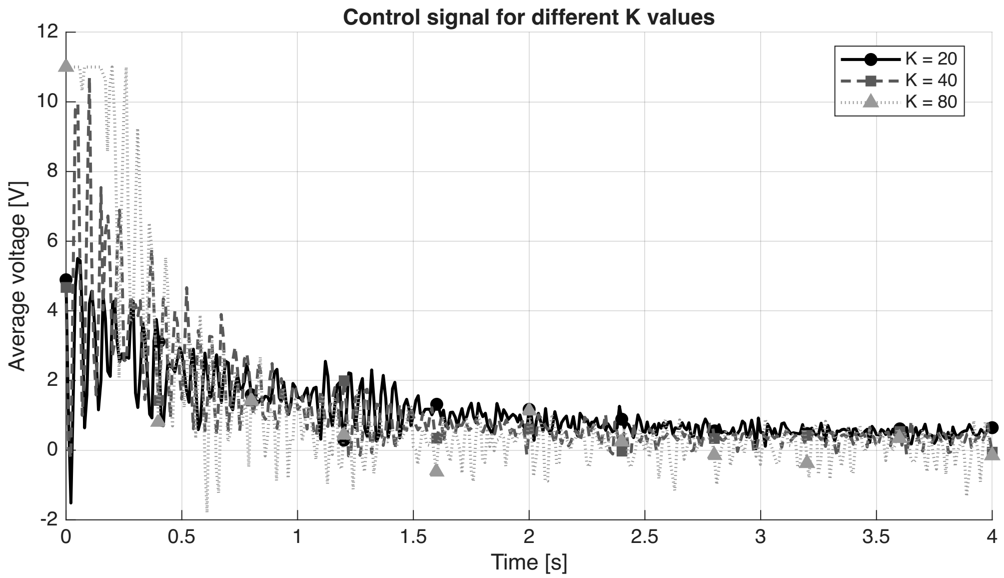


Figure 5: Motor voltage commands for the responses in Figure 4.

Interpretation: The experimental step responses confirm the theoretical predictions from Section 1.3: higher K yields faster settling but increases peak voltage. For $K_{\text{fast}} = 80$ rad/(s·m), the voltage briefly saturates, causing mild overshoot. The selected $K_{\text{nom}} = 40$ rad/(s·m) avoids saturation while achieving $t_s \approx 3$ s.

2.3 Combined estimator and controller: influence of estimator pole choice (2c)

In this experiment, both the state feedback controller and the state estimator are active. The controller uses the estimated position $\hat{x}[k]$ instead of the true position:

$$u[k] = K(x_{\text{ref}}[k] - \hat{x}[k]). \tag{36}$$

Per the assignment specification, the estimator is designed to be 10 times slower than the controller.

Designing the estimator to be 10× slower than the controller: With $K_{nom} = 40$ rad/(s·m), $z_{\text{cl}} = 0.9868$ (cf. Section 1.3). A discrete-time pole closer to 1 corresponds to slower dynamics. To make the estimator 10 times slower than the controller in continuous-time:

1. convert discrete-time controller pole z_{cl} to continuous-time: $s_{\text{cl}} = \frac{\ln(z_{\text{cl}})}{T_s} = \frac{\ln(0.9868)}{0.01} \approx -1.33$ rad/s
2. estimator pole is 10× slower: $s_{\text{est}} = \frac{s_{\text{cl}}}{10} = -0.133$ rad/s
3. convert back to discrete-time: $z_{\text{est}} = e^{s_{\text{est}} \cdot T_s} = e^{-0.00133} \approx 0.9987$

Solving for L using $z_{\text{est}} = 1 + L$:

$$L = z_{\text{est}} - 1 = 0.9987 - 1 = -0.0013. \tag{37}$$

Selected slow estimator gain:

$$\boxed{L_{\text{slow}} = -0.0013\ [-].} \tag{38}$$

This value for L will be used in the remainder of Section 2.3 (2c) during the experiments.

