An Efficient Algorithm for Bandit Linear Optimization

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(in between jobs...)

Coauthors

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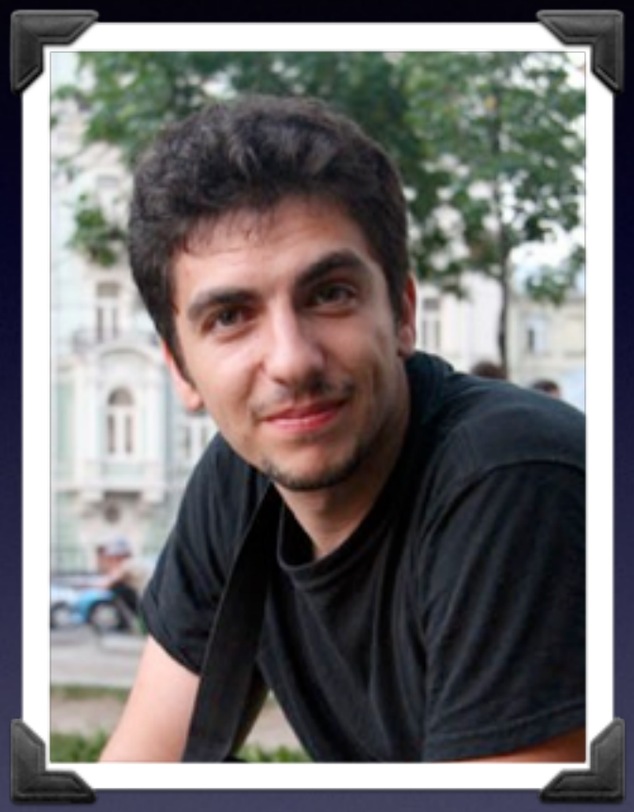

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Many thanks to my advisor Peter Bartlett

Act I: Online Linear Optimization

Online Linear Optimization (OLO)

Decision Space: $\mathcal{K} \overset{\text{cvx}}{\subset} \mathbb{R}^d$ compact, convex

For $t = 1, \ldots, T$:

- Player chooses $\mathbf{x}_t \in \mathcal{K}$
- Adversary chooses linear (convex) $\mathbf{f}_t(\cdot)$
- Player suffers $\mathbf{f}_t(\mathbf{x}_t)$ (or, $\mathbf{f}_t \cdot \mathbf{x}_t$)

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$$\underbrace{\sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}_t) - \min_{\mathbf{x}^* \in \mathcal{K}} \sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}^*)}$$

Regret

Why Minimize Regret? And Applications…

- **Modeling**: very few assumptions
- **Competitive Framework**: provides guarantees *relative* to a benchmark
- **Finance**: "Universal" portfolios [Cover '91]
- **Probability-One**: gives deterministic guarantees to the decision-maker, like options contracts [DeMarzo et al. '06]
- **Analytical Tool**: is used to study sequential optimization, and e.g. convergence of game play to equilibrium

Warmup:
Two Sequential Decision Problems

Example 1: The "Expert Setting"

- There are a set of N experts (or "actions")
- For all t:
  - Player chooses distribution $\mathbf{w}_t$ over experts
  - The experts' costs $\mathbf{c}_t$ are revealed
  - Player suffers expected cost $\mathbf{w}_t \cdot \mathbf{c}_t$
- Want to minimize $\sum_t \mathbf{w}_t \cdot \mathbf{c}_t - \min_i \sum_t \mathbf{c}_t(i)$
- Typical Alg: $\mathbf{w}_{t+1}(i) \propto \mathbf{w}_t(i) \exp(\mathbf{c}_t(i))$

Example 2:
Driving to Work

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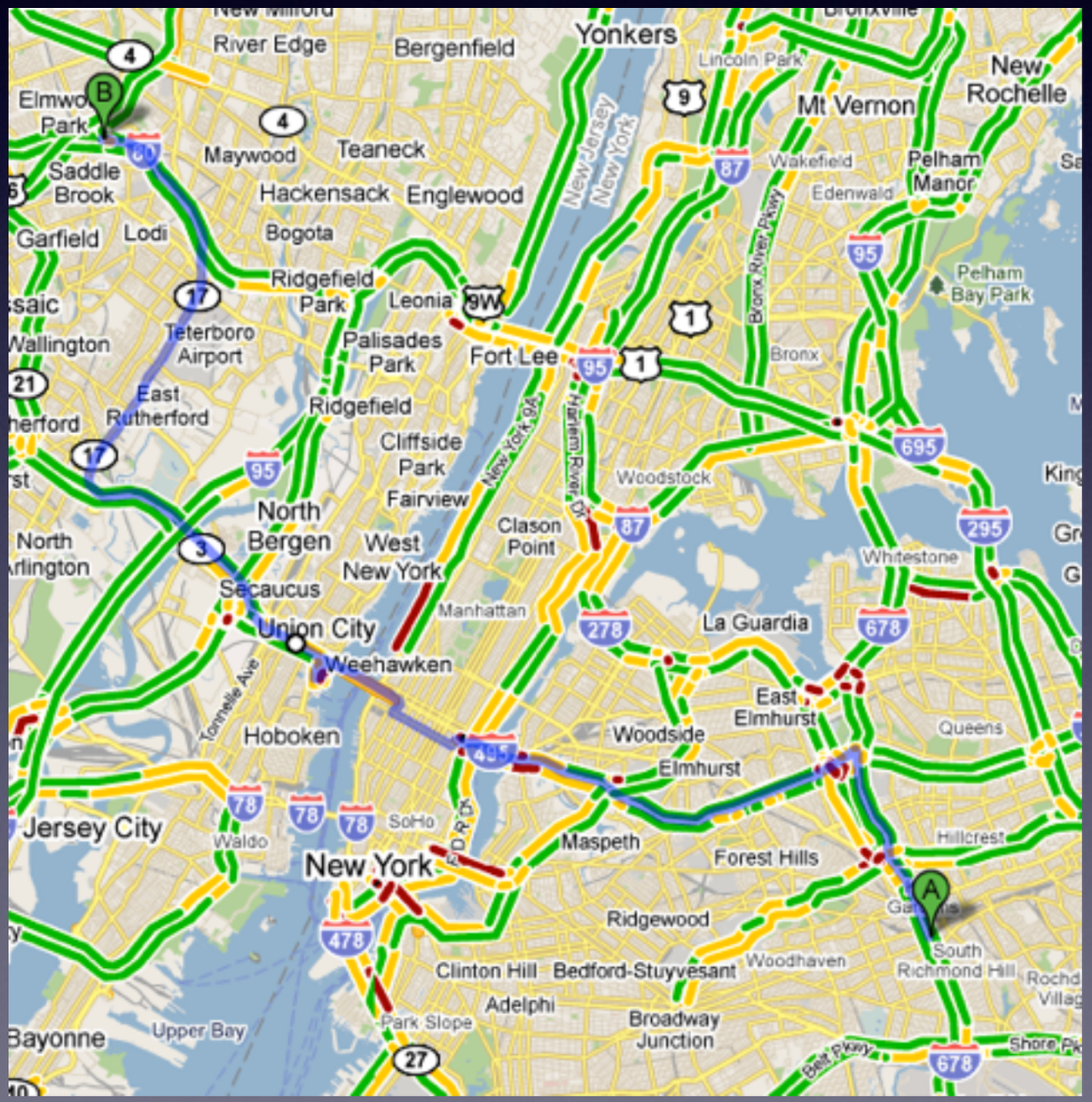

Efficient Approach: Flow Polytope

- **Flow Polytope:** convex hull of all paths from Source to Sink
- Problem dim = # edges
- Efficient decomposition from flows to randomized paths

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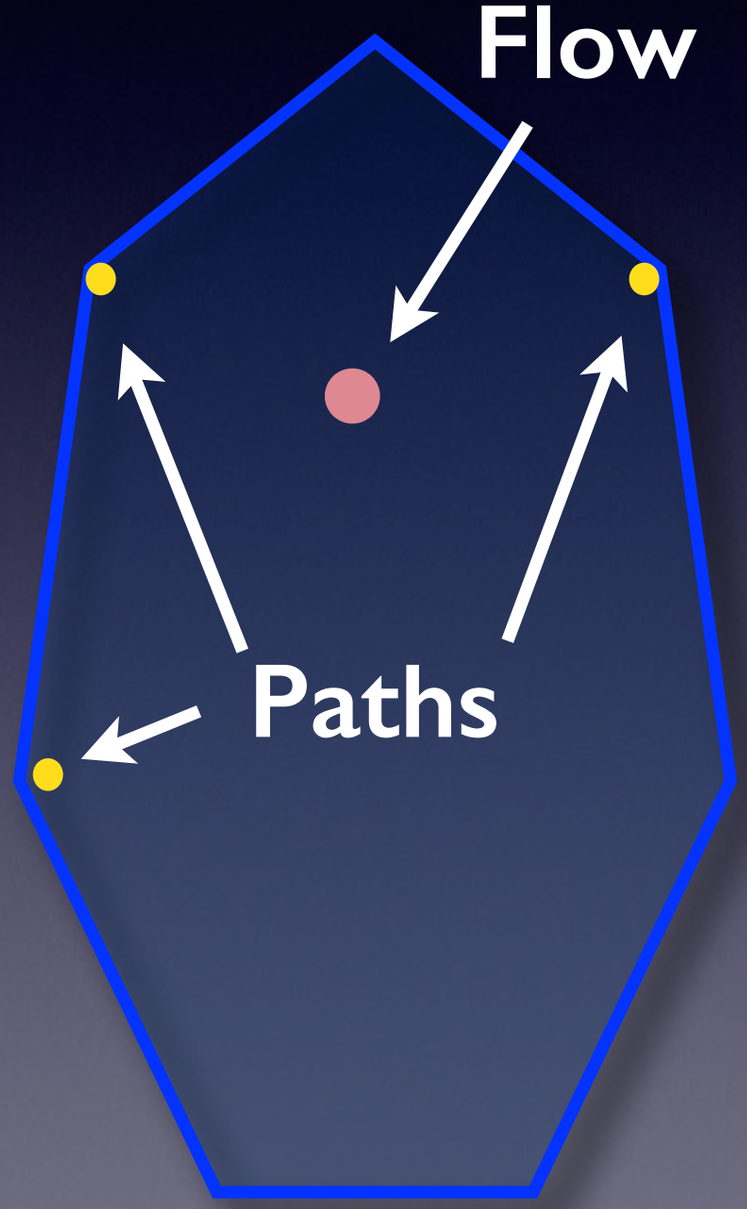

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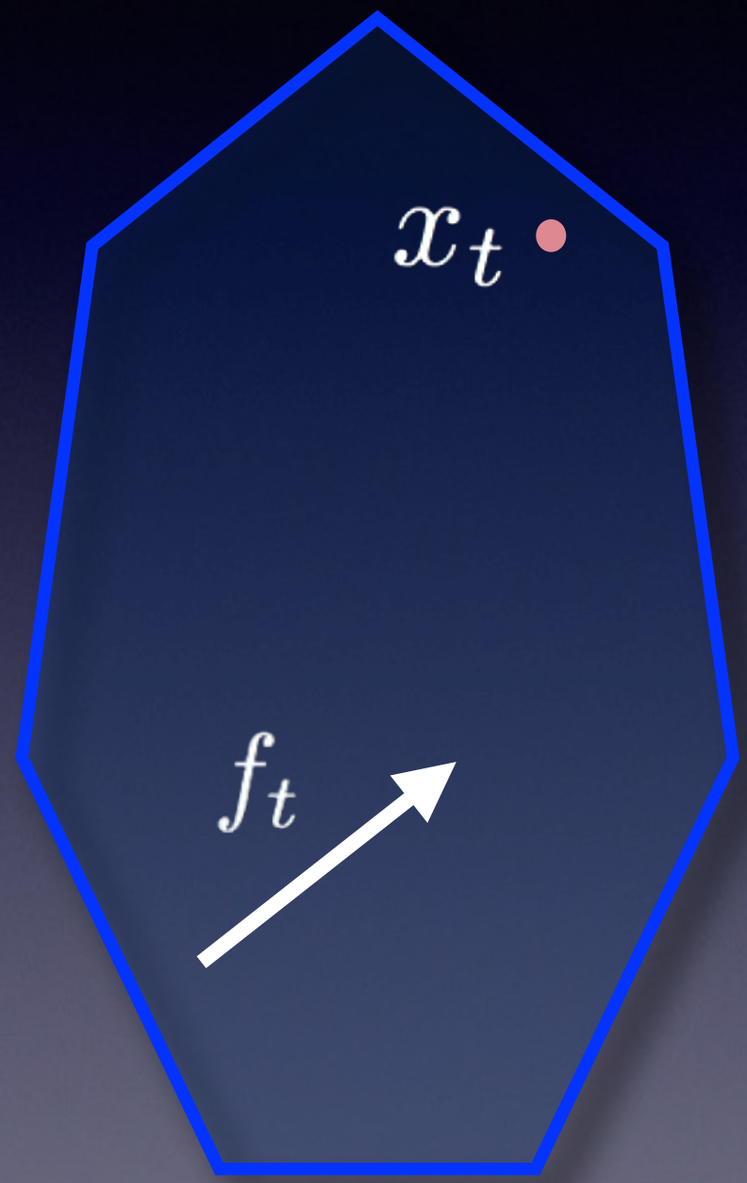

