Chapter 8
- Phasor

Outline

- Sinusoidal signals
- Circuit response to a sinusoidal input
- Phasor

Sinusoidal Signal (Current or Voltage)

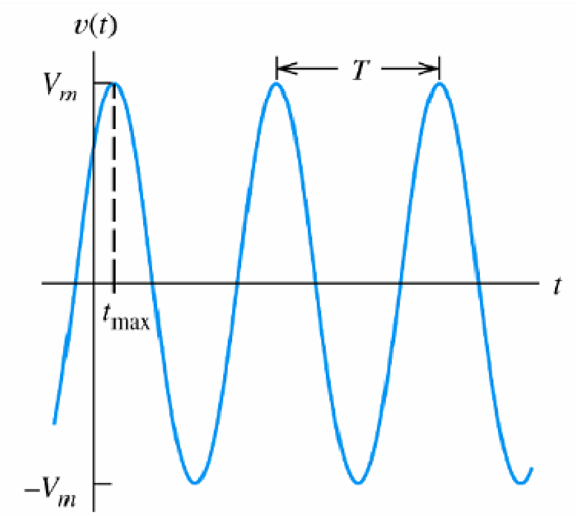
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the **peak value**

ω is the **angular frequency** in radians per second

$(\omega t + \theta)$ is the **phase angle**

T is the **period**

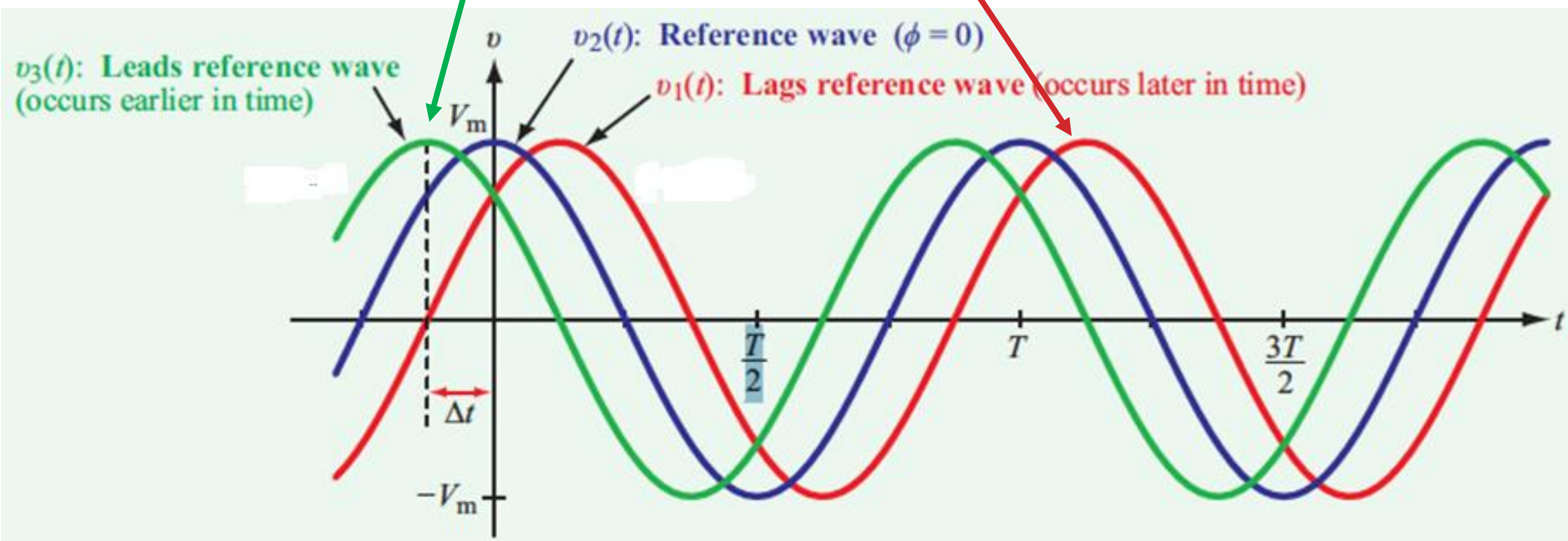


$$f = \frac{1}{T} \qquad \omega = 2\pi f$$

It is more appropriate to call ω angular velocity.

