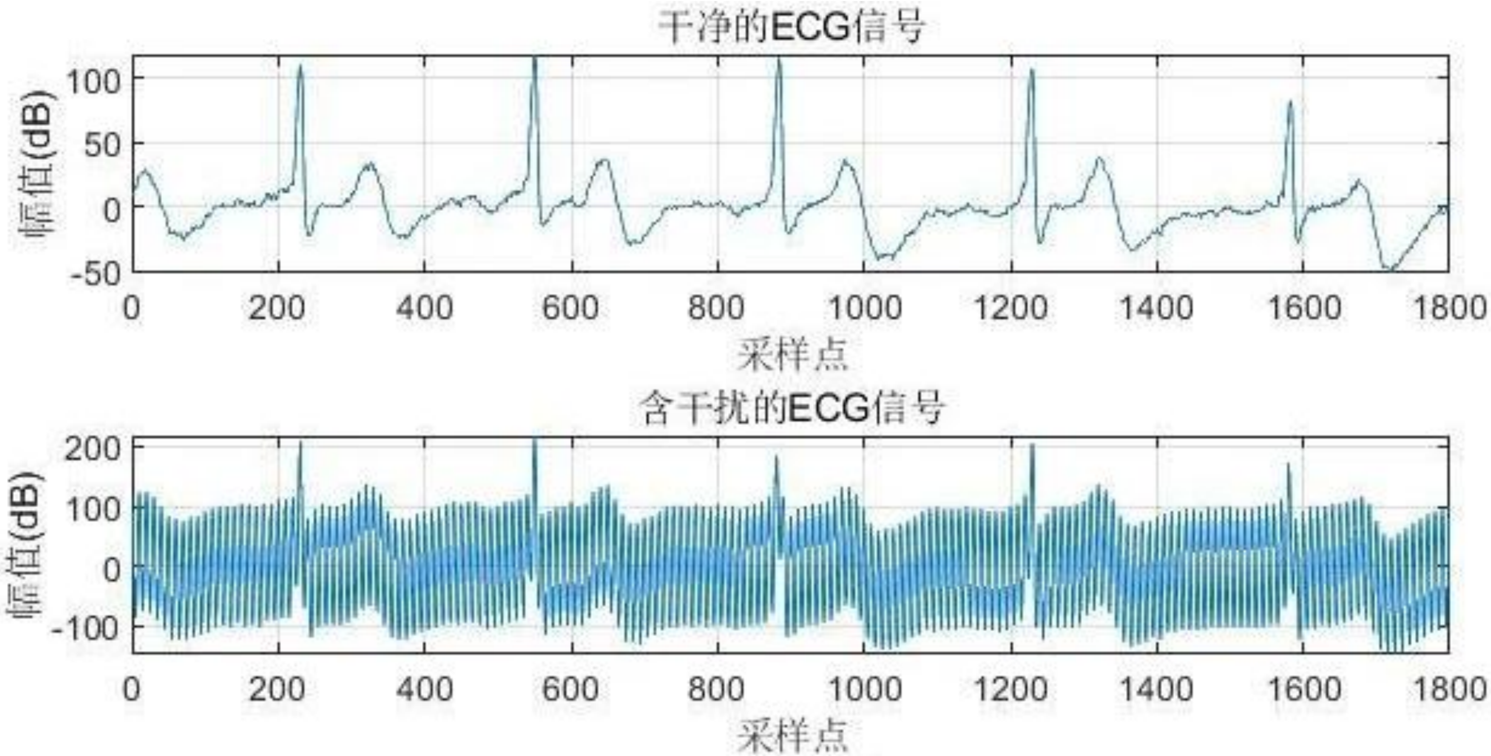
Lecture 10

- Frequency Response

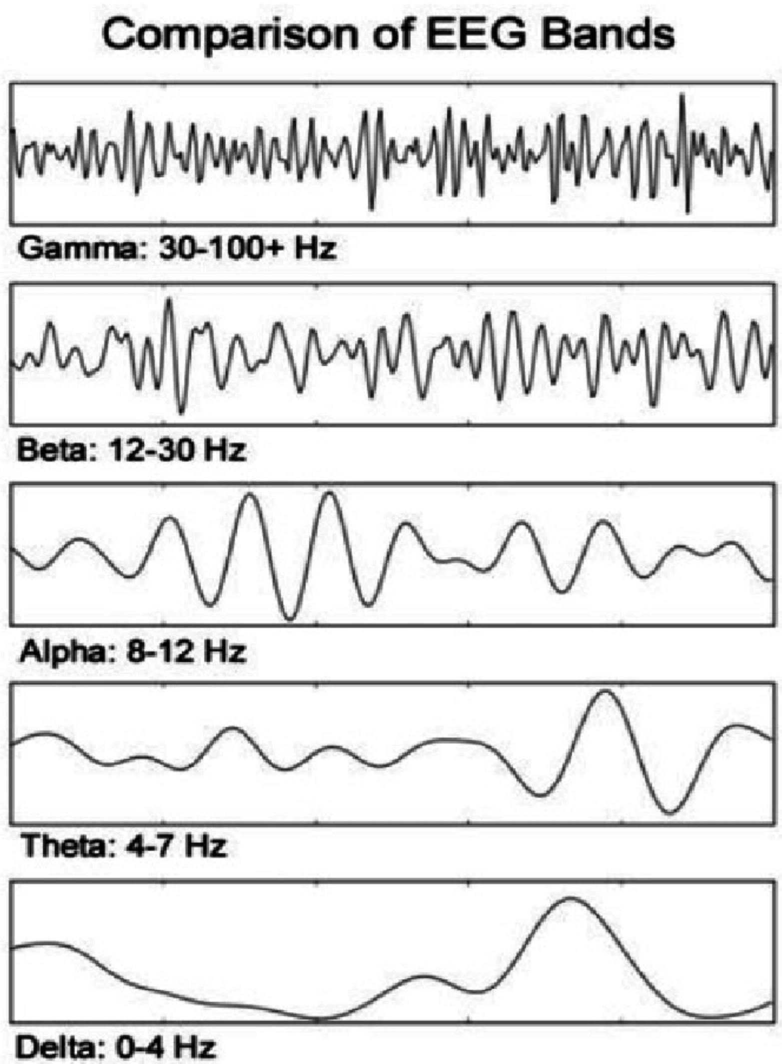
Relevance to BME

ECG信号工频干扰去除



EEG frequency bands

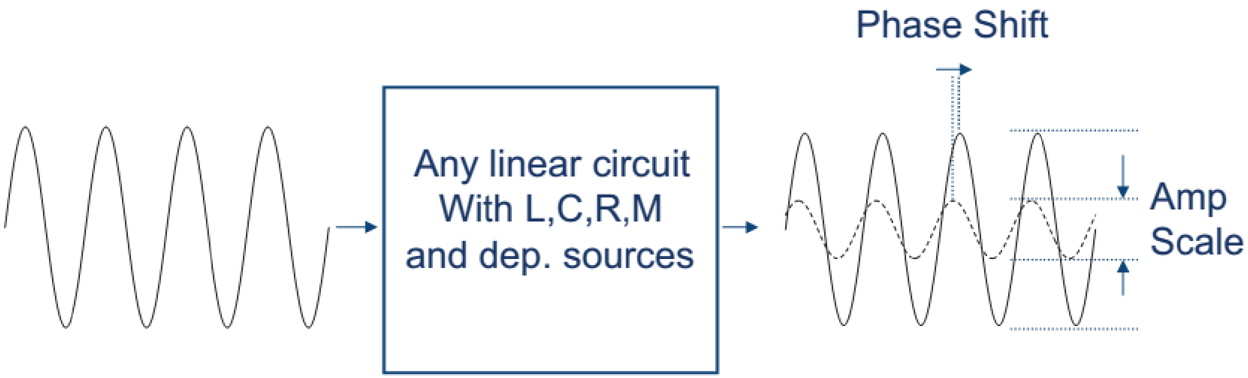
脑电信号采集



Outline

- Frequency response
- Transfer function
- Bode plots
- Resonance
- Filters

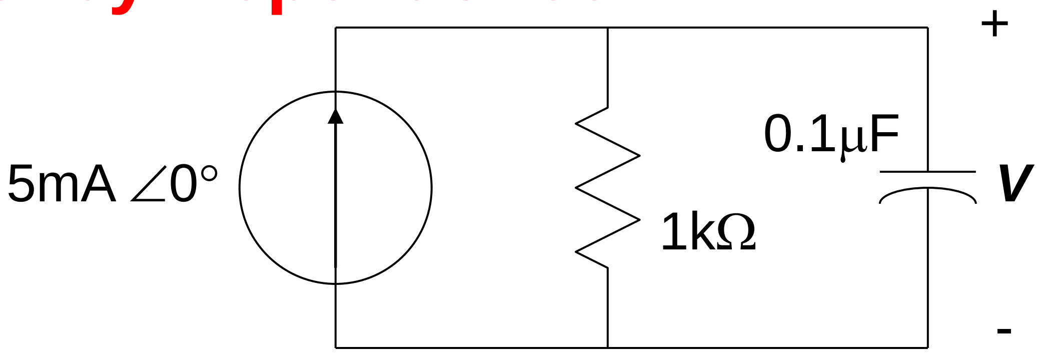
Frequency Response



- When a linear circuit is excited by a sinusoid, its steady-state output is a sinusoid at the *same* frequency.
  - Only the <u>magnitude</u> and <u>phase</u> of the output differ from the input.
- The "Frequency Response" is a characterization of the variation in input-output relationship for sinusoidal inputs with frequency.

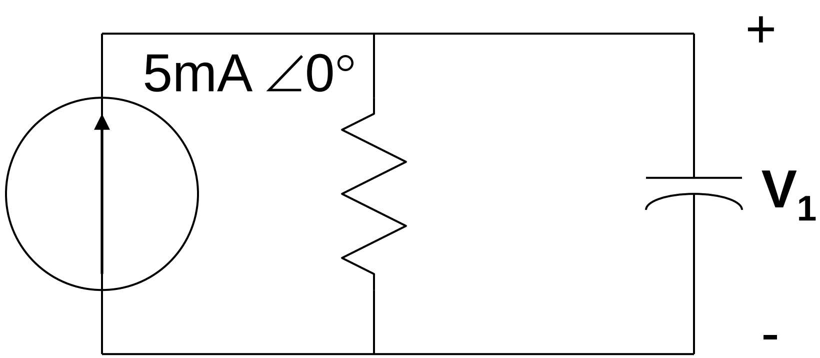
[Source: Berkeley]

Frequency Dependence

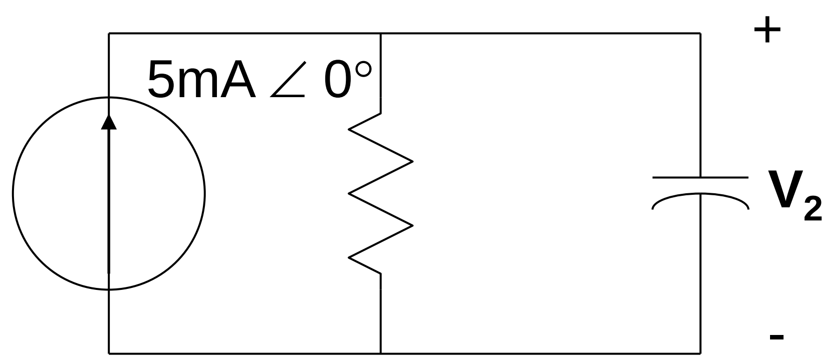


Find $v(t)$

Case 1: ω=2π x 3000



Case 2: ω=2π x 455000



$$\mathbf{Z}_{eq} = 468.2\angle -62.1°\Omega$$

$$\mathbf{V}_1 = 2.34\angle -62.1°\text{V}$$

$$\mathbf{Z}_{eq} = 3.5\angle -89.8°\Omega$$

$$\mathbf{V}_2 = 17.5\angle -89.8°\text{mV}$$

Outline

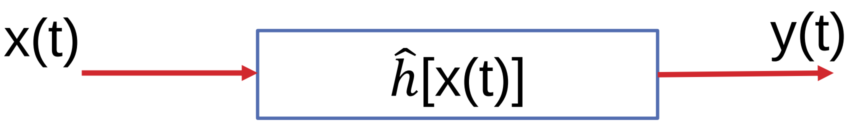
- Frequency response

- Transfer function
  - Definition
  - Capacitor voltage in RC circuit
  - Inductor voltage in RL circuit
  - Human impedance model
  - Output prediction

- Bode plots
- Resonance
- Filters

Transfer Function

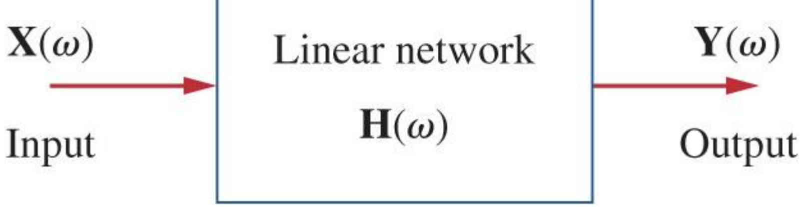
- The transfer function *H(ω)* is the frequency-dependent ratio of a phasor output *Y(ω)* to a phasor input *X(ω)*.



$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display



$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

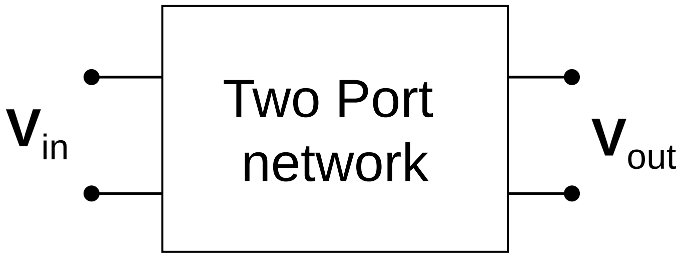
$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Transfer impedance is equal to the impedance if the input and output ports are the same.