Online Linear Optimization

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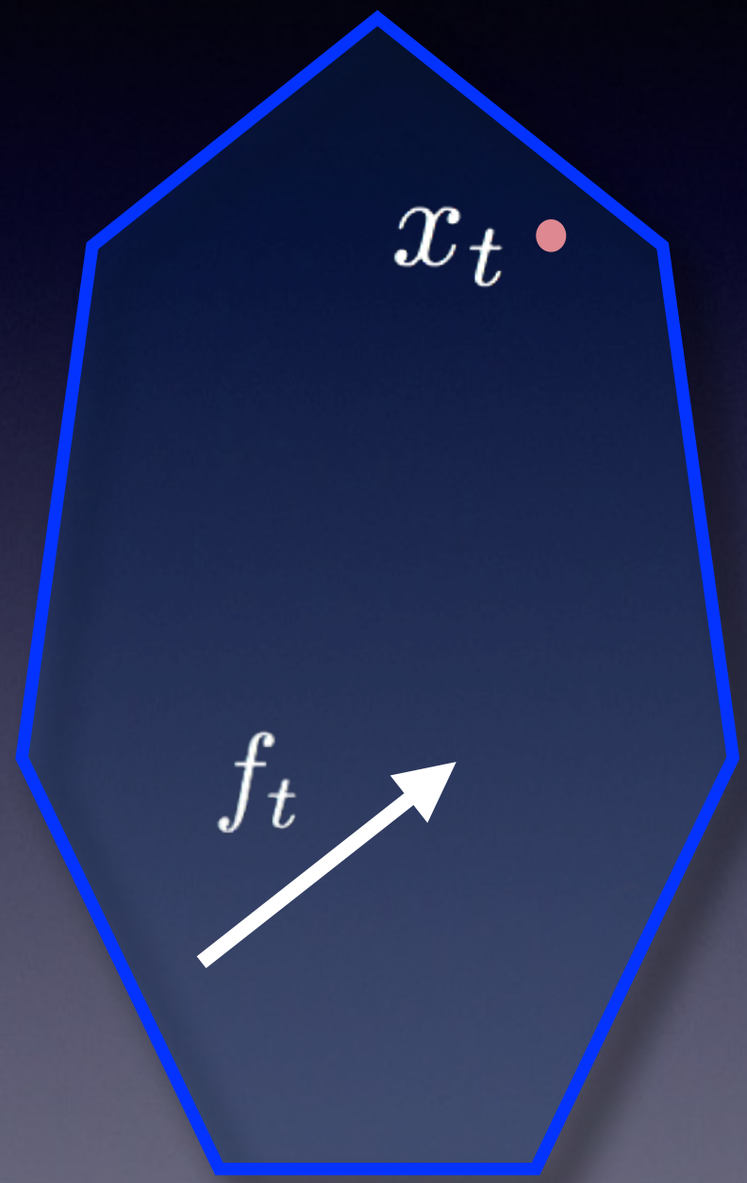
  - Player pays $f_t \cdot x_t$, *the "expected" cost*



Tight Regret Bounds for "Full Info" setting

Regret: $$\sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}_t) - \min_{\mathbf{x}^* \in \mathcal{K}} \sum_{t=1}^{T} \mathbf{f}_t(\mathbf{x}^*)$$

Tight Regret Bounds for "Full Info" setting

Setting	Regret rate
$\mathcal{K} := \Delta_d, \mathbf{f}_t \in [0,1]^d$	$\sqrt{T \log d}$
$\mathcal{K} := B_2, \mathbf{f}_t \in B_2$	$\sqrt{T}$
$\mathcal{K} := [0,1]^d, \mathbf{f}_t \in [0,1]^d$	$d\sqrt{T}$
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$\nabla \mathbf{f}_t \in B_2, \nabla^2 \mathbf{f}_t \succeq \alpha I$	$\frac{\log T}{\alpha}$

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- Decision set is $[-1, 1]$
- Adversary randomly samples $f_t \in \{-1, 1\}$
- For any player strategy:

$$\begin{aligned} \text{Worst case regret} &\geq \mathbb{E}\left[\sum f_t \cdot x_t - \min_{x \in [-1,1]} \sum f_t \cdot x\right] \\ &= 0 + \mathbb{E}\left|\sum f_t\right| \\ &= \Theta(\sqrt{T}) \end{aligned}$$

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Conclusion: at least $T^{1/2}$ regret is unavoidable

Algorithm 1: Follow the Leader

$$\mathbf{x}_t := \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{s=1}^{t-1} \mathbf{f}_s \cdot \mathbf{x}$$

Algorithm 2: Follow the *Regularized* Leader (FTRL)

$$\mathbf{x}_t := \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{s=1}^{t-1} \mathbf{f}_s \cdot \mathbf{x} + \lambda R(\mathbf{x})$$

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"Regularize" with a "curved" convex function

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Choices of R leads to different algs:

$R(\mathbf{x}) = \sum x_i \log x_i \implies$ "Exponentiated Gradient" alg

$R(\mathbf{x}) = \|\mathbf{x}\|_2^2 \implies$ "Online Gradient Descent" alg

Useful Property: Strong Convexity

$$\underbrace{R(x) - R(y) - \nabla R(y) \cdot (x - y)}_{D_R(x,y)} \geq \frac{\|x - y\|^2}{2}$$

$D_R(x, y)$ ("Bregman" divergence of R)

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(“Bregman” divergence of R)

$$R(x) = \|x\|_2^2 \implies D_R(x,y) = \|x - y\|_2^2$$

$$R(x) = \sum_i x_i \log x_i \implies D_R(x,y) = \sum x_i \log \frac{x_i}{y_i}$$

FTRL Regret Bound

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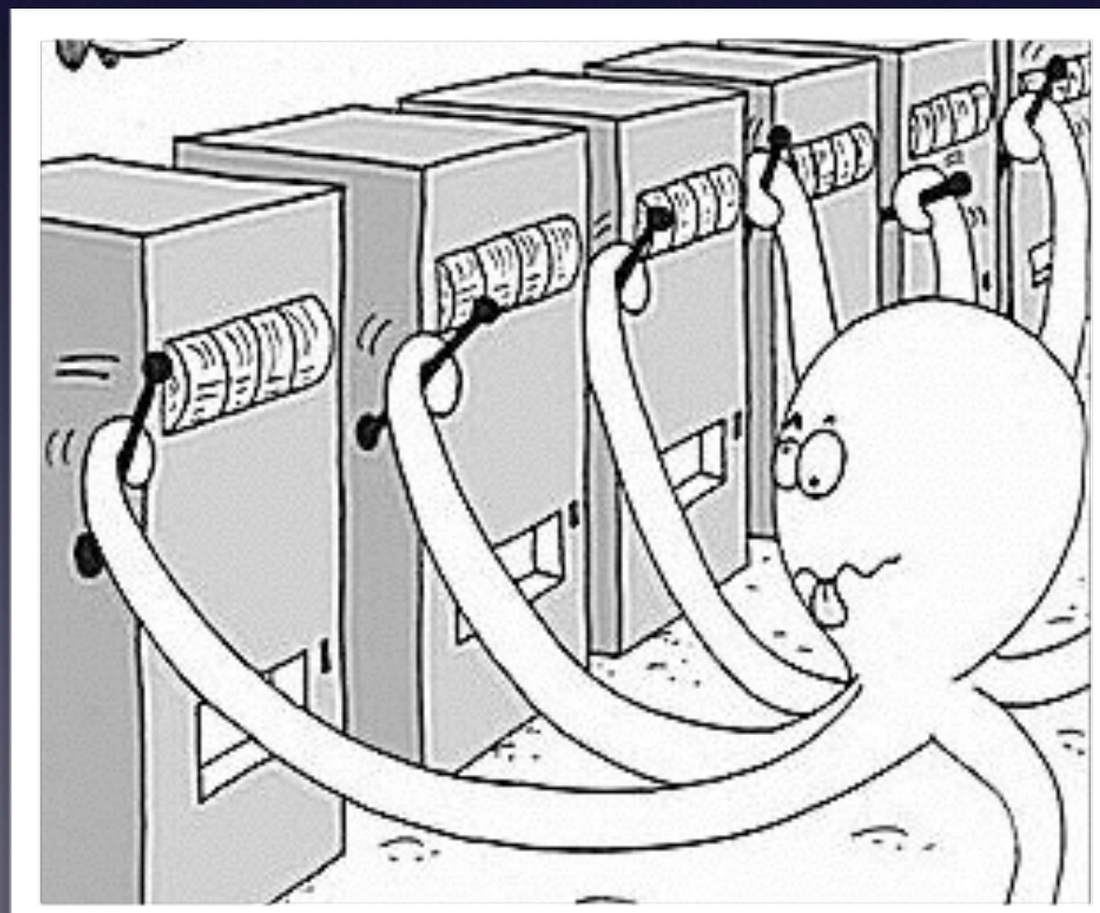
$$\leq \frac{T \cdot G}{\lambda} + \lambda D \leq 2\sqrt{T \cdot G \cdot D}$$

Online Linear Opt. in the "Bandit" Setting



OLO for *Full Info* and *Bandit* Feedback

- Two feedback settings:

- **Full-Information**:
  Player sees the full loss function f_t

- **Bandit**:
  Player sees cost of chosen action $f_t \cdot x_t$

FTRL in Bandit Setting?

$$\mathbf{x}_t := \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{s=1}^{t-1} \mathbf{f}_s \cdot \mathbf{x} + \lambda R(\mathbf{x})$$

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Major Bug:
we never observe this vector

Solution: One-sample Estimates

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- Input: hidden vector $\ell := \langle \ell_1, \ldots, \ell_n \rangle$
- Sample $I \in \{1, \ldots, n\}$ uniformly at random
- Witness random coordinate ℓ_I
- Output: estimate $\hat{\ell} := \langle 0, \ldots, 0, n\ell_I, 0, \ldots, 0 \rangle$

One-Sample Estimates Part II

- In previous example, I had to sample uniformly at random from coordinates
- BUT I want low regret, and ALG tells me to sample according to **p**. OK, modification:

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This Gets Big!!

Full Information to Bandit Feedback

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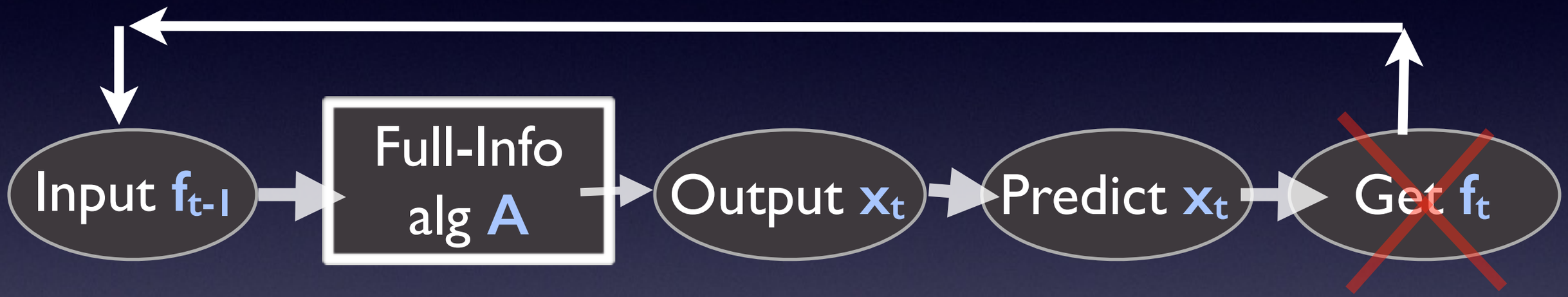

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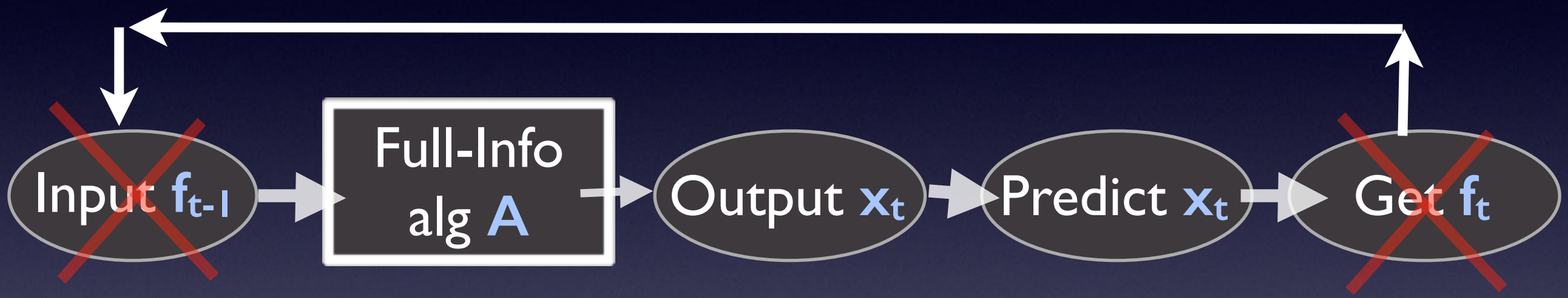

