RELATIONSHIP BETWEEN BOOLEAN ALGEBRA AND LATTICE
(Boolean algebra as lattice)

A lattice L is a partially ordered set in which every pair of elements x, y ∈ L has a least upper bound denoted by l u b (x, y) and a greatest lower bound denoted by g l b (x, y).

The two operations of meet and join denoted by and∧∨ respectively defined for any pair of elementsx, y ∈ L as

$$x \vee y = \mathrm{l\,u\,b}\,(x,y) \text{ and } x \wedge y = \mathrm{g\,l\,b}\,(x,y)$$

A lattice L with two operations of meet and join shall be a Boolean algbera if L is

1. Complemented: i.e.

(i) If must have a least element 0 and a greatest element 1 and
(ii) For every element x ∈ L these must exist an element x′∈ L such that

$$x \vee x' = 1 \text{ and } x \wedge x' = 0$$

2. Distributed: i.e. $\forall\, x, y, z \in L$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and, $$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Boolean Algebra:

A complemented distributive lattice is known as a Boolean Algebra. It is denoted by (B, ∧,∨,',0,1), where B is a set on which two binary operations ∧ (*) and ∨(+) and a unary operation (complement) are defined. Here 0 and 1 are two distinct elements of B.

Since (B,∧,∨) is a complemented distributive lattice, therefore each element of B has a unique complement.

Ex: (P(A), *, +, ‵, 0,1) where A ={1,2} and P(A) – Power set of A –Boolean Algebra

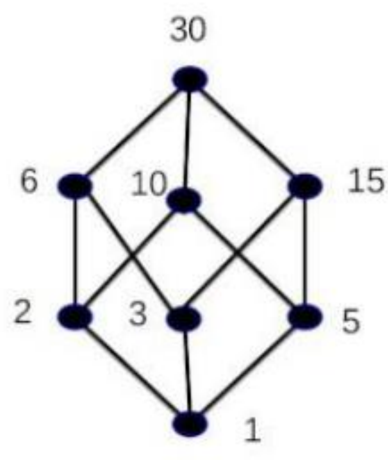
Ex: S6 is boolean algebra or not
S6={1,2,3,6}

Element	Complement	GLB (0) - 1	LUB (1) - 6
1	6	1	6
2	3	1	6
3	2	1	6
6	1	1	6

From above table we can observe that all elements have only one complement. So S6 is complemented and distributed lattice so it is also called boolean algebra.

- D_{30} is complemented lattice

Element	Its Complement
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1



D_{30}

Properties of Boolean Algebra:

0′ = 1 and 1′ = 0

Law/Theorem	Law of Addition	Law of Multiplication
Identity Law	$x + 0 = x$	$x \cdot 1 = x$
Complement Law	$x + x' = 1$	$x \cdot x' = 0$
Idempotent Law	$x + x = x$	$x \cdot x = x$
Dominant Law	$x + 1 = 1$	$x \cdot 0 = 0$
Involution Law	$(x')' = x$	
Commutative Law	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative Law	$x+(y+z) = (x+y)+z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Distributive Law	$x \cdot (y+z) = x \cdot y+x \cdot z$	$x+y \cdot z = (x+y) \cdot (x+z)$
Demorgan's Law	$(x+y)' = x' \cdot y'$	$(x \cdot y)' = x' + y'$
Absorption Law	$x + (x \cdot y) = x$	$x \cdot (x + y) = x$

Examples:

1) Find the values of $1.0 + (0 + \overline{1}) + \overline{0}.0$
2) Show that $(1.1) + [(0 . \overline{1}) + 0] = 1$
3) Find the values of $(\overline{1} . \overline{0}) + (1 . \overline{0})$

- Note:
- The complement, Boolean sum and Boolean product correspond to the logic operators ~ , ˅ and ˄ respectively, where 0 corresponds to F (False) and 1 corresponds to T (True)
- Equalities in Boolean algebra can be considered as equivalences of compound propositions.

