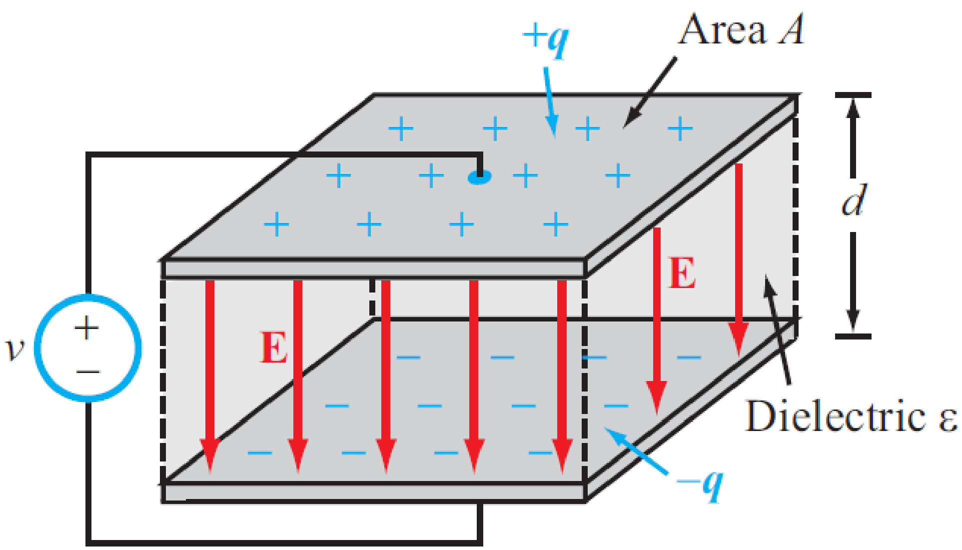
第六章 一阶电路

大纲

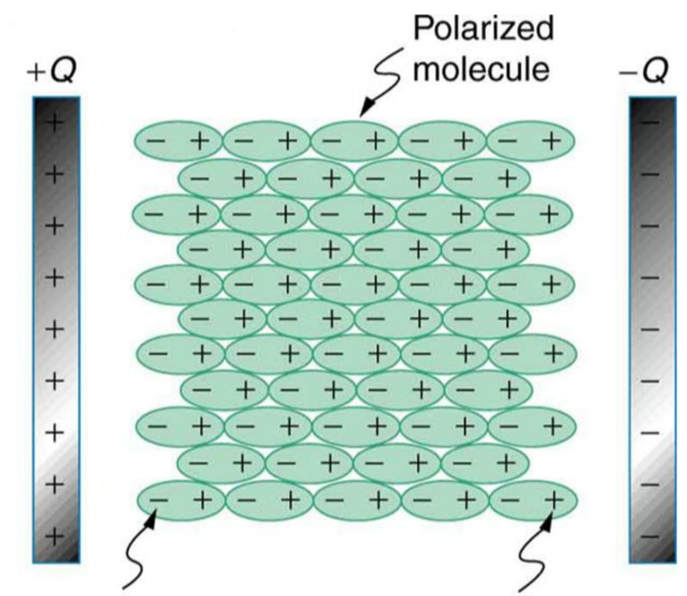
- 电容与电感
- 一阶电路方程和求解
- 一阶电路实例分析

Capacitors

An element that stores energy in electric field



Parallel plate capacitor



平板电容器公式

电场 $\mathrm{E} = \dfrac{电荷面密度}{\varepsilon_0} = \dfrac{自由电荷面密度}{\varepsilon}$

ε_0：真空介电常数 （8.85*10^{-12} F/m）； ε：介质的介电常数

自由电荷 Q =自由电荷面密度×面积 = εAE

电压 = 电场对路径的积分：V = Ed

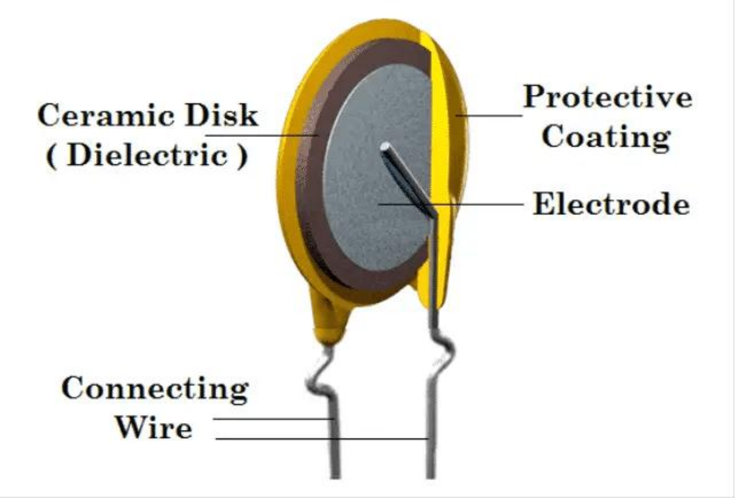
电容 $C = \dfrac{Q}{V} = \dfrac{\varepsilon \mathrm{AE}}{\mathrm{Ed}} = \dfrac{\varepsilon \mathrm{A}}{\mathrm{d}}$

自由电荷 Q = CV = εAE

Energy = $\int_0^Q V dq = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C} = \frac{CV^2}{2}$

Practice

- What is the capacitance of a capacitor with diameter of 4 mm, electrode distance of 1 mm; $\varepsilon = 100\ \varepsilon_0$?



$$\frac{100 \times 8.85\mathrm{F/m} \times 10^{-12} \pi (0.002\ \mathrm{m})^2}{0.001\ \mathrm{m}}$$

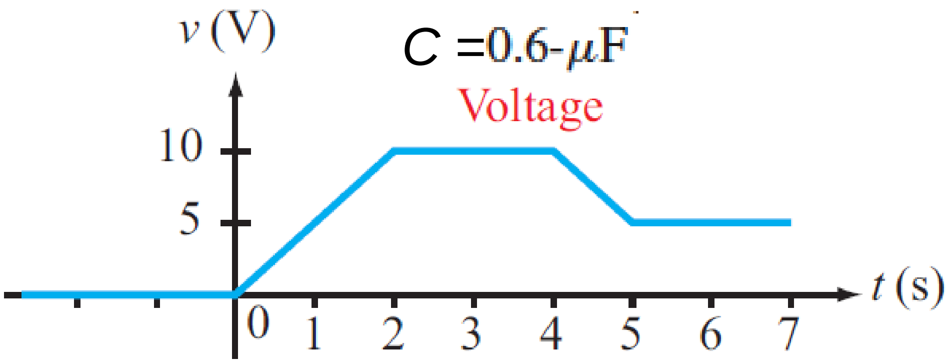
$$= 11.1\ \mathrm{pF}$$

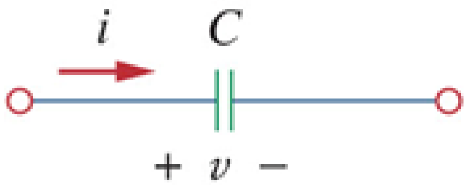
What is the energy stored in the capacitor when the voltage is 5 V?

$$\frac{11.1 \cdot 10^{-12}\mathrm{F} \cdot 25\mathrm{V}^2}{2} = 0.139\ \mathrm{nJ}$$