Transfer Function

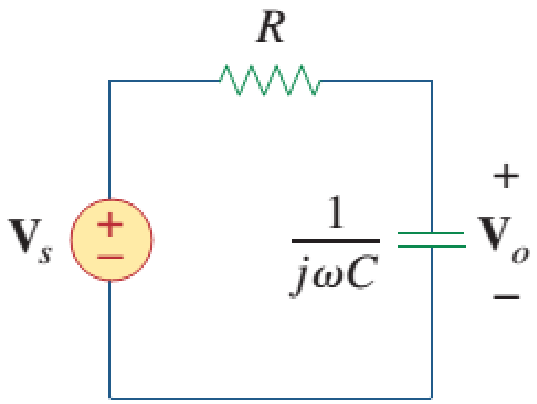
- Complex quantity; ratio of two phasors (but not a phasor)
- Both magnitude and phase are functions of frequency



$$\mathbf{H}(f) = \frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{\text{out}}}{V_{\text{in}}} \angle(\theta_{\text{out}} - \theta_{\text{in}})$$

$$\mathbf{H}(\mathbf{f}) = H(f)\angle\theta(\mathrm{f})$$

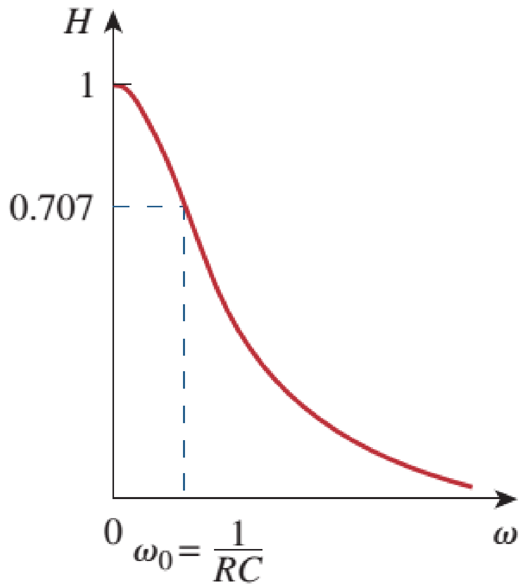
Example

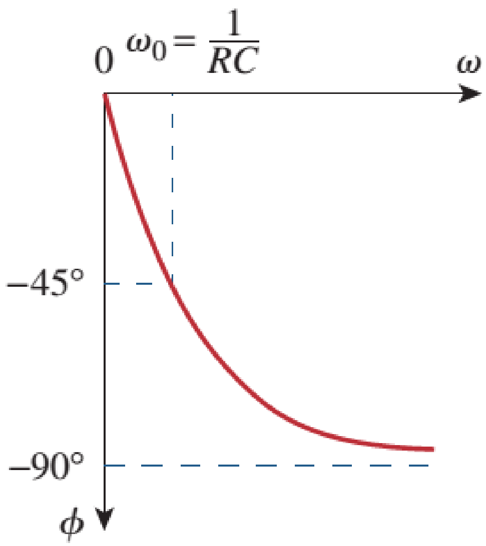


$$V_0 = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} V_s = \frac{1}{j\omega CR+1} V_s = \frac{1}{j\omega/\omega_0+1} V_s$$

$$\boxed{\omega_0 = \frac{1}{RC}}$$

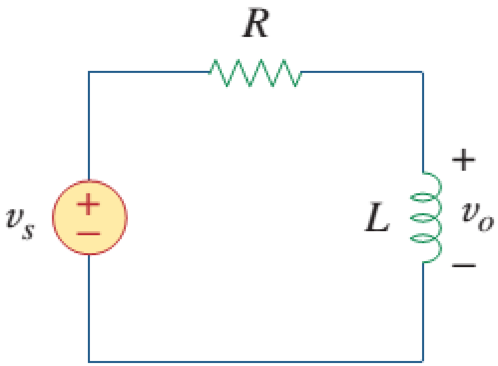
$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

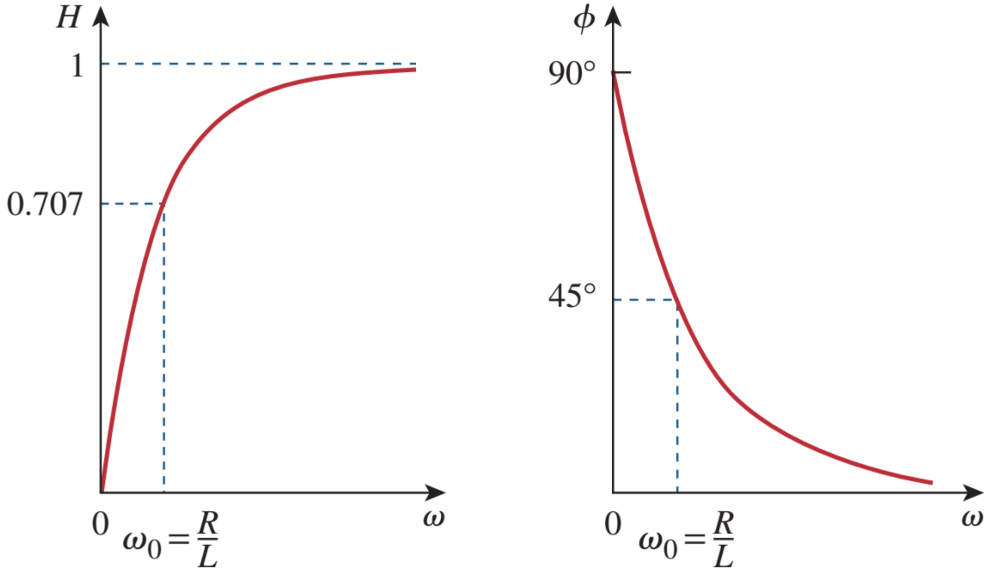




Exercise

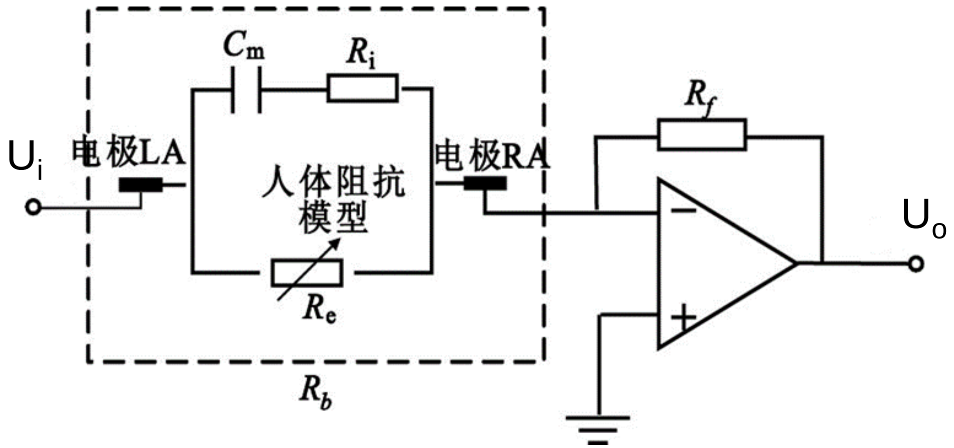
- Obtain the transfer function $\mathbf{V_o}/\mathbf{V_s}$ of the RL circuit.





Human Impedance Model

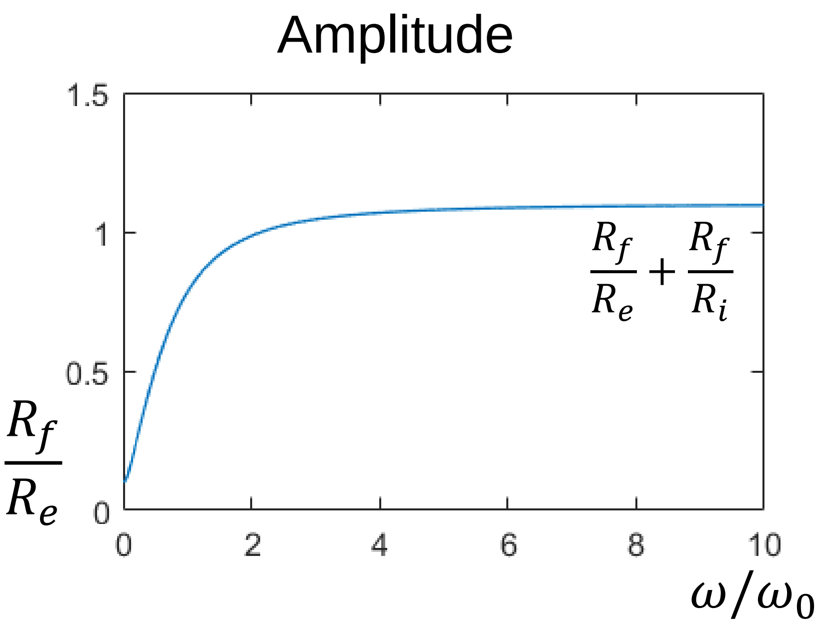
Obtain the transfer function of the following circuit which can be used to measure the impedance of the human body.

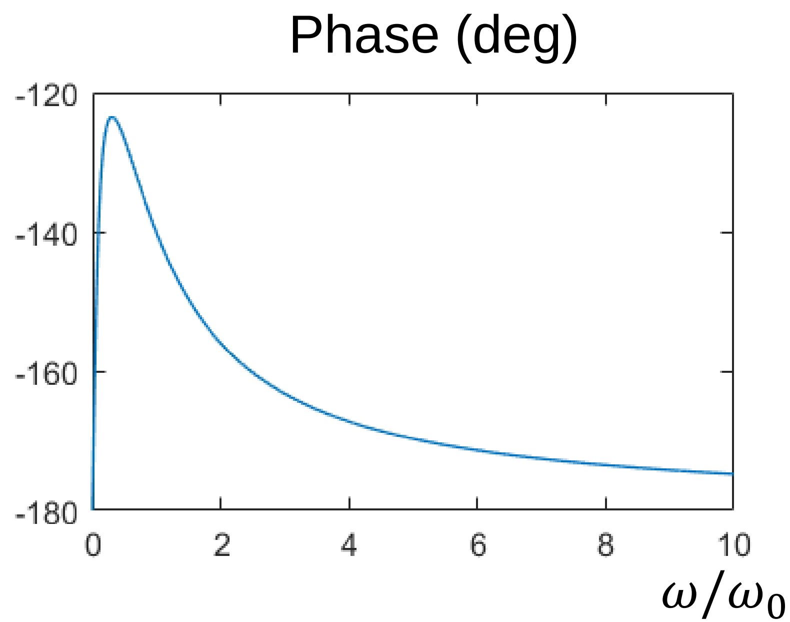


$$\frac{U_o}{U_i} = -R_f\left(\frac{1}{R_e} + \frac{j\omega C_m}{j\omega R_i C_m + 1}\right) = -\frac{R_f}{R_e} - \frac{R_f}{R_i}\frac{j\frac{\omega}{\omega_0}}{j\frac{\omega}{\omega_0}+1} = -\frac{R_f}{R_e} - \frac{R_f}{R_i}\left[1 - \frac{1}{j\frac{\omega}{\omega_0}+1}\right]$$

$$= -\frac{R_f}{R_e} - \frac{R_f}{R_i} + \frac{R_f}{R_i}\frac{1}{j\frac{\omega}{\omega_0}+1} \qquad \omega_0 = \frac{1}{R_i C_m}$$

Human Impedance Model

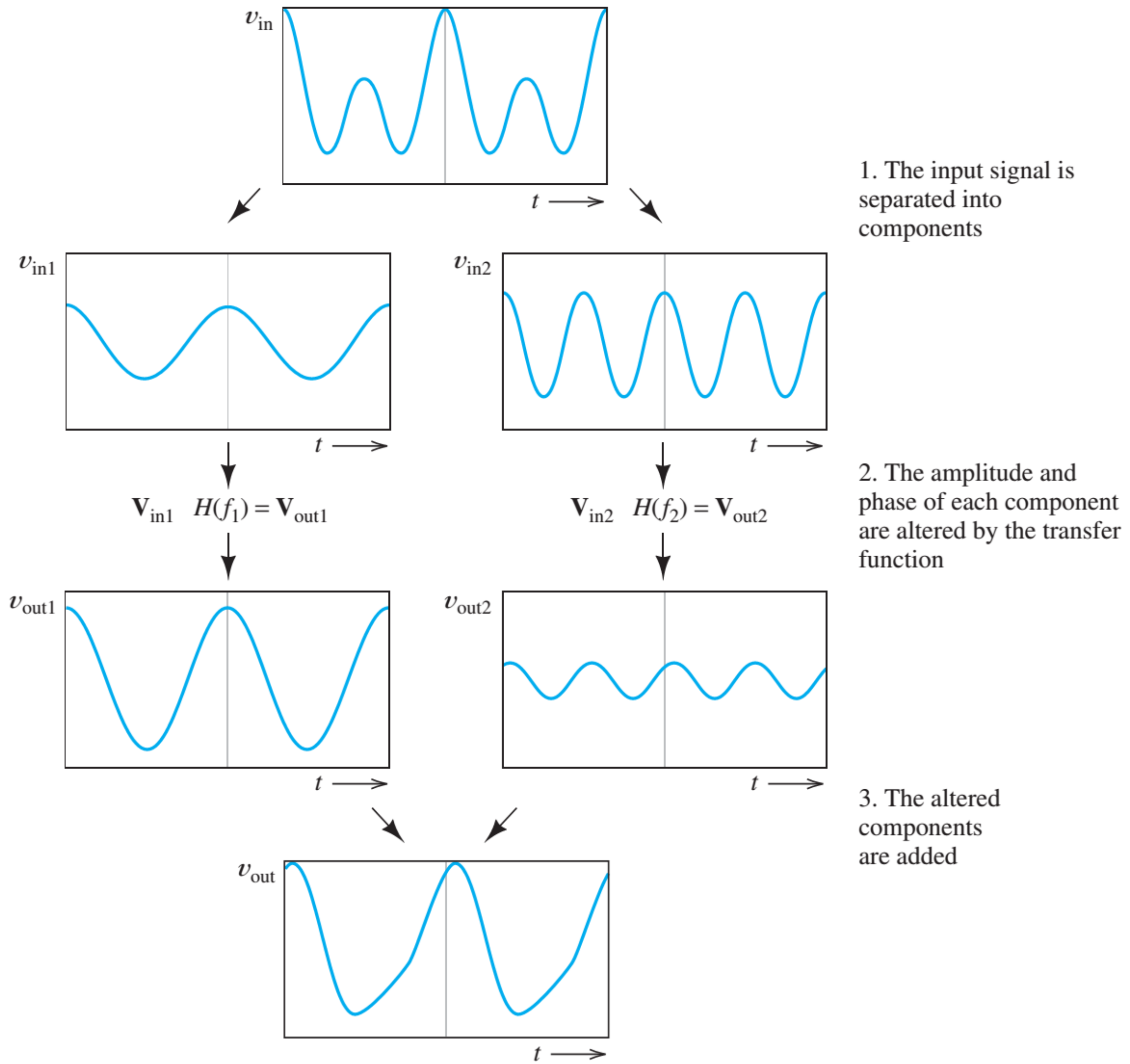




$$\sqrt{\left(\frac{R_f}{R_e}+\frac{R_f}{R_i}\frac{(\frac{\omega}{\omega_0})^2}{1+(\frac{\omega}{\omega_0})^2}\right)^2+\left(\frac{R_f}{R_i}\frac{\frac{\omega}{\omega_0}}{1+(\frac{\omega}{\omega_0})^2}\right)^2}$$