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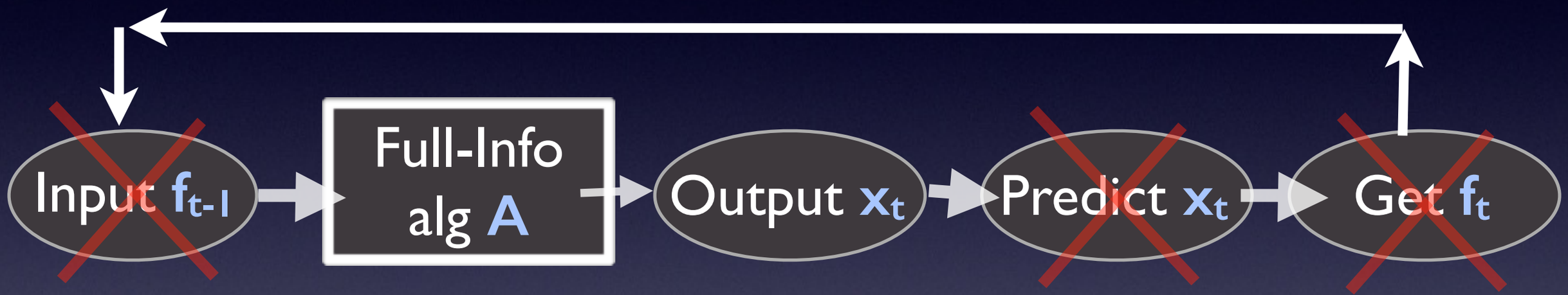

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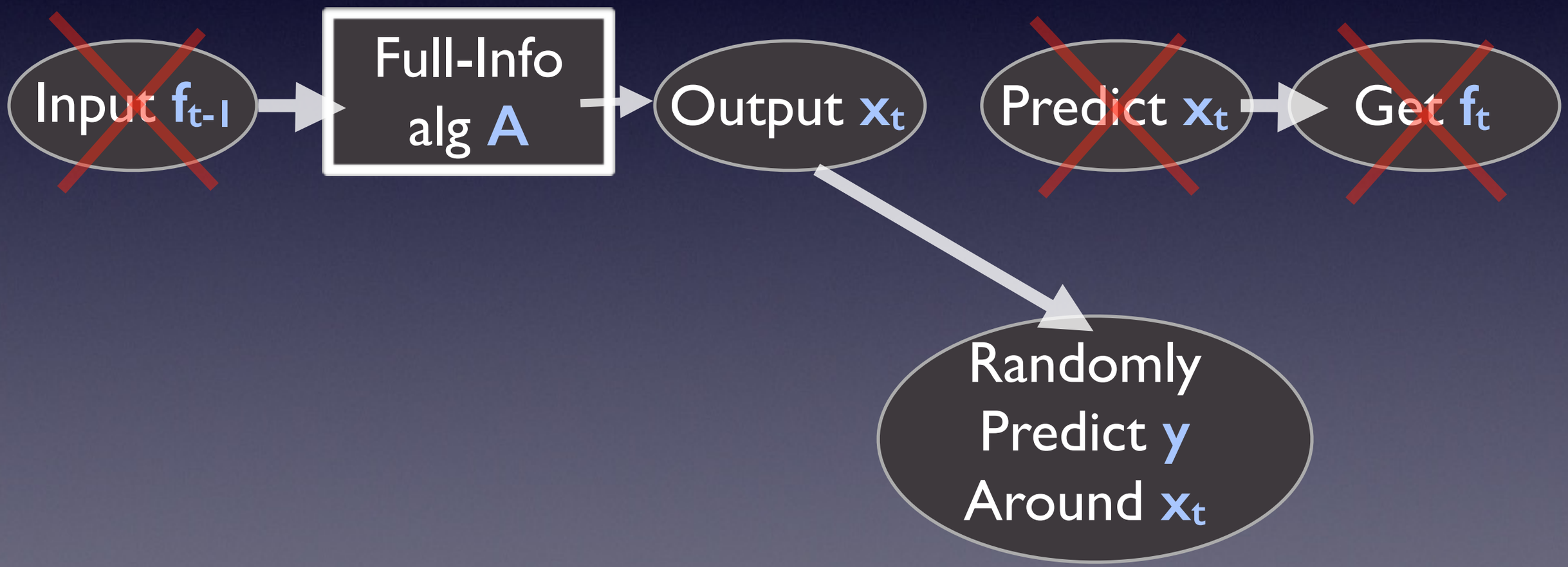

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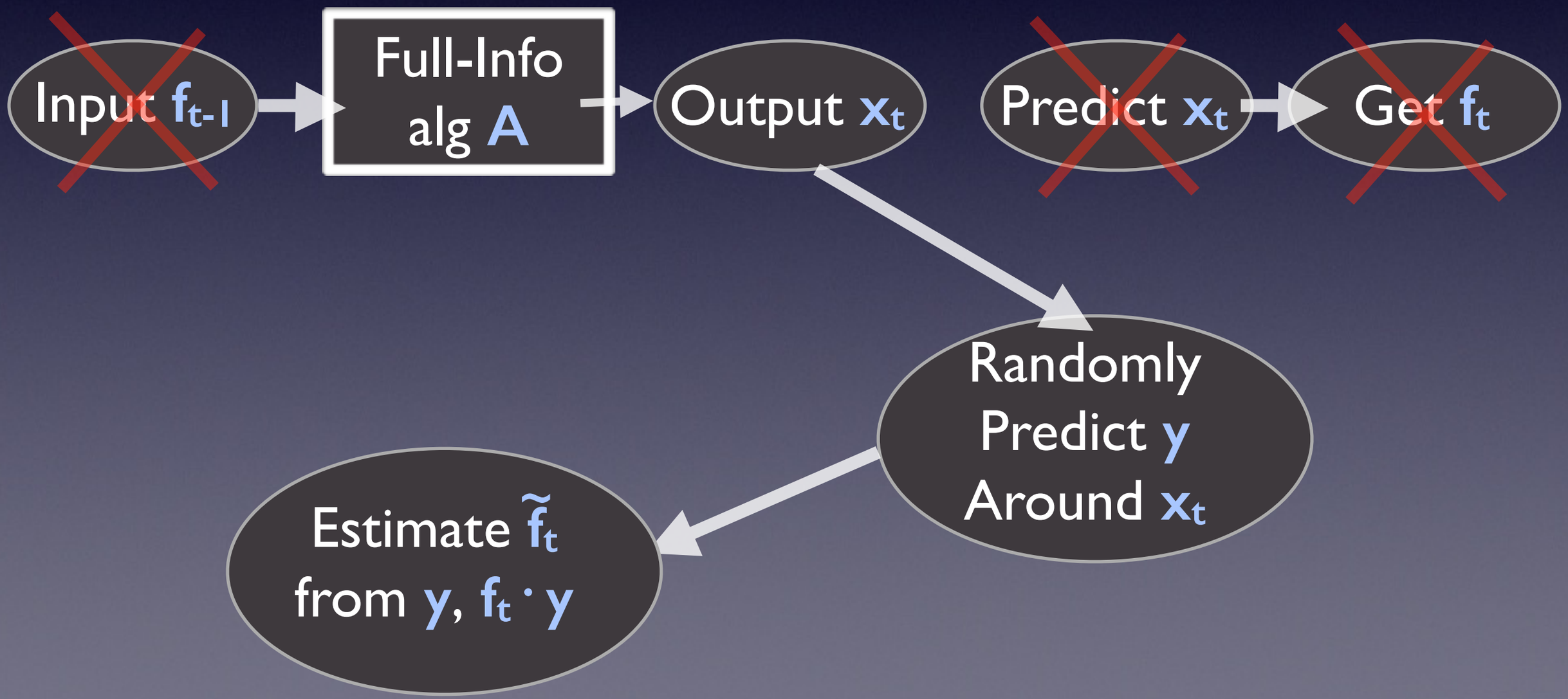

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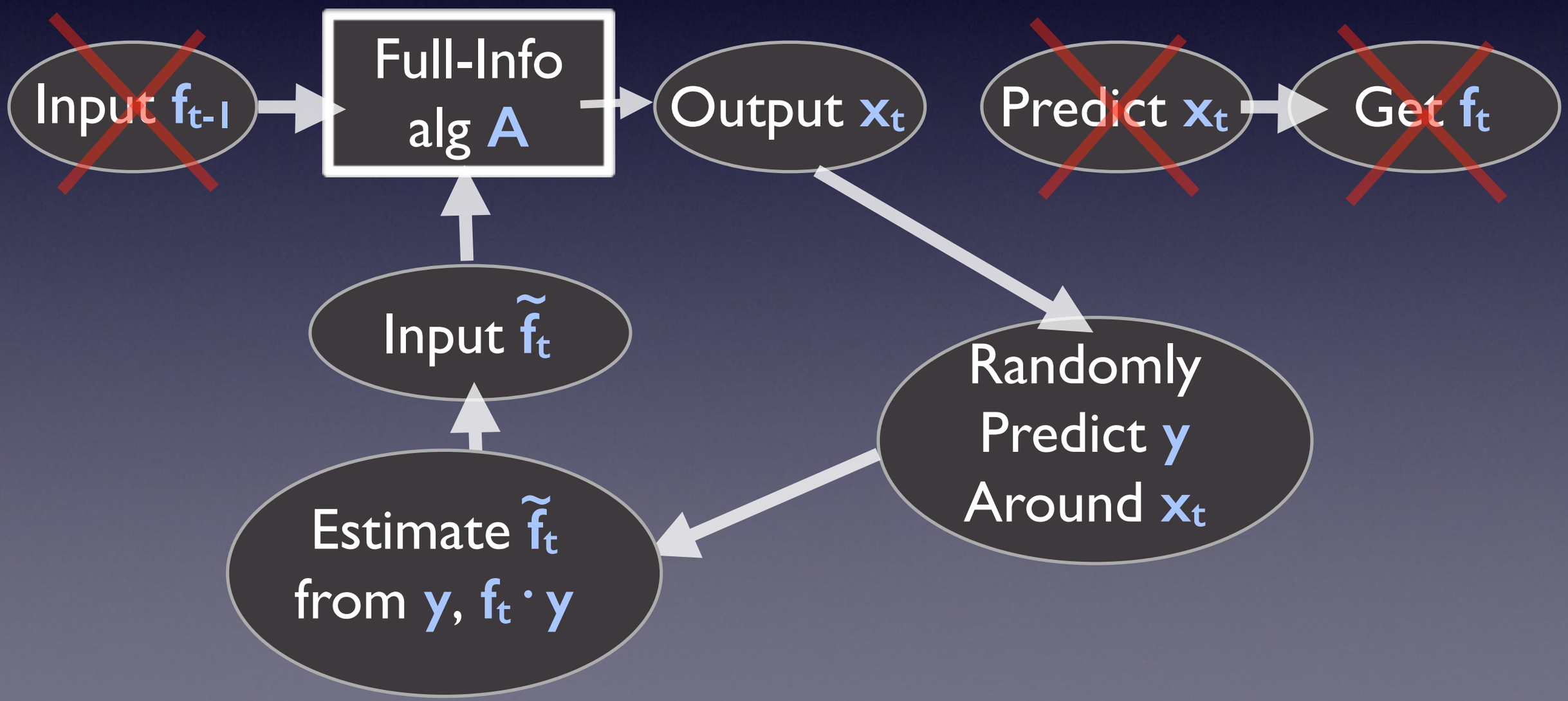

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FTRL in Bandit Setting?

$$\mathbf{x}_t := \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{s=1}^{t-1} \tilde{\mathbf{f}}_s \cdot \mathbf{x} + \lambda R(\mathbf{x})$$

Solution:

Use the estimated functions!

Estimates are Enough?

Fortunately, it is sufficient to compete with an unbiased estimate of the loss functions

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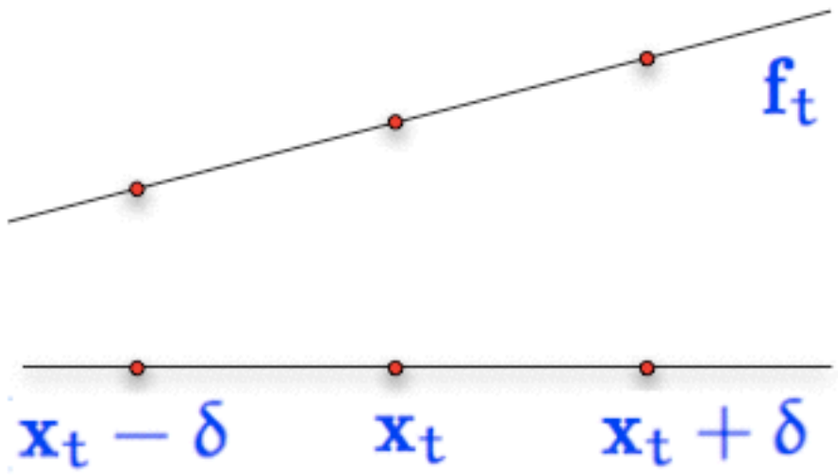
$$\begin{aligned} \mathbb{E}\,\mathrm{Regret}(\tilde{f}_1, \ldots, \tilde{f}_T) &\geq \mathrm{Regret}(\mathbb{E}\tilde{f}_1, \ldots, \mathbb{E}\tilde{f}_T) \\ &= \mathrm{Regret}(f_1, \ldots, f_T) \end{aligned}$$

Easy to check: Regret is convex in f_t's

The Problem with One-Sample Estimates



Predict randomly $\mathbf{y}_t = \mathbf{x}_t \pm \delta$.

