Relation on a set

A relation between two sets is **a collection of ordered pairs containing one object from each set**. If the object x is from the first set and the object y is from the second set, then the objects are said to be related if the ordered pair (x,y) is in the relation. A function is a type of relation.

Definition: Relation

A relation in mathematics defines the relationship between two different sets of information. If two sets are considered, the relation between them will be established if there is a connection between the elements of two or more non-empty sets.

In the morning assembly at schools, students are supposed to stand in a queue in ascending order of the heights of all the students. This defines an ordered relation between the students and their heights.

A ***relation*** from a set A to a set B is a subset of A×B (Cartesian Product). Hence, a relation R consists of ordered pairs (a,b), where a∈A and b∈B. If (a,b)∈R, we say that ***is related to*** , and we also write **aRb**.

aRb : a is related to b

$a \not R b$. a is not related to b

A={1,2,6}

B={1,3,5}

A X B= {(1,1),(1,3),(1,5),(2,1),(2,3),(2,5),(6,1),(6,3),(6,5)}

There is a “<” Relation from set A to set B

Relation ‘<’

R= {(1,3),(1,5),(2,3),(2,5)}

Relation ‘>’

R= {(2,1),(6,1),(6,3),(6,5)}

Relation ‘<=’

R= {(1,1),(1,3),(1,5),(2,3),(2,5)}

1R1

1R3

1R5

2R3

2R5

6R5 is in R. False

Remark

We can also replace R by a symbol, especially when one is readily available. This is exactly what we do in, for example, $a<b$. To say it is not true that $a<b$, we can write $a\not<b$. Likewise, if $(a,b)\notin R$, then a is not related to b, and we could write a/Rb. But the slash may not be easy to recognize when it is written over an uppercase letter. In this regard, it may be a good practice to avoid using the slash notation over a letter.

Example:

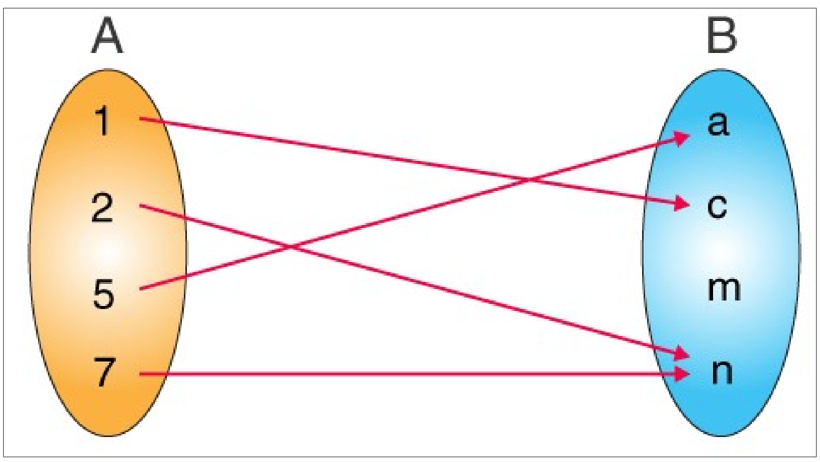
Let $A=\{1,2,3,4,5,6\}$ and $B=\{1,2,3,4\}$. Define $(a,b)\in R$ if and only if $(a-b)\bmod 2=0$. Then
AXB={(1,1)(1,2)(1,3)(1,4)(
$R=\{(1,1),(1,3),(2,2),(2,4),(3,1),(3,3),(4,2),(4,4),(5,1),(5,3),(6,2),(6,4)\}$

Definition
The ***domain*** of a relation $R \subseteq A \times B$ is defined as $$\text{domain of } R = \{a \in A \mid (a,b) \in R \text{ for some } b \in B\},$$ and the ***range*** is defined as $$\text{range of } R = \{b \in B \mid (a,b) \in R \text{ for some } a \in A\}.$$

This mapping depicts a relation from set A into set B. A relation from A to B is a subset of A x B. The ordered pairs are (1,c),(2,n),(5,a),(7,n). For defining a relation, we use the notation where,

set {1, 2, 5, 7} represents the domain.

set {a, c, n} represents the range.



