Rajiv Gandhi University of Knowledge Technologies- AP.

(Catering to the Educational Needs of Gifted Rural Youth of Andhra Pradesh)

PROBABILITY AND SATATISTICS

Probability Distributions

Discrete Probability distributions:

1. Bernoulli distribution
2. Binomial distribution
3. Poisson distribution
4. Negative Binomial distribution
5. Geometric distribution
6. Hyper geometric distribution.

Continuous Probability distributions:

7. Uniform distribution
8. Exponential distribution
9. Beta and Gamma distributions
10. Normal distribution.

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Bernoulli trail: Any trail of random experiment is called Bernoulli trail if it satisfies the following conditions

(a) Trail should be finite

(b) Trail should be independent

(c) There are only two possible outcomes success and failure

(d) The probability of a success is each trail is same.

I.Bernoulli distribution: A random variable X is said to follow the Bernoulli distribution if its probability mass function given by $P(X = x) = p^x(1 - p)^{1-x}$, $x = 0\ \&1\ only$

$P(X = x) = p^x(q)^{1-x}$, $x = 0,1$

Note:

- Mean of the Bernoulli distribution is p
- Variance of the Bernoulli distribution is pq

Ex: What is the probability of getting a score of not less than 5 in a throwing a die?

Sol: There are six possible scores S={1, 2, 3, 4, 5, 6}

Any score in {1, 2, 3, 4} is a failure and any score in {5, 6} is a success.

The probability of getting a score of not less than 5 in a throw of a die is $p = \frac{2}{6} = \frac{1}{3}$

The Bernoulli trial $P(X = x) = p^x(1 - p)^{1-x}$, $x = 0,1$

$$P(X = 1) = P(success) = p^1(q)^{1-1} = p = \frac{1}{3}$$

II.Binomial Distribution: Let an experiment consists of "n" **Bernoulli trails** such that for each trail p is the probability of success and q that of a failure then the probability of **x** success in a series of **n** trails is given by $\mathbf{nc_x\, p^x q^{n-x}}$ where p+q=1

i.e; A random variable X is said to follow the binomial distribution if its probability mass function given by $P(X = x) = B(n, p, x) = nc_x\, p^x(1 - p)^{n-x} = \binom{n}{x} p^x q^{n-x}$, $x = 0,1,2, .. n$ Where p+q=1

Notation: $X \sim B(n, p)$ Read as follows binomial distribution with parameters n and p.

- n ⟶ number of trails
- P ⟶ Probability of success
- q⟶ Probability of failure (1-p)
- x ⟶ number of successes out of n trails. x=0,1,2….

Note: This distribution is used when we need x successes out of n trails.

- Mean of the binomial distribution is np
- Variance of the binomial distribution is npq

• Mode of the binomial distribution is

$$\text{Mode}=\begin{cases}\text{intigral part of } (n+1)p; & \text{if } (n+1)p \text{ is not an intiger}\\ (n+1)p \text{ and } (n+1)p-1; & \text{if} (n+1)p \text{ is an intiger}\end{cases}$$

1. In tossing a coin 10 times simultaneously. Find the probability of getting

(a) Exactly 6 heads.

(b) At least 7 heads

(c) At most 3 heads

Sol: Given n = 10, Probability of getting a head in tossing a coin $p = \frac{1}{2}$

Probability of getting no head $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

The probability of getting x successes among n trails is $P(X = x) = nc_x\ p^x q^{n-x}$

(a) Probability of getting exactly six heads is given by

$$P(X = 6) = 10c_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6} = 0.205$$

(b) Probability of getting at least seven heads is given by

P(X ≥7) =P (x=7) + P(x=8) + P(x=9) + P(x=10)

$$P(X \geq 7) = 10c_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + 10c_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + 10c_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + 10c_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$$

$$P(X \geq 7) = \left(\frac{1}{2}\right)^{10} (10c_7 + 10c_8 + 10c_9 + 10c_{10}) = \frac{176}{1024}$$

(c) Probability of getting at most 3 heads is given by

P(X ≤ 3) = P(X =0) + P(X =1) + P(X=2) + P(X=3)

$$P(X \leq 3) = 10c_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10-0} + 10c_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{10-1} + 10c_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{10-2} + 10c_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{10-3}$$

$$P(X \leq 3) = \left(\frac{1}{2}\right)^{10} (10c_0 + 10c_1 + 10c_2 + 10c_3) = \frac{176}{1024}$$

2. The mean and variance of binomial distribution are 4 and 4/3 respectively, find P(X≥1).

Sol: Given that Mean of binomial distribution=np=4

Variance binomial distribution =npq=4/3

$$\frac{npq}{np} = \frac{\frac{4}{3}}{4} \Longrightarrow q = \frac{1}{3}$$

$$p = 1 - q = \frac{2}{3}$$

Since $np = 4 \Longrightarrow n = \frac{4}{p} = \frac{4}{1} * \frac{3}{2} = 6$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 6c_0 p^0 q^{6-0} \qquad P(X = x) = nc_x\ p^x q^{n-x}$$

$$P(X \geq 1) = 1 - 1\left(\frac{1}{3}\right)^6 = 0.99$$

3. The probability of a man hitting a target is 1/4.

i.If he fires 7 times, what is the probability of his hitting the target

(a) At least twice.

(b) At most 3 times?

ii.How many times must he fires so that the probability of his hitting the target at least once is more than 2/3?

Sol: n= number of trials = 7,

p= the probability of hitting a target =1/4

q=The probability of not hitting a target=3/4

(i)

(a) At least twice

$$P(X \geq 2) = 1 - P(X < 2) = 1 - \{P(X = 0) + P(X = 1)\}$$

$$P(X \geq 2) = 1 - \left\{7c_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{7-0} + 7c_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{7-1}\right\} = \frac{4547}{8192} = 0.555$$

(b) At most 3 times:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X \leq 3) = 7c_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^7 + 7c_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^6 + 7c_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^5 + 7c_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^4 = \frac{3087}{4096} = 0.929$$

(ii) Given that at least once is more than 2/3

$$\Rightarrow P(X \geq 1) > \frac{2}{3}$$

$$\Rightarrow 1 - P(X < 1) > \frac{2}{3}$$

$$\Rightarrow 1 - P(X = 0) > \frac{2}{3}$$

$$\Rightarrow 1 - nc_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{n-0} > \frac{2}{3}$$

$$\Rightarrow 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$\Rightarrow 1 - \frac{2}{3} > \left(\frac{3}{4}\right)^n$$

$\Rightarrow \frac{1}{3} > \left(\frac{3}{4}\right)^n$ This is satisfied for n=4

4. Out of 800 families with 5 children each, how many would you expect to have

(a) 3 Boys

(b) 5 Girls

(c) Either 2 or 3 boys

(d) At least one boy? Assume equal probabilities for boys and girls.

Sol: No. of children n=5

p=Probability of each boy =1/2

q= Probability of each girl=1/2

The probability distribution is $P(X=x)=nc_x\, p^x q^{n-x}$

(a) 3 Boys:

$P(X=3)=5c_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{5}{16}=0.3125$ per family

Thus for 800 families the probability of number of families having 3 boys =800*0.3125=250 families

(b) 5 Girls:

$P(5\text{ girls})=P(X=5)=5c_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0=\frac{1}{32}=0.03125$ per family

Thus for 800 families the probability of number of families having 5 girls =800*0.3125=25 families

(c) Either 2 or 3 boys

$P(\text{Either 2 or 3 boys})=P(X=2)+P(X=3)$

$P(X=2)+P(X=3)=5c_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3+5c_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2=\frac{20}{32}=\frac{5}{8}$ per family

Expected number of families with 2or 3 boys= $800*\frac{5}{8}=500$ families

(d) At least one boy:

$P(\text{At least 1 boys})=P(X\geq 1)=1-P(X=0)$

$P(X\geq 1)=1-5c_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5=\frac{31}{32}=0.968$ per family

Expected number of families with one boy=800*0.9687=775

5. Six dice are thrown 729 times. How many times do you expect at least three dice to show a 5 or 6?

Sol: Given that n=6

p=Probability of occurrence of 5 or 6 in one throw $p=\frac{2}{6}=\frac{1}{3}$

$q=1-\frac{1}{3}=\frac{2}{3}$

The probability of getting at least three dice to show a 5 or 6

$P(X\geq 3)=P(X=3)+P(X=4)+P(X=5)+P(X=6)$

$P(X\geq 3)=6c_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3+6c_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2+6c_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1+6c_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0=\frac{233}{729}=0.319$

The expected number of such cases in 729 times = 729*0.319= 232.55

6. In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

Sol: p=Probability that the bomb strikes the target =50% =1/2

Probability that out of n bombs, at least two strike the target, is greater than 0.99. i.e; $P(X \geq 2) \geq 99\%$

Let X be a random variable representing the number of bombs striking the target.

Then $P(X = x) = nc_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x} = nc_x \frac{1}{2^n}$, $x = 0,1,2 \ldots \ldots$

We should have $P(X \geq 2) \geq 0.99$

$1 - P(X \leq 1) \geq 0.99$

$P(X \leq 1) \leq 0.01$

$P(X = 0) + P(X = 1) \leq 0.01$

$$nc_0 \frac{1}{2^n} + nc_1 \frac{1}{2^n} \leq 0.01$$

$$\frac{1}{2^n}(1 + n) \leq 0.01$$

$$\frac{1}{0.01}(1 + n) \leq 2^n$$

$100(1 + n) \leq 2^n$

By trial method, n=11.

Hence the minimum number of bombs needed to destroy the target completely is 11.

Tutorial Questions:

1. 12% of the tablets produced by a tablet machine are defective. What is the probability that out of a random sample of 20 tablets produced by the machine, 5 are defective? Answer: $\mathbf{0.0567}$

2. 20% Of the bulbs produced are defective. Find probability that at most 2 bulbs out of 4 bulbs are defective. Answer: $\mathbf{0.9728}$

3. If 3 of 12 car drivers do not carry driving license, what is the probability that a traffic inspector who randomly checks 3 car drivers, will catch 1 for not carrying driving license. (Use binomial dist.) **Answer**: $\mathbf{27}/64$

4. The probability that India wins a cricket test match against Australia is given to be 1/3. If India and Australia play 3 tests matches, what is the probability that

(a) India will lose all the three test matches.

(b) India will win at least one test match.

