Unit-4

Functions, Limits and Continuity of a Function

$F(X) = 5x+x^3$

$A=\{2,3,4,6,8\}$

$f(2) = 5(2) +(2)^3 = 10+8 = 18$

$f(3) = 5(3) +(3)^3 = 15+27= 42$

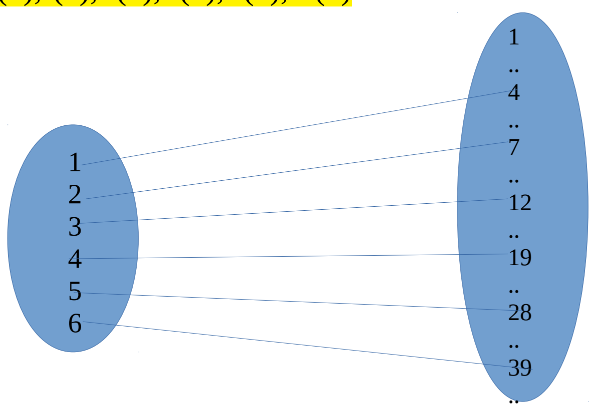
$f(4) = 5(4) +(4)^3 = 20+64= 84$

$f(6) = 246$

$f(8) = 552$

Range : $R =\{18,42,84,246,552\}$
Domain : $A =\{2,3,4,6,8\}$
Co-Domain: $B =\{1...600\}$

2)
$f:A \rightarrow N$
$f(x)=x^2 +3.$
$A= \{1,2,3,4,5,6\}$
$B=\{1,2,3,4,5,6,----\}$
f(1),f(2),F(3),F(4), f(5), F(6)



Co-domain : B
Domain : A
And Range : $R_f = \{4,7,12,19,28,39\}$

f:B->A // function from B to A
B set input (domain)
A set genral output (co domain)
Range
$f(1) = 2(1) + 1 =3$
$f(2)= 2(2) + 1= 5$

$f(3)= 2(3) + 1= 7$
$f(4)= 2(4) + 1= 9$
Range R f = {3,5,7,9}
Domain= set A = {1,2,3,4}
Co Domain= set B = { 1,2,3,4,5,6,7,8,9,10}
Range = { 3,5,7,9}
A function is a rule that assigns each element of a set, called the domain, to exactly one element of a second set, called the codomain.
Notation: $f:X\rightarrow Y$ is our way of saying that the function is called f, the domain is the set X, and the codomain is the set Y
$f(x)=y$ means the element x of the domain (input) is assigned to the element y of the codomain. We say y is an output. Alternatively, we call y the image of x under f.
The range is a subset of the codomain. It is the set of all elements which are assigned to at least one element of the domain by the function. That is, the range is the set of all outputs.
We would write $f:A\rightarrow B$ to describe a function with name f, domain X and codomain Y.
$f:N\rightarrow N$
$f(x)=x^2 +3$.
A= {1,2,3,4,5,6,----}
B= {1,2,3,4,5,6,----}
Example 1:
If $f(x) = 2x^2 +3x-7$ then find f(1), f(2), f(3), f(-2).
Examle 2:
A= { 2,4,3}
B= { 1,2,3,4,5,6,7,8,9,10----------------------------50}
$f(x)= 2x^2 +3x-7$
f: A->B
find domain, co dmain and range
domain= A, codomain= B, range={ 7, 20, 37}
$f(2) = 2.2^2 + 3.2 – 7 = 8+6-7=7$
$f(4) = 2.4^2 + 3.4 – 7 = 32 + 12 – 7= 20$
$f(3)= 2.3^2 + 3.3 – 7 = 18 + 9 +12 = 37$

Types of function

Constant
Identity
Moduls
Greatest Integer

Constant function:
A constant function is a linear function for which the range does not change no matter which member of the domain is used.
f(x)=C it is constant value.
A={1,2,3,4} f:A->A, f(x)= 3. find the range.
f(x)=3, f(1)= 3, f(2)= 3, f(3)=3, f(4)=3, Rf= {3,3,3,3}