Let, A = { 1, 2, 9 } and B = { 1, 3, 7 }

- **Case 1** − If relation R is 'equal to' then R = { (1, 1) }

  Dom(R) = { 1} , Ran(R) = { 1 }

- **Case 2** − If relation R is 'less than' then R = { (1, 3), (1, 7), (2, 3), (2, 7) }

  Dom(R) = { 1, 2 } , Ran(R) = { 3, 7 }

- **Case 3** − If relation R is 'greater than' then R = { (2, 1), (9, 1), (9, 3), (9, 7) }

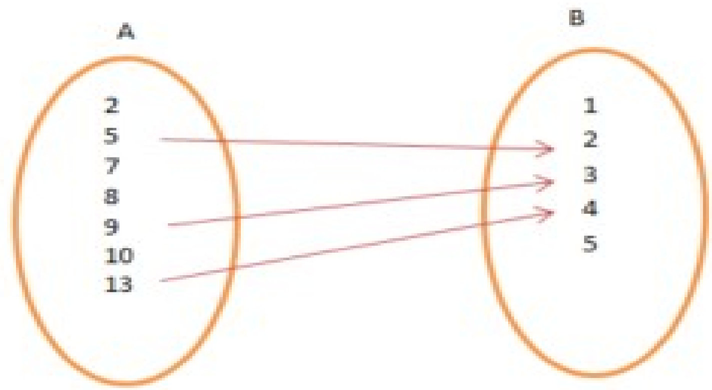
  Dom(R) = { 2, 9 }, Ran(R) = { 1, 3, 7 }

Example:

Let A and B be two sets such that A = {2, 5, 7, 8, 10, 13} and B = {1, 2, 3, 4, 5}. Then,

R = {(x, y): x = 4y – 3, x ∈A and y ∈B} (Set-builder form)

R = {(5, 2), (10, 3), (13, 4)} (Roster form)



Representation of relation

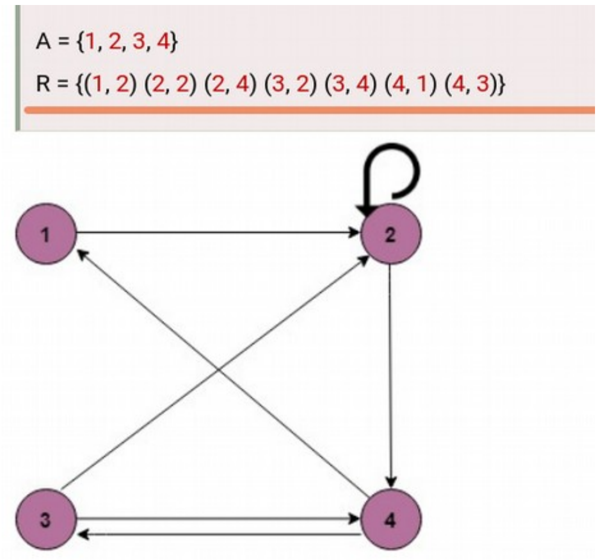
Relation as Matrix

Let P = {1, 2, 3, 4}, Q = {a, b, c, d}

and R = {(1, a), (1, b), (1, c), (2, b), (2, c), (2, d)}.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left\{ \begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\} \end{array}$$

Relation as a Directed Graph: There is another way of picturing a relation R when R is a relation from a finite set to itself.



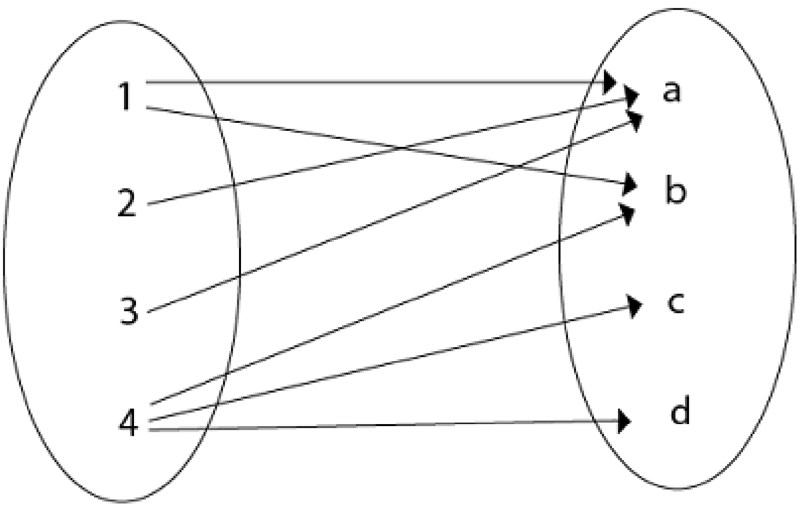
Relation as an Arrow Diagram

Let P = {1, 2, 3, 4}

Q = {a, b, c, d}

R = {(1, a), (2, a), (3, a), (1, b), (4, b), (4, c), (4, d)}

The arrow diagram of relation R is shown in fig:



Inverse Relation

In simple words, if (x, y) ∈ R, then (y, x) ∈ R^{-1} and vice versa. i.e., If R is from A to B, then R^{-1} is from B to A. Thus, if R is a subset of A x B, then R^{-1} is a subset of B x A.