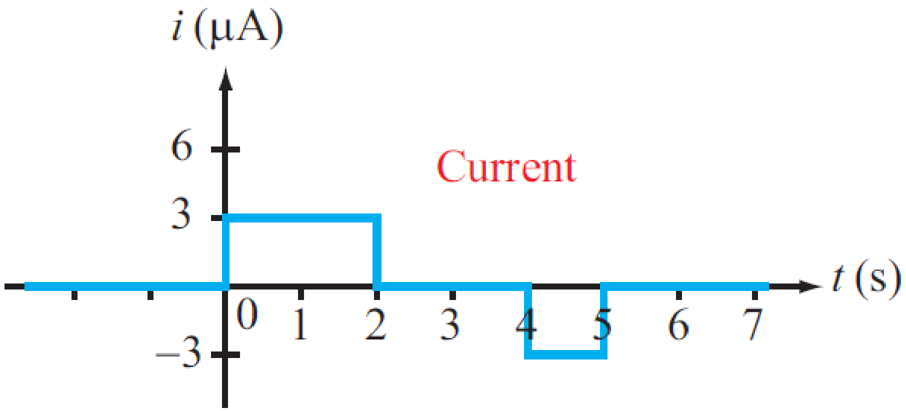
Capacitor Response

C =0.6-μF



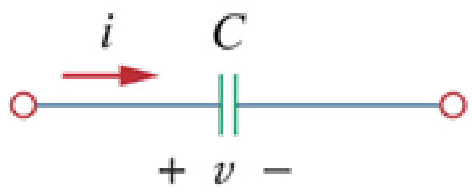


$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$



Capacitor Response

$C = 0.6\text{-}\mu\text{F}$



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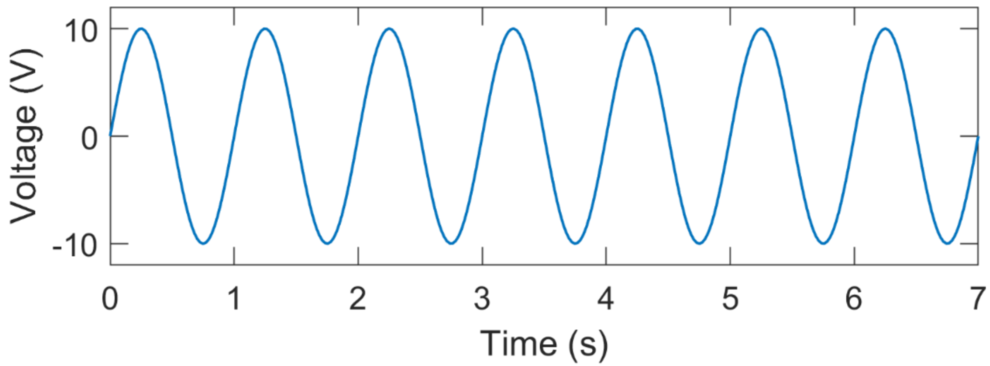
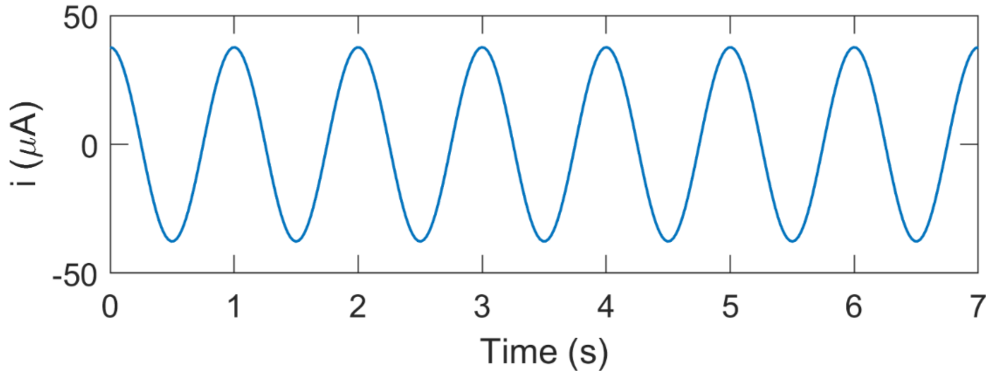
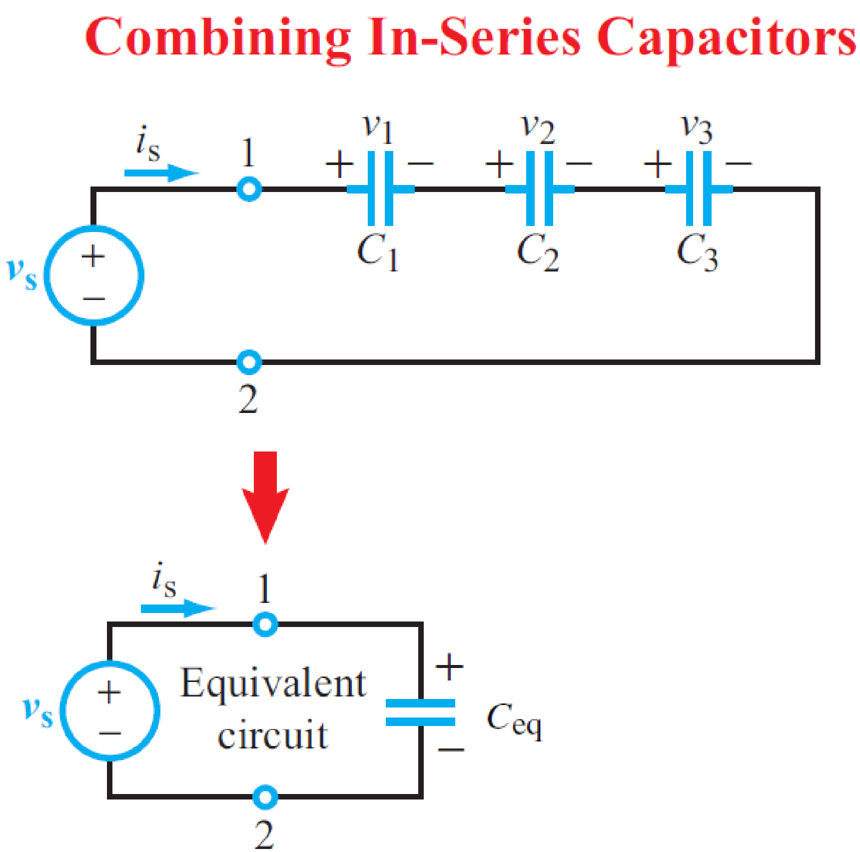




$$\text{Amplitude} = 2\pi C 10 = 37.7\ \mu\text{A}$$

Capacitors in Series



$$I = C_1\frac{dV_1}{dt} = C_2\frac{dV_2}{dt} = C_3\frac{dV_3}{dt}$$

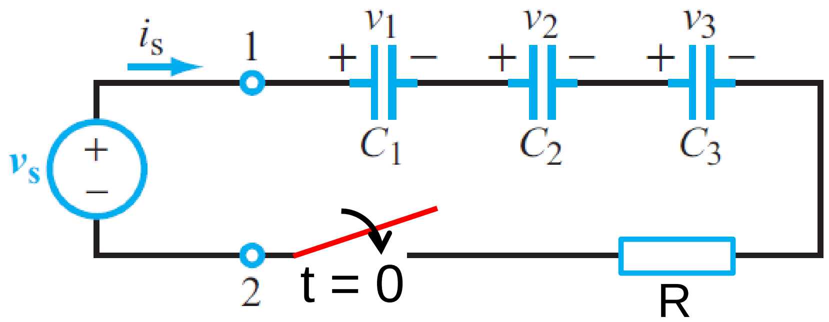
$$I = C_{eq}\frac{dV}{dt} = C_{eq}\frac{d(V_1+V_2+V_3)}{dt}$$

$$= C_{\text{eq}}\left(\frac{I}{C_1}+\frac{I}{C_2}+\frac{I}{C_3}\right)$$

$$C_{\text{eq}} = \left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)^{-1}$$

与并联电阻的等效电阻公式类似。

稳态时各电容的分压



假设连接电源之前各电容所存储的电荷为0，求当开关闭合且达到稳态时各电容的电压。

KVL: $v_s = v_1 + v_2 + v_3$ (1)

KCL: $C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} = C_3 \frac{dv_3}{dt}$

