Pattern Recognition - Summary

Friedrich-Alexander-Universität Erlangen-Nürnberg

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1 Support Vector Machines (SVM)

Just like other classifiers (Neural Nets, Nearest Neighbor, etc.), the goal of SVMs is to draw a linear line (decision boundary) to separate classes. But instead of drawing any sufficient line to separate the classes, SVMs aim to find a unique decision boundary that **maximizes the margin (distance)** between each class. The solution to this problem is unique and depends only on the features that are close to the decision boundary.

1.1 Hard Margin Problems

The hard margin SVM needs linearly seperable classes. Lets assume there is an affine function

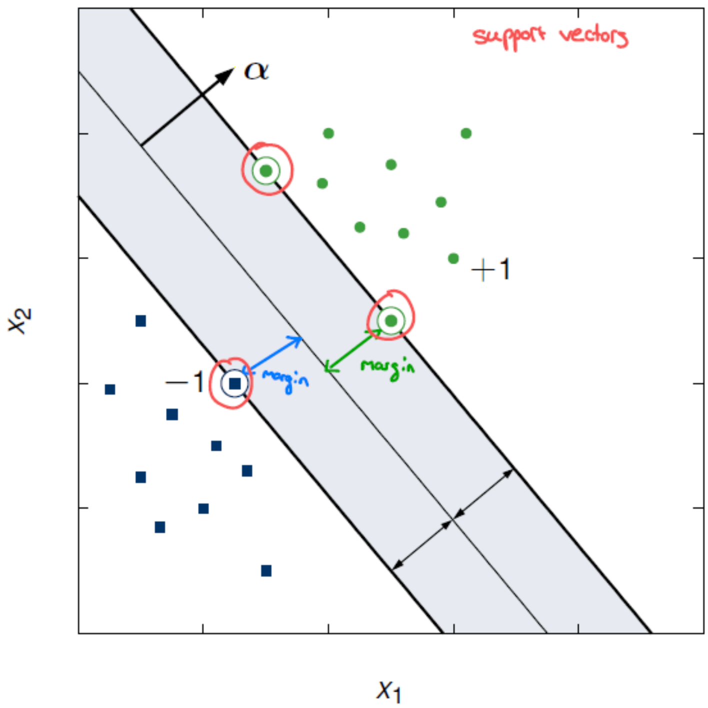

Figure 1: Hard margin SVM

defined as:

$$f(x) = \vec{\alpha}^T x + \alpha_0 \tag{1}$$

Where $\vec{\alpha}$ is the normal vector to the decision boundary and α_0 is some sort of bias. For any point x on the decision boundary, it holds that $f(x) = 0$.
There are three major points we need to think about to end up with a nice optimization problem:

1. Introduce margin constraints:
We need to ensure that all points are classified correctly and therefore lie outside the margin. Therefore, we introduce the following constraints:

- For points of class $+1$: $f(x) \geq 1$
- For points of class -1: $f(x) \leq -1$

This means that for a sample point x_i with the label $y_i = +1$, it holds that $f(x_i) \geq 1$. Similarly, for a sample point x_j with the label $y_j = -1$, it holds that $f(x_j) \leq -1$. These two constraints can be combined into one single constraint ($y \in \{+1, -1\}$):

$$y_i f(x_i) - 1 = y_i(\vec{\alpha}^T x_i + \alpha_0) - 1 \geq 0 \quad \forall i \tag{2}$$

2. Define the margin:
The margin is defined as the distance between the decision boundary and the closest points from either class. To compute the margin width, we take a sample from each class that lies exactly on the margin (i.e., satisfies $y_i f(x_i) - 1 = 0 \quad \forall i$) and subtract them from each other. When we project the resulting vector onto tho normalized normal vector of the hyperplane, we get the margin width:

$$\text{width} = \frac{\vec{\alpha}}{\|\vec{\alpha}\|_2} \cdot (\vec{x}_{y=+1} - \vec{x}_{y=-1}) \tag{3}$$

Now we multiply this out:

$$\text{width} = \frac{1}{\|\vec{\alpha}\|_2}(\vec{\alpha}^T \vec{x}_{y=+1} - \vec{\alpha}^T \vec{x}_{y=-1}) \tag{4}$$

We know from our margin constraints defined in step 1 that for support vectors (points on the margin), the inequality becomes an equality:

- For the positive support vector $\vec{x}_{y=+1}$:

$$\vec{\alpha}^T \vec{x}_{y=+1} + \alpha_0 = 1 \quad \Rightarrow \quad \vec{\alpha}^T \vec{x}_{y=+1} = 1 - \alpha_0$$

- For the negative support vector $\vec{x}_{y=-1}$:

$$\vec{\alpha}^T \vec{x}_{y=-1} + \alpha_0 = -1 \quad \Rightarrow \quad \vec{\alpha}^T \vec{x}_{y=-1} = -1 - \alpha_0$$

Substituting these expressions back into the width equation:

$$\begin{aligned} \text{width} &= \frac{1}{\|\vec{\alpha}\|_2}((1 - \alpha_0) - (-1 - \alpha_0)) \quad (5) \\ &= \frac{1}{\|\vec{\alpha}\|_2}(1 - \alpha_0 + 1 + \alpha_0) \quad (6) \\ &= \frac{2}{\|\vec{\alpha}\|_2} \quad (7) \end{aligned}$$

3. Minimize the norm:
Since we want to **maximize** the margin width $\frac{2}{\|\vec{\alpha}\|_2}$, this is mathematically equivalent to **minimizing** the length of the normal vector $\|\vec{\alpha}\|_2$. For mathematical convenience (to make derivatives easier later), we minimize the squared norm:

Primal Optimization Problem (Hard Margin)

$$\text{minimize} \quad \frac{1}{2}\|\vec{\alpha}\|_2^2 \quad \text{subject to} \quad y_i(\vec{\alpha}^T x_i + \alpha_0) \geq 1 \quad \forall i$$