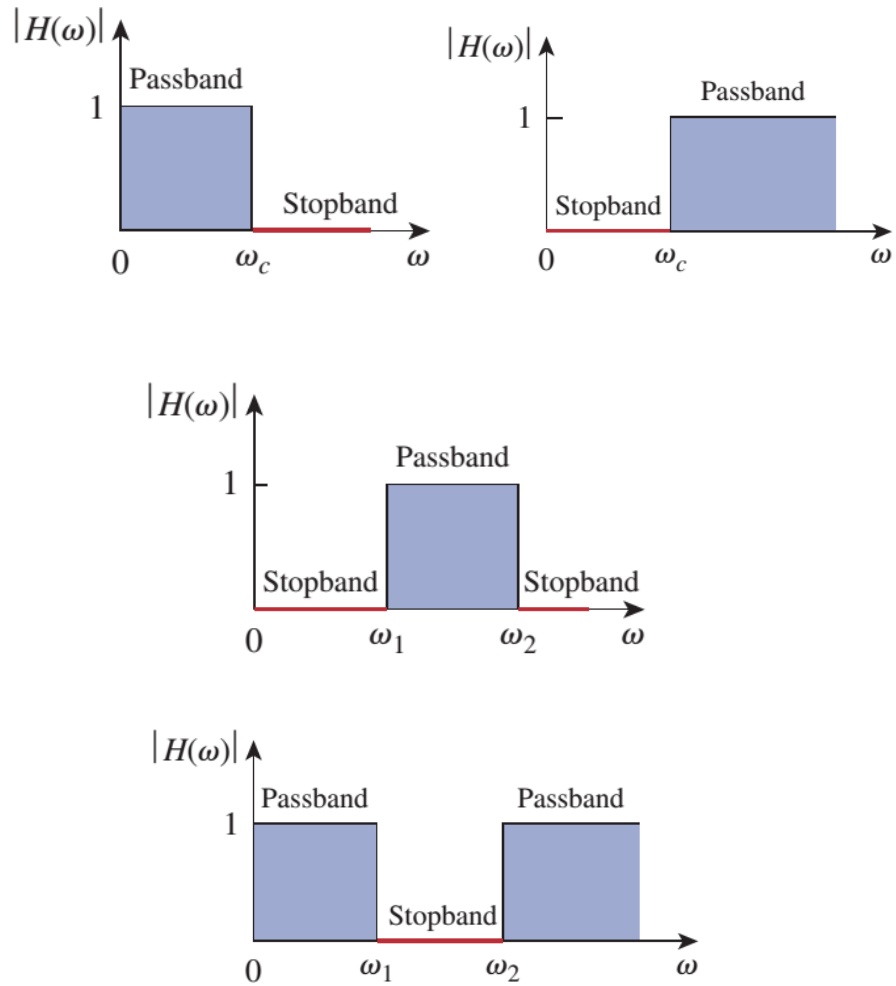
Chapter 10
- Filters

Outline

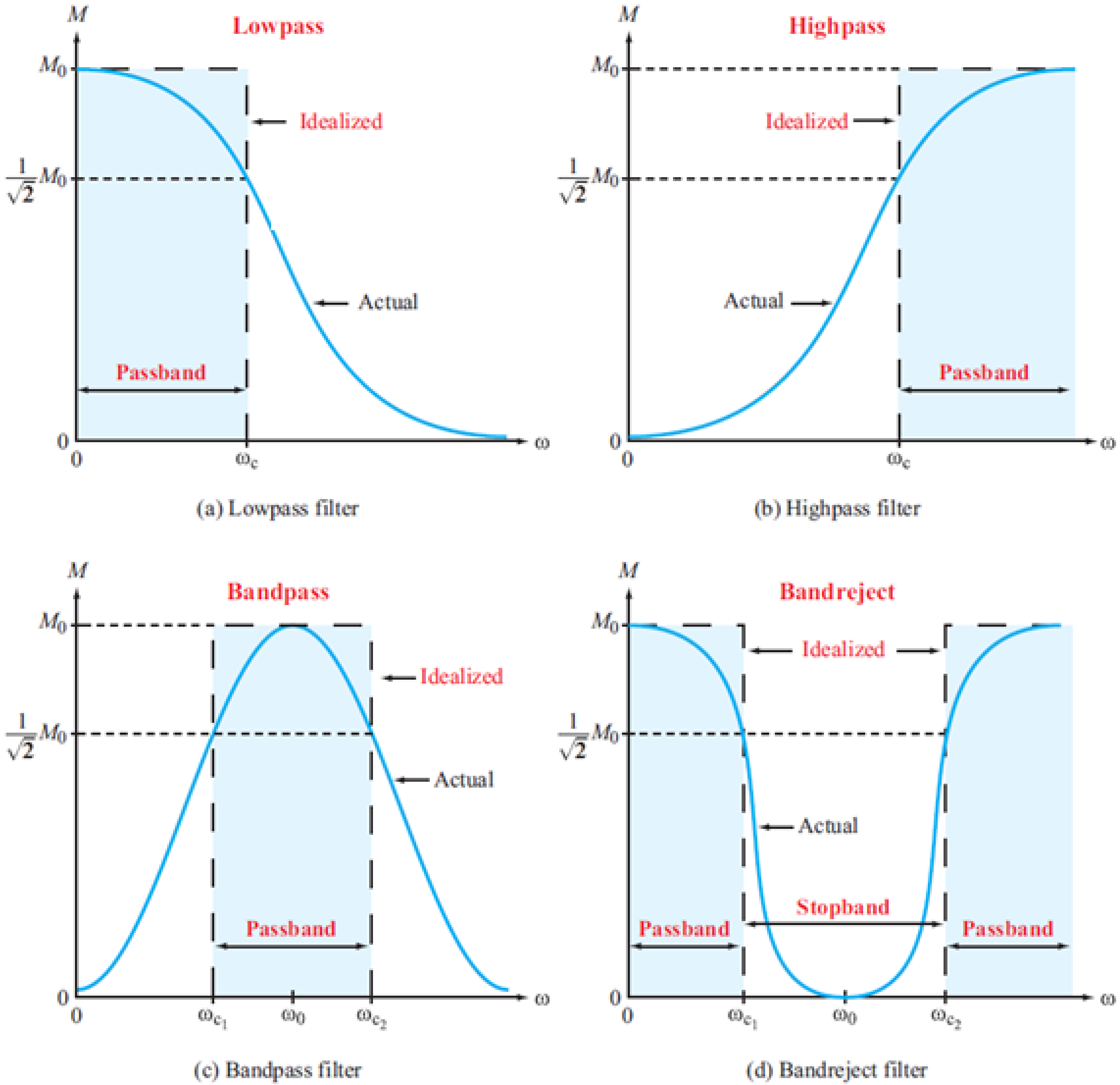
- Filter types and parameters
- Passive filters: RC, RL, RLC
- Active filters
- 50 Hz notch filter

Filter types

- Circuit designed to retain certain frequency range but discard or attenuate others
  - *Low-pass*: pass low frequencies and reject high frequencies
  - *High-pass*: pass high frequencies and reject low frequencies
  - *Band-pass*: pass some particular range of frequencies, reject other frequencies outside that band
  - *Band-reject (notch)*: reject a range of frequencies and pass all other frequencies



Filters – Realistic Curves



Filter parameters

- Characteristic frequencies (cutoff frequencies)
- Bandwidth
- Quality factor (Q factor)/ Damping factor (1/Q)
- Sensitivity to component value changes
- Group delay

Group delay

Delay $\tau(\omega) = -\dfrac{d\varphi(\omega)}{d\omega}$

Prove that if both the amplitude and group delay of the transfer function are constant in the frequency range of interest, then the output signal is a delayed version of the input signal.

Proof:

$$S_{\text{in}}(t) = \int_{\omega_1}^{\omega_2} \bar{f}(\omega) e^{j\omega t} d\omega$$

$$S_{\text{out}}(t) = \int_{\omega_1}^{\omega_2} \bar{f}(\omega) |H(\omega)| e^{j\varphi(\omega)} e^{j\omega t} d\omega$$

For constant $|H(\omega)|$ and $\tau(\omega)$ in $[\omega_1, \omega_2]$, i.e.

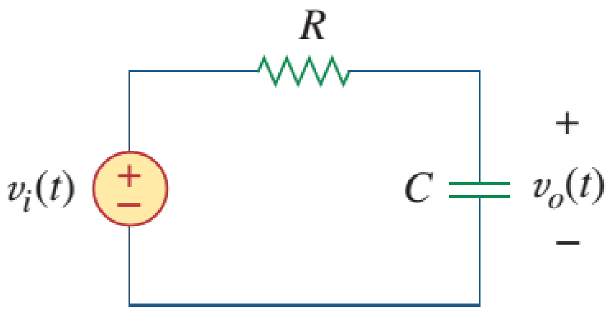
$$|H(\omega)| = H_0$$

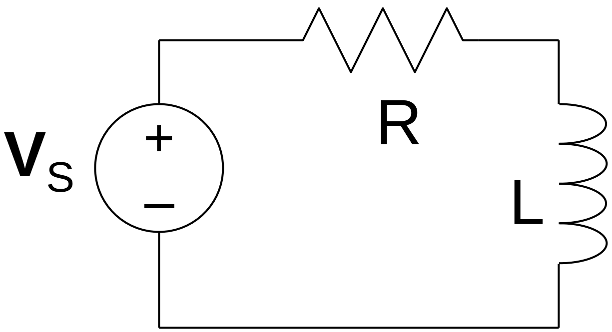
$$\varphi(\omega) = -\tau_0 \omega + \varphi_0$$

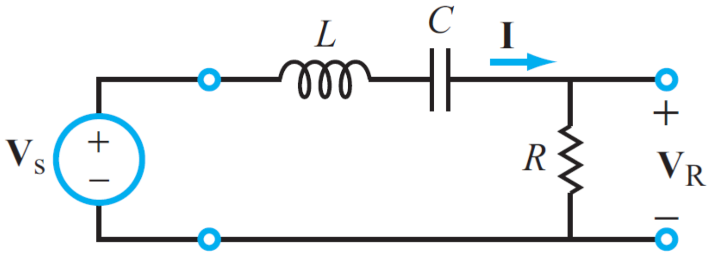
$$S_{\text{out}}\,(t) = H_0 \int_{\omega_1}^{\omega_2} \bar{f}(\omega) e^{j(-\tau\omega + \varphi_0)} e^{j\omega t} d\omega = H_0 e^{j\varphi_0} S_{\text{in}}(t - \tau_0)$$

Passive Filters

- A filter is passive if it consists only of <u>passive</u> elements
  - R, L and C

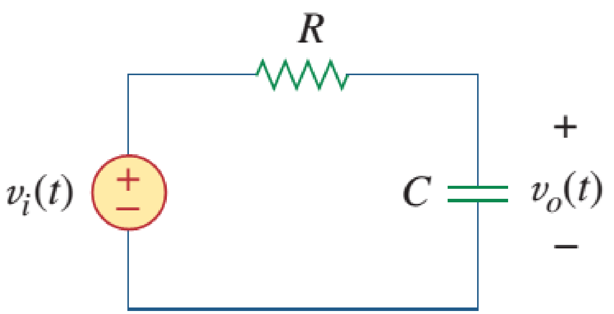






First-Order RC Low-pass Filter

- A typical low-pass filter is formed wher the output of a RC circuit is taken off the capacitor.

