Unit 2 - Mathematical Logic

Propositional Logic is concerned with statements to which the truth values, “true” and “false”, can be assigned. The purpose is to analyze these statements either individually or in a composite manner.

Prepositional Logic – Definition

A proposition is a collection of declarative statements that has either a truth value "true” or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, etc). The connectives connect the propositional variables.

Some examples of Propositions are given below −

- "Man is Mortal", it returns truth value “TRUE”
- "12 + 9 = 3 – 2", it returns truth value “FALSE”
- The sun rises in the East and sets in the West, it returns truth value “TRUE”
- 1 + 1 = 2, it returns truth value “TRUE”
- 'b' is a vowel.it returns truth value “FALSE”

The following is not a Proposition −

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true or false.

Propositions: Examples

- The following are propositions
  - — Today is Monday *M*
  - — The grass is wet *W*
  - — It is raining *R*
- The following are not propositions
  - — C++ is the best language *Opinion*
  - — When is the pretest? *Interrogative*
  - — Do your homework *Imperative*

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Trenton is the capital of New Jersey.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

- **Examples – Non propositions:**

1. It is good
2. He is the tallest person in this class

4 2+x=10

5 What a beautiful morning!

6 Get up and do your exercises

7 The number *x* is an integer.

8 Are you busy?

Connectives

In propositional logic generally we use five connectives which are –

❿ OR (∨)

❿ AND (∧)

❿ Negation/ NOT (¬/~)

❿ NAND

❿ NOR

❿ Implication / if-then (→)

❿ If and only if (⇔)

Truth Table

Since we need to know the truth value of a proposition in all possible scenarios, we consider all the possible combinations of the propositions which are joined together by Logical Connectives to form the given compound proposition. This compilation of all possible scenarios in a tabular format is called a **truth table**.

1. Negation – If p is a proposition, then the negation of p is denoted by $\neg p$, which when translated to simple English means-

"It is not the case that p" or simply "not p".

The truth value of $\neg p$ is the opposite of the truth value of p.

The truth table of $\neg p$ is-

Negation can be represented by (¬/~).

p	$\neg p$
T	F
F	T

Example:

The negation of "It is raining today", is "It is not the case that is raining today" or simply "It is not raining today".

P: Agra is the capital of India
~P : Agra is not the capital of India

P: All girls are intelligent.
~P : some girls are not intelligent.

2. Conjunction – For any two propositions p and q, their conjunction is denoted by $p \wedge q$, which means "p and q". The conjunction $p \wedge q$ is True when both p and q are True, otherwise False.

The truth table of $p \wedge q$ is-

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

P: Today is Sunday.
Q: It is raining today.
P ^ Q : Today is Sunday and It is raining today.

3. Disjunction – For any two propositions p and q, their disjunction is denoted by $p \vee q$, which means "p or q". The disjuction $p \vee q$ is True when either p or q is True, otherwise False.

The truth table of $p \vee q$ is-

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example,

The disjunction of the propositions p– "Today is Friday" and q– "It is raining today", $p \vee q$ is "Today is Friday or it is raining today". This proposition is true on any day that is a Friday or a rainy day(including rainy Fridays) and is false on any day other than Friday when it also does not rain.

P: It is cold
Q: It is raining
P v Q: It is cold or raining.

nand and ***nor*** gates

- Most common gates are ***nand*** gates and ***nor*** gates. Their truth tables are given by

Truth tables for nand *and* nor			
p	*q*	*p nand q*	*p nor q*
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	T

Conditional Opearator (If...then)

A statement that is of the form "If p, then q" is a conditional statement. Here 'p' refers to 'hypothesis' and 'q' refers to 'conclusion'.

For example, "If Cliff is thirsty, then she drinks water." This is a conditional statement. It is also called an implication. '→' is the symbol used to represent the relation between two statements. For example, A→B. It is known as the logical connector. It can be read as A implies B.

Example 1: If a number is divisible by 4, then it is divisible by 2.