$$\frac{d(C_1 v_1 - C_2 v_2)}{dt} = 0$$

$$\frac{d(C_2 v_2 - C_3 v_3)}{dt} = 0$$

因此，$C_1 v_1 - C_2 v_2$ 和 $C_2 v_2 - C_3 v_3$ 不随时间变化。

$$C_1 v_1 - C_2 v_2 = 0 \quad (2)$$
$$C_2 v_2 - C_3 v_3 = 0 \quad (3)$$

方程(1)-(3)联立可得，

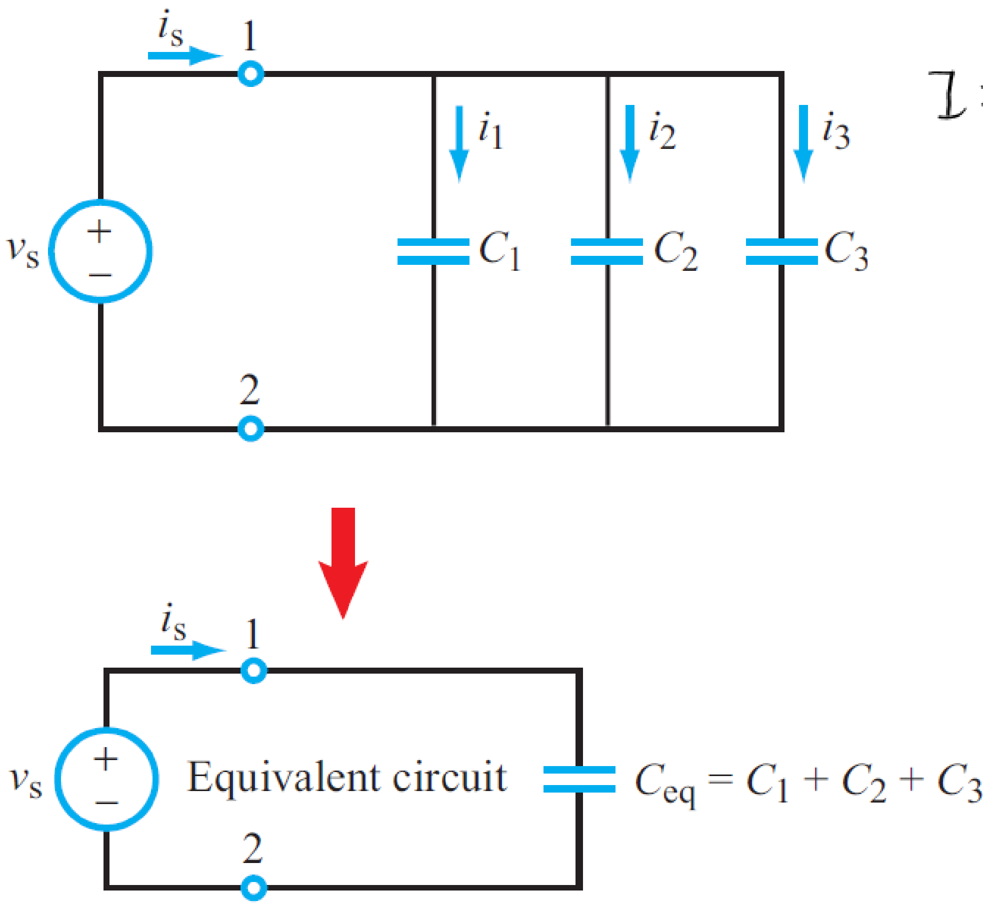
$$v_1 = \frac{v_s C_2 C_3}{C_1 C_2 + C_1 C_3 + C_2 C_3} = v_s \frac{\frac{1}{C_1}}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

$$v_2 = \frac{v_s C_1 C_3}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

$$v_3 = \frac{v_s C_1 C_2}{C_1 C_2 + C_1 C_3 + C_2 C_3}$$

串联电容的分压公式和并联电阻的分流公式类似。

Capacitors in Parallel

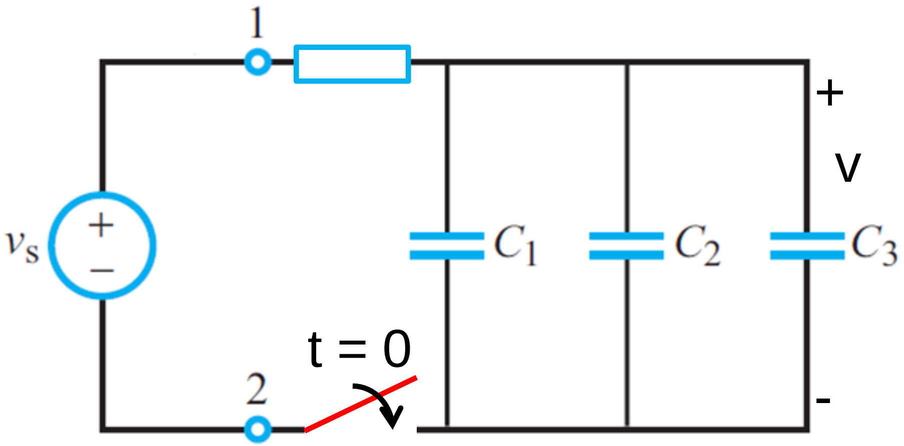


$$I = [C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}]$$
$$= (C_1 + C_2 + C_3) \frac{dv}{dt}$$
$$= C_{eff} \frac{dv}{dt}$$

$$C_{eq} = C_1 + C_2 + C_3$$

与串联电阻的等效电阻公式相同

Capacitors in Parallel



求当开关闭合且达到稳态时各电容的所存储的电荷。

因为电容并联，它们两端的电压相等且均为v_s。

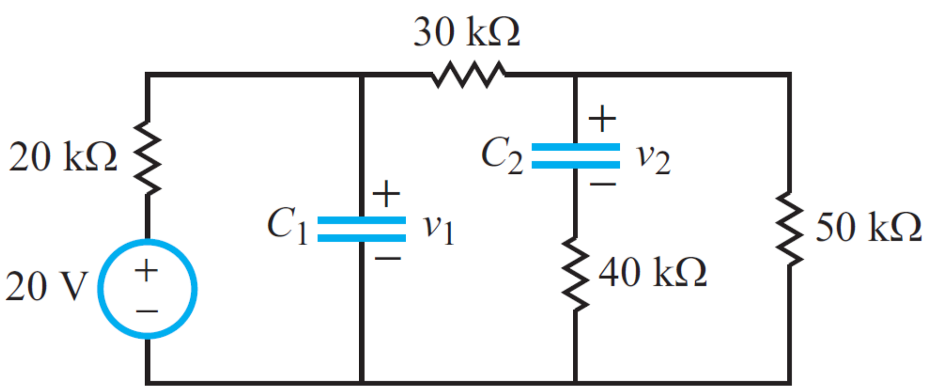
$Q_1 = v_s C_1$
$Q_2 = v_s C_2$
$Q_3 = v_s C_3$

电路的动态响应

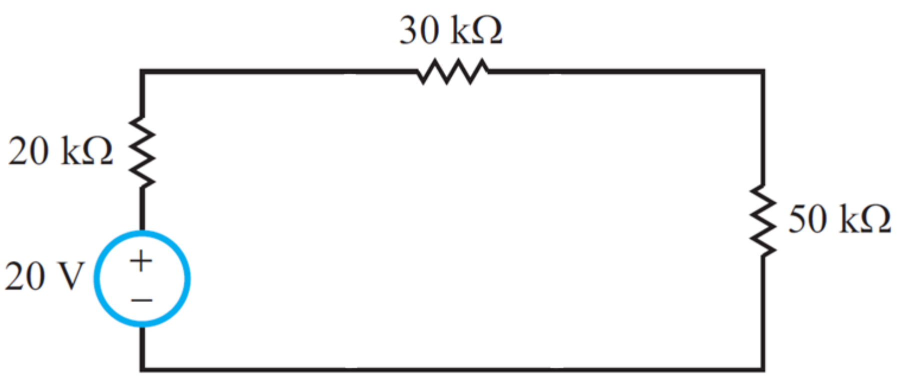
- 当电路的结构或内部元件特性发生改变时，电路的瞬态响应可分解为自然响应和稳态值之和（线性系统的可叠加性）
- 自然响应描述电路中不存在电源时的动态变化过程，电压电流值在t→∞时趋向0。
- 稳态值对应在电源作用下电路的最终状态。
- 稳态时流过电容器的电流为0 （开路）。

Exercise

Find the voltages across C_1 and C_2 at steady state.



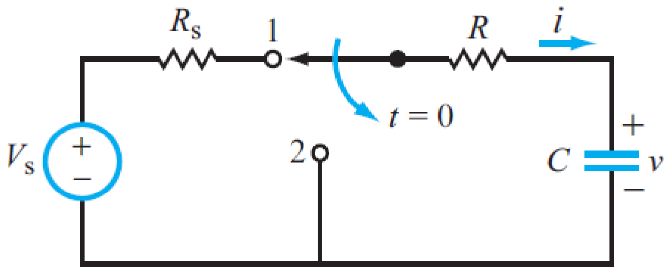
Equivalent circuit at steady state



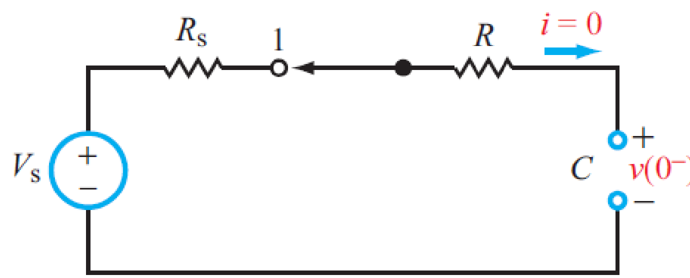
V_1 = 16 V; V_2=10 V;

自然响应

Natural response: The behavior (in terms of voltage and current) of the circuit itself, with no external sources of excitation.

