# UNIT – 4 POSET and Lattices

## Partially Ordered Sets

Consider a relation R on a set S satisfying the following properties:

1. R is reflexive, i.e., xRx for every $x \in S$.
2. R is antisymmetric, i.e., if xRy and yRx, then $x = y$.
3. R is transitive, i.e., xRy and yRz, then xRz.

Then R is called a partial order relation, and the set S together with partial order is called a partially order set or POSET and is denoted by $(S, \leq)$.

Example:

1. The set N of natural numbers form a poset under the relation '≤' because firstly $x \leq x$, secondly, if $x \leq y$ and $y \leq x$, then we have $x = y$ and lastly if $x \leq y$ and $y \leq z$, it implies $x \leq z$ for all $x, y, z \in N$.
2. The set N of natural numbers under divisibility i.e., 'x divides y' forms a poset because x/x for every $x \in N$. Also if x/y and y/x, we have $x = y$. Again if x/y, y/z we have x/z, for every $x, y, z \in N$.
3. Consider a set S = {1, 2} and power set of S is P(S). The relation of set inclusion $\subseteq$ is a partial order. Since, for any sets A, B, C in P (S), firstly we have $A \subseteq A$, secondly, if $A \subseteq B$ and $B \subseteq A$, then we have $A = B$. Lastly, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Hence, $(P(S), \subseteq)$ is a poset.
4. A= {1,2,3,4,6,12,36,48} Relation " divides"
   A is poset, Divides is partial order relation

**Hasse diagram**: A **Hasse diagram** is a *graphical representation* of the relation of elements of a **partially ordered set (poset)** with an implied *upward orientation*.

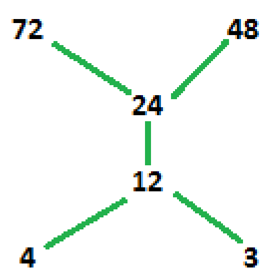
A point is drawn for each element of the partially ordered set (poset) and joined with the line segment according to the following rules:

- If $p<q$ in the poset, then the point corresponding to *p appears lower* in the drawing than the point corresponding to q.
- The two points p and q will be joined by line segment *iff p is related to q*.

Few examples of Hasse Diagram are given below:

**Example-1:** Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, /)

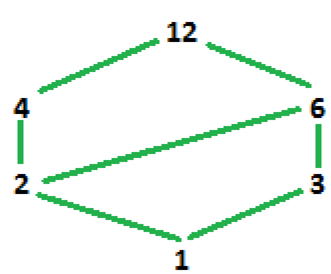
**Explanation –** According to above given question first, we have to find the poset for the divisibility.
Let the set is A.
A={(3 12), (3 24), (3 48), (3 72), (4 12), (4 24), (4 48), (4 72), (12 24), (12 48), (12 72), (24 48), (24 72)}
So, now the Hasse diagram will be:




In above diagram, 3 and 4 are at same level because they are not related to each other and they are smaller than other elements in the set. The next succeeding element for 3 and 4 is 12 i.e, 12 is divisible by both 3 and 4. Then 24 is divisible by 3, 4 and 12. Hence, it is placed above 12. 24 divides both 48 and 72 but 48 does not divide 72. Hence 48 and 72 are not joined.
We can see transitivity in our diagram as the level is increasing.

**Example-2:** Draw Hasse diagram for (D , /)

**Explanation –** Here, D means set of positive integers divisors of 12.
So, D ={1, 2, 3, 4, 6, 12} now the Hasse diagram will be-



In above diagram, 1 is the only element that divides all other elements and smallest. Hence, it is placed at the bottom. Then the elements in our set are 2 and 3 which do not divide each other hence they are placed at same level separately but divisible by 1 (both joined by 1). 4 is divisible by 1 and 2 while 6 is divisible by 1, 2 and 3 hence, 4 is joined by 2 and 6 is joined by 2 and 3. 12 is divisible by all the elements hence, joined by 4 and 6 not by all elements because we have already joined 4 and 6 with smaller elements accordingly.