Phase Lead/Lag

$$V_m \cos \frac{2\pi t}{T} \qquad V_m \cos\left(\frac{2\pi t}{T} + \frac{\pi}{4}\right) \qquad V_m \cos\left(\frac{2\pi t}{T} - \frac{\pi}{4}\right)$$



Phase lead/lag comparison should be made after constraining the phase difference to a range of [-π, π).

Useful Relations

$$\sin x = \pm\cos(x \mp 90^\circ)$$
$$\cos x = \pm\sin(x \pm 90^\circ)$$
$$\sin x = -\sin(x \pm 180^\circ)$$
$$\cos x = -\cos(x \pm 180^\circ)$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2\sin x \cos y = \sin(x + y) + \sin(x - y)$$
$$2\cos x \cos y = \cos(x + y) + \cos(x - y)$$

The product of 2 sinusoidal signals can be decomposed into 2 signals with frequencies of f_1+f_2 and f_1-f_2.

Why Sinusoids?

- Many motion are sinusoidal in nature
  - Motion of a pendulum, electromagnetic waves, ripples on water surface
- A very easy signal to generate and transmit
  - Dominant form of signal in communication/electric power industries
- They are very easy to handle mathematically.
  - Derivative and integral are also sinusoids.
- Through *Fourier analysis*, any periodic function can be represented as sum of sinusoids.

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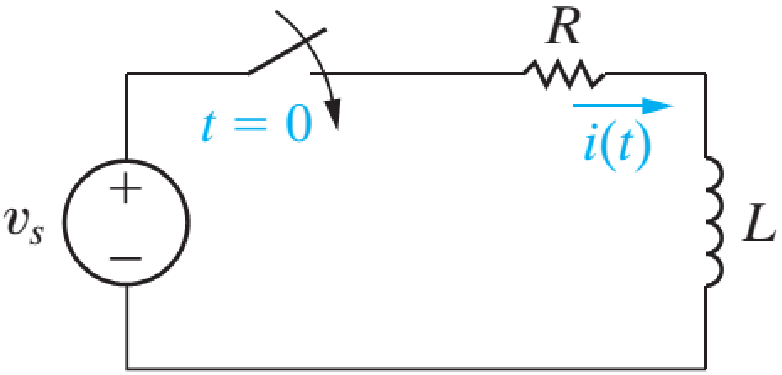
The Sinusoidal Response

$v_s = V_m \cos(\omega t + \phi)$, $i(0^-) = 0$.

Find $i(t)$, $t \geq 0$.



$$\boxed{L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi)}$$

Ordinary differential equation

Try a sinusoidal solution: $i(t) = i_m \cos(\omega t + \beta)$

$$-L i_m \omega \sin(\omega t + \beta) + R i_m \cos(\omega t + \beta) = V_m \cos(\omega t + \varphi)$$

$$i_m \sqrt{(L\omega)^2 + R^2}\left[\frac{-L\omega \sin(\omega t + \beta)}{\sqrt{(L\omega)^2 + R^2}} + \frac{R \cos(\omega t + \beta)}{\sqrt{(L\omega)^2 + R^2}}\right] = V_m \cos(\omega t + \varphi)$$

$$\theta = \operatorname{atan}\left(\frac{L\omega}{R}\right)$$

$$i_m \sqrt{(L\omega)^2 + R^2}\,[\cos\theta \cos(\omega t + \beta) - \sin\theta \sin(\omega t + \beta)] = V_m \cos(\omega t + \varphi)$$

$$i_m \sqrt{(L\omega)^2 + R^2} \cos(\omega t + \theta + \beta) = V_m \cos(\omega t + \varphi)$$

To satisfy the above equation, we should have

$i_m = \frac{V_m}{\sqrt{(L\omega)^2 + R^2}}$, $\beta = \varphi - \theta$, Therefore, the solution is

$$i(t) = \frac{V_m}{\sqrt{(L\omega)^2 + R^2}} \cos(\omega t + \varphi - \theta)$$

However, the above is the steady-state response and does not in general satisfy the initial condition.

