

§1.8 Introduction to Linear Transformations

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, and define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.
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With T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector \mathbf{x} whose image under T is \mathbf{b} and determine whether \mathbf{x} is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

7. Let A be a 6×5 matrix. What must a and b be in order to define $T : \mathbb{R}^a \rightarrow \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and how many columns must a matrix A have in order to define a mapping from \mathbb{R}^5 into \mathbb{R}^4 by the rule $T(\mathbf{x}) = A\mathbf{x}$?
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Find all \mathbf{x} in \mathbb{R}^4 that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A .

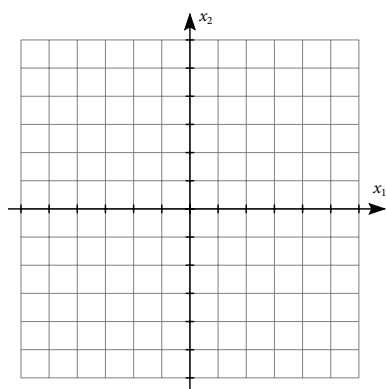
9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

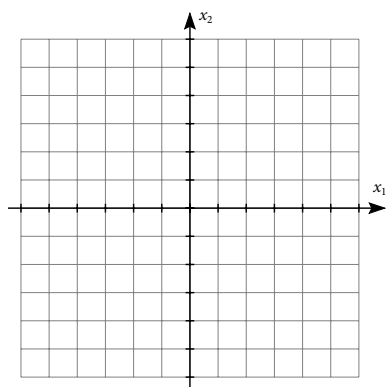
11. Let $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is \mathbf{v} in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?
12. Let $\mathbf{b} = \begin{bmatrix} -1 & 3 & -1 & 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is \mathbf{b} in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?
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Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T . Describe geometrically what T does to each vector in \mathbb{R}^2 .

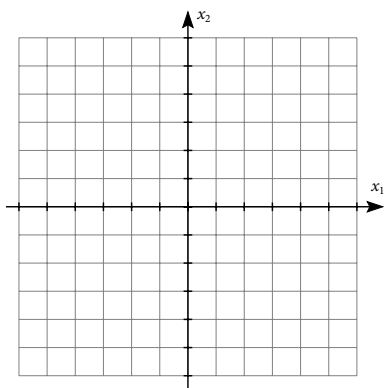
13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



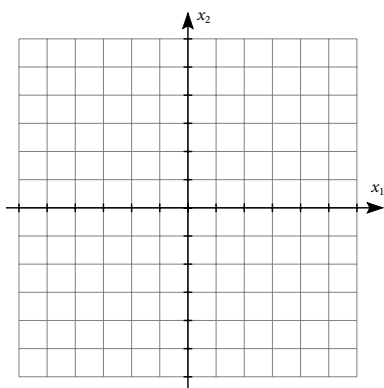
14. $T(\mathbf{x}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



17. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 & 3 \end{bmatrix}$. Use the fact that T is a *linear* transformation to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u} + 2\mathbf{v}$.

18. insert

19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps \mathbf{e}_1 into \mathbf{y}_1 and maps \mathbf{e}_2 into \mathbf{y}_2 . Find the images of $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps \mathbf{x} into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each \mathbf{x} .

Mark each statement True or False (**T/F**). Justify each answer.

21. (**T/F**) A linear transformation is a special type of function.

22. (**T/F**) Every matrix transformation is a linear transformation.

23. (**T/F**) If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is \mathbb{R}^3 .

24. (T/F) The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A .
25. (T/F) If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation and if \mathbf{c} is in \mathbb{R}^m , then a uniqueness question is “Is \mathbf{c} in the range of T ?”
26. (T/F) Every linear transformation is a matrix transformation.
27. (T/F) A linear transformation preserves the operations of vector addition and scalar multiplication.
28. (T/F) A linear transformation preserves the operations of vector addition and scalar multiplication.
29. (T/F) A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all \mathbf{v}_1 and \mathbf{v}_2 in the domain of T and for all scalars c_1 and c_2 .
30. (T/F) The superposition principle is a physical description of a linear transformation.
31. (T/F) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that reflects each point through the x_1 -axis. (See Practice Problem 2.). Make two sketches similar to Figure 6 that illustrates properties (i) and (ii) of a linear transformation.
32. (T/F) Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span \mathbb{R}^n , and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is a zero transformation. That is, show that if \mathbf{x} is any vector in \mathbb{R}^n , then $T(\mathbf{x}) = \mathbf{0}$.
33. (T/F) Given $\mathbf{v} \neq \mathbf{0}$ and \mathbf{p} in \mathbb{R}^n , the line through \mathbf{p} in the direction of \mathbf{v} has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).
34. (T/F) Let \mathbf{u} and \mathbf{v} be linearly independent vectors in \mathbb{R}^3 , and let P be the plane through \mathbf{u} , \mathbf{v} , and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in \mathbb{R}). Show that a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, or onto a line through $\mathbf{0}$, or onto just the origin in \mathbb{R}^3 . What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order for the image of the plane P to be a plane?
35. (T/F)