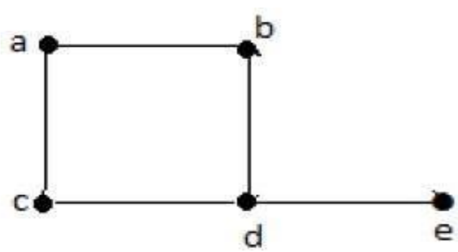
Graph Theory - Introduction

In the domain of mathematics and computer science, *graph theory is the study of graphs that concerns with the relationship among edges and vertices*. It is a popular subject having its applications in computer science, information technology, biosciences, mathematics, and linguistics to name a few. Without further ado, let us start with defining a graph.

What is a Graph?

A graph is a pictorial representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by points termed as **vertices**, and the links that connect the vertices are called **edges**.

Formally, a graph is a pair of sets **(V, E)**, where **V** is the set of vertices and E is the set of edges, connecting the pairs of vertices. Take a look at the following graph −



In the above graph,

V = {a, b, c, d, e}

E = {ab, ac, bd, cd, de}

Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering −

Electrical Engineering − The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.

Computer Science − Graph theory is used for the study of algorithms. For example,

- Kruskal's Algorithm
- Prim's Algorithm
- Dijkstra's Algorithm

Computer Network − The relationships among interconnected computers in the network follows the principles of graph theory.

Science − The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.

Linguistics − The parsing tree of a language and grammar of a language uses graphs.

General − Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.

Graph Theory - Fundamentals

A graph is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself. The concept of graphs in graph theory stands up on some basic terms such as point, line, vertex, edge, degree of vertices, properties of graphs, etc. Here, in this chapter, we will cover these fundamentals of graph theory.

Point

A point is a particular position in a one-dimensional, two-dimensional, or three-dimensional space. For better understanding, a point can be denoted by an alphabet. It can be represented with a dot.

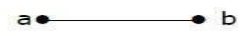
Example

Here, the dot is a point named ‘a’.

Line

A **Line** is a connection between two points. It can be represented with a solid line.

Example



Here, ‘a’ and ‘b’ are the points. The link between these two points is called a line.

Vertex

A vertex is a point where multiple lines meet. It is also called a **node**. Similar to points, a vertex is also denoted by an alphabet.

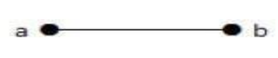
Example

Here, the vertex is named with an alphabet ‘a’.

Edge

An edge is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.

Example

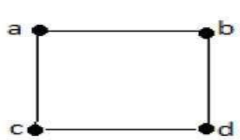


Here, ‘a’ and ‘b’ are the two vertices and the link between them is called an edge.

Graph

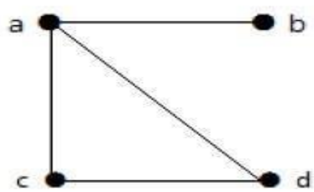
A graph ‘G’ is defined as G = (V, E) Where V is a set of all vertices and E is a set of all edges in the graph.

Example 1



In the above example, ab, ac, cd, and bd are the edges of the graph. Similarly, a, b, c, and d are the vertices of the graph.

Example 2



In this graph, there are four vertices a, b, c, and d, and four edges ab, ac, ad, and cd.

Loop

In a graph, if an edge is drawn from vertex to itself, it is called a loop.

Example 1

In the above graph, V is a vertex for which it has an edge (V, V) forming a loop.

Example 2

In this graph, there are two loops which are formed at vertex a, and vertex b.

Degree of Vertex

It is the number of vertices adjacent to a vertex V.

Notation – deg(V).

In a simple graph with n number of vertices, the degree of any vertices is −

deg(v) ≤ n – 1 ∀ v ∈ G

A vertex can form an edge with all other vertices except by itself. So the degree of a vertex will be up to the **number of vertices in the graph minus 1**. This 1 is for the self-vertex as it cannot form a loop by itself. If there is a loop at any of the vertices, then it is not a Simple Graph.

Degree of vertex can be considered under two cases of graphs −

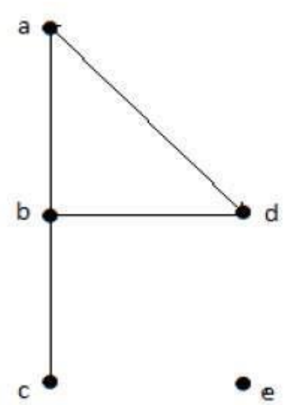
- Undirected Graph
- Directed Graph

Degree of Vertex in an Undirected Graph

An undirected graph has no directed edges. Consider the following examples.

Example 1

Take a look at the following graph −

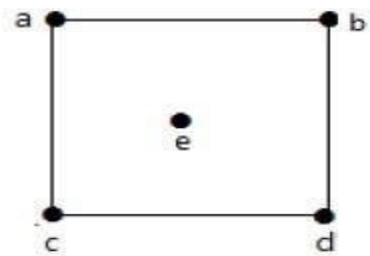


In the above Undirected Graph,

- deg(a) = 2, as there are 2 edges meeting at vertex 'a'.
- deg(b) = 3, as there are 3 edges meeting at vertex 'b'.
- deg(c) = 1, as there is 1 edge formed at vertex 'c'
- So 'c' is a **pendent vertex**.
- deg(d) = 2, as there are 2 edges meeting at vertex 'd'.
- deg(e) = 0, as there are 0 edges formed at vertex 'e'.
- So 'e' is an **isolated vertex**.

Example 2

Take a look at the following graph −



In the above graph,

deg(a) = 2, deg(b) = 2, deg(c) = 2, deg(d) = 2, and deg(e) = 0.

The vertex 'e' is an isolated vertex. The graph does not have any pendent vertex.

Degree of Vertex in a Directed Graph

In a directed graph, each vertex has an **indegree** and an **outdegree**.

Indegree of a Graph

- Indegree of vertex V is the number of edges which are coming into the vertex V.
- **Notation** − deg−(V).