Total Response

$$i(t) = Ce^{-\left(\frac{R}{L}\right)t} + \frac{V_m}{\sqrt{(L\omega)^2+R^2}}\cos(\omega t + \varphi - \theta)$$

also satisfies the equation

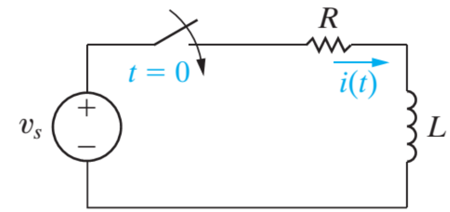
$$L\frac{di}{dt} + Ri = V_m\cos(\omega t + \phi)$$

Proof:

Let $f_1(t) = Ce^{-\left(\frac{R}{L}\right)t}, f_2(t) = \frac{V_m}{\sqrt{(L\omega)^2+R^2}}\cos(\omega t + \varphi - \theta)$

$$L\frac{di}{dt} + Ri = Lf_1{}'(t) + R\ f_1(t) + Lf_2{}'(t) + R\ f_2(t) =$$
$$0 + V_m\cos(\omega t + \phi) = V_m\cos(\omega t + \phi)$$

Total Response



$$v_s = V_m \cos(\omega t + \phi)$$

Note: The coefficient before $e^{-\left(\frac{R}{L}\right)t}$ is determined by the initial condition $i(0) = 0$.

$$i(0) = C + \frac{V_m}{\sqrt{(L\omega)^2 + R^2}} \cos(\varphi - \theta) = 0$$

$$C = \frac{-V_m}{\sqrt{(L\omega)^2 + R^2}} \cos(\varphi - \theta)$$

$$i = \boxed{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \boxed{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)}$$

Transient response

Steady-state response

- For $L\frac{di}{dt} + Ri = V_m\cos(\omega t + \phi)$, we have derived the steady-state solution,

$i(t) = \frac{V_m}{\sqrt{(L\omega)^2+R^2}}\cos(\omega t + \varphi - \theta)$, with$\theta = \mathrm{atan}(\frac{L\omega}{R})$.

- Similarly, for $CR\frac{dV}{dt} + V(t) = V_m\cos(\omega t + \phi)$, we have (replacing R by 1 and L by CR in the above formula for $i(t)$):

$V(t) = \frac{V_m}{\sqrt{(CR\omega)^2+1}}\cos(\omega t + \varphi - \theta)$, with $\theta = \mathrm{atan}(CR\omega)$.

- Adding the transient solution and using the initial condition V(0)=0, we have

$$\mathrm{V}(t) = \frac{V_m}{\sqrt{(RC\omega)^2+1}}\cos(\omega t + \varphi - \theta) - \frac{V_m}{\sqrt{(RC\omega)^2+1}}\cos(\varphi - \theta)e^{-\frac{t}{RC}}$$

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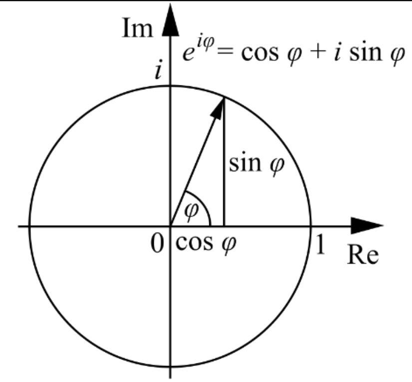
Phasor



- The idea of phasor representation is based on Euler's identity:

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

- From this we can represent a sinusoid as the real component of a vector in the complex plane.

$$v(t) = V\cos(\omega t + \varphi) = \mathrm{Re}\left\{V e^{j\varphi}\, e^{j\omega t}\right\} = \mathrm{Re}\{v_c(t)\}$$

- What makes the complex representation attractive is the fact that the derivatives of the $v_c(t)$ have the same functional form as $v_c(t)$.

