



Chapter 7

- Second-Order Circuits

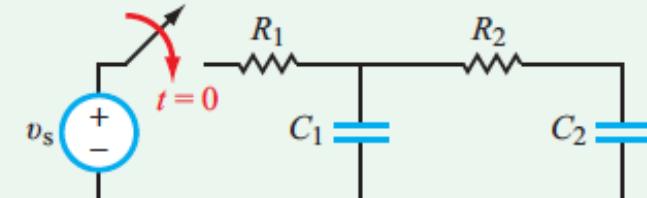
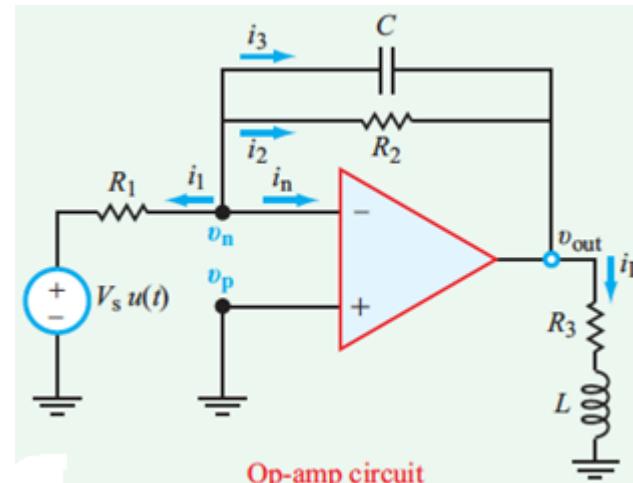


Outline

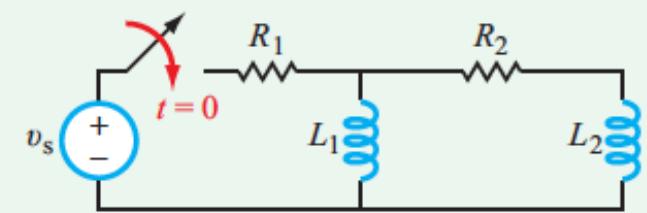
- 二阶电路的结构
- 二阶电路的方程形式
- 二阶电路的初态和终态求解
- 二阶电路响应的求解
- 实际二阶电路分析

Second-Order Circuits

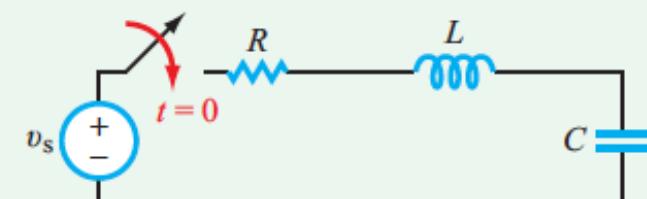
- Two *independent* energy storage elements
- Independent: The voltage/current of one element cannot be inferred from that of the other element.
- The voltage/current can be described by a second order differential equation.



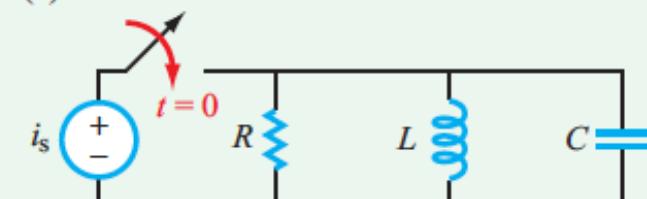
(a) 2 capacitors



(b) 2 inductors

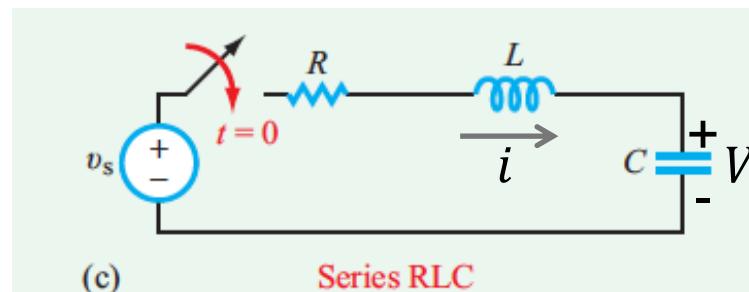


(c) Series RLC



(d) Parallel RLC

Second-Order Circuits



$$Ri + L \frac{di}{dt} + V = v_s \quad (1)$$

$$i = C \frac{dV}{dt} \quad (2)$$

Plug (2) into (1), we have

$$RC \frac{dV}{dt} + LC \frac{d^2V}{dt^2} + V = v_s$$

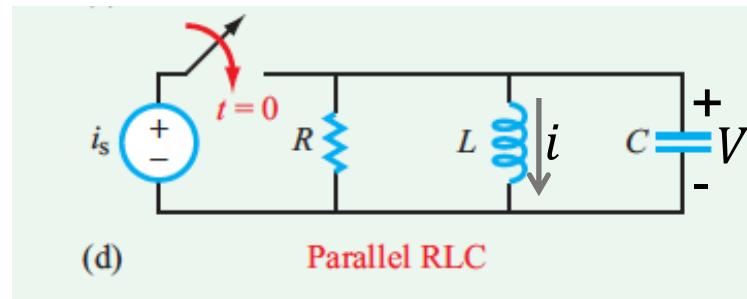
To obtain the equation for the current i , rewrite (1) as

$$V = v_s - Ri - L \frac{di}{dt} \quad (3)$$

Plug (3) into (2), we have

$i = -CR \frac{di}{dt} - CL \frac{d^2i}{dt^2} \rightarrow LC \frac{d^2i}{dt^2} + RC \frac{di}{dt} + i = 0$, i.e. i follows the same equation as V , except for the constant term.

Source-Free Parallel RLC Network



$$i + C \frac{dV}{dt} + \frac{V}{R} = i_s \quad (1)$$

$$V = L \frac{di}{dt} \quad (2)$$

Plug (2) into (1), we have

$$\frac{L}{R} \frac{di}{dt} + LC \frac{d^2i}{dt^2} + i = i_s$$

To obtain the equation for the current V , rewrite (1) as

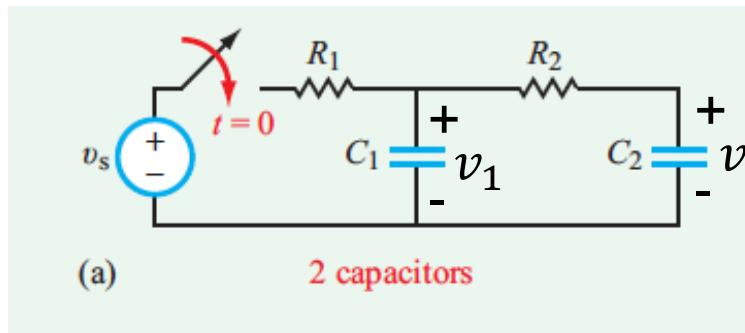
$$i = i_s - C \frac{dV}{dt} - \frac{V}{R} \quad (3)$$

Plug (3) into (2), we have

$$V = -CL \frac{d^2V}{dt^2} - \frac{L}{R} \frac{dV}{dt} \rightarrow LC \frac{d^2V}{dt^2} + \frac{L}{R} \frac{dV}{dt} + V = 0,$$

i.e. i follows the same equation as V , except for the constant term.

Double Capacitors Network



$$\left\{ \begin{array}{l} C_1 \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} + \frac{v_1 - v_s}{R_1} = 0 \\ \frac{v_1 - v_2}{R_2} - C_2 \frac{dv_2}{dt} = 0 \end{array} \right.$$

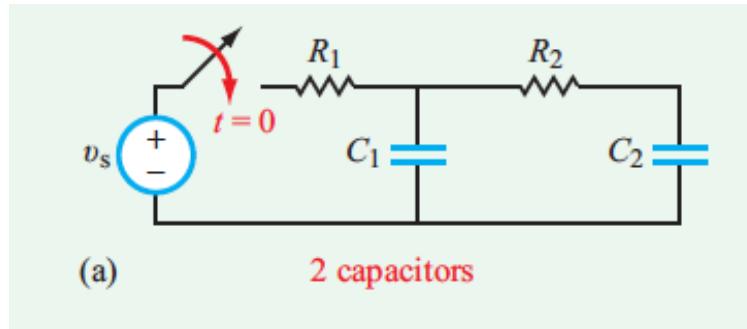
$$C_1 \frac{dv_1}{dt} + \frac{v_1 - v_2}{R_2} + \frac{v_1 - v_s}{R_1} = 0$$

$$v_2 = R_2 C_1 \frac{dv_1}{dt} + v_1 + R_2 \frac{v_1 - v_s}{R_1}$$

Plug the last equation into the first, we have:

$$\left(C_1 + C_2 + \frac{R_2}{R_1} C_2 \right) \frac{dv_1}{dt} + R_2 C_1 C_2 \frac{d^2 v_1}{dt^2} + \frac{v_1 - v_s}{R_1} = 0$$

Equation for v_1



$$\left\{ \begin{array}{l} C_1 \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} + \frac{v_1 - v_s}{R_1} = 0 \\ \frac{v_1 - v_2}{R_2} - C_2 \frac{dv_2}{dt} = 0 \end{array} \right.$$

$$v_1 = R_2 C_2 \frac{dv_2}{dt} + v_2$$

Plug the last equation into the first, we have:

$$\left(C_2 + C_1 + \frac{R_2}{R_1} C_2 \right) \frac{dv_2}{dt} + R_2 C_1 C_2 \frac{d^2 v_2}{dt^2} + \frac{v_2 - v_s}{R_1} = 0$$



