6 Dynamical Systems

Solution 6.1: Yes, the continuous-time component modeling the simple pendulum has Lipschitz-continuous dynamics. The rate of change of the angle φ is given by the expression ν, which is linear, and hence, Lipschitz-continuous function from `real` to `real`. The rate of change of the angular velocity ν is a linear combination of $\texttt{sin}\,\varphi$ which is Lipschitz-continuous since $\|\texttt{sin}\,\varphi\| \leq 1$ for all values of φ, and the input u which is also Lipschitz-continuous, and thus, is Lipschitz-continuous. ■

Solution 6.3: Suppose the dynamics of the component H_1 is given by $\dot{x} = f_1(x, u)$ and $v = h_1(x, u)$, and the dynamics of the component H_2 is given by $\dot{y} = f_2(y, v)$ and $w = h_2(y, v)$. We know that the functions f_1, h_1, f_2, and h_2 are all Lipschitz-continuous.

The parallel composition $H = H_1 \| H_2$ has input variable u, state variables x and y, and output variables v and w. Its dynamics is specified by

$\dot{x} = f_1(x, u)$;
$\dot{y} = f_3(x, y, u) = f_2(y, h_1(x, u))$;
$v = h_1(x, u)$; and
$w = h_3(x, y, u) = h_2(y, h_1(x, u))$.

To show that the component H is Lipschitz-continuous we need to show that the functions f_1, f_3, h_1, and h_3 are Lipschitz-continuous. We already know that f_1 and h_1 are Lipschitz-continuous. Below we show that the function f_3 is Lipschitz-continuous. The proof for the Lipschitz-continuity of h_3 is analogous.

To prove Lipschitz-continuity of f_3, we need to find a constant K such that for all (x, y, u) and (x', y', u'), $\|f_3(x, y, u) - f_3(x', y', u')\| \leq K\|(x, y, u) - (x', y', u')\|$. We know that the functions h_1 and f_2 are Lipschitz-continuous. Let the corresponding Lipschitz constants be K_1 and K_2 respectively.

$$
\begin{aligned}
\|f_3(x,y,u) - f_3(x',y',u')\| &= \|f_2(y, h_1(x,u)) - f_2(y', h_1(x',u'))\| \\
&\leq K_2\,\|(y, h_1(x,u)) - (y', h_1(x',u'))\| \\
&\leq K_2\,[\|y - y'\| + \|h_1(x,u) - h_1(x',u')\|] \\
&\leq K_2\,[\|y - y'\| + K_1\,\|(x,u) - (x',u')\|] \\
&\leq K_2\,[\|(x,y,u) - (x',y',u')\| + K_1\,\|(x,y,u) - (x',y',u')\|] \\
&= K_2(K_1 + 1)\|(x,y,u) - (x',y',u')\|
\end{aligned}
$$

■

Solution 6.4: The continuous-time component of figure 6.8 has two state variables. To calculate equilibria of the system, we set the rate of change of each state variable to be 0. This gives the equations $v = 0$ and $(-kv - m\,g\,\texttt{sin}\,\theta)/m = 0$ (since the input force F is also 0). These equations can be satisfied only when $\texttt{sin}\,\theta$ equals 0. When the grade θ is 5 degrees, this does not hold, and hence, the model does not have any equilibrium state. ■

Solution 6.5: To find equilibrium states, we set $3\,s_1 + 4\,s_2 = 0$ and $2\,s_1 + s_2 = 0$. Since these two equations are linearly independent, the system of equations has a unique solution, namely, $s_1 = s_2 = 0$. Thus, the origin $(0, 0)$ is the sole equilibrium of the system. To check whether this equilibrium is stable, suppose we perturb the state to, say, $s_0 = (\delta, \delta)$ for a small value of $\delta > 0$. Then, the system response $\overline{S}_0(t)$ is the solution to the linear differential equation $\dot{s}_1 = 3\,s_1 + 4\,s_2$ and $\dot{s}_2 = 2\,s_1 + s_2$ with the initial state s_0. While it is possible to solve this initial value problem, for current purpose simply observe that at time 0, both the quantities $\dot{s}_1$ and $\dot{s}_2$ are positive values causing the initial state s_0 to flow away from the origin. The magnitudes of the rates only increase as a result causing the resulting signal $\overline{S}_0(t)$ to diverge. That is, no matter how small a value of $\delta > 0$ we choose, for the initial state $s_0 = (\delta, \delta)$, the quantity $\|\overline{S}_0(t)\|$ grows unboundedly. Thus, the equilibrium $(0, 0)$ is unstable. ■

Solution 6.6: Setting $x^2 - x = 0$ gives us two equilibria: $x = 0$ and $x = 1$. To analyze the stability of the equilibrium $x = 0$, let us consider the behavior of the system starting from the initial state $x_0 = \delta$, for $0 < \delta < 0.5$. Then the rate of change $\delta^2 - \delta$ is negative, and furthermore, the magnitude of this rate decreases as the state gets closer to the equilibrium. The resulting response signal $\overline{x}(t)$ converges to 0 with $0 \leq \overline{x}(t) \leq \delta$ for all t. Analogously, if the initial state is $x_0 = -\delta$, for $0 < \delta < -0.5$, Then the rate of change is positive, and the resulting response signal converges to 0 with $-\delta \leq \overline{x}(t) \leq 0$ for all t. Thus, the equilibrium 0 is asymptotically stable.