Set $\tilde{\mathbf{f}}_t = \pm(\mathbf{f}_t \cdot \mathbf{y}_t)/\delta$.

Unbiased:

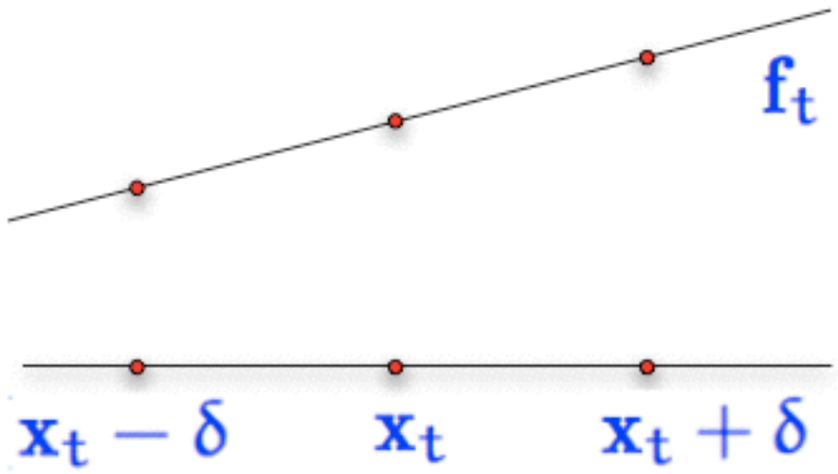
$$\mathbb{E}\tilde{\mathbf{f}}_t = \frac{1}{2}\frac{\mathbf{f}_t \cdot (\mathbf{x}_t + \delta)}{\delta} - \frac{1}{2}\frac{\mathbf{f}_t \cdot (\mathbf{x}_t - \delta)}{\delta} = \mathbf{f}_t$$

and

$$\mathbb{E}\mathbf{y}_t = \mathbf{x}_t$$

The Problem with One-Sample Estimates

Size of Estimate
≈
Distance to Boundary



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Grows large!

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Timeline of Results on Adversarial Bandit

	Regret	Efficient?	Convex Sets?	W.H.P?
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McMahan and Blum 04	$T^{3/4}$	✓	All	✗
Flaxman, Kalai, McMahon 05	$T^{3/4}$	✓	All	✗
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Two Crucial Dilemmas

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- How to sample around x_t and simultaneously construct unbiased estimates?

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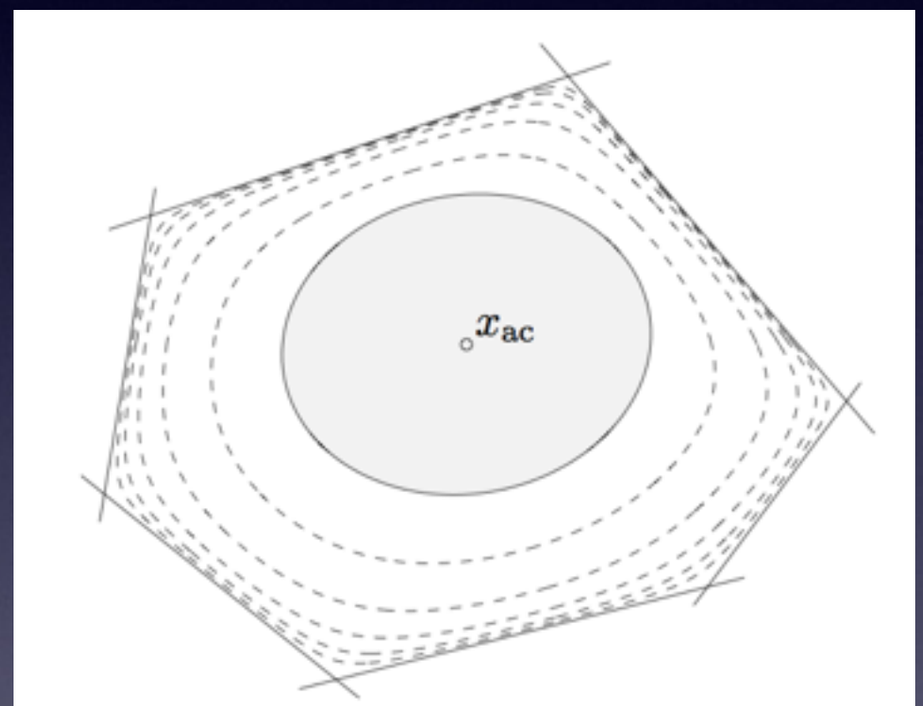
- How to deal with estimates that get very large?

The Magic Trick: Self-Concordant Functions

- Long-standing open question: How to use **Newton's Method** for poly-time convex optimization?
- Nesterov and Nemirovsky: add "self-concordant barrier" to objective
- Restrictions on 2nd + 3rd derivatives
- Only efficient method for general convex optimization



(Self-concordant is roughly "log-barrier")

A Bonus Prize: The Dikin Ellipsoid

$$\{\mathbf{z} \ : \ (\mathbf{z} - \mathbf{x}_t)\nabla^2\mathcal{R}(\mathbf{x}_t)(\mathbf{z} - \mathbf{x}_t) \leq 1\}$$

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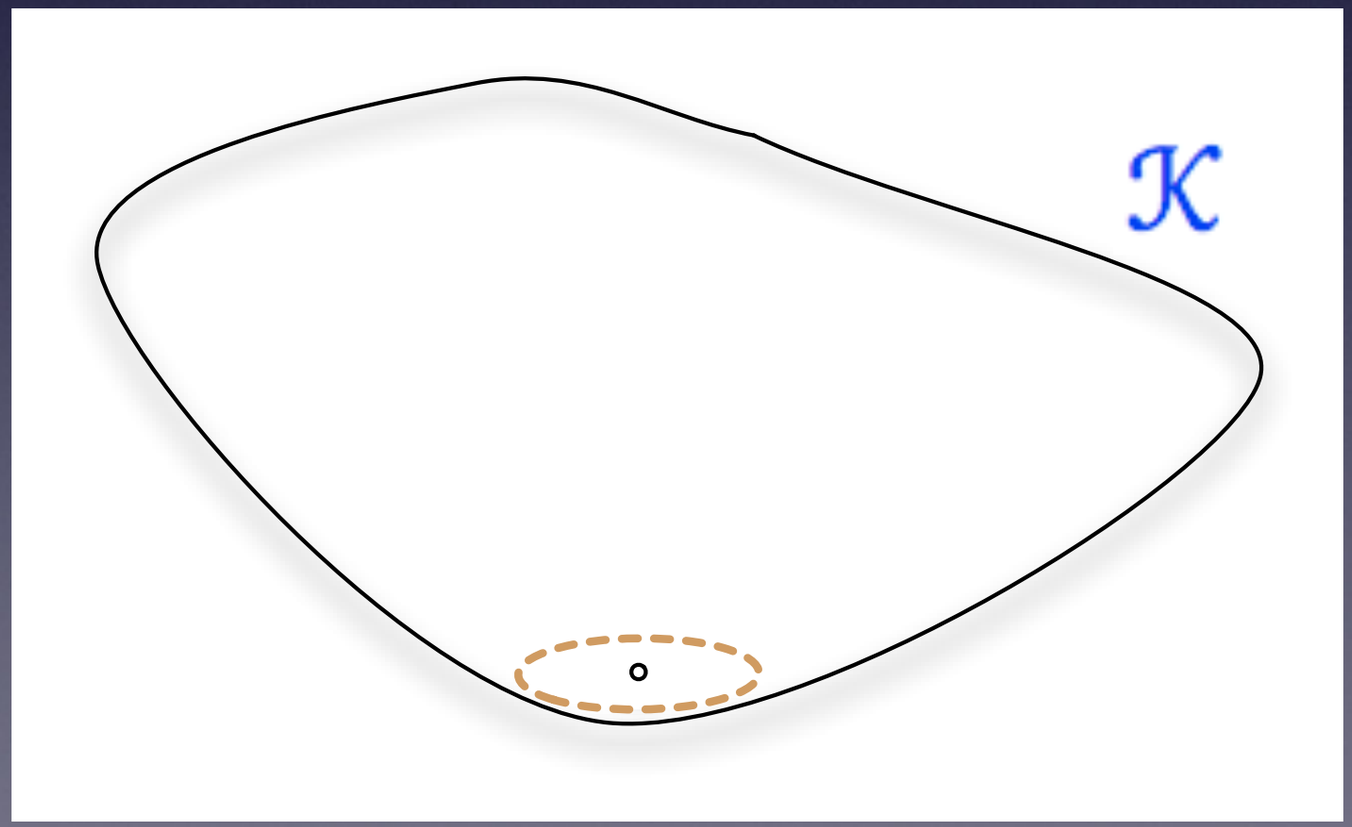

A Bonus Prize:
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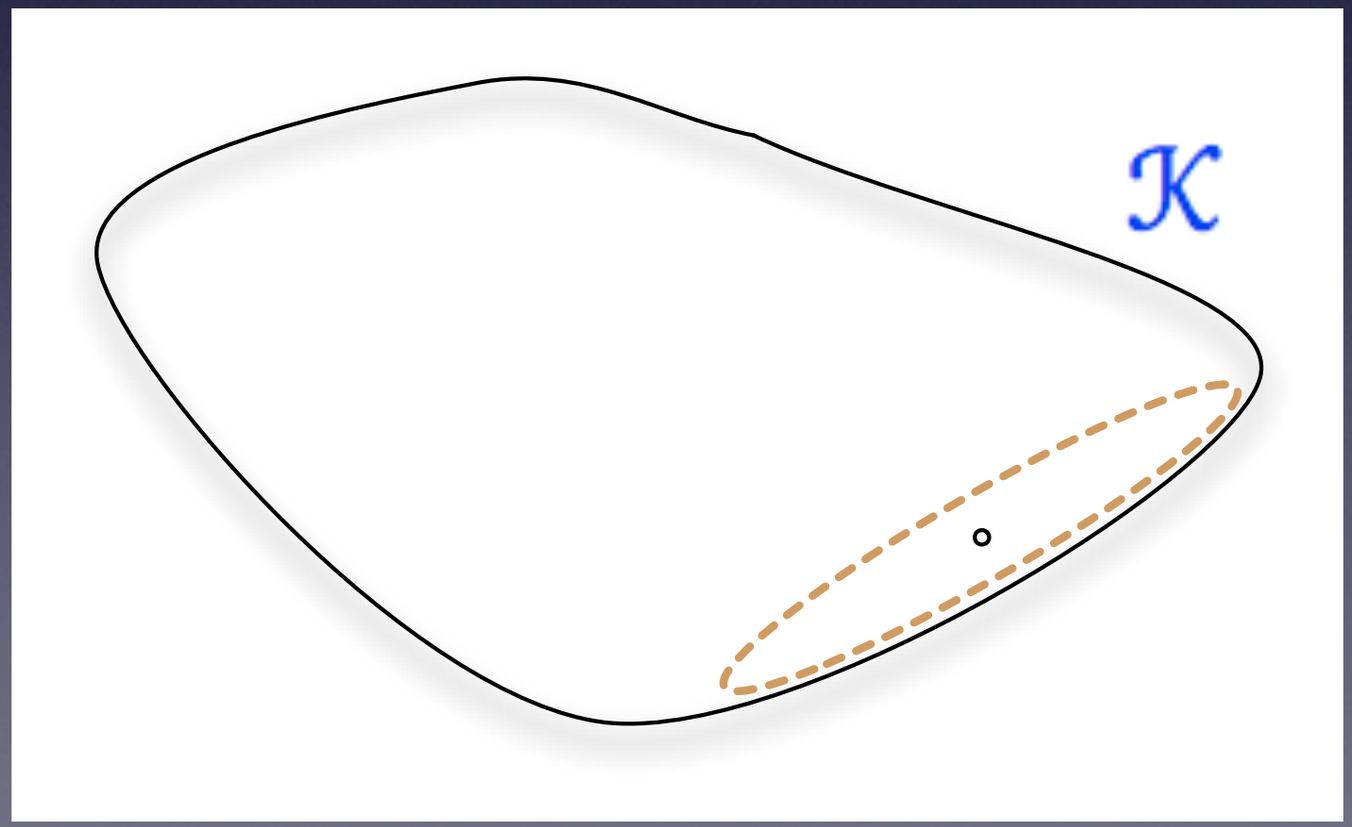