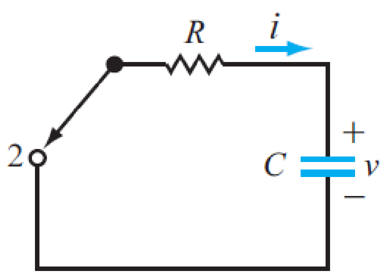


(a) RC circuit



$$v(0^-) = v_s$$

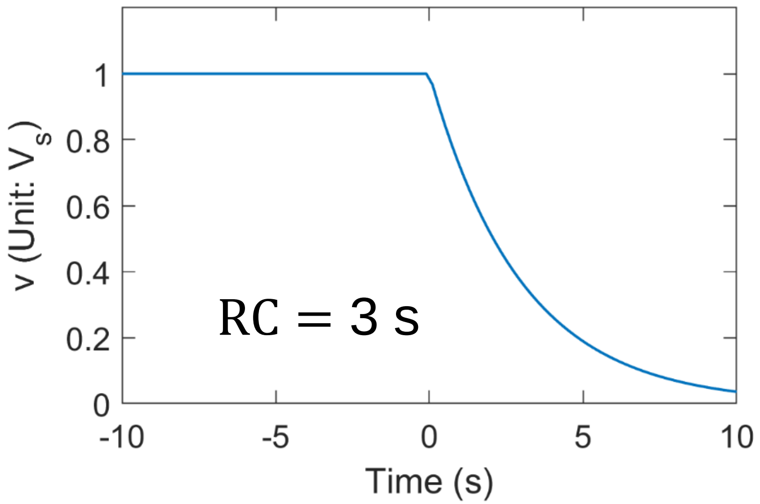
(b) At $t = 0^-$



(c) At $t \geq 0$

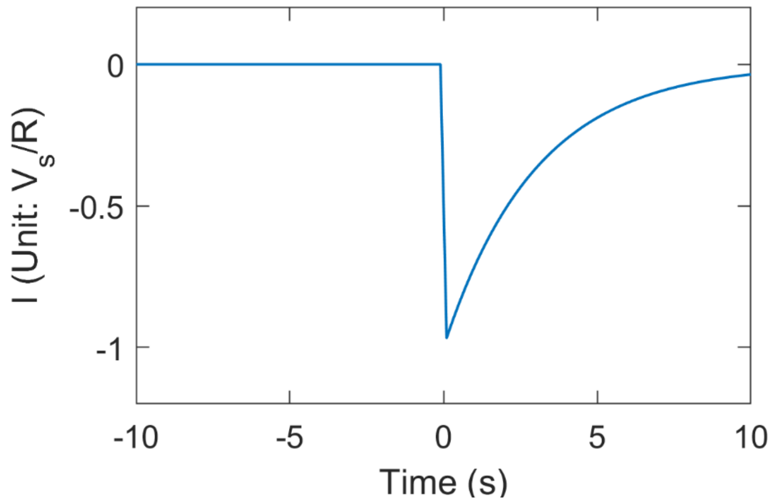
RC电路的自然响应

电压响应



连续

电流响应



不连续

电容电压电流的瞬时变化特性

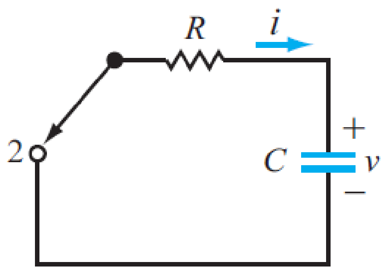
- 电容的电压具有时间连续性

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i(t)dt; \quad v(0^-) = v(0^+) = v_s$$

- 电容的电流可以发生突变

自然响应的求解

$$v(0^+) = v_s$$



(c) At $t \geq 0$

节点电压法

KCL: $C\frac{dv}{dt} + \frac{v}{R} = 0$

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

$$d\ln(v) = -\frac{1}{RC}dt$$

$$\ln v - \ln v(0) = \ln\frac{v}{V_s} = -\frac{1}{RC}t$$

$$\frac{v}{V_s} = e^{-\frac{1}{RC}t}$$

$$v = V_s e^{-\frac{1}{RC}t}$$

$$[RC] = \Omega \cdot F = s$$

General First Order Constant Coefficient Differential Equation

$$a\frac{dx}{dt}+bx+c=0 \qquad (1)$$

$$x(t=0)=x_0$$

$$a\frac{d(x+\frac{c}{b})}{dt}+b\left(x+\frac{c}{b}\right)=0$$

Let $x'=x+\frac{c}{b}$, then $\frac{a}{b}\frac{dx'}{dt}+x'=0$, which has the same form as the natural response. Therefore,

$$x'=x'(0)e^{-\frac{b}{a}t}$$

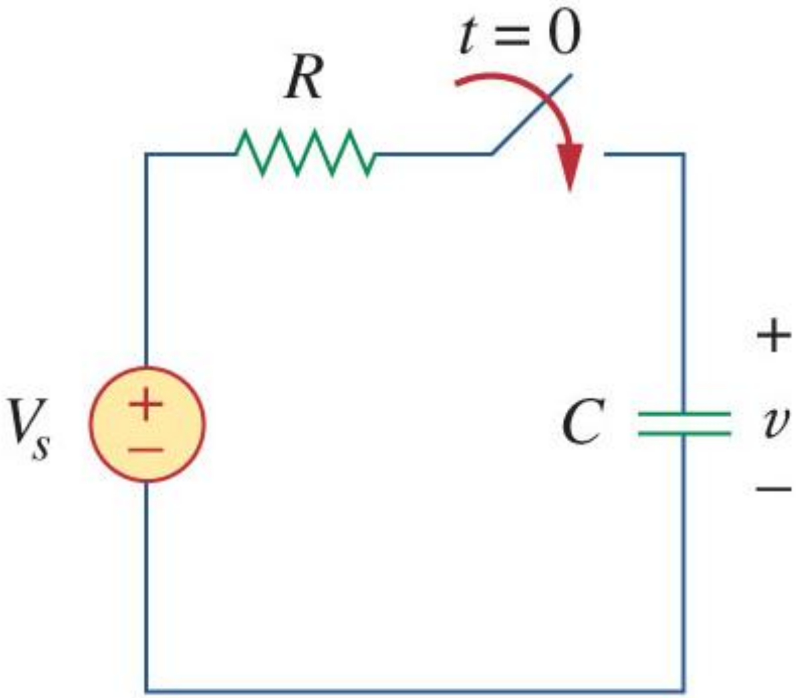
$$x=\left[x(0)+\frac{c}{b}\right]e^{-\frac{b}{a}t}-\frac{c}{b}$$

$$=\underbrace{[x(0)-x(\infty)]e^{-\frac{b}{a}t}}_{\text{自然响应}}+\underbrace{x(\infty)}_{\text{稳态}}.$$

Note:

1. $x(\infty)=-\frac{c}{b}$ is the solution of Eq. (1) when $\frac{dx}{dt}=0$.
2. 自然响应的等效初始值为真实初始值与稳态值之差。

Step Response of an RC circuit



Method 1

$t = 0^+$: $v = 0$.

$t > 0$: $\frac{v - V_s}{R} + C\frac{dv}{dt} = 0$

Then solve the equation following the general procedure.

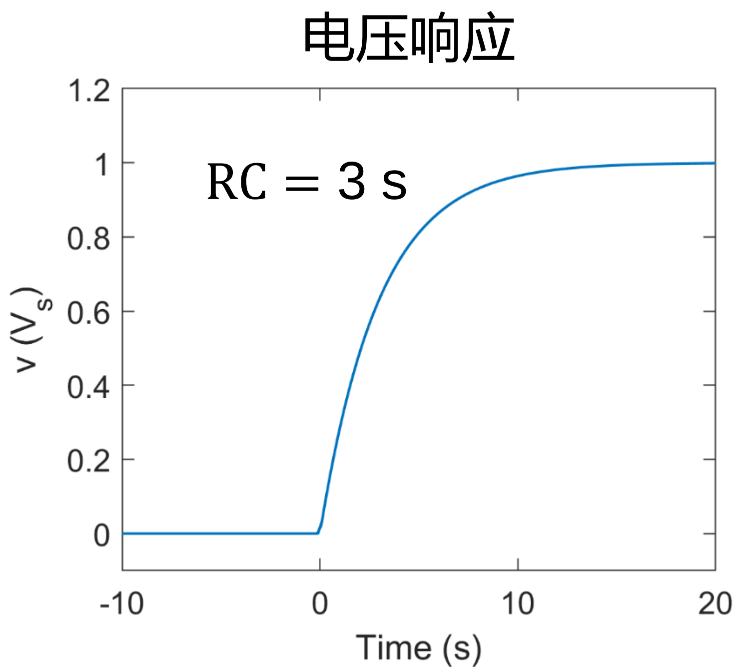
Method 2

$t = 0^+$: $v = 0$.