Example 2: If today is Monday, then yesterday was Sunday.

p implies q
if p, then q
q if p
p only if q
p is sufficient for q
q whenever p
q is necessary for p

ex:
p:It is below feezing
q: It is snowing

If it is below freezing, it is also snowing ----- p->q
It is not snowing if it is below freezing. ------ p->~q
ex:

Given:	p: I do my homework.
	q: I get my allowance.
Problem:	What does p→q represent?

Ans:
If I do my homework, then I get an allowance."

p	q	p->q
T	T	T
T	F	F
F	T	T
F	F	T

Statements to symbolic form:

p:Ravi is poor
q:Ravi is happy

1) Ravi is poor and happy
2) Ravi is poor but happy
3) Ravi is neither poor nor happy
4) Ravi is rich and unhappy
1) p ^ q
2) p ^ q
3) ~p ^ ~q
4) ~p ^ ~q

p	q	~p	~q	~p ^ ~q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P: Ravi is tall
q: Ravi is handsome

1) Ravi is short or not handsome
2) Ravi is tall or handsome
3) It is not true that Ravi is short or not handsome

1) ~p V ~q
2) pVq
3) ~ (~pV~q)

P: You drive over 65 miles per hour
Q: You get a speeding ticket
P-> Q: If You drive over 65 miles per hour then You get a speeding ticket

Symbolic form using logical connectives:

- You don't drive over 65 miles per hour.
  ~P
- You drive over 65 miles per hour but you don't get a speeding ticket.
  P ^ ~Q
- You will get a speeding ticket if you drive over 65 miles per hour .
  P->Q
- You get a speeding ticket but you don't drive over 65 miles per hour
  Q ^ ~P

- If You don’t drive over 65 miles per hour, then You will not get a speeding ticket
  ~P -> ~Q

Ex: P: it is cold
Q: It is raining.

~P ^Q – It is not cold and it is raining.
Q -> P – If it is raining then it is cold
Neither its raining nor its cold - ~Q ^ ~P
Its raining if it is not cold. - ~P->Q
Either its raining or its cold - q v p

Biconditional Statement

A biconditional statement is a combination of a conditional statement and its converse written in the *if and only if* form.

It is a combination of two conditional statements, “if two line segments are congruent then they are of equal length” and “if two line segments are of equal length then they are congruent”.

Bi-conditionals are represented by the symbol ↔ or ⇔ .

Ex: A rectangle is a square if and only if the adjacent sides have equal length.

Ex: Two lines are parallel if and only if they have equal slope.

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

Ex: P : India will win the cricket series
Q: Indian players are in good form
P<=>Q : India will win the cricket series if and only if Indian players are in good form.

In propositions, if the last column of their truth tables contain only T then such propositions are called tautologies.

OR

Tautology: A compound statement which is always true is called tautology.

Truth Table for $(p \to q) \vee (q \to p)$				
p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

On the other hand, a proposition is called a contradiction if it contains only F in the last column of its truth table.

OR

Contradiction: A compound statement which is always false is called contradiction.

	p	q	$p \to q$	$\neg(p \to q)$	$q \wedge \neg(p \to q)$
(a)	T	T	T	F	F
(b)	T	F	F	T	F
(c)	F	T	T	F	F
(d)	F	F	T	F	F

Proposition that is neither a tautology nor a contradiction is called a contingency.

P	Q	$\neg P$	$P \to Q$	$\neg P \wedge (P \to Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Examples :
p: Babu is rich
q: Babu is happy

1) Babu is rich or babu is happy
   p Vq

2) Babu is rich and babu is happy

p^q

3) If babu is rich then babu is happy
p->q

4)Babu is happy then babu is rich
q->p

5)Babu is rich or babu is not happy
p V ~q

6) Babu is happy if and only if babu is not rich
q <-> ~p

7)Babu is not rich then babu is happy
~p->q

8)It is not true that Babu is not rich
~(~p)

9) If babu is not rich and happy then Babu is rich
(~p^q)->p

Practice Examples:
P: Ramen is coward.
Q: Ramen is lazy
r: Ramen is rich

1) Ramen is either coward or poor.
2) Ramen is neither coward nor lazy
3) It is false that Ramen is coward but not lazy
4) Ramen is coward or lazy but not rich.
5) It is false that Ramen is coward or lazy but rich
6) It is not true that Ramen is not rich
7) Ramen is rich or else Ramen is both coward and lazy.