Answer: (**a**) $\mathbf{0.2963}$, (**b**) $\mathbf{0.7037}$

5. What are the properties of Binomial Distribution? The average percentage of failure in a certain examination is 40. What is the probability that out of a group of 6 candidates, at least 4 passed in examination? Answer: $\mathbf{0.5443}$

6. The probability that in a university, a student will be a post-graduate is 0.6. Determine the probability that out of 8 students (a) None (b) Two (c) At least two will be post-graduate Answer: (**a**) $\mathbf{0.0007}$, (**b**) $\mathbf{0.0413}$, (**c**) $\mathbf{0.9914}$

7. The probability that an infection is cured by a particular antibiotic drug within 5 days is 0.75. Suppose 4 patients are treated by this antibiotic drug. What is the probability that (a) no patient is cured (b) exactly two patient are cured (c) At least two patients are cured. Answer: (**a**) $\mathbf{0.0039}$, (**b**)$\mathbf{0.2109}$, (**c**)$\mathbf{0.9492}$

8. Assume that on the average one telephone number out of fifteen called between 1 p.m. and 2 p.m. on weekdays is busy. What is the probability that if 6 randomly selected telephone numbers were called (a) not more than three, (b) at least three of them would be busy? Answer: $\mathbf{0.9997}$, $\mathbf{0.0051}$

9. A dice is thrown 6 times getting an odd number of success, Find probability

(a) Five success

(b) At least five success

(c) At most five success. **Answer**: (**a**) $\mathbf{3}/$, (**b**) $\mathbf{7}/\mathbf{64}$, (**c**) $\mathbf{63}/\mathbf{6}4$

10. A multiple choice test consists of 8 questions with 3 answer to each question (of which only one is correct). A student answers each question by rolling a balanced dice & checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 & the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction? **Answer**: $\mathbf{P(X \geq 6) = 0.0197}$

11. Obtain the binomial distribution for which mean is 10 and variance is 5.

12. Determine binomial distribution whose mean is 4 and variance is 3 and hence evaluate $P(X \geq 2)$.

Ans:$\mathbf{(X \geq 2) = 0.9365}$

13. For the binomial distribution with n = 20, p = 0.35. Find Mean, Variance and Standard deviation. **Answer**: (**a**)$\mathbf{7.0000}$ (**b**)$\mathbf{4.55}$, (**c**)$\mathbf{2.1331}$

14. If the probability of a defective bolt is 0.1 Find mean and standard deviation of the distribution of defective bolts in a total of 400. **Answer**: $\boldsymbol{\mu = 40, \sigma = 6}$

15. Find the maximum n such that probability of getting no head in tossing a fair coin n times is greater than 0.1 [Ans: n=3]

16. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution [Ans: p=0.2]

17. A multiple-choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction? [Ans: 0.0197]

[Hint: 75% correct answers implies 6 of 8 questions should be correct)

18. A coffee connoisseur claims that he can distinguish between a cup of instant coffee and a cup of percolator coffee 75% of the time. It is agreed that his claim will be accepted if he correctly identifies at least 5 of the 6 cups. Find his chances of having the claim (i) accepted, (ii) rejected, when he does have the ability he claims.

[Ans: (i) 0.534, (ii) 0.466]

19. Determine the binomial distribution for which the mean is 4 and variance 3. [Ans: n=16, p=1/4]

20. In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails. [Ans:31]

21. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts. [Ans: 323]

22. An irregular six-faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many items in 10,000 sets of 10 throws each, would you expect it to give no even number? [Ans: 1]

23. Find the probability that at most 5 defective components will found in a lot of 200. Experience shows that 2% of such components are defective. Also find the probability of more than five defective components. [Ans: 0.7867, 0.2133]

24. Assume that 50% of all engineering students are good in mathematics. Determine the probabilities that among 18 engineering students (i) exactly 10 (ii) at least 10 (iii)at most 8 (iv) at least 2 and at most 9are good in mathematics.

III.Poisson Distribution: A random variable X is said to follow Poisson distribution if its probability mass function given by

$$\mathbf{P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \dots\dots\dots}$$

Notation: Read as $\boldsymbol{X \sim P(\lambda)}$ as follows poisson distribution with parameter λ .

Conditions for Applicability of Poisson Distributions: This distribution is as similar Binomial distribution where

(a) The Number of trials n are very large. $\mathrm{i.e};\ n \to \infty$

(b) The probability of occurrence of an event p is very small. $\mathrm{i.e};\ p \to 0$

(c) np= λ(say) =finite

Examples of Poisson Distribution:

1) Number of defective bulbs produced by a reputed company.

2) Number of telephone calls per minute at a switchboard.

3) Number of cars passing a certain point in one minute.

4) Number of printing mistakes per page in a large text

Note:

- Mean of the Poisson distribution is λ
- Variance of the Poisson distribution is λ
- Mode of the Poisson distribution is Mode= $\begin{cases} \lambda \text{ and } \lambda - 1; & \text{if } \lambda \text{ is an intiger} \\ \text{intigral part of } \lambda; & \text{if } \lambda \text{ is not an intiger} \end{cases}$

Type1: When λ is not given

1. If 5% of the electrical bulbs manufacturing by a company are defective, find the probability that in a sample of 100 bulbs (a) none is defective (b) 5 bulbs will be defective.

Sol: Given that n=100, p=5%=0.05

$\lambda = np = 100*0.05 = 5$

Let X be the number of defective bulbs $P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$

(a) $P(X=0) = \frac{(5)^0}{0!}e^{-5} = e^{-5}$

(b) $P(X=5) = \frac{(5)^5}{5!}e^{-5}$

2. Assuming that one in 80 births is a case of twins. Calculate the probability of 2 or more sets of twins on a day when 30 births occur.

Sol: Given that number of births n=30,

Probability of getting twins $p = \frac{1}{80}$

$$\lambda = np = 30 * \frac{1}{80} = \frac{3}{8}$$

Let X be the number of twins, we have $P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$P(X \geq 2) = 1 - \frac{\left(\frac{3}{8}\right)^0}{0!}e^{-\frac{3}{8}} - \frac{\left(\frac{3}{8}\right)^1}{1!}e^{-\frac{3}{8}} = 0.055$$

(OR)

$$P(X \geq 2) = 1 - P(X=0) - P(X=1)$$

$$P(X \geq 2) = 1 - 30c_0\left(\frac{1}{80}\right)^0\left(\frac{79}{80}\right)^{30} - 30c_1\left(\frac{1}{80}\right)^1\left(\frac{79}{80}\right)^{29} = 0.056$$

3. If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals

(a) **exactly 3**

(b) **more than 2 individuals**

(c) **None individual suffer a bad reaction**

Sol: Given p=0.001 and n=2000 $\lambda = np = 2000 * 0.001 = 2$

$$P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$$

(a) $P(X=3) = \frac{2^3}{3!}e^{-2} = 0.1804$

(b) $P(X>2) = 1 - P(X \leq 2) = 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$$P(X>2) = 1 - e^{-2}\left\{\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right\} = 1 - 0.6766 = 0.323$$

(c) $P(X=0)=\frac{2^0}{0!}e^{-2}=0.135$

4. A manufactures of the blades knows that 5% of his product is defective. If he sells blades in boxes of 100, and guarantees that not more than 10 blades will be defective, what is the probability that a box will fail to meet the guarantee quality?

Sol: Given that number of blades n=100

Probability of getting defective blades p=5%=0.05

$\lambda=np=100*0.05=5$

Let X be the number of defective blades $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}$

$P(X>10)=1-P(X\le 10)$

$$P(X>10)=1-\sum_{i=0}^{10}P(X=i)$$

$$P(X>10)=1-\sum_{x=0}^{10}\frac{5^x}{x!}e^{-5}$$

5. Suppose on an average 1 house in a1000 in a certain district has a fire during a year. If there are 2000 houses in that district, what is the probability that exactly 5 houses will have a fire during the year?

Sol: Given that number of houses n=2000

Probability of fire $p=\frac{1}{1000}$ and $\lambda=np=2000*\frac{1}{1000}=2$

Let X be the number of houses fire $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}\quad \Rightarrow P(X=5)=\frac{2^5}{5!}e^{-2}$

Type2: When λ is given

6. The average number of phone calls / minute coming into a switch board between 2p.m and 4p.m is 2.5. Determine the probability that during one particular minute there will be (i) 4 or fewer (ii) more than 6 calls.

Sol: Let X be the number of phone calls / minute coming into a switch board.

Given mean λ=2.5, We have $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}=\frac{(2.5)^x}{x!}e^{-2.5}$

(a) $P(X\le 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$$P(X\le 4)=e^{-2.5}\left\{\frac{(2.5)^0}{0!}+\frac{(2.5)^1}{1!}+\frac{(2.5)^2}{2!}+\frac{(2.5)^3}{3!}+\frac{(2.5)^4}{4!}\right\}=0.891$$

(b) $P(X>6)=1-p(x\le 6)=1-\{P(X=0)+P(X=1)+\cdots\ldots.+P(X=5)+P(X=6)\}$

$$P(X>6)=1-e^{-2.5}\left\{\frac{(2.5)^0}{0!}+\frac{(2.5)^1}{1!}+\frac{(2.5)^2}{2!}+\frac{(2.5)^3}{3!}+\frac{(2.5)^4}{4!}+\frac{(2.5)^5}{5!}+\frac{(2.5)^6}{6!}\right\}=0.014$$

In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.

Sol:

Average number of typo-graphical errors/page

Let X be the Number of errors per page.

Average number of typo-graphical errors/page=λ=390/520=0.75,

We have $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}$

$$P(X=0)=\frac{\lambda^o}{0!}e^{-0.75}=e^{-0.75}$$

probability that a random sample of 5 pages will contain no error=$[P(0)]^5=(e^{-0.75})^5=e^{-3.75}$

7. A car-hire firm has two cars which it hires out, day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days

(a) On which there is no demand

(b) On which demand is refused.