$$\omega_0=\frac{1}{R_iC_m}$$

Response to multiple frequency components

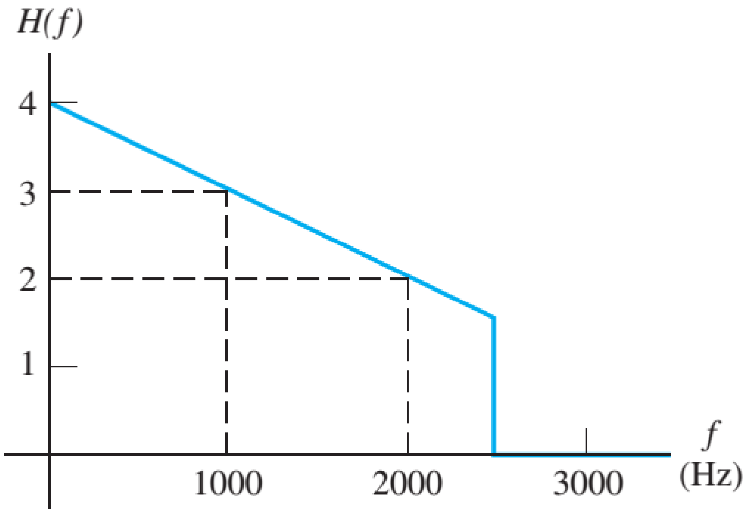


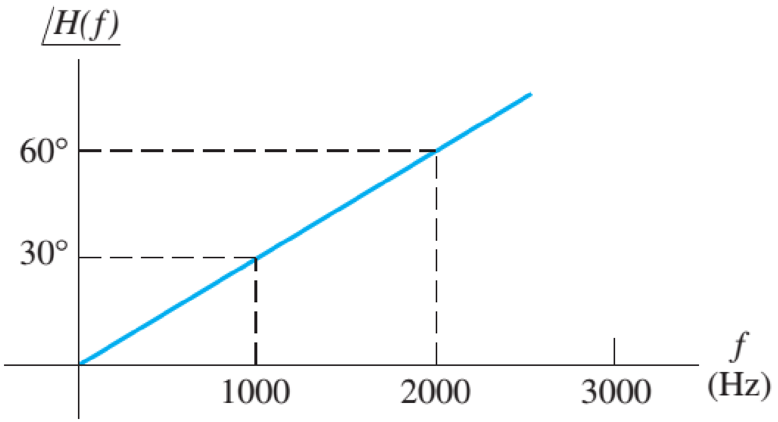
Exercise II – output prediction

- If the input signal is

$$v_{\text{in}}(t) = 3 + 2\cos(2000\pi t) + \cos(4000\pi t - 70°),$$

find an expression as a function of time for the output.





Answer: $v_{\text{out}}(t) = 12 + 6\cos(2000\pi t + 30^0) + 2\cos(2000\pi t - 10^0)$

Outline

- Frequency response

- Transfer function

- Bode plots (or diagram)
  - Motivation
  - General form of transfer functions
  - Bell and Decibel
  - Bode plots for 0^{th}, 1^{st}, 2^{nd} order polynomials

- Resonance

- Filters

General Form of the Transfer Function

$$H(f) = \frac{\sum_{i=0}^{m} a_i (j2\pi f)^i}{\sum_{i=0}^{n} b_i (j2\pi f)^i}$$

The circuit unknown variables (*x*; voltage or current) can be expressed in matrix form as

$$Ax = c$$

Input and output are one of the components in *c* and x, respectively.

Therefore, $x = A^{-1}c$. The transfer function is an element of the A^{-1} matrix.

$[A^{-1}]_{i,j} = \frac{\det(A_{j,i}^{*})}{\det(A)}$, $A_{i,j}$* is the cofactor (代数余子式) of $A_{i,j}$.

All determinants can be expressed as polynomials of the impedances.

$\det(A) = K \sum_{i=0}^{n} a_i (j\omega)^{i-m} = \frac{K}{(j\omega)^m} \sum_{i=0}^{n} a_i (j\omega)^i$, where

m and n depend on the number of independent inductors and capacitors.

Similarly, $\det\left(A_{j,i}^{*}\right) = \frac{K'}{(j\omega)^{m'}} \sum_{i=0}^{n'} a_i (j\omega)^i$

Bode Plots

- Bode plot is a tool for visualizing (understanding) the transfer function.

$$H(f)=\frac{\sum_{i=0}^{m} a_i (j\omega)^i}{\sum_{i=0}^{n} b_i (j\omega)^i}=\frac{K(j\omega)^{\pm 1}(1+j\omega/z_1)[1+j2\zeta_1\omega/\omega_k+(j\omega/\omega_k)^2]\cdots}{(1+j\omega/p_1)[1+j2\zeta_2\omega/\omega_n+(j\omega/\omega_n)^2]\cdots}$$

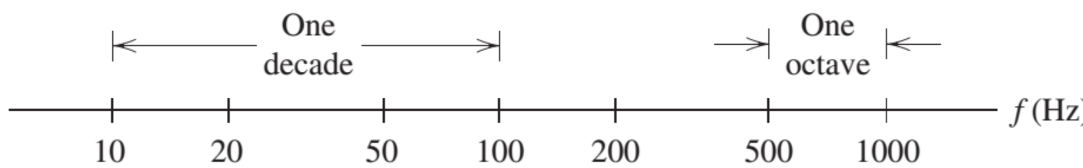
$$\log|H(f)|=\log|K|\pm\log\omega+\log\left|1+j\frac{\omega}{z_1}\right|+\log\left|1+\frac{j2\zeta_1\omega}{\omega_k}+(j\omega/\omega_k)^2\right|+\ldots$$

$$\angle(H(f))=\angle(K)\pm\angle(j\omega)+\angle\left(1+j\frac{\omega}{z_1}\right)+\angle\left(1+\frac{j2\zeta_1\omega}{\omega_k}+\left(\frac{j\omega}{\omega_k}\right)^2\right)-\angle\left(1+j\frac{\omega}{p_1}\right)-\angle\left(1+\frac{j2\zeta_2\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right)\ldots$$

- Each of the above terms are not always present.

- Each term can be considered separately and then summed up.

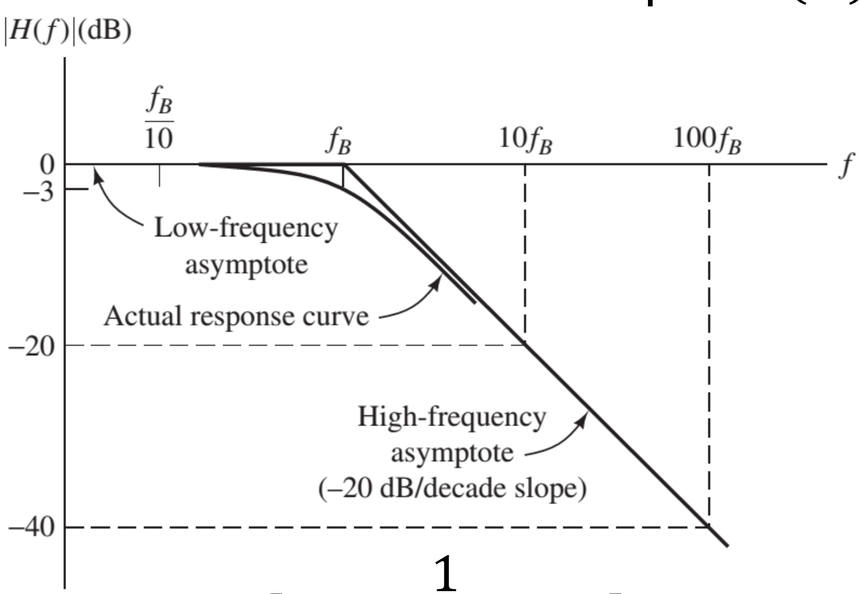
Bode Plots



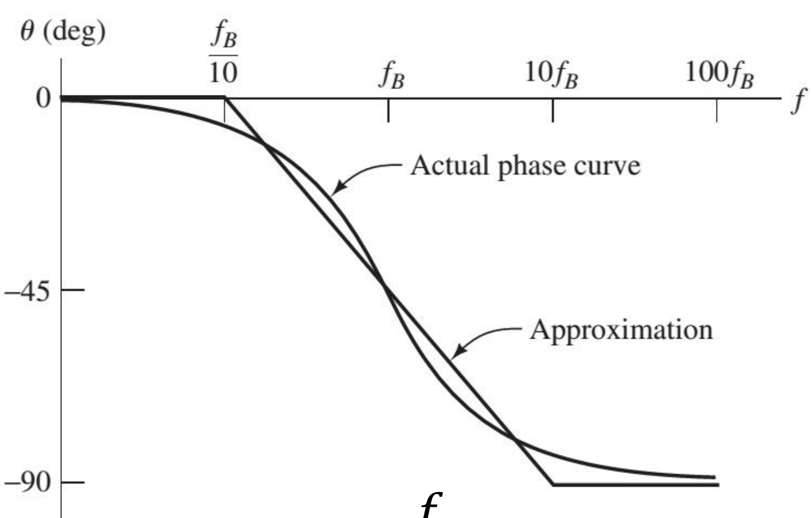
Plotting the frequency response, magnitude or phase, on plots with

- Frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)

Example: $H(w) = \frac{1}{1+j(f/fB)}$



$$20\log[\frac{1}{\sqrt{1+(f/fB)^2}}]$$



$$-\operatorname{atan}(\frac{f}{f_B})$$

Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of *power* levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
  - Definition of bel: Ratio = $\log_{10}(P_1/P_2)$ B,
    where P_1 and P_2 are *power levels*.

- One bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

$$\text{Ratio} = \log_{10}(P_1/P_2)\ \text{B} = 10\log_{10}(P_1/P_2)\ \text{dB}$$

1 B = 10 dB
1 dB = 0.1 B

dB for Power

- To express a power in terms of decibels, one starts by choosing a reference power, $P_{reference}$, and write

$$\text{Power P in decibels} = 10 \log_{10}(P/P_{reference})$$

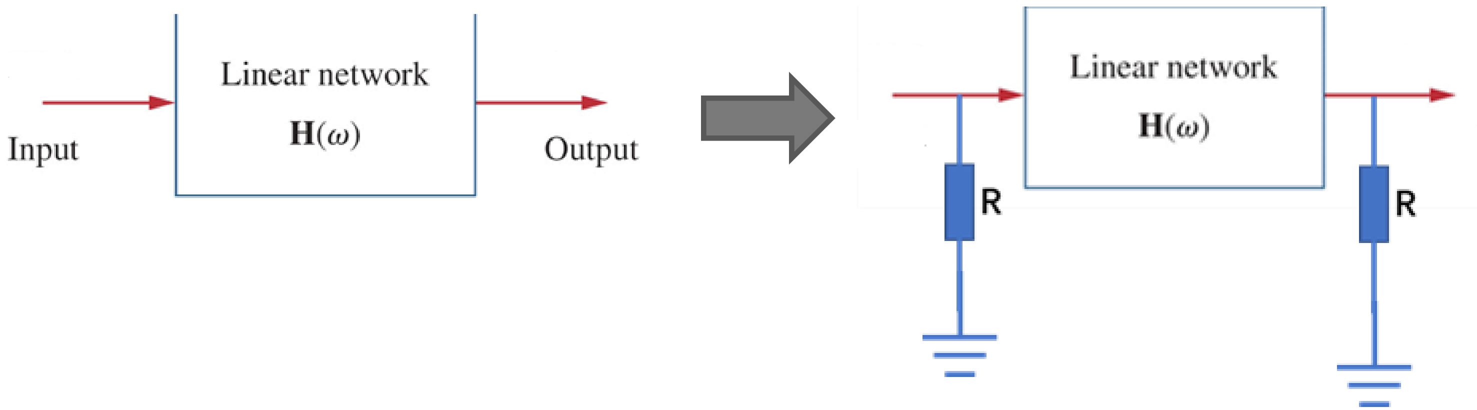
- When the reference is 1 mW, the dB value is called dBm. dBm is a measure of absolute power. e.g. 0 dBm = 1 mW; 10 dBm = 10 mW

- Exercise: Express a power of 50 mW in decibels relative to 1 watt and 1mW.

P (dB) = -13 dB when $P_{reference}$ = 1 W.
P (dBm) = 17 dBm

dB for transfer function

To convert transfer function into power ratio, imagine connecting two loads with the same impedance to the input and output.



$$G_{dB} = 10\log_{10}\frac{P_{\text{output}}}{P_{\text{input}}} = 10\log_{10}|H(\omega)|^2 = 20\log_{10}|H(\omega)|$$

dB for Voltage or Current

Question: How many decibels larger is the voltage of a 9-volt battery than that of a 1.5-volt AA battery?

Answer: $20\log_{10}(9/1.5) = 15.6$ dB

Question: What is the output/input voltage(power) ratio of an amplifier with a gain of 34 dB?

Answer: $10^{34/20} = 50$ (voltage); $10^{34/10} = 2512$ (power)

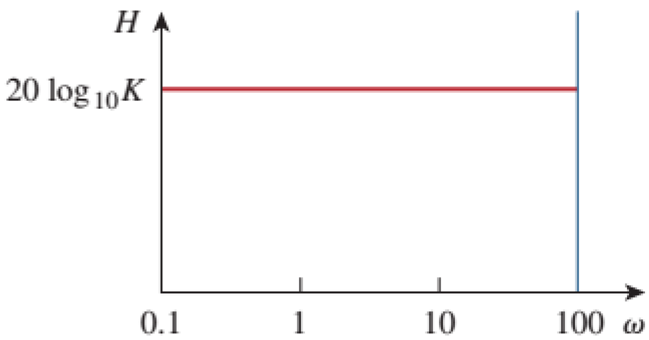
Question: How many decibels larger is the voltage(power) if the voltage (power) is doubled?