For the equilibrium $x = 1$, if we set the initial state to $x_0 = 1 + \delta$, for $\delta > 0$, then no matter how small δ we choose, the rate of change is positive with its magnitude increasing as the state moves away from the equilibrium 1. Thus, the response signal $\overline{x}(t)$ diverges, and the equilibrium $x = 1$ is unstable. ■

Solution 6.7: We model the tuning fork as a closed continuous-time component H with two state variables x—denoting the displacement, and v—denoting the velocity. Initially, the displacement x equals x_0 and the velocity v equals 0. The dynamics is given by $\dot{x} = v$ and $\dot{v} = (-k/m)\,x$. The output of the system can be the displacement x.

We want to find the solution to the differential equation $\ddot{x} = (-k/m)\,x$ with the initial condition $\overline{x}(0) = x_0$. If we set $\overline{x}(t) = b\,\texttt{cos}\,(a\,t)$, then $\overline{x}(0) = b$, $(d/dt)\,\overline{x}(t) = -b\,a\,\texttt{sin}\,(a\,t)$, and $(d^2/dt^2)\,\overline{x}(t) = -b\,a^2\,\texttt{cos}\,(a\,t)$ which equals $-a^2\,\overline{x}(t)$. Thus, if we choose $b = x_0$ and $a = \sqrt{k/m}$, we have the desired solution. That is, for the initial displacement x_0, the state response of the tuning fork is described by the signal $\overline{x}(t) = x_0\,\texttt{cos}\,(\sqrt{k/m}\ t)$. This corresponds to the fork oscillating between x_0 to $-x_0$ in a perpetual rhythmic motion.

Setting $\dot{x} = 0$ and $\dot{v} = 0$ gives the sole equilibrium of the system: $x = 0$ and $v = 0$. This corresponds to the situation where the fork is stationary and vertical. If we set the initial displacement to x_0, we know that the state response is given by $\overline{x}(t) = x_0\,\texttt{cos}\,(\sqrt{k/m}\ t)$. For such a signal, $\|\overline{x}(t)\| \leq \|x_0\|$

Verify that $\overline{s}(0)$ evaluates to s_0, and differentiating the expression for $\overline{s}(t)$ gives $(d/dt)\,\overline{s}(t) \;=\; a\,e^{at}s_0 \,+\, b\,c\,e^{at}$ which equals $a\,\overline{s}(t) \,+\, b\,c$ as desired. ■

Solution 6.13: For this system, the dynamics matrix is

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial $\texttt{det}(\,A \;-\; \lambda\,\mathbf{I})$ turns out to be $\lambda^2 - 1$. As a result, we have two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. To obtain the eigenvector $x_1 = [x_{11} \;\; x_{12}]^T$ corresponding to the eigenvalue 1, we need to solve the equation $A\,x_1 = x_1$. This gives $x_{11} = x_{12}$, and we choose $[1 \;\; 1]^T$ as the eigenvector x_1. To obtain the eigenvector $x_2 = [x_{21} \;\; x_{22}]^T$ corresponding to the eigenvalue -1, we need to solve the equation $A\,x_2 = x_2$, which gives $x_{22} = 0$, and we choose $[1 \;\; 0]^T$ as the eigenvector x_2.

To compute the state response by diagonalization using theorem 6.2, the transformation matrix is:

$$P \;=\; [\,x_1 \;\; x_2\,] \;=\; \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

To compute the inverse P^{-1}, we can view the entries in the desired matrix as unknowns and solve the system of simultaneous linear equations given by $P\,P^{-1} = \mathbf{I}$. This gives

$$P^{-1} \;=\; \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

Starting in the initial state s_0, the state of the system at time t is described by $P\,\mathbf{D}(e^t, e^{-t})\,P^{-1}s_0$. If the initial state vector s_0 is $[s_{01} \;\;\; s_{02}]^T$, then by calculating the matrix products, we get a closed-form solution for the state of the system at time t:

$$\begin{aligned} \overline{S}_1(t) \;&=\; e^{-t}\,s_{01} \;+\; (e^t - e^{-t})\,s_{02} \\ \overline{S}_2(t) \;&=\; e^t\,s_{02}. \end{aligned}$$

■

Solution 6.14: For this exercise, the dynamics matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

The characteristic polynomial then is $\lambda^2 + 3\lambda + 2$. This gives two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. By solving the equation $A\,x_1 = -x_1$, we get the first eigenvector $[1 \;\;\; -1]^T$, and by solving the equation $A\,x_2 = -2\,x_2$, we get the second eigenvector $[1 \;\;\; -2]^T$.