Sol: Given mean λ=1.5

We have $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}$

(a) No demand:

$$P(X=0)=\frac{1.5^0}{0!}e^{-1.5}=0.2231$$

Number of days in a year there is no demand of car = 365*0.2231 = 81 days

(b) Demand refused:

Some demand is refused if the number of demands is more than two i.e., X>2

$$P(X>2)=1-P(X\le 2)=1-\{P(X=0)+P(X=1)+P(X=2)\}$$

$$P(X>2)=1-e^{-1.5}\left\{\frac{1.5^0}{0!}+\frac{1.5^1}{1!}+\frac{1.5^2}{2!}\right\}=1-0.808=0.1913$$

Number of days in a year when some demand is refused 365* 0.1913= 69.82=70 days

8. If a Poisson distribution is such that $P(X=1)\frac{3}{2}=P(X=3)$ find (a)P(x≥1) b)P(x≤3) (c)P(2≤x≤5)

Sol: Given $P(X=1)\frac{3}{2}=P(X=3)$

$$\frac{3}{2}*\frac{\lambda^1}{1!}e^{-\lambda}=\frac{\lambda^3}{3!}e^{-\lambda} \qquad \Rightarrow \frac{3\lambda}{2}=\frac{\lambda^3}{6} \qquad \Rightarrow \lambda^3-9\lambda=0$$

$\lambda=0,3,-3$ But $\lambda>0$ so $\lambda=3$

Hence $P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda}=\frac{3^x}{x!}e^{-3}$

(a) $P(X\ge 1)=1-P(X<1)=1-P(X=0)=1-\frac{3^0}{0!}e^{-3}=0.9502$

(b) $P(X\le 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$P(X \leq 3) = e^{-3}\left\{\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right\} = 13e^{-3} = 0.6472$$

(c) $P(2 \leq X \leq 5) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$P(2 \leq X \leq 5) = = e^{-3}\left\{\frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right\} = 0.7169$$

9. If X and Y are independent Poisson variates such that P(X=1) = P(X=2) and P(Y=2) = P(Y=3), find the mean and variance of X-2Y.

Sol: $P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}$

$P(X = 1) = P(X = 2)$

$$\frac{\lambda^1}{1!}e^{-\lambda} = \frac{\lambda^2}{2!}e^{-\lambda} \Longrightarrow \lambda^2 - 2\lambda = 0 \Longrightarrow \lambda = 0,2 \Longrightarrow \lambda = 2 = E(X) = V(Y)$$

$P(Y = 2) = P(Y = 3)$

$$\frac{k^2}{2!}e^{-k} = \frac{k^3}{3!}e^{-k} \Longrightarrow k^3 - 3k = 0 \Longrightarrow k = 0,3 \Longrightarrow k = 3 = E(Y) = V(Y)$$

$E(X - 2Y) = E(X) - 2E(Y) = 2 - 2 * 3 = -4$

$V(X - 2Y) = V(X) + (-2)^2 V(Y) = 2 + 4 * 3 = 14$

Tutorial Questions:

1. In a company, there are 250 workers. The probability of a worker remain absent on any one day is 0.02. Find the probability that on a day seven workers are absent. **Answer**: $\mathbf{P(X = 7) = 0.104}$

2. A book contains 100 misprints distributed randomly throughout its 100 pages. What is the probability that a page observed at random contains at least two miss prints? Assume Poisson Distribution.

Answer: $\mathbf{0.2642}$

3. For Poisson variant X, if P(X =3) =P(X=4) then, Find P(X=0). **Ans**: $\mathbf{(X = 0) = e^{-4}}$

4. In a bolt manufacturing company, it is found that there is a small chance of 1 500 for any bolt to be defective. The bolts are supplied in a packed of 30 bolts. Use Poisson distribution to find approximate number of packs, (a) Containing no defective bolt and (b) Containing two defective bolt, in the consignment of 10000 packets. **Answer**: **(a) 0 (b) 1**7

5. In sampling a large number of parts manufactured by a machine, the mean number and of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain exactly two defective parts? **Answer**: $\mathbf{P(X = 2) = 0.270}$

6. Potholes on a highway can be serious problems. The past experience suggests that there are, on the average, 2 potholes per mile after a certain amount of usage. It is assumed that the Poisson process applies to the random variable "no. of potholes". What is the probability that no more than four potholes will occur in a given section of 5 miles? **Answer**: $\mathbf{P(X \leq 4) = 0.0315}$

7. 100 Electric bulbs are found to be defective in a lot of 5000 bulbs. Find the probability that at the most 3 bulbs are defective in a box of 100 bulbs. **Answer**: $\mathbf{P(X \leq 3) = 0.8567}$

8. Certain mass produced articles of which 0.5% are defective, are packed in cartons each containing 100. What proportion of cartons are free from defective articles, and what proportion contain 3 or more defectives? **Answer**: $\mathbf{P(X = 0) = 60.65}$, $\mathbf{P(X \geq 3) = 0.0144}$

9. The probability that a person catch swine flu virus is 0.001. Find the probability that out of 3000 persons (a) exactly 3, (b) more than 2 person will catch the virus. (F(2; λ)=0.42) Answer: (a) 0.2242 (b) 0.5769

T 12 Suppose 1% of the items made by machine are defective. In a sample of 100 items find the probability that the sample contains All good, 1 defective and at least 3 defective.

Answer: $\mathbf{P(X = 0) = 0.3679}$; $\mathbf{P(X = 1) = 0.3679}$; $\mathbf{P(X \geq 3) = 0.080}$3

10. A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and generates that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality? [Ans: 0.0136]

11. An insurance company insures 4,000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00,000 will have car accident each year that result in this type of injury. What is the probability that more than 3 of the insured will collect on their policy in a given year? [Ans: 0.0008]

12. In a frequency distribution, frequency corresponding to 3 successes is $2/3$ times frequency corresponding to 4 successes. Find the mean and standard deviation of the distribution. [Ans: 6]

13. The average number of phone calls arriving at a telephone exchange is 30 per hour. What is the probability that (i) no calls arrive in a 3 minute period (ii) more than 5 calls arrive in 5 minute period [Ans: (i) 0.2231, (ii) 0.642]

14. A manufacture accepts the work submitted by his typist only when there is no mistake in the work. The typist has to type on an average 20 letters per day of about 200 words each. Find the chance of her making a mistake

(i) if less than 1% of the letters submitted by her are rejected. [Ans: 0.0000506]

(ii) if on 90% days all the letters submitted by her are accepted. [Ans: 0.0000263]

15. In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error. [Ans: 0.0235]

16. Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents is (i) at least one (ii) at most one [Ans: (i) 0.8347, (ii) 0.4628]

17. If a bank receives on the average 6 bad cheques per day. What is the probability that it will receive (i) 4 bad cheques on any given day (ii) 10 bad cheques over any consecutive days. **Answer**: (**a**) $\mathbf{P(X=4)=0.1338}$ (**b**) $\mathbf{2P(X=10)=0.0826}$

18. Using Poisson distribution find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.

19. If X is Poisson variate such that $P(X=0)=P(X=2)+3P(X=4)$, find (i)the mean of X (ii) $P(X\le 2)$

IV.Geometric Distribution: This distribution is used when experiment is repeated until we get the **first successes.**

A random variable X is said to follow the Geometric distribution if its probability mass function given by $P(X = x) = (q)^{x-1} * p$, where $x = 1,2 \ldots..$

Notation: $X \sim GD(x, p)$ Read as follows Geometric distribution with parameters x and p.

- x$\rightarrow$ number of attempts to make first successes
- p $\rightarrow$ Probability of success
- q $\rightarrow$ Probability of failure

Examples:

1.Tossing a coin repeatedly until the first head appears.

2.Shot the target until it hits.

3.Give the test until he will pass it

4.Throwing a die repeatedly until first time six appears.

Note:

- Mean of the Geometric distribution is $\frac{1}{p}$
- Variance of the Geometric distribution is $\frac{q}{p^2}$

1.In a certain manufacturing process is known that on the average one in every 100 items is defective. What is the probability that the 5th item inspected is the first defective item found?

Sol: Given that p=1/100=0.01

$P(X = x) = (q)^{x-1} * p$

$P(X = 5) = (1 - 0.01)^{5-1} * 0.01 = 0.0096$

2.Suppose that a trainee solder shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7,

(a) What is the probability that the target would be hit on 10th attempt?

(b) Probability that it takes him less than 4 shots?

(c) Probability that it takes him even number of shots?

Sol: Given that probability of hitting p=0.7 and $P(X = x) = q^{x-1} * p$

(a) $P(X = 10) = q^9 * p$

$P(X = 10) = (0.3)^9 * (0.7) = 0.0000137$

(b) $P(X < 4) = P(X = 1) + P(X = 2) + P(x = 3)$

$P(X < 4) = p + qp + q^2p$

$P(X < 4) = 0.7 + (0.3)(0.7) + (0.3)^2(0.7) = 0.9730$

(c) $P(X = even) = P(X = 2) + P(X = 4) + P(x = 6) + \ldots\ldots$

$$P(X = \text{even}) = qp + q^3p + q^5p + \cdots.$$

$$P(X = \text{even}) = qp\{1 + q^2 + q^4 + \cdots\}$$

$$P(X = \text{even}) = qp\left\{\frac{1}{1-q^2}\right\}$$

$$P(X = \text{even}) = (0.3)(0.7)\left\{\frac{1}{1-(0.3)^2}\right\} = 0.2308$$

3.The probability that an applicant for a driver's license will pass the road test on any given attempt is 0.8, what is the probability that he will finally pass the test

(i) On the 4^th attempt.

(ii) In fewer than 4^th attempt

Sol: Let X denote the number of trials required to achieve the first success

Then X follows a geometric distribution given by $(X = x) = q^{x-1}p$, x= 1,2,3, …

Here p= 0.8, q=0.2

(i) P(First success on 4_{th} trial)=$P(X=4)=q^{4-1}p$

$P(X=4)=q^{4-1}p=(0.2)^3(0.8)=0.0064$

(ii) P(First success in less than 4 trials) =P(X<4) = P(X=1)+ P(X=2)+ P(X=3)

$P(X<4) = p(q^0+q^1+q^2)$

$P(X<4) = 0.8(1+0.2+0.04) = 0.992$

4.The probability that a machine produces a defective item is 0.02. Each item is checked as it is produced. Assuming that these are independent trials, what is the probability that at least 100 items must be checked to find one that is defective?

Sol: Given that p=0.02

Let X denotes the trial number on which the first defective item is observed.

$$P(X \geq 100) = \sum_{x=100}^{\infty}(q)^{x-1} * p$$

$$P(X \geq 100) = \sum_{x-99=1}^{\infty}(q)^{x-1} * p$$

Let x-99=y so x=y+99

$$P(X \geq 100) = \sum_{y=1}^{\infty}(q)^{y+99-1} * p$$

$$P(X \geq 100) = (q)^{99}\sum_{y=1}^{\infty}(1-p)^{y-1} * p \qquad \text{Since } \sum_{y=1}^{\infty}(q)^{y-1} * p = 1$$

$$P(X \geq 100) = (q)^{99}(1) = (0.98)^{99} = 0.1353$$

Hence the probability that at least 100 items must be checked to find one that is defective is 0.1353.