Full closed-loop system poles: The combined system (controller + estimator) has two poles:

$$\boxed{\lambda_1 = z_{\rm cl} = 0.9868, \qquad \lambda_2 = z_{\rm est} = 0.9987,} \tag{39}$$

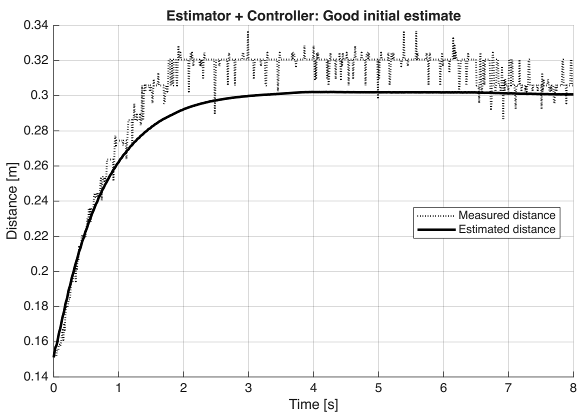
which corresponds to

$$\boxed{K = 40 \text{ rad/(s·m)}, \qquad L = -0.0013} \tag{40}$$

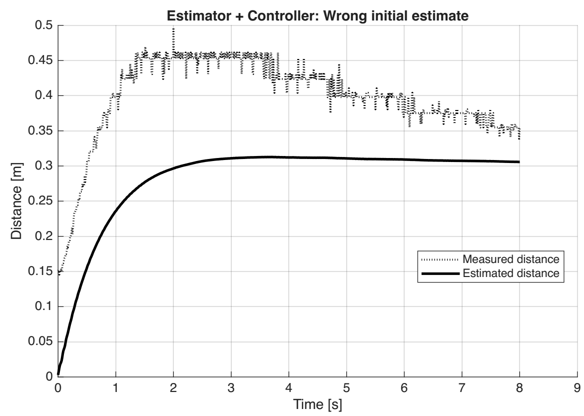
Experiments (correct vs. wrong initial estimate): Two runs are performed:

1. correct initial estimate: $\hat{x}[0] = x[0]$.
2. incorrect initial estimate: $\hat{x}[0] = x[0] + \Delta x$, with Δx several cm.

In both cases, the measured signal $y[k] = -x[k]$ and the estimated output $\hat{y}[k] = -\hat{x}[k]$ are plotted.



(a) Good initial estimate



(b) Wrong initial estimate

Figure 6: Combined estimator and controller. Left: good initial estimate; right: wrong initial estimate showing slow estimator convergence.

Observed behavior:

1. **Correct initial estimate:** If $\hat{x}[0] \approx x[0]$, the estimator error is nearly zero, so:
   - the estimator output $\hat{x}[k]$ is immediately accurate,
   - the controller behaves almost exactly as in the no-estimator case,
   - the step response closely matches the ideal closed-loop behavior governed by z_{cl}.
2. **Incorrect initial estimate:** Because the chosen estimator pole is very slow:

$$e[k+1] = z_{est} \cdot e[k], \quad \text{with} \quad z_{est} \approx z_{cl}^{1/10} \approx 0.9987, \tag{41}$$

   the error decays only gradually. During this long transient, the controller acts on:

$$x_{ref}[k] - \hat{x}[k], \tag{42}$$

   which may be very different from the true distance error.
   Consequences of the incorrect initial estimate are:

   - delayed motion toward the setpoint,
   - possible motion in the wrong direction initially,
   - slow convergence to the correct position,

- degraded control performance even though the controller gain K is unchanged.

This demonstrates that a too-slow estimator harms the transient performance, especially when initial estimation erros are present.

3. **When designing a state estimator yourself (using pole placement):** Instead of making the estimator slower, the closed-loop estimator pole would be placed faster than the controller pole, typically 2–5 times faster in continuous-time terms. For example, if the controller pole is $z_{cl} = 0.2$, a good choice for the estimator pole is around $z_{est} \approx 0.1$ (or even smaller), which corresponds to an L that gives very quick convergence of $\hat{x}[k]$ without excessive noise amplification.

Separation principle and full closed-loop poles: The combined system has eigenvalues $\lambda_1 = z_{\text{cl}}$ and $\lambda_2 = z_{\text{est}}$, confirming the separation principle: controller and estimator poles appear independently in the closed-loop system. However, although the poles are decoupled, the controller acts on $\hat{x}[k]$ rather than $x[k]$. A slow estimator with wrong initialization causes $\hat{x}[k]$ to lag behind $x[k]$, degrading transient performance despite unchanged pole locations. The closed-loop transfer function describes only asymptotic behavior; it does not capture the effect of initial estimation error $e[0]$.

Conclusion: Choosing z_{est} much slower than z_{cl} severely degrades transient performance and introduces long settling times when $\hat{x}[0]$ is wrong—demonstrating why **estimators are typically placed faster than controllers**, not slower.

References

[1] Jan Swevers and Goedele Pipeleers. *C9. Control Design Using State Space Methods*. KU Leuven, 2024.