Answer: $20\log_{10}(2) = 6$ dB (voltage)
$10\log_{10}(2) = 3$ dB (power)

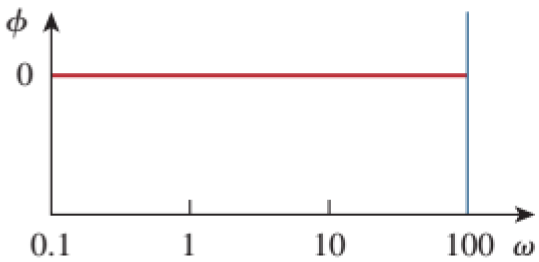
Constant term K

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

K>0

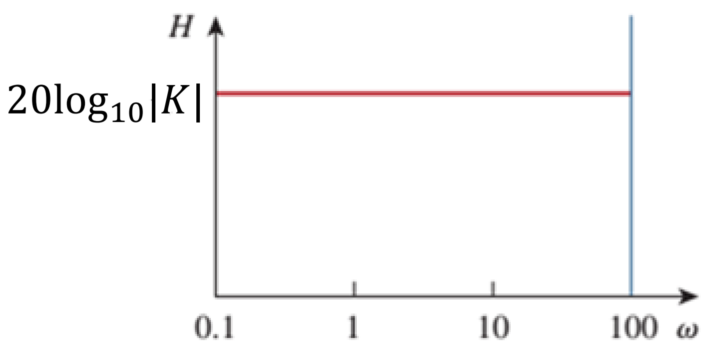


(a)

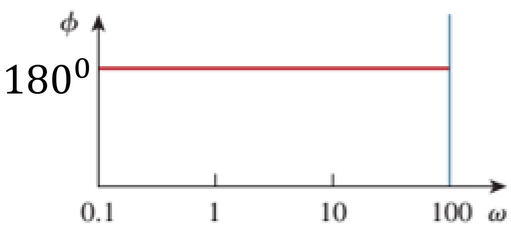


(b)

K<0



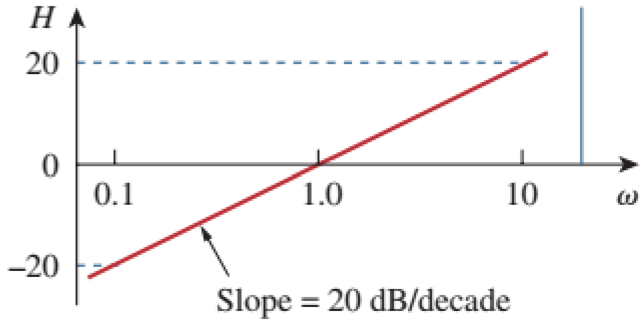
(a)



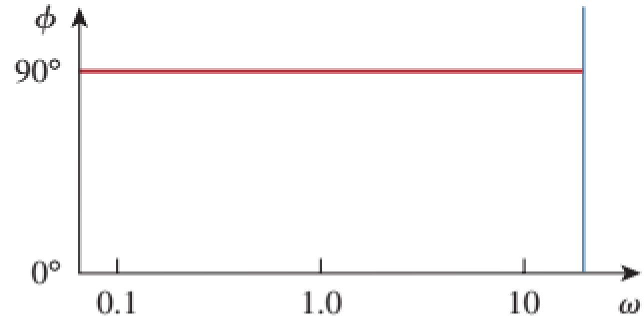
(b)

$j\omega$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

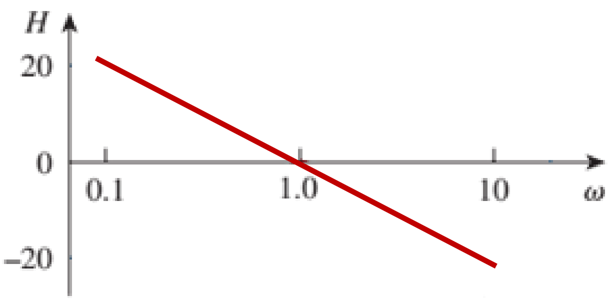


(a)

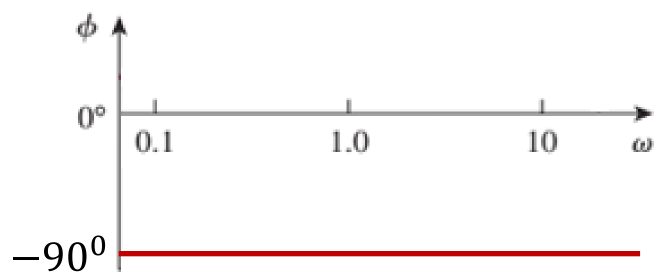


(b)

$(j\omega)^{-1}$

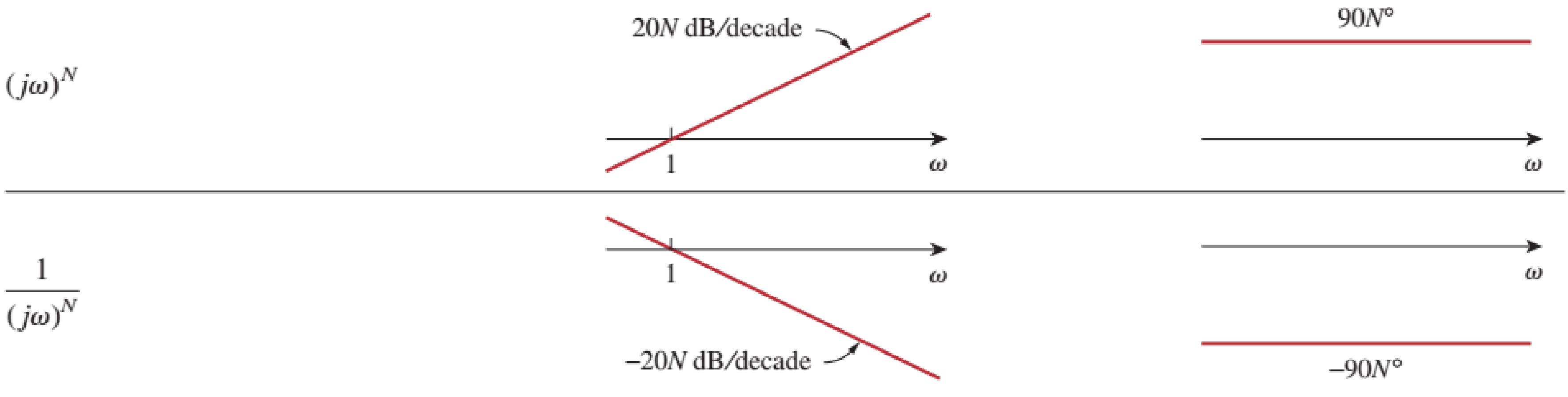


(a)



(b)

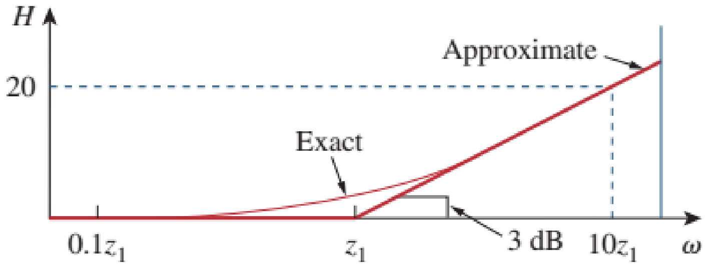
$(j\omega)^{\pm N}$



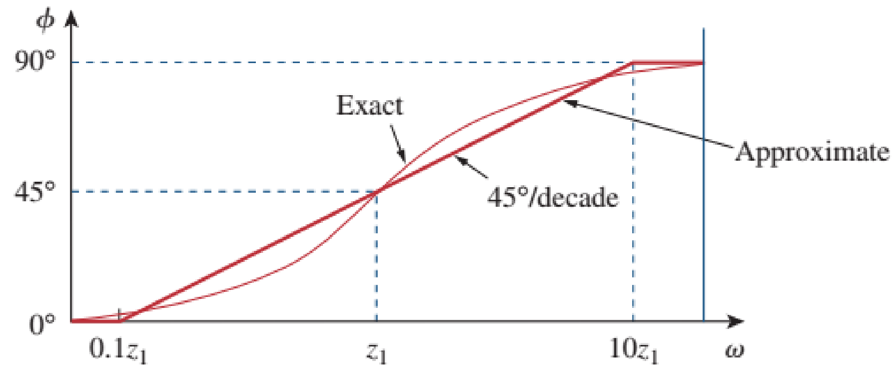
$1+j\omega/z_1$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

	H_{dB}	φ
$\omega \to 0$	0	0
$\omega \gg z_1$	$20(\log_{10}\lvert\omega\rvert - \log_{10} z_1)$	90^o



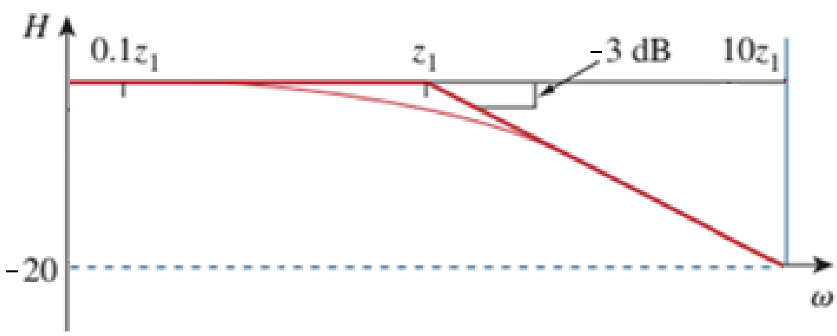
(a)

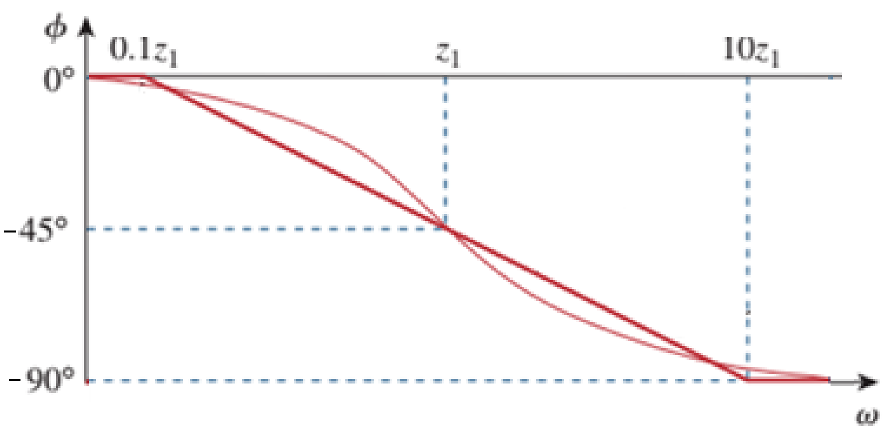


(b)

$1/(1+j\omega/p_1)$

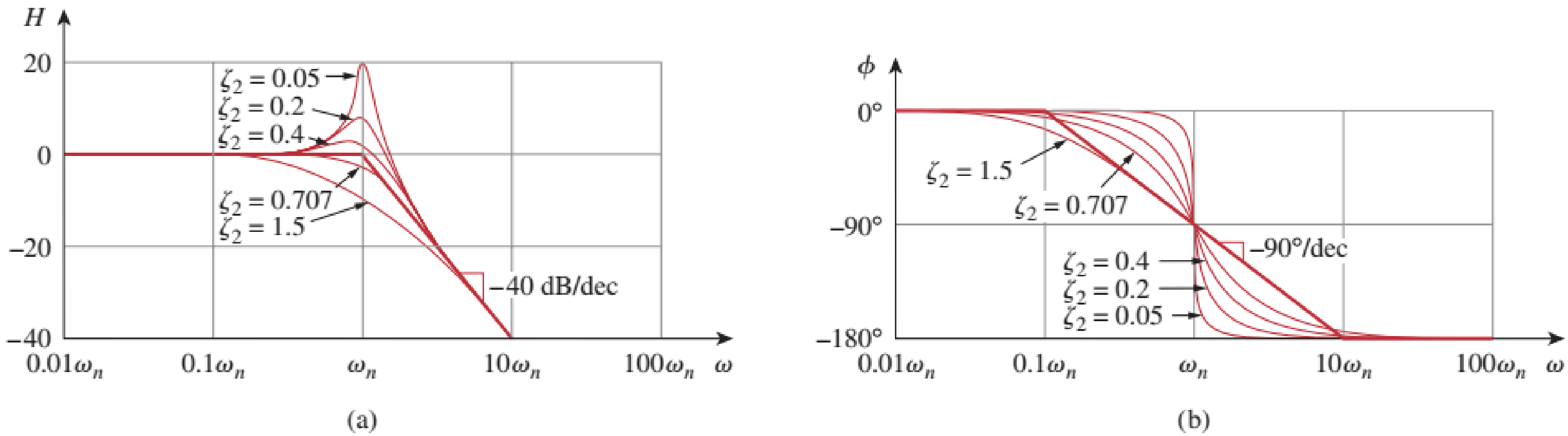
	H_{dB}	φ
$\omega \to 0$	0	0
$\omega \gg p_1$	$-20(\log_{10}\lvert\omega\rvert - \log_{10}p_1)$	90^o





$1/[1+2j\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$

	H_{dB}	φ
$\omega \to 0$	0	0
$\omega \gg \omega_n$	$-40(\log_{10}\lvert\omega\rvert - \log_{10}\omega_n)$	-180^o