Why Second Order Equations

2b method

- 变量数 2b (v and I for each branch)
- KCL 独立方程数 $n-1$
- KVL 独立方程数 $b-n+1$
- $V-I$ 关系方程数 b (contain two first-order derivatives)



Why Second Order Equations

Expressing the equations in matrix form: $A x = b$, the solution is $x=A^{-1}b$.
The two first order derivatives are only present in b . Denoting the variables in the derivatives as x_1 and x_2 , their solutions will be of the following form :

$$x_1 = a_{11} \frac{dx_1}{dt} + a_{12} \frac{dx_2}{dt} + c_1 \quad (1)$$

$$x_2 = a_{21} \frac{dx_1}{dt} + a_{22} \frac{dx_2}{dt} + c_2 \quad (2)$$

Solving the above two equations for x_1 and $\frac{dx_1}{dt}$, we have

$$x_1 = \frac{a_{11}}{a_{21}} x_2 + \frac{a_{21}a_{12}-a_{11}a_{22}}{a_{21}} \frac{dx_2}{dt} + c_1 - \frac{a_{11}}{a_{21}} c_2 \quad (3)$$

Plug (3) into (2), we have

$$(a_{21}a_{12}-a_{11}a_{22}) \frac{d^2x_2}{dt^2} + (a_{11}+a_{22}) \frac{dx_2}{dt} - x_2 + c_2 = 0$$

From symmetry, we also have

$$(a_{21}a_{12}-a_{11}a_{22}) \frac{d^2x_1}{dt^2} + (a_{11}+a_{22}) \frac{dx_1}{dt} - x_1 + c_1 = 0$$



Initial Conditions

To solve a 2nd order differential equation,

$$\frac{d^2x}{dt^2} + a \frac{dx}{dt} + bx + c = 0$$

the initial conditions of x and $\frac{dx}{dt}$ at $t = 0$ are needed.



Steps to Obtain Initial Condition

At $t = 0^+$:

0th order: $V_c(0^+) = V_c(0^-); I_L(0^+) = I_L(0^-)$.

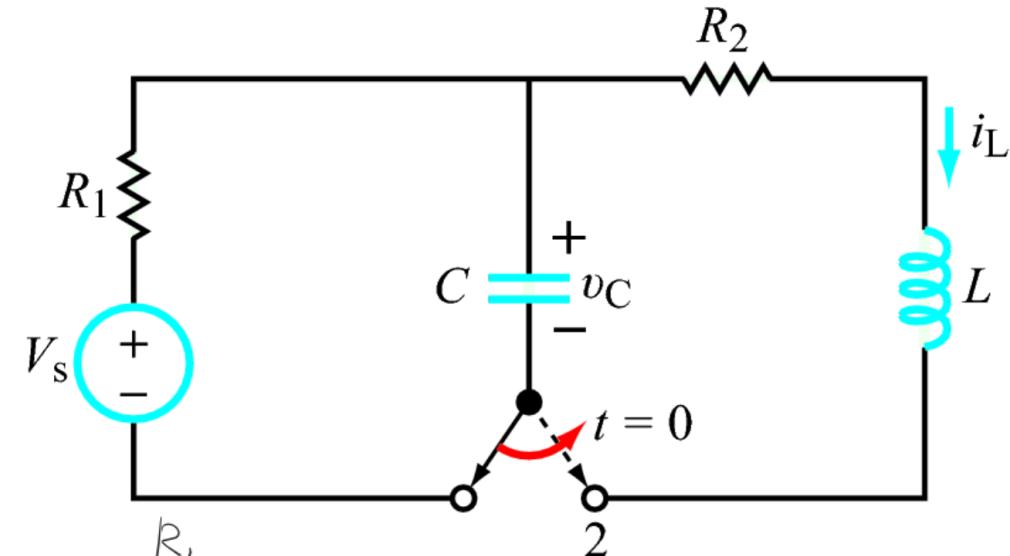
1st order: $\frac{dV_c(0^+)}{dt} = \frac{1}{C} I_c(0^+); \frac{dI_L(0^+)}{dt} = \frac{1}{L} V_L(0^+)$.

To obtain $I_c(0^+)$ and $V_L(0^+)$, we treat

- Capacitor as a voltage source with $V = V_c(0^+)$
- Inductor as a current source with $I = I_L(0^+)$
- Then solve regular (without 1st and 2nd order terms) circuit equations

Example One

The switch has been closed for a long time. It is open at $t = 0$. Find $v_C(0^+)$, $dv_C(0^+)/dt$, $i_L(0^+)$, $di_L(0^+)/dt$

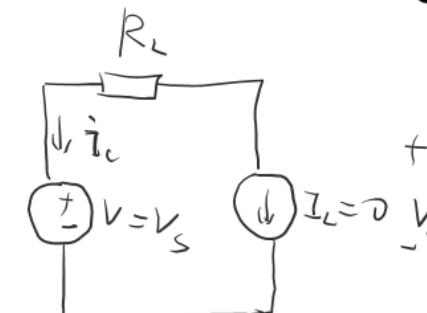


$$V_C(0^+) = V_{C(0^-)} = V_s$$

$$I_L(0^+) = I_{L(0^-)} = 0$$

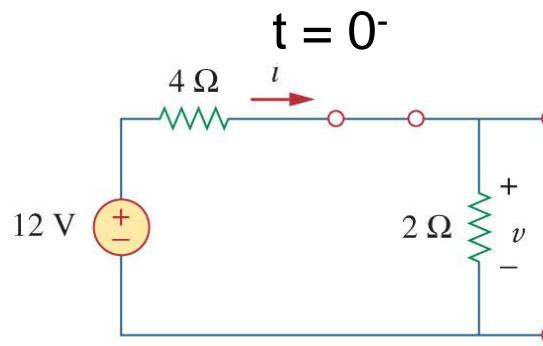
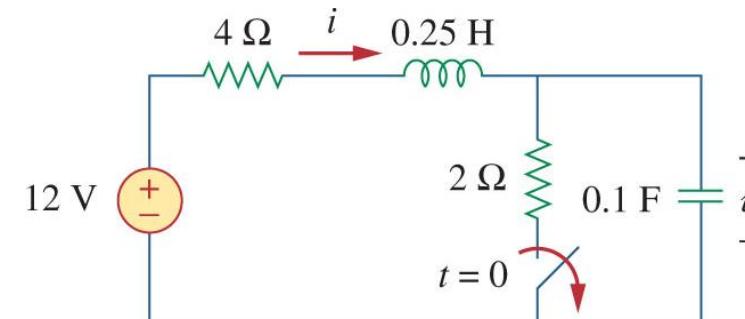
$$i_C(0^+) = 0 \Rightarrow \frac{dV_C(0^+)}{dt} = 0$$

$$V_L(0^+) = V_s \Rightarrow \frac{dI_L(0^+)}{dt} = \frac{V_s}{L}$$

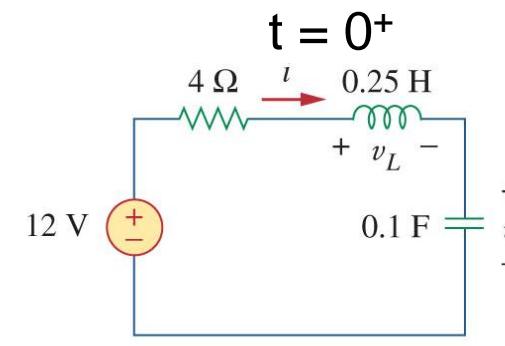


Example Two

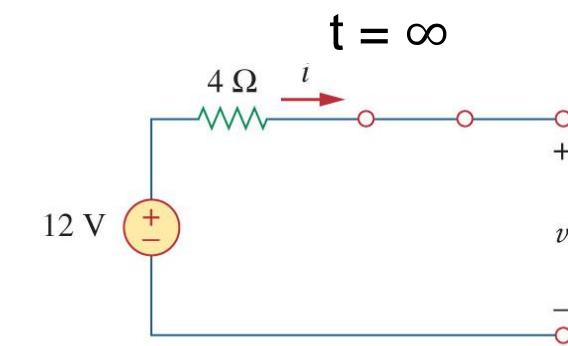
- The switch has been closed for a long time. It is open at $t = 0$. Find
 - $i(0^+), v(0^+)$
 - $di(0^+)/dt, dv(0^+)/dt$
 - $i(\infty), v(\infty)$



(a)



(b)



(c)



Solution of Example Two

$$i(0^+) = i(0^-) = \frac{12}{6} = 2A$$

$$V(0^+) = V(0^-) = 12 \cdot \frac{2}{6} = 4V$$

$$\frac{di(0^+)}{dt} = \frac{V_L}{L} = 0$$

$$\frac{dV(0^+)}{dt} = \frac{i(0^+)}{C} = 20 \frac{V}{S}$$

$$i(\infty) = 0A$$

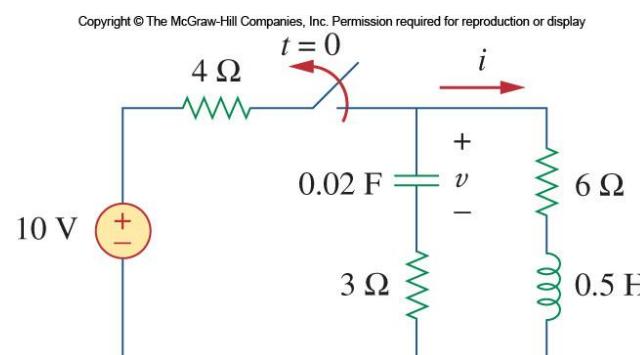
$$V(\infty) = 12V$$

Exercise

- Assume the circuit has reached steady state at $t = 0^-$.

