Homework X

CS XXXX

First Last
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Problem 1

[16 points]

Vikings always tell the truth and Saxons always lie. Given the following information, use a truth table to determine what type each person is or if their status cannot be determined. Be sure to provide a conclusion based on your work.

Person A says: Nothing.
Person B says: "C is not a Viking if I am a Viking."
Person C says: "If I or B is a Viking, then A is a Viking."

Solution:

Let a be the proposition "Person A is a Viking", let b be the proposition "Person B is a Viking" and let c be the proposition "Person C is a Viking".

We can then represent the statement of Person B as $(b \to \neg c)$, and the statement of Person A as $((c \vee b) \to a)$.

Thus, we can construct a truth table to examine the possible scenarios, as follows:

a	b	c	$\neg c$	$(c \vee b)$	$(b \to \neg c)$	$(c \vee b) \to a$
T	T	T	F	T	F	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	T	F	T	T	T	F
F	F	T	F	T	T	F
F	F	F	T	F	T	T

Under the assumption that there are people that are neither Vikings nor Saxons, we cannot determine the types of either Person A, Person B, or Person C. However, if we assume that Vikings and Saxons are the only people to exist, we can make the following observation:

We note that if "Person X is a Viking" is true, then the statement that Person X makes must also be true. Similarly, if "Person X is not a Viking" is true, then Person X must be a Saxon, and thus, the statement that Person X makes must be false. In other words, the only possibilities are those such that $b \equiv (b \to \neg c)$ and $c \equiv ((c \vee b) \to a)$. We observe that this is true only in row 7, leading to the conclusion that

$$\boxed{\text{Person A is a Saxon, Person B is a Viking, and Person C is a Saxon}}$$

Please turn over...

Problem 2

[6 points each]

For each of the following statements, push all negations ($\neg$) as far as possible so that no negation is to the left of a quantifier (in other words, the negation is immediately to the left of a predicate). Make sure to cite all steps.

(a) $\neg\forall x(\exists y A(x,y) \land \neg\exists z B(z,x))$

(b) $\neg\exists x(\forall y A(y,x) \to \neg\forall z C(z,x))$

(c) $\neg\exists x(\neg\forall y(B(y,x) \to \exists z D(z,y)) \land \forall w E(w,x))$

Solution:

(a)

$$\begin{aligned}
&\neg\forall x(\exists y A(x,y) \land \neg\exists z B(z,x)) \\
&\equiv \exists x \neg(\exists y A(x,y) \land \neg\exists z B(z,x)) && \text{(De Morgan's Law for Quantifiers)} \\
&\equiv \exists x(\neg\exists y A(x,y) \lor \neg\neg\exists z B(z,x)) && \text{(De Morgan's Law for Propositions)} \\
&\equiv \exists x(\neg\exists y A(x,y) \lor \exists z B(z,x)) && \text{(Double Negation Law)} \\
&\equiv \boxed{\exists x(\forall y \neg A(x,y) \lor \exists z B(z,x))} && \text{(De Morgan's Law for Quantifiers)}
\end{aligned}$$

(b)

$$\begin{aligned}
&\neg\exists x(\forall y A(y,x) \to \neg\forall z C(z,x)) \\
&\equiv \forall x \neg(\forall y A(y,x) \to \neg\forall z C(z,x)) && \text{(De Morgan's Law for Quantifiers)} \\
&\equiv \forall x \neg(\neg\forall y A(y,x) \lor \neg\forall z C(z,x)) && \text{(Conditional Disjunction)} \\
&\equiv \forall x(\neg\neg\forall y A(y,x) \land \neg\neg\forall z C(z,x)) && \text{(De Morgan's Law for Propositions)} \\
&\equiv \forall x(\forall y A(y,x) \land \neg\neg\forall z C(z,x)) && \text{(Double Negation Law)} \\
&\equiv \boxed{\forall x(\forall y A(y,x) \land \forall z C(z,x))} && \text{(Double Negation Law)}
\end{aligned}$$

(c)

$$\begin{aligned}
&\neg\exists x(\neg\forall y(B(y,x) \to \exists z D(z,y)) \land \forall w E(w,x)) \\
&\equiv \forall x \neg(\neg\forall y(B(y,x) \to \exists z D(z,y)) \land \forall w E(w,x)) && \text{(De Morgan's Law for Quantifiers)} \\
&\equiv \forall x \neg(\neg\forall y(\neg B(y,x) \lor \exists z D(z,y)) \land \forall w E(w,x)) && \text{(Conditional Disjunction)} \\
&\equiv \forall x(\neg\neg\forall y(\neg B(y,x) \lor \exists z D(z,y)) \lor \neg\forall w E(w,x)) && \text{(De Morgan's Law for Propositions)} \\
&\equiv \forall x(\forall y(\neg B(y,x) \lor \exists z D(z,y)) \lor \neg\forall w E(w,x)) && \text{(Double Negation Law)} \\
&\equiv \boxed{\forall x(\forall y(\neg B(y,x) \lor \exists z D(z,y)) \lor \exists w \neg E(w,x))} && \text{(De Morgan's Law for Quantifiers)}
\end{aligned}$$

Please turn over...