Condition for the presence of a peak/dip

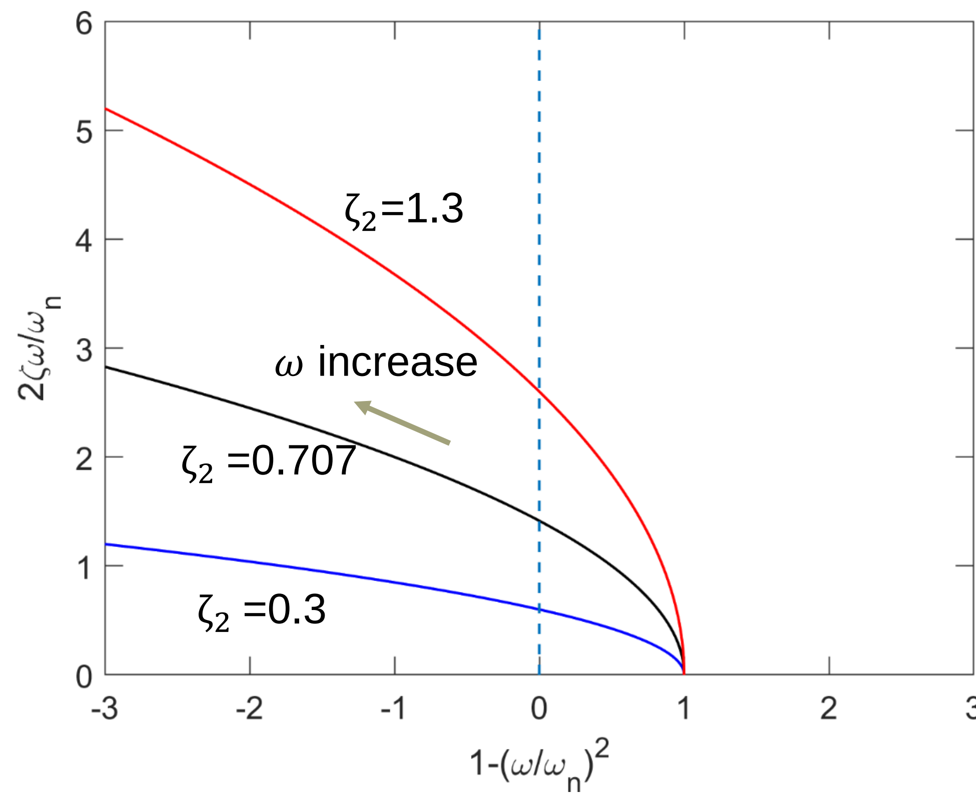
$$|H|^{-2} = \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2$$

$$\frac{d|H|^{-2}}{d\omega} = -2\left(1 - \frac{\omega^2}{\omega_n^2}\right)\frac{2\omega}{\omega_n^2} + \frac{8\zeta^2\omega}{\omega_n^2} = 0$$

$$-\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2\zeta^2 = 0$$

$$\omega^2 = \omega_n^2(1 - 2\zeta^2) > 0$$

$$|\zeta| < 0.707$$

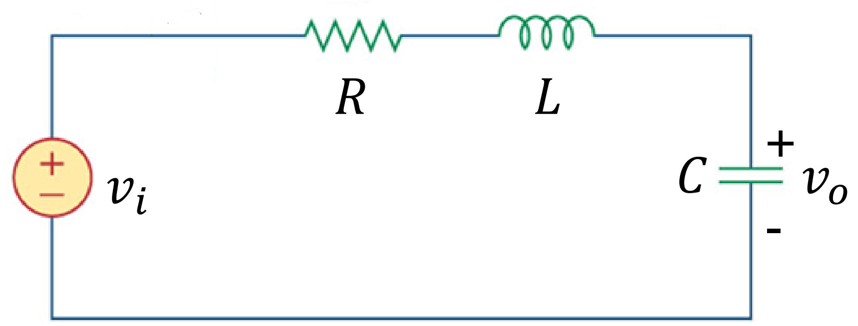


$|H|^2$ shows a peak $at\ \omega = \omega_n\sqrt{1-2\zeta^2}$, when $\zeta^2 < \frac{1}{2}$ or $|\zeta| < 0.707$

When $\zeta^2 = \frac{1}{2}$, $|H|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_n}\right)^4}$, 2nd order Butterworth low pass filter.

Exercise

Compare the condition for the presence of a peak and the underdamping condition in the series RLC circuit.

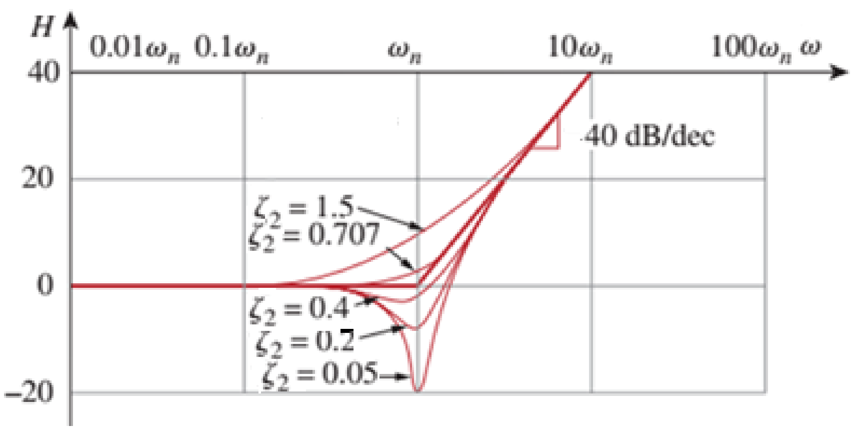


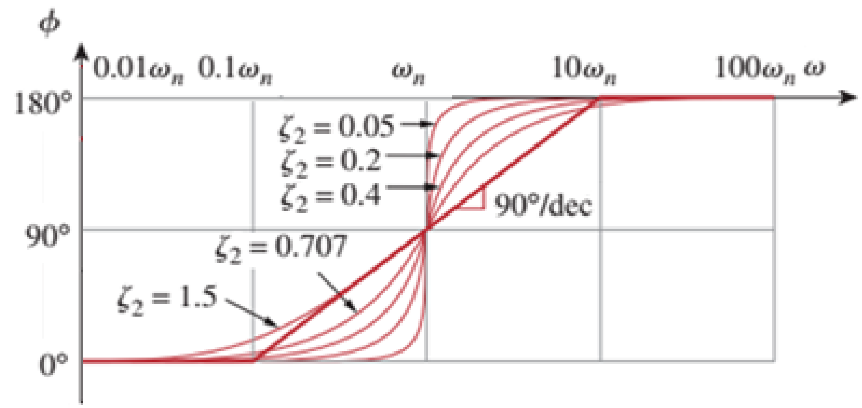
$$H(\omega) = \frac{\frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C} + R} = \frac{1}{1 + j\omega RC + (j\omega LC)^2}$$

$$\omega_n = \frac{1}{\sqrt{LC}}; \frac{2\zeta}{\omega_n} = RC \rightarrow \zeta = \frac{\omega_n RC}{2} \le \frac{\sqrt{2}}{2}$$

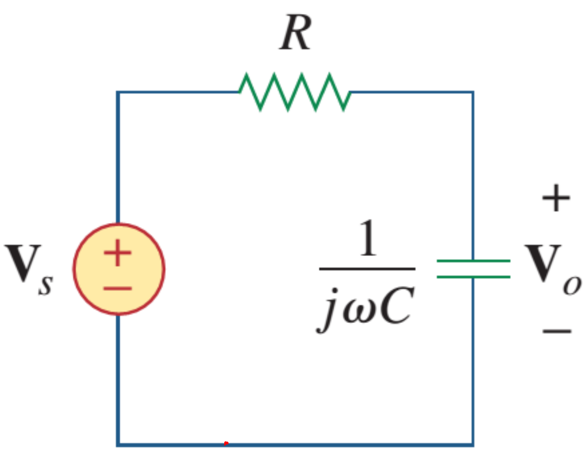
Underdamping for RLC circuit: $\frac{R}{2L} < \omega_n \rightarrow \frac{\omega_n RC}{2} < 1$. No ripple

$1+2j\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2$





Example: Bode Plot of First-Order Lowpass Filter

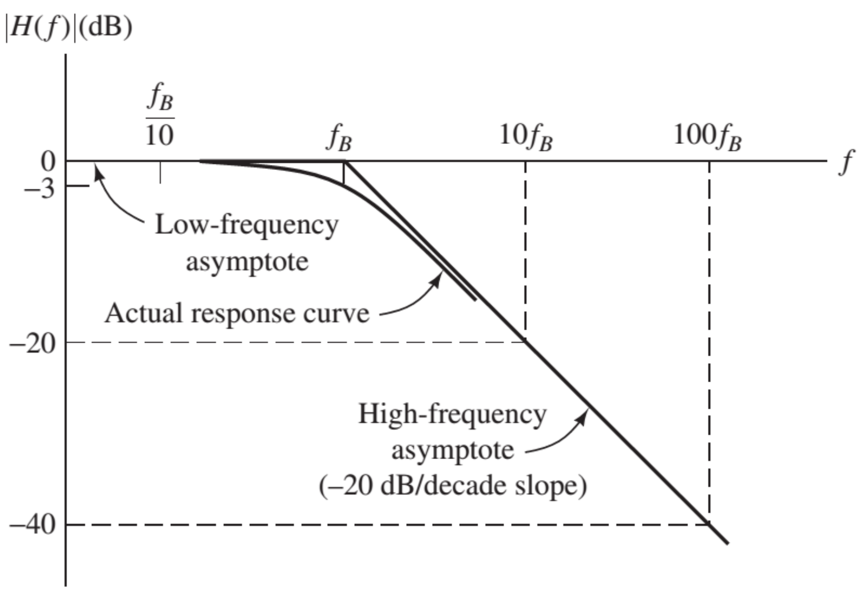


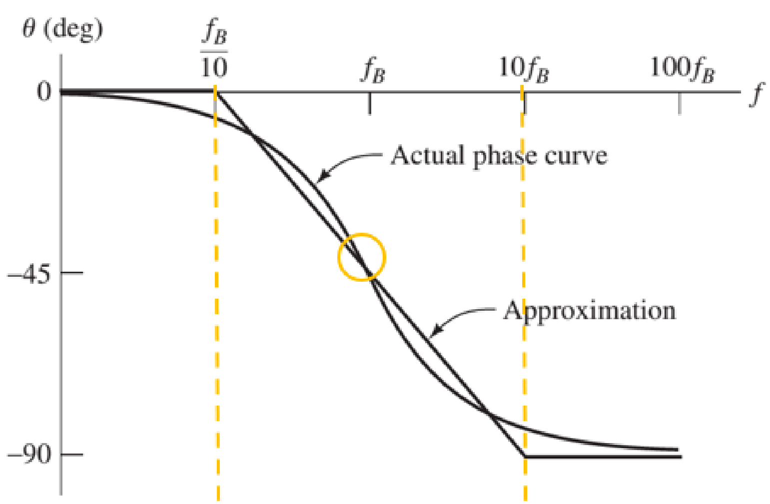
$$V_0 = \frac{1}{j\omega/\omega_B + 1} V_s$$

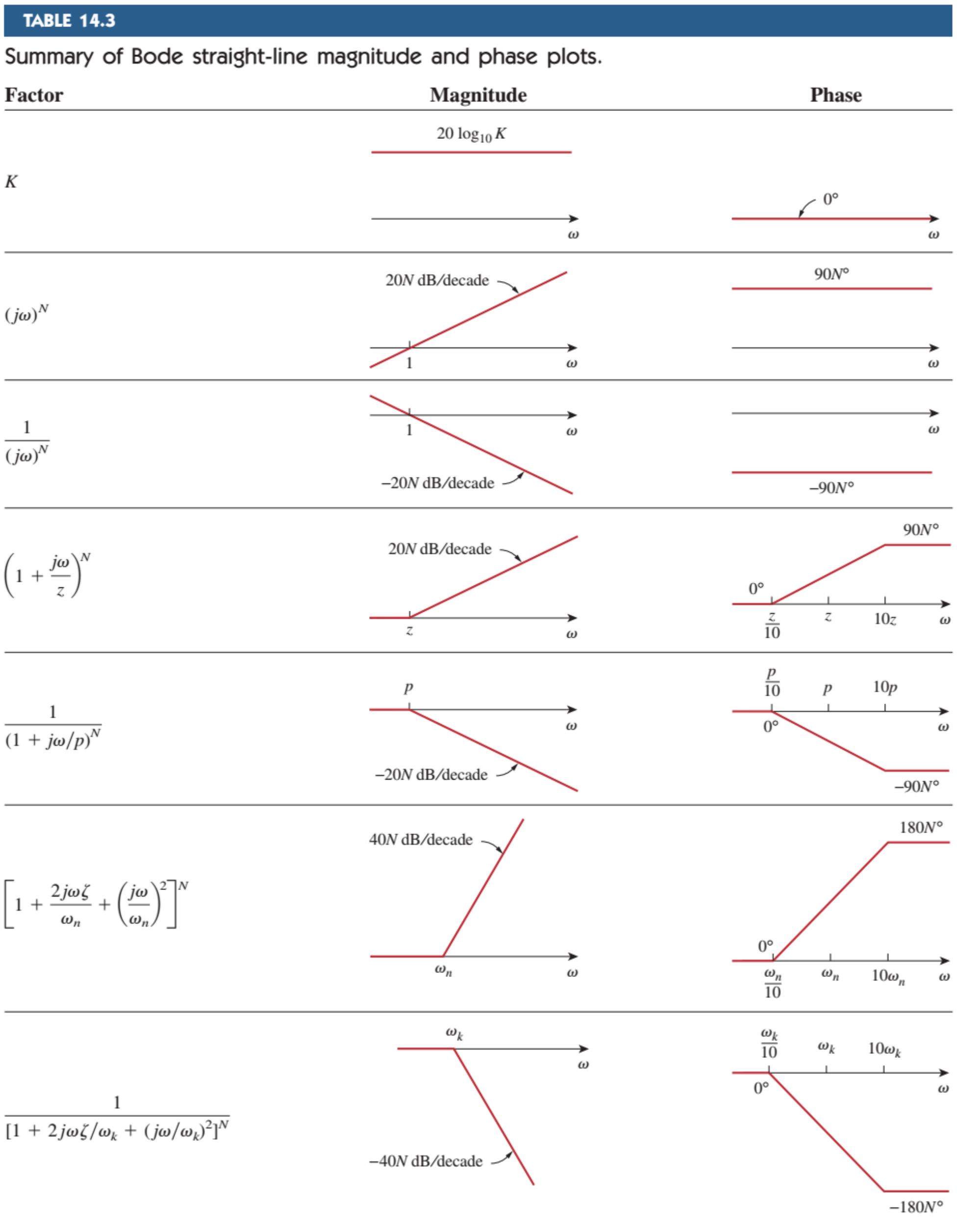
$$= \frac{1}{jf/f_B + 1} V_s$$

$$\omega_B = \frac{1}{RC}$$

$$f_B = \frac{1}{2\pi RC}$$







Constant; K = vertical coordinate at $\omega = 1$

Linearly increase/decrease; determine N from initial slope in magnitude plot

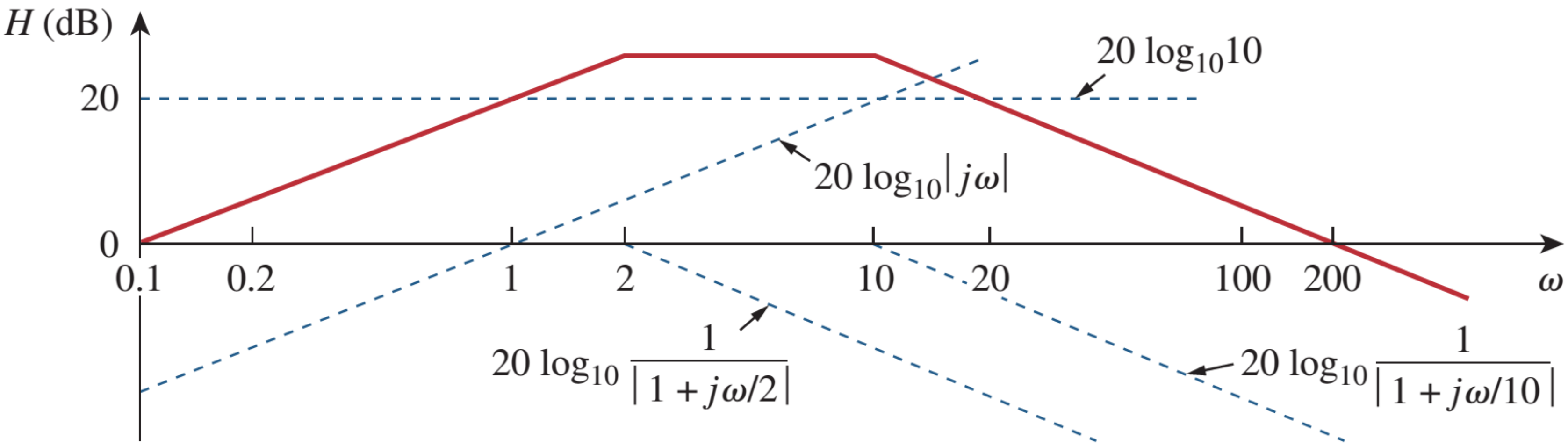
Flat for ω → 0, determine N from change in slope and determine ω_n from turning point in magnitude plot