Find

- $i(0^+), v(0^+)$
- $di(0^+)/dt, dv(0^+)/dt$
- $i(\infty), v(\infty)$



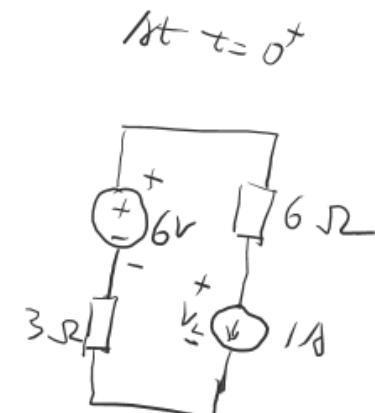
$$i(0^+) = i(0^-) = 1A$$

$$v(0^+) = v(0^-) = 6V$$

$$\frac{di(0^+)}{dt} = \frac{V_L}{L} = 6A/s$$

$$V_L + 6 + 3 - 6 = 0 \Rightarrow V_L = -3V$$

$$\frac{dv(0^+)}{dt} = -\frac{i}{C} = -50V/s$$





Solutions of the 2nd Order Equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by + c = 0$$

In RLC circuits, we have $a>0$ and $b>0$.

Let $y = x - \frac{c}{b}$

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

Initial conditions: $x(0) = y(0) + \frac{c}{b}$ and $\frac{dx(0)}{dt} = \frac{dy(0)}{dt}$.

The solution for y is a sum of the steady state solution ($y = -\frac{c}{b}$) and a solution when $c = 0$, i.e. natural response with all voltage/current sources removed.



Solution of the 2nd Order Equation

First, try a solution of the form $x = e^{-pt}$

$$(p^2 - ap + b)e^{-pt} = 0$$

$$p^2 - ap + b = 0$$

When $a/2 > \sqrt{b}$, two real solutions: $p = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$

When $\frac{a}{2} < \sqrt{b}$, two complex solutions: $p = \frac{a}{2} \pm i\sqrt{b - \left(\frac{a}{2}\right)^2}$

When $\frac{a}{2} = \sqrt{b}$, one real solution: $p = \frac{a}{2}$



The Second Solution for Critical Damping

When critically damped, we have $\frac{a}{2} = \sqrt{b}$

$$\frac{d^2x}{dt^2} + a \frac{dx}{dt} + \frac{a^2}{4} x = 0 \quad (1)$$

In addition to $x = e^{-\frac{a}{2}t}$, $x = te^{-\frac{a}{2}t}$ also satisfies the above equation.

Proof:

$$\frac{dx}{dt} = -\frac{a}{2}te^{-\frac{a}{2}t} + e^{-\frac{a}{2}t}$$

$$\frac{d^2x}{dt^2} = -\frac{a}{2}\left[-\frac{a}{2}te^{-\frac{a}{2}t} + e^{-\frac{a}{2}t}\right] - \frac{a}{2}e^{-\frac{a}{2}t}$$

Therefore, the left side of Eq. (1) is

$$-\frac{a}{2}\left[-\frac{a}{2}t + 1\right] - \frac{a}{2} + a\left[-\frac{a}{2}t + 1\right] + \frac{a^2}{4}t = 0$$



The General Solution

- The general solution is a linear combination of the two solutions that we find

$$x(t) = Ax_1(t) + Bx_2(t)$$

- To uniquely determine A and B, we apply the initial conditions for $\frac{dx(0)}{dt}$ and $x(0)$:

$$\left\{ \begin{array}{l} Ax_1(0) + Bx_2(0) = x(0) \\ A \frac{dx_1(0)}{dt} + B \frac{dx_2(0)}{dt} = \frac{dx(0)}{dt} \end{array} \right. \quad \rightarrow \quad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) \\ \frac{dx_1(0)}{dt} & \frac{dx_2(0)}{dt} \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ \frac{dx(0)}{dt} \end{bmatrix}$$



Overdamping

When $a/2 > \sqrt{b}$, two real solutions: $x = e^{-pt}$ with $p = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} > 0$

$$x(t) = Ae^{-p_1 t} + Be^{-p_2 t}$$

$$\begin{aligned}[A]_B &= \begin{bmatrix} x_1(0) & x_2(0) \\ x'_1(0) & x'_2(0) \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -p_1 & -p_2 \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \frac{1}{p_1 - p_2} \begin{bmatrix} -p_2 & -1 \\ p_1 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix}\end{aligned}$$

$$x(t) = \frac{[p_2 x(0) + x'(0)]}{p_2 - p_1} e^{-p_1 t} - \frac{[p_1 x(0) + x'(0)]}{p_2 - p_1} e^{-p_2 t}$$



Underdamping

When $\frac{a}{2} < \sqrt{b}$, two complex solutions: $x = e^{-pt}$ with $p = \frac{a}{2} \pm i\sqrt{b - \left(\frac{a}{2}\right)^2}$

Define $\delta = \frac{a}{2}$, $\omega_0 = \sqrt{b}$, $\omega = \sqrt{\omega_0^2 - \delta^2}$

$$\begin{aligned}x(t) &= Ae^{-\delta t+i\omega t} + Be^{-\delta t-i\omega t} = e^{-\delta t}(Ae^{i\omega t} + Be^{-i\omega t}) \\&= e^{-\delta t}(C \cdot \cos\omega t + D \cdot \sin\omega t)\end{aligned}$$

$$C = x(0)$$

$$-C\delta + D\omega = x'(0)$$

$$D = [x'(0) + \delta x(0)]/\omega$$

$$x(t) = [x(0)\cos(\omega t) + \frac{x'(0) + \delta x(0)}{\omega} \sin \omega t]e^{-\delta t}$$



Critical damping

When $\frac{a}{2} = \sqrt{b}$, two solutions: $e^{-\frac{a}{2}t}$ and $te^{-\frac{a}{2}t}$

Define $\delta = \frac{a}{2}$ $x(t) = [A + Bt]e^{-\delta t}$

$$A = x(0)$$

$$-A\delta + B = x'(0)$$

$$B = x'(0) + \delta x(0)$$

$$x(t) = [x(0)(1 + \delta t) + x'(0)t]e^{-\delta t}$$



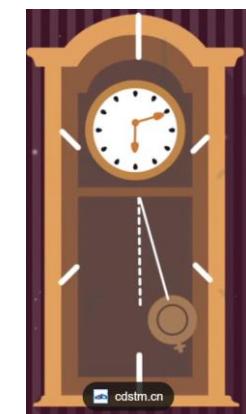
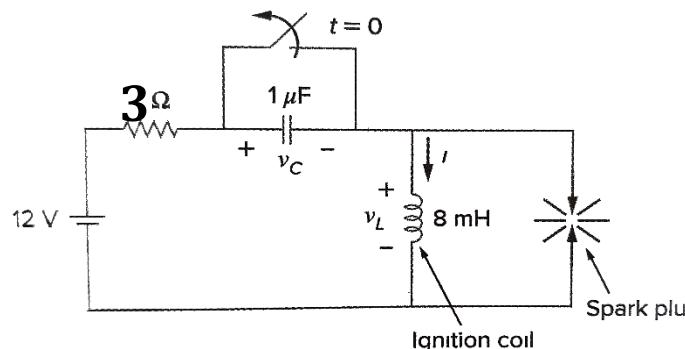
Outline (Lesson 2)

- RLC 串联电路的自然响应
 - 阻尼特性：欠，过，临界
 - 初始条件： $x(0) = 0$ 或 $x'(0) = 0$
- RLC串联电路的阶跃响应
- RLC串联电路的全响应
- 其它二阶电路的分析
- 冲击响应的分析



二阶电路时域分析的重要性

- 二阶电路应用广泛 (金属探测器, 打火塞, MRI线圈...)。
- 频域分析虽然简单, 但无法得到每一时刻的电路特性。
- 理论分析同样适用于其他系统





Damping conditions

$$\frac{d^2y}{dt^2} + a \frac{dy}{dt} + by + c = 0 \quad \xrightarrow{y = x - \frac{c}{b}} \quad \frac{d^2x}{dt^2} + a \frac{dx}{dt} + bx = 0$$

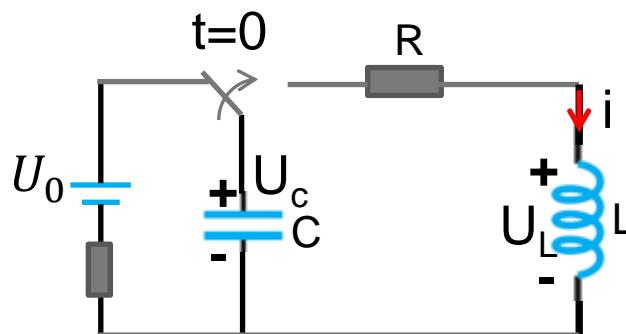
$$\delta = \frac{a}{2}, \omega_0 = \sqrt{b}$$

Three damping conditions can be identified depending on the relative values between δ and ω_0 :