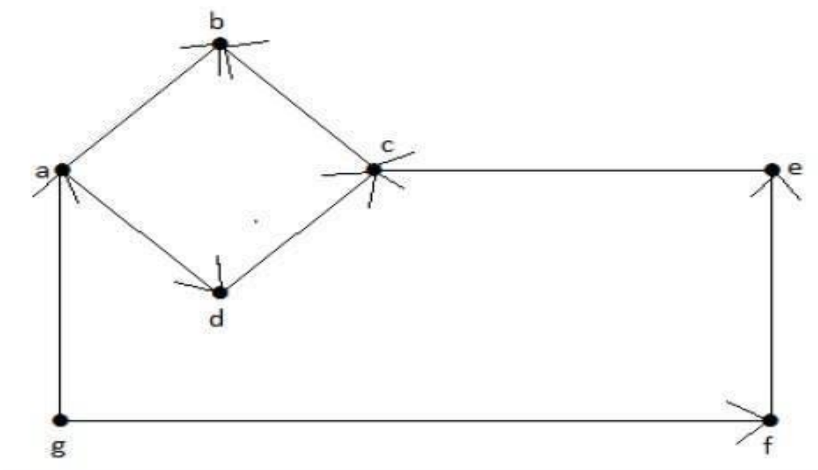
Outdegree of a Graph

- Outdegree of vertex V is the number of edges which are going out from the vertex V.
- **Notation** − deg+(V).

Consider the following examples.

Example 1

Take a look at the following directed graph. Vertex 'a' has two edges, 'ad' and 'ab', which are going outwards. Hence its outdegree is 2. Similarly, there is an edge 'ga', coming towards vertex 'a'. Hence the indegree of 'a' is 1.

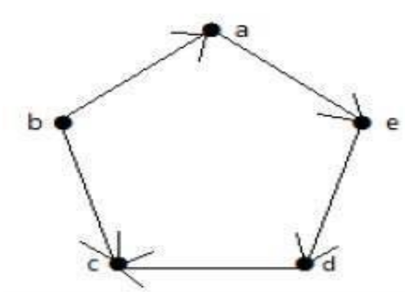


The indegree and outdegree of other vertices are shown in the following table −

Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

Example 2

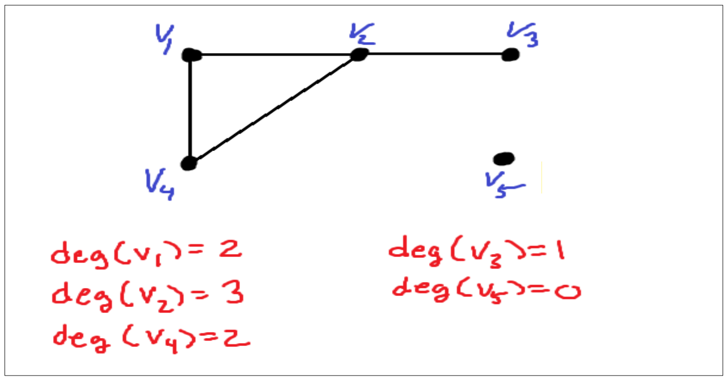
Take a look at the following directed graph. Vertex ‘a’ has an edge ‘ae’ going outwards from vertex ‘a’. Hence its outdegree is 1. Similarly, the graph has an edge ‘ba’ coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.

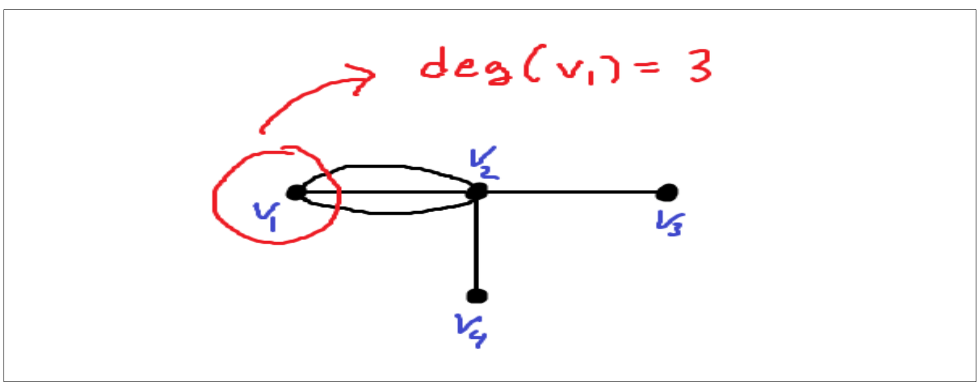


The indegree and outdegree of other vertices are shown in the following table −

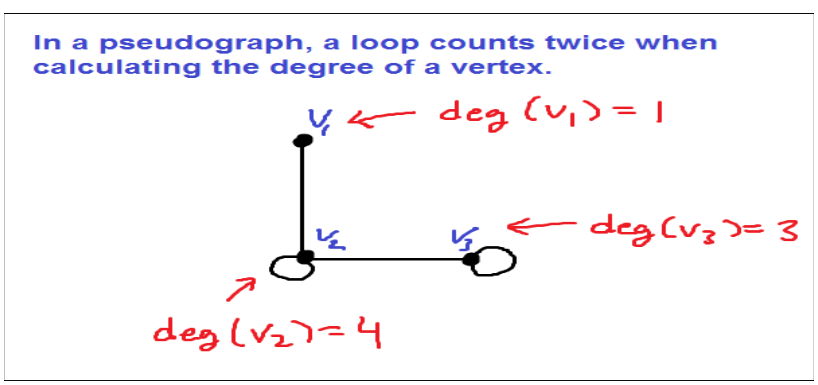
Vertex	Indegree	Outdegree
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

Example:





Example:



Pendent Vertex

By using degree of a vertex, we have a two special types of vertices. A vertex with degree one is called a pendent vertex.

Example

Here, in this example, vertex ‘a’ and vertex ‘b’ have a connected edge ‘ab’. So with respect to the vertex ‘a’, there is only one edge towards vertex ‘b’ and similarly with respect to the vertex ‘b’, there is only one edge towards vertex ‘a’. Finally, vertex ‘a’ and vertex ‘b’ has degree as one which are also called as the pendent vertex.

Isolated Vertex

A vertex with degree zero is called an isolated vertex.

Example

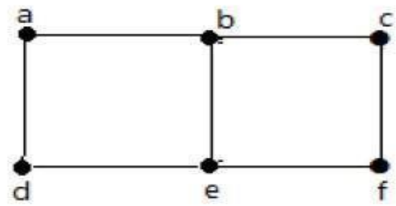
Here, the vertex ‘a’ and vertex ‘b’ has a no connectivity between each other and also to any other vertices. So the degree of both the vertices ‘a’ and ‘b’ are zero. These are also called as isolated vertices.