To compute the state response by diagonalization using theorem 6.2, the transformation matrix is:

$$P \;=\; [\,x_1 \;\; x_2\,] \;=\; \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}.$$

The inverse of this matrix is:

$$P^{-1} \;=\; \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

Starting in the initial state s_0, the state of the system at time t is described by $P\,\mathbf{D}(e^t, e^{-t})\,P^{-1}s_0$. This gives a closed-form solution for the state of the system at time t:

$$\begin{aligned}
\overline{S}_1(t) &= (\,2\,e^{-t} \;-\; e^{-2t}\,)\,s_{01} \;+\; (\,e^{-t} \;-\; e^{-2t}\,)\,s_{02} \\
\overline{S}_2(t) &= 2\,(\,-e^{-t} \;+\; e^{-2t}\,)\,s_{01} \;+\; (\,-e^{-t} \;+\; 2\,e^{-2t}\,)\,s_{02}.
\end{aligned}$$

■

Solution 6.15 : For this example, the dynamics matrix is

$$A = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 9 \end{bmatrix}.$$

The characteristic polynomial then is $(3-\lambda)(2-\lambda)(9-\lambda)$. This gives three eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 9$. By solving the equation $A\,x_1 = 2\,x_1$, we get the first eigenvector $[-28 \;\; 7 \;\; 16]^T$, by solving the equation $A\,x_2 = 3\,x_2$, we get the second eigenvector $[3 \;\; 0 \;\; -2]^T$, and by solving the equation $A\,x_3 = 9\,x_3$, we get the third eigenvector $[0 \;\; 0 \;\; 1]^T$.

To compute the state response by diagonalization using theorem 6.2, the transformation matrix is:

$$P \;=\; [\,x_1 \;\; x_2 \;\; x_3\,] \;=\; \begin{bmatrix} -28 & 3 & 0 \\ 7 & 0 & 0 \\ 16 & -2 & 1 \end{bmatrix}.$$

The inverse of this matrix is:

$$P^{-1} \;=\; \begin{bmatrix} 0 & 1/7 & 0 \\ 1/3 & 4/3 & 0 \\ 2/3 & 8/21 & 1 \end{bmatrix}.$$

Starting in the initial state s_0, the state of the system at time t is described by $P\,\mathbf{D}(e^{2t}, e^{3t}, e^{9t})\,P^{-1}s_0$. This gives a closed-form solution for the state of the system at time t:

$$\begin{aligned}
\overline{S}_1(t) &= e^{3t}\,s_{01} \;+\; 4\,(e^{3t} - e^{2t})\,s_{02} \\
\overline{S}_2(t) &= e^{2t}\,s_{02} \\
\overline{S}_3(t) &= 2/3\,(e^{9t} - e^{3t})\,s_{01} \;+\; 1/21\,(48\,e^{2t} - 56\,e^{3t} + 8\,e^{9t})\,s_{02} \;+\; e^{9t}\,s_{03}.
\end{aligned}$$

■

$$
\begin{aligned}
&= P^{-1} A\, \overline{S}(t) \\
&= P^{-1} A\, P\, \overline{S}'(t) \\
&= J\, \overline{S}'(t).
\end{aligned}
$$

Thus, the signal $\overline{S}'(t)$ is the state response of the system H' starting from the initial state s_0'. Now for all times t,

$$
\begin{aligned}
\|\overline{S}'(t)\| &= \|P^{-1}\overline{S}(t)\| \\
&\leq \Delta_{P^{-1}} \|\overline{S}(t)\| \\
&< \Delta_{P^{-1}}\ \epsilon \\
&= \Delta_{P^{-1}}\ \epsilon' / \Delta_{P^{-1}} \\
&= \epsilon'.
\end{aligned}
$$

Thus, we have proved that the state of the system H' stays ϵ'-close to the origin at all times provided its initial state is δ'-close to the origin. The proof that if the system H' is stable, then so is the system H is identical with the roles of the transformations P and P^{-1} reversed. The proof that the system H is asymptotically stable if and only if the system H' is asymptotically stable is analogous. ■

Solution 6.17: For the system in exercise 6.13, one of the eigenvalues is 1, and hence, the system is unstable. For the system in exercise 6.14, all the eigenvalues are negative, and hence, the system is asymptotically stable. For the system in exercise 6.15, all the eigenvalues are positive, and hence, the system is unstable. ■

Solution 6.18: For the given dynamical system

$$
A = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \mathbf{C}(A, B) = \begin{bmatrix} 1 & 3/2 \\ 1 & 3 \end{bmatrix}.
$$

For the matrix A, the characteristic polynomial is $\lambda^2 - 5\,\lambda/2$. Thus the eigenvalues are 0 and 5/2. From the stability test for linear systems, we can conclude that the system is unstable.