For regular Hasse Diagram:

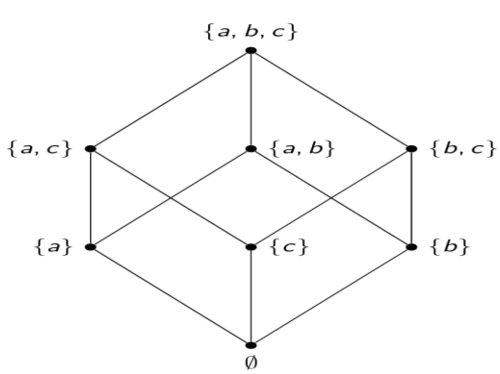
- **Maximal elements** are those which are not succeeded by another element.
- **Minimal elements** are those which are not preceded by another element.
- **Greatest element** (if it exists) is the element succeeding all other elements.
- **Least element** is the element that precedes all other elements.

**Note –** Greatest and Least element in Hasse diagram are only one.

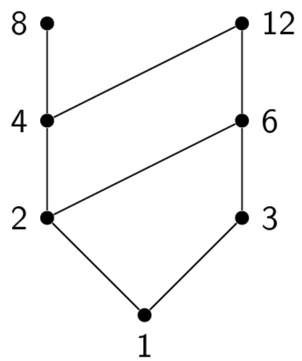
***In Example-1,***
Maximal elements are 48 and 72 since they are succeeding all the elements.
Minimal elements are 3 and 4 since they are preceding all the elements.
Greatest element does not exist since there is no any one element that succeeds all the elements.
Least element does not exist since there is no any one element that precedes all the elements.

***In Example-2,***
Maximal and Greatest element is 12 and Minimal and Least element is 1.

**Example-3 : <P(S) ,** ⊆ > where S = {a,b,c}



**Example-4 : Poset ({1,2,3,4,6,8,12}, D)**




**Least Upper Bound (SUPREMUM):**

Let A be a subset of a partially ordered set S. An element M in S is called an upper bound of A if M succeeds every element of A, i.e. if, for every x in A, we have x ≤ M

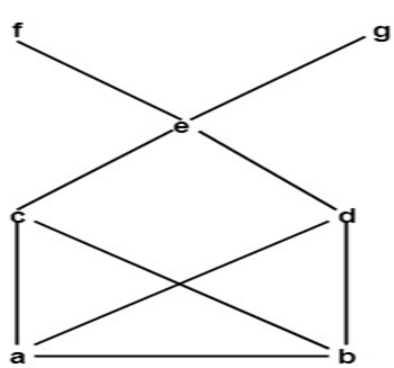
If an upper bound of A precedes every other upper bound of A, then it is called the supremum of A and is denoted by Sup (A)

**Greatest Lower Bound (INFIMUM):**

An element m in a poset S is called a lower bound of a subset A of S if m precedes every element of A, i.e. if, for every y in A, we have m ≤ y