Adjacency

Here are the norms of adjacency −

- In a graph, two vertices are said to be **adjacent**, if there is an edge between the two vertices. Here, the adjacency of vertices is maintained by the single edge that is connecting those two vertices.
- In a graph, two edges are said to be adjacent, if there is a common vertex between the two edges. Here, the adjacency of edges is maintained by the single vertex that is connecting two edges.

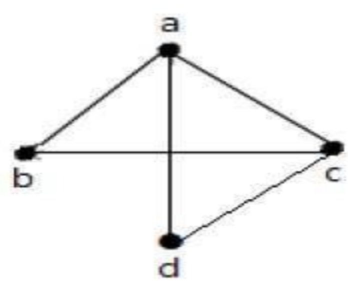
Example 1



In the above graph −

- ‘a’ and ‘b’ are the adjacent vertices, as there is a common edge ‘ab’ between them.
- ‘a’ and ‘d’ are the adjacent vertices, as there is a common edge ‘ad’ between them.
- ab’ and ‘be’ are the adjacent edges, as there is a common vertex ‘b’ between them.
- be’ and ‘de’ are the adjacent edges, as there is a common vertex ‘e’ between them.

Example 2

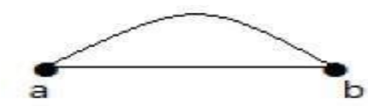


In the above graph −

- ‘a’ and ‘d’ are the adjacent vertices, as there is a common edge ‘ad’ between them.
- ‘c’ and ‘b’ are the adjacent vertices, as there is a common edge ‘cb’ between them.
- ‘ad’ and ‘cd’ are the adjacent edges, as there is a common vertex ‘d’ between them.
- ‘ac’ and ‘cd’ are the adjacent edges, as there is a common vertex ‘c’ between them.

Parallel Edges

In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.

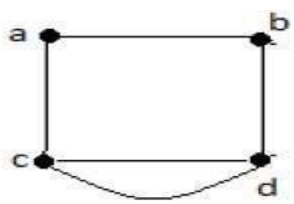


In the above graph, ‘a’ and ‘b’ are the two vertices which are connected by two edges ‘ab’ and ‘ab’ between them. So it is called as a parallel edge.

Multi Graph

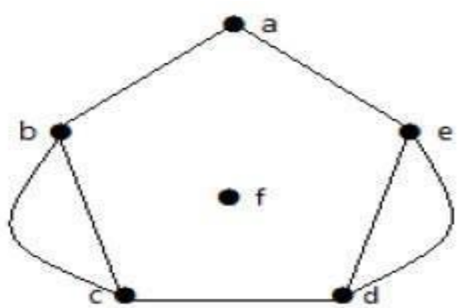
A graph having parallel edges is known as a Multigraph.

Example 1



In the above graph, there are five edges ‘ab’, ‘ac’, ‘cd’, ‘cd’, and ‘bd’. Since ‘c’ and ‘d’ have two parallel edges between them, it a Multigraph.

Example 2



In the above graph, the vertices ‘b’ and ‘c’ have two edges. The vertices ‘e’ and ‘d’ also have two edges between them. Hence it is a Multigraph.

Graph Theory - Basic Properties

Graphs come with various properties which are used for characterization of graphs depending on their structures. These properties are defined in specific terms pertaining to the domain of graph theory. In this chapter, we will discuss a few basic properties that are common in all graphs.

Distance between Two Vertices

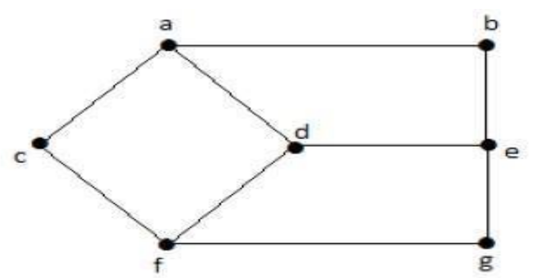
It is number of edges in a shortest path between Vertex U and Vertex V. If there are multiple paths connecting two vertices, then the shortest path is considered as the distance between the two vertices.

Notation − d(U,V)

There can be any number of paths present from one vertex to other. Among those, you need to choose only the shortest one.

Example

Take a look at the following graph −



Here, the distance from vertex 'd' to vertex 'e' or simply 'de' is 1 as there is one edge between them. There are many paths from vertex 'd' to vertex 'e' −

- da, ab, be
- df, fg, ge
- de (It is considered for distance between the vertices)
- df, fc, ca, ab, be
- da, ac, cf, fg, ge

Types of Graphs

There are various types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure. We will discuss only a certain few important types of graphs in this chapter.

Null Graph

A graph having no edges is called a Null Graph.

Example

In the above graph, there are three vertices named 'a', 'b', and 'c', but there are no edges among them. Hence it is a Null Graph.

Trivial Graph

A **graph with only one vertex** is called a Trivial Graph.

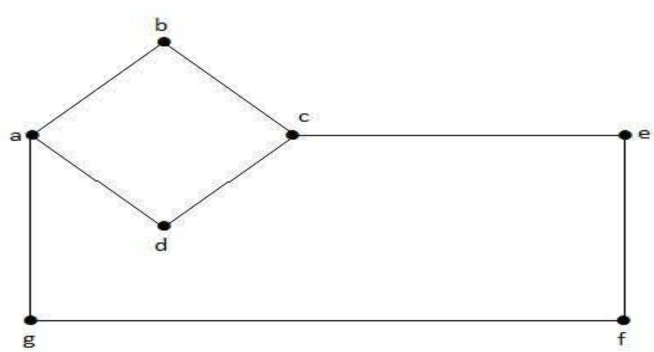
Example

In the above shown graph, there is only one vertex 'a' with no other edges. Hence it is a Trivial graph.

Non-Directed Graph

A non-directed graph contains edges but the edges are not directed ones.