Example--Standard Form

$$\mathbf{H}(\omega) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)} = \frac{10\,j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

Example - Magnitude

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$



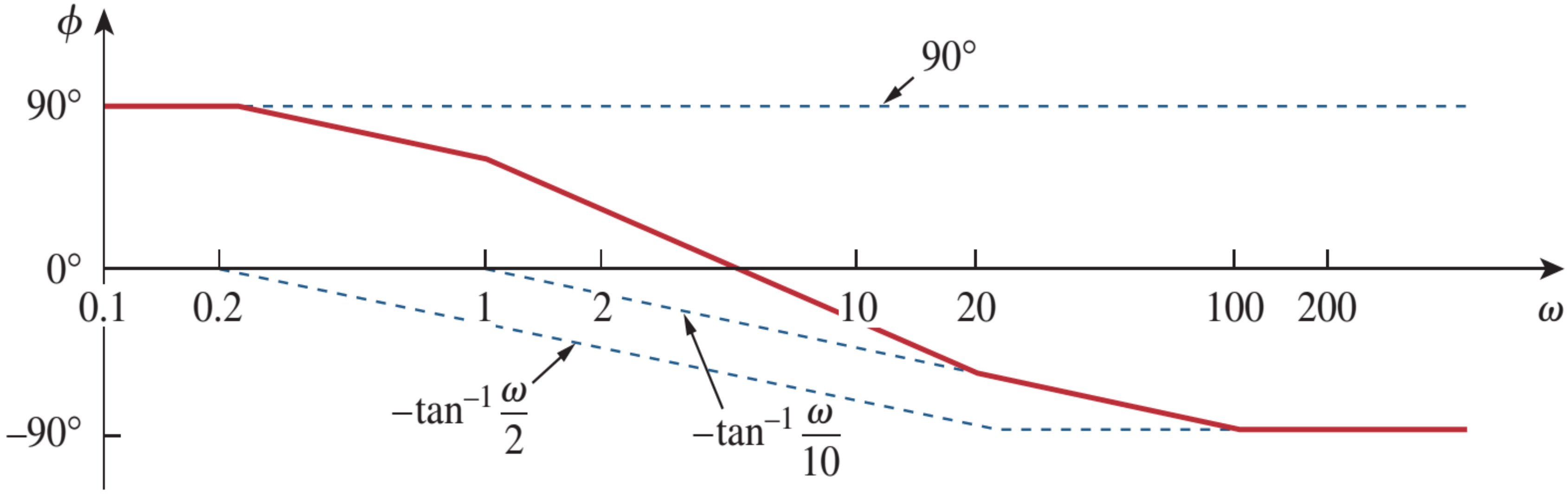
$$H_{\mathrm{dB}} = 20\log_{10}10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1 + \frac{j\omega}{2}\right| - 20\log_{10}\left|1 + \frac{j\omega}{10}\right|$$

Example - Phase

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \angle 90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$

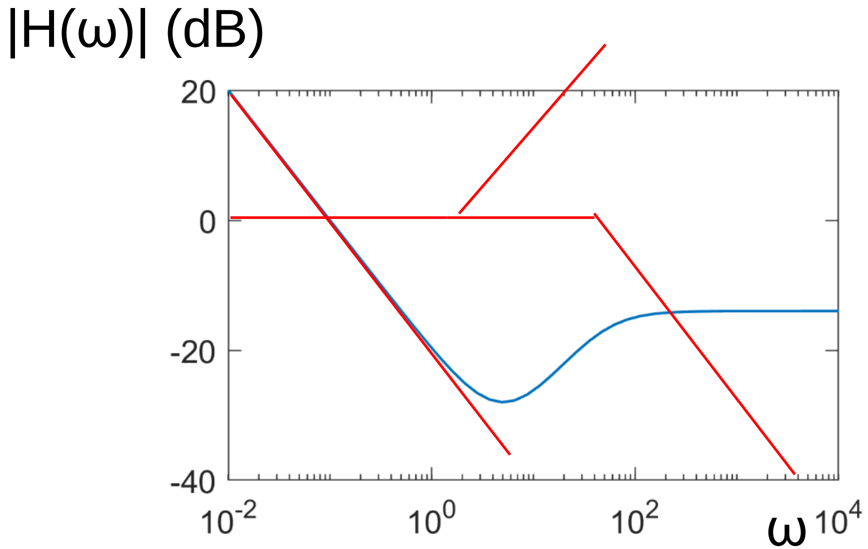


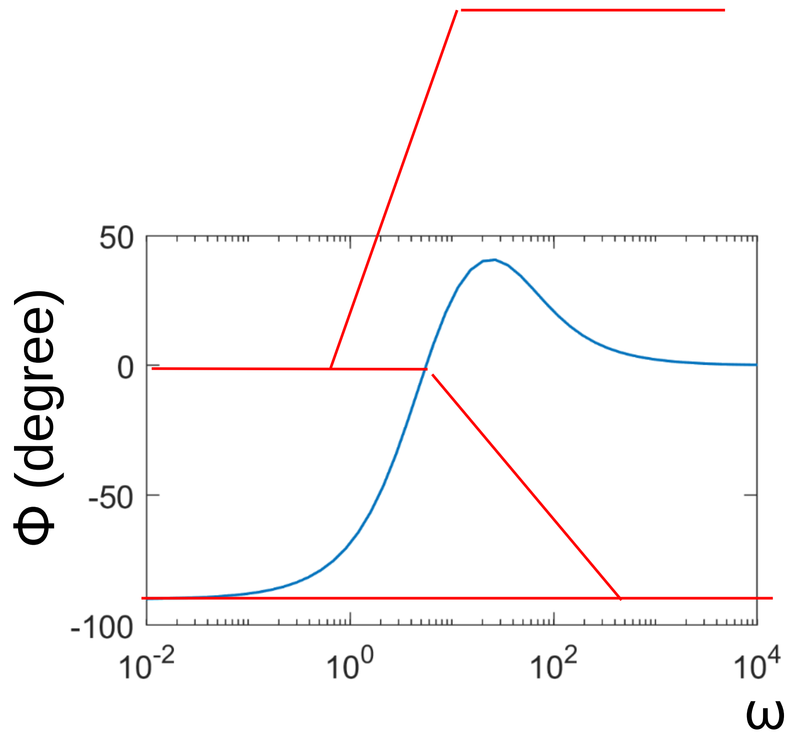
Exercises

- $\mathbf{H}(\omega) = \dfrac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$

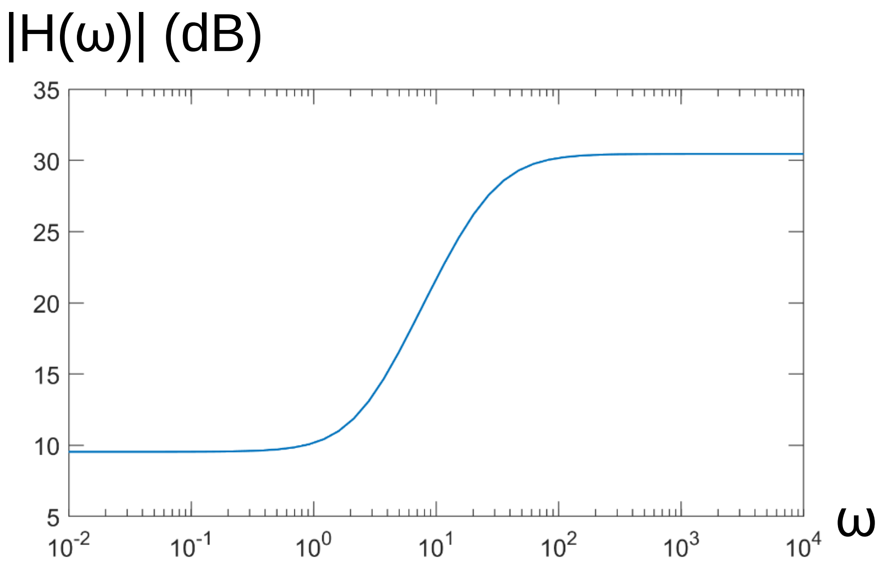
- $\mathbf{H}(\omega) = \dfrac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}$

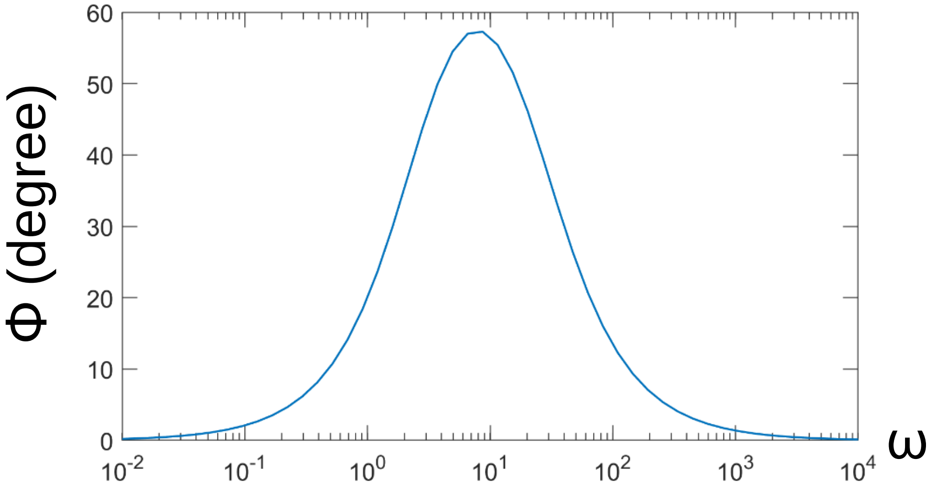
$$\mathbf{H}(\omega) = \frac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)} = \frac{\left(1 + \frac{j\omega}{5}\right)^2}{j10\omega\left(1 + \frac{j\omega}{50}\right)}$$





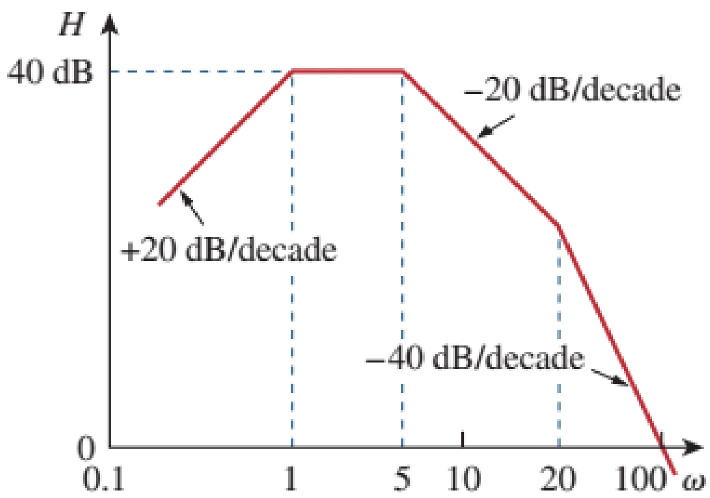
$$\mathbf{H}(\omega)=\frac{(j10\omega+30)^2}{(300-3\omega^2+j90\omega)}=\frac{3(1+\frac{j\omega}{3})^2}{1+j\frac{3\omega}{10}+\left(\frac{j\omega}{100}\right)^2}$$





Obtain the transfer function

Exercise: find the transfer function represented by the following plot.



$$H(\omega) = \frac{100 j\omega}{(1 + j\omega)(1 + \frac{j\omega}{5})(1 + \frac{j\omega}{20})}$$

Outline

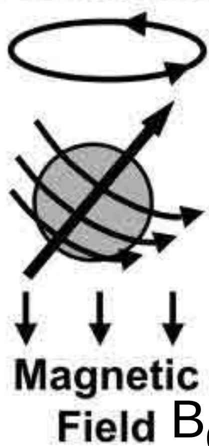
- Frequency response
- Transfer function
- Bode plots (or diagram)
- Resonance
  - Introduction
  - Series RLC (Resonance frequency, Q factor, bandwidth)
  - Parallel RLC
- Filters

What is resonance

- From Latin resonantem: to sound again, resound, re-echo
- Chinese: 共振/谐振 (describes the mechanism for resound to occur)；
  - 激发信号的相位与系统响应的相位一致。
  - 激发能量的持续累积可大大增加其震荡幅度。

B_1 field generation

核磁共振



B_0

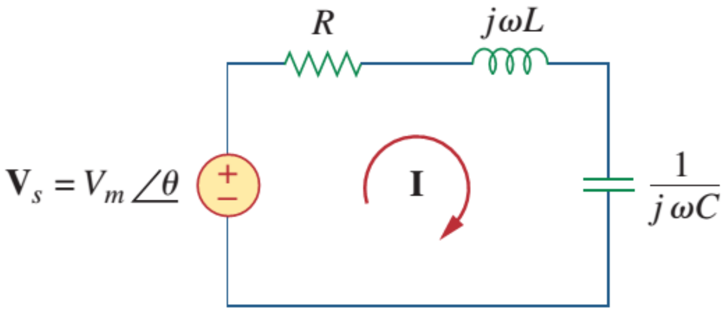
B_0~1000000B_1

1940 Tacoma Narrows Bridge collapse

Series Resonance

- A series resonant circuit consists of an inductor and capacitor in series.

$$H(\omega) = \frac{V}{I} = \mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$



  - Resonance occurs when the imaginary part of *Z* is zero (|Z| is minimum).
  - The value of ω that satisfies this is called the resonant frequency.

