Inverse of a 3x3 Matrix

A method for finding the inverse of a 3×3 matrix is described in this document.

The matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ will be used to illustrate the method.

1. Matrix of Minors

If we go through each element of the matrix and replace it by the determinant of the 2×2 matrix that results from deleting the element's row and column.

For the example matrix, starting with the element on row 1 and column 1:

$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ \cancel{1} & 0 & 1 \\ \cancel{1} & 2 & 1 \end{pmatrix}$, $\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$=-2 gives the first element of the matrix of minors $\begin{pmatrix} -2 & & \\ & & \\ & & \end{pmatrix}$.

For the example matrix, starting with the element on row 1 and column 2:

$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{2} \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$, $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$=-2 gives the first element of the matrix of minors: $\begin{pmatrix} -2 & 0 & \\ & & \\ & & \end{pmatrix}$.

Eventually, the following matrix of minors is obtained: : $\begin{pmatrix} -2 & 0 & 2 \\ -2 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$.

2. Matrix of Cofactors

In order to determine the matrix of cofactors, the signs of the matrix of minors are changed by applying the following $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$.

For the example, the matrix of minors is : $\begin{pmatrix} -2 & 0 & 2 \\ 2 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix}$.

3. Determinant

The determinant can be found by the sum of an element-by-element multiplication of the original matrix with the cofactor matrix. It gives the same value whichever row or column is used.

For the example, choosing the top row gives determinant of $1 \times (-2) + 2 \times 0 + 2 \times 2 = 2$.

Alternatively, choosing the middle column determinant of $2 \times 0 + 0 \times (-1) + 2 \times 1 = 2$.

Note that if the determinant is zero then the matrix does not have an inverse. The matrix is said to be singular.

4. Adjoint

The adjoint matrix is the transpose of the matrix of cofactors.

For the example the adjoint matrix is : $\begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$.

5. Inverse

The inverse is simply the adjoint matrix, multiplied by the reciprocal of the determinant:

For the example, the inverse is $\frac{1}{2}\begin{pmatrix} -2 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \end{pmatrix}$.

Spreadsheet solution

The accompanying spreadsheet finds the inverse by elimination.

			Finding the inverse of a 3x3 matrix							www.mathematics.me.uk			
Matrix			1	2	2		Place 3x3 matrix in yellow area						
			1	0	1								
			1	2	1								
Matrix of minors			-2	0	2		For each element the corresonding row and column is struck out and the determinant of the remaining 2x2 is found						
			-2	-1	0								
			2	-1	-2								
Matrix of cofactors			-2	0	2		The diamond elements change sign.			+	-	+	
			2	-1	0					-	+	-	
			2	1	-2					+	-	+	
Determinant			2										
Adjoint			-2	2	2								
			0	-1	1								
			2	0	-2								
Inverse	0.5	x	-2	2	2		=	-1	1	1		solution	
			0	-1	1			0	-0.5	0.5			
			2	0	-2			1	0	-1			
heck			1	2	2			-1	1	1		=	1	0	0
			1	0	1		x	0	-0.5	0.5			0	1	0
			1	2	1			1	0	-1			0	0	1
													Identity matrix		