Example

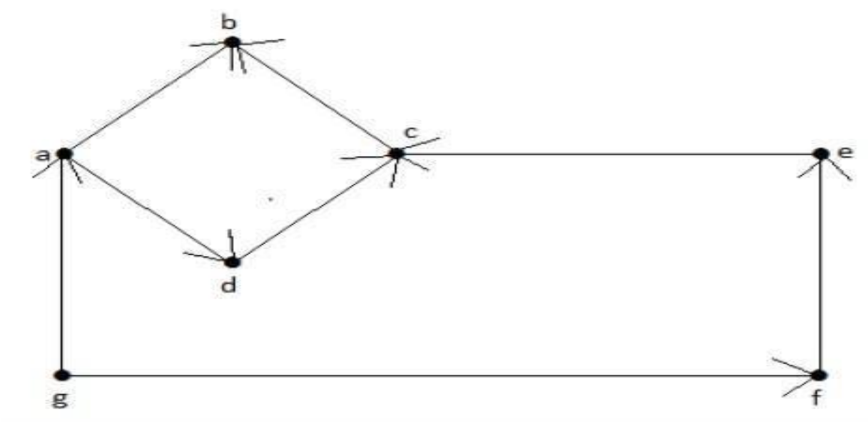


In this graph, 'a', 'b', 'c', 'd', 'e', 'f', 'g' are the vertices, and 'ab', 'bc', 'cd', 'da', 'ag', 'gf', 'ef' are the edges of the graph. Since it is a non-directed graph, the edges 'ab' and 'ba' are same. Similarly other edges also considered in the same way.

Directed Graph

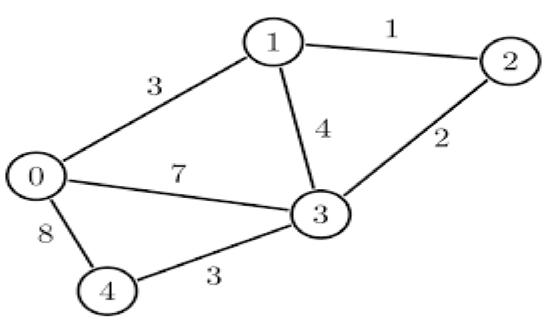
In a directed graph, each edge has a direction.

Example



In the above graph, we have seven vertices 'a', 'b', 'c', 'd', 'e', 'f', and 'g', and eight edges 'ab', 'cb', 'dc', 'ad', 'ec', 'fe', 'gf', and 'ga'. As it is a directed graph, each edge bears an arrow mark that shows its direction. Note that in a directed graph, 'ab' is different from 'ba'.

Weighted Graph : A *weighted graph* is a graph such that each edge is labeled with a number, called the weight of that edge.

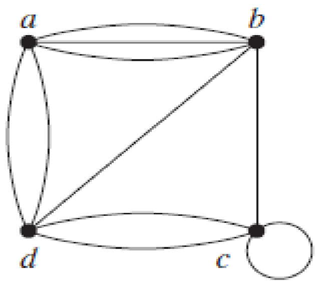


Simple Graph

A graph **with no loops** and no **parallel edges** is called a simple graph.

- The maximum number of edges possible in a single graph with 'n' vertices is ${}^nC_2$ where ${}^nC_2 = n(n – 1)/2$.
- The number of simple graphs possible with 'n' vertices = $2^{{}^n c_2} = 2^{n(n-1)/2}$.

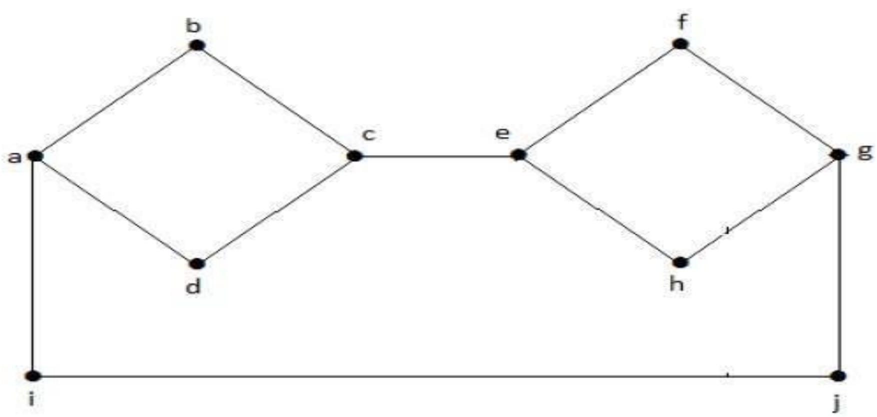
Pseudo Graph : A graph having loop and parallel edges is called Pseudo graph.



Connected Graph

A graph G is said to be connected **if there exists a path between every pair of vertices**. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

Example :In the following graph, each vertex has its own edge connected to other edge. Hence it is a connected graph.

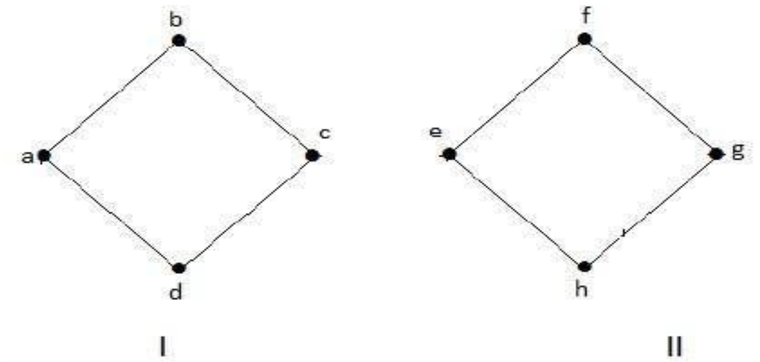


Disconnected Graph

A graph G is disconnected, if it does not contain at least two connected vertices.

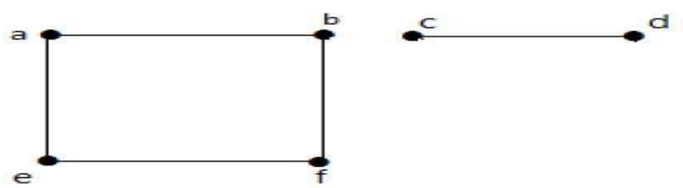
Example 1

The following graph is an example of a Disconnected Graph, where there are two components, one with 'a', 'b', 'c', 'd' vertices and another with 'e', 'f', 'g', 'h' vertices.



The two components are independent and not connected to each other. Hence it is called disconnected graph.

Example 2



In this example, there are two independent components, a-b-f-e and c-d, which are not connected to each other. Hence this is a disconnected graph.