For the controllability matrix $\mathbf{C}(A, B)$, the two columns are independent and its rank is 2. From theorem 6.5, we can conclude that the system is controllable.

The desired gain matrix F is a (1×2)-matrix, and let its entries be f_1 and f_2. Consider the matrix $A - BF$:

$$
\begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ \ f_2] = \begin{bmatrix} 1/2 - f_1 & 1 - f_2 \\ 1 - f_1 & 2 - f_2 \end{bmatrix}.
$$

The characteristic polynomial for this matrix is

$$
\begin{aligned}
P(\lambda, f_1, f_2) &= (1/2 - f_1 - \lambda)(2 - f_2 - \lambda) - (1 - f_2)(1 - f_1); \\
&= \lambda^2 + (f_1 + f_2 - 5/2)\lambda + (f_2/2 - f_1).
\end{aligned}
$$

The roots of this characteristic polynomial are λ_1 and λ_2 exactly when

$$\begin{aligned} f_1 + f_2 - 5/2 &= -\lambda_1 - \lambda_2; \\ f_2/2 - f_1 &= \lambda_1 \lambda_2. \end{aligned}$$

If we want the eigenvalues to be $-1+j$ and $-1-j$, then we need to solve

$$f_1 + f_2 - 5/2 \;=\; 2; \quad f_2/2 - f_1 \;=\; 2.$$

Solving these equations give us $f_1 = 1/6$ and $f_2 = 13/3$. This means that with the choice of the gain matrix F to be $[1/6 \;\; 13/3]$, the resulting closed-loop system is asymptotically stable with eigenvalues $-1+j$ and $-1-j$. ■

Solution 6.19: For the dynamical system in this exercise

$$A \;=\; \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}; \; B \;=\; \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \; \mathbf{C}(A,B) \;=\; \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}.$$

For the matrix A, the characteristic polynomial is $\lambda^2 + 3\lambda + 2$. Thus the eigenvalues are -2 and -1. For the controllability matrix $\mathbf{C}(A,B)$, the two rows are identical, and thus, the rank is 1 and the system is not controllable.

Setting the entries of the gain matrix to be f_1 and f_2, consider the matrix $A - BF$:

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \;\; f_2] = \begin{bmatrix} -f_1 & -2-f_2 \\ 1-f_1 & -3-f_2 \end{bmatrix}.$$

The characteristic polynomial for this matrix is

$$\begin{aligned} P(\lambda, f_1, f_2) &= (f_1 + \lambda)(3 + f_2 + \lambda) + (2 + f_2)(1 - f_1); \\ &= \lambda^2 + (f_1 + f_2 + 3)\lambda + (f_1 + f_2 + 2). \end{aligned}$$

The roots of this characteristic polynomial are λ_1 and λ_2 exactly when $-\lambda_1 - \lambda_2 = f + 1$ and $\lambda_1\lambda_2 = f$, where $f = f_1 + f_2 + 2$. For any given f, $\lambda_1 = -1$ and $\lambda_2 = -f$ is the solution to this system of equations. This means that one of the eigenvalues of the matrix $A - BF$ is always -1. If we wish the other eigenvalue to be e, we can choose the entries f_1 and f_2 of the gain matrix so that $e = -(f_1 + f_2 + 2)$. ■

Solution 6.22: The sequence of values computed by Euler's method for simulation is given by: for every $i \geq 0$, $s_{i+1} = s_i + \Delta\, s_i = (1+\Delta)\, s_i$. Thus, $s_1 = (1+\Delta)\, s_0$, $s_2 = (1+\Delta)\, s_1 = (1+\Delta)^2\, s_0$, and after n steps of simulation, the state s_n equals $(1+\Delta)^n\, s_0$. For $s_0 = 2$, $\Delta = 0.1$, and $n = 50$, we get $s_{50} = 2\,(1.1)^{50}$, which equals 234.78171. ■

Solution 6.23: Consider the calculation of the state s_{i+1} from state s_i using the second-order Runge-Kutta method: $k_1 = s_i$, $k_2 = s_i + \Delta\, s_i$, and $s_{i+1} = s_i + \Delta\,(k_1 + k_2)/2$. This gives $s_{i+1} = (1 + \Delta + \Delta^2/2)\, s_i$. Thus, after n steps of simulation, the state s_n equals $(1 + \Delta + \Delta^2/2)^n\, s_0$. For $s_0 = 2$, $\Delta = 0.1$, and $n = 50$, we get $s_{50} = 2\,(1.105)^{50}$, which equals 294.53974. ■