Exercise

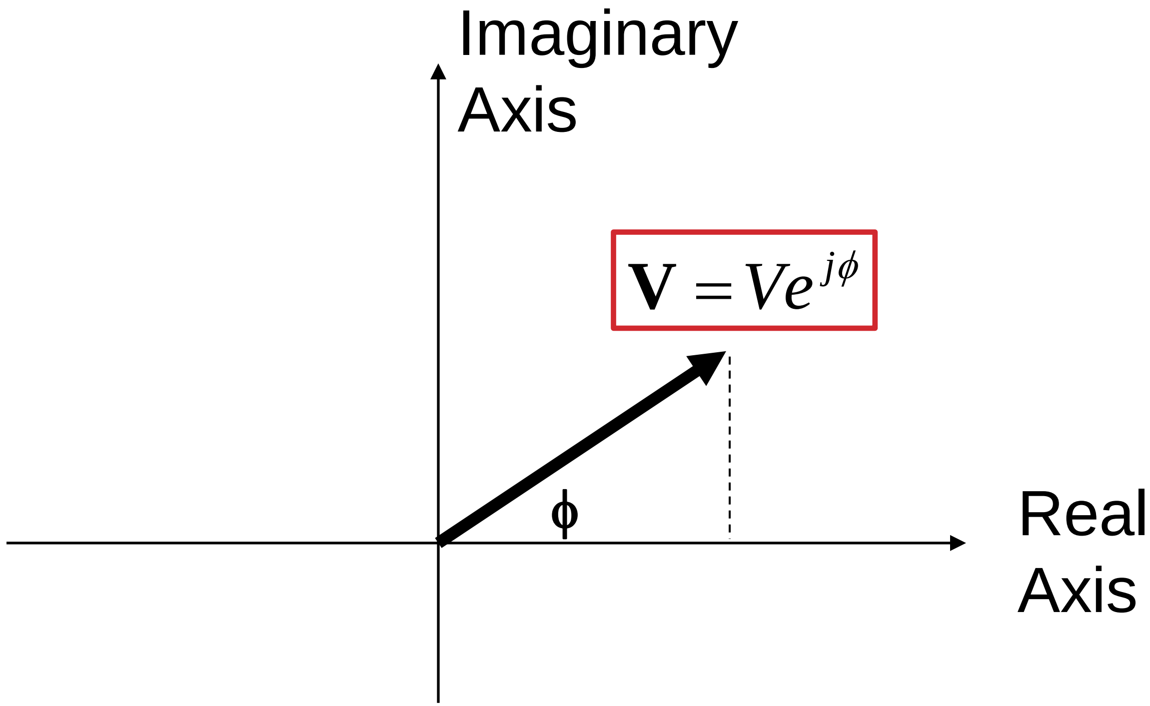
Calculate the first-order derivatives of $v(t) = V\cos(\omega t + \varphi)$ and $v_c(t) = Ve^{j(\varphi+\omega t)}$.

$$\frac{dv}{dt} = -V\omega\sin(\varphi + \omega t); \frac{dv_c}{dt} = j\omega Ve^{j(\varphi+\omega t)}.$$

Phasor definition

$$v(t) = V\cos(\omega t + \phi) = \mathrm{Re}\left\{V e^{j\varphi} e^{j\omega t}\right\} = \mathrm{Re}\left(\mathbf{V} e^{j\omega t}\right) \quad \boxed{\mathbf{V} = V e^{j\varphi}}$$

Complex representation of the magnitude and phase of a sinusoid



Phasor transformation

$$v(t) = V_m \cos(\omega t + \phi) \quad \Leftrightarrow \quad \mathbf{V} = V_m \underline{/\phi}$$

(Time-domain representation) (Phasor-domain representation)

- To obtain the phasor for a sinusoidal signal, take its amplitude (V) and initial phase (φ) and combine them as $Ve^{j\varphi}$.

- To obtain the sinusoidal signal, multiply the phasor by $e^{j\omega t}$ and take the real part.