Regular Graph

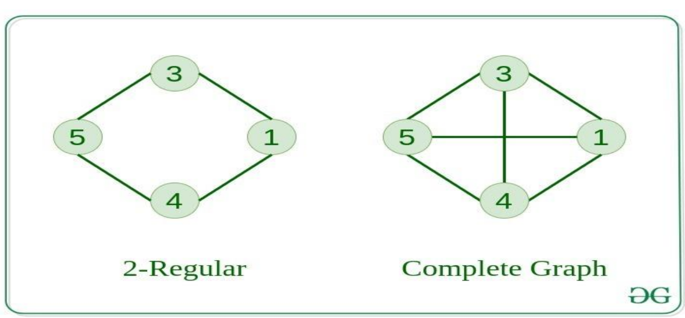
The graph in which the degree of every vertex is equal to the other vertices of the graph.

1 – 2 edges

3 – 2 edges

4 – 2 edges

5 – 2 edges



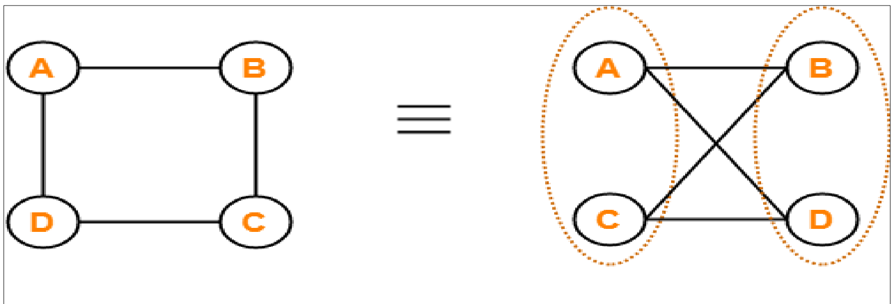
Let the degree of each vertex be **K** then the graph is called K-regular.

Complete Graph
The graph in which from each node there is an edge to each other node.

Cyclic Graph
A graph containing at least one cycle is known as a Cyclic graph

Bipartite Graph
A graph in which vertex can be divided into two sets such that vertex in each set does not contain any edge between them.

- It consists of two sets of vertices X and Y.
- The vertices of set X join only with the vertices of set Y.
- The vertices within the same set do not join.



Example of Bipartite Graph

A,B,C,D

Set X = {A,C} Set Y= {B,D}

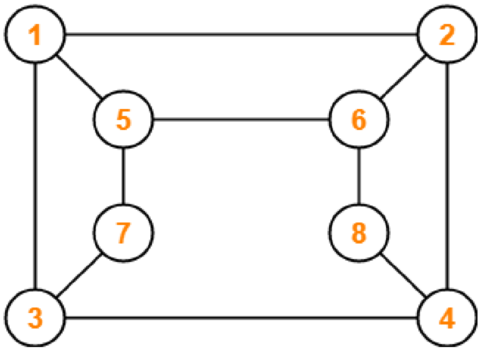
Vertices =A,B,C,D

Set X= {A,C}

Set Y = {B,D}

Here,

- The vertices of the graph can be decomposed into two sets.
- The two sets are X = {A, C} and Y = {B, D}.
- The vertices of set X join only with the vertices of set Y and vice-versa.
- The vertices within the same set do not join.
- Therefore, it is a bipartite graph.

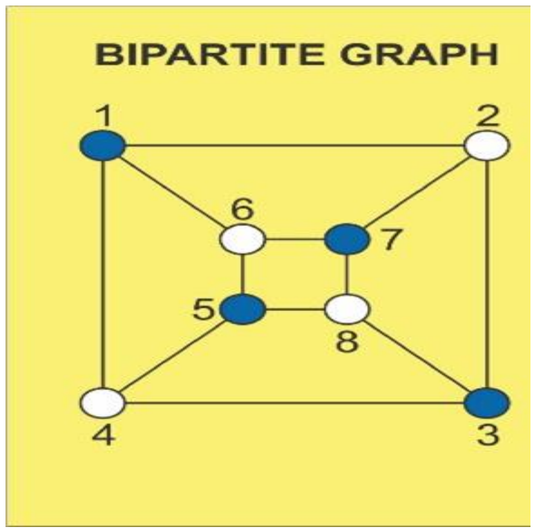


Vertices={1,2,3,4,5,6,7,8}

X={1,4,6,7}

Y = {2,3,5,8}

Example

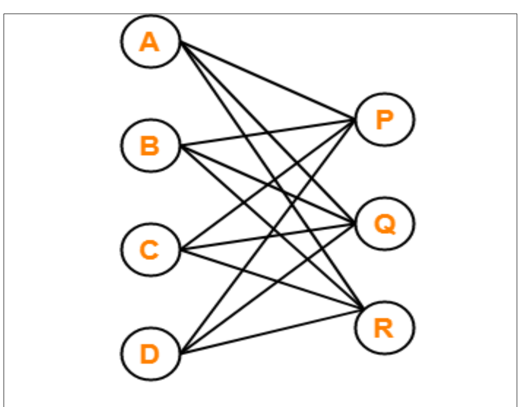


Set X = {1,3,5,7}

Set Y = {2,4,6,8}

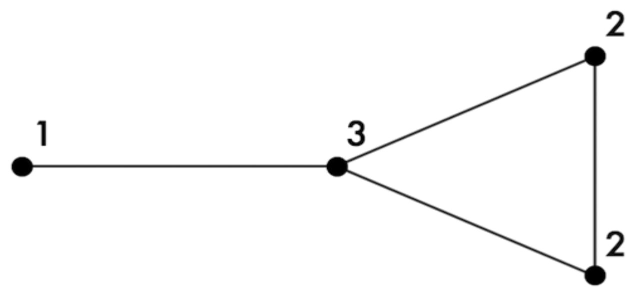
Complete Bipartite Graph-

A bipartite graph where every vertex of set X is joined to every vertex of set Y is called as complete bipartite graph.



Example of Complete Bipartite Graph

Degree Sequence



Vertex 1 – 1
Vertex 2 – 2
Vertex 3 – 3
Vertex 4 - 2

(3,2,2,1)

graph have degree sequence (3, 2, 2, 1).

Graph Theory - Isomorphism

A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs. Note that we label the graphs in this chapter mainly for the purpose of referring to them and recognizing them from one another.

Isomorphic Graphs

Two graphs G_1 and G_2 are said to be isomorphic if −

- Their number of components (vertices and edges) are same.
- Their edge connectivity is retained.