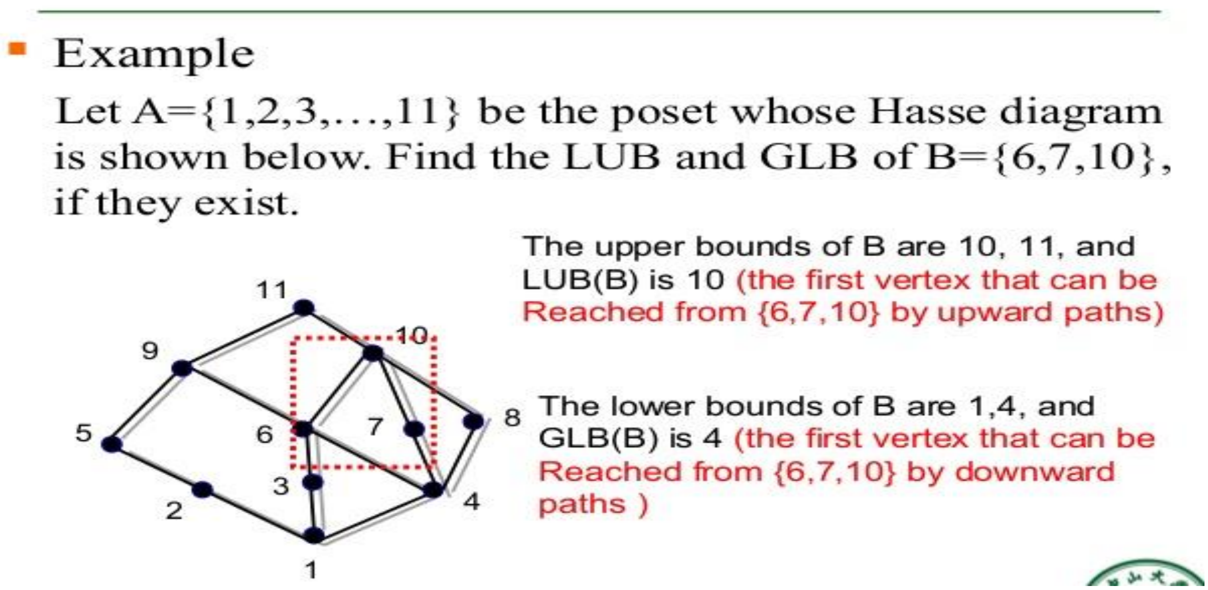
If a lower bound of A succeeds every other lower bound of A, then it is called the infimum of A and is denoted by Inf (A)

**Example:** Consider the poset A = {a, b, c, d, e, f, g} be ordered shown in fig. Also let B = {c, d, e}. Determine the upper and lower bound of B.

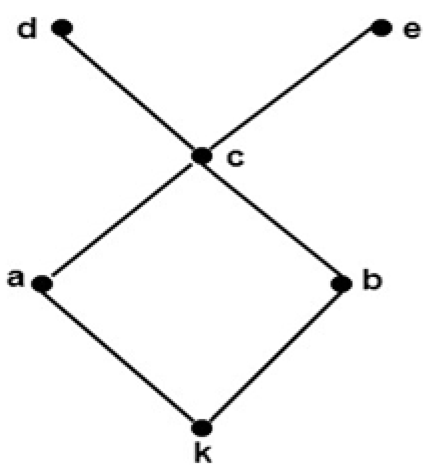


The upper bound of B is e, f, and g because every element of B is '≤' e, f, and g.

The lower bounds of B are a and b because a and b are '≤' every elements of B.




**Example:** Determine the least upper bound and greatest lower bound of B = {a, b, c} if they exist, of the poset whose Hasse diagram is shown in fig:




**Solution:** The least upper bound is c.

The greatest lower bound is k.

**Lattices:**

**Lattices –** A Poset in which every pair of elements has both, a least upper bound and a greatest lower bound is called a lattice.
There are two binary operations defined for lattices –

1. **Join –** The join of two elements is their least upper bound. It is denoted by V.

   **LUB(a,b) = LUB(a,b) = a V b = a $\oplus$ $\boldsymbol{b}$**

2. **Meet –** The meet of two elements is their greatest lower bound. It is denoted by ^.

   **GLB(a,b) = GCD(a,b) = a ^ b = a $*$ $\boldsymbol{b}$**

**Properties:**

Let L be a non-empty set closed under two binary operations called meet and join, denoted by ∧ and ∨. Then L is called a lattice if the following axioms hold where a, b, c are elements in L:

**1) Idempotent laws:**

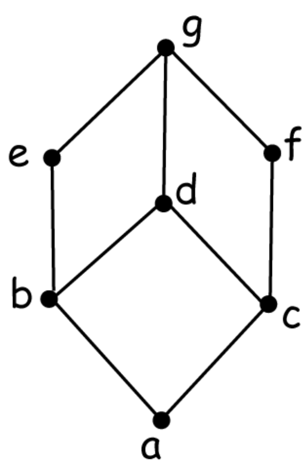
(a) $a \wedge a = a$ (b) $a \vee a = a$

**2) Commutative Law: -**
(a) $a \wedge b = b \wedge a$ (b) $a \vee b = b \vee a$

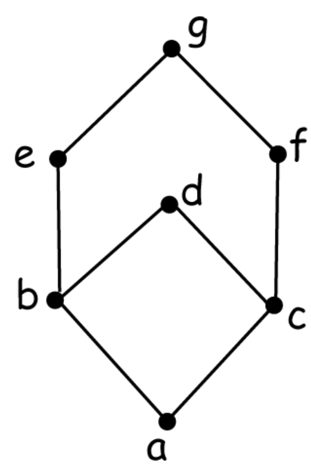
**3) Associative Law:-**
(a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (b) $(a \vee b) \vee c = a \vee (b \vee c)$