Exercise I

- Transform these sinusoids to phasors

(a) $i = 6\cos(50t - 40^\circ)$ A

(b) $v = -4\sin(30t + 50^\circ)$ V

(c) i=0 A

Answers: $6\angle(-40^o)$

$4\angle 140^o$

0

Outline

- Review of the basics of phasor
- Phasor transformations
- Solution of the second-order differential equations using phasors
- KCL and KVL of phasors
- I-V relationships of R/L/C in phasor domain (Impedance)

Exercise II

- Find the sinusoids represented by these phasors

$$\mathbf{I} = -3 + j4 \text{ A}$$

$$\mathbf{V} = j8e^{-j20^\circ} \text{ V}$$

$$\mathbf{V} = -25\underline{/40^\circ} \text{ V}$$

$$\mathbf{I} = j(12 - j5) \text{ A}$$

Answers:

$i = 5\cos(\omega t + 126.9^o)$ A

$\text{v} = 8\cos(\omega t + 70^o)$ V

$\text{v} = 25\cos(\omega t - 140^o)$ V

$i = 13\cos(\omega t + 67.4^o)$ A

Time Domain - Phasor Domain Transformation

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\Leftrightarrow$	
$A\cos(\omega t+\phi)$	$\Leftrightarrow$	
$-A\cos(\omega t+\phi)$	$\Leftrightarrow$	
$A\sin\omega t$	$\Leftrightarrow$	
$A\sin(\omega t+\phi)$	$\Leftrightarrow$	
$-A\sin(\omega t+\phi)$	$\Leftrightarrow$	
$\frac{d}{dt}(x(t))$	$\Leftrightarrow$	
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\Leftrightarrow$	
$\int x(t)\,dt$	$\Leftrightarrow$	
$\int A\cos(\omega t+\phi)\,dt$	$\Leftrightarrow$	

Time Domain - Phasor Domain Transformation

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\Leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\Leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\Leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\Leftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\Leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\Leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}(x(t))$	$\Leftrightarrow$	$j\omega\mathbf{X}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\Leftrightarrow$	$j\omega Ae^{j\phi}$
$\int x(t)\,dt$	$\Leftrightarrow$	$\frac{1}{j\omega}\mathbf{X}$
$\int A\cos(\omega t+\phi)\,dt$	$\Leftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

Some Proofs

Time domain	Phasor
$s = A\cos(\omega t + \varphi)$	$Ae^{j\varphi}$
$A\sin(\omega t + \varphi) = A\cos\left(\omega t + \varphi - \frac{\pi}{2}\right)$	$Ae^{j\left(\varphi - \frac{\pi}{2}\right)} = -jAe^{j\varphi}$
$\frac{ds}{dt} = -A\omega\sin(\omega t + \varphi)$	$j\omega Ae^{j\varphi}$
$\int A\cos(\omega t + \varphi)\,dt = \frac{A}{\omega}\sin(\omega t + \varphi)$	$\frac{1}{j\omega}Ae^{j\varphi}$

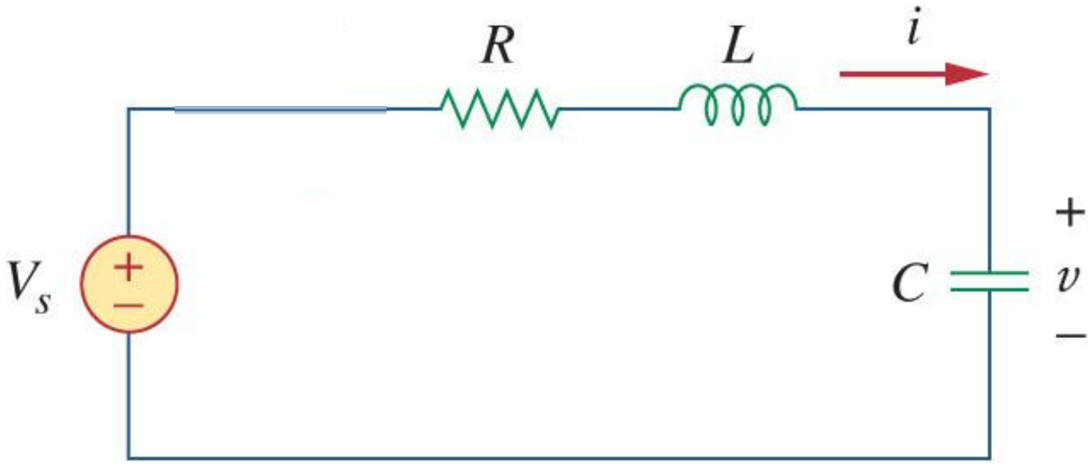
Exercise III

- For $s = A\cos(\omega t + \varphi)$, find the phasor of $\frac{d^n s}{dt^n}$.