1.0+(0+1’)+0’.0
=0+(0+0)+1.0
=0+0+0
=0

Boolean Operations

- The **complement** is denoted by a bar. It is defined by

$$\overline{0} = 1 \quad \text{and} \quad \overline{1} = 0.$$

- The **Boolean sum**, denoted by + or by OR, has the following values:

$$1 + 1 = 1, \quad 1 + 0 = 1, \quad 0 + 1 = 1, \quad 0 + 0 = 0$$

- The **Boolean product**, denoted by $\cdot$ or by AND, has the following values:

$$1 \cdot 1 = 1, \quad 1 \cdot 0 = 0, \quad 0 \cdot 1 = 0, \quad 0 \cdot 0 = 0$$

What are Boolean Functions?

A special mathematical function with n degrees and where X={0,1} is the Boolean domain with a being a non-negative integer.

$f : X^n \rightarrow X$ The Boolean function helps in describing the way in which the Boolean output is derived from Boolean inputs.

Example – Let, **F(A,B)=A’B’**

2+3-5*4/10 – arithmetic expression

Boolean function means it has boolean expression

Boolean Expression:

Consider a Boolean algebra (B, ∨,∧,',0,1).A Boolean expression over Boolean algebra B is defined as

1. Every element of B is a Boolean expression.
2. Every variable name is a Boolean expression.
3. If a_1 and a_2 are Boolean expression, then a_1,'∨ a_2 and a_1∧ a_2 are Boolean expressions.

Example: Consider a Boolean algebra ({0, 1, 2, 3},∨,∧,',0,1).

1. 0 ∨ x
2. (2∧3)
3. $(x_1 \vee x_2) \wedge (x_1 \wedge x_3)$ = (x1+x2) * (x1*x3)

are Boolean expressions over the Boolean Algebra.

A Boolean expression that contains n distinct variables is usually referred to as a Boolean expression of n variables.

Canonical Forms:

There are two types of canonical forms:

1. Disjunctive Normal Forms or Sum of Products or (SOP).
2. Conjunctive Normal Forms or Products of Sums or (POS).

A	B	C	Term	Minterm
0	0	0	x'y'z'	m_0
0	0	1	x'y'z	m_1
0	1	0	x'yz'	m_2
0	1	1	x'yz	m_3
1	0	0	xy'z'	m_4
1	0	1	xy'z	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

Sum of Products (SOP) form

The product of all the variables either in direct or complemented form is known as minterm.

Example

Let, **F(x,y,z)=x’y’z’+xy’z+xyz’+xyz**

Or, **F(x,y,z)=m0+m5+m6+m7**

Hence,

F(x,y,z)=Σ(0,5,6,7)

The complement form of F(x,y,z)

F’(x,y,z)=x’yz+x’y’z+x’yz’+xy’z’

Or, **F’(x,y,z)=m3+m1+m2+m4**

Hence,

F’(x,y,z)=Σ(3,1,2,4)=Σ(1,2,3,4)

Minterms:

1. f(A,B,C) = A’B’C’ + A’BC’ + A’BC + ABC

   $= m_0 + m_2 + m_3 + m_7$

   $= \Sigma m(0,2,3,7)$

2. f(A,B,C)= A’B’C + A’BC + AB’C + ABC

   $= m_1 + m_3 + m_5 + m_7$

   $= \Sigma m(1,3,5,7)$

3. f(A,B,C)= A’B’C’ + A’BC’ + A’BC + ABC’

   $= m_0 + m_2 + m_3 + m_6$

   $= \Sigma m(0,2,3,6)$

Product of Sums (POS) form

All the variables in direct or complemented form when added is a maxterm.