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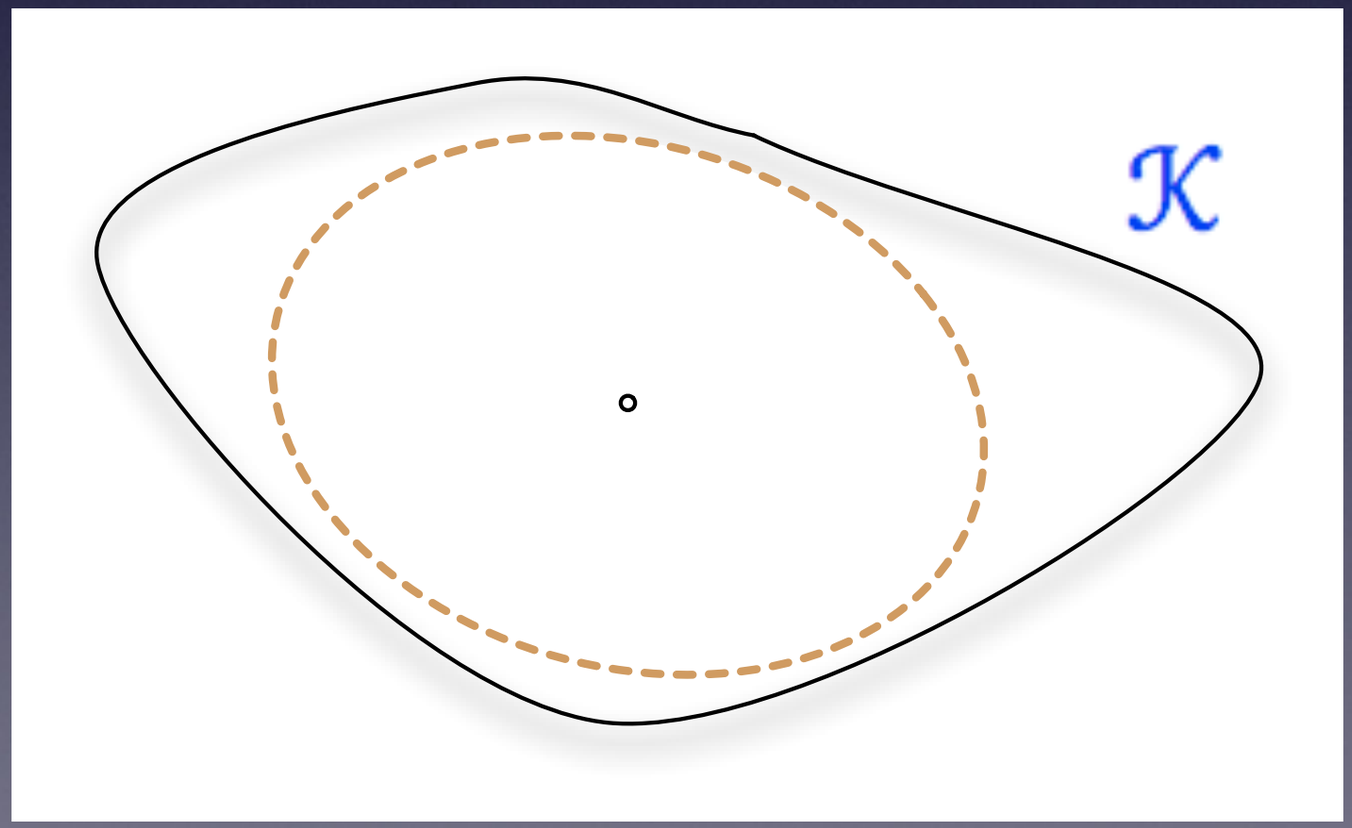

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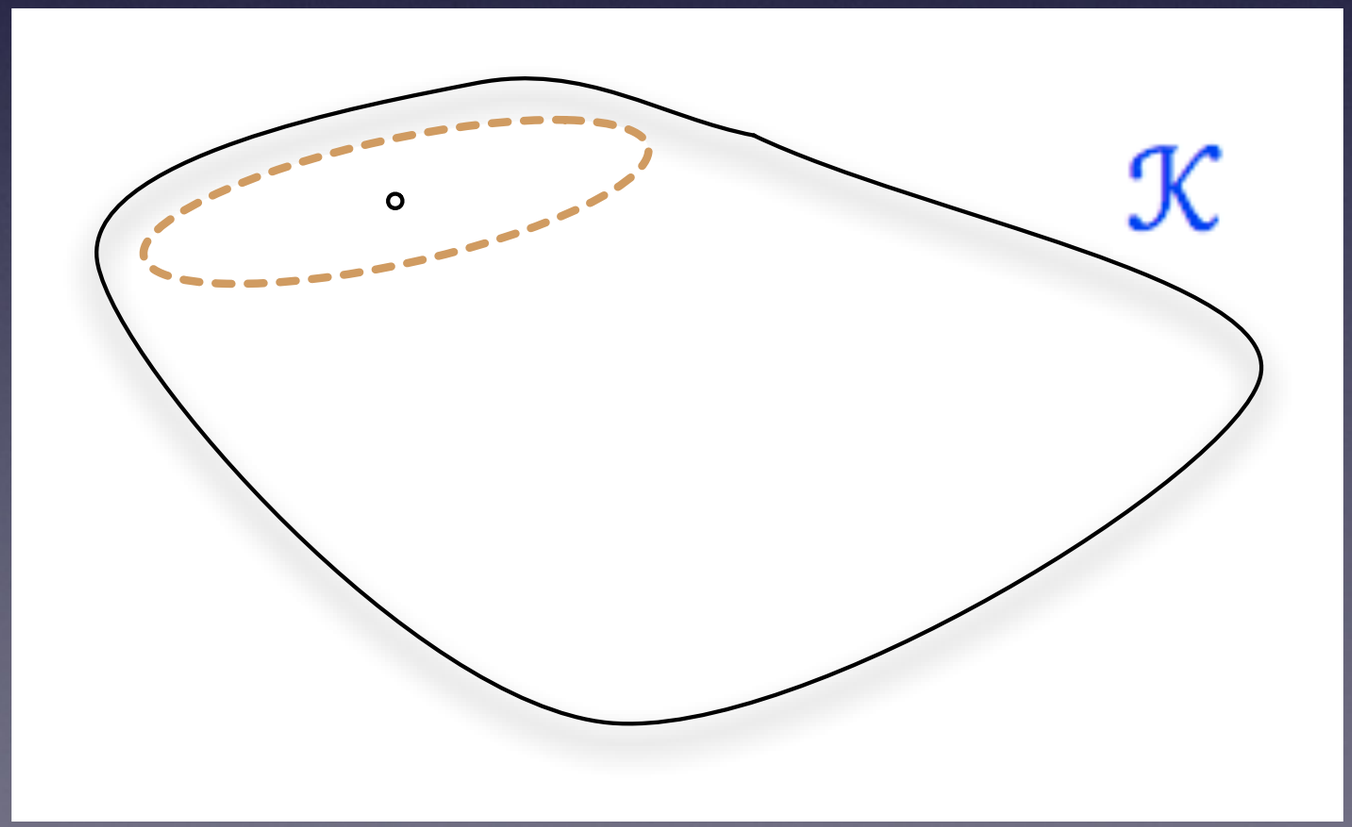

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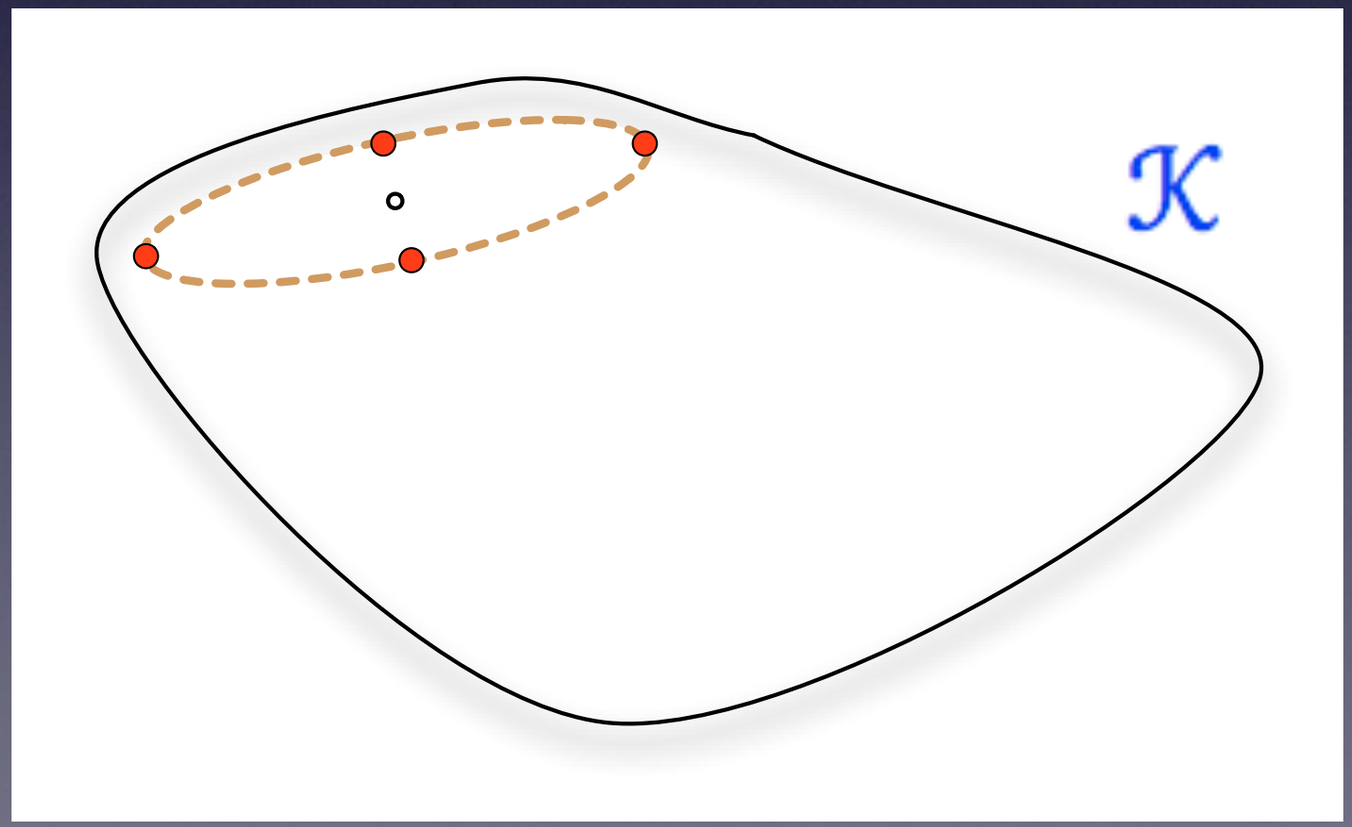

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Orig. Objective: $\min_{x \in D} g(x)$

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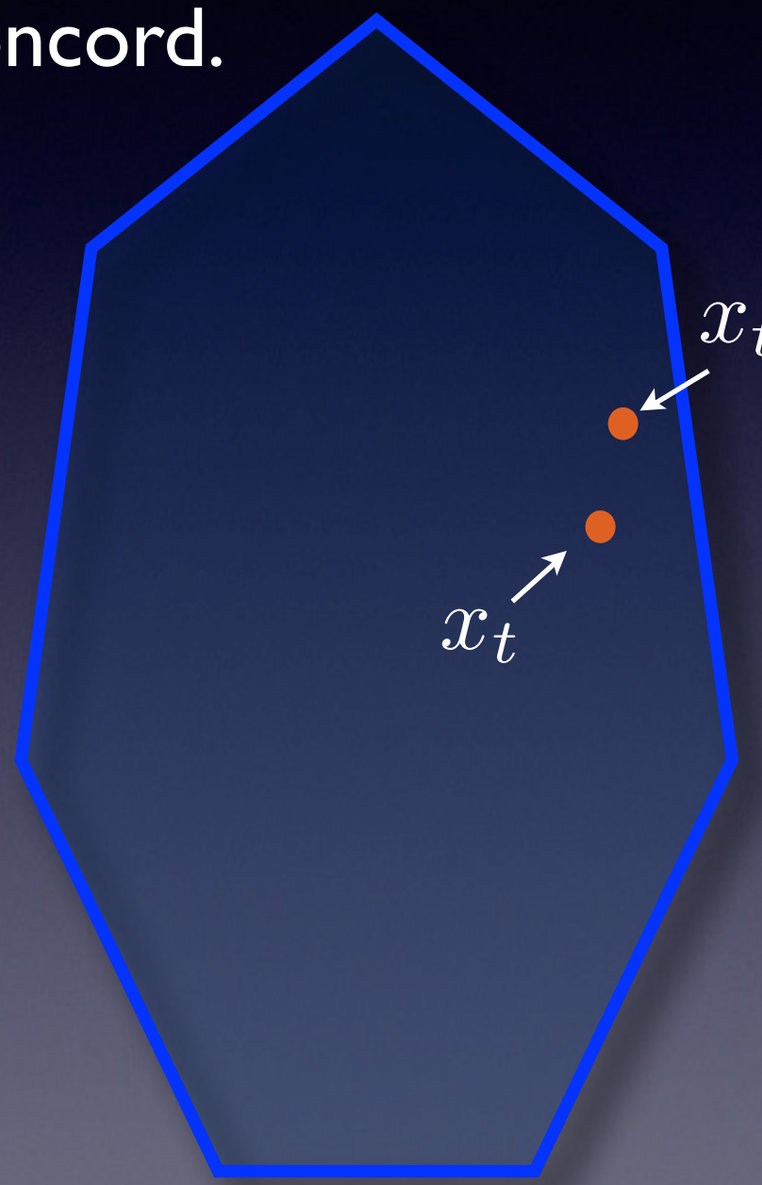