$$P\left(\frac{d^n s}{dt^n}\right) = j\omega P\left(\frac{d^{n-1} s}{dt^{n-1}}\right) = (j\omega)^2 P\left(\frac{d^{n-2} s}{dt^{n-2}}\right) \dots$$
$$= (j\omega)^n P(s)$$

Exercise IV

- Using the phasor approach, determine the voltage $v(t)$ in the series RLC circuit below with R=5 Ω, L=3 μH, C=3 nF, and $V_s = 50\cos(1.0 \times 10^7 t + 70^o)$ V.



$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

$$\left(-\omega^2 + \frac{R}{L}j\omega + \frac{1}{LC}\right)v_p = \frac{V_{s,p}}{LC}$$

$$v_p = \frac{V_{s,p}}{-\omega^2 LC + j\omega RC + 1}$$

$$v_p = 277.5\angle 13.3^o$$

$$v(t) = 277.5\cos(1.0 \times 10^7 t + 13.3^o)\text{ V}$$

Kirchhoff’s Laws in the Phasor Domain

- Let $v_1, v_2, \cdots v_n$ be the voltages around a closed loop. Then according to KVL and KCL,

$$v_1 + v_2 + \cdots + v_n = 0$$

$$i_1 + i_2 + \cdots + i_n = 0$$

The same equations hold for their phasors, i.e.:

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

Linearity of Phasor Transformation

$$v_1 + v_2 + \cdots + v_n$$

$$\parallel$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n)$$

$$\parallel$$

$$\mathrm{Re}\left(V_{m1}e^{j\theta_1}\cdot e^{j\omega t}\right) + \cdots + \mathrm{Re}\left(V_{mn}e^{j\theta_n}\cdot e^{j\omega t}\right)$$

$$\parallel$$

$$\mathrm{Re}\left((\mathbf{V}_1 + \cdots + \mathbf{V}_n)\cdot e^{j\omega t}\right), \text{ where } \mathbf{V}_k = V_{mk}e^{j\theta_k}$$

$\Updownarrow$ Phasor transformation

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n$$

Therefore, the phasor of $v_1 + v_2 + \cdots + v_n$ is equal to the sum of phasors of v_i. 由可叠加性可知相量变换是一种线性变换。

Exercise – Verifying the Linearity

If $y_1 = 20\cos(\omega t - 30°)$ and $y_2 = 40\cos(\omega t + 60°)$,
Find the phasor of $y = y_1 + y_2$.

1. Use trigonometric identities
2. Use the linearity of phasor transformation.
3. Confirm that the phasor of y is equal to the sum of the phasors of y_1 and y_2.

1.
$$y = (20\cos 30 + 40\cos 60)\cos\omega t$$
$$+ (20\sin 30 - 40\sin 60)\sin\omega t$$
$$= 37.32\cos\omega t - 24.64\sin\omega t.$$
$$y = 44.72\cos(\omega t + 33.43^\circ)$$
$$Y = 44.72\angle 33.43^o$$

2.
$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$
$$= 20\angle{-30^\circ} + 40\angle{60^\circ}$$
$$= (17.32 - j10) + (20 + j34.64)$$
$$= 37.32 + j24.64$$
$$= 44.72\angle{33.43^\circ}.$$

Proof of KVL(KCL) in the phasor domain

$$v_1 + v_2 + \cdots + v_n = 0$$

- Apply phasor transformation (denote by P) on both sides

$$P(v_1 + v_2 + \cdots + v_n) = P(0) = 0$$
$$P(v_1) + P(v_2) + \cdots + P(v_n) = 0$$
$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Similarly,

$$P(i_1 + i_2 + \cdots + i_n) = P(0) = 0$$
$$P(i_1) + P(i_2) + \cdots + P(i_n) = 0$$
$$\boldsymbol{I}_1 + \boldsymbol{I}_2 + \cdots + \boldsymbol{I}_n = 0$$