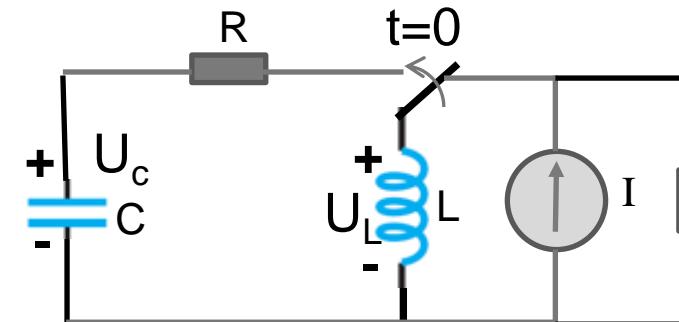
- Overdamping ($\delta > \omega_0$)
- Underdamping ($\delta < \omega_0$)
- Critical damping ($\delta = \omega_0$)

Initial conditions

$$U_c(0) = U_0, U_c'(0) = 0$$



$$U_c(0) = 0, U_c'(0) = -\frac{I}{C}$$





Overdamping ($\delta > \omega_0$)

$$x(t) = \frac{[p_2 x(0) + x'(0)]}{p_2 - p_1} e^{-p_1 t} - \frac{[p_1 x(0) + x'(0)]}{p_2 - p_1} e^{-p_2 t},$$

$$\text{where } p_1 = \delta + \sqrt{\delta^2 - \omega_0^2}, \quad p_2 = \delta - \sqrt{\delta^2 - \omega_0^2}$$

	$x'(0) = 0$	$x(0) = 0$
$x(t)$	$\frac{x(0)}{p_1 - p_2} (p_1 e^{-p_2 t} - p_2 e^{-p_1 t})$	$\frac{x'(0)}{p_1 - p_2} (e^{-p_2 t} - e^{-p_1 t})$
$x'(t)$	$\frac{x(0)p_1 p_2}{p_1 - p_2} (e^{-p_1 t} - e^{-p_2 t})$	$\frac{x'(0)}{p_1 - p_2} (p_1 e^{-p_1 t} - p_2 e^{-p_2 t})$
$x''(t)$	$\frac{x(0)p_1 p_2}{p_1 - p_2} (p_2 e^{-p_2 t} - p_1 e^{-p_1 t})$	$\frac{x'(0)}{p_1 - p_2} (p_2^2 e^{-p_2 t} - p_1^2 e^{-p_1 t})$



Source-Free Series RLC Circuit

$$\frac{d^2U_c}{dt^2} + \frac{R}{L}\frac{dU_c}{dt} + \frac{1}{LC}U_c = 0$$

$$\delta = \frac{R}{2L}; \omega_0 = \frac{1}{\sqrt{LC}}$$

Unit of δ : neper per s (Np/s)

Unit of ω_0 : radian per s (rad/s)

x neper: signal ratio = e^x

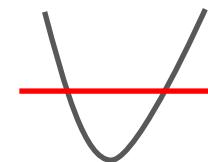
$$p_1 p_2 = \frac{1}{LC}$$



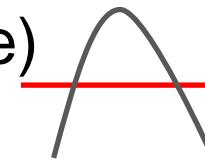
时间曲线分析规则

- $f'(t) > 0$: $f(t)$ 单调递增
- $f'(t) < 0$: $f(t)$ 单调递减
- $f'(t) = 0$: $f(t)$ 达到峰值或谷底

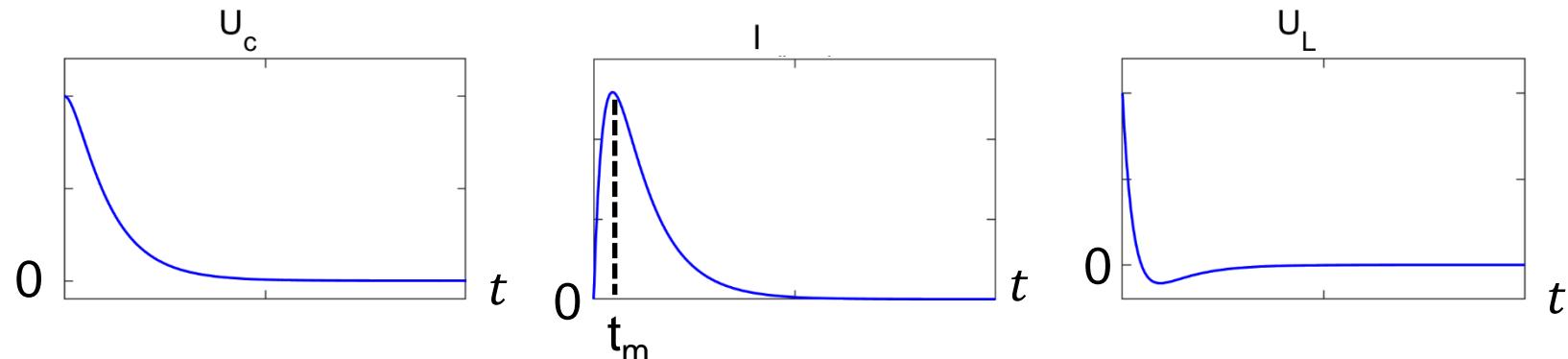
- $f''(t) > 0$: $f(t)$ 为凹函数 (Convex)



- $f''(t) < 0$: $f(t)$ 为凸函数 (Concave)



Time course when $x'(0) = 0$



$$i = -C \frac{dU_c}{dt}$$

$$p_2 e^{-p_2 t_m} = p_1 e^{-p_1 t_m}$$

$$t_m = \frac{\ln p_2/p_1}{p_2 - p_1}$$

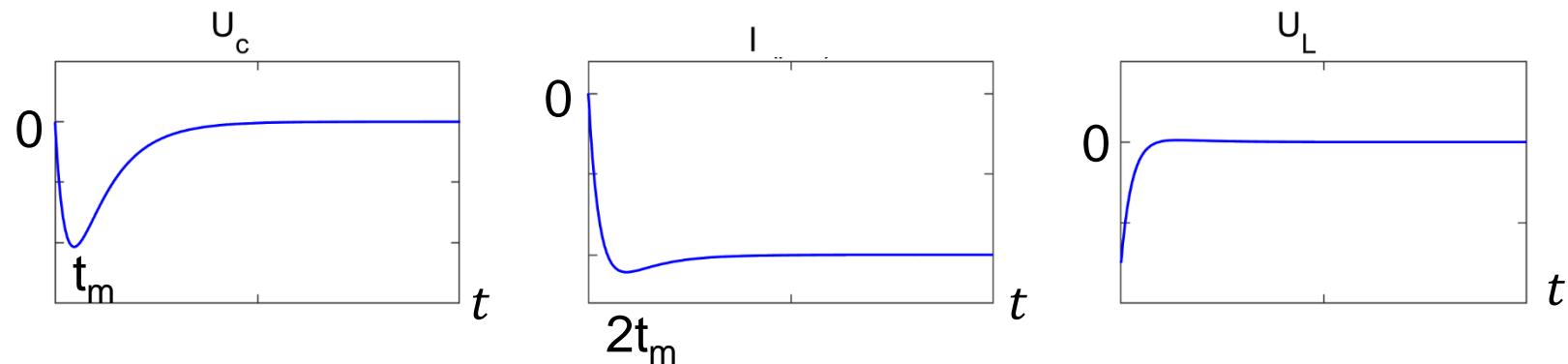
$$U_L = L \frac{di}{dt} = -LCU_c''(t)$$

电容释放能量: $t > 0$

电感吸收能量: $0 < t < t_m$

电感释放能量: $t > t_m$

Time course when $x(0) = 0$



电容吸收能量: $0 < t < t_m$

电容释放能量: $t > t_m$

电感能量

$0 < t < t_m$ 释放,

$t_m < t < 2t_m$ 吸收

$t > 2t_m$ 释放

$$p_2^2 e^{-2p_2 t_m} = p_1^2 e^{-2p_1 t_m}$$



Underdamping ($\delta < \omega_0$)

$$x(t) = [x(0)\cos(\omega t) + \frac{x'(0)+\delta x(0)}{\omega}\sin \omega t]e^{-\delta t}, \text{ where } \omega = \sqrt{\omega_0^2 - \delta^2}$$

	$x'(0) = 0$	$x(0) = 0$
$x(t)$	$\frac{x(0)\omega_0}{\omega} e^{-\delta t} \sin(\omega t + \beta)$	$\frac{x'(0)}{\omega} e^{-\delta t} \sin \omega t$
$x'(t)$	$-\frac{x(0)\omega_0^2}{\omega} e^{-\delta t} \sin \omega t$	$-\frac{x'(0)\omega_0}{\omega} e^{-\delta t} \sin(\omega t - \beta)$
$x''(t)$	$\frac{x(0)\omega_0^3}{\omega} e^{-\delta t} \sin(\omega t - \beta)$	$\frac{x'(0)\omega_0^2}{\omega} e^{-\delta t} \sin(\omega t - 2\beta)$

