

MATRIX

A matrix is a collection of **numbers arranged into a fixed number of rows and columns**. Usually the numbers are real numbers.

$$A = \begin{array}{c} \text{m-by-n matrix} \\ \hline a_{i,j} \quad \text{n columns} \xrightarrow{\text{j changes}} \\ \text{m rows} \downarrow \text{i changes} \\ \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots \\ a_{3,1} & a_{3,2} & a_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{array}$$

$A = M \times N = \text{no. of rows} \times \text{no. of columns}$

Here is an example of a matrix with three rows and three columns:

$$A = \begin{array}{c} \text{col 1} \\ \vdots \\ \text{row 1} \dots \end{array} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & 4.6 \\ 4 & -1 & 0 \end{pmatrix}$$

A **matrix** is a rectangular arrangement of numbers into rows and columns. For example, matrix A has three **rows** and three **columns**.

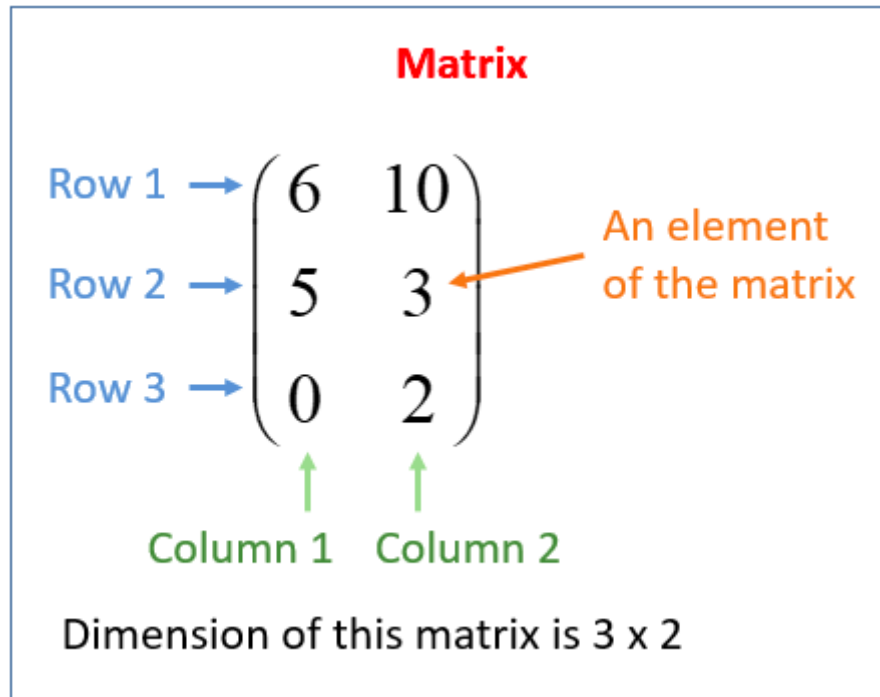
Each number that makes up a matrix is called an **element** of the matrix. The elements in a matrix have specific locations.

The upper left corner of the matrix is row 1 column 1. In the above matrix the element at row 1 col 1 is the value 1. The element at row 2 column 3 is the value 4.6.

The top row is row 1. The leftmost column is column 1. This matrix is a 3x3 matrix because it has three rows and three columns. In describing matrices, the format is:

rows X columns

The following diagram shows the rows and columns of a 3 by 2 matrix.



Dimension of matrix = order of matrix

no. of rows X no. of columns

Notation

A matrix is usually shown by a **capital letter** (such as A, or B)

Each entry (or "element") is shown by a **lower case letter** with a "subscript" of **row,column**:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

Here are some sample entries: order of matrix B=2X3

A Matrix

(This one has 2 Rows and 3 Columns)

$b_{1,1} = 6$ (the entry at row 1, column 1 is 6)

$b_{1,3} = 24$ (the entry at row 1, column 3 is 24)

$b_{2,3} = 8$ (the entry at row 2, column 3 is 8)

$$C = \begin{pmatrix} 5 & 9 & 6 & 0 \\ 8 & 1 & 12 & 3 \end{pmatrix}$$

Rows=2

Columns= 4

Order of Matrix C= 2X 4

Elements of matrix C= 1st row 1st column: 5

1st row 2nd column : 9

12: 2nd row 3rd column

0: 1st row 4th column

6: 1st row 3rd column

Adding

To add two matrices: add the numbers in the matching positions:

These are the calculations:

$$3+4=7 \quad 8+0=8$$

$$4+1=5 \quad 6-9=-3$$

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

$$6 + (-9) = 6 - 9 = -3$$

If Order of matrix A = Order of Matrix B

Then

addition of two matrices A and B can be possible

A= order 2X2

B= order 2X2

$$6 + (-9) = 6 - 9 = -3$$

Subtracting

To subtract two matrices: subtract the numbers in the matching positions:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

$$6 - (-9) = 6 + 9 = 15$$

These are the calculations:

$$3-4=-1 \quad 8-0=8$$

$$4-1=3 \quad 6-(-9)=15$$

Negative

The negative of a matrix is also simple:

$$-\begin{bmatrix} 2 & -4 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -7 & -10 \end{bmatrix}$$