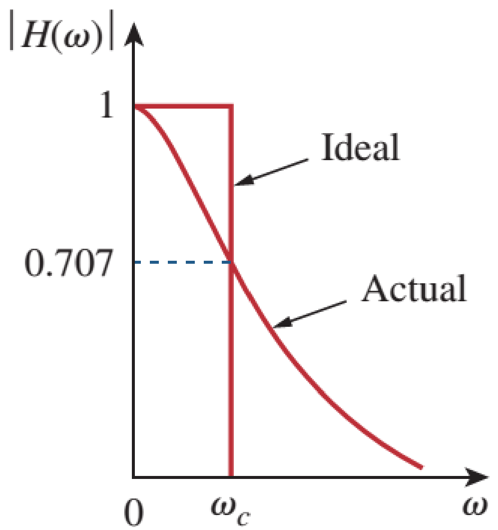


$$\mathbf{H}(\omega) = \frac{1}{\frac{1}{j\omega C}+R}\,\frac{1}{j\omega C} = \frac{1}{1+j\omega/\omega_c}$$

- The cutoff frequency is:

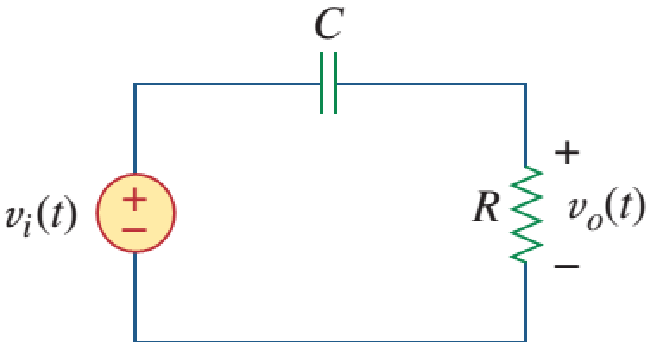
$$\omega_c = \frac{1}{CR},\ |H(\omega_c)| = 0.707$$

  - Filter is designed to pass from DC up to ω_c.



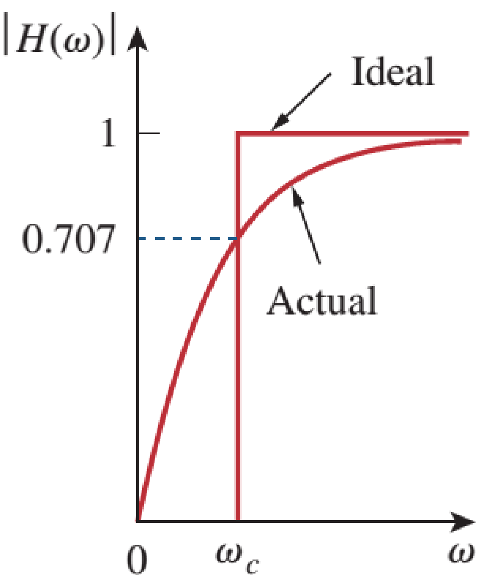
First-Order RC High-pass Filter

$$H(\omega) = \frac{R}{\frac{1}{j\omega C} + R} = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$



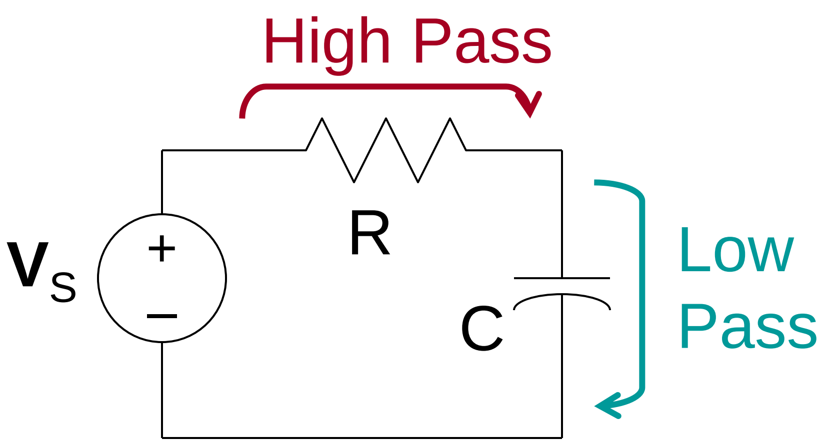
- The cutoff frequency.

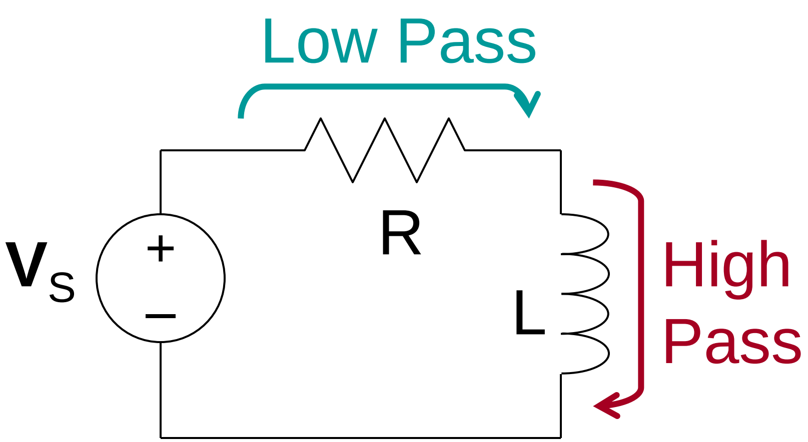
$$\omega_c = \frac{1}{CR}, \ |H(\omega_c)| = 0.707$$



- The difference being that the frequencies passed go from ω_c to infinity.

How about RL Circuits?





$$H_R(\omega) = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$

$$H_C(\omega) = \frac{1}{1 + j\omega/\omega_c}$$

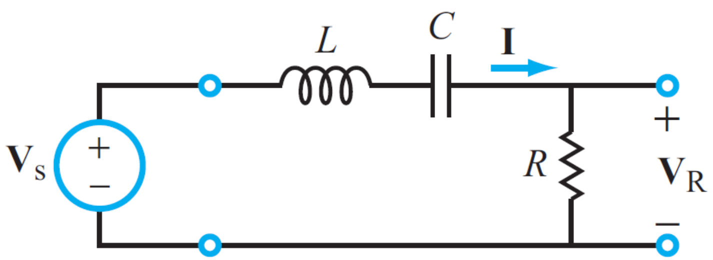
$$\omega_c = \frac{1}{RC}$$

$$H_R(\omega) = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega/\omega_c}$$

$$H_L(\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$$

$$\omega_c = \frac{R}{L}$$

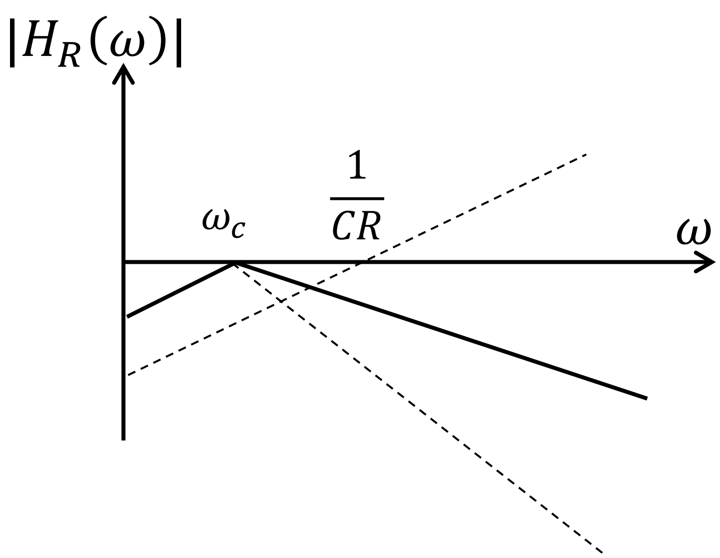
Bandpass RLC Filter

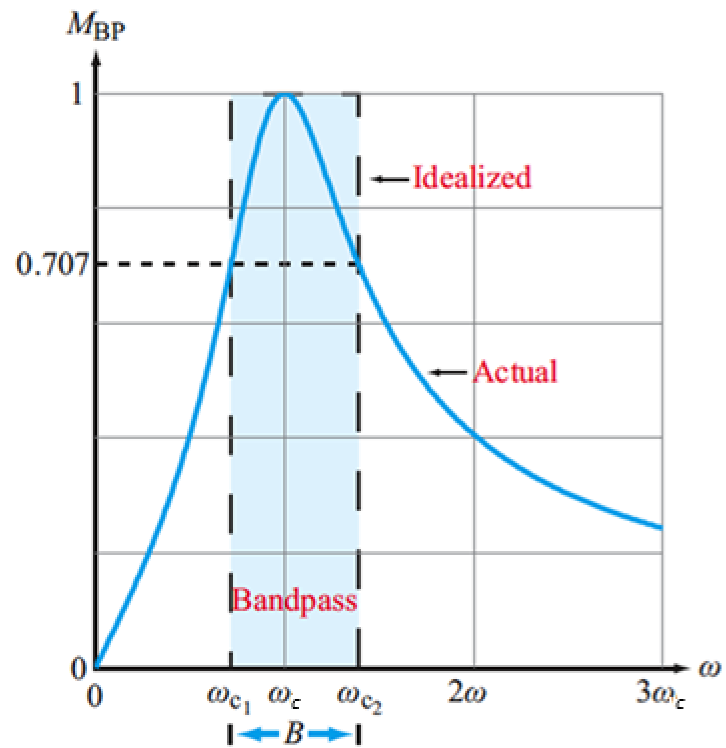


$$H_R(\omega) = \frac{R}{R + 1/j\omega C + j\omega L}$$

$$= \frac{\frac{1}{Q}\frac{j\omega}{\omega_c}}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$$

$$|H_R(\omega_c)| = 1$$





Bandwidth of the Pass Band

$$H_R(\omega) = \frac{\frac{1}{Q}\frac{j\omega}{\omega_c}}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$$

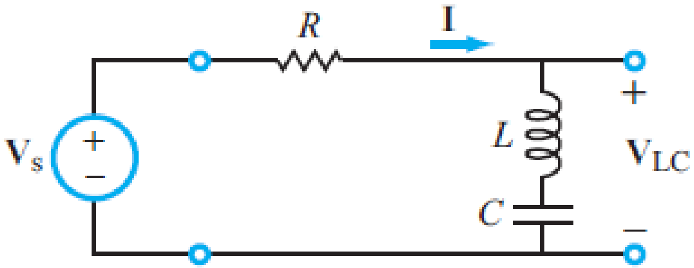
$$|H_R(\omega)|^2 = \frac{\left(\frac{1}{Q}\frac{\omega}{\omega_c}\right)^2}{[1 - \frac{\omega^2}{\omega_c^2}]^2 + \left(\frac{1}{Q}\frac{\omega}{\omega_c}\right)^2} = \frac{1}{2}$$

$$1 - \frac{\omega^2}{\omega_c^2} = \pm\frac{1}{Q}\frac{\omega}{\omega_c}$$

$\omega_{1,2} = \left[\mp\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}\right]\omega_c$, same as the cutoff frequencies for the resonance peak.