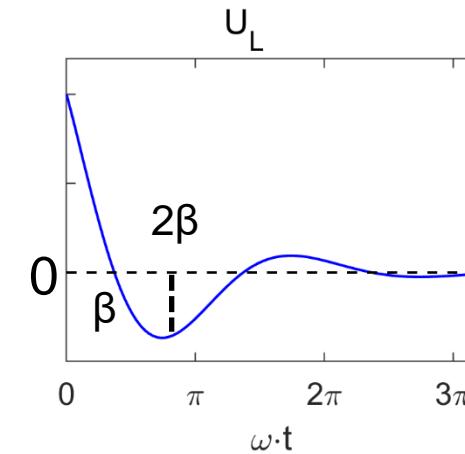
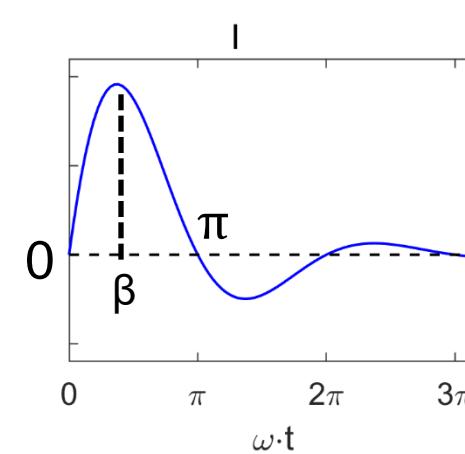
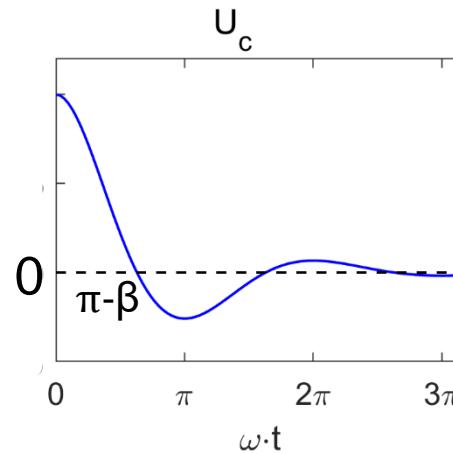
$$\frac{d[e^{-\delta t} \sin (\omega t + \alpha)]}{dt} = -\omega_0 e^{-\delta t} \sin (\omega t + \alpha - \beta)$$

$$\omega = \sqrt{\omega_0^2 - \delta^2}$$

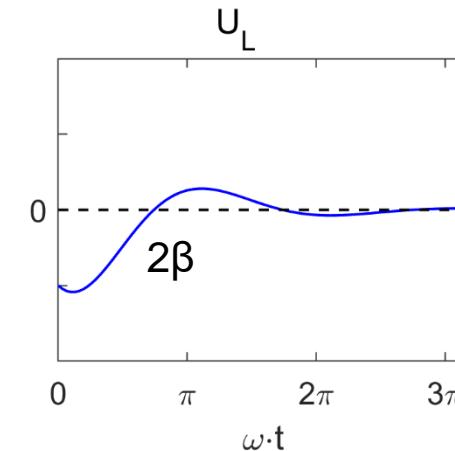
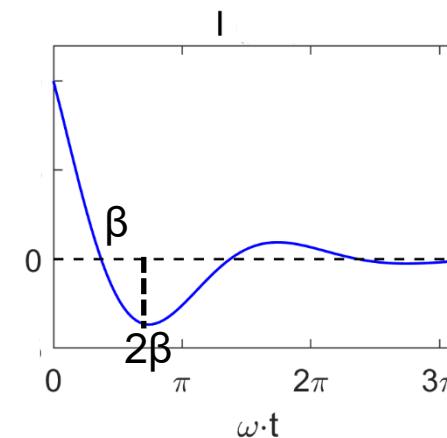
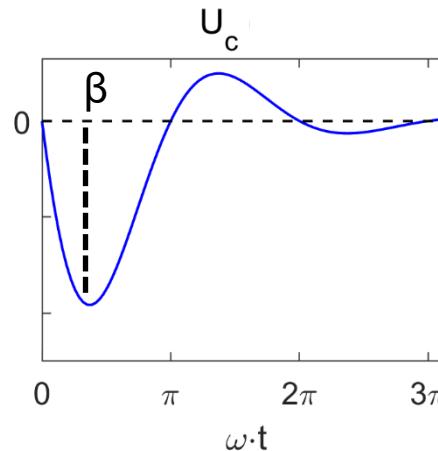
$$\beta = \arctan\left(\frac{\omega}{\delta}\right)$$



Time course when $x'(0) = 0$



Time course when $x(0) = 0$



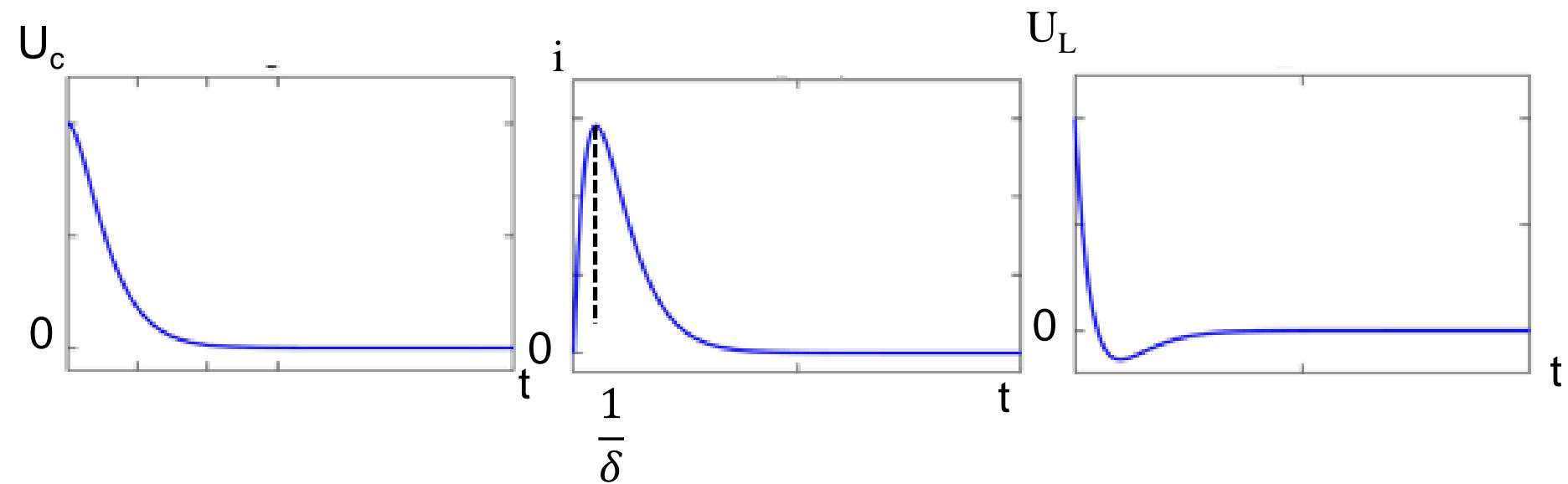


Critical damping ($\delta = \omega_0$)

$$x(t) = [x(0)(1 + \delta t) + x'(0)t]e^{-\delta t}$$

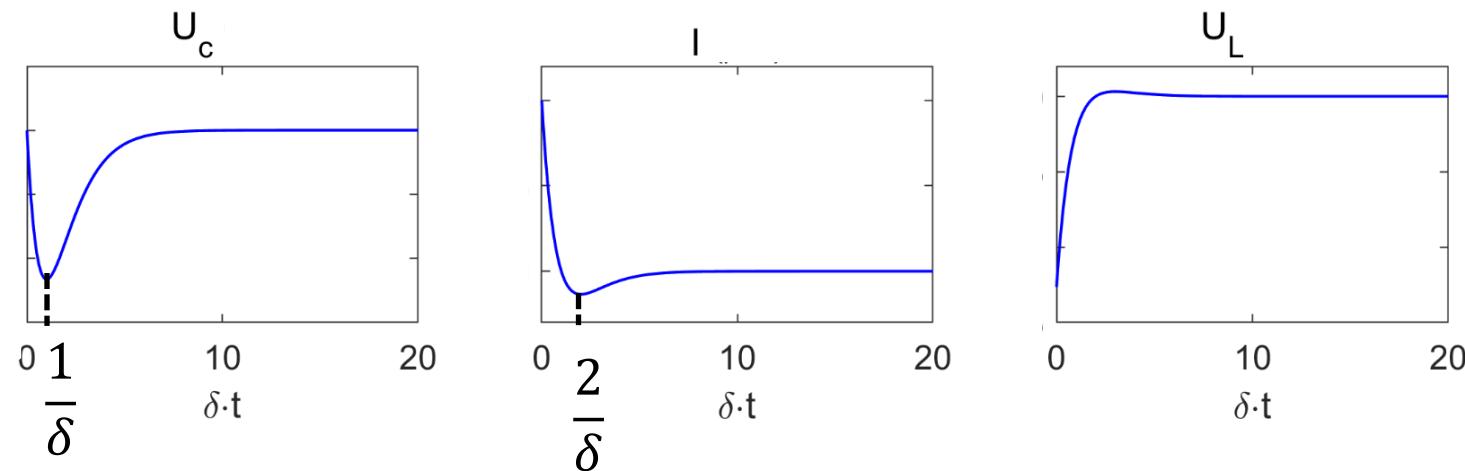
	$x'(0) = 0$	$x(0) = 0$
$x(t)$	$x(0)(1 + \delta t) e^{-\delta t}$	$x'(0)te^{-\delta t}$
$x'(t)$	$-x(0)\delta^2 t e^{-\delta t}$	$x'(0)(1 - \delta t)e^{-\delta t}$
$x''(t)$	$-x(0)\delta^2(1 - \delta t)e^{-\delta t}$	$-\delta x'(0)(2 - \delta t)e^{-\delta t}$

Time course when $x'(0) = 0$





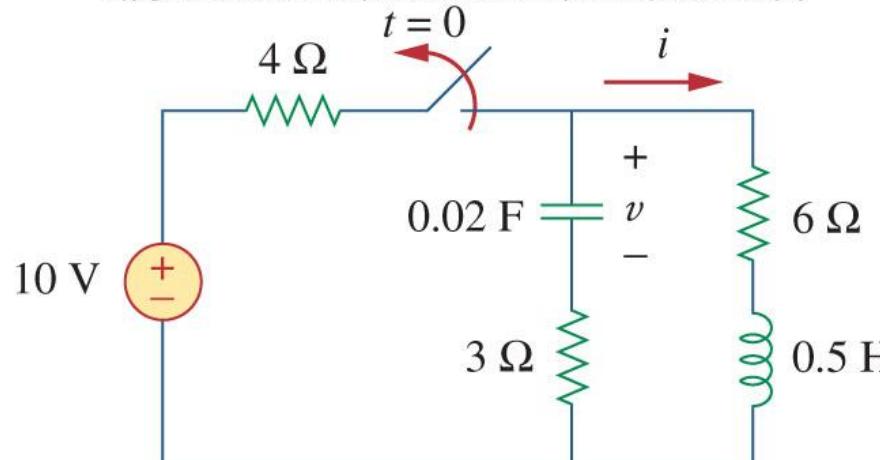
Time course when $x(0) = 0$



Example 3 (both $v(0)$ and $v'(0)$ nonzero)

- Find $v(t)$ & $i(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.