These are the calculations:

$$-(2)=-2 \quad -(-4)=+4$$

$$-(7)=-7 \quad -(10)=-10$$

Multiply by a Constant

We can multiply a matrix by a **constant** (*the value 2 in this case*):

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

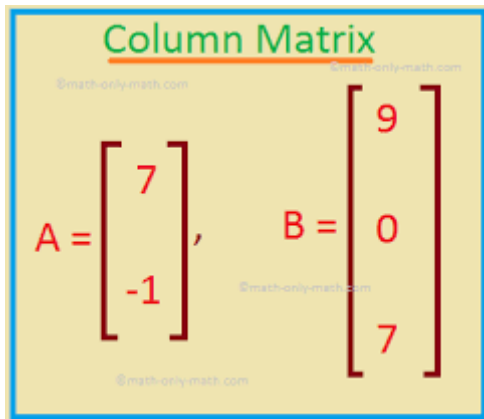
These are the calculations:

$$2 \times 4 = 8 \quad 2 \times 0 = 0$$

$$2 \times 1 = 2 \quad 2 \times -9 = -18$$

Types of matrices

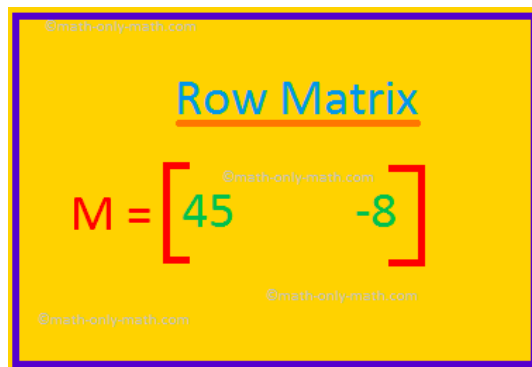
Column Matrix: A matrix which has exactly one column is called a column matrix.



The diagram shows two column matrices, A and B, on a light yellow background with a blue border. Matrix A is a 2x1 matrix with elements 7 and -1. Matrix B is a 3x1 matrix with elements 9, 0, and 7. The title "Column Matrix" is at the top in green. The matrices are separated by a comma. There are several small watermarks "@math-only-math.com" scattered around the matrices.

$$A = \begin{bmatrix} 7 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 0 \\ 7 \end{bmatrix}$$

Row Matrix: A matrix which has exactly one row is called a row matrix.



The diagram shows a row matrix M on a yellow background with a purple border. Matrix M is a 1x2 matrix with elements 45 and -8. The title "Row Matrix" is at the top in blue. There are several small watermarks "@math-only-math.com" scattered around the matrix.

$$M = \begin{bmatrix} 45 & -8 \end{bmatrix}$$

Square

A **square** matrix has the same number of rows as columns.

$$\begin{bmatrix} 2 & 0 \\ 1 & 8 \end{bmatrix}$$

above matrix is of order 2

A square matrix (2 rows, 2 columns)

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \\ 3 & 0 & 7 \end{bmatrix}$$

above matrix is of order 3

Also a square matrix (3 rows, 3 columns)

The **Main or leading Diagonal** starts at the top left and goes down to the right

$$\begin{bmatrix} 7 & 6 & 4 \\ 4 & 2 & -2 \\ 3 & 0 & 9 \end{bmatrix}$$

Transpose

A **Transpose** is where we swap entries across the main diagonal (rows become columns) like this:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 4 & -9 \\ 24 & 8 \end{bmatrix}$$

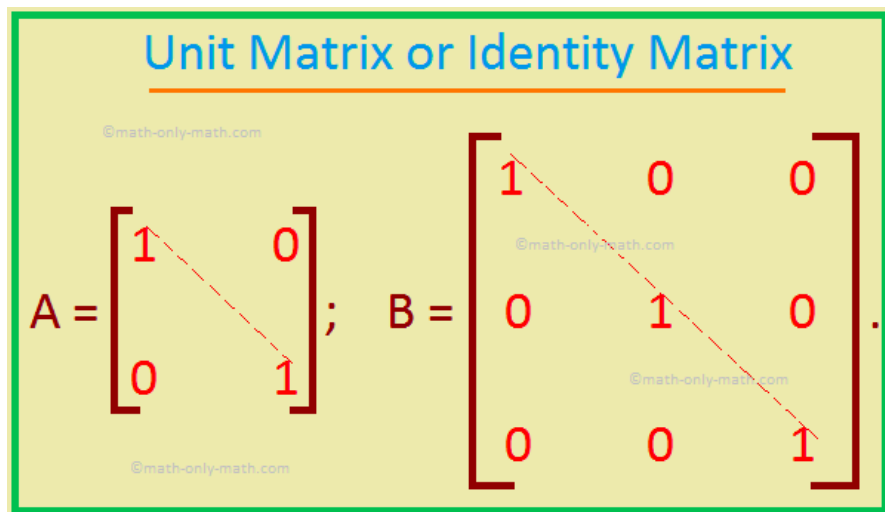
Identity Matrix

An **Identity Matrix** has **1s** on the main diagonal and **0s** everywhere else:

we can represent identity matrix using I_2 =unit matrix of order 2

I_3 = unit matrix of order 3

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



It is the matrix equivalent of the number "1", when we multiply with it the original is unchanged:

$$A \times I = A$$

$$I \times A = A$$

Diagonal Matrix

A square matrix is called a diagonal matrix when it has zero anywhere not on the main diagonal:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Scalar Matrix