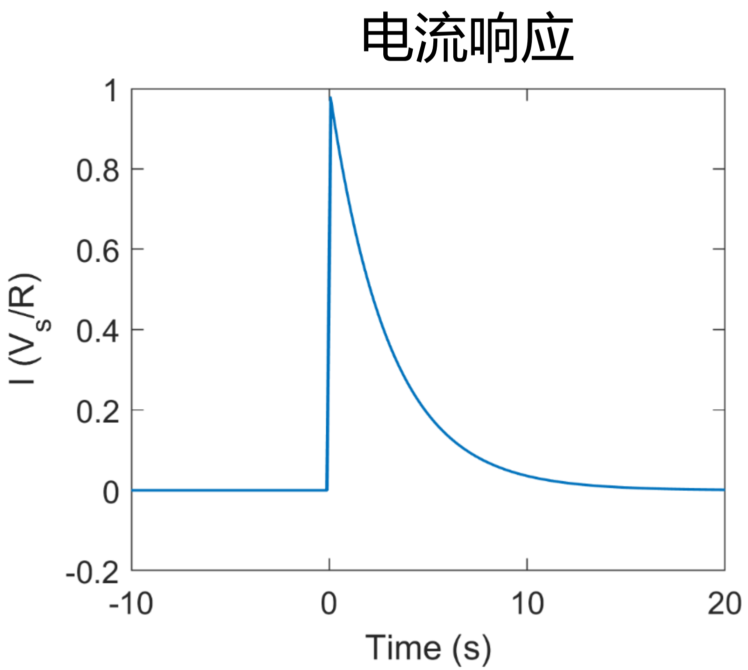
$t = \infty$: $v = V_s$

$$V(t) = -V_s e^{-\frac{t}{RC}} + V_s$$

RC电路的阶跃响应

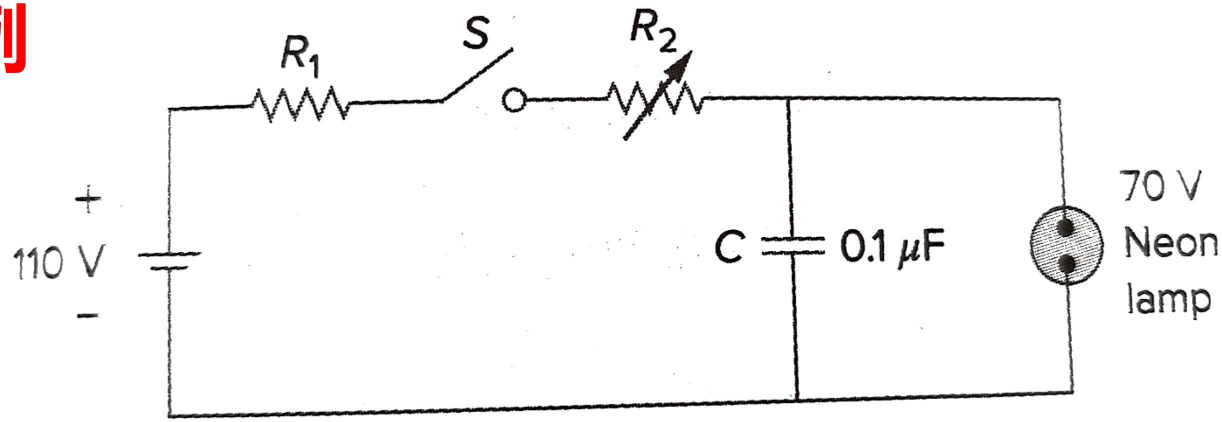


连续



不连续

RC电路应用实例



当电容两端电压达到70 V时，霓虹灯开启。开启后的低电阻使电容两端电压迅速降为零，霓虹灯关闭，电容重新开始充电。求霓虹灯的闪烁频率。当闪烁频率为1 Hz时，电阻值应为多少?

After lamp is turned off at t=0, $v = -110e^{-\frac{t}{(R_1+R_2)C}} + 110.$

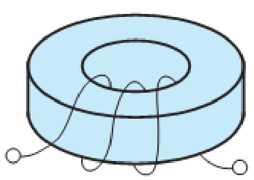
When lamp is on at $t = t_{\mathrm{on}}$, $-110e^{-\frac{t_{\mathrm{on}}}{(R_1+R_2)C}} + 110 = 70.$

$t_{\mathrm{on}} = (R_1+R_2)C \ln\frac{11}{4}$, $f = \frac{1}{t_{\mathrm{on}}} = 0.99\frac{1}{(R_1+R_2)C}.$

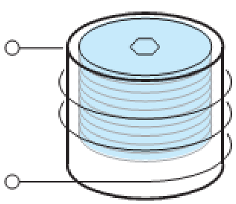
当$0.99\frac{1}{(R_1+R_2)C} = 1$，$(R_1+R_2) = 9.9\ \mathrm{M\Omega}.$

Inductors

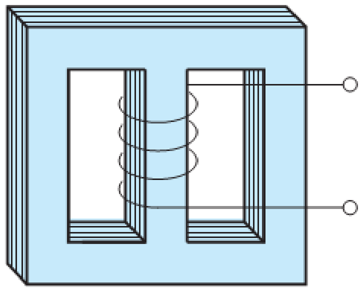
- A storage element that stores energy in magnetic field.
  - They have applications in lamps, transformers, radios, MRI machines, defibrillator, and electric motors.
- Any conductor has inductance, but the effect is typically enhanced by coiling the wire up around high permeability cores.



(a) Toroidal inductor



(b) Coil with an iron-oxide slug that can be screwed in or out to adjust the inductance



(c) Inductor with a laminated iron core

螺旋线圈电感器相关公式

磁场 B $= \mu \cdot$ 单位长度电流密度 $= \mu \frac{N}{l} I$;

磁通量$\Phi = BSN$， S 为线圈的截面积

$$V = \frac{d\varphi}{dt} = \frac{\mu N^2 S}{l}\frac{dI}{dt} = L\frac{dI}{dt}$$

所以，电感 $L = \frac{N^2 \mu S}{l}$

Δ Energy = $\int_0^t VIdt = \int_0^t L\frac{dI}{dt}Idt = \int_0^t LIdI = \frac{1}{2}LI^2|_0^t$

E= $\frac{1}{2}LI^2$; (单位换算：H A^2=J)

N为线圈的圈数， l为长度， μ − magnetic permeability;
磁导率 (真空 $\mu_0=4\pi \cdot 10^{-7}$ N/A^2 or $4\pi \cdot 10^{-7}$ H/m)

电感与电容比较

	电容	电感
储存量	$Q = CV$	$\Phi = LI$
电磁场	$\mathrm{E} = \dfrac{\text{自由电荷面密度}}{\varepsilon}$	$\mathrm{B} = \mu \cdot \text{单位长度电流密度} = \mu \dfrac{N}{l} I = \dfrac{\Phi}{SN}$
$I - V$关系	$I = \dfrac{dQ}{dt} = C\dfrac{dV}{dt}$	$V = \dfrac{d\Phi}{dt} = L\dfrac{dI}{dt}$
电容电感公式	$C = \dfrac{Q}{V} = \dfrac{\varepsilon \mathrm{AE}}{\mathrm{Ed}} = \dfrac{\varepsilon \mathrm{A}}{\mathrm{d}}$	$L = \dfrac{N^2 \mu S}{l}$
能量	$\dfrac{1}{2}CV^2$	$\dfrac{1}{2}LI^2$

Practice

- What is the inductance of a coil with diameter = 4 mm, length = 10 mm, 50 loops and $\mu = 100\ \mu_0$?

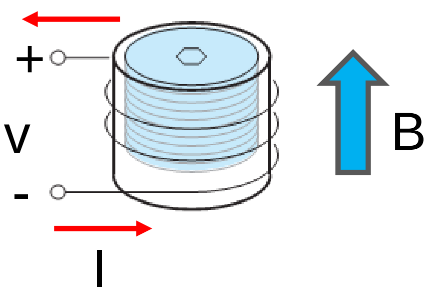
$$\frac{100 \times 4\pi \times \frac{10^{-7}\mathrm{H}}{\mathrm{m}} 50^2 \pi (0.002\ \mathrm{m})^2}{0.01\ \mathrm{m}} = 395\ \mu\mathrm{H}$$

- What is the stored energy if the current is 1 A?