If $(x, y) \in R \Leftrightarrow (y, x) \in R^{-1}$

Relation Inverse Relation

Thus

If $R = \{(x, y) : x \in A, y \in B\}$

Then $R^{-1} = \{(y, x) : y \in B, x \in A\}$

Example:

Relation: {(a, 1), (b, 2), (c, 3), (d, 4)}

Inverse Relation: {(1, a), (2, b), (3, c), (4, d)}

Inverse Relation Examples

Have a look at the following relations and their inverse relations on two sets A = {a, b, c, d, e} and B = {1, 2, 3, 4, 5}.

- If R = {(a, 2), (b, 4), (c, 1)} ⇔ R^{-1} = {(2, a), (4, b), (1, c)}
- If R = {(c, 1), (b, 2), (a, 3)} ⇔ R^{-1} = {(1, c), (2, b), (3, a)}
- If R = {(b, 3), (c, 2), (e, 1)} ⇔ R^{-1} = {(3, b), (2, c), (1, e)}

Example:

A={1,2,3,4,5}

B={1,2,3,4,5,6,7,8,9,10}

Relation : is equal to (<=)

Find out Relation set R, Domain set, Range set. Also represent your relation set R using graph, arrow and matrix representation. Find $R^{-1.}$

A={1,2,3,4,5}

B={1,2,3,4,5,6,7,8,9,10}

AxB={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,8),(1,9),(1,10) (2,1),(2,2),(2,3)(2,4)(2,5)(2,6)(2,7)(2,8)(2,9)(2,10) (3,1),(3,2),(3,3)(3,4)(3,5) (3,6)(3,7)(3,8)(3,9)(3,10) (4,1),(4,2),(4,3)(4,4)(4,5)(4,6)(4,7)(4,8)(4,9)(4,10) (5,1),(5,2),(5,3)(5,4)(5,5)(5,6)(5,7)(5,8)(5,9)(5,10)}

R={ (1,1),(2,2),(3,3),(4,4),(5,5)}

Domain set ={1,2,3,4,5}

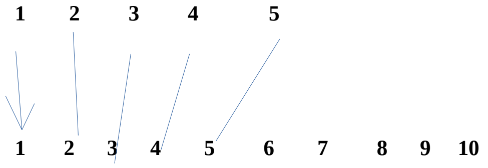
Range set = {1,2,3,4,5}

Matrix

R={ (1,1),(2,2),(3,3),(4,4),(5,5)}

	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	0	1	0	0	0	0	0	0
5	0	0	0	0	1	0	0	0	0	0

Graph



R={ (1,1),(2,2),(3,3),(4,4),(5,5)}

R-1 = { (1,1),(2,2),(3,3),(4,4),(5,5)}

Example:

A={1,2,3,4,5}

B={1,2,3,4,5,6,7,8,9,10}

Relation : is less than (<)

Find out Relation set R, Domain set, Range set.

Also represent your relation set R using graph, arrow and matrix

representation. Find R^{-1}.

AxB={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,8),(1,9),(1,10) (2,1),(2,2),(2,3)(2,4)(2,5)(2,6)(2,7)(2,8)(2,9)(2,10) (3,1),(3,2),(3,3)(3,4)(3,5)(3,6)(3,7)(3,8)(3,9)(3,10) (4,1),(4,2),(4,3)(4,4)(4,5)(4,6)(4,7)(4,8)(4,9)(4,10) (5,1),(5,2),(5,3)(5,4)(5,5)(5,6)(5,7)(5,8)(5,9)(5,10)}