Note

If $G_1 \equiv G_2$ then −

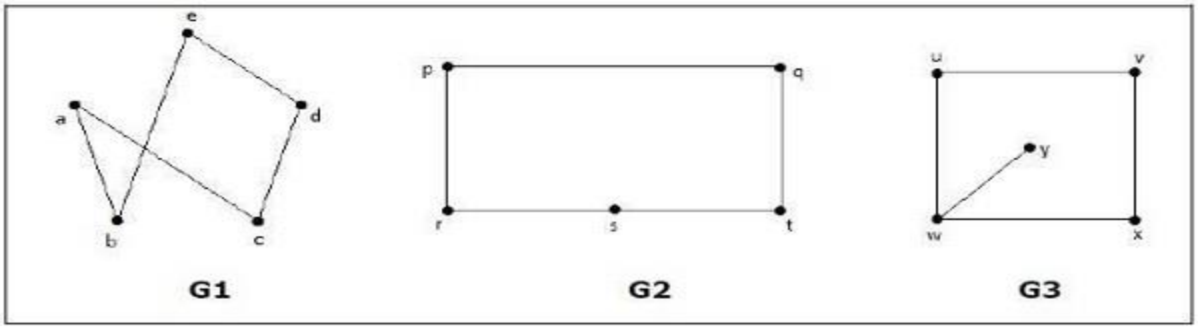
$|V(G_1)| = |V(G_2)|$

$|E(G_1)| = |E(G_2)|$

Degree sequences of G_1 and G_2 are same.

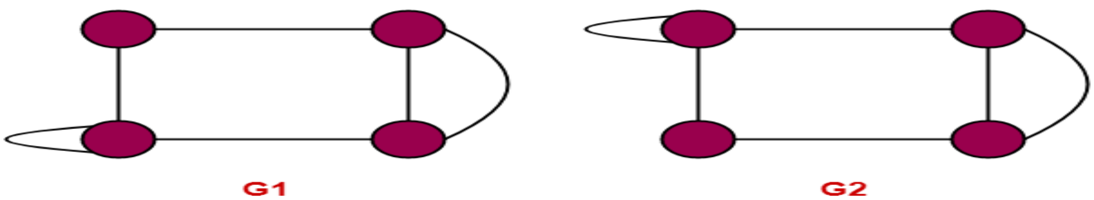
Example

Which of the following graphs are isomorphic?



In the graph G_3, vertex 'w' has only degree 3, whereas all the other graph vertices has degree 2. Hence G3 not isomorphic to G1 or G2. Here (G1 ≡ G2).

Example:



Checking Necessary Conditions-

Condition-01:

- Number of vertices in graph G1 = 4
- Number of vertices in graph G2 = 4

Here, Both the graphs G1 and G2 have same number of vertices.

So, Condition-01 satisfies.

Condition-02:

- Number of edges in graph G1 = 6
- Number of edges in graph G2 = 6

Here, Both the graphs G1 and G2 have equal number of edges.

So, Condition-02 satisfies.

Condition-03: Degree Sequence

G1={2,3,3,4}

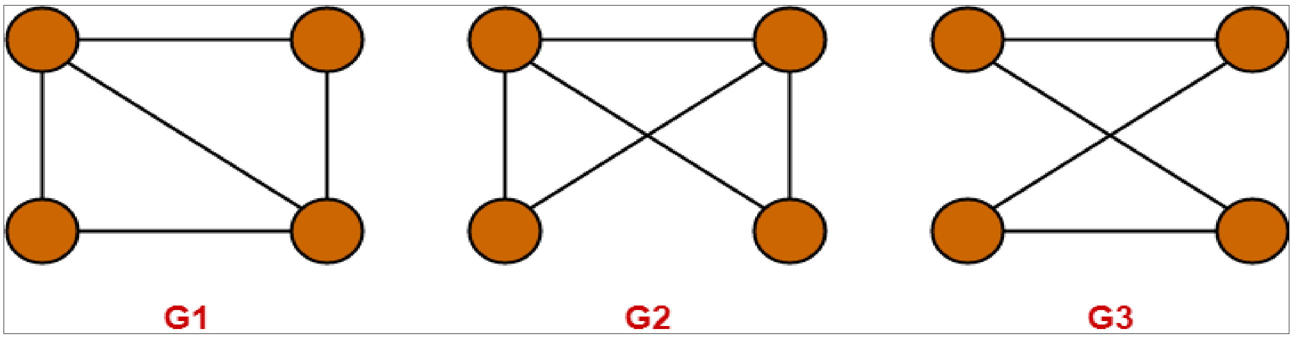
G2={2,3,3,4}

Here,Both the graphs G1 and G2 have equal degree sequence.

So, Condition-03 satisfies.

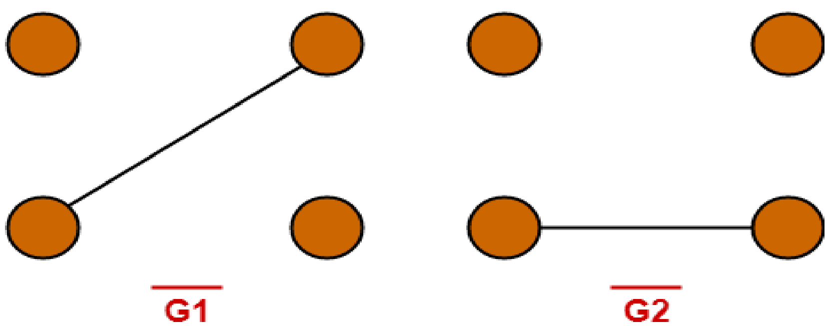
All conditions are satisfied so both graphs G1 and G2 are isomorphoc graphs.

