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$$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} \cdot (3x+1)^{3/2} + C$$

Ex:- $\int x^2 (x^3+1)^{3/2} \, dx$

Let $x^3+1 = u$

$\therefore \frac{d}{dx}(x^3+1) = \frac{d}{dx} u$

$\therefore 3x^2 = \frac{du}{dx} \Rightarrow dx = \frac{du}{3x^2}$

Now $\int x^2 (x^3+1)^{3/2} dx = \int \cancel{x^2} (u)^{3/2} \cdot \frac{du}{3\cancel{x^2}}$

$$= \frac{1}{3} \int u^{3/2} du$$

$$= \frac{1}{3} \cdot \frac{u^{3/2+1}}{3/2+1} = \frac{1}{3} \cdot \frac{u^{5/2}}{5/2} + C$$

$$= \frac{2}{(3)\cdot(5)} \cdot u^{5/2} + C$$

$$= \frac{2(x^3+1)^{5/2}}{15} + C$$

Ex:- $\int e^{-3x} dx$

Let $u = -3x \quad \therefore \frac{d}{dx} u = \frac{d}{dx}(-3x)$

$\therefore \frac{du}{dx} = -3 \Rightarrow \frac{du}{-3} = dx$

Now $\int e^{-3x} dx = \int e^u \cdot \frac{du}{-3} = \frac{-1}{3} \int e^u du$

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$, $c$ is constant

2. $\int [f(x) \pm g(x) \pm h(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx \pm \int h(x)\, dx$

In general,

$$\int [f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)]\, dx$$
$$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \int f_3(x)\, dx \pm \ldots \pm \int f_n(x)\, dx$$

**One Important Note:** If the power of the numerator of the integrand is greater than or equal to that of its denominator, then before integration the numerator must be divided by the denominator.

## ILLUSTRATIVE EXAMPLES

1. ***Find the following integrals:***

(i) $\int (8x^2 + 3)\, dx$ (ii) $\int x^{\frac{3}{2}}\, dx$

(iii) $\int (3x^2 + 2)^3\, x dx$ (iv) $\int \left(\frac{2+x}{x}\right) dx$

(v) $\int \left(2^x + \frac{1}{2}e^{-x} + \frac{3}{x} - \frac{1}{\sqrt[4]{x}}\right) dx$ (vi) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

(vii) $\int \sqrt{x}(1+x)\, dx$ (viii) $\int \left(e^{2x} + \frac{1}{e^x}\right) dx$

(ix) $\int \left(\frac{ax^3 + bx^2 + cx + d}{x}\right) dx$ (x) $\int \frac{x^3}{x+1}\, dx$

(xi) $\int (2x+1)^{\frac{1}{3}}\, dx$ (xii) $\int \frac{1}{4x-11}\, dx$

**Solution:** (*i*) $\int (8x^2 + 4)\, dx$

$= \int 8x^2\, dx + \int 4\, dx$ (property) (2)

$= 8\int x^2\, dx + 4\int 4\, dx$ (property) (1)

$= 8.\ \frac{x^{2+1}}{2+1} + 4x + C$ (formulae (2) and (1))

$= \frac{8}{3}x^3 + 4x + C$

**Note:** In fact, each integral includes one constant of integration. Thus the actual integral is $\left(\frac{8}{3}x^3 + C_1\right) + (4x + C_2)$. However, $C_1$ and $C_2$ can together be expressed as $C$ where $C = C_1 + C_2$.

(ii) $\int x^{\frac{3}{2}} dx = \dfrac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$ [Formula (2)]

$= \dfrac{2x^{\frac{5}{2}}}{5} + C$

(iii) $\int (3x^2+2)^3\, x dx$

$= \int (27x^7 + 54x^5 + 36x^3 + 8x)\, dx$

$= 27\int x^7\, dx + 54\int x^5\, dx + 36\int x^3\, dx + 8\int x dx$ [Properties (1) and (2)]

$= \dfrac{27x^8}{8} + 9x^6 + 9x^4 + 4x^2 + C$ [Formula (2)]

(iv) $\int \left(\dfrac{2+x}{x}\right) dx = \int \dfrac{2}{x}\, dx + \int dx$

$= 2\int \dfrac{1}{x}\, dx + \int dx$

$= 2\log x + x + C$

(v) $\int \left(2^x + \dfrac{1}{2}e^{-x} + \dfrac{3}{x} - \dfrac{1}{\sqrt[4]{x}}\right) dx$

$= \int 2^x\, dx + \dfrac{1}{2}\int e^{-x}\, dx + 3\int \dfrac{1}{x}\, dx - \int x^{-\frac{1}{4}}\, dx$