V-I Phasor Relationship for Resistors

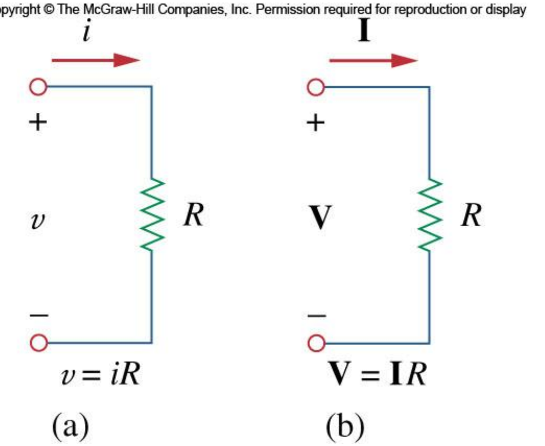
- For the resistor, the voltage and current are related via Ohm's law. As such, the voltage and current are in phase with each other.

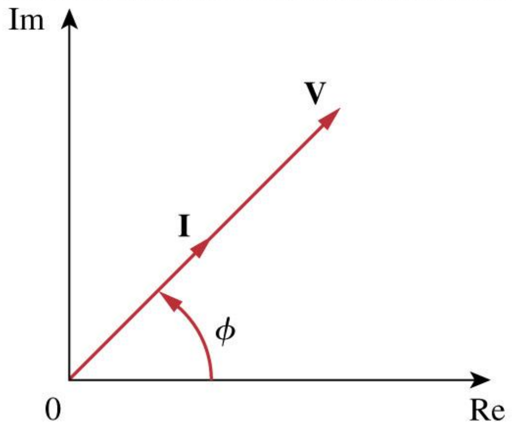
$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

$$v = Ri$$

$$\mathbf{V} = RI_m \angle \phi$$

$$\boxed{\mathbf{V} = R\mathbf{I}}$$



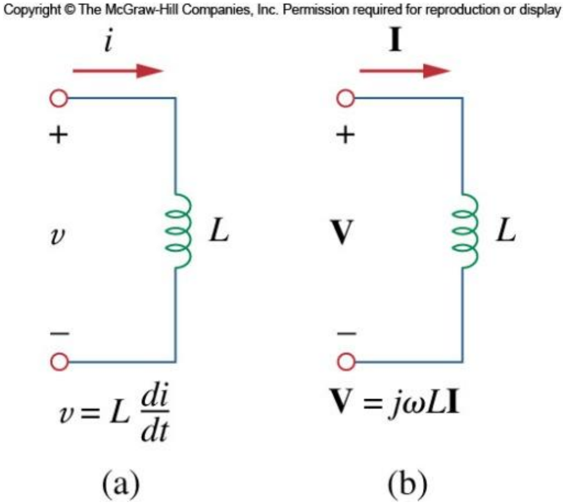


V-I Phasor Relationship for Inductors

$$i = I_m \cos(\omega t + \phi) \qquad \mathbf{I} = I_m \angle \phi$$

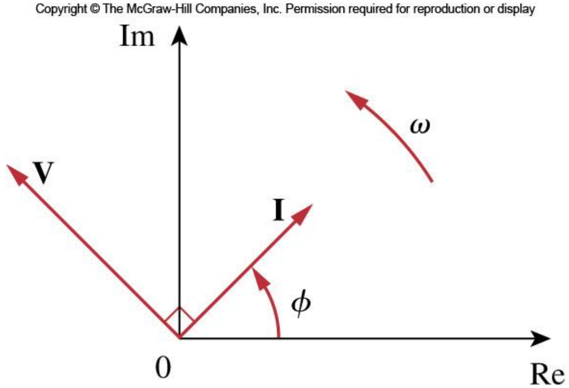
$$v = L\frac{di}{dt}$$

$$\mathbf{V} = j\omega L \cdot \mathbf{I}$$



$$\boxed{\mathbf{V} = j\omega L\mathbf{I}}$$

- The voltage leads the current by 90° (phase shift = 90°)

