SHIVAKANTH THUDI

HW 5 - ROBOT EM

$$p(z_i = 0) = \pi_0$$

$$p(x_i \mid z_i = 1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ - \frac{(x_i - \mu)^2}{2\sigma^2} \right\}$$

$$p(x_i \mid z_i = 0) = \frac{1}{5} \mathbb{1}\{0 \le x_i \le 5\}$$

1)

Marginal log-likelihood is :

$$\boxed{\begin{aligned} &\log p(x_1, x_2, \ldots\ldots, x_n \mid \mu, \sigma^2, \pi_0) \\ &= \sum_{i=1}^{n} \log\left(\frac{\pi_0}{5} \cdot \mathbb{1}\{0 \le x_i \le 5\} + \frac{1-\pi_0}{\sqrt{2\pi}\,\sigma} \exp\left\{ - \frac{(x_i - \mu)^2}{2\sigma^2} \right\} \right) \end{aligned}}$$

2) E-Step → Posterior Probabilities

$$p(z_i = 0 \mid x_i, \mu, \sigma^2, \pi_0)$$

$$= \frac{p(x_i \mid z_i = 0)\, p(z_i = 0)}{\sum_z p(x_i \mid z_i)\, p(z_i)}$$

$$= \boxed{\frac{\frac{\pi_0}{5} \mathbb{1}(0 \leq x_i \leq 5)}{\frac{\pi_0}{5} \mathbb{1}(0 \leq x_i \leq 5) + \frac{1-\pi_0}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x_i-\mu)^2}{2\sigma^2} \right\}}}$$

3) M step for μ

$$\frac{\partial}{\partial \mu} \log p(x_1, x_2, \ldots x_n \mid \mu, \sigma^2, \pi_0) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\frac{1-\pi_0}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{-(x_i-\mu)^2}{2\sigma^2}\right\} \cdot \frac{2(x_i-\mu)}{2\sigma^2}}{\frac{\pi_0}{5} \mathbb{1}\{0 \le x_i \le 5\} + \frac{1-\pi_0}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{-(x_i-\mu)^2}{2\sigma^2}\right\}} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\left(1 - p(z_i = 0 \mid x_i, \sigma^2, \pi_0)\right)(x_i - \mu)}{\sigma^2} = 0$$

$$\Rightarrow \boxed{\mu = \frac{\sum_{i=1}^{n} \left(1 - p(z_i=0 \mid x_i)\right) x_i}{\sum_{i=1}^{n} \left(1 - p(z_i = 0 \mid x_i)\right)}}$$

4) M step for π_0

$$\frac{\partial}{\partial \pi_0} \log p(x_1, x_2, \ldots x_n | \mu, \sigma^2, \pi_0) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\frac{1}{5}\mathbb{1}\{0 \le x_i \le 5\} - \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}}{\frac{\pi_0}{5}\mathbb{1}\{0 \le x_i \le 5\} + \frac{1-\pi_0}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}} = 0$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\frac{\pi_0(1-\pi_0)}{5}\mathbb{1}\{0 \le x_i \le 5\} - \frac{\pi_0(1-\pi_0)}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\}}{\frac{\pi_0}{5}\mathbb{1}\{0 \le x_i \le 5\} + \frac{1-\pi_0}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}$$

$$\Rightarrow \sum_{i=1}^{n} (1-\pi_0)\, p(z_i=0|x_i) - \sum_{i=1}^{n} \pi_0\, p(z_i=1|x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} p(z_i=0|x_i) = \pi_0 \left[\sum_{i=1}^{n} \big(p(z_i=1|x_i) + p(z_i=0|x_i)\big)\right]$$

$$(\equiv n * 1)$$

$$\Rightarrow \boxed{\pi_0 = \frac{1}{n}\sum_{i=1}^{n} p(z_i=0|x_i)}$$