A	B	C	Term	Maxterm
0	0	0	$x+y+z$	M_0
0	0	1	$x+y+z'$	M_1
0	1	0	$x+y'+z$	M_2
0	1	1	$x+y'+z'$	M_3
1	0	0	$x'+y+z$	M_4
1	0	1	$x'+y+z'$	M_5
1	1	0	$x'+y'+z$	M_6
1	1	1	$x'+y'+z'$	M_7

Let **F(x,y,z)=(x+y+z).(x+y+z’).(x+y’+z).(x’+y+z)**

Or, **F(x,y,z)=M0.M1.M2.M4**

Hence,

F(x,y,z)=п(0,1,2,4)

F’’(x,y,z)=(x+y’+z’).(x’+y+z’).(x’+y’+z).(x’+y’+z’)

Or, F(x,y,z)=M3.M5.M6.M7

Hence,

F’(x,y,z)=п(3,5,6,7)

Convert SOP form into Standard form/complete sop form/canonical sop

Ex 1 : F(a,b,c) = a.b

= ab . 1

= ab (c +c′)

= abc + abc′

Ex2: F(a,b,c) = a + b

= a.1 + b.1

= a(b+b′) + b. (a+a′)

= (ab +ab′).1 +(ab +ba′).1

= (ab +ab′)(c+c′) + (ab +ba′) (c+c′)

= abc +ab′c′ +ab′c +abc′ + abc + abc′ +a′bc + a′bc′

=ab′c′ +ab′c + abc + abc′ +a′bc + a′bc′

Ex 3
first term = A′
Second term=BC
variables= A,B,C

F= A′ + BC

= A′ .1.1 + BC .1

=A′. (B+B′) . (C+C′) + (A+A′)BC

= (A′B +A′B′) (C+C′)+ ABC+A′BC

=A′BC+A′BC′+A′B′C+A′B′C′+ABC+A′BC

= A′BC+A′BC′+A′B′C+A′B′C′+ABC

Ex 4 : F= X(XY′+X′Y+Y′Z)

$=XXY'+XX'Y+XY'Z$
$=XXY'+0.Y+XY'Z$
$=XXY'+XY'Z$
$=XY'+XY'Z$
$=XY'.1 + XY'Z$
$=XY'(Z+Z')+XY'Z$
$=XY'Z+XY'Z'+XY'Z$
$=XY'Z+XY'Z'$

Ex: 5 $F=Z(X'+Y)+Y'$
$=ZX'+ZY+Y'$
$=(Y+Y')ZX'+(X+X')ZY+(Z+Z')(X+X')Y'$
$=X'YZ+X'Y'Z+XYZ+X'YZ+(Z+Z')XY'+X'Y'$
$=X'YZ+\underline{X'Y'Z}+XYZ+X'YZ+XY'Z+XY'Z'+\underline{X'Y'Z}+X'Y'Z'$

$=X'YZ+X'Y'Z+XYZ+XY'Z+XY'Z'+X'Y'Z'$

Ex 6: $F= (X'+Y)'+X'Y$
$=XY'+X'Y$ [demorgan's law]
$=XY'(Z+Z')+X'Y(Z+Z')$
$= XY'Z+ XY'Z'+ X'YZ+ X'YZ'$

Ex : 7 $F=Y(X+YZ)'$

$=Y(X'(YZ)')$
$=YX'(Y'+Z')$
$=YX'Y'+YX'Z'$
$=X'YZ'$

Ex 8:$F=X(XY+Y'+X'Y)$
$=XXY+XY'+XX'Y$

Ex: 9 $E= (x+y)'.(xy')'$
$= (x'y').(x'+y)$
$=x'y'x'+x'y'y$
$=x'y'+0$
$=x'y'(z+z')$
$=x'y'z+x'y'z'$

Ex 10: $E=x(xy+x'y+y'z)$
$=xxy+xx'y+xy'z$

=xy+xy′z
=xy(z+z′)+xy′z
=xyz+xyz′+xy′z

Complete POS form

Given,

$$
\begin{aligned}
F(A, B, C) &= (A+B)(B+C) \\
&= (A+B) + (C \,.\, C')\,(B+C) + (A \,.\, A') \\
&= (A+B+C)\,(A+B+C')\,(B+C+A)\,(B+C+A') \\
&= (A+B+C)\,(A+B+C')\,(A+B+C)\,(A'+B+C) \\
&= (A+B+C)\,(A+B+C')\,(A'+B+C)
\end{aligned}
$$