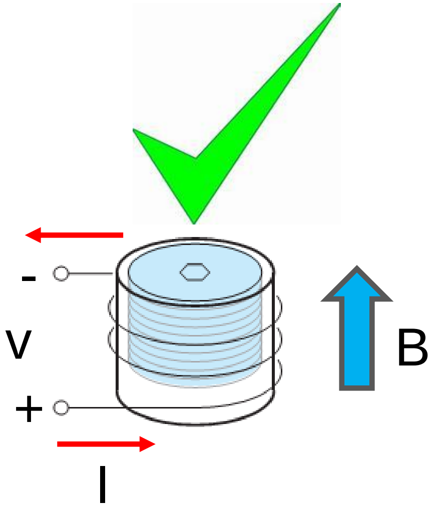
$$E = \frac{1}{2} L I^2 = 197.5\ \mu\mathrm{J}$$

Direction of the induced voltage

楞次定律：感应电压具有这样的方向，它的效应总要阻碍引起感应电压的磁通量的变化。



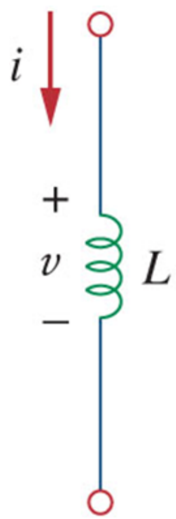
or



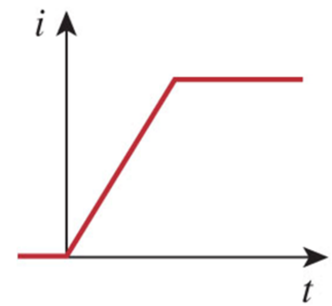
What is the direction of v when I increases with time?

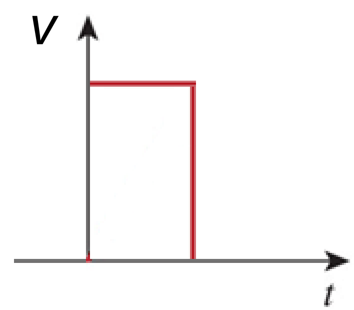
该电压施加于外部电路所造成的电流方向与I 的变化方向相反。

Important Property of Inductors



$$v = L\frac{di}{dt}$$

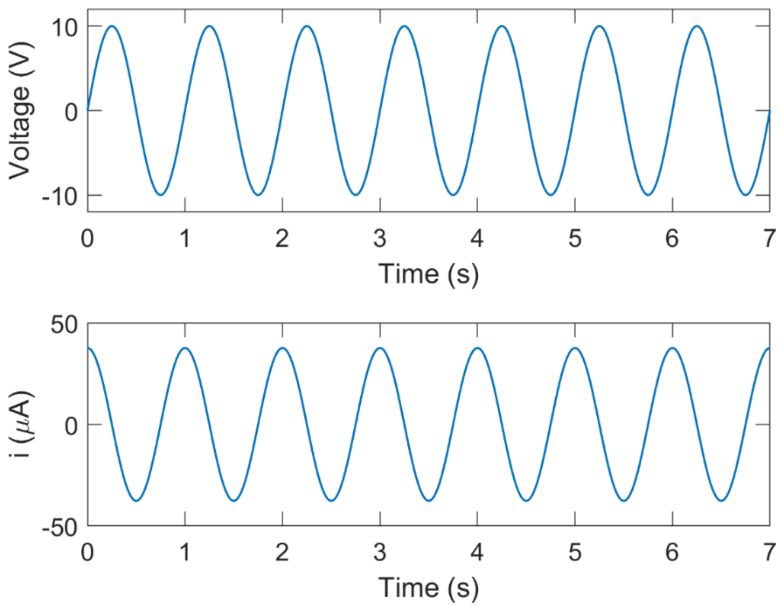




At static state, v = 0 V， R = 0 Ω;
Inductor can be treated as short circuit at steady state.

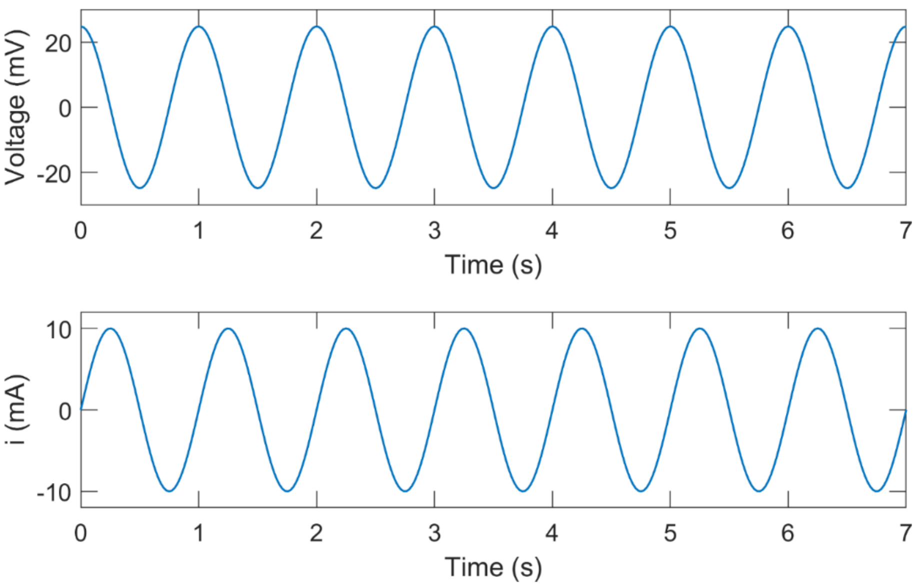
Inductor response to a sinusoidal current

电容



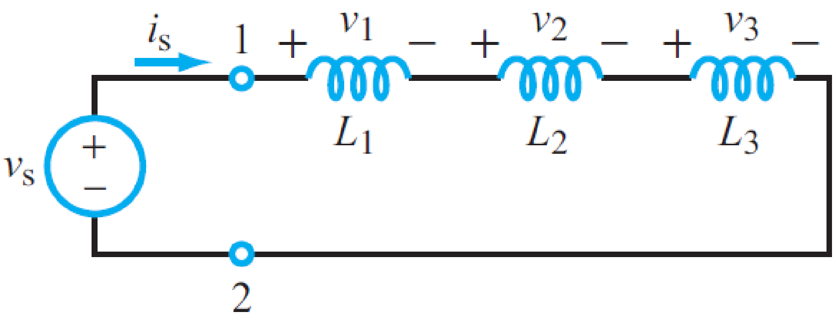
电流相位提早π/2

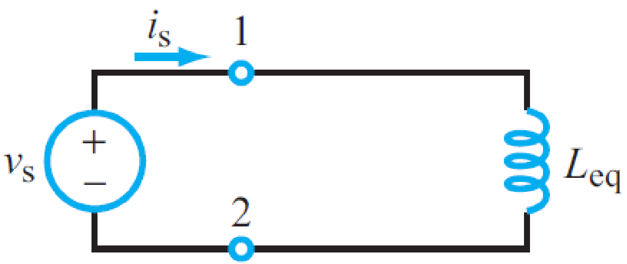
电感



电压相位提早π/2

Inductors in Series





$$V_s = V_1 + V_2 + V_3 = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} + L_3 \frac{dI}{dt}$$

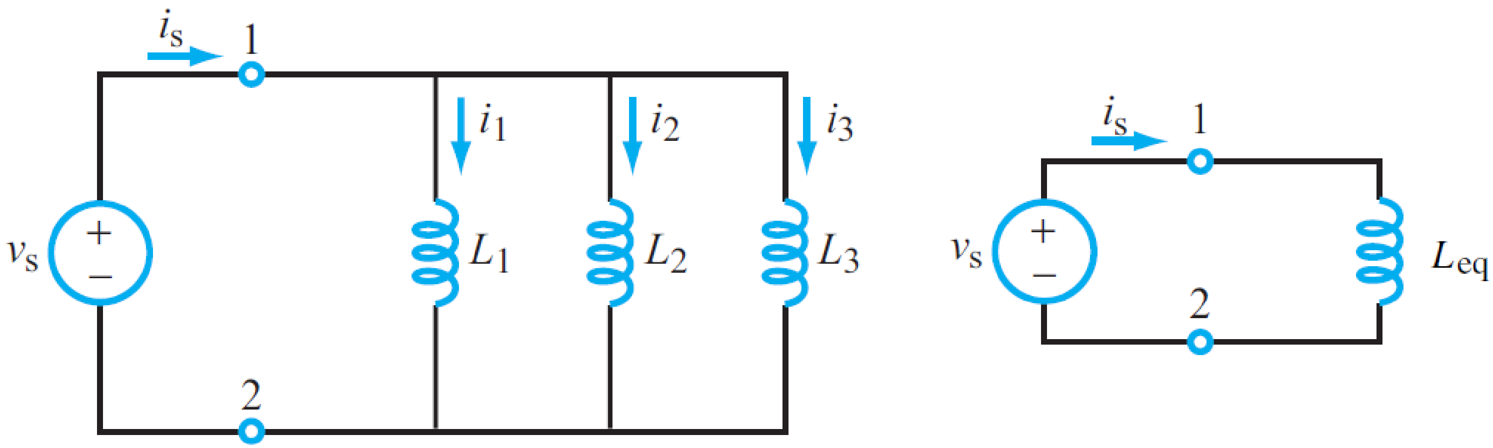
$$= (L_1 + L_2 + L_3) \frac{dI}{dt}$$

$$= L_{eq} \frac{dI}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3$$

Inductors in Parallel

Combining In-Parallel Inductors



$$I = i_1 + i_2 + i_3$$

$$V_s = L_1 \frac{di_1}{dt} = L_2 \frac{di_2}{dt} = L_3 \frac{di_3}{dt}$$

$$V_s = L_{eq} \frac{dI}{dt} = L_{eq}\left(\frac{di_1}{dt} + \frac{di_2}{dt} + \frac{di_3}{dt}\right)$$