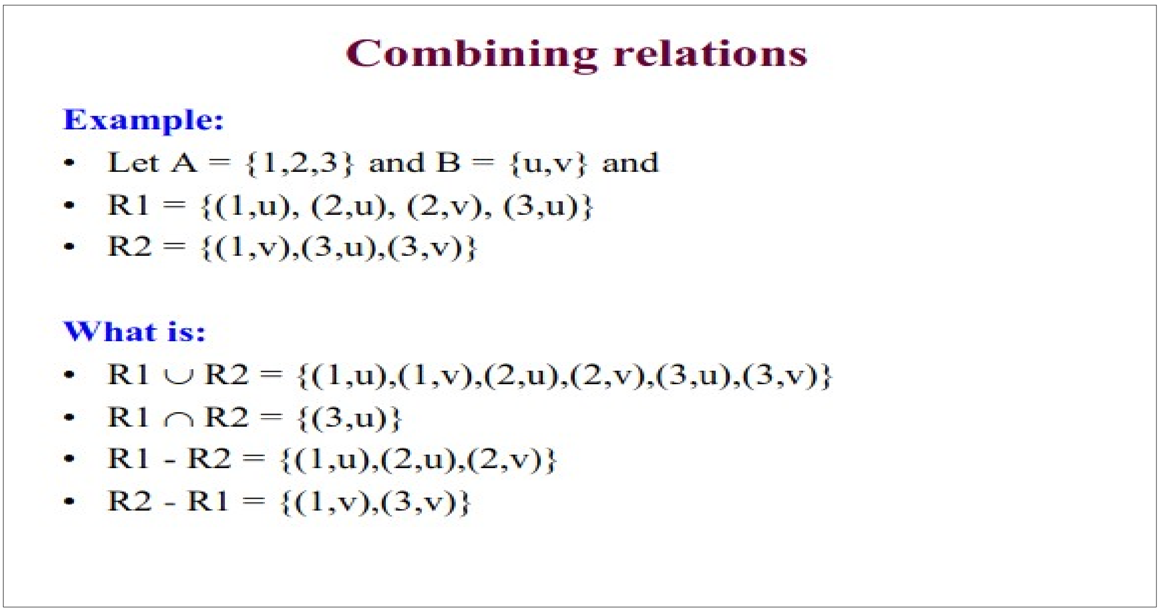
R={(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(2,3)(2,4)(2,5)(2,6)(2,7)(2,8)(2,9)(2,10),(3,4)(3,5)(3,6)(3,7)(3,8)(3,9)(3,10),(4,5)(4,6)(4,7)(4,8)(4,9)(4,10),(5,6)(5,7)(5,8)(5,9)(5,10)}

Domain={1,2,3,4,5}

Range={2,3,4,5,6,7,8,9,10}

	1	2	3	4	5	6	7	8	9	10
1	0	1	1	1	1	1	1	1	1	1
2	0	0	1	1	1	1	1	1	1	1
3	0	0	0	1	1	1	1	1	1	1
4	0	0	0	0	1	1	1	1	1	1
5	0	0	0	0	0	1	1	1	1	1

Combining Relations
A={1,2,3} B={u,v}
AXB={(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
R1UR2= {(1,u),(2,u),(2,v),(3,u),(1,v),(3,v)}
R1∩R2= {(3,u)}
R1-R2 ={(1,u),(2,u),(2,v)}
R2-R1 = {(1,v),(3,v)}
R1’ = {(1,v),(3,v)}
R2’={(1,u),(2,u),(2,v)}



LetA={1,2,3}andB={1,2,3,4}.
The relation R_1 is on A and the relation R_2 is on B:
R_1={(1,1),(2,2),(3,3)}
R_2={(1,1),(1,2),(1,3),(1,4)}.
Definition 1.Determine the following relations.
(a)$R_1 \cup R_2$ = {(1,1)(2,2),(3,3),(1,2),(1,3),(1,4)}
(b)$R_1 \cap R_2$ = {(1,1)}