Rules of inference for propositional logic

Inference: deriving conclusion from evidences.

Rules of inference: are the templates for constructing valid arguments

⓾ **Modus Pones:**

If *P*, then *Q*. - true
P. - true
Therefore, *Q*

1. Modus Ponens: $p \rightarrow q$ T, p T, $\therefore q$ OR $[(p \rightarrow q) \wedge p] \rightarrow q$

An example of an argument that fits the form *modus ponens*:

p: today is Tuesday

q: John will go to work

If today is Tuesday, then John will go to work. - true

Today is Tuesday. - true

Therefore, John will go to work.

⑩ **Modus Tollens:**

If *P*, then *Q*.

Not *Q*.
Therefore, not *P*.

2. Modus Tollens: $p \rightarrow q$ T, $\sim q$ T, $\therefore \sim p$ OR $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

For example:

p: the dog detects an intruder
q: the dog will bark

p->q : If the dog detects an intruder, the dog will bark.
~q: the Dog will not bark
~p:the dog does not detect an intruder

If the dog detects an intruder, the dog will bark.
The dog did not bark.
Therefore, no intruder was detected by the dog.

Another example:

p: Rex is a chicken
q: he is a bird

If Rex is a chicken, then he is a bird.
Rex is not a bird.

Therefore, Rex is not a chicken.

⑩ **Hypothetical Syllogism:**

If p, then q.
If q, then r.
(So) If p, then r

$$\frac{P \to Q, Q \to R}{\therefore P \to R}$$

3. Hypothetical Syllogism

$$\begin{array}{l} p \to q \quad T \\ \underline{q \to r \quad T} \\ \therefore p \to r \end{array}$$

OR

[(p -> q) ^ (q -> r)] -> (p -> r)

p:I do not wake up
q: I cannot go to work
r: I will not get paid

p-> q : If I do not wake up, then I cannot go to work.
q-> r: If I cannot go to work, then I will not get paid.
p->r : Therefore, if I do not wake up, then I will not get paid.

Here is a sensible example, illustrating each of the above:

- "If it is a car, then it has wheels. It is a car. Therefore, it has wheels." (Modus Ponens - CORRECT)
- "If it is a car, then it has wheels. It does not have wheels. Therefore, it is not a car." (Modus Tollens - CORRECT)
- "If it is a car, then it has wheels. It has wheels. Therefore, it is a car." (Affirming the Consequent - INCORRECT.)
- "If it is a car, then it has wheels. It is not a car.

  therefore, it does not have wheels."

  P->Q,

  ~P

  so ~Q – not valid.

- "If Red is 9, Grey is 1.

  Grey is 2.

  Therefore, Red is not 9."

P->Q,

~Q

So ~P – valid through modus tollens

Predicate Calculus

statement 1:

Dr. Roy is an effective teacher.

statement 2:

Dr. Gupta is an effective teacher.

Common part.

Statement:

x > 5

x is variable------ is the subject of the statement

is greater than 5 ----- is property that the subject of statement can have

p(x) = x>5

p denotes the predicate 'is greater than 5' and x is a variable.

Example:

Assusme P(x) as the statement 'x>5'. Find the truth values of p(6) and p(4).

p(x)=x>5

p(6)=6>5 ---- truth value is 'true'

p(4)=4>5 ---- truth value is 'false'

Example:

Let Q(x,y) denote the statement 'x=y+5'. Find the truth values of Q(2,3) and Q(5,0)

x=2, y=3

‘2=3+5’ ---false

x=5 and y =0

‘5=0+5’----true

Example: Determine the truth value for the following statements.

A={1,2,3,4,5}

- x+3<10

p(1)= 1+3<10 ---- truth value = true

p(2)= 2+3<10 ---- truth value = true

p(3)= 3+3<10 ---- truth value = true

p(4)= 4+3<10 ---- truth value = true

p(5)= 5+3<10 ---- truth value = true

- x+3<=7

Truth value for p(1), p(2), p(3) and p(4) are true but for p(5) truth value is false

- x+3=7
- x+3>10