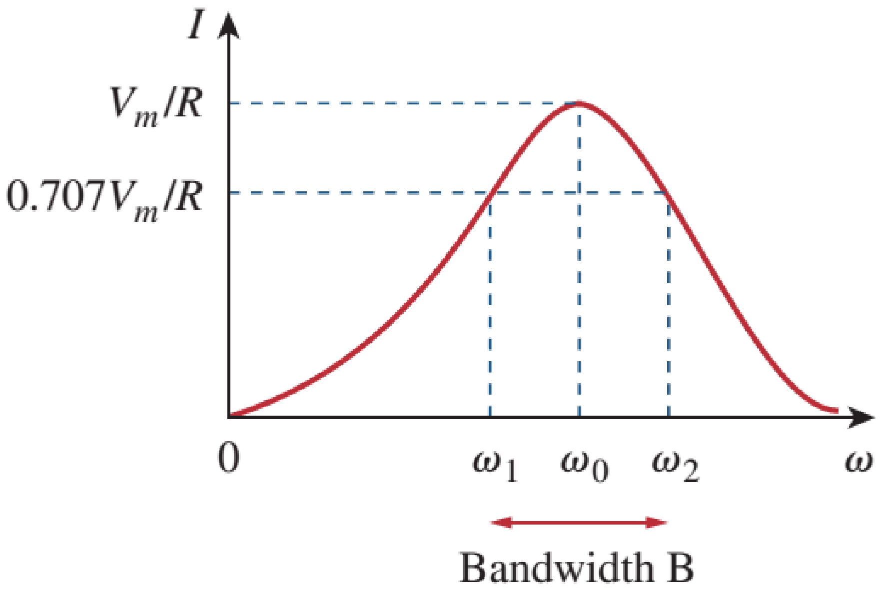
$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

Series Resonance

- At resonance:
  - The impedance is purely resistive
  - The voltage V_s and the current I are in phase
  - The magnitude of the impedance is minimum
  - The inductor and capacitor voltages can be much larger than that of the source

$$|\mathbf{V}_L| = |\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C}$$

Bandwidth



$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$P = \frac{1}{2} I^2 R = \frac{1}{2} \frac{V_m^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

- Bandwidth: the difference between the two half-power frequencies

$$P_{\max} = \frac{1}{2} \frac{V_m^2}{R}$$

Bandwidth

When $P = \frac{1}{2P_{\max}}$,

$$(\omega L - \frac{1}{\omega C})^2 = R^2$$

$$\omega L - \frac{1}{\omega C} = \pm R$$

$$\omega^2 LC \mp \omega CR - 1 = 0$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

The other two solutions are negative.

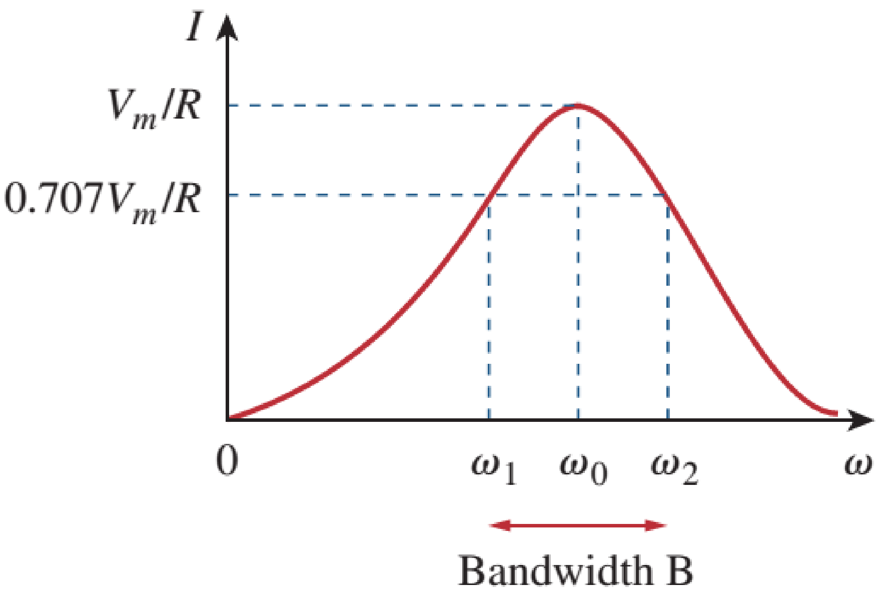
$$\boxed{B = \omega_2 - \omega_1 = \frac{R}{L}}$$

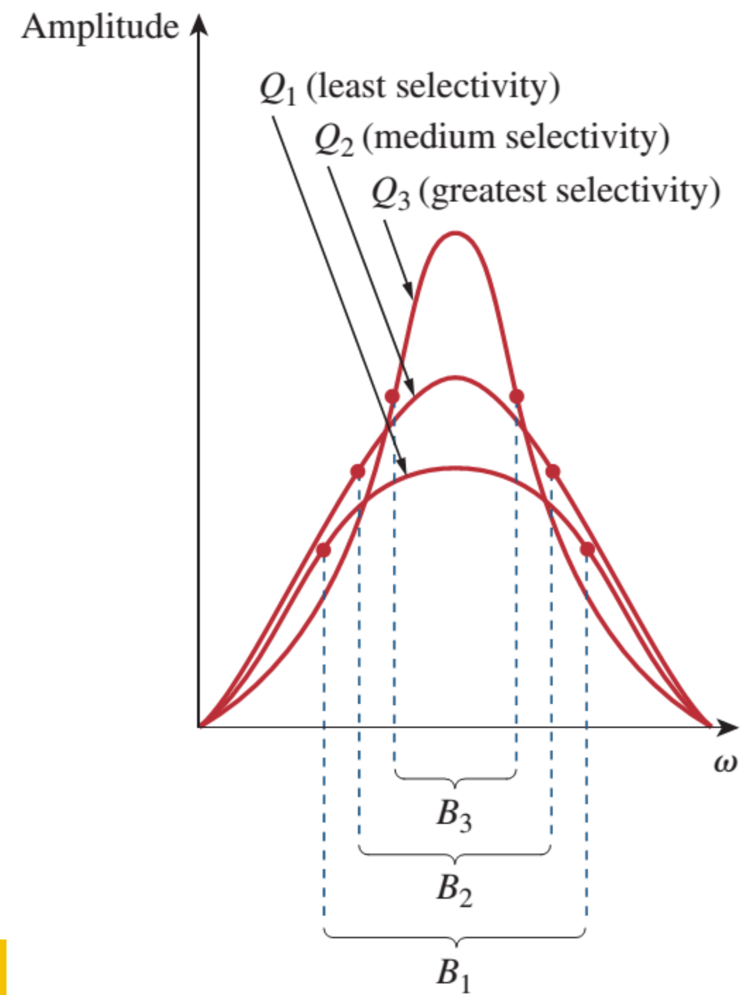
Useful relationship:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\omega_1 \omega_2}$$

Quality Factor *Q*

- Quality factor *Q*: measure the "sharpness" of the resonance.





The smaller the *B*, the higher the *Q*.

$$Q = \frac{\omega_0}{B}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad B = \omega_2 - \omega_1 = \frac{R}{L}$$

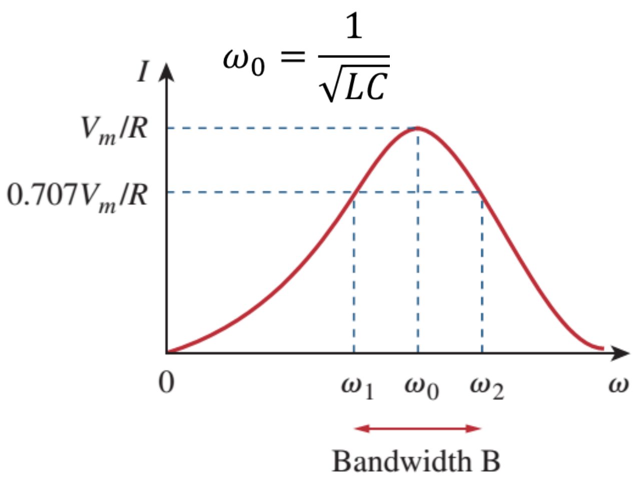
$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$$

Definition of *Q* in terms of energy dissipation

Q =Peak stored energy: Energy dissipated in $\frac{1}{2\pi}$ period at resonance.

$$Q = \frac{\frac{1}{2}LI^2}{\left[\left(\frac{1}{2}I^2R\right)\frac{1}{\omega_0}\right]} = \frac{L\omega_0}{R}$$

Positions of Half-Power Frequencies



$$\omega_{1,2} = \mp\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

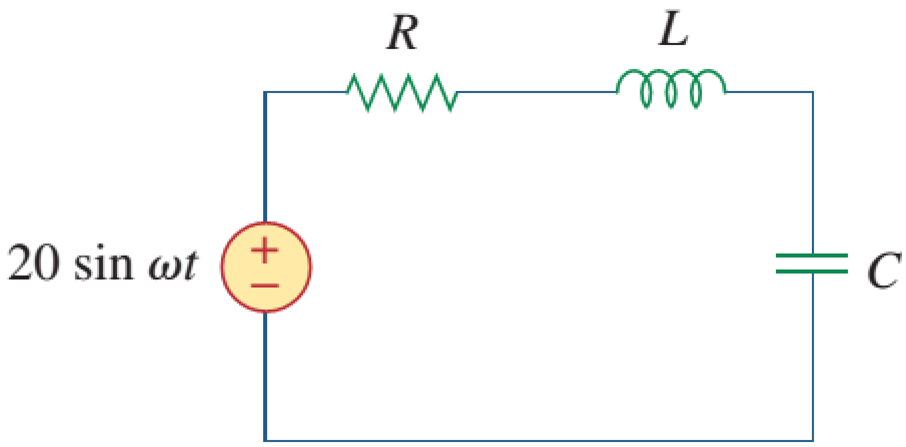
Note that the half power frequencies are centered around $\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$, but not the resonance frequency ω_0.

$$\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \omega_0\sqrt{1 + \frac{1}{4Q^2}}$$

- For high-Q ($Q \geq 10$) circuits,

$\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \approx \omega_0$ and $\omega_{1,2} \approx \omega_0 \mp \frac{\omega_0}{2Q}$, i.e. approximately center around ω_0.

Example



In the circuit, $R = 2\Omega$, $L = 1\text{mH}$ and $C = 0.4\mu\text{F}$

- Find resonant frequency ω_0.
- Calculate *Q* and bandwidth *B.*
- Find half-power frequencies.
- Determine the amplitude of the current at $\omega_{0,}$ ω_1 and ω_2.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$B = R/L = 2 \text{ krad/s}$$

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

$$\omega_1 = 49 \text{ krad/s}, \ \omega_2 = 51 \text{ krad/s}$$

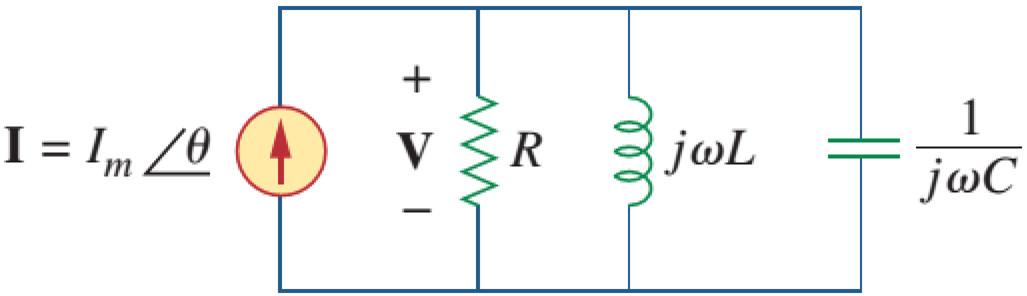
At $\omega = \omega_0$,

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

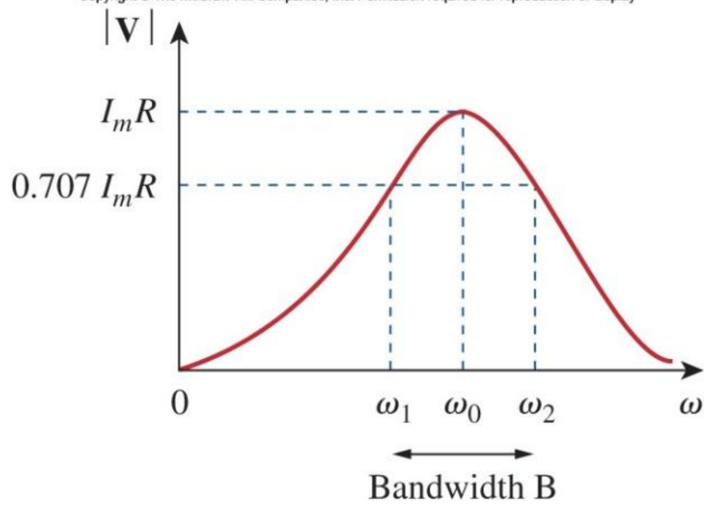
At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

Parallel resonance



Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display



Parallel resonance

Parallel circuit

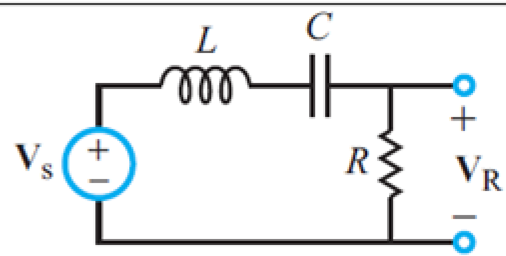
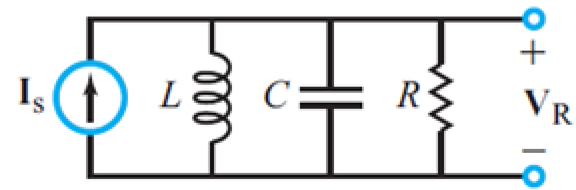
$$Y = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} \qquad V = \frac{I}{Y}, \qquad P = \frac{V^2}{R}$$