Example:

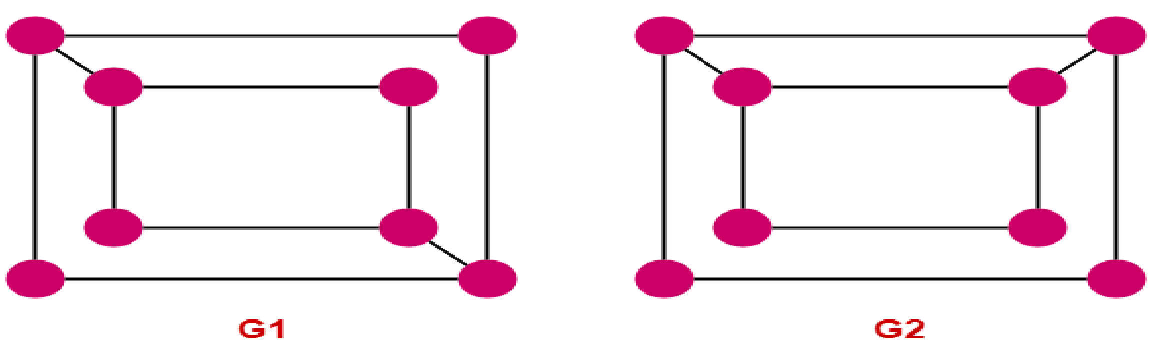


Check for G1 and G2, G2 and G3, G1 and G3

Example:

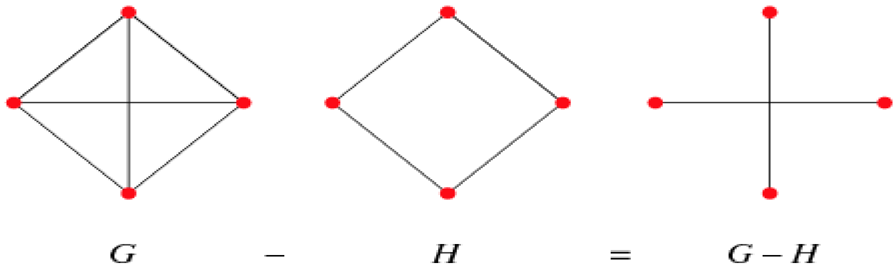


Example



Graph Operations:

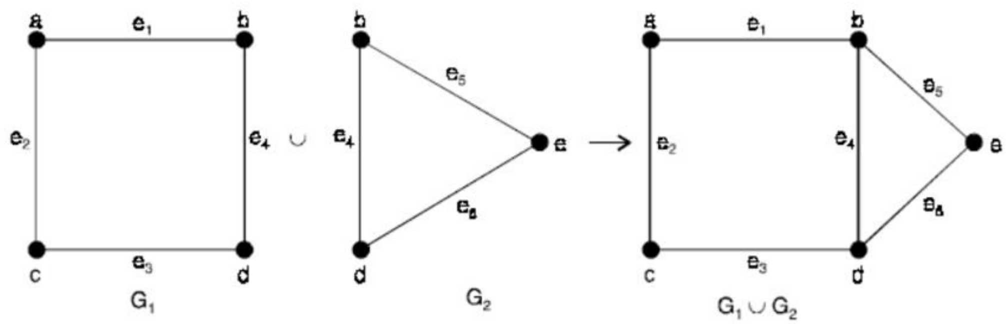
Graph Difference:



Union: Given two graphs G1 and G2, their union will be a graph such that

V(G1 ∪ G2) = V(G1) ∪ V(G2)

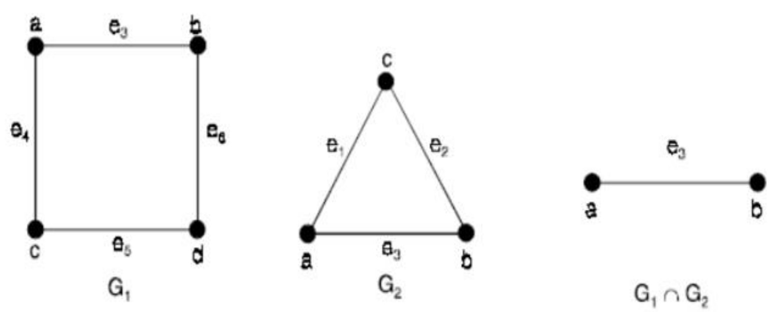
and E(G1 ∪ G2) = E(G1) ∪ E(G2)



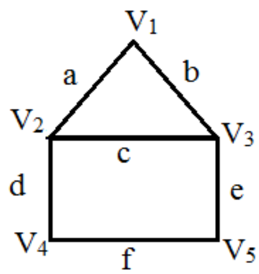
Intersection: Given two graphs G1 and G2 with at least one vertex in common then their intersection will be a graph such that

V(G1 ∩ G2) = V(G1) ∩ V(G2)

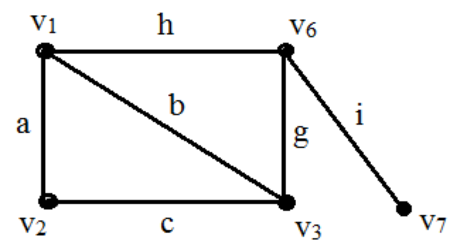
and E(G1 ∩ G2) = E(G1) ∩ E(G2)



Example :- Let G_1 and G_2 be any two graphs then Union , Intersection and ring sum of these graphs as shown below $G_1 \cup G_2$, $G_1 \cap G_2$, $G_1 \oplus G_2$



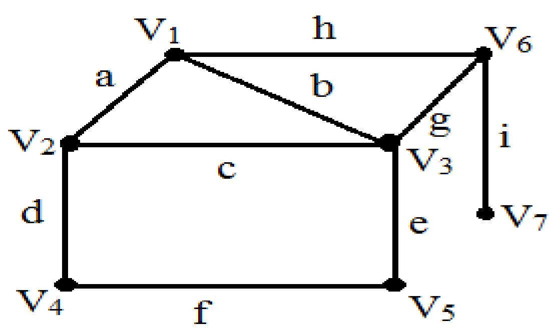
Graph G_2



Graph G_2

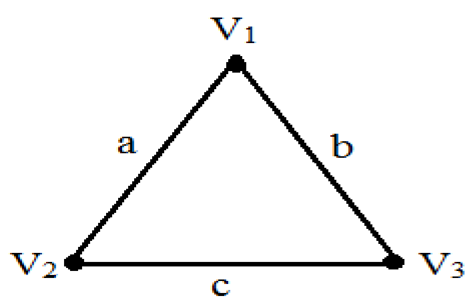
Solution :- We know ,

(1) $G_1 \cup G_2 \Leftrightarrow v_1 \cup v_2$ and $E_1 \cup E_2$



Graph $G_1 \cup G_2$

(2) $G_1 \cap G_2 \Leftrightarrow v_1 \cap v_2$ and $E_1 \cap E_2$



Graph $G_1 \cap G_2$.