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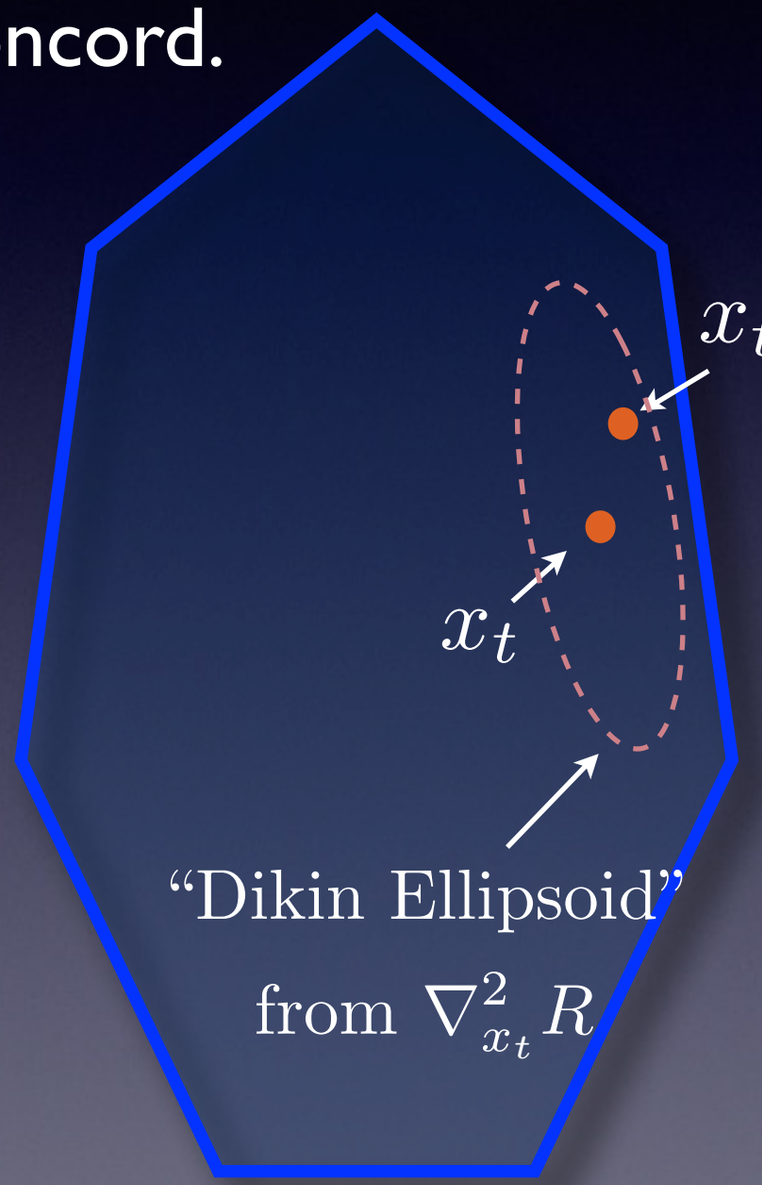

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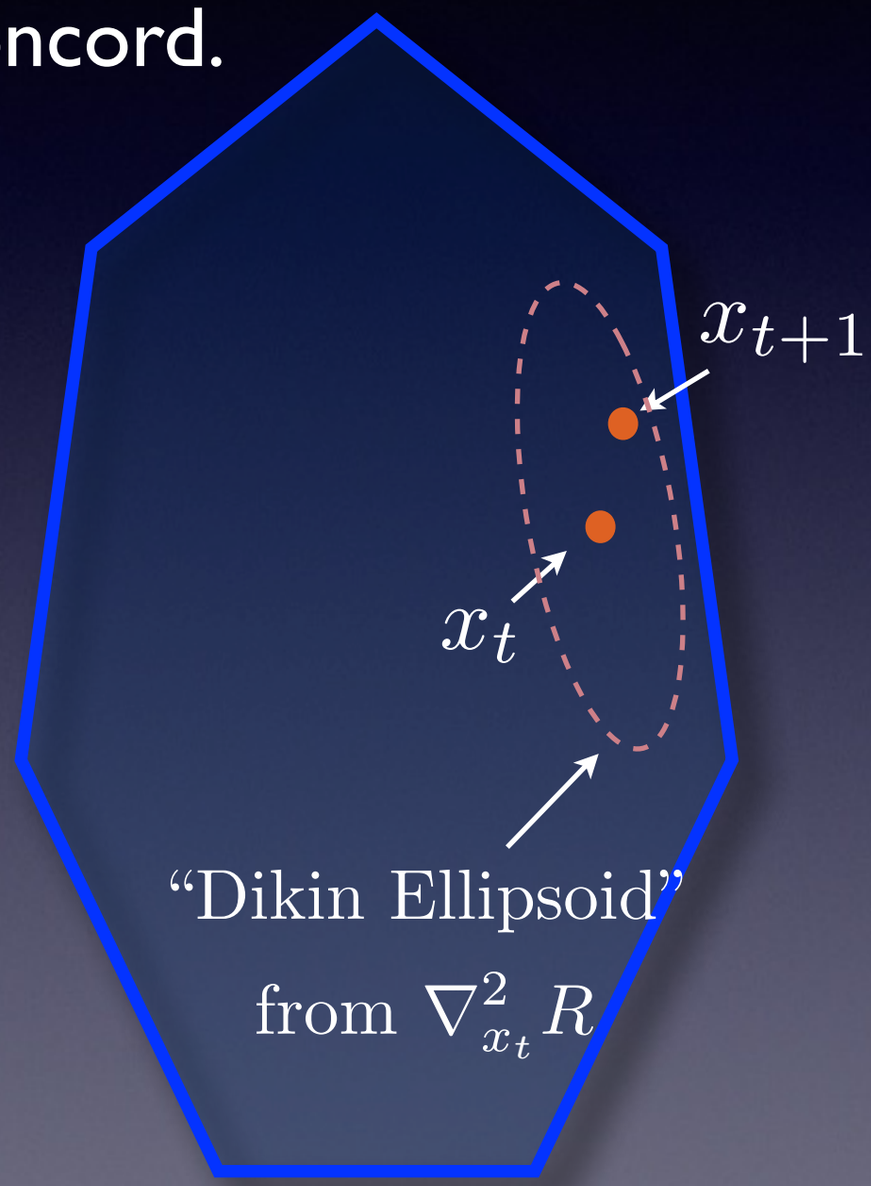
Important Fact:

$$x_{t+1} \in (\nabla^2_{x_t} R)\text{-ellipsoid} \subset D$$



Back to Bandit Optimization: Estimation Troubles

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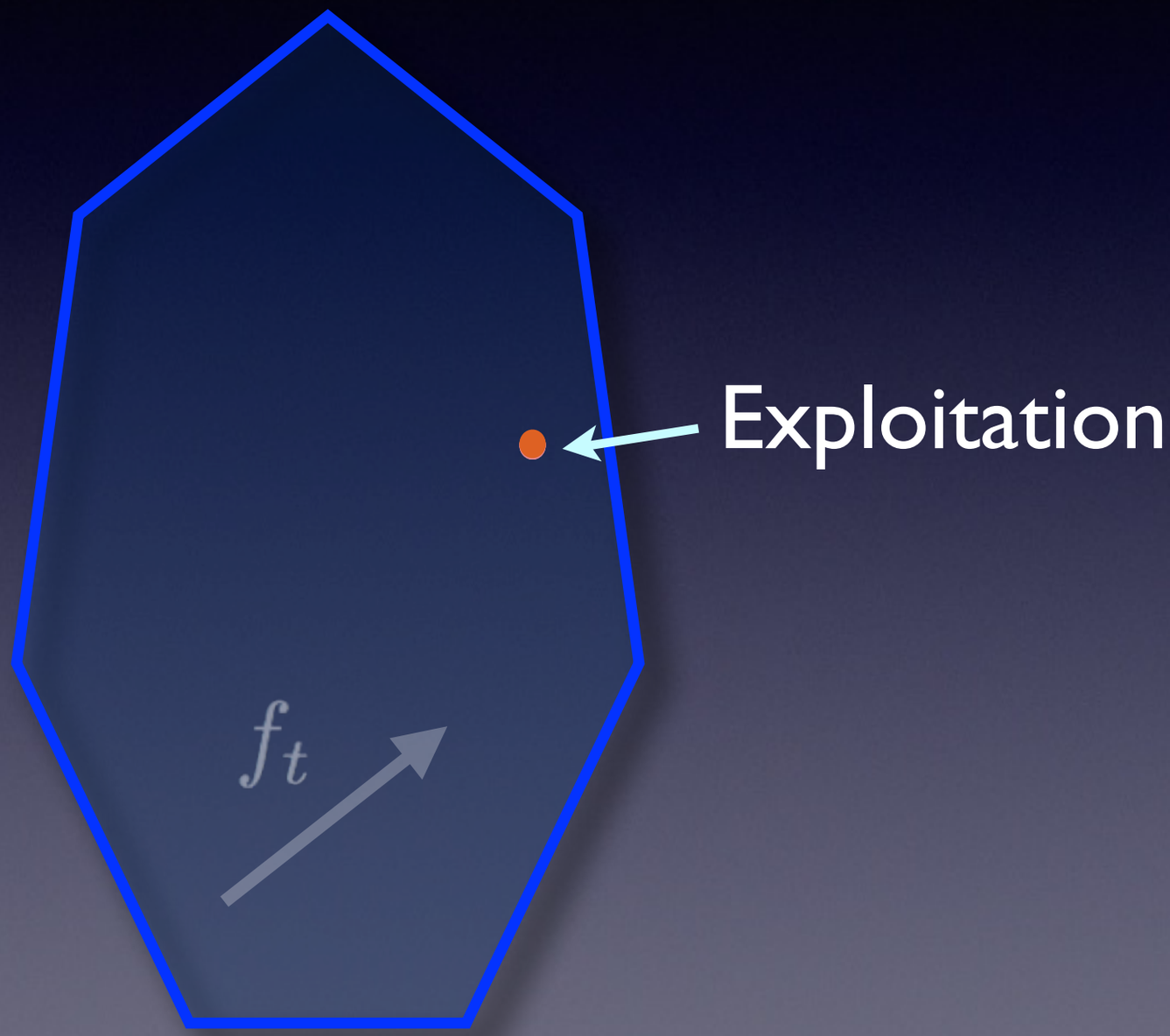