Problem 3

[18 points]

Prove the following, noted first by Cantor himself.

`The power set of the natural numbers` $\mathcal{P}(\mathbb{N})$ `is not countable.`

Solution: by contradiction

We proceed with a proof by contradiction. We thus assume that there exists a bijection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Let $n \in \mathbb{N}$ and $S \in \mathcal{P}(\mathbb{N})$. Since f is a bijection, $\forall S(\exists n f(n) = S)$. We now consider the set $B = \{n \in \mathbb{N} \mid n \notin f(n)\}$. Since B is constructed by choosing elements of $\mathbb{N}$, $B \in \mathcal{P}(\mathbb{N})$. Additionally, $\forall n(n \in B \iff n \notin f(n))$. Hence, $\forall n(f(n) \neq B)$. Therefore, by construction, we have shown that $\exists S(\forall n f(n) \neq S)$. This is equivalent to $\neg(\forall S(\exists n f(n) = S))$.

Hence, we have shown $\forall S(\exists n f(n) = S)$ and $\neg(\forall S(\exists n f(n) = S))$, that is, P and $\neg P$ – a contradiction. We can thus assume that our initial assumption was false and that there does not exist a bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$. This allows us to conclude that the power set of the natural numbers $\mathcal{P}(\mathbb{N})$ is not countable. ■

Please turn over...

Problem 4

[16 points]

Let $n \in \mathbb{Z}$. Prove or disprove that if n is odd, then $64 \mid (2n^2+22)(n^2+15)$.

Solution: **by cases**

We use a proof by cases with a direct proof in each case to demonstrate that if n is odd, then $64 \mid (2n^2+22)(n^2+15)$. We assume that $n = 2k+1, k \in \mathbb{Z}$.

We perform a series of algebraic simplifications on $(2n^2+22)(n^2+15)$, as follows:

$$\begin{aligned}
(2n^2+22)(n^2+15) &= (2(2k+1)^2+22)((2k+1)^2+15) \\
&= (2(4k^2+4k+1)+22)(4k^2+4k+1+15) \\
&= (8k^2+8k+2+22)(4k^2+4k+1+15) \\
&= (8k^2+8k+24)(4k^2+4k+16) \\
&= 8(k^2+k+3)\cdot 4(k^2+k+4) \\
&= 32(k^2+k+3)(k^2+k+4)
\end{aligned}$$

We now consider two cases.

Case 1: k is odd

We assume $k = 2q+1, q \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
32(k^2+k+3)(k^2+k+4) &= 32(k^2+k+3)((2q+1)^2+(2q+1)+4) \\
&= 32(k^2+k+3)(4q^2+4q+1+2q+1+4) \\
&= 32(k^2+k+3)(4q^2+6q+6) \\
&= 32(k^2+k+3)\cdot 2(2q^2+3q+3) \\
&= 64(k^2+k+3)(2q^2+3q+3)
\end{aligned}$$

Therefore, in this case, by definition, $64 \mid (2n^2+22)(n^2+15)$.

Case 2: k is even

We assume $k = 2q, q \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
32(k^2+k+3)(k^2+k+4) &= 32(k^2+k+3)((2q)^2+(2q)+4) \\
&= 32(k^2+k+3)(4q^2+2q+4) \\
&= 32(k^2+k+3)\cdot 2(2q^2+q+2) \\
&= 64(k^2+k+3)(2q^2+q+2)
\end{aligned}$$

Therefore, in this case, by definition, $64 \mid (2n^2+22)(n^2+15)$.

As we shown by direct proof in each case that $(2n^2+22)(n^2+15)$ is a multiple of 64, we can conclude that $64 \mid (2n^2+22)(n^2+15)$. ■

Please turn over...

Problem 5

[16 points]

We say a function f is big-Theta of another function g if it holds that $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$ simultaneously. Show that $n\log(n)$ is $\Theta(\log n!\,)$.