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$$\begin{aligned}v(0) &= 6 \text{ V} \\v'(0) &= -i(0)/C = -1 \text{ A}/0.02 \text{ F} \\&= -50 \text{ V/s}\end{aligned}$$

$$\delta = \frac{R}{2L} = 9 \text{ Np/s} \quad \omega_0 = \frac{1}{\sqrt{LC}} = 10 \text{ rad/s}$$

Therefore, underdamping

$$\omega = \sqrt{\delta^2 - \omega_0^2} = 4.359 \text{ rad/s} \quad \beta = \tan\left(\frac{\omega}{\delta}\right) = 0.451$$



Solution

$$\begin{aligned}v(t) &= [v(0)\cos(\omega t) + \frac{v'(0) + \delta v(0)}{\omega} \sin \omega t] e^{-\delta t} \\&= v(0)[\cos(\omega t) + \frac{\delta}{\omega} \sin \omega t] e^{-\delta t} + v'(0) \frac{1}{\omega} \sin \omega t e^{-\delta t}\end{aligned}$$

线性系统的可叠加性：

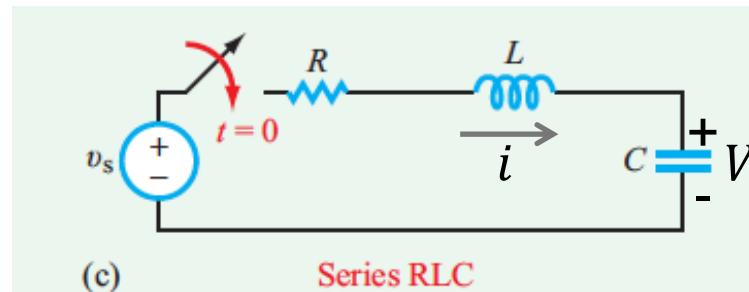
$v(t)$ is the sum of responses to initial conditions
[$v(0), 0$] (电容放电) and [$0, v'(0)$] (电感消磁).



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- 其它二阶电路的分析
- 冲击响应的分析

Step response



$$RC \frac{dV}{dt} + LC \frac{d^2V}{dt^2} + V = v_s$$

$$\frac{d^2V}{dt^2} + \frac{R}{L} \frac{dV}{dt} + \frac{1}{LC} V = \frac{v_s}{LC}$$

$$x = V - v_s$$

$$\frac{d^2x}{dt^2} + \frac{R}{L} \frac{dx}{dt} + \frac{1}{LC} x = 0$$

$$x(0) = V(0) - v_s = -v_s$$

$$x'(0) = V'(0) = 0$$



Solution of $V(t)$

	$x(t)$	$V(t)$
Overdamping	$\frac{x(0)}{p_1 - p_2} (p_1 e^{-p_2 t} - p_2 e^{-p_1 t})$	$\frac{-v_s}{p_1 - p_2} (p_1 e^{-p_2 t} - p_2 e^{-p_1 t}) + v_s$
Underdamping	$\frac{x(0)\omega_0}{\omega} e^{-\delta t} \sin(\omega t + \beta)$	$\frac{-v_s \omega_0}{\omega} e^{-\delta t} \sin(\omega t + \beta) + v_s$
Critical damping	$x(0)(1 + \delta t) e^{-\delta t}$	$-v_s(1 + \delta t) e^{-\delta t} + v_s$

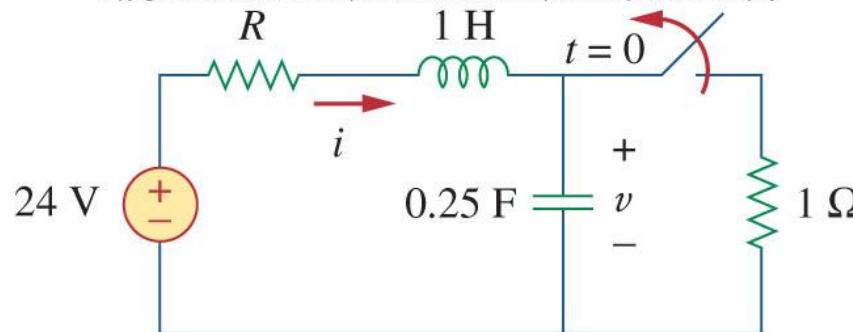


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Full Response

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$$\begin{aligned}v(0) &\neq 0 \text{ and/or } v'(0) \neq 0 \\v'(\infty) &\neq 0\end{aligned}$$

$$t > 0 : \frac{d^2v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{v}{LC} = \frac{24}{LC}$$

- Find $v(t)$ and $i(t)$ for $t > 0$ when:
 - $R = 5\Omega$
 - $R = 4\Omega$
 - $R = 1\Omega$



Full response

$$\frac{d^2v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{1}{LC} v = \frac{24}{LC}$$

$$x = v - 24$$

$$\frac{d^2x}{dt^2} + \frac{R}{L} \frac{dx}{dt} + \frac{1}{LC} x = 0$$

$$x(0) = v(0) - 24$$

$$x'(0) = v'(0)$$



Full Response When $R = 5\Omega$

For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4A = C \frac{dv(0)}{dt}, \quad v(0) = 4V, \quad \frac{dv(0)}{dt} = 16 \text{ V/s}, \quad v(\infty) = 24 \text{ V}$$

For $t > 0$, switch open, a series RLC network

$$\delta = \frac{R}{2L} = 2.5 \frac{\text{Np}}{\text{s}}, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 2 \frac{\text{rad}}{\text{s}}$$

Overdamped.

$$v(t) = v_{ss} + (-21.33e^{-t} + 1.33e^{-4t})$$

$$v_{ss} = 24 \text{ V}$$



Full Response When $R = 4\Omega$

- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 4.8 \text{ A} = C \frac{d\nu(0)}{dt}, \quad \nu(0) = 4.8 \text{ V}, \quad \frac{d\nu(0)}{dt} = 19.2 \text{ V/s}, \quad \nu(\infty) = 24 \text{ V}$$

- For $t > 0$, switch open, a series RLC network

$$\delta = \frac{R}{2L} = 2 \frac{\text{Np}}{\text{s}}, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 2 \text{ rad/s}, \quad p_{1,2} = -2 \text{ Np/s} \quad \text{Critically damped}$$

$$\nu(t) = \nu_{ss} + (A_1 + A_2 t)e^{-2t}$$

$$\nu_{ss} = 24 \text{ V}, \quad A_1 = -19.2 \text{ V}, \quad A_2 = -19.2 \text{ V/s}$$



Full Response When $R = 1 \Omega$

- For $t < 0$, switch closed, capacitor open, inductor shorted.

$$i(0) = 12A = C \frac{dv(0)}{dt}, \quad v(0) = 12V, \quad \frac{dv(0)}{dt} = 48$$