$$= L_{eq}\left(\frac{V_s}{L_1} + \frac{V_s}{L_2} + \frac{V_s}{L_3}\right) \Rightarrow L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}}$$

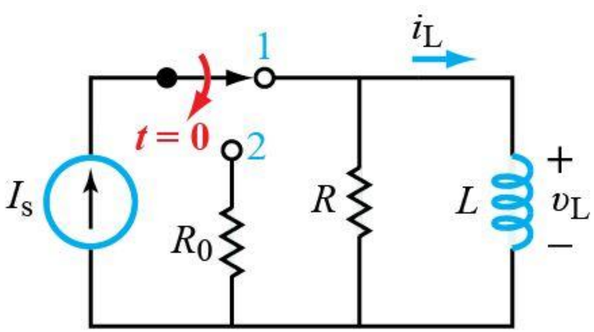
RLC串并联的等效公式

	串联	并联
电阻	$R_1 + R_2$	$\dfrac{R_1 R_2}{R_1 + R_2}$
电容	$\dfrac{C_1 C_2}{C_1 + C_2}$	$C_1 + C_2$
电感	$L_1 + L_2$	$\dfrac{L_1 L_2}{L_1 + L_2}$

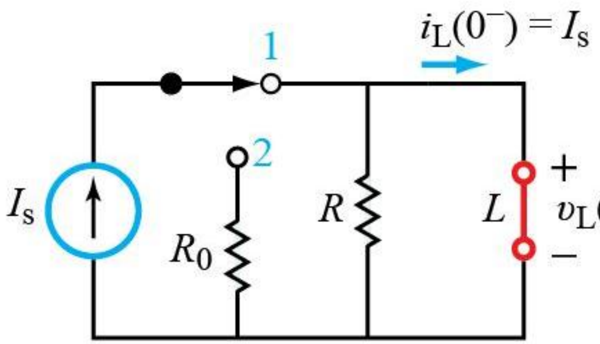
	串联分压	并联分流
电阻	$\dfrac{R_\mathrm{i}}{R_1 + R_2}$	$\dfrac{1/R_\mathrm{i}}{1/R_1 + 1/R_2}$
电容	$\dfrac{1/C_i}{1/C_1 + 1/C_2}$	$\dfrac{C_i}{C_1 + C_2}$
电感	$\dfrac{L_i}{L_1 + L_2}$	$\dfrac{1/L_i}{1/L_1 + 1/L_2}$

稳态值

求当开关位于1和2位置时电感的稳态电压和电流分别时多少?

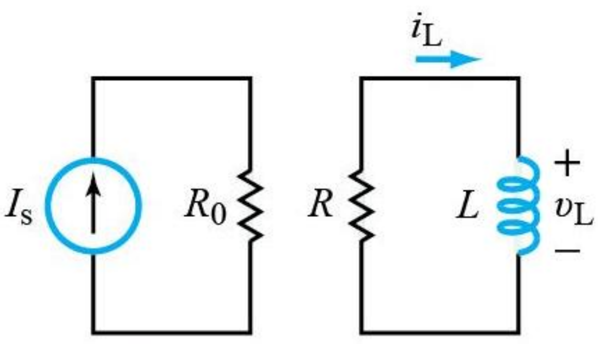


开关位置1



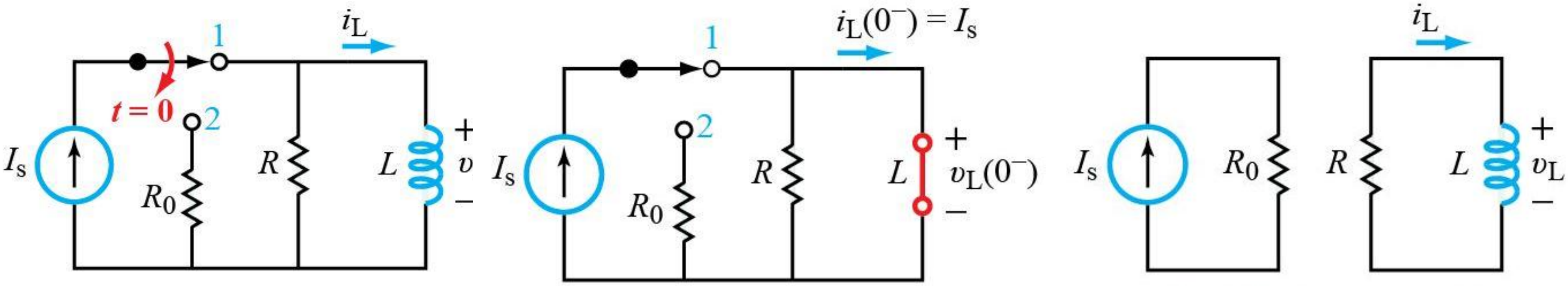
$$i_L = I_s$$
$$v_L = 0$$

开关位置2



$$i_L = 0$$
$$v_L = 0$$

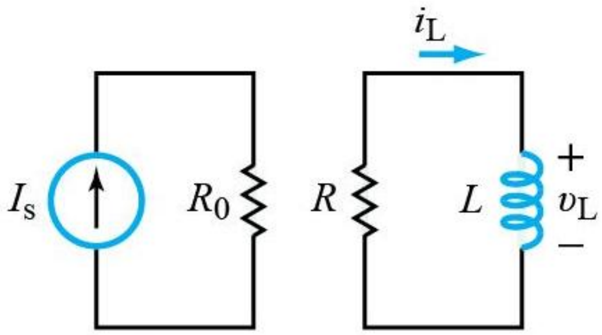
Natural Response of the RL Circuit



求开关从1切换至2后电感电压电流的自然响应?

$$i_L(0^+) = i_L(0^-) = I_s$$

Natural Response of the RL Circuit



$$i_L(0^+) = I_s$$

$$Ri_L + L\frac{di_L}{dt} = 0$$

$a = L,\ b = R,\ c = 0$

$$i_L(t) = I_s e^{-\frac{R}{L}t} = I_s e^{-\frac{t}{\tau}}$$

$$\tau = \frac{L}{R}$$

若 $R = 100k\Omega$，$L = 395mH$

$$\tau = \frac{L}{R} = 3.95ns$$

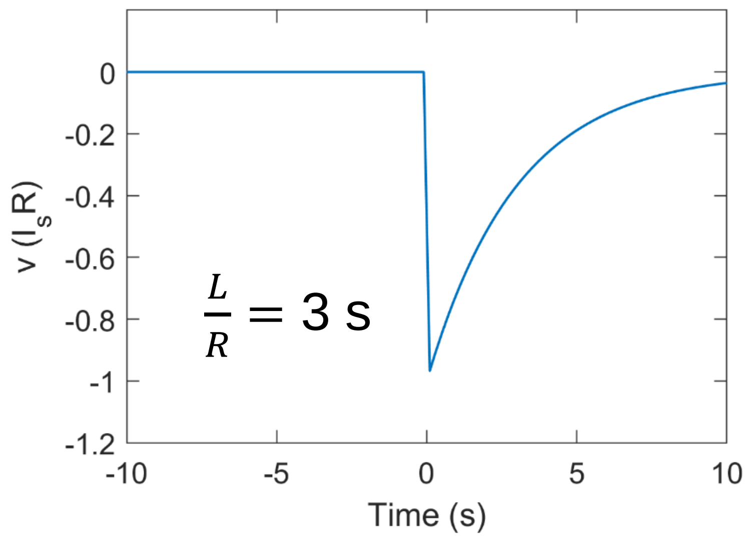
Initial energy stored in the inductor: $\frac{1}{2}LI_s^2$