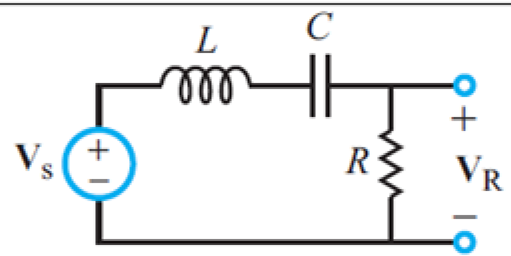
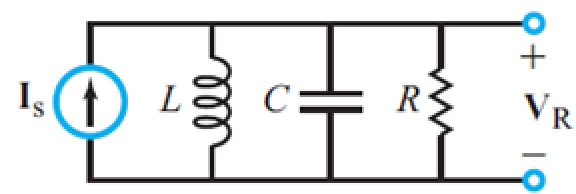
Series circuit

$$Z = R + j\omega L + \frac{1}{j\omega C} \qquad I = \frac{V}{Z}, \qquad P = RI^2$$

We can get the results for parallel resonance after the following replacements in the series resonance results

R → 1/R; L → C; C → L, I→V, V→I

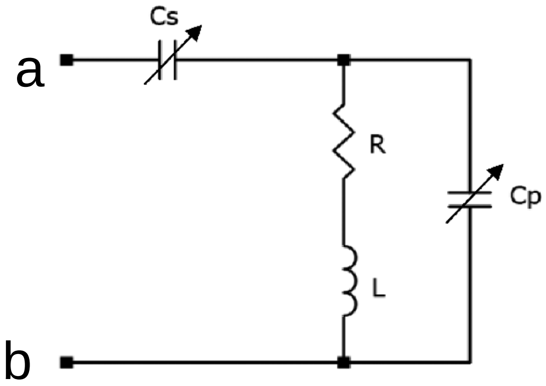
RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	
Bandwidth, B	$\dfrac{R}{L}$	
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	

RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 RC}$	$\dfrac{\omega_0}{B} = \omega_0 RC = \dfrac{R}{\omega_0 L}$
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

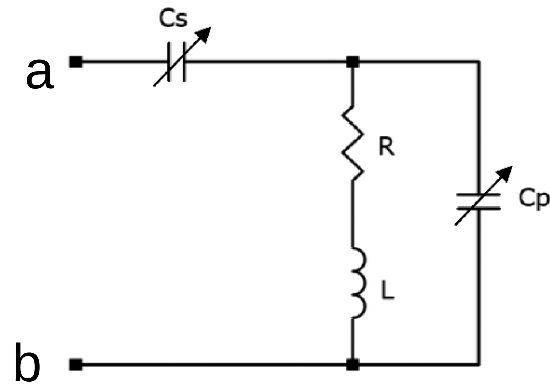
Notes: (1) The expression for Q of the series RLC circuit is the inverse of that for Q of the parallel circuit. (2) For $Q \geq 10$, $\omega_1 \simeq \omega_0 - \frac{B}{2}$, and $\omega_2 \simeq \omega_0 + \frac{B}{2}$.

Resonance frequency of RF coil (transmit mode)

The following is the equivalent circuit of an MRI coil, find the resonance frequency of the coil.



Solution



$$Z = -j\frac{1}{\omega C_s} + \frac{R + j\omega L}{j\omega C_p R - \omega^2 C_p L + 1}$$

$$\mathrm{Im}(Z) = -\frac{1}{\omega C_s} + \frac{-\omega C_p R^2 + \left(1-\omega^2 C_p L\right)\omega L}{\left(1-\omega^2 C_p L\right)^2 + \left(\omega C_p R\right)^2} = 0$$

$$\left(1-\omega^2 C_p L\right)^2 + \left(\omega C_p R\right)^2 = \left(1-\omega^2 C_p L\right)\omega^2 L C_s - \omega^2 C_s C_p R^2$$

$$C_p\left(C_p + C_s\right)L^2\omega^4 + \left(C_p^2 R^2 + C_p C_s R^2 - LC_s - 2LC_p\right)\omega^2 + 1 = 0$$

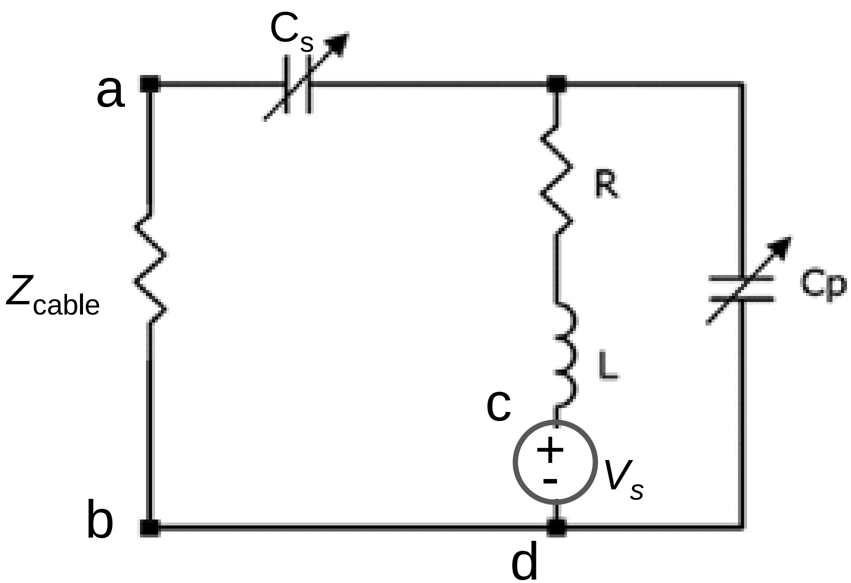
When $C_p^2 R^2 + C_p C_s R^2 \ll LC_s + 2LC_p, \omega_{res} \approx 1/\sqrt{L(C_p + C_s)}$

$$\mathrm{Re}(Z) = \frac{1-2\omega^2 C_p RL}{\left(1-\omega^2 C_p L\right)^2 + \left(\omega C_p R\right)^2}R.$$

At resonance, $\mathrm{Re}(Z) \approx (1 + \frac{C_p}{C_s})^2$, when $\left(\omega C_p R\right)^2 \ll 1$.

One can adjust C_s and C_p values to make the circuit resonate at a specific frequency with equivalent impedance at terminals *a*-*b* matched to that of the cable (typically 50 Ω).

Resonance frequency of RF coil (receive mode)



One can show that when the circuit resonates at frequency *f* with impedance matched with the cable impedance at terminal a-b. It is also on resonance at frequency *f* in the receive mode, i.e. the equivalent impedance at the two terminals of V_s is real. Therefore, the same coil can be used as an efficient receiver coil.

When impedance matched, $Z_{\text{cable}} = \frac{R(1-2\omega^2 C_p RL)}{(1-\omega^2 C_p L)^2 + (\omega C_p R)^2}$

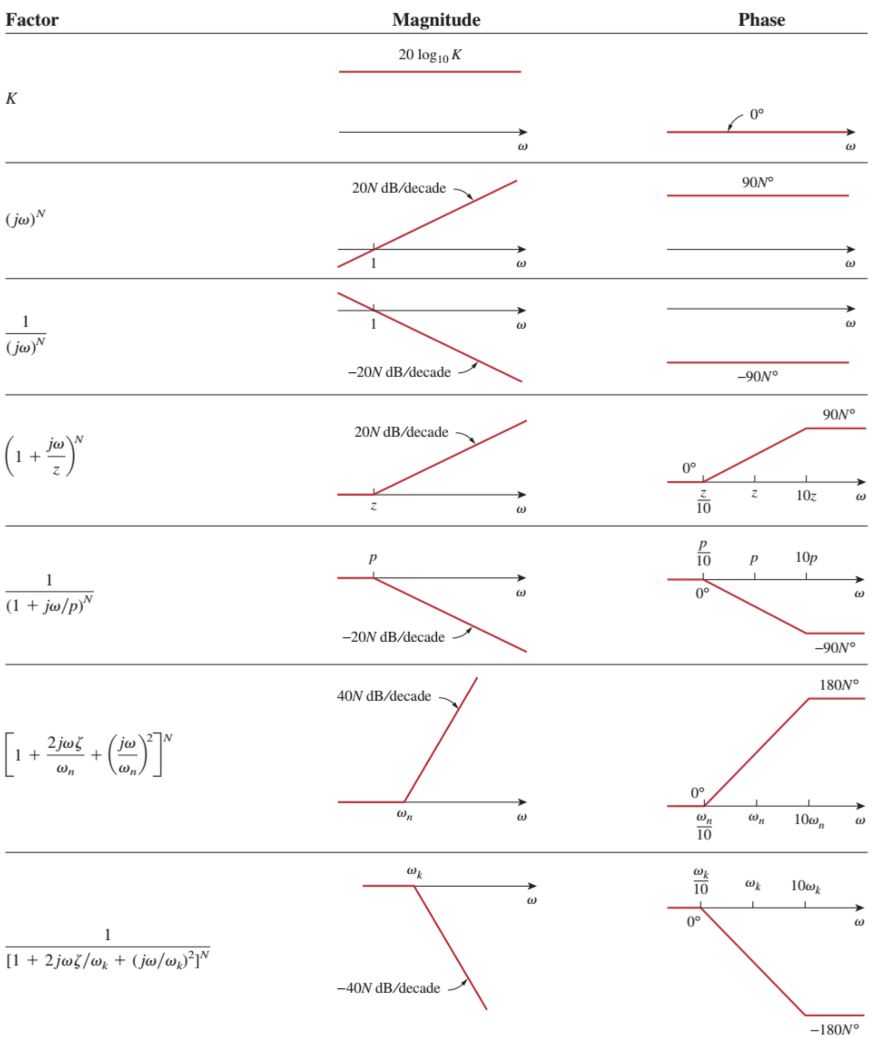
$$Z_{\text{cd}} = R + j\omega L + \frac{\left(-j\frac{1}{\omega C_s} + Z_{\text{cable}}\right)\left(-j\frac{1}{\omega C_p}\right)}{-j\frac{1}{\omega C_s} + Z_{\text{cable}} - j\frac{1}{\omega C_p}}.$$

One can show that $\text{Im}(Z_{\text{cd}}) = 0$ at resonance frequency ω_{res}, where ω_{res} is the frequency that satisfies $\text{Im}(Z) = 0$ in the transmit mode.

Summary

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

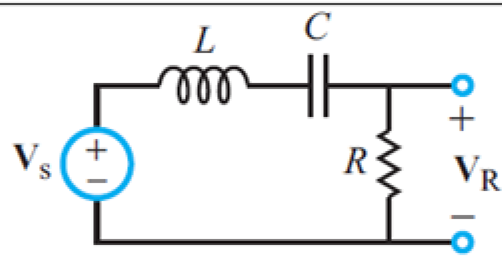
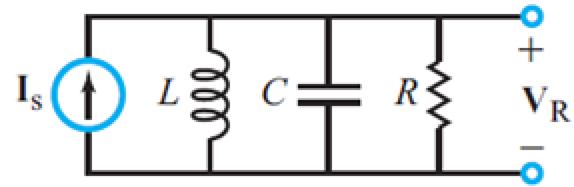
Summary of Bode straight-line magnitude and phase plots.



Power P in decibels = $10\log_{10}\frac{P}{P_{\text{reference}}}$

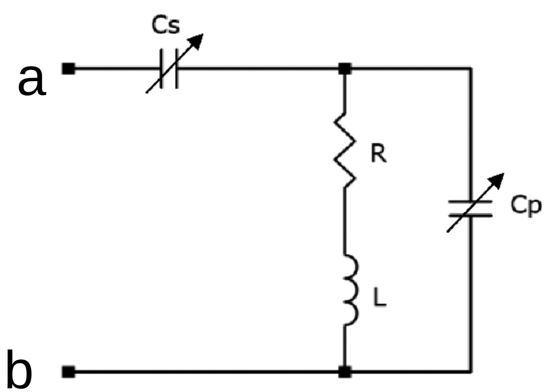
$H(\omega)$ in decibels = $20\log_{10}(|H(\omega)|)$

Resonance condition: $\mathrm{Im}(Z) = 0$

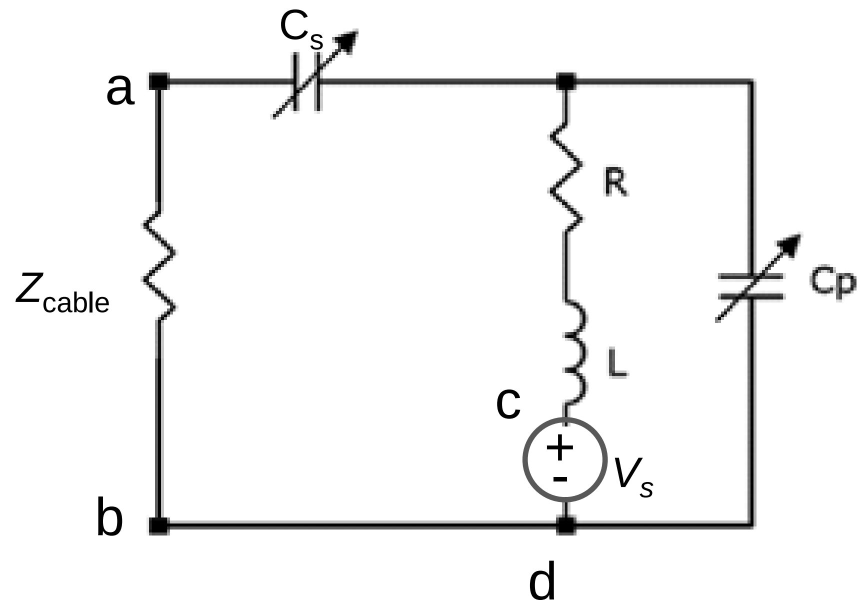
RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	$\dfrac{\omega_0}{B} = \dfrac{R}{\omega_0 L}$
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

RF coil

Transmit mode



Receive mode



$\omega_{res} \approx \frac{1}{\sqrt{L(C_p+C_s)}}$ in both case when R is small, i.e.

$$C_p^2R^2 + C_pC_sR^2 \ll LC_s + 2LC_p$$

$$R \ll \sqrt{\frac{LC_s + 2LC_p}{C_p(C_p + C_s)}}$$