**4) Absorption Law: -**
(a) $a \wedge (a \vee b) = a$ (b) $a \vee (a \wedge b) = a$

**Which of the given Hasse diagram are lattices**

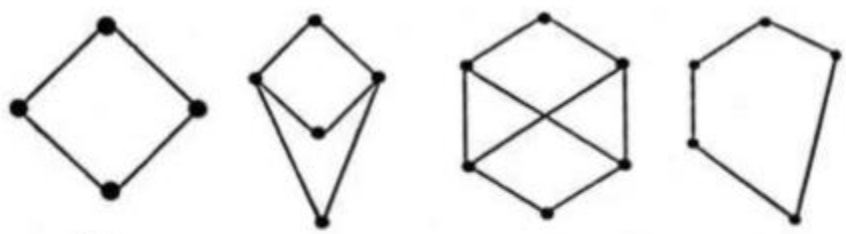



Lattice




Not a Lattice

Consider the following four Hasse diagrams of posets:



(i) (ii) (iii) (iv)

Which of these posets correspond to lattices? (Select as many as apply.)

Answer : (i) and (iv)

**Example : Prove that <P(S) , ⊆ > where S = {a,b,} is a lattice**

| $\wedge$ | $\phi$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ |
|---|---|---|---|---|
| $\phi$ | $\phi$ | $\phi$ | $\phi$ | $\phi$ |
| $a$ | $\phi$ | $\{a\}$ | $\phi$ | $\{a\}$ |
| $b$ | $\phi$ | $\phi$ | $\{b\}$ | $\{b\}$ |
| $\{a,b\}$ | $\phi$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ |

Table-1: Composition table for meet

| $\vee$ | $\phi$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ |
|---|---|---|---|---|
| $\phi$ | $\phi$ | $\{a\}$ | $\{b\}$ | $\{a,b\}$ |
| $a$ | $\{a\}$ | $\{a\}$ | $\{a,b\}$ | $\{a,b\}$ |
| $b$ | $\{b\}$ | $\{a,b\}$ | $\{b\}$ | $\{a,b\}$ |
| $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ |

Table-2: Composition table for join

For any pair of element GLB and LUB exists so <P(S) , ⊆ > is a lattice .

**Sub-Lattices:**

Consider a non-empty subset $L_1$ of a lattice L. Then $L_1$ is called a sub-lattice of L if $L_1$ itself is a lattice i.e., the operation of L i.e., $a \vee b \in L_1$ and $a \wedge b \in L_1$ whenever $a \in L_1$ and $b \in L_1$.

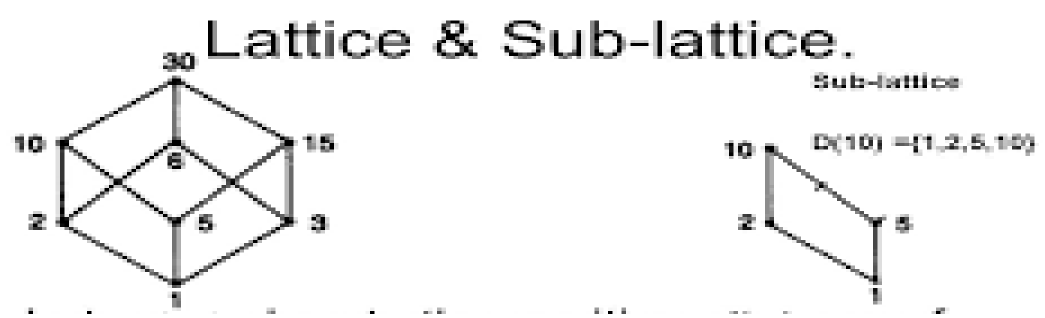
If we can find lub and glb of all the elements of posets than it is called as lattice and if they are not connected to each other then it is not called a sub-latice

**Example:** Consider the lattice of all +ve integers $I_+$ under the operation of divisibility. The lattice $D_n$ of all divisors of n > 1 is a sub-lattice of $I_+$.