5.A gambler plays roulette at Monte Carlo and continues gambling, wagering the same amount each time on "Red", until he wins for the first time. If the probability of "Red" is 18/38 and the gambler has only enough money for 5 trials, then

(a) What is the probability that he wins on the second trial?

(b) What is the probability that he will win before he exhausts his funds;

Sol: $p = P(\text{Red}) = \frac{18}{38}$

(a) The probability that he wins on the second trial is given by

$$P(X = 2) = (q)^{2-1}p = \left(1 - \frac{18}{38}\right)\left(\frac{18}{38}\right) = 0.2493$$

(b) Hence the probability that he will win before he exhausts his funds is given by

$$P(X \leq 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$P(X \leq 5) = p + qp + q^2p + q^3p + q^4p = 0.956$$

V.Negative Binomial Distributions:

Negative Binomial Distributions is applicable when we need to performed an experiment **until a total of r successes** are obtained.

A random variable X is said to follow the Negative Binomial distribution if its probability mass function given by

$$P(X = x) = P\{(r-1)\text{success in the first } (x-1)\text{trail } \& \textit{ a success in the } x^{th} \text{ trail}\}$$

$$P(X = x) = (x-1)c_{r-1}\ q^{x-r}P^r = \binom{x-1}{r-1}P^r\, q^{x-r} \text{ where } x = r, r+1, r+2, \ldots$$

Notation: $X \sim NBD(r,p)$ Read as follows binomial distribution with parameters n and p.

- r ⟶ number of success
- p ⟶ Probability of success
- q ⟶ Probability of failure

Note:

(a) If r=1, means we perform an experiment till we obtain first success which is the case of Geometric distribution.

(b) It is the generalization of the Geometric distribution

(c) This distribution is also known as Pascal distribution.

Note:

- Mean of the Negative Binomial distribution is $\frac{r}{p}$
- Variance of the Negative Binomial distribution is $\frac{rq}{p^2}$

1. What is the probability that the fifth head is observed on the 10th independent flip of a coin?

Sol: Let X denote the number of trials needed to observe 5th head.

Hence X has a negative binomial distribution

$$P(X = x) = (x-1)c_{r-1}\, p^r\, q^{x-r}$$

$$p = \frac{1}{2}, x = 10, r = 5$$

$$P(X = 10) = 9c_4\, p^5\, q^5 = 9c_4 \left(\frac{1}{2}\right)^{10} = \frac{63}{512}$$

OR

$$P(X = x) = P\{(r-1)\text{success in the first } (x-1)\text{trail } \& \textit{ a success in the } x^{th} \text{ trail}\}$$

$$P(X = 10) = (9c_4\ p^4q^5)p$$

$$P(X = 10) = \left\{9c_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^5\right\}\left(\frac{1}{2}\right) = \frac{63}{512}$$

2. If the probability is 0.40 that a child exposed to a certain disease will contain, what is the probability that the 10^{th} child exposed to the disease will be the 3^{rd} to catch it?

Sol: Let X be the disease child

Given that x=10 and p=0.40 and r=3

$(X = x) = (x - 1)c_{r-1}\ q^{x-r}p^r$

Probability that the 10^{th} child exposed to the disease will be the 3^{rd} to catch it is

$P(X = 10) = (10 - 1)c_{3-1}\ (0.6)^{10-3}(0.4)^3$

OR $P(X = 10) = (9c_2\ p^2q^7)p \quad \Rightarrow P(X = 10) = 9c_2\ (0.6)^7(0.4)^3 = 0.0645$

3. Let X is the number of births in a family until the second daughter is born. If the probability of having a male child is ½. Find the Probability that the sixth child in the family is the second daughter.

Sol: Let X be the child daughter

Given that x=6, p=1/2 and r=2

$P(X = x) = P\{(r - 1)\text{success in the first } (x - 1)\text{trail } \& \textit{ a success in the } x^{th} \text{ trail}\}$

$P(X = 2) = (5c_1\ p^1q^4)p$

$P(X = 2) = 5c_1\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{5}{2^6}$

4. In a company 5% defective components are produced. What is the probability that at least 5 components are to be examined in order to get 3 defectives?

Sol: Given that x=5, p=5%=0.05 and r=3

$P(X \geq 5) = 1 - P(X < 5) = 1 - P(X = 3) - P(X = 4)$

$P(X = x) = P\{(r - 1)\text{success in the first } (x - 1)\text{trail } \& \textit{ a success in the } x^{th} \text{ trail}\}$

$P(X \geq 5) = 1 - (2c_2\ p^2q^0)p - (3c_2\ p^2q^1)p$

$P(X \geq 5) = 1 - 2c_2\ (0.05)^3 - 3c_2\ (0.05)^3(0.95)^1 = 0.995$

5. An item is produced in large numbers. The machine is known to produce 5% defective. A quality control inspector is examining the items by taking them at random. What is the probability that at least 4 items are to be examined in order to get 2 defective?

Given that p=5%=0.05, x=4, and r=2

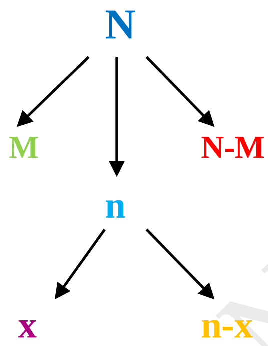
$P(X \geq 4) = 1 - P(X < 4) = 1 - P(X = 2) - P(X = 3)$

$P(X = x) = P\{(r - 1)\text{success in the first } (x - 1)\text{trail } \& \textit{ a success in the } x^{th} \text{ trail}\}$

$P(X \geq 4) = 1 - (1c_1\ p^1q^0)p - (2c_2\ p^2q^1)p = 0.995$

VI.Hyper geometric Distribution:

A random variable X is said to have a hyper geometric distribution if its probability mass function is given by $P(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}=hyp(M,N-M,n)$ where x=0,1,2,…….n



Notations:

- $N \rightarrow$ Population Size
- $M \rightarrow$ Number of success states in the Population.
- $n \rightarrow$ Number of draws
- $x \rightarrow$ Number of observed success

Conditions for Applicability of Hyper Geometric Distributions:

i) The result of each draw can be classified in to one of two mutually exclusive categories (Success/ Failure)

ii) The probability of success changes on each draw, as each draw decreases the population.(sampling without replacement from a finite population)

Note:

(a) Mean $E(X)=\frac{nM}{N}$

(b) Variance $V(X)=\frac{NM(N-M)(N-n)}{N^2(N-1)}$

1. A crate contains 50 light bulbs of which 5 are defective and 45 are not. A quality control inspector randomly samples 4 bulbs without replacement. Let X be the number of defective bulbs selected, find the p.m.f of the discrete random variable X

Sol: The Probability mass Function is $P(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, where $x=0,1,2\dots.n$

$$P(X=x)=\frac{5C_x * 45C_{4-x}}{50C_4}, \text{where } x=0,1,2,3,4.$$

2. A lake contains 600 fish, 80 of which have been tagged by scientists. A researcher randomly catches 15 fish from the lake. Find a formula for the probability mass function of X, the number of fish in the researcher's sample which are tagged.

Sol: The Probability mass Function is $P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, where $x = 0,1,2 \dots . n$

$$P(X = x) = \frac{80C_x * 520C_{15-x}}{600C_{15}}, \text{where } x = 0,1,2, \dots ,15$$

3. Let the random variable X denote the number of aces in a five-card hand dealt from a standard 52 cards deck. Find a formula for the probability mass function of X.

Sol: The Probability mass Function is $P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, where $x = 0,1,2 \dots . n$

$$P(X = x) = \frac{4C_x * 48C_{5-x}}{52C_5}, \text{where } x = 0,1,2, \dots .$$

4. Ravi likes to play cards. He draws 5 cards from a pack of 52 cards. What is the Probability of that from the 5 cards drawn Ravi draws only 2 face cards.

Sol: Total number of cards N=52

Total face cards M=16

Non-face cards N-M=36

Number of draws n=5

Number of observed success draws x=2

The Probability mass Function is $P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, where $x = 0,1,2 \dots . n$

$$P(X = 2) = \frac{16C_2 * 36C_3}{52C_5} = 0.0533$$

5. Find the expectation of a Hyper Geometric Distribution such that the probability that a 4-trail Hyper geometric experiment results in exactly 2 successes, when the population consists of 16 items.

Sol: Let X denotes the number of successes

Expected value of X is $E(X) = \frac{nM}{N}$

N=16, n=4, M=2

$$E(X) = \frac{2*4}{16} = 0.5$$

6. Consider Raju draws 3 cards from a pack of 52 cards. What is the probability of getting no king?

Sol: Given that N=52, M=4, n=3, x=0

The Probability mass Function is $P(X = x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, where $x = 0,1,2 \dots . n$

$$P(X=0) = \frac{4C_0 * 48C_3}{52C_3} = 0.7826$$

7. A random sample of 5 students is drawn without replacement from among 300 seniors, and each of these 5 seniors is asked if she/he has tried a certain drug. Suppose 50% of the seniors actually have tried the drug. What is the probability that two of the students interviewed have tried the drug?

Sol: Let X denote the number of students interviewed who have tried the drug.

Hence the probability that two of the students interviewed have tried the drug is

$$P(X=2) = \frac{\binom{150}{2}\binom{150}{3}}{\binom{300}{5}} = 0.3146$$

8. A radio supply house has 200 transistor radios, of which 3 are improperly soldered and 197 are properly soldered. The supply house randomly draws 4 radios without replacement and sends them to a customer. What is the probability that the supply house sends 2 improperly soldered radios to its customer?

Sol: The probability that the supply house sends 2 improperly soldered radios to its customer is

$$P(X=2) = \frac{\binom{3}{2}\binom{197}{2}}{\binom{200}{4}} = 0.000895$$

9. Suppose there are 3 defective items in a lot of 50 items. A sample of size 10 is taken at random and without replacement. Let X denotes the number of defective items in the sample. What is the probability that the sample contains at most one defective item?

Sol: Clearly, X is Hyp(3, 47, 10).