$= \dfrac{2^x}{\log 2} + \dfrac{1}{2}.\dfrac{e^{-x}}{-1} + 3\log x - \dfrac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + C$

$= \dfrac{2^x}{\log 2} - \dfrac{1}{2}e^{-x} + 3\log x - \dfrac{4}{3}x^{\frac{3}{4}} + C$ [See formulae (5) and (6)]

(vi) $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right) dx$

$= \int x^{\frac{1}{2}}\, dx - \int x^{-\frac{1}{2}}\, dx$

$= \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= \dfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$

(vii) $\int \sqrt{x}(1+x)\, dx = \int x^{\frac{1}{2}}\, dx + \int x^{\frac{3}{2}}\, dx$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$= \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + C$$

(*viii*) $\int\left(e^{2x} + \frac{1}{e^x}\right)dx = \int e^{2x}\,dx + \int e^{-x}\,dx$

$$= \frac{e^{2x}}{2} + \frac{e^{-x}}{-1} + C \qquad \text{[Formula (5)]}$$

$$= \frac{e^{2x}}{2} - e^{-x} + C$$

**Note :** $\int e^{-x}\,dx = \int e^{(-1)x}\,dx = \frac{e^{(-1)x}}{-1} = -e^{-x} \quad \left[\because \int e^{mx}\,dx = \frac{e^{mx}}{m}\right]$

(*ix*) $\int\left(\frac{ax^3 + bx^2 + cx + d}{x}\right)dx = \int\left(ax^2 + bx + c + \frac{d}{x}\right)dx$

$$= a\int x^2 dx + b\int x dx + c\int dx + d\int \frac{1}{x}\,dx$$

$$= \frac{ax^3}{3} + b\frac{x^2}{2} + cx + d\log x + c$$

(*x*) $\int \frac{x^3}{x+1}\,dx$

[When the power of the numerator of the integrand is greater than or equal to that of its denominator, then before integration the numerator is to be divided by the denominator.]

$$= \int\left[(x^2 - x + 1 - \frac{1}{x+1}\right]dx$$

$$= \int x^2\,dx - \int x dx + \int dx - \int \frac{1}{x+1}\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log(x+1) + C$$

$x+1)\,x^3\,(x^2 - x + 1$

$x^3 + x^2$

$-x^2$

$-x^2 - x$

$x$

$x + 1$

$-1$

$\therefore \frac{x^3}{x+1} = (x^2 - x + 1) - \frac{1}{x+1}$

[Formula (6)]

**N.B.** $\int \frac{1}{ax+b}\,dx = \frac{\log(ax+b)}{a} + C$

$\therefore \int \frac{1}{x+1}\,dx = \frac{\log(x+1)}{1} + C$

(xi) $\int (2x+1)^{\frac{1}{3}}\, dx = \dfrac{(2x+1)^{\frac{1}{3}+1}}{\frac{1}{3}+1} \times \dfrac{1}{2} + C$ [Form

$$= \frac{3}{8}(2x+1)^{\frac{4}{3}} + C$$

(xii) $\int \dfrac{1}{4x-11}\, dx = \dfrac{\log(4x-11)}{4} + C$ [Formu

2. ***Evaluate the following integrals:***

***(i)*** $y = \int (x^3 + 2x^{\frac{5}{2}} + 5x^{\frac{3}{2}} + 10x)\, dx$***, if*** $y = 0$ ***when*** $x = 0$

***(ii)*** $y = \int (3x^2 + 2x + 6)\, dx$***, if*** $y = 5$ ***when*** $x = 0$

***(iii)*** $P = \int \dfrac{1}{\sqrt{2ax}}\, dx, P = 2a$ ***when*** $x = \dfrac{1}{3}a^3$***. If*** $x = 2a^3$***, evaluate P.***

**N.B.** In such examples the actual value of the constant of integration is determined fro given condition.

**Solution: (*i*)**
$$y = \int (x^3 + 2x^{\frac{5}{2}} + 5x^{\frac{3}{2}} + 10x)\, dx$$

$$= \int x^3\, dx + 2\int x^{\frac{5}{2}}\, dx + 5\int x^{\frac{3}{2}}\, dx + 10\int x dx$$

$$= \frac{1}{4}x^4 + \frac{4}{7}x^{\frac{7}{2}} + 2x^{\frac{5}{2}} + 5x^2 + C$$

Given that, $y = 0$ when $x = 0$

$\therefore \quad 0 = 0 + 0 + 0 + 0 + C \Rightarrow C = 0$

$\therefore \quad y = \dfrac{1}{4}x^4 + \dfrac{4}{7}x^{\frac{7}{2}} + 2x^{\frac{5}{2}} + 5x^2$