Ex : 1 (A+B)(B+C)
=(A+B+0)(B+C+0)
=(A+B+C.C′)(B+C+A.A′)
=(A+B+C)(A+B+C′)(A+B+C)(A′+B+C)
=(A+B+C)(A+B+C′)(A′+B+C)

$$
\begin{aligned}
&(\overline{A}+B+\overline{C})(A+\overline{C}) \\
&=(\overline{A}+B+\overline{C})(A+\overline{C}+B\overline{B}) \\
&=(\overline{A}+B+\overline{C})(A+\overline{C}+B)(A+\overline{C}+\overline{B}) \\
&=(\overline{A}+B+\overline{C})(A+B+\overline{C})(A+\overline{B}+\overline{C})
\end{aligned}
$$

Ex : 2 F= A′+BC

=(A′+B)(A′+C)
=(A′+B+0)(A′+C+0)
=(A′+B+C.C′)(A′+C+B.B′)
=(A′+B+C)(A′+B+C′)(A′+C+B)(A′+C+B′)
=(A′+B+C)(A′+B+C′)(A′+C+B′)

Simplify the Boolean expression:

Ex : 1 Prove that (A+B′)(C+B′) = AC +B′

LHS (A+B′)(C+B′)

AC+AB′+B′C+B′B′

AC+AB′+B′C+B′

AC+AB′+B′(C+1)

AC+AB′+B′.1

AC+B′(A+1)

AC+B′

EX 2 :Prove that A + A′.B = A+B

LHS -

(A+A′)(A+B) // DISTRIBUTIVE LAW

1.(A+B) //COMPLEMENT LAW

A+B

Ex : 3 xy + x′z+yz = xy +x′z

xy + x′z+yz

xy+x′z+yz(x+x′)

Xy +x′z+yzx+yzx′

Xy + xyz + x′z +x′yz

xy(1+z) + x′z(1+y)

xy+x′z

Equivalent Boolean expression:
Each table has the same first two columns. The true/false values in the last column of each table are the same, which shows that the two boolean expressions are equivalent. One expression can be used in place of the other in a program

Example 1 :

a	b	a ∨ b	p = ~(a ∨ b)
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

a	b	~a	~b	q = ~a ∧ ~b
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

Example 2
Expression1:

A’B + A’B’C’+ABC’+AB’C’

Expression2:

A’B+A’BC+A’B’C’+ABC’+AB’C’+A’BC’

Expression1
A’B + A’B’C’+ABC’+AB’C’

A	B	C	A’	B’	C’	A’B	A’B’C’	ABC’	AB’C’	EXP1
0	0	0	1	1	1	0	1	0	0	1
0	0	1	1	1	0	0	0	0	0	0
0	1	0	1	0	1	1	0	0	0	1
0	1	1	1	0	0	1	0	0	0	1
1	0	0	0	1	1	0	0	0	1	1
1	0	1	0	1	0	0	0	0	0	0
1	1	0	0	0	1	0	0	1	0	1
1	1	1	0	0	0	0	0	0	0	0

A′B+A′BC+A′B′C′+ABC′+AB′C′+A′BC′

A	B	C	A′	B′	C′	A′B	A′B′C′	A′BC	ABC′	AB′C′	A′BC′	EX2
0	0	0	1	1	1	0	1	0	0	0	0	1
0	0	1	1	1	0	0	0	0	0	0	0	0
0	1	0	1	0	1	1	0	0	0	0	1	1
0	1	1	1	0	0	1	0	1	0	0	0	1
1	0	0	0	1	1	0	0	0	0	1	0	1
1	0	1	0	1	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	0	0	1	0	0	1
1	1	1	0	0	0	0	0	0	0	0	0	0

Compare expression1 and expression2, since both expressions have same truth table, the expressions are equivalent.