$$B = \frac{\omega_c}{Q} = \alpha\omega_c$$

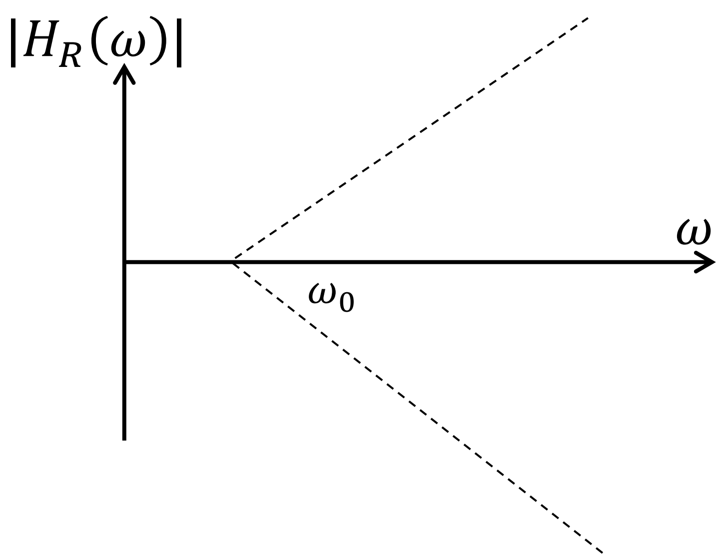
Band-reject (Band-stop) Filter

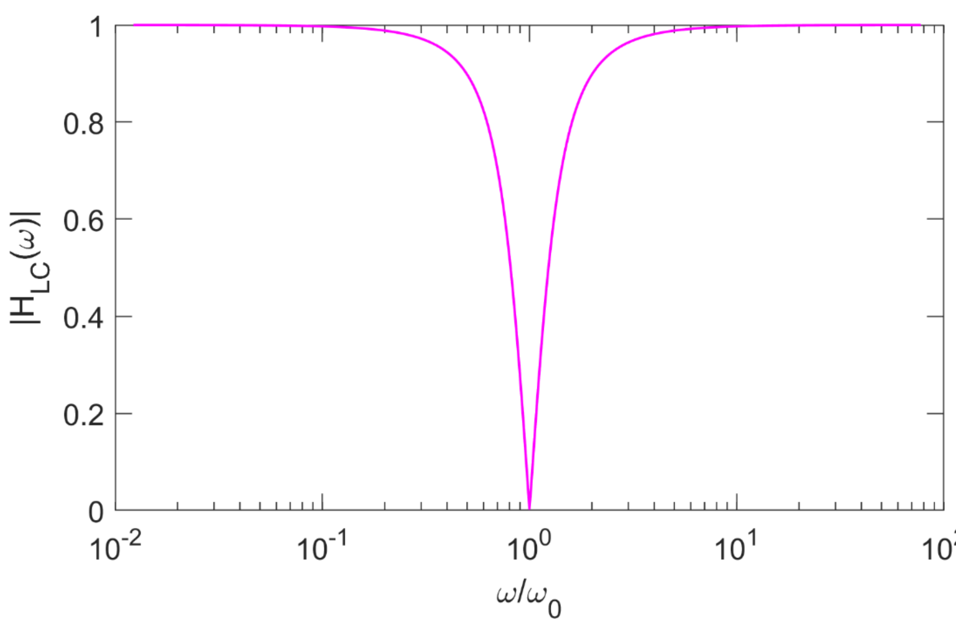


$$H_{LC}(\omega) = \frac{1+(jw)2LC}{1+jwCR+(jw)2LC}$$

$$= \frac{1+(j\omega)^2/\omega_c^2}{1+\frac{1}{Q}\frac{j\omega}{\omega_c}+(j\omega)^2/\omega_c^2}$$

$$|H_R(\omega_c)| = 0$$





Bandwidth of the Stop Band

$$H_{LC}(\omega) = \frac{1 + (j\omega)^2/\omega_c^2}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$$

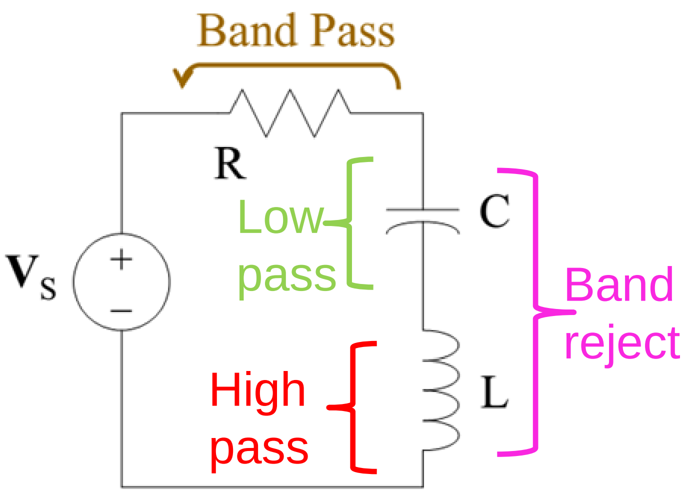
$$|H_{Lc}(\omega)|^2 = \frac{[1 - \frac{\omega^2}{\omega_c^2}]^2}{[1 - \frac{\omega^2}{\omega_c^2}]^2 + \left(\frac{1}{Q}\frac{\omega}{\omega_c}\right)^2} = \frac{1}{2}$$

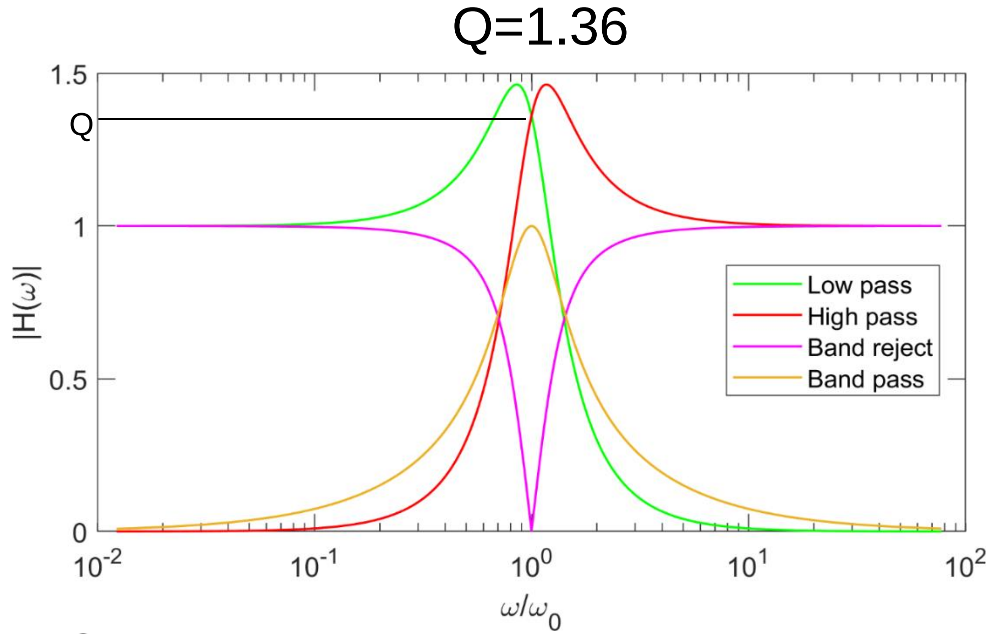
$$1 - \frac{\omega^2}{\omega_c^2} = \pm\frac{1}{Q}\frac{\omega}{\omega_c}$$

$\omega_{1,2} = \left[\mp\frac{1}{2Q} + \sqrt{1 + \frac{1}{4Q^2}}\right]\omega_c$, same as the cutoff frequencies for the resonance peak.

$$B = \frac{\omega_c}{Q} = \alpha\omega_c$$

Second-Order RLC Filter Circuits





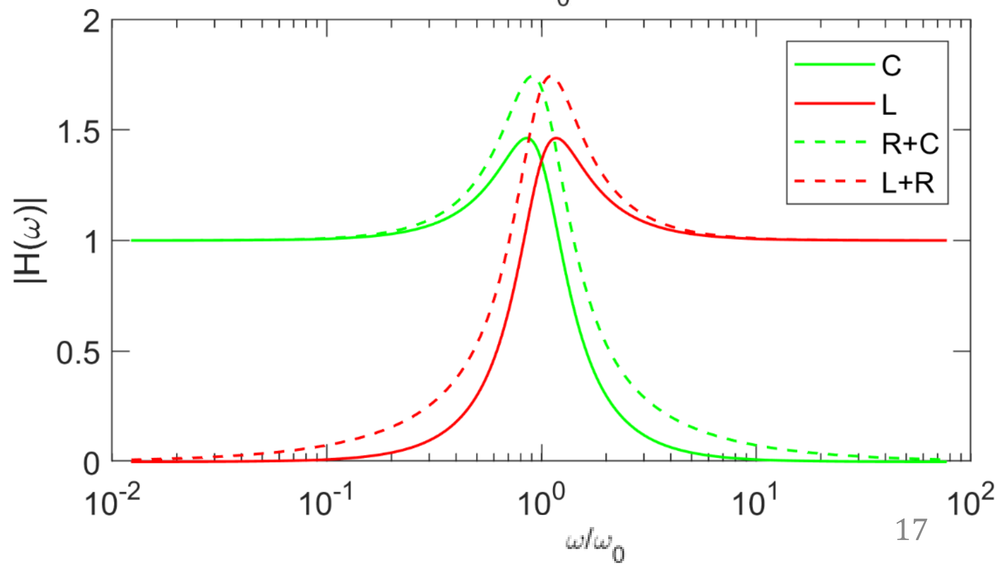
Z=R + 1/*jωC*+ *jwL*

$$H_{BP} = R/Z$$

$$H_{LP} = \frac{1}{j\omega CZ}$$

$$H_{HP} = \frac{j\omega L}{Z}$$

$$H_{BR} = H_{LP} + H_{HP}$$



Summary (passive filters)

First order (Series RC/RL):

Low pass: $H(\omega) = \frac{1}{j\omega/\omega_c + 1}$

High pass: $H(\omega) = \frac{j\omega/\omega_c}{j\omega/\omega_c + 1}$

Second order (series RLC):

Low pass: $H(\omega) = \frac{1}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$

High pass: $H(\omega) = \frac{\left(\frac{j\omega}{\omega_c}\right)^2}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$

Band pass: $H(\omega) = \frac{\frac{1}{Q}\frac{j\omega}{\omega_c}}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$

Band reject: $H(\omega) = \frac{1 + (j\omega)^2/\omega_c^2}{1 + \frac{1}{Q}\frac{j\omega}{\omega_c} + (j\omega)^2/\omega_c^2}$

- $\omega_c = \sqrt{LC}$
- $Q = \frac{\omega_c L}{R} = \frac{1}{\omega_c RC}$
- Bandwidth = $\frac{\omega_c}{Q}$ for BP and BR.
- Ripple in the pass band when Q>0.707 for LP.

Summary

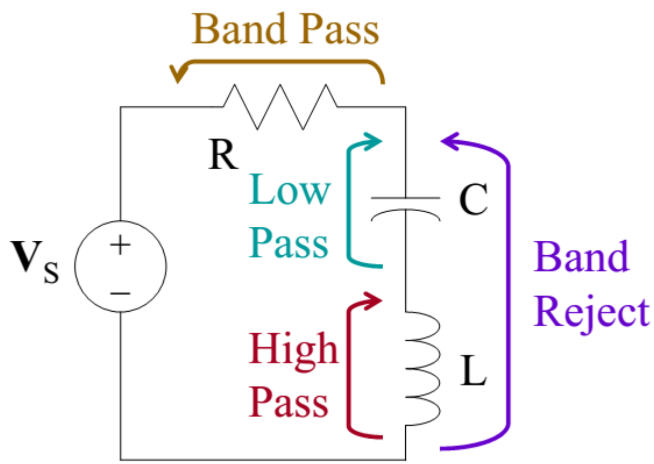
	Filter type	特征频率 (ω_c)	$\lvert H_R(\omega_c)\rvert$
V_R in RC	High-pass	$\frac{1}{RC}$	0.707
V_C in RC	Low-pass	$\frac{1}{RC}$	0.707
V_R in RL	Low-pass	$\frac{R}{L}$	0.707
V_L in RL	High-pass	$\frac{R}{L}$	0.707
V_R in RLC	Band pass	$\frac{1}{\sqrt{LC}}$	1
V_C in RLC	Low-pass	$\frac{1}{\sqrt{LC}}$	Q
V_L in RLC	High-pass	$\frac{1}{\sqrt{LC}}$	Q
V_{LC} in RLC	Band reject	$\frac{1}{\sqrt{LC}}$	0

Outline

- Filter types
- Passive filters: RC, RL, RLC
- Active filters
  - First order active LP/HP filters
  - Cascaded filters
  - Single stage 2nd order filters
- 50 Hz notch filter

Active Filters

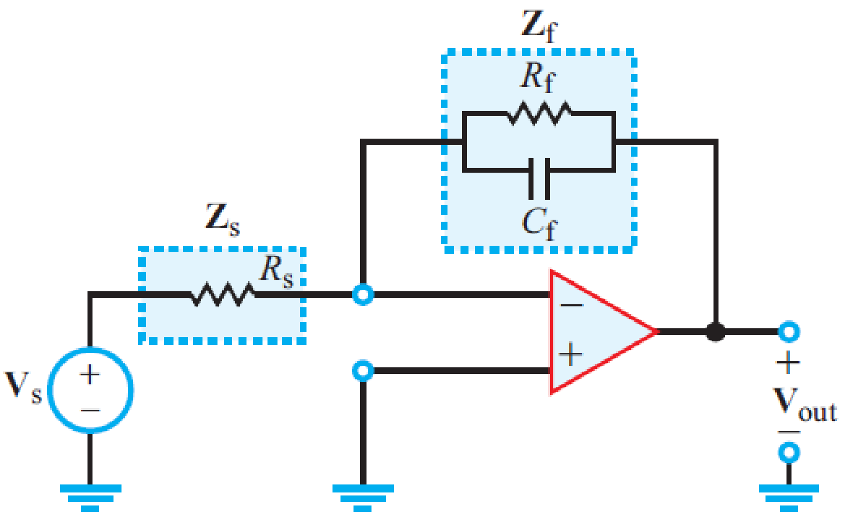
- Passive filters have a few drawbacks.
  - Generally, they cannot create gain greater than 1.
  - They require inductors, which tend to be bulky and more expensive than other components.
  - Input load changes with output load.