Determine all the sub-lattices of $D_{30}$ that contain at least four elements, $D_{30}$={1,2,3,5,6,10,15,30}.

**Solution:** The sub-lattices of $D_{30}$ that contain at least four elements are as follows:

1. {1, 2, 6, 30} 2. {1, 2, 3, 30} 3. {1, 5, 15, 30} 4. {1, 3, 6, 30}
5. {1, 5, 10, 30} 6. {1, 3, 15, 30} 7. {2, 6, 10, 30}



**Types of Lattices:**

**Bounded Lattices:**

A lattice L is called a bounded lattice if it has greatest element 1 and a least element 0.

**Example:**

1. The power set P(S) of the set S under the operations of intersection and union is a bounded lattice

   since ∅ is the least element of P(S) and the set S is the greatest element of P(S).

2. The set of +ve integer $I_+$ under the usual order of $\leq$ is not a bounded lattice since it has a least element 1 but the greatest element does not exist.

**Properties of Bounded Lattices:**

If L is a bounded lattice, then for any element $a \in L$, we have the following identities:

1. $a \vee 1 = 1$
2. $a \wedge 1 = a$
3. $a \vee 0 = a$
4. $a \wedge 0 = 0$

**Distributive Lattice:**

A lattice L is called distributive lattice if for any elements a, b and c of L,it satisfies following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

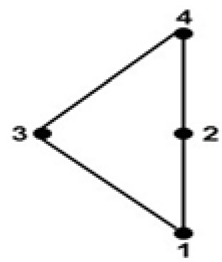
If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

**Example:**

1. The power set P (S) of the set S under the operation of intersection and union is a distributive function. Since,

   $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

   and, also $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ for any sets a, b and c of P(S).
2. The lattice shown in fig II is a distributive. Since, it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3, and 4.




II

$1 \wedge (2 \vee 3) = 1 \wedge 4 = 1$ LHS

$(1 \wedge 2) \vee (1 \wedge 3) = 1 \vee 1 = 1$ RHS

By principle of duality, $1 \vee (2 \wedge 3) = (1 \vee 2) \wedge (1 \vee 3)$

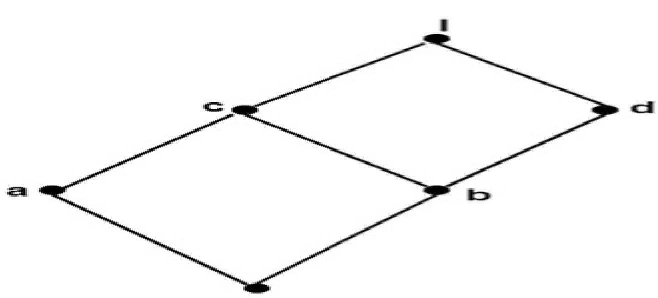
**Complements and complemented lattices:**

Let L be a bounded lattice with least element 0 and greatest element 1. Let x be an element if L. An element x’ in L is called a complement of x

if $x \vee x' = 1$ and $x \wedge x' = 0$

A lattice L is said to be complemented if L is bounded and every element in L has a complement.

**Example:** Determine the complement of a and c in fig:



**Solution:** The complement of a is d. Since, a ∨ d = 1 and a ∧ d = 0

The complement of c does not exist. Since, there does not exist any element c such that c ∨ c'=1 and c ∧ c'= 0.

***Find complement of each element of $D_{42}$***

*D*42={1,2,3,6,7,14,21,42}. The Hasse diagram is shown in fig. has the greatest element I=42 and least element O=1.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| X | 1 | 2 | 3 | 6 | 7 | 14 | 21 | 42 |
| X' | 42 | 21 | 14 | 7 | 6 | 3 | 2 | 1 |