(*ii*) $y = \int (3x^2 + 2x + 6)\, dx$

$$= 3\int x^2\, dx + 2\int x dx + 6\int dx$$

$$= 3.\frac{x^3}{3} + 2.\frac{x^2}{2} + 6x + C$$

$$= x^3 + x^2 + 6x + C$$

Given that $y = 5$ when $x = 0$

$\therefore \quad 5 = 0 + 0 + 0 + C \Rightarrow C = 5$

$\therefore \quad y = x^3 + x^2 + 6x + 5$

...e integral, *i.e.*, the function

(i) $I = \int \sqrt{5x + 7}\, dx$

**Solution:** Let $5x + 7 = z$

Integrating both sides w.r.t. $x$, we get,

$$5 \frac{dx}{dz} = 1$$

$$5dx = dz$$

$\Rightarrow$

$\Rightarrow \quad dx = \frac{dz}{5}$

$\therefore \quad I = \int \sqrt{z}.\left(\frac{dz}{5}\right)$

$$= \frac{1}{5}\int z^{\frac{1}{2}}\,dz$$

$$= \frac{1}{5}\frac{z^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{2}{15}z^{3/2} + C$$

$$= \frac{2}{15}(5x+7)^{3/2} + C$$

**N.B.** Students should integrate $5x + 7 = z$ mentally w.r.t. $z$ and write $5dx = dz$ directly.

or, $\quad \int \sqrt{5x+7\,dx} = \int (5x+7)^{1/2}\,dx$

$$= \frac{(5x+7)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\left(\frac{1}{5}\right) + C$$

$$= \frac{2}{15}(5x+7)^{3/2} + C$$

(*ii*) $I = \int x\sqrt{2x+1}\,dx$

**Solution :** Let $2x + 1 = z^2$

Integrating both sides w.r.t. $x$ we get,

$$2dx = 2z\,dz$$

$\Rightarrow \quad dx = zdz$

Again, $2x + 1 = z^2$

$\Rightarrow \quad 2x = z^2 - 1$

$\Rightarrow \quad x = \frac{z^2-1}{2}$

$\therefore \quad I = \int \frac{z^2-1}{2}\sqrt{z^2}\,.zdz$

$$= \frac{1}{2}\int (z^2-1)\,z^2\,dz$$

$$= \frac{1}{2}\int (z^4 - z^2)\,dz$$

$$= \frac{1}{2}\left(\frac{z^5}{5} - \frac{z^3}{3}\right) + C$$

$$= \frac{1}{2}\left[\frac{(2x+1)^{5/2}}{5} - \frac{(2x+1)^{3/2}}{3}\right] + C$$

$[\because z^2 = 2x + 1 \Rightarrow z = (2x+1)^{1/2}]$

(iii) $I = \int \frac{2x-2}{x^2-2x+3}\,dx$

**Solution:** Let $x^2 - 2x + 3 = t$

Integrating w.r.t. $x$ we get,

$$2x - 2 = \frac{dt}{dx}$$

$$\Rightarrow\ (2x-2)\,dx = dt$$

$$\therefore I = \int \frac{1}{t}\,dt = \log t + c$$

$$= \log(x^2 - 2x + 3) + c$$

**Note:** In calculus (differential or integral) $\log x$ implies $\log_e x$ which is also denoted by $ln\,x$.

(iv) $I = \int e^{2x^2-5x+6}\,.\,(4x-5)\,dx$

**Solution:** Let $2x^2 - 5x + 6 = t$

Integrating w.r.t. $x$ we get,

$$(4x-5)\,dx = dt$$

$$\therefore\quad I = \int e^t\,dt = e^{-t} + c = e^{2x^2-5x+6} + c$$

(v) $I = \int \frac{4x-5}{4x^2-10x+8}\,dx$

**Solution :** Let $4x^2 - 10x + 8 = z$

Integrating w.r.t. $x$, $(8x-10)\,dx = dz$

Now, $I = \frac{1}{2}\int \frac{8x-10}{4x^2-10x+8}\,dx$ ( Note this step)

$$= \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\log z + C$$

$$= \frac{1}{2}\log(4x^2 - 10x + 8) + C$$

(vi) $I = \int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

**Solution :** Let $t = e^x - e^{-x}$

$$\Rightarrow\quad \frac{dt}{dx} = e^x - (-e^{-x}) = e^x + e^{-x}$$

$$\Rightarrow\quad dt = (e^x + e^{-x})\,dx$$

$$\therefore\quad I = \int \frac{dt}{t} = \log t + c = \log(e^x - e^{-x}) + C.$$

**Important Note:** If the integrand is such that its numerator is the derivative of the denom
of the denominator ... of the integrand plus the consta
... integrand is the derivative of the den
... though the numera