A scalar matrix has all main diagonal entries the same, with zero everywhere else:

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

A scalar matrix

Triangular Matrix

Lower triangular is when all entries above the main diagonal are zero:

$$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 7 & 6 & -3 \end{bmatrix}$$

A lower triangular matrix

Upper triangular is when all entries below the main diagonal are zero:

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 4 & 11 \\ 0 & 0 & 5 \end{bmatrix}$$

An upper triangular matrix

Zero Matrix (Null Matrix)

Zeros just everywhere:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrix

Zero Matrix / Null Matrix

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$$A = \begin{bmatrix} 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Rules for multiplication


$$\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -4 \\ 6 & 8 & 0 \end{bmatrix}$$

$(2 \times \textcolor{red}{2}) \cdot (\textcolor{red}{2} \times 3)$

 **same**

$$\begin{bmatrix} 5 & -3 \\ 4 & 1 \\ 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$(3 \times \textcolor{red}{2}) \cdot (\textcolor{red}{2} \times 1)$

 **same**

Matrix B X Matrix A

no. of columns of first matrix = no. Of rows of second matrix

Then matrix multiplication is possible.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2} \times \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{2 \times 2} \quad \checkmark$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \times \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{2 \times 2} \quad \times$$

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3} \times \begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix}_{2 \times 3} \quad \times$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_{2 \times 1} \times \begin{bmatrix} x & y & z \end{bmatrix}_{1 \times 3} \quad \checkmark$$

First matrix	Second matrix	Product matrix
$\begin{pmatrix} 3 & 1 & 1 & 4 \\ 5 & 3 & 2 & 1 \\ 6 & 2 & 9 & 5 \end{pmatrix}$	$\begin{pmatrix} 4 & 9 \\ 6 & 8 \\ 9 & 7 \\ 7 & 6 \end{pmatrix}$	$= \begin{pmatrix} 55 & 66 \\ 63 & 89 \\ 152 & 163 \end{pmatrix}$
3×4	4×2	3×2

Matrices Multiplication

$$\begin{pmatrix} 4 & 2 & 4 \\ 8 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ 2 & 8 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 44 & 72 \\ 37 & 73 \end{pmatrix}$$

- $\rightarrow 4 \times 3 + 2 \times 2 + 4 \times 7 = 44$
- $\rightarrow 4 \times 5 + 2 \times 8 + 4 \times 9 = 72$
- $\rightarrow 8 \times 3 + 3 \times 2 + 1 \times 7 = 37$
- $\rightarrow 8 \times 5 + 3 \times 8 + 1 \times 9 = 73$

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Order of first matrix = 2X3

Order of Second Matrix = 3X2

Multiplying a Matrix by Another Matrix

But to multiply a matrix **by another matrix** we need to do the "[dot product](#)" of rows and columns ... what does that mean? Let us see with an example:

To work out the answer for the **1st row** and **1st column**:

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & \end{bmatrix}$$

The "Dot Product" is where we **multiply matching members**, then sum up:

$$(1, 2, 3) \cdot (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 \\ = 58$$

We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \end{bmatrix}$$

We can do the same thing for the **2nd row** and **1st column**:

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 \\ = 139$$

And for the **2nd row** and **2nd column**:

$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 \\ = 154$$

And we get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix} \quad \checkmark$$

DONE!

For multiplication to be defined, the "inner" numbers must match. The result will be determined by the "outer" numbers.

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & 3 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ -2 & 5 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 27 & -2 & 12 \\ -1 & 6 & 0 & 6 \end{bmatrix}$$

2×3 3×4 2×4

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 10 + 2 \times 20 + 3 \times 30 & 1 \times 11 + 2 \times 21 + 3 \times 31 \\ 4 \times 10 + 5 \times 20 + 6 \times 30 & 4 \times 11 + 5 \times 21 + 6 \times 31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 5 \\ -3 \end{bmatrix} \\
 = (2)(2) + (-3)(4) + (0)(5) + (4)(-3) \\
 = -20.$$

$$\begin{matrix} \mathbf{A} & \mathbf{B} & \mathbf{A} * \mathbf{B} \end{matrix} \\
 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 3 \\ 5 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1*6 + 2*5 + 3*4 & 1*3 + 2*2 + 3*1 \\ 4*6 + 5*5 + 6*4 & 4*3 + 5*2 + 6*1 \end{pmatrix}$$

$$\begin{matrix} 28 & 10 \\ 73 & 28 \end{matrix}$$

Dividing

And what about division? Well we **don't** actually divide matrices, we do it this way:

$$A/B = A \times (1/B) = A \times B^{-1}$$

where **B⁻¹** means the "inverse" of B.