Total energy dissipated by the resistor:

$$\int_0^\infty i_L^2 R dt = I_s^2 R \int_0^\infty e^{-\frac{2t}{\tau}} dt = \frac{\tau}{2} I_s^2 R = \frac{1}{2} L I_s^2$$

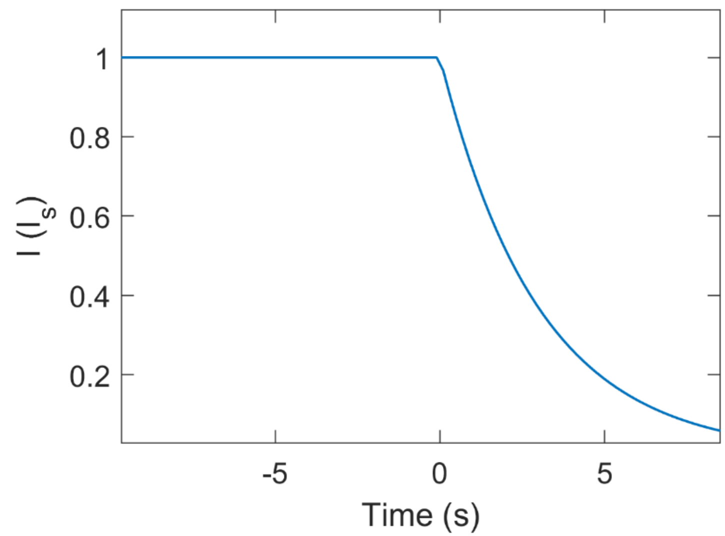
RL电路的自然响应

电压响应



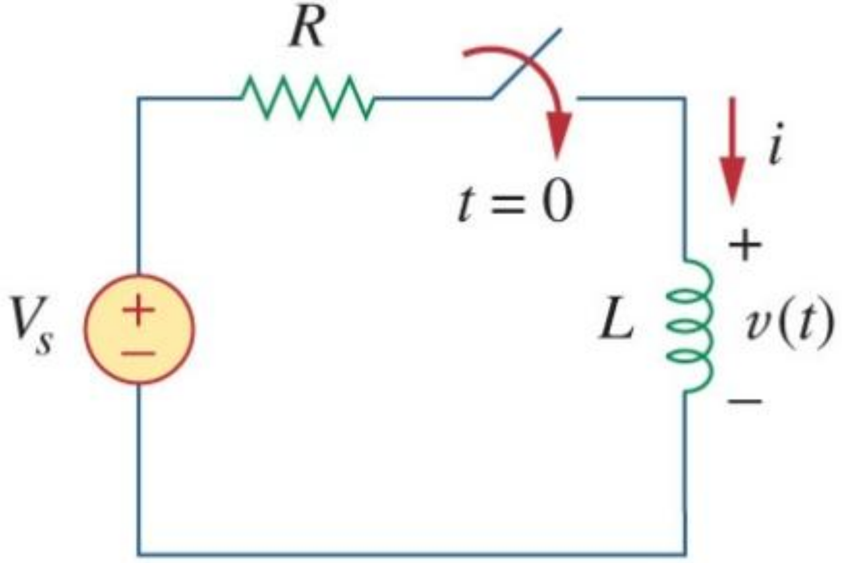
不连续

电流响应



连续

Step Response of the RL Circuit



Method 2

$$i(0^+) = i(0^-) = 0$$

$$i(\infty) = \frac{V_s}{R}$$

$$i = -\frac{V_s}{R}e^{-\frac{R}{L}t} + \frac{V_s}{R}$$

Method 1

$$i(0^+) = i(0^-) = 0$$

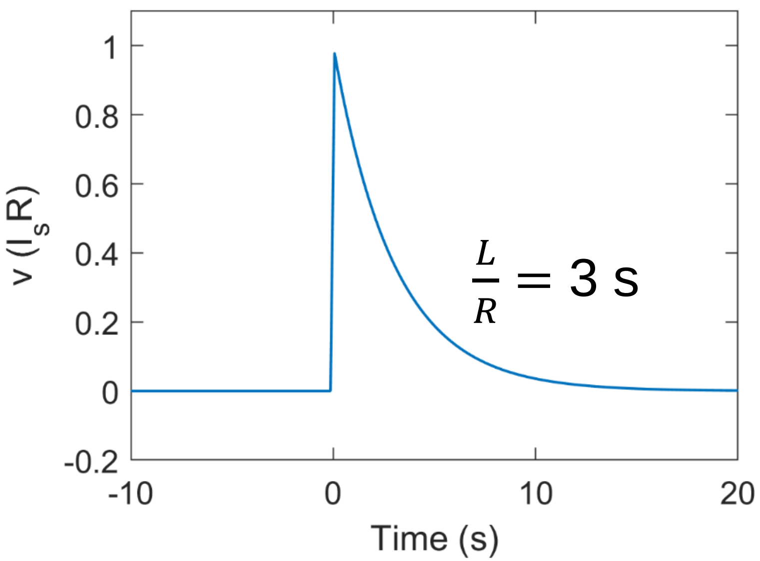
$$-V_s + Ri + L\frac{di}{dt} = 0$$

$$a = L,\ b = R,\ c = -V_s$$

$$\tau = \frac{L}{R}$$

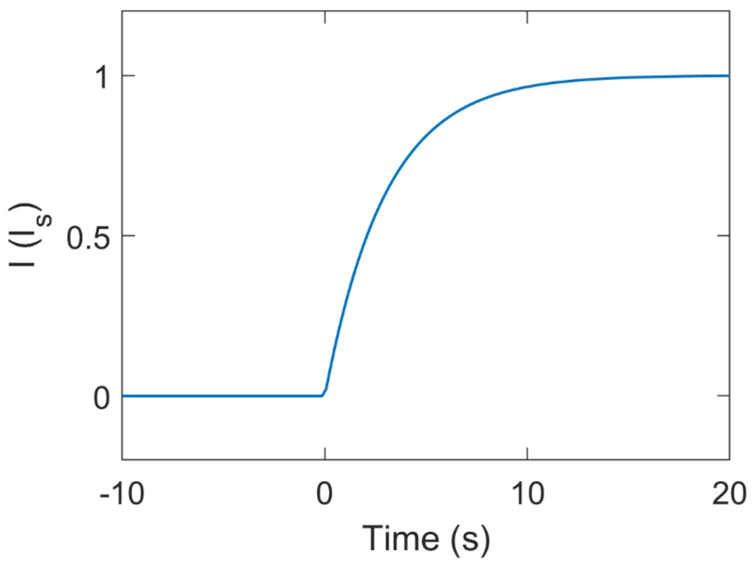
RL电路的阶跃响应

电压响应



不连续

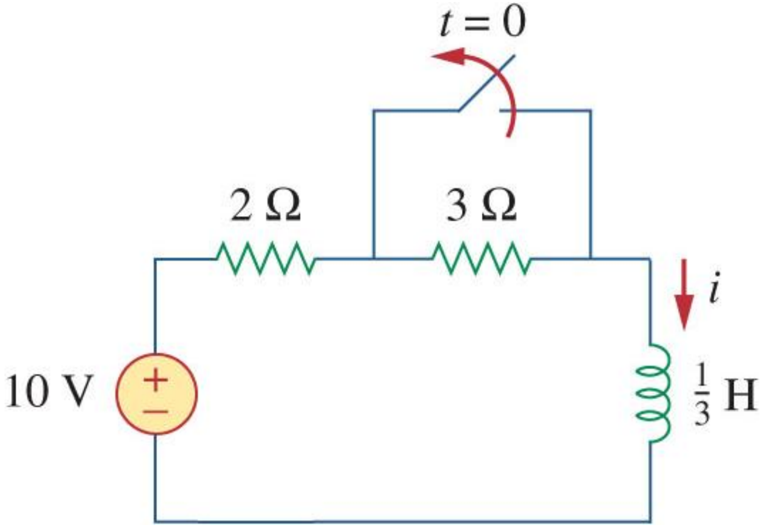
电流响应



连续

Exercise

- Find $i(t)$ in the circuit for $t > 0$. Assume that the switch has been closed for a long time.



$i(0) = 5A$

$i(\infty) = 2A$

$\tau = \frac{L}{R} = \frac{\frac{1}{3}}{5} = 66.7\,ms$

$t>0$: $i(t) = (5-2)\,e^{-\frac{t}{66.7\,ms}} + 2$ unit A