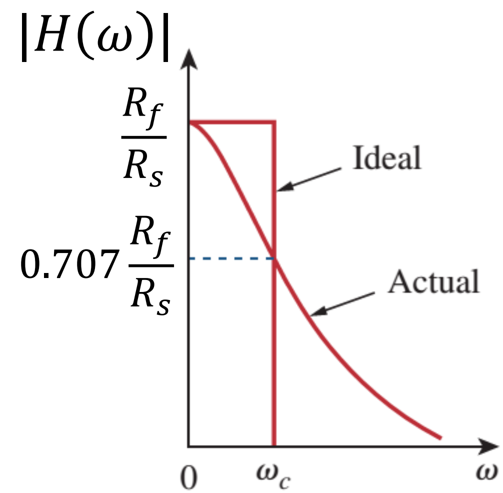


Active Filters

- It is possible, using op-amps, together with resistors and capacitors, to create all the common filters.
  - Their ability to isolate input and output also makes them very desirable. (V_i load does not change with V_o load)
  - Limited to frequency less than 1MHz.

Active 1st–order Low-pass Filter (FB type)

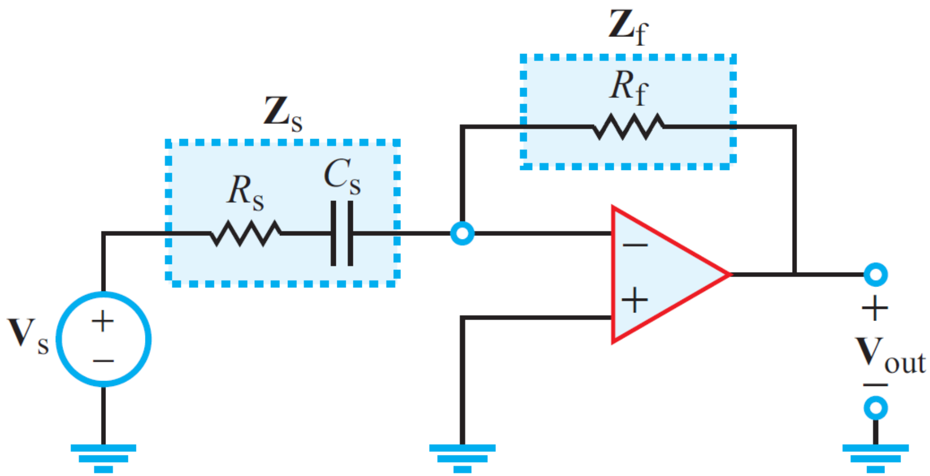


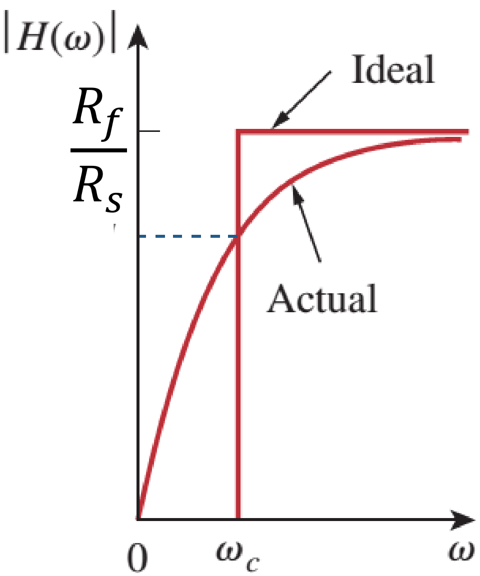


$$H(\omega) = -\frac{Z_f}{Z_s} = -\frac{R_f}{R_s}\frac{1}{1 + j\omega R_f C_f} \qquad \omega_c = \frac{1}{R_f C_f}$$

Question: What if Z_f and Z_s are switched? Do we get a high-pass filter?

Active 1st–order High-pass Filters (FB type)





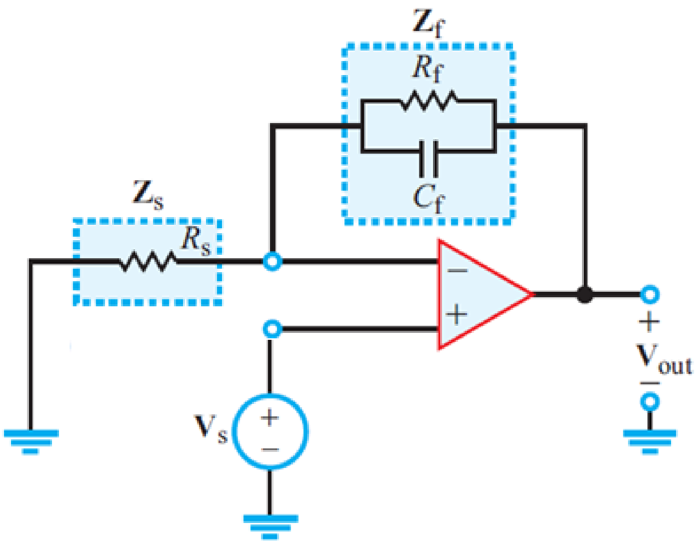
$$H(\omega) = -\frac{Z_f}{Z_s} = -\frac{R_f}{R_s + \frac{1}{j\omega C_s}} = -\frac{j\omega R_f C_s}{j\omega R_s C_s + 1} =$$

$$-\frac{R_f}{R_s}\frac{j\omega/\omega_c}{j\omega/\omega_c + 1},$$

where $\omega_c = \frac{1}{R_s C_s}$.

Question: What if Z_f and Z_s are switched? Do we get a low-pass filter?

Active 1st–order Low-pass Filters (VCVS type)



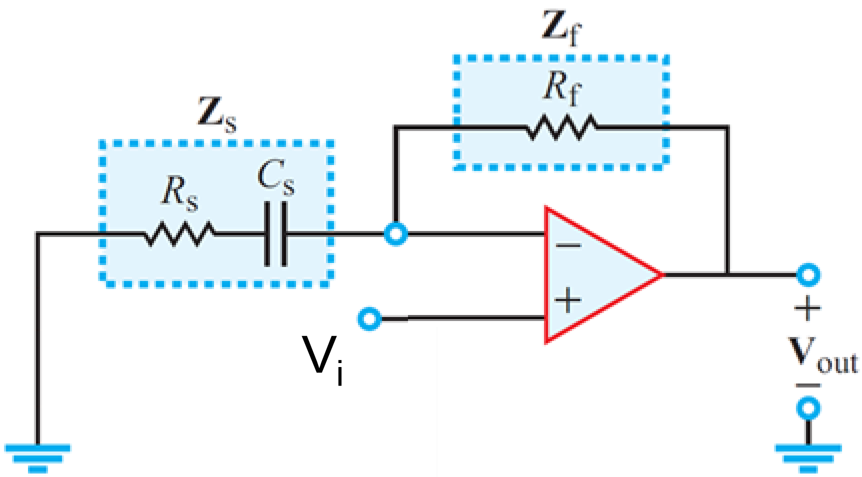
$$V_i = \frac{R_s V_o}{R_s + \dfrac{R_f}{1 + j\omega C_f R_f}}$$

$$H(\omega) = 1 + \frac{R_f/R_s}{1 + j\omega C_f R_f}$$

$$\omega \to 0: \quad H(\omega) \to 1 + R_f/R_s$$

$$\omega \to \infty: \quad H(\omega) \to 1$$

Active 1st–order Low-pass Filters (VCVS type)



$$H(\omega) = 1 + \frac{Z_f}{Z_s} = 1 + \frac{R_f}{R_s + \frac{1}{j\omega C_s}} =$$

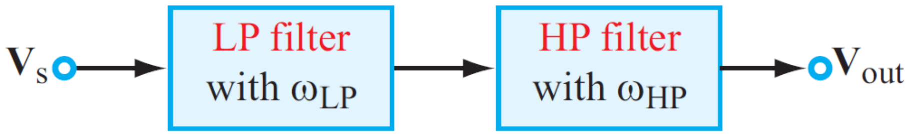
$$1 + \frac{j\omega R_f C_s}{j\omega R_s C_s + 1} = 1 + \frac{R_f}{R_s}\frac{j\omega/\omega_c}{j\omega/\omega_c + 1},$$

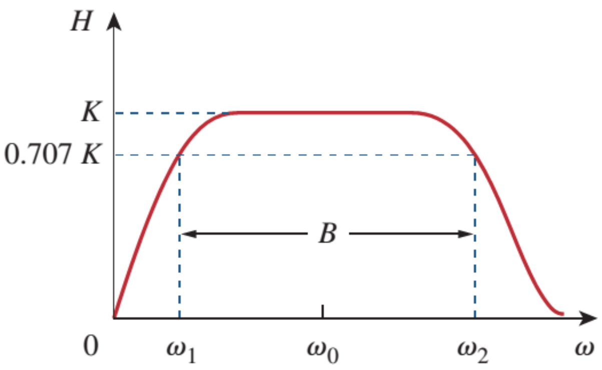
where $\omega_c = \frac{1}{R_s C_s}$.