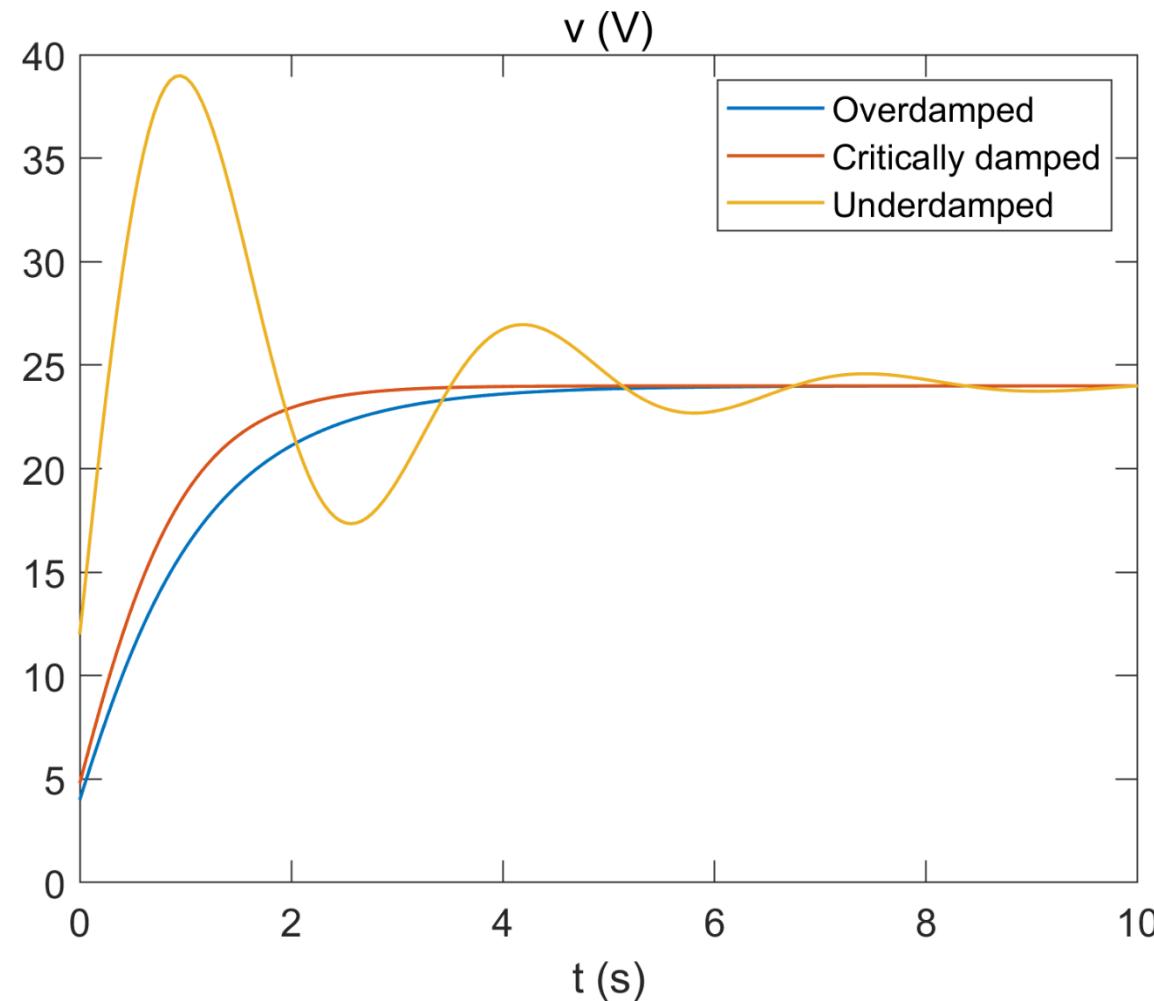
- For $t > 0$, switch open, a series RLC network

$$\delta = \frac{R}{2L} = 0.5 \frac{Np}{s}, \quad \omega_0 = \frac{1}{\sqrt{LC}} = 2 \text{ rad/s}, \quad p_{1,2} = -0.5 \mp 1.936i \text{ s}^{-1}$$

Underdamped

$$v(t) = v_{ss} + (A_1 \sin 1.936t + A_2 \cos 1.936t)e^{-0.5t}$$

$$v_{ss} = 24 \text{ V}, \quad A_1 = 21.7 \text{ V}, \quad A_2 = -12.0 \text{ V}$$



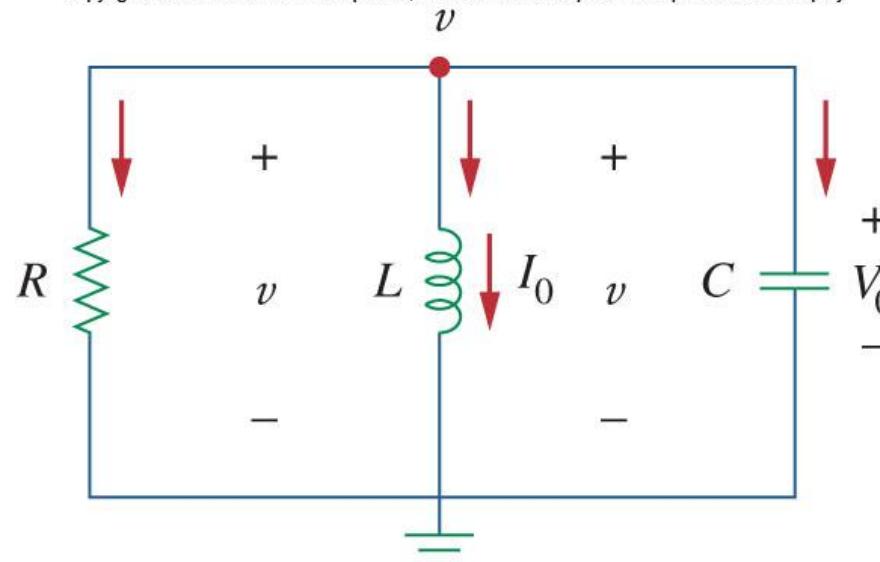


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Parallel RLC Network

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$$\begin{cases} 0 = \frac{v}{R} + C \frac{dv}{dt} + i_L \\ v = L \frac{di_L}{dt} \end{cases}$$

$$\text{For } i_L: i_L + \frac{L}{R} \frac{di_L}{dt} + L C \frac{d^2 i_L}{dt^2} = 0$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC} \frac{di_L}{dt} + \frac{i_L}{LC} = 0$$

$$\text{For } v: \frac{1}{R} \frac{dv}{dt} + C \frac{d^2 v}{dt^2} + \frac{1}{LC} v = 0$$

$$\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0$$

$$\text{solution: } \zeta = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

General RLC Circuits

- Find the complete response v for $t > 0$ in the circuit.

1. Initial and final conditions

$$v(0^+) = v(0^-) = 12V, i(0^+) = i(0^-) = 0$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-12/2}{0.5} = -12 V/s; \quad v_{ss} = 4V$$

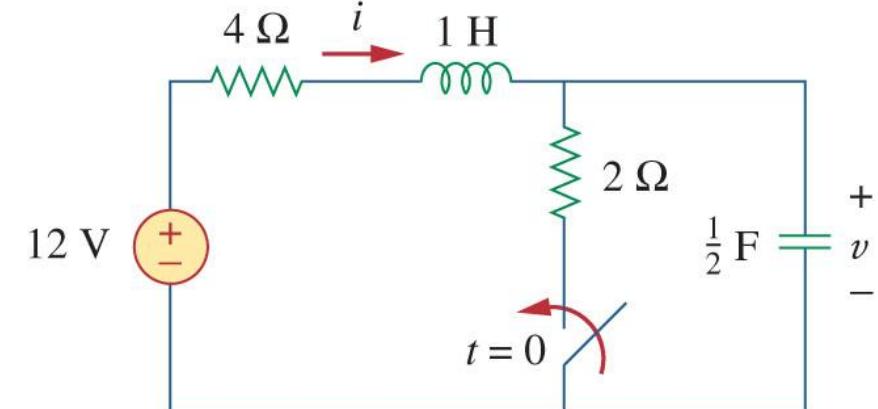
2. Transient response (remove power sources)

$$\text{KCL at node } a: i = \frac{v}{2} + 0.5 \frac{dv}{dt}$$

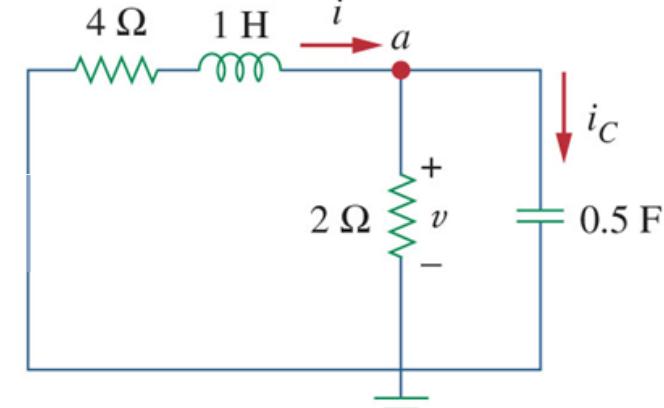
$$\text{KVL on left mesh: } 4i + 1 \frac{di}{dt} + v = 0$$

$$\frac{d^2v}{dt^2} + 5 \frac{dv}{dt} + 6v = 0 \quad v_t(t) = A_1 e^{-3t} + A_2 e^{-2t} = -4e^{-3t} + 12e^{-2t} \text{ (V)}$$

3. Add transient and steady-state responses: $v(t) = v_t(t) + v_{ss}$



Circuit for transient response



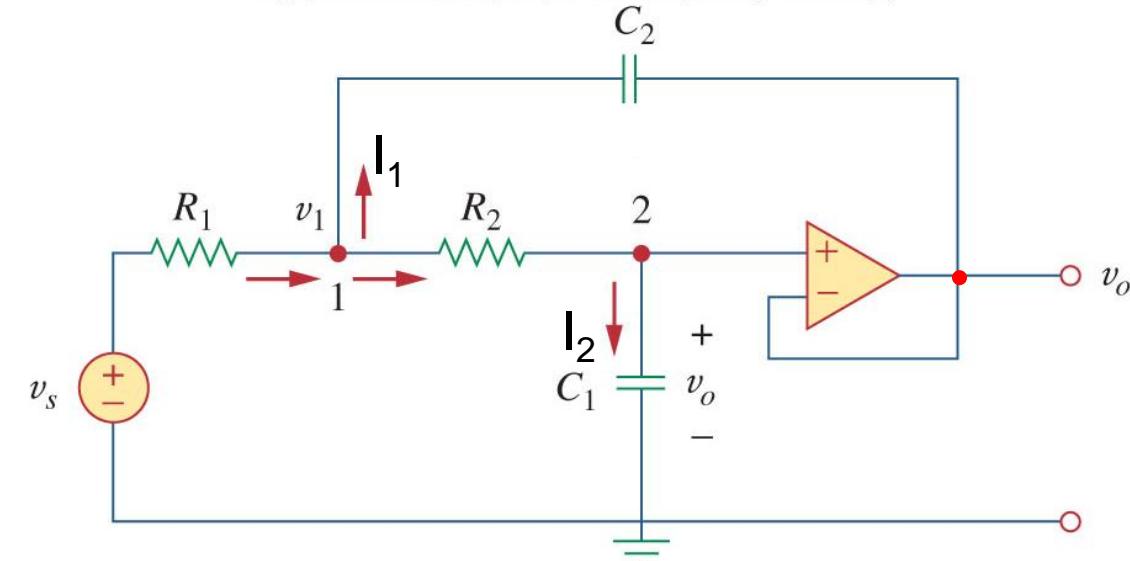
Example of 2nd-order op-amp circuits

Find v_o for $t > 0$

when $v_s = 10u(t)$ mV.