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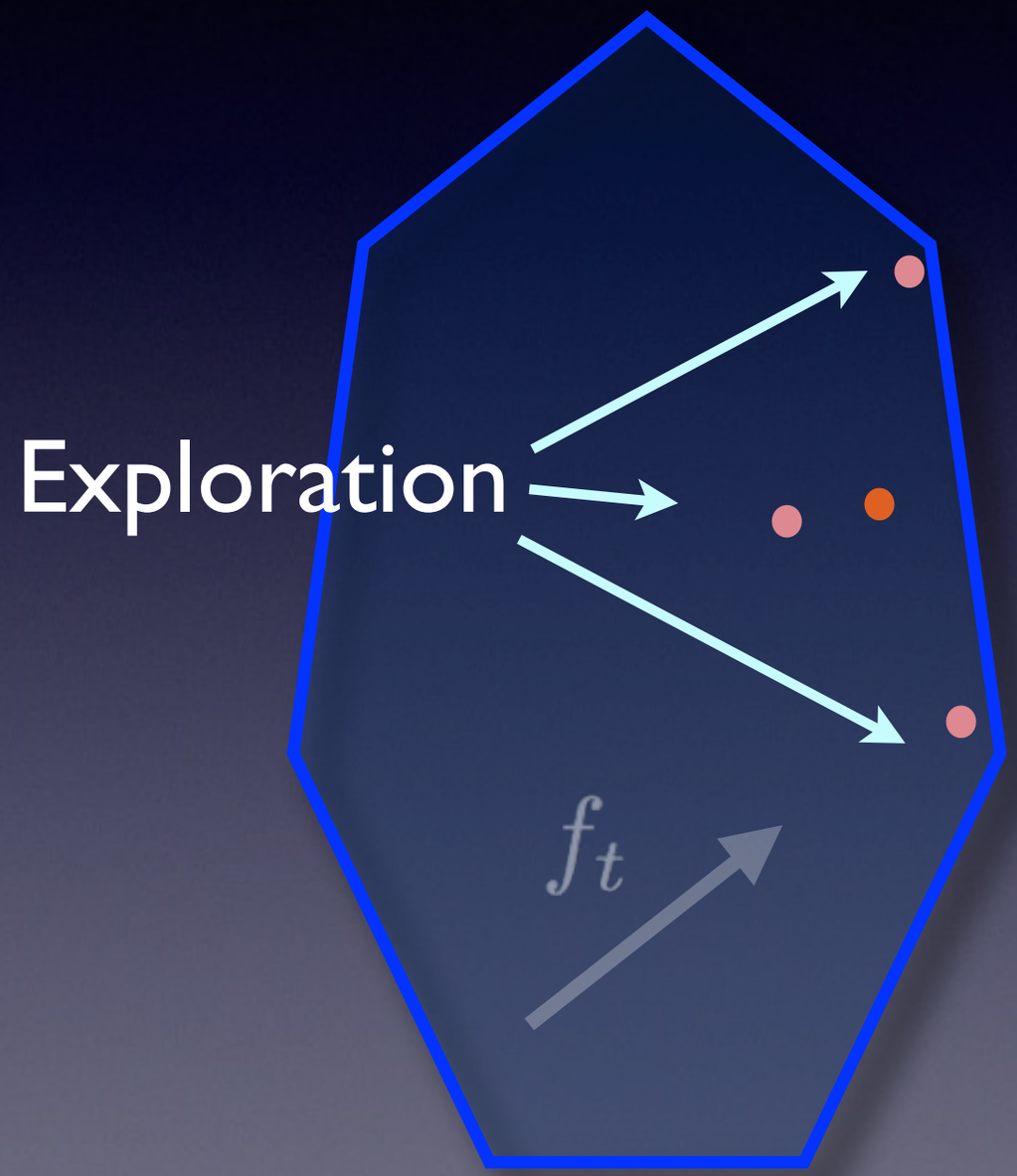

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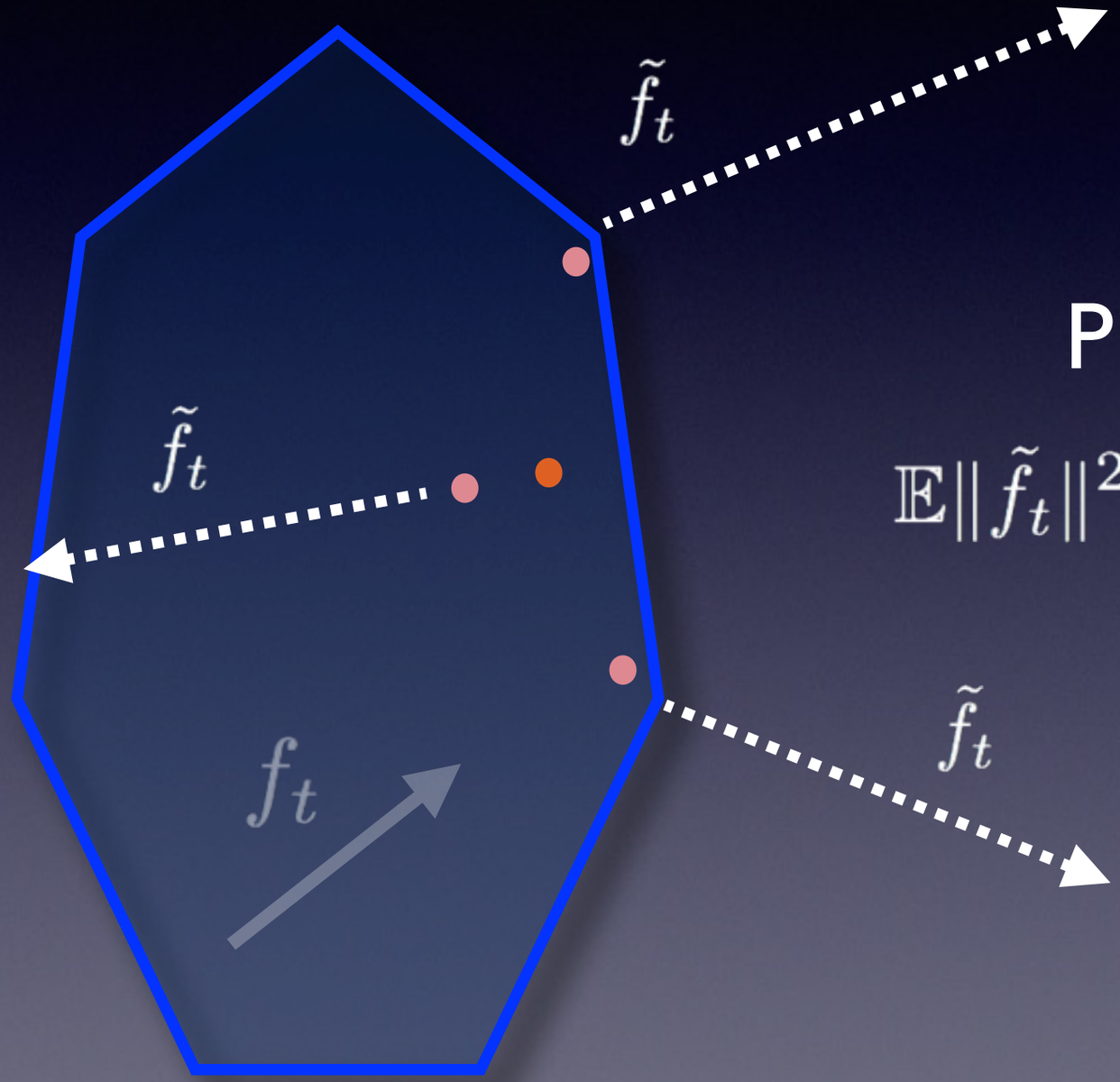


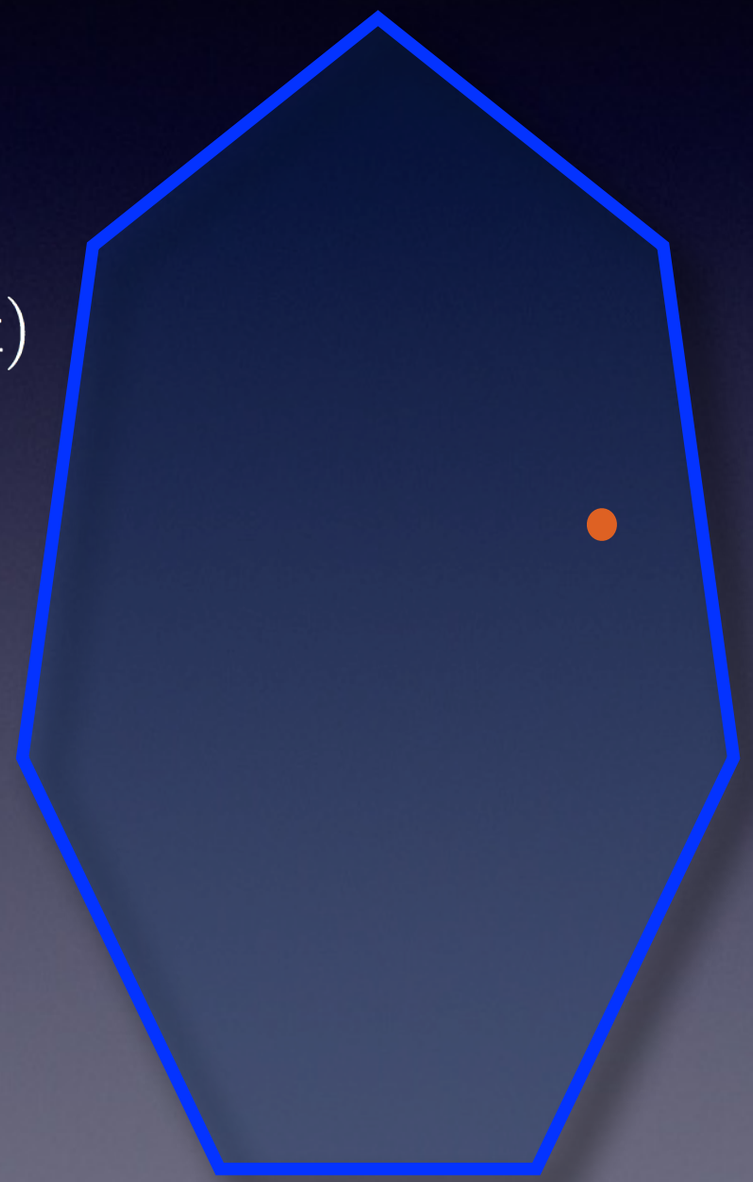
Problem:

$$\mathbb{E}\|\tilde{f}_t\|^2 \approx \frac{1}{\mathrm{dist}(x_t, \mathrm{bndry})}$$

Solution: Regularize w/ Self-concordant Function

R self-concordant barrier fun. on $\mathcal{K}$

Update: $\mathbf{x}_t := \arg\min_{\mathbf{x}\in\mathcal{K}} \sum_{s=1}^{t-1} \mathbf{f}_s \cdot \mathbf{x} + \lambda R(\mathbf{x})$



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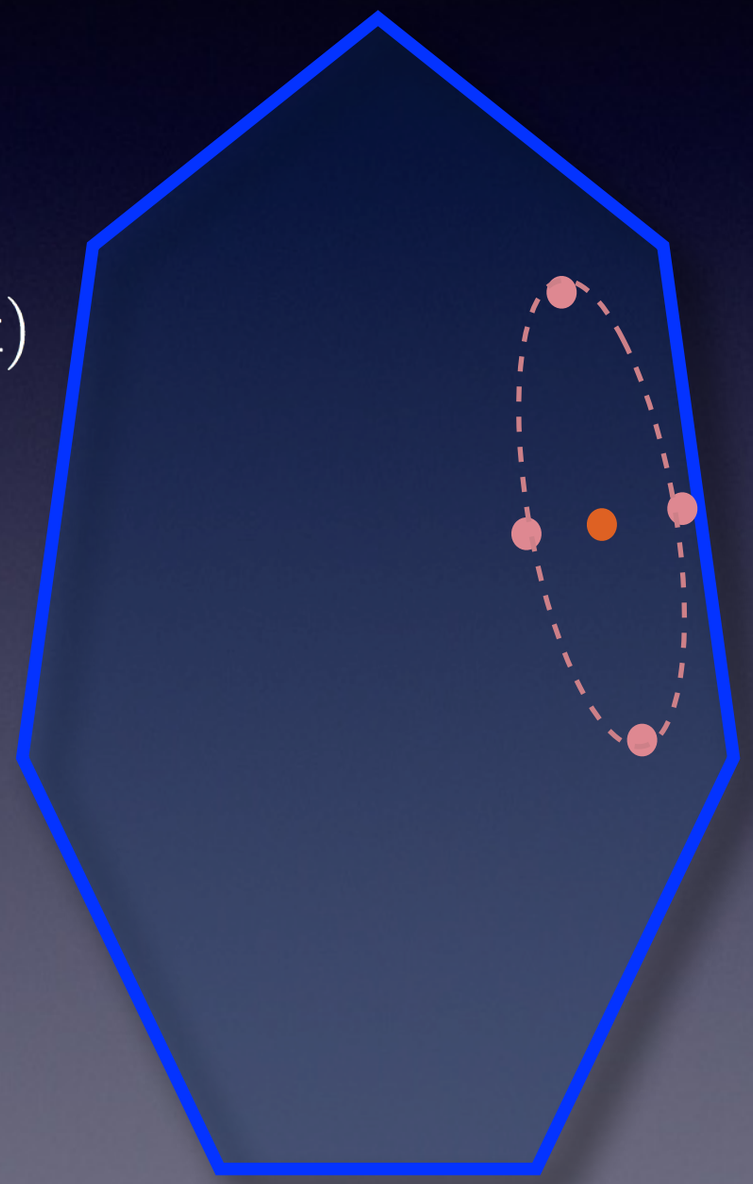
eigenpoles of $(\nabla^2_{x_t} R)$ – ellipsoid