Solution: **by witnesses**

Let $f(n) = n\log(n)$ and $g(n) = \log n!$, where $n \in \mathbb{N}^+$. We have

$$\begin{aligned}
\log n! &= \log\left(\prod_{i=1}^{n} i\right) && \text{(Definition of } n!\text{)}\\
&= \sum_{i=1}^{n} \log(i) && (\log(ab) = \log(a) + \log(b))\\
&\leq \sum_{i=1}^{n} \log(n) && (i \in \{1,2,3,\ldots,n\} \implies \log(i) \leq \log(n))\\
&\leq \log(n)\sum_{i=1}^{n} 1 && (\log(n) \text{ is constant})\\
&\leq \log(n) \times n && (\textstyle\sum_{i=1}^{n} 1 = n)\\
&\leq n\log(n) && \text{(Commutativity of multiplication)}
\end{aligned}$$

Given $n \in \mathbb{N}^+$, we have demonstrated that $\forall n(\log n! \leq n\log n)$. Hence, with witnesses $C = 1, k = 1$, $\log n!$ is $\mathcal{O}(n\log n)$.

$$\begin{aligned}
2\log(n!\,) &= \log\left((n!\,)^2\right) && (c\log(a) = \log(a^c))\\
&= \log(n!\cdot n!\,) && (a^2 = a\cdot a)\\
&= \log\left(\left(\prod_{i=1}^{n} i\right)\cdot\left(\prod_{k=0}^{n-1}(n-k)\right)\right) && \text{(Definition of } n!\text{)}\\
&= \log\left(\left(\prod_{i=1}^{n} i\right)\cdot\left(\prod_{i=1}^{n-1}(n-i+1)\right)\right) && \text{(Setting } k = i-1\text{)}\\
&= \log\left(\left(\prod_{i=1}^{n} i\right)\cdot\left(\prod_{i=1}^{n}(n-i+1)\right)\right) && (n-n+1 = 1)\\
&= \log\left(\prod_{i=1}^{n} i(n-i+1)\right) && \text{(Same indices for products)}\\
&\geq \log\left(\prod_{i=1}^{n} n\right) && (i(n-i) \geq n-i \implies i(n-i+1) \geq n)\\
&\geq \log(n^n) && (\textstyle\prod_{i=1}^{n} n = n^n)\\
&\geq n\log n && (\log(a^c) = c\log(a))
\end{aligned}$$

Given $n \in \mathbb{N}^+$, we have demonstrated that $\forall n(2\log n! \geq n\log n)$. Hence, with witnesses $C = 2, k = 1$, $n\log n$ is $\mathcal{O}(\log n!\,)$.

As we have shown that f is $\mathcal{O}(g)$, and g is $\mathcal{O}(f)$, we can conclude that f is $\Theta(g)$, that is, $n\log(n)$ is $\Theta(\log n!\,)$. ■

Problem 6

[16 points]

Let f and g be functions such that $f(x) = 3^{5x}$, where $f : \mathbb{R} \to \mathbb{R}$ and $g(x) = x^5$, where $g : \mathbb{R}^+ \to \mathbb{R}$.

Determine whether $g(x)$ is $O(f(x))$. Justify your answer using witnesses. If $g(x)$ is not $O(f(x))$ then show an argument using a proof by contradiction using witnesses as to why $g(x)$ is not $O(f(x))$.

Solution: **by witnesses**

We proceed using a proof by witnesses to show that $g(x)$ is $\mathcal{O}(f(x))$ where $f(x) = 3^{5x}$, $f : \mathbb{R} \to \mathbb{R}$ and $g(x) = x^5$, $g : \mathbb{R}^+ \to \mathbb{R}$.

1)	$\log(x) \leq x$	$\forall x > 1$	x grows faster than $\log(x)$ and $1 > \log(1)$
2)	$5\log(x) \leq 5x$	$\forall x > 1$	Multiplying both sides in 1) by 5
3)	$5\log(x) \leq (5\log(3))x$	$\forall x > 1$	Multiplying the RHS in 2) by $\log(3) > 1$
4)	$\log(x^5) \leq (\log(3^5))x$	$\forall x > 1$	Applying the property $a\log(b) = \log(b^a)$ on 3)
5)	$2^{\log(x^5)} \leq 2^{(\log(3^5))x}$	$\forall x > 1$	Raising both sides of 4) to the power of 2
6)	$2^{\log(x^5)} \leq \left(2^{\log(3^5)}\right)^x$	$\forall x > 1$	Rewriting the RHS in 5) using $a^{bc} = \left(a^b\right)^c$
7)	$x^5 \leq \left(3^5\right)^x$	$\forall x > 1$	Applying $2^{\log(f(x))} = f(x)$ on 6)
8)	$x^5 \leq 3^{5x}$	$\forall x > 1$	Rewriting the RHS in 7) using $\left(a^b\right)^c = a^{bc}$
$\therefore$	x^5 is $\mathcal{O}(3^{5x})$		Witnesses $C = 1, k = 1$

We have demonstrated that x^5 is $\mathcal{O}(3^{5x})$ using witnesses $C = 1, k = 1$, that is, we have shown that there exists C, k such that for all $x > k$, $|g(x)| \leq C|f(x)|$. Hence, we can conclude by the definition of Big-O that $g(x)$ is $\mathcal{O}(f(x))$. $\blacksquare$