Identity Function is defined as the real valued function.
f(x) = x, A={1,2,3}, f(x)= x, find the range.
F(1)=1 f(2)=2 f(3)=3 {1,2,3}

The Modulus Function
The modulus of any number gives us the magnitude of that number(i.e either negative or positive value in modulus gives a positive output). Using the modulus operation, we can define the modulus function as follows:
f(x)=|x|, A= {1,-1,2,-3}, f(x)= |x|
f(1)= |1| = 1 f(-1)= |-1| = 1 f(2)= |2|=2 f(-3)=|-3|=3
range = {1,2,3}

Greatest Integer Function
Greatest Integer Function [X] indicates an integral part of the real number smaller integer to . It is also known as floor of X .
f(x)=[x]
Input: X = 2.3= 2-3 Output: [2.3] = 2
Input: X = -8.0725 = -8 to -9 Output: [-8.0725] = -9
Input: X = 2 Output: [2] = 2
1.5 = 1 to 2 =1
[6.2] = 6 to 7 = 6
[-6.2] = -6 to -7 = -7
in negative, -6 > -7, - 5 > - 6

Examples:

If f(x)=[x], then f(-4.89)=-5
If f(x)= x, then f(4)=4
If f(x)=|x|, then f(-4)=4
If f(x)= 3, then f(-3)=3
0.75 = 0 to 1
0.25 = -1 to 0

Limits

Rule 1 of limit:

$$\lim_{x\to a} f(x)\pm g(x) = \lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

lim x->2 3x-4x
lim x->2 3x - lim x->2 4x
3(2) - 4(2)= 6-8= -2

example:

$$\lim_{x\to 2} x^2+x^4 = \lim_{x\to 2} x^2+\lim_{x\to 2} x^4 = 2^2+2^4 = 4+16 = 20$$

Example 2:

$$\lim_{x\to 2} x^2-x^4 = \lim_{x\to 2} x^2-\lim_{x\to 2} x^4 = 2^2-2^4 = 4-16 = -12$$

Rule 2:

$$\lim_{x\to a} c*f(x) = c\lim_{x\to a} f(x)$$

lim x->2 $4x^2$

4 . lim x->2 x^2

4. $(2)^2$

= 4. 4=16
Example:
$\lim_{x \to 4} 2 * x^2 = 2 \lim_{x \to 4} x^2$ = 2 * 4^2 = 2*16 = 32

Example2: $\lim_{x \to 1} 4x$

Rule 3:

$$\lim_{x \to a} f(x) * g(x) = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$$

Example :

$$\lim_{x \to 2} (2x) * (x^2 + 5) = \lim_{x \to 2} (2x) * \lim_{x \to 2} (x^2 + 5)$$

$$= 2 \lim_{x \to 2} (x) * (\lim_{x \to 2} (x^2) + \lim_{x \to 2} (5))$$

= 2*2 *(4+5)

= 4*9 = 36

Example2:
$$\lim_{x \to 4} (4x^2) * (x^3 + 8x + 4) * x^3$$

= 4*16 *(64+32+4) *64
= 64*(100)*64
= 409600

Rule 4:
$$\lim_{x \to a} \frac{f(x)}{g(X)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(X)}$$
lim x->2 3x/5x
lim x->2 3x / lim x->2 5x
3.2 / 5.2
6/10
3/5
Example:

$$\frac{\lim\limits_{x\to 2} 2x}{\lim\limits_{x\to 2} 23x^2} = \frac{\lim\limits_{x\to 2} 2}{\lim\limits_{x\to 2} 23x} = \frac{2}{23*2} = \frac{2}{46} = \frac{1}{23}$$