(c)$R_1 - R_2$ ={(2,2),(3,3)}

(d)$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

Let $A=\{1,2,3\}$ and $B=\{1,2,3,4\}$

Universal Relation

$AXB= \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4)\}$

$R1=\{(1,1),(2,2),(3,3)\}$

$R1' =\{ (1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4)\}$

Example:
set x = {4,5,6)
set y = {a,b,c}
set z = {l,m,n}

Relation R1 : X->y
Relation R2: y->z

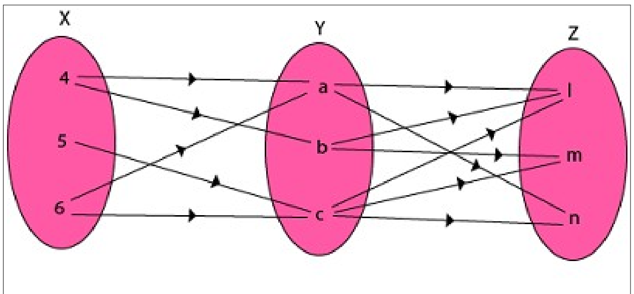


Fig : R_1 o R_2

$R_1 = \{(4, a), (4, b), (5, c), (6, a), (6, c)\}$

$R_2 = \{(a, l),(a, n), (b, l), (b, m), (c, l), (c, m), (c, n)\}$

Find the composition of relation R_1 o R_2

R_1 o R_2 $= \{(4, l), (4, n), (4, m), (5, l), (5, m), (5, n), (6, l), (6, m), (6, n)\}$

Composite of relations

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite of R and S** is the relation consisting of the ordered pairs (a,c) where $a \in A$ and $c \in C$, and for which there is a $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by S o R.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- S o R = {(1,b),(3,a),(3,b)}

Let P = {2, 3, 4, 5}. Consider the relation R and S on P defined by

R = {(2, 2), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 3)}

S = {(2, 3), (2, 5), (3, 4), (3, 5), (4, 2), (4, 3), (4, 5), (5, 2), (5, 5)}.

Find the matrices of the above relations.

Use matrices to find the following composition of the relation R and S.

(i)RoS (ii)RoR (iii)SoR

P={2,3,4,5} Relation R : P->P
P={2,3,4,5} Relation S: P->P

R={(2,2),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),(5,3)}

S={(2,3),(2,5),(3,4),(3,5),(4,2),(4,3),(4,5)(5,2),(5,5)}

RoS={(2,3),(2,5),(2,4),(2,2),(3,2),(3,3),(3,5),(4,2),(4,5),(5,4),(5,5)}

R={(2,2),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),(5,3)}

R={(2,2),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),(5,3)}

RoR= {(2,2),(2,3),(2,4),(2,5),(3,5),(3,3),(4,3),(5,4),(5,5)}

S={(2,3),(2,5),(3,4),(3,5),(4,2),(4,3),(4,5)(5,2),(5,5)}
R={(2,2),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5),(5,3)}

SoR =

R o S = {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)}.

R o R = {(2, 2), (3, 2), (3, 3), (3, 4), (4, 2), (4, 5), (5, 2), (5, 3), (5, 5)}

S o R = {(2, 4) , (2, 5), (3, 3), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5), (5, 2), (5, 3), (5, 4), (5, 5)}.

Types of relations:

Universal Relation :

A relation R from set A to set B is said to be a universal relation if R=AXB.

A={1,2}
B={3}

AXB= {(1,3,),(2,3)}

R= “is less than” (<)
Find R?
R={(1,3),(2,3)}
Here R= AXB, so R is an universal relation.

Empty relation:

Example:

A={1,2}
B={3}

AXB= {(1,3),(2,3)}
Relation: (>)
Find R?
R= {}

Here R= {}, so R is a empty relation or null relation or void relation

Reflexive Relation
Example:
A={1,2,3,4}
Relation is (divides)
AXA= {(1,1),(1,2)(1,3)(1,4)(2,1)(2,2)(2,3),(2,4),(3,1)(3,2)(3,3),(3,4),(4,1)(4,2)(4,3),(4,4)}
R={(1,1),(1,2)(1,3)(1,4),(2,2),(2,4),(3,3),(4,4)}

Universal Relation:

R is not equal to AXA so R is not an universal relation.

Empty Relation:

R is not an empty relation.

Reflexive Relation

for all a, (a,a) ∈ R, so R is a reflexive relation.

Irreflexive Relation

for all a, (a,a) ∈ R, so R is a reflexive relation, so it cant be irreflexive relation

Symmetric Relation

for all a, (a,b)∈ R but (b,a) not belongs to R, so it can not be a symmetric relation.

Antisymmetric Relation

for all a, (a,b)∈ R but (b,a) not belongs to R, so it can be an anti- symmetric relation.
(a,a) allows

Asymmetric Relation

for all a, (a,b)∈ R but (b,a) not belongs to R, but (a,a) ∈ R so it can not be called an asymmetric relation.

Transitive Relations

for all (a,b)∈ R and (b,c)∈ R, we have (a,c)∈ R, so R is a transitive relation.

Example:
A= {1,2,3,4,5}
Relation is (<=)
AxA={1,2,3,4,5}X{1,2,3,4,5}

AxA ={(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)}

R={(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5)}

Find out types of relations.