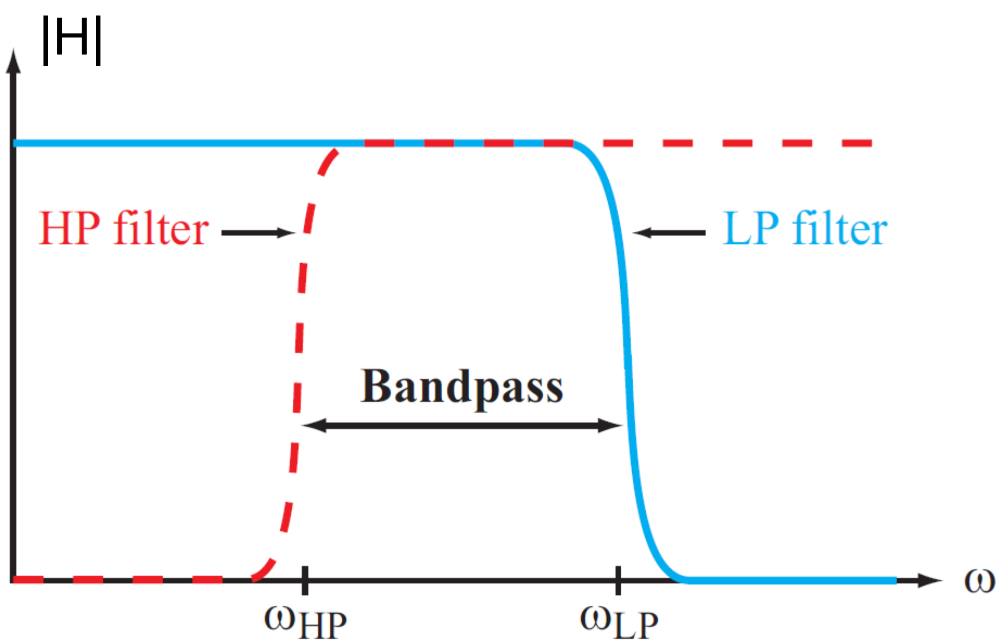
$\omega \to 0: \quad H(\omega) \to 1$

$\omega \to \infty: \quad H(\omega) \to 1 + R_f/R_s$

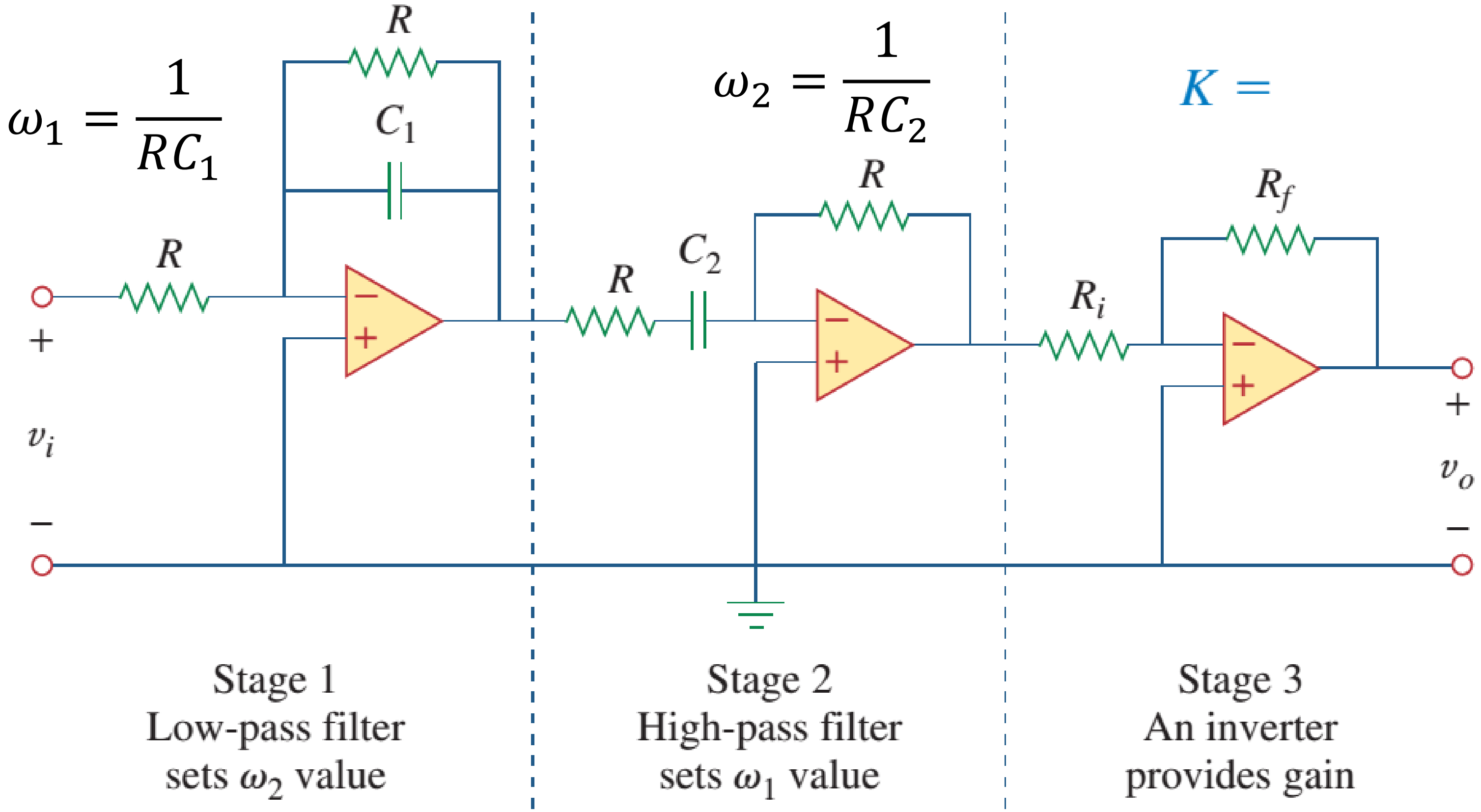
Band-pass filter







Active Band-pass Filter

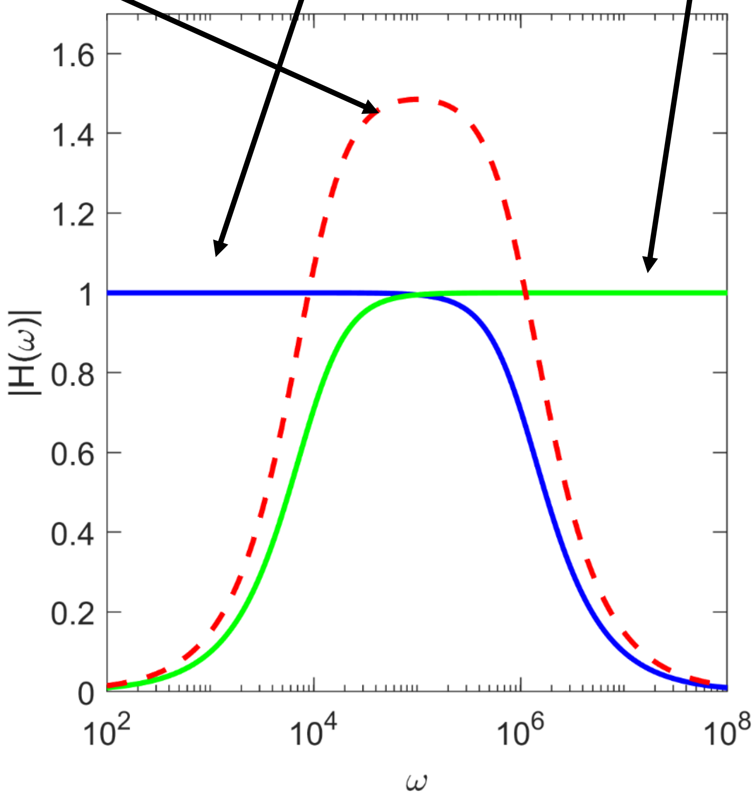


$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \left(-\frac{1}{1 + j\omega C_1 R}\right)\left(-\frac{j\omega C_2 R}{1 + j\omega C_2 R}\right)\left(-\frac{R_f}{R_i}\right)$$

$$\omega_1 > \omega_2$$

Active Band-pass Filter

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o}{\mathbf{V}_i} = \left(-\frac{1}{1 + j\omega C_1 R}\right)\left(-\frac{j\omega C_2 R}{1 + j\omega C_2 R}\right)\left(-\frac{R_f}{R_i}\right)$$



R_i = 2 kΩ

R_f = 3 kΩ

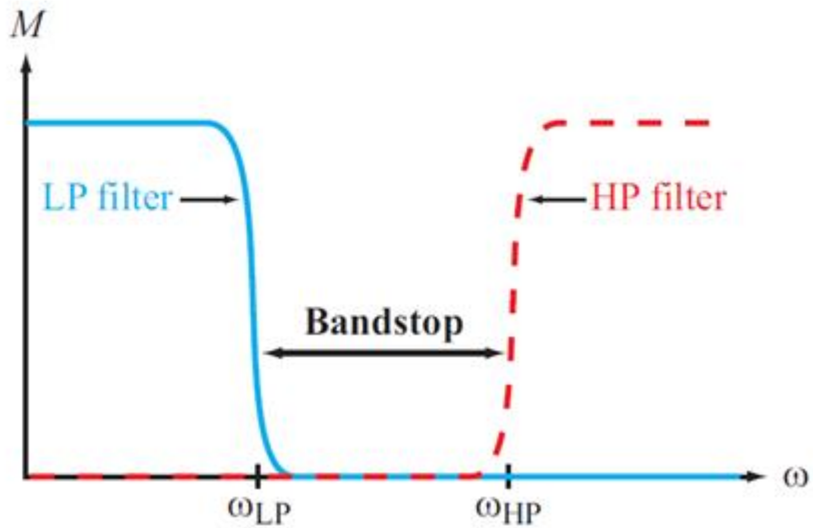
R = 1 kΩ

C_1 = 1 nF

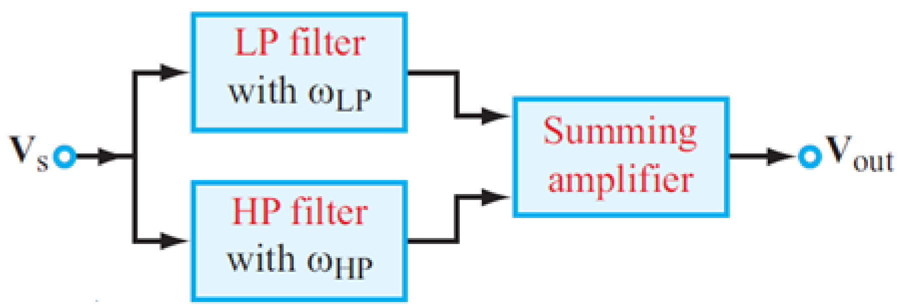
C_2 = 100 nF

Question: Do we get a band reject filter if C_1>>C_2?

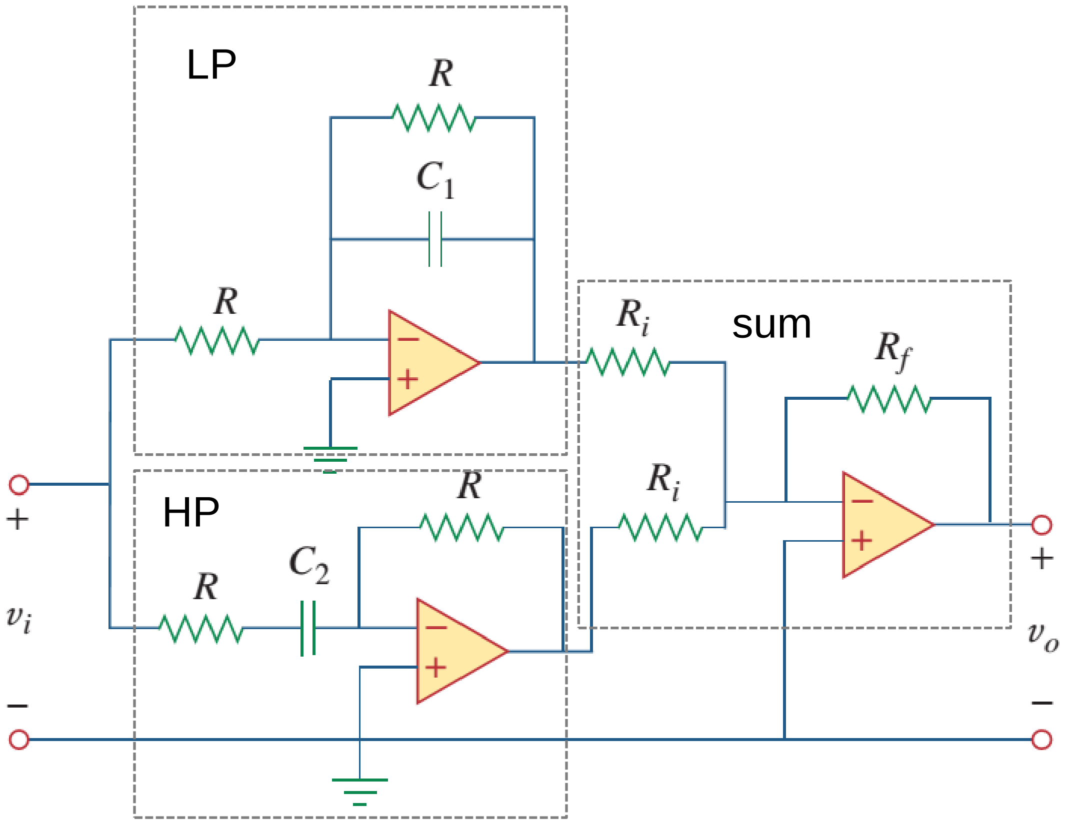
Active Band-reject Filter



(b) Bandreject filter



Active Band-reject Filter



$$H(\omega) = -\frac{R_f}{R_i}\left[-\frac{1}{1+j\omega RC_1} - \frac{j\omega RC_2}{1+j\omega RC_2}\right]$$