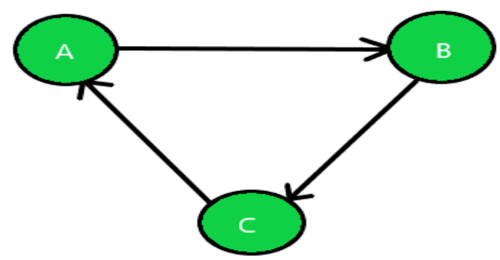
Connectedness in graph:

Strongly Connected: A graph is said to be **strongly connected** if every pair of vertices(u, v) in the graph contains a path between each other. In an unweighted directed graph G, every pair of vertices u and v should have a path in each direction between them i.e., bidirectional path.

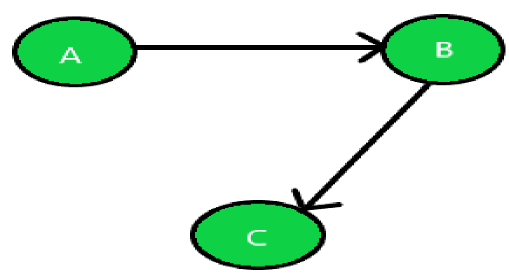
Unilaterally Connected: A graph is said to be **unilaterally connected** if it contains a directed path from u to v OR a directed path from v to u for every pair of vertices u, v. Hence, at least for any pair of vertices, one vertex should be reachable form the other.

Weakly Connected: A graph is said to be **weakly connected** if there doesn't exist any path between any two pairs of vertices. Hence, if a graph G doesn't contain a directed path (from u to v or from v to u for every pair of vertices u, v) then it is weakly connected.

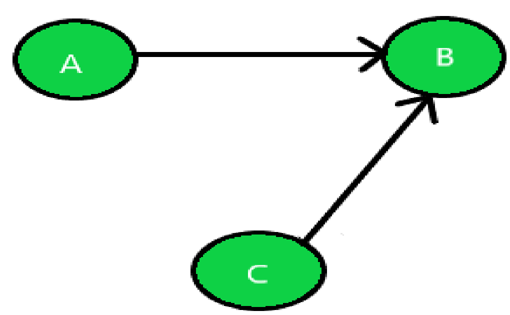
Strongly Connected Graph -------→



Unilaterally Connected Graph ----→



Weakly Connected Graph -------→



Graph representation

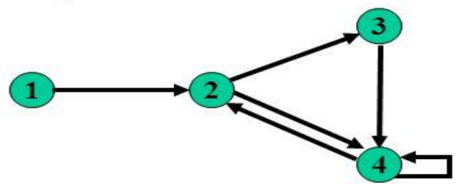
Adjacency Matrix

- Definition: Let $G = (V, E)$ be a simple digraph. Let $V = \{v_1, v_2, \ldots v_n\}$ be the vertices (nodes). Order the vertices from v_1 to v_n. The $n \times n$ matrix A whose elements are given by

$$a_{ij} = \begin{cases} 1, \text{ if } (v_i, v_j) \in E \\ 0, \text{ otherwise} \end{cases}$$

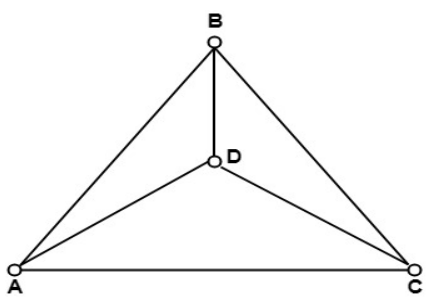
is the *adjacency matrix* of the graph G.

- Example:



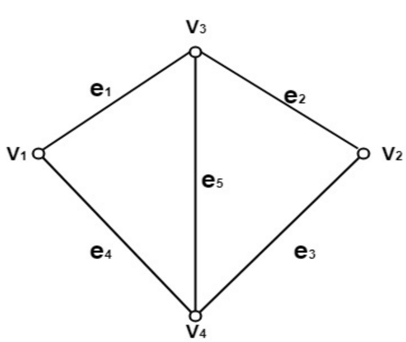
$$\mathbf{A} = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Example:



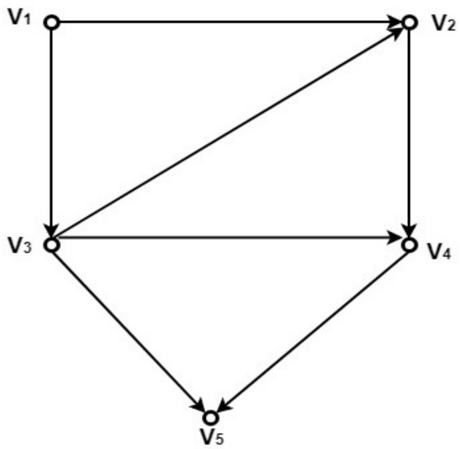
	A	B	C	D
A	0	1	1	1
B	1	0	1	1
C	1	1	0	1
D	1	1	1	0

Example:



	v1	v2	v3	v4
v1	0	0	1	1
v2	0	0	1	1
v3	1	1	0	1
v4	1	1	1	0

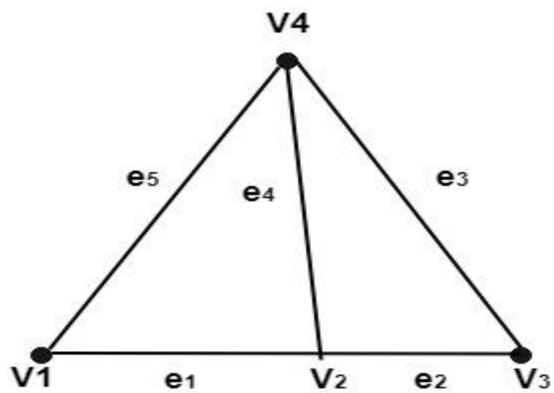
Example:



	v1	v2	v3	v4	v5
v1	**0**	**1**	**1**	**0**	**0**
v2	**0**	**0**	**0**	**1**	**0**
v3	**0**	**1**	**0**	**1**	**1**
v4	**0**	**0**	**0**	**0**	**1**
v5	**0**	**0**	**0**	**0**	**0**

Example:

	v1	v2	v3	v4
v1	**0**	**1**	**0**	**1**
v2	**1**	**0**	**1**	**1**
v3	**0**	**1**	**0**	**1**
v4	**1**	**1**	**1**	**0**



Linked representation of graph