KCL at node 2:

$$C_1 \frac{d v_o}{dt} = \frac{v_1 - v_o}{R_2}$$



$$\begin{aligned} \text{KCL at node 1: } \frac{v_s - v_1}{R_1} &= C_2 \frac{d(v_1 - v_o)}{dt} + \frac{v_1 - v_o}{R_2}, \quad \frac{v_s - v_0 - v_1 + v_o}{R_1} = C_2 \frac{d(v_1 - v_o)}{dt} + \frac{v_1 - v_o}{R_2} \\ \frac{v_s - v_0}{R_1} - \frac{R_2}{R_1} C_1 \frac{d v_o}{dt} &= C_1 R_2 C_2 \frac{d^2 v_o}{dt^2} + C_1 \frac{d v_o}{dt} \\ \frac{d^2 v_o}{dt^2} + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} \right) \frac{d v_o}{dt} + \frac{v_o}{R_1 R_2 C_1 C_2} &= \frac{v_s}{R_1 R_2 C_1 C_2} \end{aligned}$$

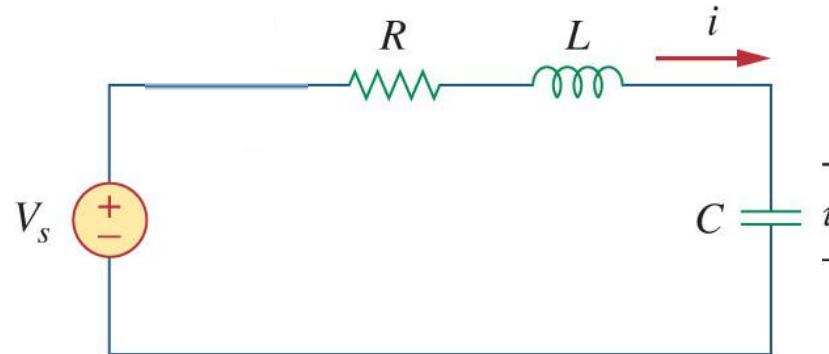
$$v_o(0) = 0; v'_o(0) = 0; v_o(\infty) = 10 \text{ mV};$$



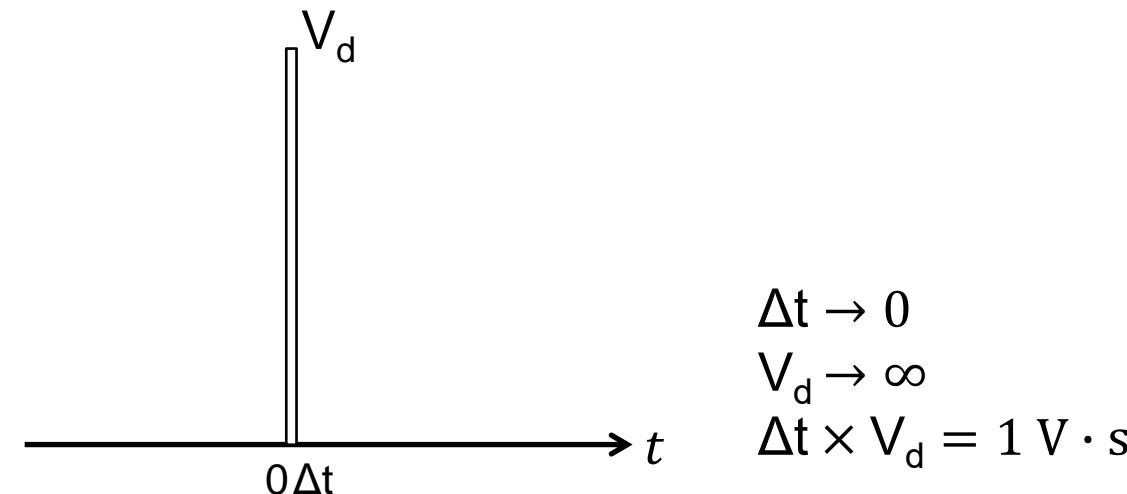
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一阶和二阶电路的单位冲击响应



$$V_s(t) = \delta(t) \quad \text{Unit of } \delta: \text{V}$$





系统响应与单位冲击响应的关系



Given the response to $V_s(t) = \delta(t)$, denoted as $h(t)$, the response to any $V_s(t)$ can be obtained via convolution as

$$v(t) = h(t) * V_s(t) = \frac{1}{V \cdot s} \int_{-\infty}^t h(t - \tau) V_s(\tau) d\tau$$

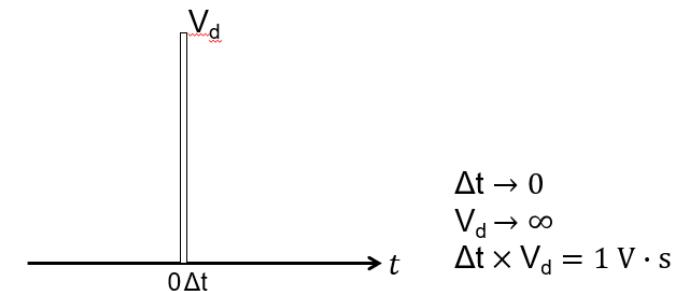
二阶电路的单位冲击响应

$$\frac{1}{b} \frac{d^2v}{dt^2} + \frac{a}{b} \frac{dv}{dt} + v = \delta(t)$$

Take the integral $\int_{0^-}^{0^+} dt$ on both side:

$$\frac{1}{b} v'(0^+) + \frac{a}{b} v(0^+) = 1 \text{ V}\cdot\text{s}$$

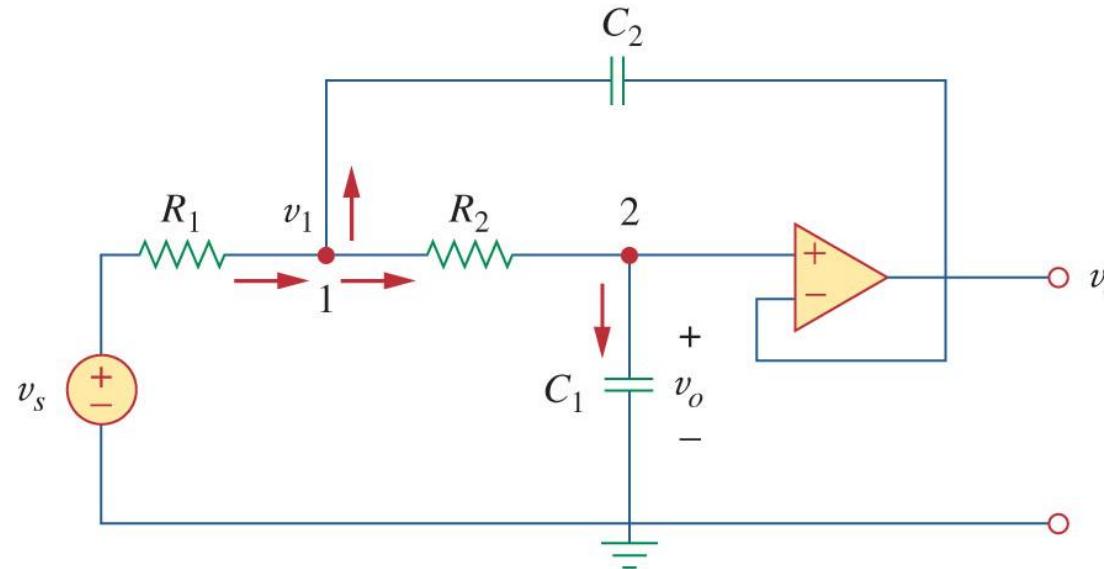
$$V_s(t) = \delta(t) \quad \text{Unit of } \delta: \text{V}$$



$v(0^+) = 0$ since $v'(t)$ is finite.

$v'(0^+) \neq 0$, i.e. $v'(t)$ exhibits abrupt change at $t = 0$.

Exercise: Find the initial condition for v_0 at $t=0^+ \text{ s}$ if $v_s = 10\delta(t) \text{ V}$.



$$\frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} \right) \frac{dv_o}{dt} + \frac{v_o}{R_1 R_2 C_1 C_2} = \frac{v_s}{R_1 R_2 C_1 C_2}$$

$$v_o(0^+) = 0;$$
$$v_o'(0^+) = \frac{10 \text{ V} \cdot \text{s}}{R_1 R_2 C_1 C_2};$$