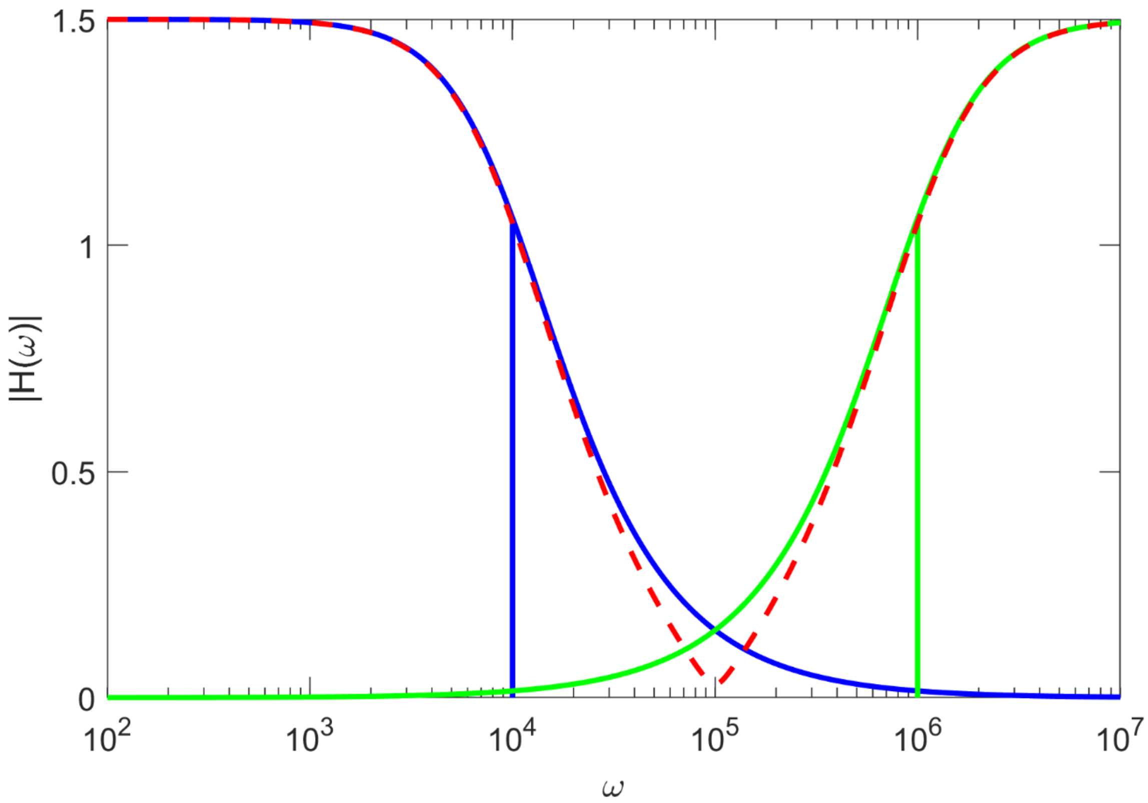
$$= \frac{R_f}{R_i}\left[\frac{1}{1+j\omega/\omega_{c1}} + \frac{j\omega/\omega_{c2}}{1+j\omega/\omega_{c2}}\right]$$

$$\omega_{c1} = \frac{1}{C_1 R}$$

$$\omega_{c2} = \frac{1}{C_2 R}$$

$$\omega_{c1} < \omega_{c2}$$

Active Band-reject Filter



R_i = 2 kΩ

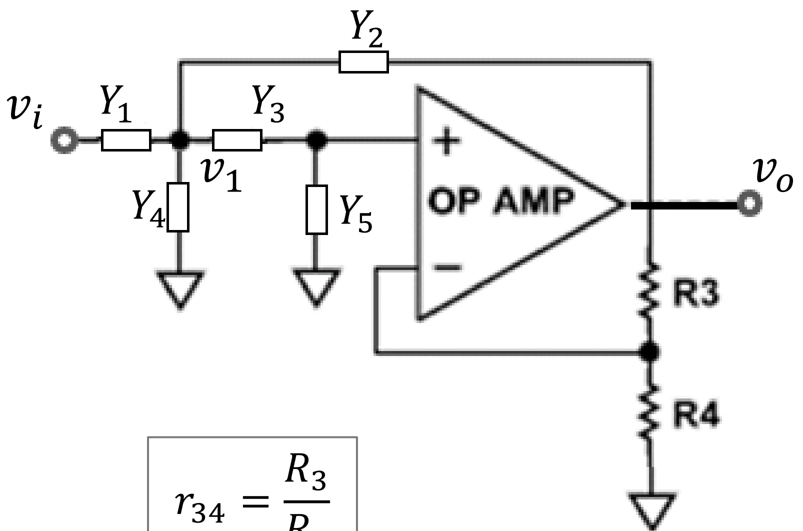
R_f = 3 kΩ

R=1 kΩ

C_1 = 100 nF

C_2 = 1 nF

电压控制电压源型滤波器



$$r_{34} = \frac{R_3}{R_4}$$

When $Y_4 = 0$, the structure is called Sallen-Key filter (proposed in 1955）.

$$\frac{1}{r_{34}+1} v_0 = \frac{Y_3 v_1}{Y_3 + Y_5}$$

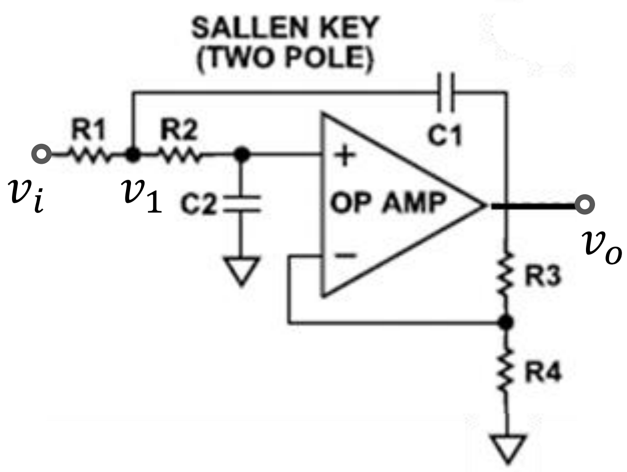
$$Y_1 v_i + Y_2 v_o + Y_3 \frac{v_0}{r_{34}+1} - v_1 (Y_1 + Y_2 + Y_3 + Y_4) = 0$$

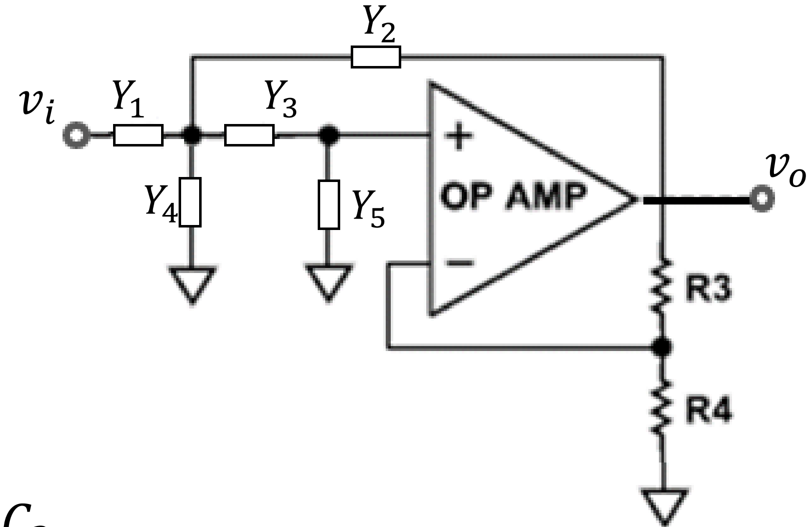
$$Y_1 v_i + Y_2 v_o + Y_3 \frac{v_0}{r_{34}+1} - \frac{(Y_3+Y_5)(Y_1 + Y_2 + Y_3 + Y_4)}{Y_3 (r_{34}+1)} v_0 = 0$$

$$\frac{v_0}{v_i} = \frac{Y_1 Y_3 (r_{34}+1)}{Y_3 (Y_1 + Y_4 - r_{34} Y_2) + Y_5 (Y_1 + Y_2 + Y_3 + Y_4)}$$

Exercise

Find the frequency response of the following circuit.





$$Y_1 = \frac{1}{R_1},\ Y_2 = j\omega C_1, Y_3 = \frac{1}{R_2},\ Y_4 = 0,\ Y_5 = j\omega C_2$$

$$\frac{v_0}{v_i} = \frac{Y_1 Y_3 (r_{34} + 1)}{Y_3(Y_1 + Y_4 - r_{34} Y_2) + Y_5(Y_1 + Y_2 + Y_3 + Y_4)}$$

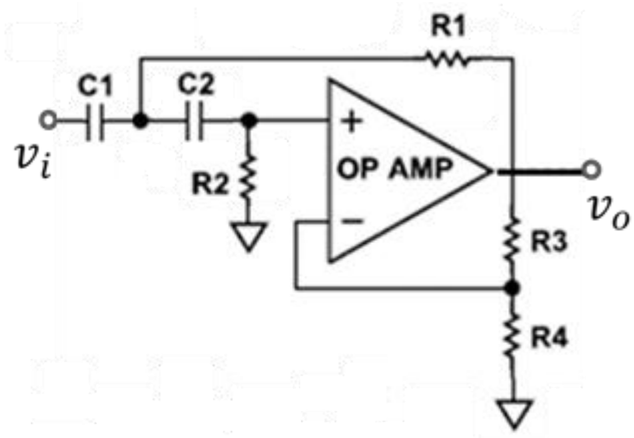
$$= \frac{r_{34} + 1}{1 + \alpha\left(\frac{j\omega}{\omega_0}\right) + \left(\frac{j\omega}{\omega_0}\right)^2}$$

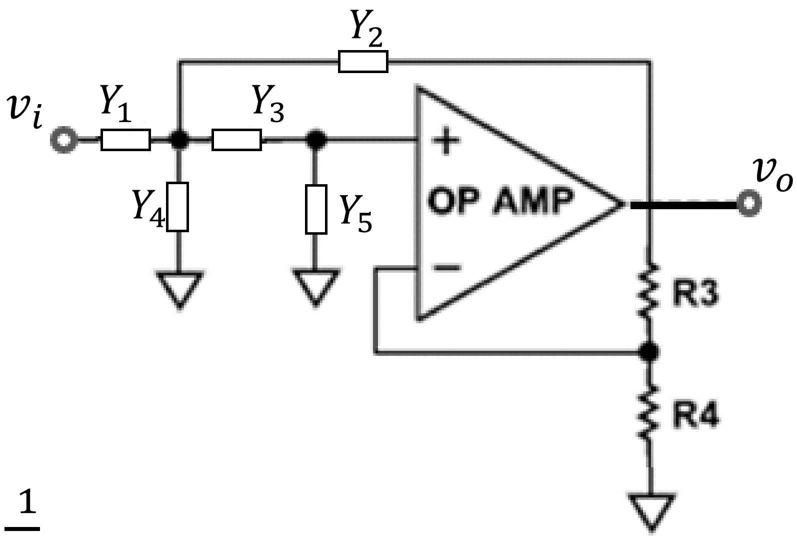
$$r_{34} = \frac{R_3}{R_4},\ \omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}}$$

$$\alpha\omega_0 = [\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) - \frac{r_{34}}{R_2 C_2}]$$

Exercise II

Find the frequency response of the following circuit.





$$Y_1 = j\omega C_1,\ Y_2 = \frac{1}{R_1},\ Y_3 = j\omega C_2,\ Y_4 = 0,\ Y_5 = \frac{1}{R_2}$$

$$\frac{v_0}{v_i} = \frac{Y_1 Y_3 (r_{34} + 1)}{Y_3(Y_1 + Y_4 - r_{34} Y_2) + Y_5(Y_1 + Y_2 + Y_3 + Y_4)}$$

$$= \frac{(r_{34} + 1)(j\omega/\omega_0)^2}{(j\omega/\omega_0)^2 + \alpha j\omega/\omega_0 + 1}$$