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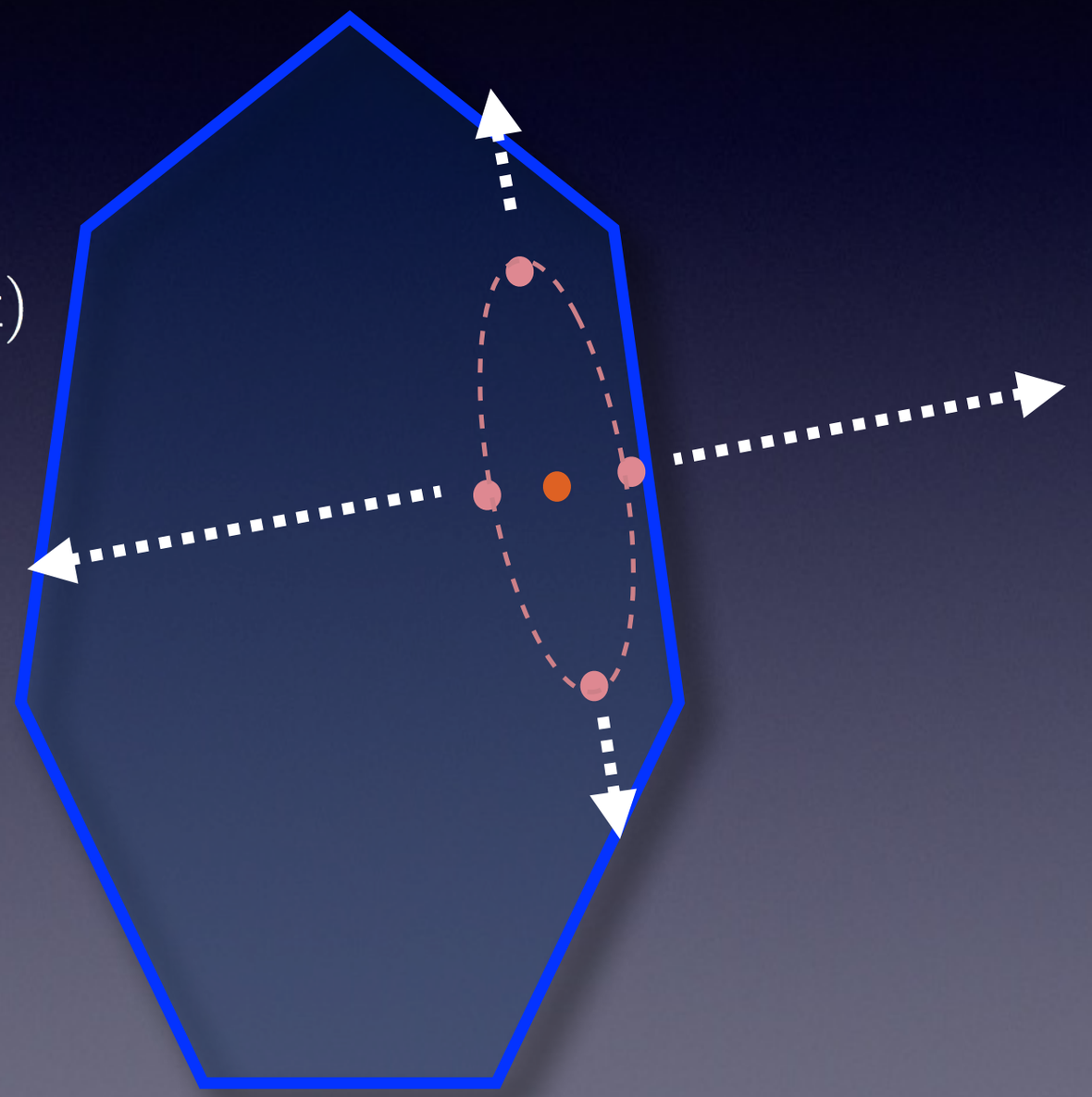
Estimate sizes: $\tilde{f}_t \approx \lambda_i^{1/2} e_i$

$\{\lambda_j\}$ eigenvals of $\nabla^2_{x_t} R$

$\{e_j\}$ unit eigenvecs of $\nabla^2_{x_t} R$



FTRL Regret Bound

$$\text{Regret}_T \leq \sum_{t=1}^{T} \lambda D_R(\mathbf{x}_t, \mathbf{x}_{t+1}) + \lambda R(\mathbf{x}^*)$$

$$\leq \sum_{t=1}^{T} \frac{\|\tilde{\mathbf{f}}_t\|_*^2}{\lambda} + \lambda R(\mathbf{x}^*)$$

$$\leq \frac{T \cdot G}{\lambda} + \lambda D \leq 2\sqrt{T \cdot G \cdot D}$$

FTRL Regret Bound

$$\text{Regret}_T \leq \sum_{t=1}^{T} \lambda D_R(\mathbf{x}_t, \mathbf{x}_{t+1}) + \lambda R(\mathbf{x}^*)$$

Grows large!

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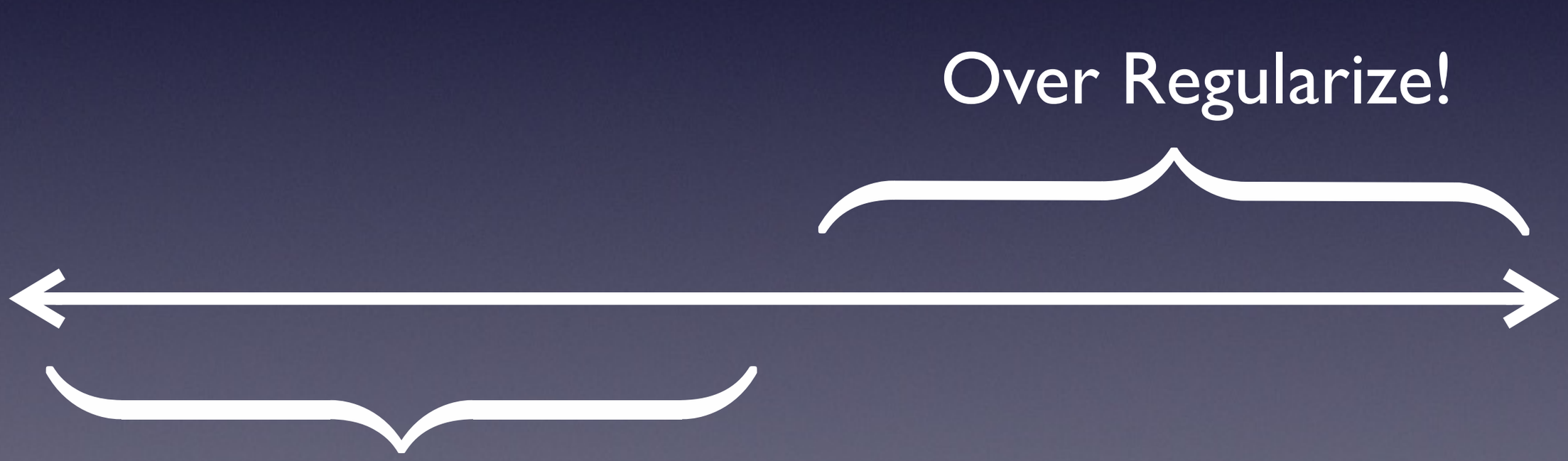
FTRL Regret Bound

$$\begin{aligned}
\mathrm{Regret}_T &\leq \sum_{t=1}^{T} \lambda D_R(\mathbf{x}_t, \mathbf{x}_{t+1}) + \lambda R(\mathbf{x}^*) \\
&\leq \sum_{t=1}^{T} \frac{\tilde{\mathbf{f}}_t^\top (\nabla_{\mathbf{x}_t}^{-2} R)\, \tilde{\mathbf{f}}_t}{\lambda} + \lambda R(\mathbf{x}^*) \\
&\leq \sum_{t=1}^{T} \frac{nG\sqrt{\sigma_{i_t}}\sigma_{i_t}^{-1}\sqrt{\sigma_{i_t}}}{\lambda} + \lambda D\theta \log T \\
&\leq 2\sqrt{n \cdot G \cdot D \cdot \theta \cdot T \log T}
\end{aligned}$$

Little Flexibility in Selecting R

X-axis: how fast does $\|\nabla^2 R\|$ grow w.r.t. 1/dist-bndry?



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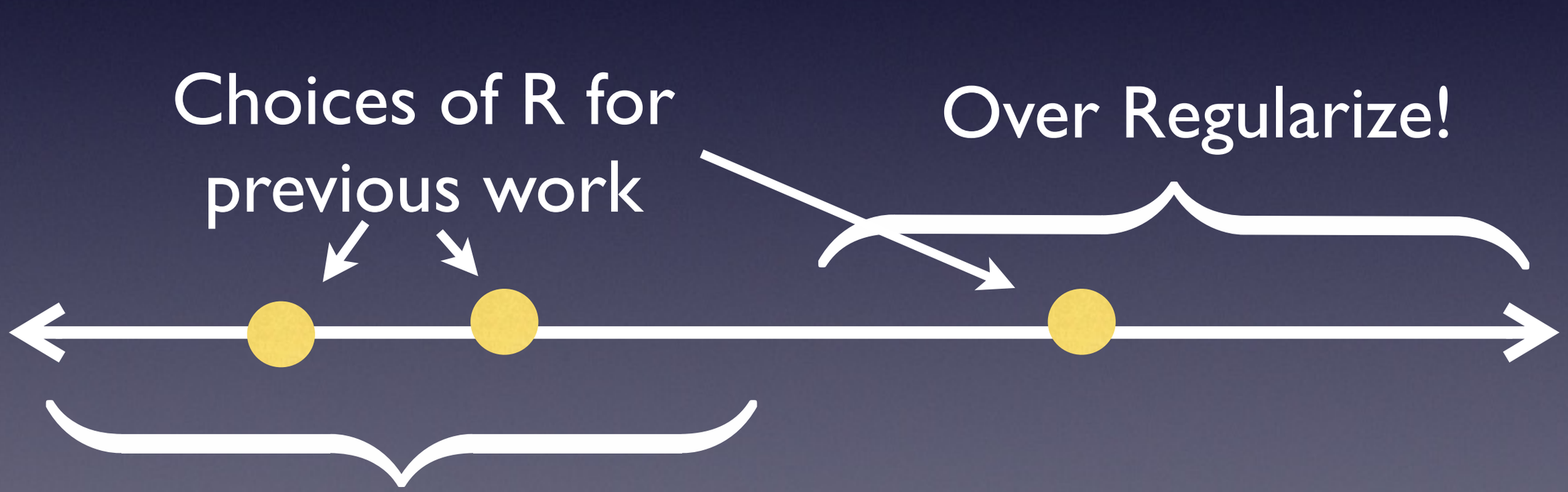

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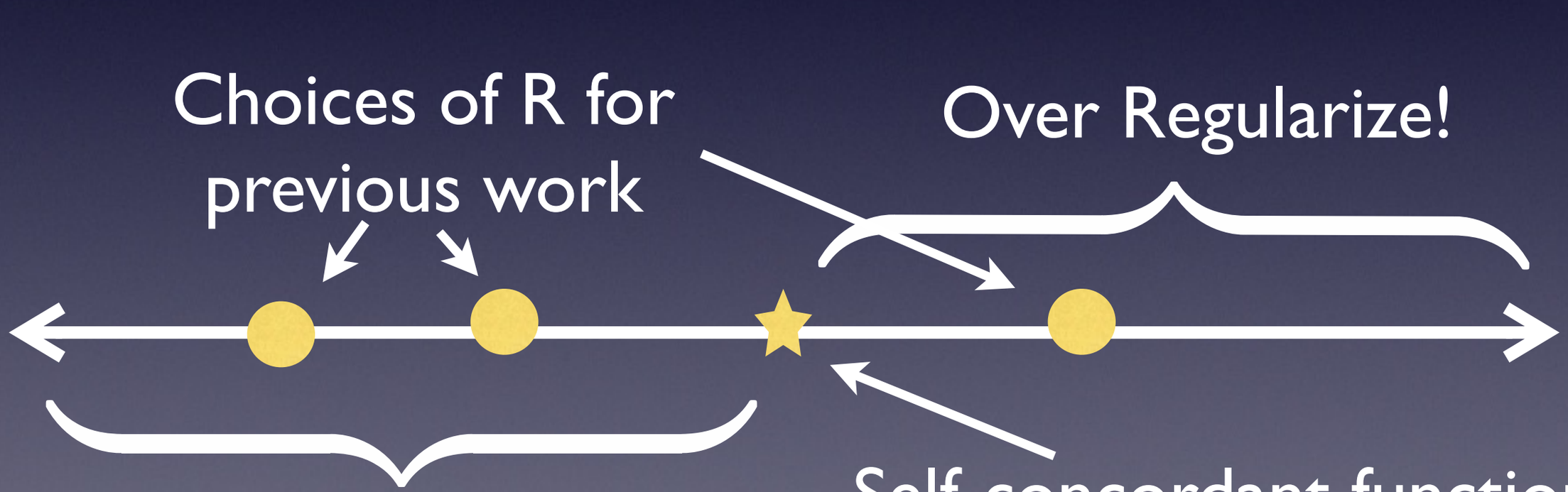

High Probability and Adaptive Adversaries

- These results are "in expectation" only! And only work against "oblivious adversaries"
- Achieving bounds with **high probability** and against **adaptive adversaries** is much more challenging
- We've made some progress on this (Abernethy/Rakhlin COLT 2009)
- Still, the general problem is hard...

Open Problem 1

- For the full-information setting, several "online learning games" have been fully solved. Minimax solution is very nice.
- Can the same be achieved for bandit setting?

Open Problem 2

- Another problem, which *looks* like Online Convex Opt, is the bandit multiclass setting
- On each round, a Learner sees an example and predicts a class. Told "right" or "wrong"
- Current best algorithms get $O(T^{2/3})$ regret
- Should be an efficient method for $O(T^{1/2})$

Thanks!