Hence the probability that the sample contains at most one defective item is

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$P(X \le 1) = \frac{\binom{3}{0}\binom{47}{10}}{\binom{50}{10}} + \frac{\binom{3}{1}\binom{47}{9}}{\binom{50}{10}} = 0.904$$

VII.Uniform distribution: A continuous random variable X is said to be uniform on the interval [a, b] if its probability density function is of the form $f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & other\ wise \end{cases}$

Note:

- Mean $E(X) = \frac{a+b}{2}$
- Variance $V(X) = \frac{(a-b)^2}{12}$

1.Buses arrive at a specified stop at 15 minutes' intervals starting at 7:00 a.m. (i.e.) they arrive at 7:00, 7:15, 7:30. If the passenger arrives at the stop at a random time (i.e.) uniformly distributed between 7:00 and 7:30 a.m. find the probability that he waits (i) less than 5 minutes for a bus (ii) at least 12 minutes for a bus.

Sol: Let X denote the arrival time in minutes of a passenger to bus stop past 7.00 am until 7.30 am.

Then X follows Uniform distribution in [0,30] with p.d.f $f(x) = \begin{cases} \frac{1}{30-0}, & 0 < x < 30 \\ 0, & other\ wise \end{cases}$

(i) P(he waits less than 5 min) = P(he arrives between 10 to 15 min or 25 to 30 minutes past 7 am)= $P(10 < x < 15) + P(25 < x < 30)$

$$\int_{10}^{15} f(x)dx + \int_{25}^{30} f(x)dx = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{30}(5+5) = \frac{1}{3}$$

(ii) P(he waits at least 12 min) = P(he arrives between 0 to 3 min or 15 to 18 minutes past 7 am) $=P(0 < X < 3) + P(15 < X < 18)$

$$\int_{0}^{3} f(x)dx + \int_{15}^{18} f(x)dx = \int_{0}^{3} \frac{1}{30} dx + \int_{15}^{18} \frac{1}{30} dx = \frac{1}{30}(3+3) = \frac{1}{5}$$

VIII.Exponential distribution: In probability theory, an Exponential Distribution is a continuous probability distribution, which is commonly used to model the time between independent events that occur at a constant average rate.

A continuous random variable X is said to have an Exponential distribution with parameter $\lambda > 0$(Known as rate parameter) if its probability density function is given by

$$f(x) = Exp(\lambda) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0, & other\ wise \end{cases}$$

The cumulative distribution function of exponential distribution is

$$F(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

Note:

- Mean $E(X) = \frac{1}{\lambda}$
- Variance $V(X) = \frac{1}{\lambda^2}$

1. The time (in hours) required to repair a machine is exponential distributed with parameter 1/3. What is the probability that the repair time exceeds 3 hours?

Sol: Given that $\lambda = \frac{1}{3}$

The Probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0, & other\ wise \end{cases}$

$$P(X > 3) = \int_3^\infty f(x)dx = \int_3^\infty \lambda e^{-\lambda x}\, dx$$

$$P(X > 3) = \left(\frac{\lambda e^{-\lambda x}}{-\lambda}\right)_3^\infty = -(0 - e^{-1}) = \frac{1}{e}$$

(OR)

The cumulative distribution function of exponential distribution is

$$F(x) = P(X \leq x) = \begin{cases} 1 - e^{-\lambda x}; & x \geq 0 \\ 0; & x < 0 \end{cases}$$

$P(X > 3) = 1 - P(X \leq 3)$

$P(X > 3) = 1 - F(3)$

$P(X > 3) = 1 - \{1 - e^{-3\lambda}\} = e^{-1}$

2. The mileage which car owners get with a certain kind of radial tire is a r.v. having an exponential distribution with mean 40,000 km. Find the probabilities that one of these tires will last (i) at least 20,000 km and (ii) at most 30,000 km

Sol:

Given Mileage X of car radial tire is exponentially distributed with mean 40000 km. $\Rightarrow \frac{1}{\lambda} = 40000 \ \Rightarrow \lambda = \frac{1}{40000}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x > 0 \\ 0, & other\ wise \end{cases}$$

(i) $P(X > 20000) = \int_{20000}^{\infty} f(x)dx = \int_{20000}^{\infty} \lambda e^{-\lambda x} dx$

$P(X > 20000) = \left(-e^{-\lambda x}\right)_{20000}^{\infty} = e^{-0.5} = 0.6065$

(OR) $P(X > 20000) = 1 - P(X \le 20000) = 1 - F(20000) = 1 - \left\{1 - e^{-\frac{1}{2}}\right\} = e^{-\frac{1}{2}}$

(ii) $P(X \le 30000) = \int_{0}^{30000} f(x)dx = \left(-e^{-\lambda x}\right)_{0}^{30000} = -e^{-\frac{3}{4}} + 1$

(OR) $P(X \le 30000) = F(30000) = 1 - e^{-\frac{30000}{400000}} = -e^{-\frac{3}{4}} + 1$

3. Assume that the length of a phone call in minutes is an exponential random variable X with parameter λ =1/10, (or the expected waiting time for a phone call is 10 minutes). If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait

a. less than 5 minutes

b. greater than 10 minutes

c. between 5 and 10 minutes

d. Also compute the expected value and variance

Sol: Given that $\lambda = \frac{1}{10}$ and The Probability density functionf(x) $= \begin{cases} \lambda e^{-\lambda x}; & x \ge 0 \\ 0, & other\ wise \end{cases}$

(a) $P(X < 5) = \int_{-\infty}^{5} f(x)dx + \int_{0}^{5} f(x)dx = 0 + \int_{0}^{5} \lambda e^{-\lambda x}\, dx$

$P(X < 5) = \left(\frac{\lambda e^{-\lambda x}}{-\lambda}\right)_{0}^{5} = \left(-e^{-\lambda x}\right)_{0}^{5} = -e^{-\frac{1}{2}} + 1$

(OR) $P(X < 5) = F(5) = 1 - e^{-\frac{1}{2}}$

(b) $P(X > 10) = \int_{10}^{\infty} f(x)dx = \left(-e^{-\lambda x}\right)_{10}^{\infty} = 0 + e^{-1}$

(OR) $P(X > 10) = 1 - P(X \le 10) = 1 - F(10) = 1 - \{1 - e^{-1}\} = e^{-1}$

(c) $P(5 < X < 10) = \int_{5}^{10} f(x)dx = \left(-e^{-\lambda x}\right)_{5}^{10} = -e^{-1} + e^{-\frac{1}{2}}$

(OR) $P(5 < X < 10) = F(10) - F(5) = (1 - e^{-1}) - \left(1 - e^{-\frac{1}{2}}\right) = e^{-\frac{1}{2}} - e^{-1}$

(d) $E(x) = \frac{1}{\lambda} = 10$

Memory less Property: if X is exponential distribution then $P(X > m/X > n) = P(X > m - n)$ for any m, n>0

4. A fast food chain finds that the average time customers have to wait for service is 45 seconds. If the waiting time can be treated as an exponential random variable, what is the probability that a customer will have to wait more than 5 minutes given that already he waited for 2 minutes?

Sol: Let X denotes the waiting time at the fast food

Mean=45 seconds

$$\frac{1}{\lambda} = 45 \Rightarrow \lambda = \frac{1}{45}$$

$$P(X > 5\,\text{min}/X > 2min) = P(X > 300/X > 120) = \frac{P(X > 300 \cap X > 120)}{P(X > 120)} = \frac{P(X > 300)}{P(X > 120)}$$

$$\frac{P(X > 300)}{P(X > 120)} = \frac{1 - F(300)}{1 - F(120)} = \frac{1 - \left\{1 - e^{-\frac{300}{45}}\right\}}{1 - \left\{1 - e^{-\frac{120}{45}}\right\}} = e^{-4}$$

(OR)

Required probability $P(X > 5\,\text{min}/X > 2min)$

$P(X > m/X > n) = P(X > m - n)$

$P(X > 300/X > 120) = P(X > 180)$

$P(X > 300/X > 120) = 1 - P(X \leq 180) = 1 - F(180)$

$$P(X > 300/X > 120) = 1 - \left\{1 - e^{-\frac{180}{45}}\right\} = e^{-4}$$

5. The life length (in months) of an electric component follows an exponential distribution with parameter ½. What is the probability that the component survives at least 10 months given that already it had survived for more than 9 months?

Sol: Given that $\lambda = \frac{1}{2}$

$P(X > 10/X > 9) = P(X > 1)$

$= 1 - P(X \leq 1) = 1 - F(1)$

$$= 1 - \left\{1 - e^{--\frac{1}{2}}\right\} = e^{-\frac{1}{2}}$$

6. The time required (in hours) to repair a machine is exponentially distributed with parameter ½. What is the probability that a repair time exceeds 2 hours? What is the conditional probability that a repair time takes at least 10 hours given that its duration exceeds 9 hours?

Sol: Given that $\lambda = \frac{1}{2}$

(a) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - \{1 - e^{-1}\} = e^{-1}$

(b) $P(X \geq 10/X > 9) = P(X \geq 1)$

$= 1 - P(X < 1) = 1 - F(1)$

$$= 1 - \left\{1 - e^{--\frac{1}{2}}\right\} = e^{-\frac{1}{2}}$$

7. The length of time a person speaks over phone follows exponential distribution with mean 6, what is the probability that the person will talk for (a) more than 8 minutes (b) between 4 and 8 minutes.

Sol: Given that mean=6

$\frac{1}{\lambda} = 6 \Rightarrow \lambda = \frac{1}{6}$

(a) $P(X > 8) = 1 - P(X \leq 8) = 1 - F(8) = 1 - \left\{1 - e^{-\frac{8}{6}}\right\} = e^{-\frac{4}{3}}$

(b) $P(4 < X < 8) = F(8) - F(4)$

$$= \left\{1 - e^{-\frac{2}{3}}\right\} - \left\{1 - e^{-\frac{4}{3}}\right\} = e^{-\frac{2}{3}} - e^{-\frac{4}{3}}$$

8. The daily consumption of milk in exceeds of 20000 gallons is approximately exponential distribution with mean 3000. The city has daily stock of 35000 gallons. What is the probability that of two days selected at random, the stock is insufficient for both days?

Sol: Given that mean=3000

$\frac{1}{\lambda} = 3000 \Rightarrow \lambda = \frac{1}{3000}$

Since the city has a daily stock of 3500 gallons and daily consumption is 2000 gallons.