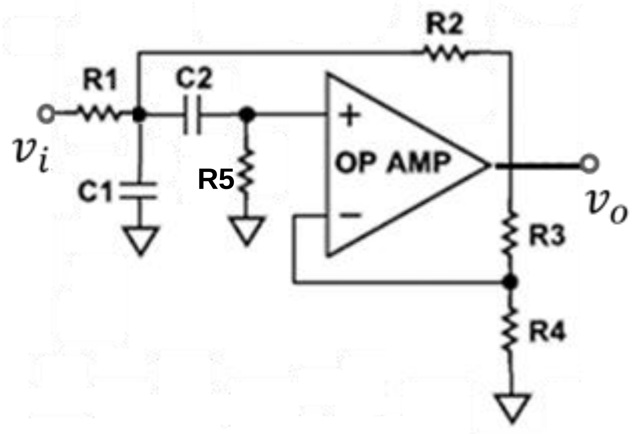
$r_{34} = \frac{R_3}{R_4},\ \omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}}$

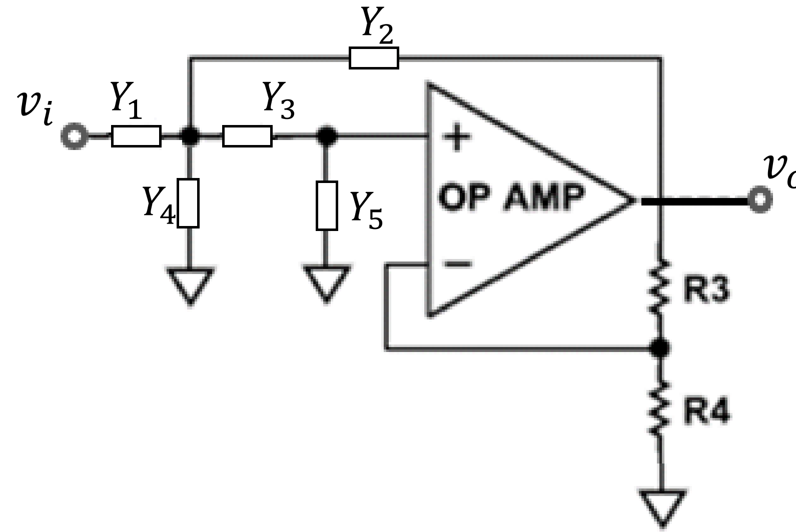
$\alpha\omega_0$

$= [\frac{1}{R_2}(\frac{1}{C_1} + \frac{1}{C_2}) - \frac{r_{34}}{R_1 C_1}]$

Exercise III

Find the frequency response of the following circuit.

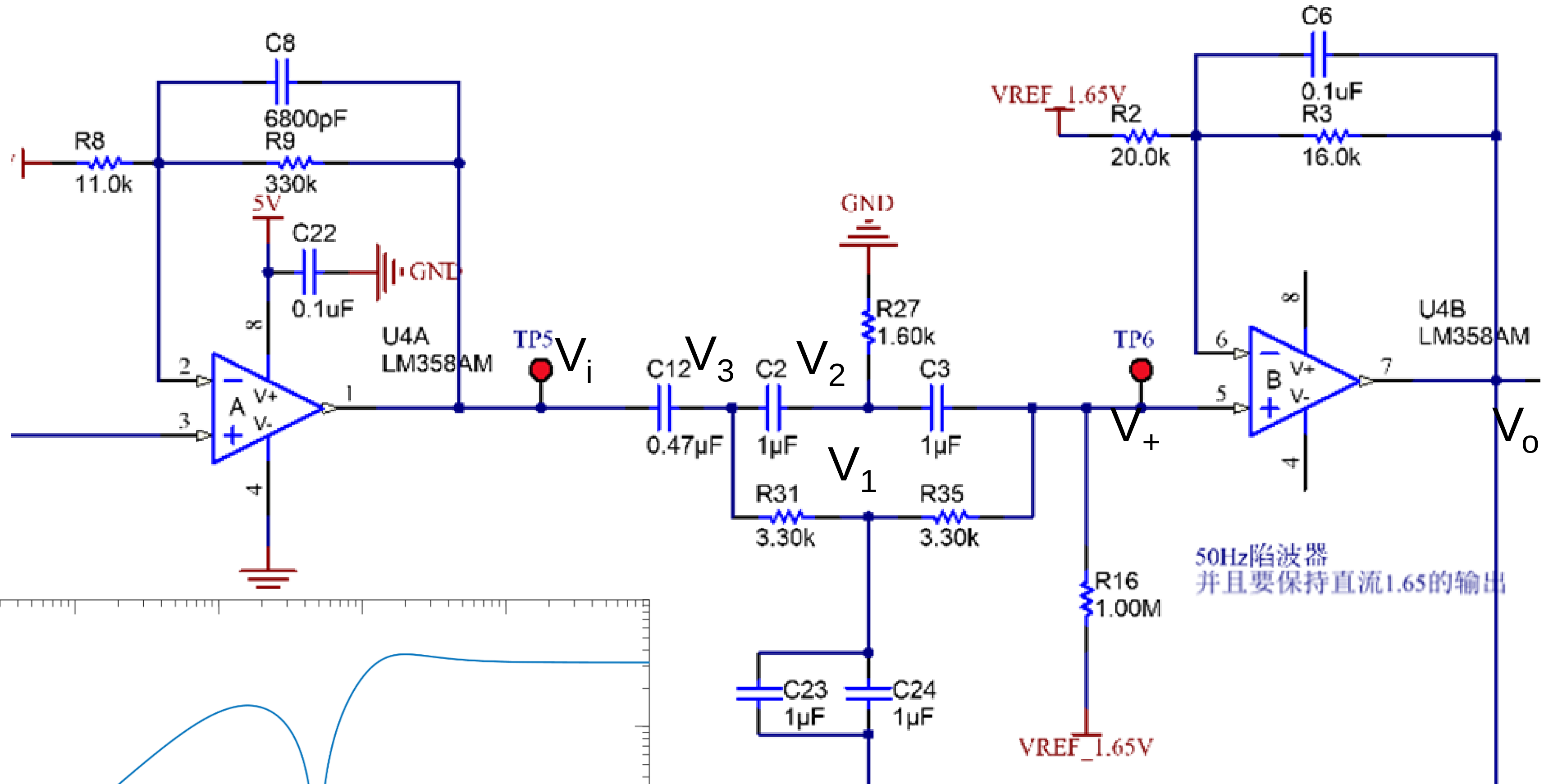


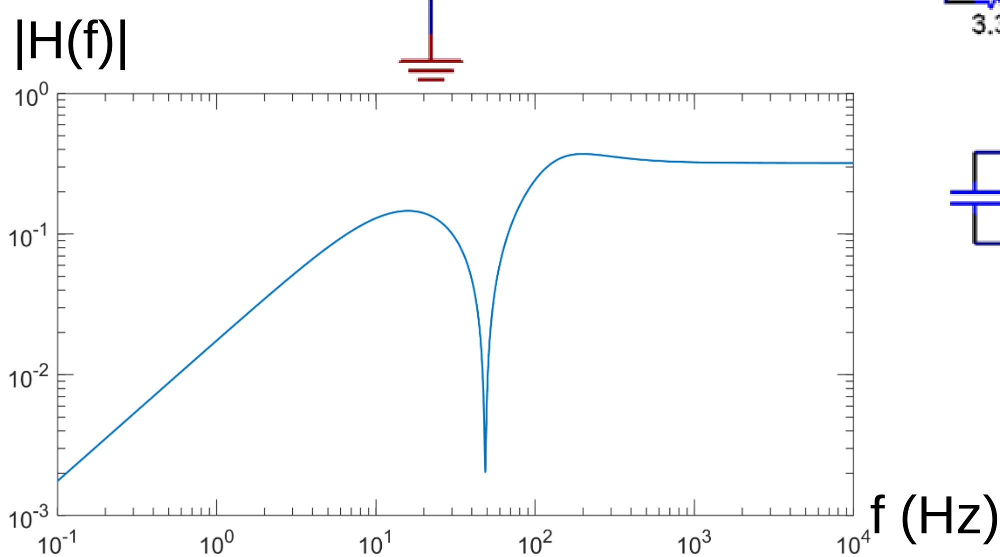


Outline

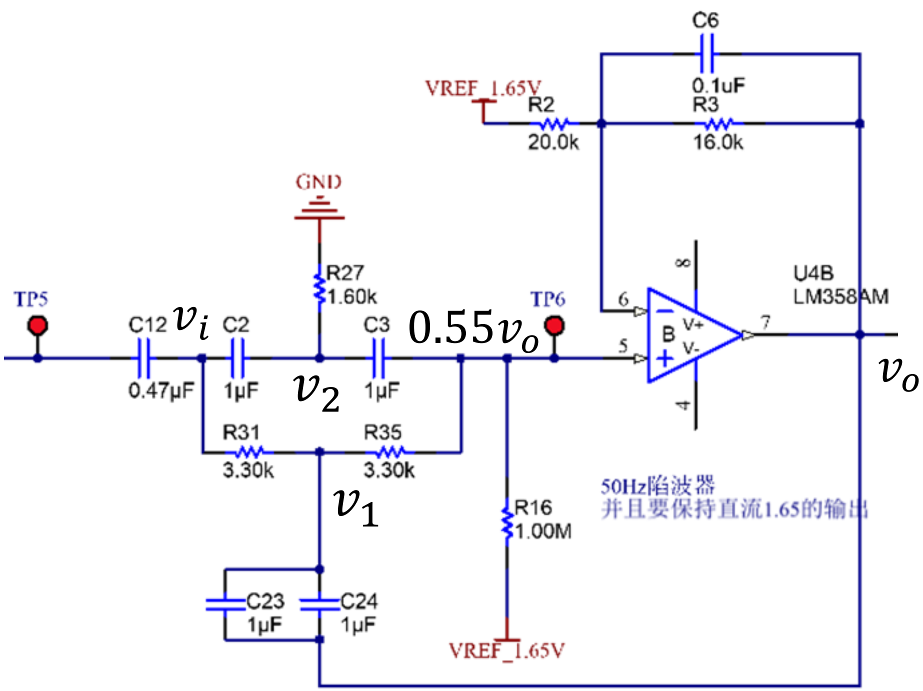
- Filter types
- Passive filters: RC, RL, RLC
- Active filters
- 50 Hz notch filter

50 Hz 陷波器


C8
6800pF
R8
11.0k
R9
330k
5V
C22
0.1uF
GND
U4A
LM358AM
TP5
Vi
C12
0.47μF
V3
C2
1μF
V2
GND
R27
1.60k
C3
1μF
V1
R31
3.30k
R35
3.30k
C23
1μF
C24
1μF
TP6
V+
R16
1.00M
VREF_1.65V
R2
20.0k
C6
0.1uF
R3
16.0k
U4B
LM358AM
Vo
50Hz陷波器
并且要保持直流1.65的输出


|H(f)|
f (Hz)

Exercise: find the frequency at the dip.



$$v_1 = \frac{j\omega 2C_{23}v_o + \dfrac{v_i + 0.55v_o}{R_{31}}}{j\omega 2C_{23} + 2/R_{31}}$$

$$v_2 = \frac{(v_i + 0.55v_o)j\omega C_3}{j\omega 2C_3 + 1/R_{27}}$$

$$(v_1 - 0.55v_o)\frac{1}{R_{31}} = (0.55v_o - v_2)j\omega C_3$$

$$\mathrm{H}(\omega) = -\frac{\dfrac{1}{j\omega 2R_{31}C_{23} + 2} + \dfrac{(j\omega C_3)^2R_{31}R_{27}}{j\omega 2R_{27}C_3 + 1}}{\dfrac{j\omega 2R_{31}C_{23} + 0.55}{j\omega 2R_{31}C_{23} + 2} - 0.55 - 0.55j\omega C_3R_{31}}$$

$$= -\frac{1 + \dfrac{(j\omega C_3)^2R_{31}R_{27}(j2\omega R_{31}C_{23} + 2)}{j2\omega R_{27}C_3 + 1}}{j\omega 2R_{31}C_{23} + 0.55 - 1.1(1 + j\omega C_3R_{31})(1 + j\omega C_{23}R_{31})}$$

$$\mathrm{H}(\omega) = -\frac{1 + \frac{(j\omega C_3)^2 R_{31} R_{27} (j2\omega R_{31} C_{23} + 2)}{j2\omega R_{27} C_3 + 1}}{j\omega 2 R_{31} C_{23} + 0.55 - 1.1(1 + j\omega C_3 R_{31})(1 + j\omega C_{23} R_{31})}$$

When $\frac{j\omega R_{31} C_{23}+1}{j2\omega R_{27} C_3+1} = 1$ and $\omega = \frac{1}{C_3\sqrt{2R_{31}R_{27}}}$, $\mathrm{H}(\omega) = 0$.