Universal Relation:
R is not equal to AXA so R is not an universal relation.

Empty Relation:
R is not an empty relation.

Reflexive Relation

for all a, (a,a) ∈ R, so R is a reflexive relation.

Irreflexive Relation
for all a, (a,a) ∈ R, so R is a reflexive relation, so it cant be irreflexive relation

Symmetric Relation

for all a, (a,b)∈ R but (b,a) not belongs to R, so it cant be a symmetric relation.

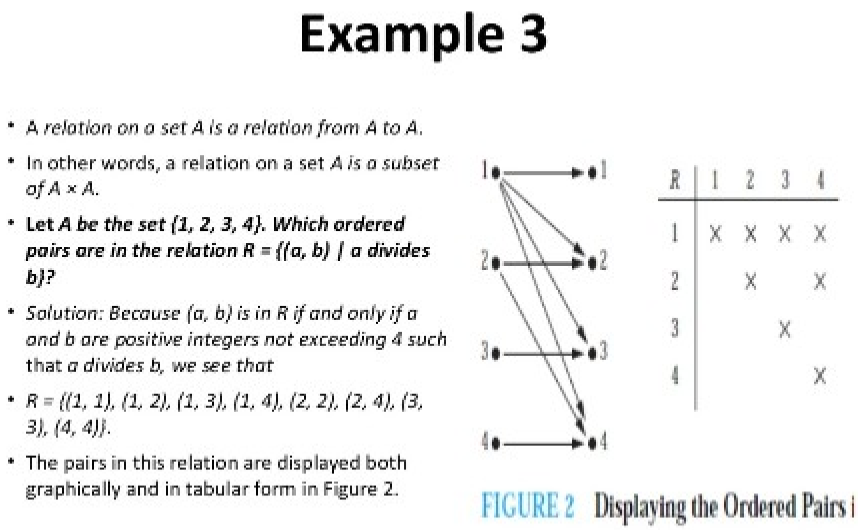
Antisymmetric Relation

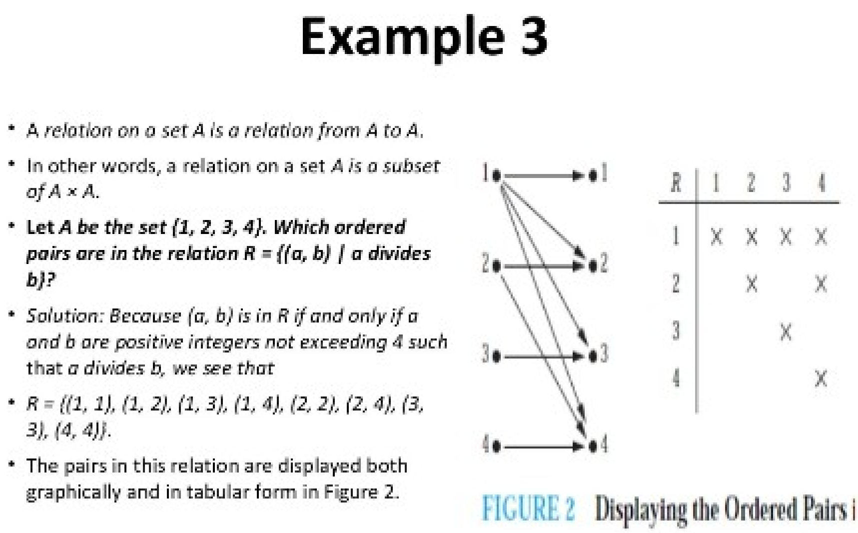
for all a, (a,b)∈ R but (b,a) not belongs to R, so it can be an anti- symmetric relation.

Asymmetric Relation

for all a, (a,b)∈ R but (b,a) not belongs to R, but (a,a) ∈ R so it can not be called an asymmetric relation.

Transitive Relations





Divides is reflexie, anti symmetric and transitive.

A= {1,2,3,4}
Relation is (>=)
AxA={1,2,3,4}X{1,2,3,4}

Compatible Relation

A relation R on set A is said to be compatible relation , if and only if it satisfies two properties−

- Reflexive
- Symmetric

Equivalence Relation

A relation R on set A is said to be an Equivalence Relation if it satisfies three properties −

- Reflexive
- symmetric
- Transitive.

Partial Order Relations

A relation R on set A is said to be an Partial Order Relation if it satisfies

- Reflexive
- Anti symmetric
- Transitive.