The probability that the city is insufficient for single day is

$$P(X > 35000/X > 20000) = P(X > 15000)$$

$$1 - P(X \leq 15000) = 1 - F(15000) = 1 - \left\{1 - e^{-\frac{15000}{3000}}\right\} = e^{-5}$$

Since both days selected at random and they are independent so required probability that the stock is insufficient for both days is P(first day) *P(2nd day)= $e^{-5} * e^{-5} = e^{-10}$

Exercise Problems:

1. The lifetime T of an alkaline battery is exponentially distributed with $\theta = 0.05$ per hour. What are mean and standard deviation of batteries lifetime? Answer: 20 hr

2. The lifetime T of an alkaline battery is exponentially distributed with $\theta = 0.05$ per hour. What are the probabilities for battery to last between 10 and 15 hours? What are the probabilities for the battery to last more than 20 hr? Answer: 0.1341, 0.3679

3. The time between breakdowns of a particular machine follows an exponential distribution with a mean of 17 days. Calculate the probability that a machine breakdown in 15 days period. Answer: 0.5862

4. The arrival rate of cars at a gas station is 40 customers per hour. (a) What is the probability of having no arrivals in 5 min. interval? (b) What is the probability for having 3 arrivals in 5 min.? Answer: (a) 0.03567, (b) 0.2202

5. 5 In a large corporate computer network, user log-on to the system can be modeled as a Poisson process with a mean of 25 log-on per hours. (a) What is the probability that there are no log-on in an interval of six min.? (b) What is the probability that time until next log-on is between 2 & 3 min.? (c) Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90? Answer: (a) 0.07788, (b) 0.152, (c) 0.25

6. The time intervals between successive barges passing a certain point on a busy waterway have an exponential distribution with mean 8 minutes. (a) Find the probability that the time interval between two successive barges is less than 5 minutes. (b) Find a time interval t such that we can be 95% sure that the time interval between two successive barges will be greater than t. Answer: 0.4647, 24.6sec

7. Accidents occur with Poisson distribution at an average of 4/week ($\lambda = 4$). (a) Calculate the probability of more than 5 accidents in any one week. (b) What is probability that at least two weeks will elapse between accidents? Answer: (a) 0.215, (b) 0.00034

IX. Normal Distribution:

A continuous random variable X is said to follow a normal distribution of its probability density function (p.d.f) is given by

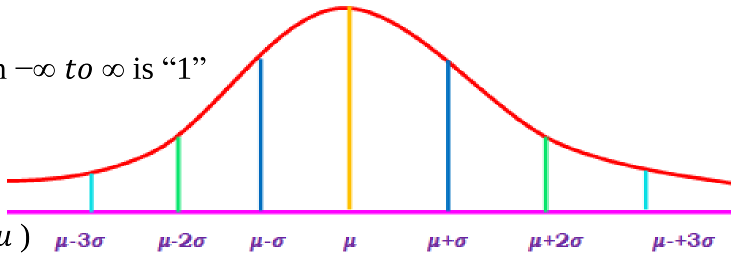
$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right];\ -\infty < x < \infty;\ \sigma > 0; -\infty < \mu < \infty \text{ Where}$$

μ = mean of the distribution and σ = Standard deviation of the distribution

Note: μ and σ^2 (variance) are called parameters of the distribution.

Properties of Normal Distribution

1. The graph of the Normal distribution in xy-plane is known as normal curve.
2. The curve is bell shaped and symmetrical (i.e; the two tails on right & left sides of mean μ extended to infinity)
3. The total area under the normal curve from $-\infty$ *to* ∞ is "1"
4. Normal curve is always centered at mean
5. It is unimodal (i.e; it has only one maximum point at x= μ)



6. Mean, median and mode coincide (i.e., equal)
7. X-axis is an asymptote to the normal curve
8. The points of inflection (maximum area) of the normal curve are at x=$\mu \pm \sigma$, which is equidistance from the mean on either side.
9. The area of the normal curve between
   (a) $\mu - \sigma$ to $\mu + \sigma = 68.27\%$
   (b) $\mu - 2\sigma$ to $\mu + 2\sigma = 95.43\%$
   (c) $\mu - 3\sigma$ to $\mu + 3\sigma = 99.73\%$

Standard Normal Variable:

Let $z = \frac{x-\mu}{\sigma}$ with mean '0' and variance is '1' then the normal variable is said to be standard normal variable.

Standard Normal Distribution:

The normal distribution with man '0' and variance '1' is said to be standard normal distribution of its probability density function is defined by

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right];\ -\infty < z < \infty;\ (\mu=0, \sigma=1)$$

- ❖ The distribution function $F(z) = P(Z \leq z) = \int f(z)dz = \frac{1}{\sqrt{2\pi}}\int e^{-\frac{z^2}{2}}dz$
- ❖ $F(-z) = 1 - F(z)$

STANDARD NORMAL (Z) TABLE , AREA BETWEEN 0 AND Z

Z	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0.004	0.008	0.012	0.016	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.091	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.148	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.17	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.195	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.219	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.258	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.291	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.334	0.3365	0.3389
1	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.377	0.379	0.381	0.383
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.399	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4115	0.4131	0.4147	0.4162
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.437	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.475	0.4756	0.4761	0.4767
2	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.483	0.4834	0.4838	0.4842	0.4846	0.485	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.489
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.492	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.494	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.496	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.497	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.498	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.499	0.499

Find the Area under Normal curve:

We know that the normal variable $z = \frac{x-\mu}{\sigma}$

If x=x₁ then $z_1 = \frac{x_1-\mu}{\sigma}$ and If x=x₂ then $z_2 = \frac{x_2-\mu}{\sigma}$

Model-1: $P(X \le x_1)$

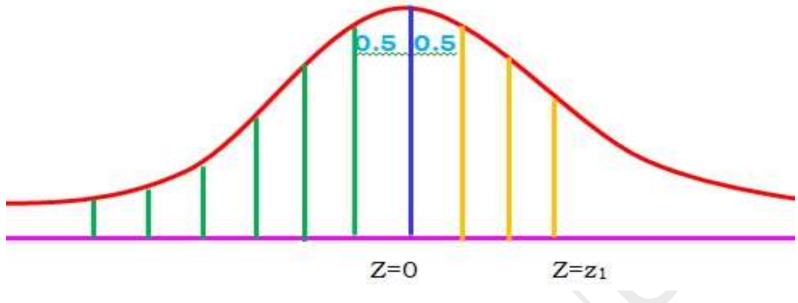
$P(X \le x_1) = P(z \le z_1)$

1. if $z_1 > 0$ then

$P(X \le x_1) = P(z \le z_1)$

$= 0.5 + P(0 < z < z_1)$

$= 0.5 + F(z_1)$



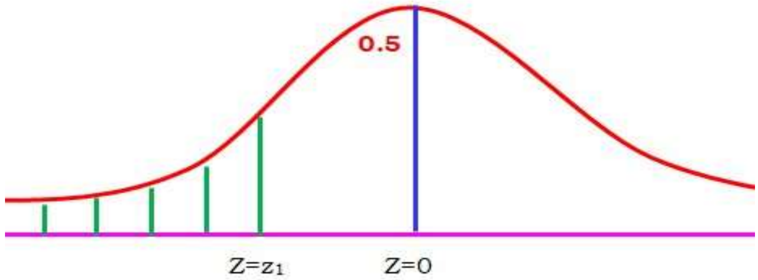
2. if $z_1 < 0$ then

$P(X \le x_1) = P(z \le z_1)$

$= 0.5 - P(z_1 < z < 0)$

$= 0.5 - P(0 < z < z_1)$

$= 0.5 - F(z_1)$



Model-2: $P(X \ge x_1)$

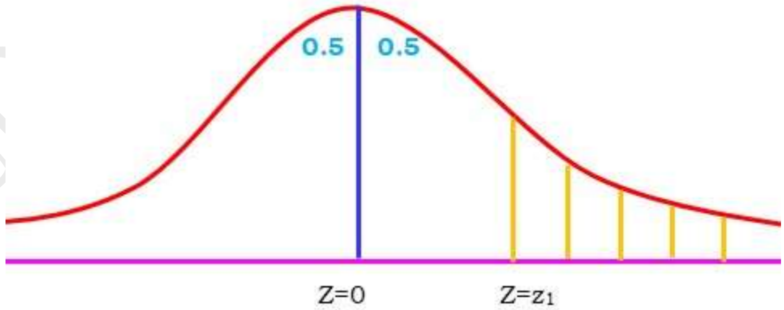
$P(X \ge x_1) = P(z \ge z_1)$

1. if $z_1 > 0$ then

$P(X \ge x_1) = P(z \ge z_1)$

$= 0.5 - P(0 < z < z_1)$

$= 0.5 - F(z_1)$



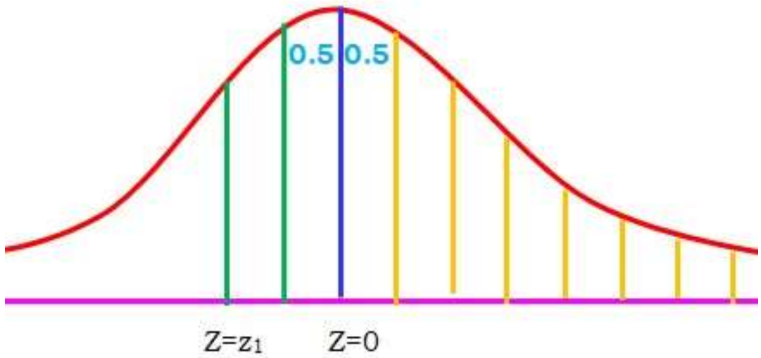
2. if $z_1 < 0$ then

$P(X \ge x_1) = P(z \ge z_1)$

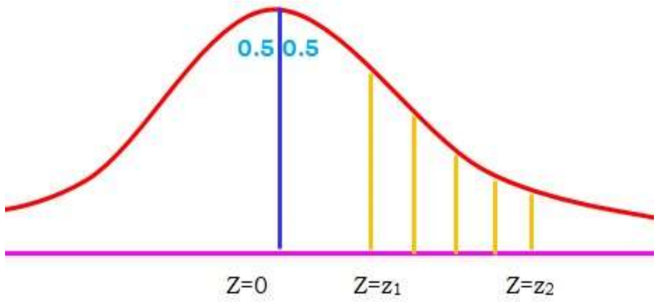
$= 0.5 + P(z_1 < z < 0)$

$= 0.5 + P(0 < z < z_1)$

$= 0.5 + F(z_1)$



Model-3: $P(x_1 \le X \le x_2)$



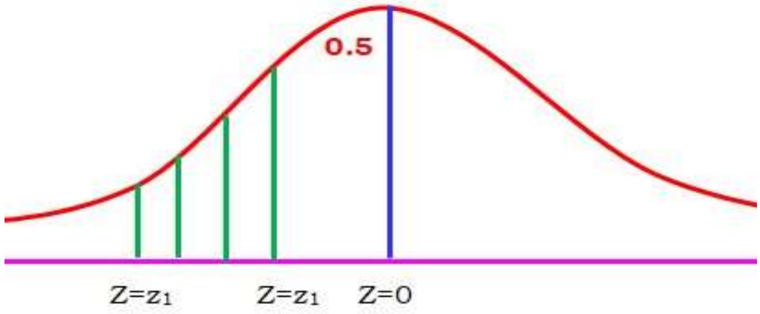
$P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