$\frac{j\omega R_{31} C_{23}+1}{j2\omega R_{27} C_3+1} = 1 \rightarrow R_{31} C_{23} = 2R_{27} C_3$, which is roughly satisfied in the circuit.

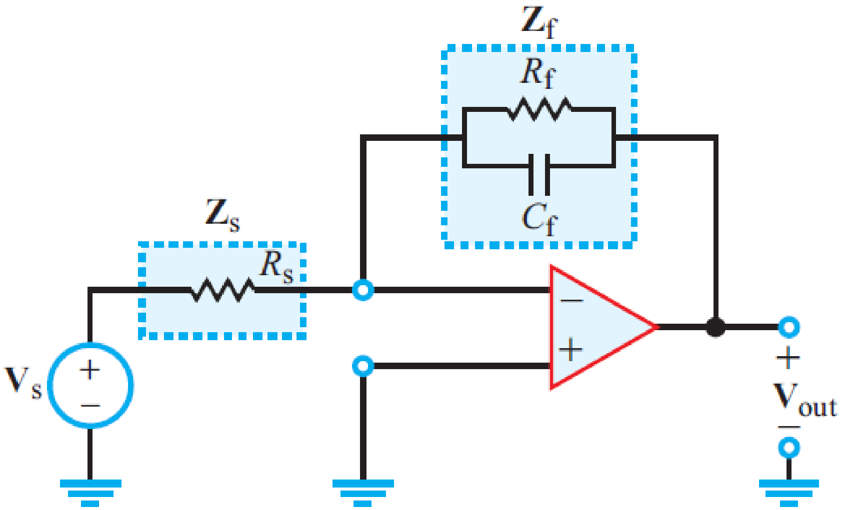
$$\omega_c = \frac{1}{C_3\sqrt{2R_{31}R_{27}}} = 307.7 \text{ rad/s}$$

$$f_c = \frac{\omega}{2\pi} = 49 \text{ Hz}$$

$$\mathrm{H}(\omega) = \frac{1}{0.55} \frac{1 + 2(j\omega C_3)^2 R_{31} R_{27}}{1 + j\omega R_{31}(1.1C_3 - 0.9C_{23})/0.55 + 2(j\omega)^2 C_3 C_{23} R_{31}^2}$$

$$= 1.82 \frac{1+(j\omega/\omega_c)^2}{1+j\omega R_{31}(1.1C_3-0.9C_{23})/0.55+2(j\omega/\omega_c)^2}$$

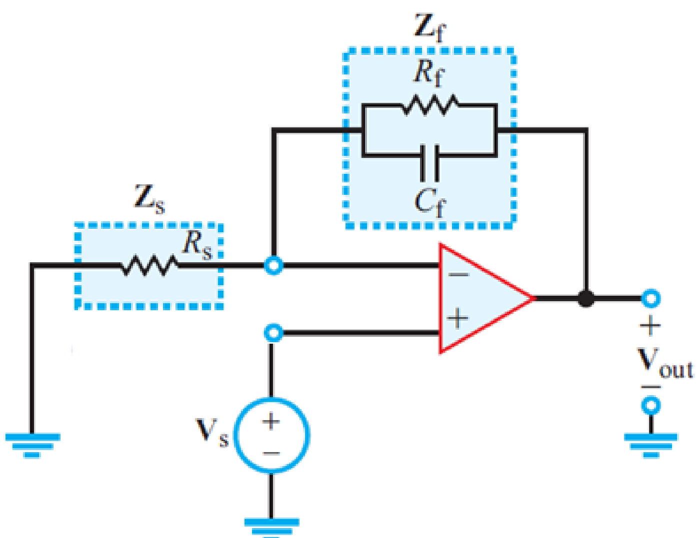
Summary (Active filters)



Low pass

$$H(\omega) = -\frac{Z_f}{Z_s} = -\frac{R_f}{R_s}\frac{1}{1 + j\omega R_f C_f}$$

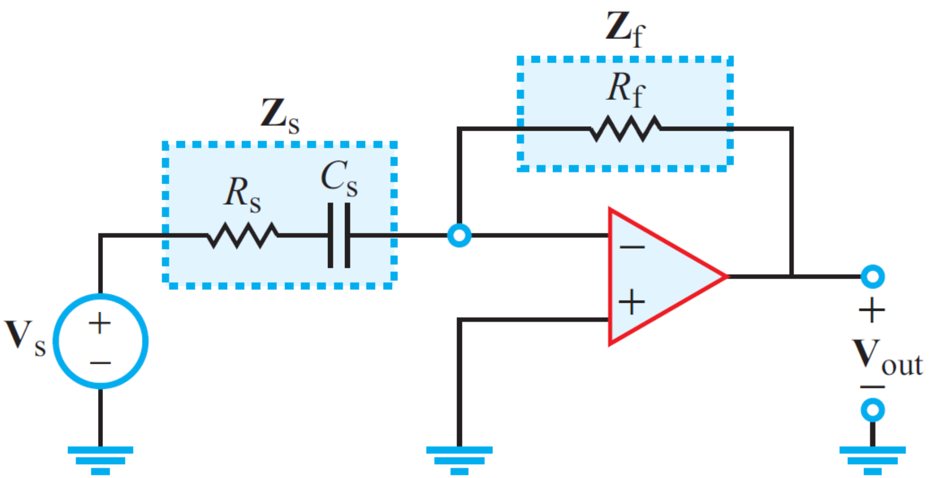
$$\omega_c = \frac{1}{R_f C_f}$$



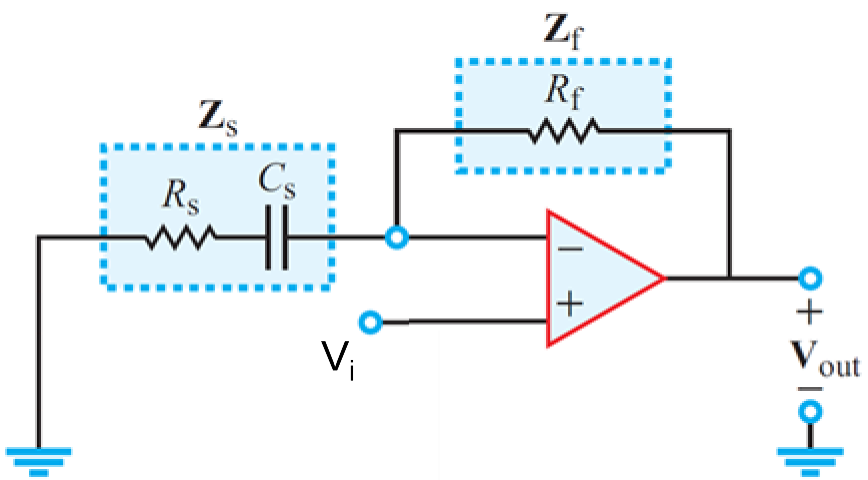
$$H(\omega) = 1 + \frac{R_f/R_s}{1 + j\omega/\omega_c}$$

Summary (Active filters)

High pass



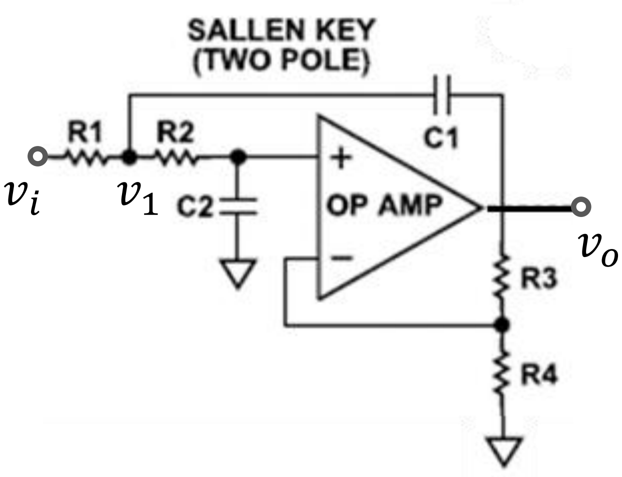
$$H(\omega) = -\frac{R_f}{R_s}\frac{j\omega/\omega_c}{j\omega/\omega_c+1},$$



$$H(\omega) = 1 + \frac{R_f}{R_s}\frac{j\omega/\omega_c}{j\omega/\omega_c + 1}$$

$$\omega_c = \frac{1}{R_s C_s}$$

Summary (Active filters)



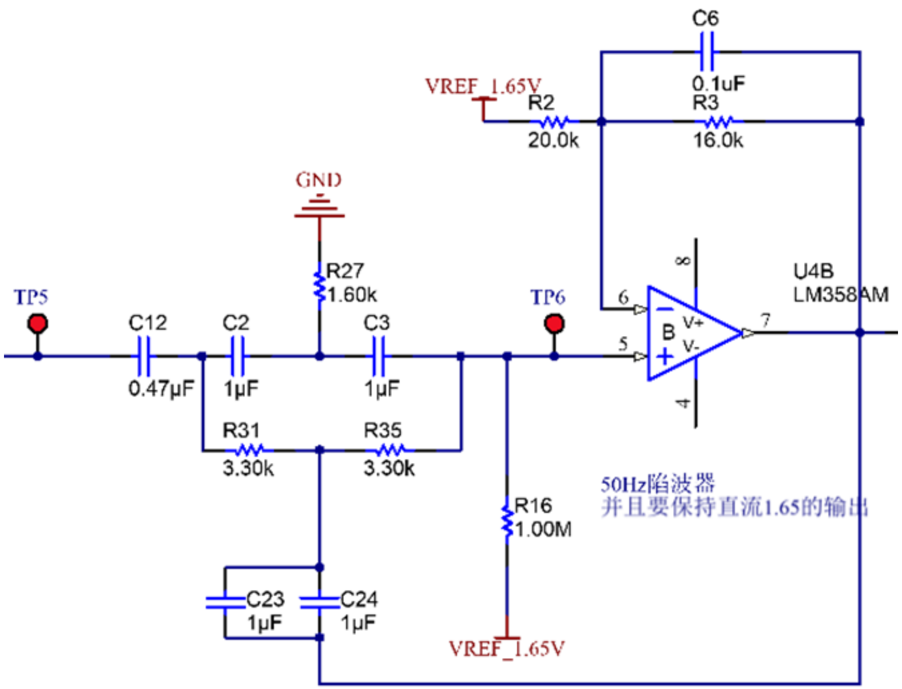
$$H(\omega) = \frac{r_{34} + 1}{1 + 1/Q\left(\frac{j\omega}{\omega_0}\right) + \left(\frac{j\omega}{\omega_0}\right)^2}$$

$$r_{34} = \frac{R_3}{R_4},\ \omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}}$$

$$Q = \frac{\sqrt{C_1 C_2 R_1 R_2}}{C_2(R_2 + R_1) - R_1 C_1 r_{34}}$$

No ripple in the pass band when $Q < 0.707$.

Summary (Active filters)



Band reject ($\mathrm{H}(\omega) = 0$) condition: $R_{31}C_{23} = 2R_{27}C_3$

$$\omega_c = \frac{1}{C_3\sqrt{2R_{31}R_{27}}} = 307.7 \text{ rad/s}$$

$$f_c = \frac{\omega}{2\pi} = 49 \text{ Hz}$$