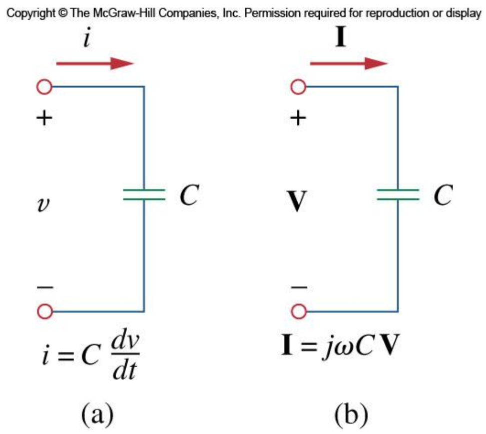


V-I Phasor Relationship for Capacitors

$$v = V_m \cos(\omega t + \phi) \qquad \mathbf{V} = V_m \angle \phi$$

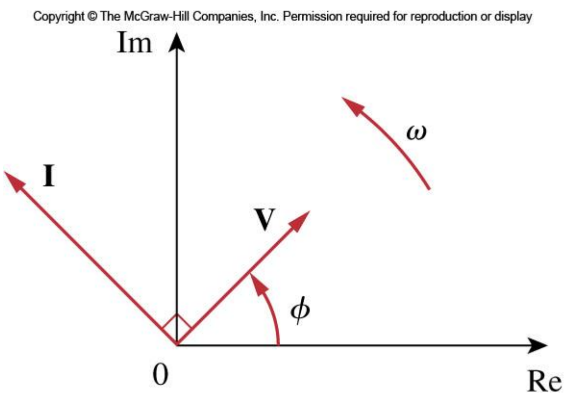
$$i = C \frac{dv}{dt}$$

$$\mathbf{I} = j\omega C \cdot \mathbf{V}$$



$$\boxed{\mathbf{V} = \frac{\mathbf{I}}{j\omega C}}$$

- The voltage lags the current by 90°.



Impedance

- The voltage-current relations for R, L and C elements are

$$\mathbf{V} = R\mathbf{I} \qquad \mathbf{V} = j\omega L\mathbf{I} \qquad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = R \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- Phasors allow us to express the relationship between current and voltage using a formula like Ohm's law:

$$\mathbf{V} = \mathbf{I}\,\boldsymbol{Z} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

- **Z** is called impedance, measured in ohms.
  - Impedance is not a phasor! But it is (often) a complex number.
  - Impedance depends on the frequency ω.

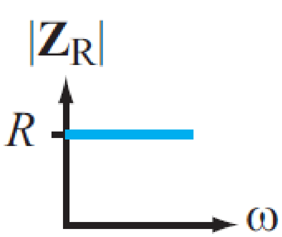
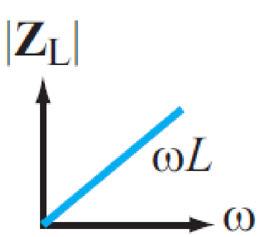
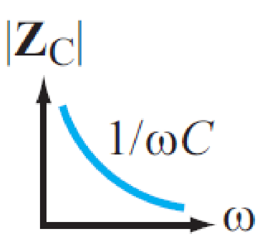
Admittance

Admittance is simply the inverse of impedance, unit: Siemens.

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}}$$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Summary of *R*, *L*, *C*

Property	R	L	C
v–i	$v = Ri$	$v = L\dfrac{di}{dt}$	$i = C\dfrac{dv}{dt}$
V–I	$\mathbf{V} = R\mathbf{I}$	$\mathbf{V} = j\omega L\mathbf{I}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$
Z	R	$j\omega L$	$\dfrac{1}{j\omega C}$
dc equivalent	R	Short circuit	Open circuit
High-frequency equivalent	R	Open circuit	Short circuit
Frequency response			

Summary

- Sinusoidal signals
- Circuit response to a sinusoidal input
  - Transient response + steady state response
- Phasor
  - Definition
  - Common formulas
  - Linearity property
  - KCL/KVL
  - Impedances of R, L, C