1. if $z_1 > 0$ & $z_2 > 0$ then

   $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

   $= P(0 < z < z_2) - P(0 < z < z_1)$

   $= F(z_2) - F(z_1)$



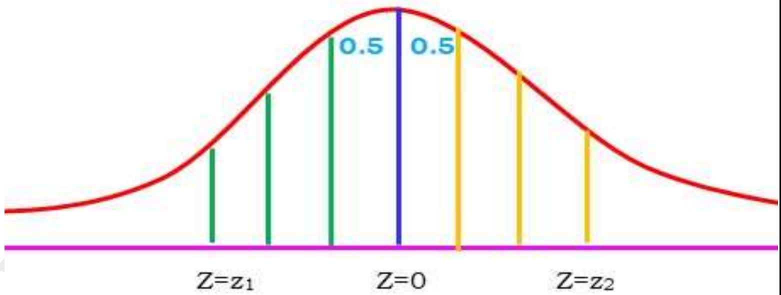
2. if $z_1 < 0$ & $z_2 < 0$ then

   $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

   $= P(z_1 \le z \le 0) - P(z_2 \le z \le 0)$

   $= P(0 < z < z_1) - P(0 < z < z_2)$

   $= F(z_1) - F(z_2)$



3. if $z_1 < 0$ & $z_2 > 0$ then

   $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

   $= P(z_1 \le z \le 0) + P(0 \le z \le z_2)$

   $= P(0 < z < z_1) + P(0 < z < z_2)$

   $= F(z_1) + F(z_2)$

1. If X is a normal variate, find the area

(a) **To the left of z = -1. 78**

(b) **To the right of $z = -1.45$**

(c) **Corresponding to $-0.8 \leq z \leq 1.53$**

(d) **To the left of $z = -2.52$**

(e) **To the right of $z = 1.83$.**

Sol:

(a) $P(z < -1.78) = 0.5 - P(-1.78 < z < 0)$

$= 0.5 - (0 < z < 1.78)$

$= 0.5 - F(1.78)$

$= 0.5 - 0.4625 = 0.0375.$

(b) $P(z > -1.45) = P(-1.45 < z < 0) + 0.5$

$= 0.5 + (0 < z < 1.4535)$

$= 0.5 + F(1.453)$

$= 0.5 + 0.4625 = 0.9265.$

(c) $P(-0.8 \leq z \leq 1.53) = P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 1.53)$

$= (0 \leq z \leq 0.8) + (0 \leq z \leq 1.53)$

$= F(0.8) + F(1.53)$

$= 0.2881 + 0.4370$

$= 0.7251.$

(d) $P(z < -2.52) = 0.5 - P(-2.52 < z < 0)$

$= 0.5 - (0 < z < 2.52)$

$= 0.5 - F(2.52)$

$= 0.0059$

(e) $P(z > 1.83) = 0.5 - P(0 < z < 1.83)$

$= 0.5 - F(1.83) = 0.036$

2. If the masses of 300 students are normally distributed with mean 68 kgs and standard deviation 3kgs.How many students have masses

(a) Greater than 72kgs.

(b) Less than or equal to 64 kgs

(c) Between 65 and 71 kgs inclusive.

Sol: Given N=300, $\mu = 68$, $\sigma = 3$.

Let X be the masses of the students.

We know that the normal variable $z = \frac{x-\mu}{\sigma}$

(a) Standard normal variate for X=72 is $z = \frac{x-\mu}{\sigma} = \frac{72-68}{3} = 1.33$

P(X>72)= (z>1.33)= 0.5− P(0< z < 1.33)

P(X>72)=0.5-F(1.33)

P(X>72)=0.5-0.4082 =0.092

Expected number of students greater than 72 is E(X>72) =300(0.092) =27.54~28 students

(b) Standard normal variate for X=64 is $z = \frac{x-\mu}{\sigma} = \frac{64-68}{3} = -1.33$

P($X \le 64$)= ($z \le -1.33$)

P($X \le 64$) = 0.5-P(-1.33<z<0)

P($X \le 64$)=0.5 −(0<z <1.33) (Using symmetry)

P($X \le 64$)=0.5-F(1.33)

P($X \le 64$)=0.5-0.4082 =0.092

Expected number of students less than or equal to 64 = E(X less than or equal to 64) =300(0.092) =27.54~28 students.

(c) Standard normal variate for X=65 is $z_1 = \frac{x-\mu}{\sigma} = \frac{65--68}{3} = -1$

Standard normal variate for X=71 is $z_2 = \frac{x-\mu}{\sigma} = \frac{71--68}{3} = 1$

P($65 \le X \le 71$) = P($-1 \le z \le 1$)

= P($-1 \le z \le 0$) +P ($0 \le z \le 1$)

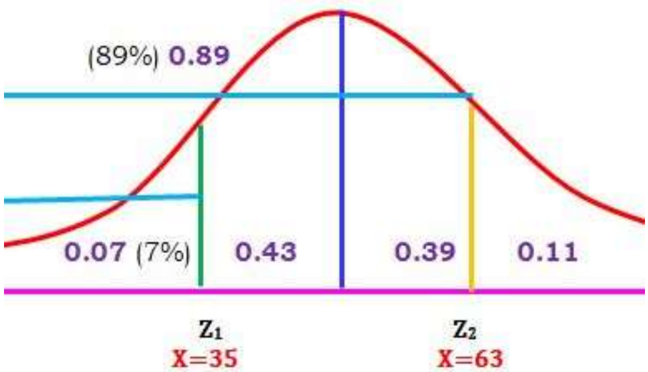
=2P($0 \le z \le 1$) ==2PF91)=2(0.341)= 0.6826

P($65 \le X \le 71$) = 300(0.6826) = 205 students.

∴Expected number of students between 65 and 71 kgs inclusive = 205 students.

3. In a normal distribution 7% of the items are under 35 and 89% of the items are under 63. Find mean and variance of the distribution

Sol: Given that P(X<35)=0.07 and P(X<63)=0.89



$z_1 = \frac{35-\mu}{\sigma}$ and $z_2 = \frac{63-\mu}{\sigma}$

Clearly $P(z_1 < z < 0) = 0.43$

$z_1 = -1.48$

$z_1 = \frac{35-\mu}{\sigma} \Rightarrow \mu - 1.48\sigma = 35$---------(1)

Clearly $P(z_2 < z < 0) = 0.39$

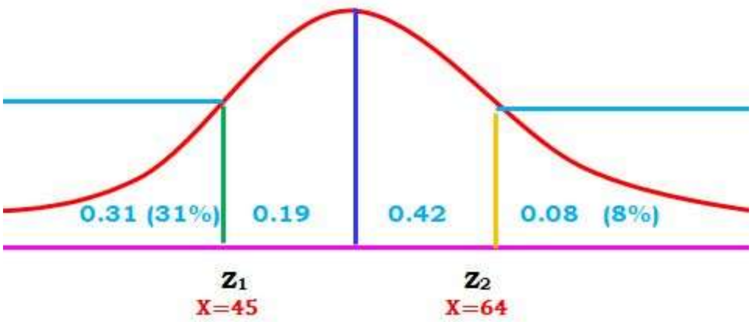
$z_2 = 1.23$

$z_2 = \frac{63-\mu}{\sigma} \Rightarrow \mu + 1.23\sigma = 63$ ---------(2)

Solve (1) and (2) we get $\mu = 50.29$ and $\sigma = 10.33$

4. In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find mean and variance of the distribution

Sol: Given P(X< 45)= 31% =0.31 and P(X >64)= 8% =0.08



$z_1 = \frac{45-\mu}{\sigma}$ and $z_2 = \frac{64-\mu}{\sigma}$

Clearly $P(z_1 < z < 0) = 0.19$

$F(z_1) = 0.19$

$z_1 = -0.5$

$z_1 = \frac{45-\mu}{\sigma} \Rightarrow \mu - 0.5\sigma = 45$---------(1)

Clearly $P(0 < z < z_2) = 0.42$

$F(z_2) = 0.42$

$z_2 = 1.42$

$z_2 = \frac{64-\mu}{\sigma} \Rightarrow \mu + 1.42\sigma = 64$ ---------(2)

Solve (1) and (2) we get $\mu = 49.9$ and $\sigma = 9.89$

5. What is the probability that there will be more than 60 heads if an unbiased coin is tossed 100 times?

Sol: The number of trials n=100

The probability of success (getting heads) p=0.5

The mean μ of a binomial distribution is given by μ=np=100·0.5=50

The standard deviation σ is given by σ^2=npq=100(0.5)(0.5)=25 hence σ=5

The standard normal variable $z = \frac{x-\mu}{\sigma}$

when x=60 then $z = \frac{60-50}{2} = 2$

$P(X > 60) = P(z > 2) = 0.5 - P(0 < z < 2) = 0.5 - F(2) = 0.5 - 0.4772 = 0.0228$

Thus, the probability of getting more than 60 heads when tossing an unbiased coin 100 times is approximately **0.0228**, or **2.28%**.

6. In a college grading system, the top 5% will be considered excellent, the next 15% will be considered an A grade, the next 30% will be considered a B grade, the next 30% will be considered a C grade, and the next 15% considered as D grade, and the rest of the students will be treated as a failure. If the scores of the students follow normal distribution with an average score is 45 and a standard deviation of 12.