Formula for Limits

1) $\lim\limits_{x\to a} C$ = C Ex: $\lim\limits_{x\to 1} 4$ = 4

where C is any constant value ex: 1,2,3,4-----

2) $\lim\limits_{x\to a} x^n = a^n$ Ex: $\lim\limits_{x\to 1} x^2$ = 1^2 = 1

$\lim\limits_{x\to 5} x^2$ = $(5)^2$ = 25

$\lim\limits_{x\to -4} x^2$ = $(-4)^2$ = 16

$\lim\limits_{x\to -2} x^5$ = $(-2)^5$ = -32

$\lim\limits_{x\to -2} x^{-5}$ = $(-2)^{-5}$ = $\frac{1}{-2^5}$ =1/ -2X-2 X -2 X- 2X -2= -1/32

3) $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ = $n*a^{n-1}$

Ex: $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$ = $\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$ = $3*2^2$ = 3*4 = 12

Examples:

1) $\lim_{x \to 1} \frac{x^7 - 1}{x - 1}$ = $\lim_{x \to 1} \frac{x^7 - 1^7}{x - 1}$ = $7*1^6$ = 7

2) $\lim_{x \to -1} \frac{x^5 + 1}{x + 1}$ = $\lim_{x \to -1} \frac{x^5 - (-1)^5}{x - (-1)}$ = $5*(-1)^4$

3) $\lim_{x \to -1} \frac{x^{21} + 1}{x^{23} + 1}$ = $\lim_{x \to -1} \frac{x^{21} - (-1)}{x^{23} - (-1)}$ = $\frac{x^{21} - (-1)}{x - (-1)} * \frac{x - (-1)}{x^{23} - (-1)}$ = 21 * 1/23 = 21/23

4) $\lim_{x \to 0} \frac{2x^2 + 1}{x^3 + 2}$ = 2*0 +1 / 0+2 = 1/2

5) $\lim_{x \to 0} \frac{4x^2 - 1}{2x - 1}$ = $\lim_{x \to 0} \frac{(2x - 1)(2x + 1)}{(2x - 1)}$ = $\lim_{x \to 0} 2x + 1$ = 2(0)+1 = 1

$x^2 - 1$ = x-1 * x+1

6) $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$

$x^2 - 5x + 6$

$x^2 -$ 3x -2x +6

x(x-3) -2(x-3)

= (x-2)(x-3) ----------------------(1)

$x^2 - 4$

$a^2 - b^2$ = (a - b) (a + b)

(x-2) (x+2)

x-3 / x+2 = -1/4

7) $\lim_{x \to 0} \frac{x^2 + 2x + 5}{x^2 + 3x + 1}$ = $\lim_{x \to 0} \frac{0^2 + 2(0) + 5}{0^2 + 3(0) + 1}$ = 5

8) $\lim_{x \to 0} \frac{x^2-3x-4}{x^2-2x-8}$

Factorizing 1:

$x^2-3x-4=0$

4 -> 1,4 -> subtraction as sign of 4 is minus. And the sign of greater value will be minus.

$x^2-4x+x-4=0$

$x^2-4x+x-4=0$

x(x-4)+(x-4) = 0
(x+1)(x-4)

Factorizing 2:
$x^2-2x-8=0$

$x^2+2x-4x-8=0$

$x(x+2)-4(x+2)=0$

$(x+2)(x-4)$

Now substituting value in equation

$\lim_{x \to 0} \frac{(x+1)(x-4)}{(x+2)(x-4)}$ = $\frac{0+1}{0+2}$ = $\frac{1}{2}$

9) $\lim_{x \to 1} \frac{(2x-3)(x-1)}{2x^2+x-3}$

10) $\lim_{x \to 0} \frac{x^3+3x^2-8x}{5x^3-4x}$

11) $\lim_{x \to -1} \frac{x^3+1}{x^2-1}$

$2x^2+x-3$

6 - > 2,3 -> +3 , -2

$2x^2+3x-2x-3$

$x(2x+3)-1(2x+3)$

$(x-1)(2x+3)$

$\lim_{x \to 1} \frac{(2x-3)(x-1)}{(2x+3)(x-1)}$ = -1/5

12) $\lim_{x \to 3} \frac{(x^3-27)}{\sqrt{(x+1)}-2}$ = $\frac{(3^3-27)}{\sqrt{(3+1)}-2}$ = 27 – 27 /2-2 = infinite = wrong

$$\frac{(x-3)(x^2+x*3+9)}{\sqrt{(x+1)}-2}*\frac{\sqrt{(x+1)}+2}{\sqrt{(x+1)}+2}$$

$$\frac{(x-3)(x^2+3x+9)*\sqrt{(x+1)}+2}{(\sqrt{(x+1)})^2-2^2}$$

$$\frac{(x-3)(x^2+3x+9)*\sqrt{(x+1)}+2}{(\sqrt{(x+1)})^2-2^2}$$

Continuity

Formula:

Example:

f(x) = $\frac{x^2-5x+6}{x^2-3x+2}$ find continuity at x= 3 and x = 2.

x^2-5x+6 = x^2 – 3x-2x+6= x(x-3) – 2(x-3) = (x-3) (x-2)
x^2-3x+2 = (x-2) (x-1)

$$\frac{(x-3)(x-2)}{(x-2)(x-1)} = \frac{(x-3)}{(x-1)}$$

x= 2 => -1
2-3/ *2-1* = -*1*/1= -1
x= 3 => 0

f(a) must exist such that $\lim_{x \to a^{+}} f(X)$ = $\lim_{x \to a^{-}} f(X)$ = $\lim_{x \to a} f(X)$

Example:
F(x) is defined as follows:check continuity at x=2

f(x) = $\frac{x^2-4}{x-2}$ **when x <2**

f(x) = 4 when x = 2
f(x) = x+2 when x >2

x>2

f(x) = x+2

$\lim_{x \to 2^{+}} x+2$ **= 2+2 = 4 -------------1)**

x<2

$\lim_{x \to 2^{-}} \frac{x^2-4}{x-2}$ **= 2^2-4/2-2 = 4-4/2-2= 0/0 (indeterminante form)**

= $\lim_{x \to 2^{-}} \frac{(x+2)(x-2)}{(x-2)}$ = $\lim_{x \to 2^{-}} x+2$ **= 4 ----------ii)**

x=2

$\lim_{x \to 2} 4$ **= 4 ---------------- iii)**

From i , 2 and 3 we can say function is continuous.

Example 2:

f(x) = 3+2x, when $x<0$

f(x) = 3-2x when $0 \leqslant x < \frac{3}{2}$

f(x) = -3-2x when $x \geq \frac{3}{2}$

Find continuity for x = 3/2

X= 0

x>0

$\lim_{x \to 0^{+}} 3-2x$ = 3

x<0

$\lim_{x \to 0^{-}} 3+2x$ = 3

x=0

$\lim_{x \to 0} 3-2x$ = 3

f(x) is continuous for x = 0

X = 3/2

$$\lim_{x \to \frac{3}{2}^{+}} -3-2x \quad = \quad -3-2\left(\frac{3}{2}\right) \quad = \quad \frac{(-6-6)}{2} \quad = -12/2 = -6$$

$$\lim_{x \to \frac{3}{2}^{-}} 3-2x \quad = \quad 3-2\left(\frac{3}{2}\right) \quad = \quad \frac{(6-6)}{2} \quad = 0$$

$$\lim_{x \to \frac{3}{2}} -3-2x \quad = -6$$

Thus f(x) is discontinious for x = 3/2

Exercise:

1) A function is defined as follows:

f(x) = x^2 $\quad 0 \leqslant x < 1$

f(x) = x $\quad 1 \leqslant x < 2$

f(x) = $\frac{1}{2}x^2$ $\quad 2 \leqslant x < 3$

Prove that f(x) is continuous at x=1 and x = 2

2) A function is defined as follows:

f(x) = x $\quad 0 \leqslant x < 1$

f(x) = 1 $\quad x = 1$

f(x) = $2-x$ $\quad 1 < x \leqslant 3$

find continuity for x=1.

3) A function is defined as follows:

f(x) = $\frac{(\sqrt{x}-2)}{x-4}$ $\quad x \neq 4$

f(X) = $\frac{1}{4}$ $\quad$ x = 4

Discuss the continuity of f(X) at x= 4