(i) What is the cutoff to get an excellent grade? (Round to nearest integer)

(ii) What is the cutoff to get an A grade? (Round to nearest integer)

(iii) If a student gets a score of 35, what is his/her grade?

(iv) What are the minimum marks a student needs to score to get a pass?

Sol: $f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]; -\infty < x < \infty; \sigma > 0; -\infty < \mu < \infty$

Given that $\mu = 45$, $\sigma = 12$

$X \sim N(45,144)$

Let X be the grading system in college.

(a) $P(X>x_1)=5\%=0.05$

$P(z>z_1)=0.05$

$0.5-P(0<z<z_1)=0.05$

$P(0<z<z_1)=0.45$

$z_1=1.645$

Since $z_1 = \frac{x_1-\mu}{\sigma} \Rightarrow x_1 = \mu + \sigma z_1$

$x_1=45+1.645*12=64.74$

$x_1=65$

(b) $P(X>x_2)=20\%=0.2$

$P(Z>z_2)=0.2$

$0.5-P(0<z<z_2)=0.2$

$P(0<z<z_2)=0.3$

$F(z_2)=0.3$

$Z_2=0.84$

Since $z_2 = \frac{x_2-\mu}{\sigma} \Rightarrow x_2 = \mu + \sigma z_2$

$x_2=45+0.84*12=55.08$

$x_2=55$

(c) We know that the normal variable $z = \frac{x-\mu}{\sigma}$

When x=35 then $z = \frac{x-\mu}{\sigma} = -0.83$

$P(X=35)=P(Z=-0.83)=P(0<Z<0.83)=F(0.83)=0.7977$

$P(X=35)=0.8=80\%$

Hence P(X=35) is “C” Grade.

(d) P(X<x_3)=5%=0.05

P(Z<z_3)=0.05

0.5-P(0<z< z_3)=0.05

F(z_3)=-0.45

Z_3=-1.645

Since $z_3 = \frac{x_3-\mu}{\sigma} \Longrightarrow x_3 = \mu + \sigma\, z_3$

x_3=45-1.645*12=25.26

x_3=25

7. Let $X\sim N(3,4)$ and $Y\sim N(5,9)$ be two independent random variables. Find the probability distribution of $Z = X + Y$. Also find mean and standard deviation of Z.

Tutorial questions:

1. Compute the value of following:

a) $P(0 \le z \le 1.43)$

b) $P(-0.73 \le z \le 0)$

c) $P(-1.37 \le z \le 2.02)$

d) $P(0.65 \le z \le 1.26)$

e) $P(z \ge 1.33)$

f) $P(|z| \le 0.5)$

Answer: $\mathbf{0.4236, 0.2673, 0.8930, 0.154, 0.0918, 0.383}$

2. What is the probability that a standard normal variate Z will be greater than 1.09, less than -1.65, lying between -1 & 1.96, lying between 1.25 & 2.75? Answer: 0.1379, 0.0495, 0.8163, 0.1026

3. The compressive strength of the sample of cement can be modeled by normal distribution with mean 6000 kg/cm2 and standard deviation of 100 kg/cm2 .

(a) What is the probability that a sample strength is less than 6250 kg/cm2 ?

(b) What is probability if sample strength is between 5800 and 5900 kg/cm2 ?

(c) What strength is exceeded by 95% of the samples?

[$P(z = 2.5) = 0.9938$, $P(z = 1) = 0.8413$, $P(z = 2) = 0.9772$, $P(z = 1.65) = 0.95$]

Answer: **(a)** $\mathbf{0.9938}$, **(b)** $\mathbf{0.1815}$, **(c)** $\mathbf{1.}$, $\mathbf{x = 6165}$

4. In a photographic process, the developing time of prints may be looked upon as a random variable having normal distribution with mean of 16.28 seconds and standard deviation of 0.12 seconds. Find the probability that it will take (a) Anywhere from 16.00 to 16.50 sec to develop one of the prints;

(b) At least 16.20 sec to develop one of the prints; (c) At most 16.35 sec to develop one of the prints.

[$P(z = 1.83) = 0.9664$, $P(z = 0.66) = 0.7454$, $P(z = 0.58) = 0.7190$]

Answer: (a) $\mathbf{0.9565}$, (b) $\mathbf{0.7475}$, (c) $\mathbf{0.7190}$

5. A random variable having the normal distribution with $\mu = 18.2$ & $\sigma = 1.25$, find the probabilities that it will take on a value (a) less than 16.5 (b) Between 16.5 and 18.8.[$F(0.48)=0.3156$, $F(-1.36)=0.0869$]

Answer: **(a)** $\mathbf{0.0885}$ **(b)** $\mathbf{0.5959}$

6. A sample of 100 dry battery cell tested and found that average life of 12 hours and standard deviation 3 hours. Assuming the data to be normally distributed what percentage of battery cells are expected to have life (a) More than 15 hour (b) Less than 6 hour (c) Between 10 & 14 hours

Answer: **(a)** $\mathbf{15.87\%}$, **(b)** $\mathbf{2.28\%}$, **(c)** $\mathbf{49.72\%}$

7. In AEC Company, the amount of light bills follows normal distribution with standard deviation 60. 11.31% of customers pay light-bill less than Rs.260. Find average amount of light bill.

Answer: $\mathbf{\mu = 332.60}$

8. The marks obtained by students in a college are normally distributed with mean of 65 & variance of 25. If 3 students are selected at random from the college, what is the probability that at least one of them would have scored more than 75 marks? **Answer**: $\mathbf{0.0668}$

9. Weights of 500 students of college is normally distributed with average weight 95 lbs. & $\sigma = 7.5$. Find how many students will have the weight between 100 and 110. **Answer**: $\mathbf{114}$

10. Distribution of height of 1000 soldiers is normal with Mean 165cm & standard deviation 15cms how many soldiers are of height

(a) Less than 138 cm,

(b) More than 198 cm,

(c) Between 138 & 198 cm **Answer**: (**a**) **36**, (**b**) **14**, (**c**) **950**

11. In a normal distribution 31% of the items are below 45 and 8% are above 64. Determine the mean and standard deviation of this distribution. **Answer**: $\mu = 49.974, \sigma = 9.95$

12. The breaking strength of cotton fabric is normally distributed with E(x) = 16 and σ(x) = 1. The fabric is said to be good if $x \geq 14$ what is the probability that a fabric chosen at random is good?

Answer: **0.9772**

13. Find the probability that at most 48 heads appear in 100 tosses of a fair coin.

X.Beta Distribution:

A random variable X is said to have the beta distribution function if its probability density function is of the form $f(x) = Bet(\alpha,\beta) = \begin{cases} x^{\alpha-1}(1-x)^{\beta-1}; & if\ 0 < x < 1 \\ 0 & ; other\ wise \end{cases}$

Note:

- mean = $\mu = \frac{\alpha}{\alpha+\beta}$
- variance $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

1. The percentage of impurities per batch in a certain chemical product is a random variable X that follows the beta distribution given by $f(x) = \begin{cases} 60\,x^3(1-x)^2; & \text{if } 0 < x < 1 \\ 0; & \text{other wise} \end{cases}$

Sol: The probability that a randomly selected batch will have more than 25% impurities is given by

$$P(X \geq 0.25) = \int_{0.25}^{1} 60\,x^3(1-x)^2 dx$$

$$P(X \geq 0.25) = 60 \int_{0.25}^{1} (x^3 - 2x^4 + x^5) dx$$

$$P(X \geq 0.25) = 60\left[\frac{x^3}{4} - \frac{2x^4}{5} + \frac{x^6}{6}\right]_{0.25}^{1} = 0.96$$

2. The proportion of time per day that all checkout counters in a supermarket are busy follows a distribution $f(x) = \begin{cases} k\,x^2(1-x)^9; & \text{if } 0 < x < 1 \\ 0; & \text{other wise} \end{cases}$, What is the value of the constant k so that f(x) is a valid probability density function?

Sol: Given that $f(x) = \begin{cases} k\,x^2(1-x)^9; & \text{if } 0 < x < 1 \\ 0; & \text{other wise} \end{cases}$

Using the definition of the beta function, we get that

$$f(x) = Bet(\alpha,\beta) = \begin{cases} x^{\alpha-1}(1-x)^{\beta-1}; & if\ 0 < x < 1 \\ 0 & ; other\ wise \end{cases} ; \alpha = 3, \beta = 10$$

$$k = Bet(\alpha,\beta) = B(3,10) = \frac{\Gamma 3.\Gamma 10}{\Gamma 13} = \frac{1}{660}$$

XI. Gamma Distribution: A random variable X is said to have a Gamma distribution with parameters $\alpha > 0$ *and* $\lambda > 0$, if its probability density function is given by

$$f(x) = \mathrm{Gam}(\alpha, \lambda) = \begin{cases} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}; & x > 0 \\ 0; & \text{other wise} \end{cases}$$

Note:

- mean $= \mu = \frac{\alpha}{\lambda}$
- variance $\sigma^2 = \frac{\alpha}{\lambda^2}$

Note:

When α = 1, we have $f(x) = \begin{cases} \lambda e^{-\lambda x}; & x > 0 \\ 0; & \text{other wise} \end{cases}$

Hence, exponential distribution Exp(λ) is a special case of Gamma distribution; i.e., Gamma(1, λ)

The gamma distribution is more flexible than the exponential distribution.

1. Given a gamma random variable X with α = 3 and λ = 2. Compute E(X), V(X) and P(X ≤ 1.5 years).
   Answer: 1.5, 0.75, 0.5768
2. Suppose you are fishing and you expect to get a fish once every 1/2 hour. Compute the probability that you will have to wait between 2 to 4 hours before you catch 4 fish. Answer: 0.124
3. The daily consumption of milk in a city, in excess of 20000 liters, is approximately distributed as a gamma variate with α =2 and λ=1/10000. The city has daily stock of 30000 liters. What is the probability that the stock is insufficient on a particular day? Answer: 0.736
4. The daily consumption of electric power in a certain city is a random variable X having probability density function $f(x) = 19xe^{-x3}$, x > 0. Find the probability that the power supply is inadequate on any given day if the daily capacity of the power plant is 12 million KW hours. Answer: 0.09758
5. Suppose that the time it takes to get service in a restaurant follows a gamma distribution with mean 8 min and standard deviation 4 minutes. (a) Find the parameters α and λ of the gamma distribution. (b) You went to this restaurant at 6:30. What is the probability that you will receive service before 6:36? Answer: 2, 1/4, 0.442