Name: ______________________________ Date: __________

§1.8 Introduction to Linear Transformations

1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, and define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

With T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$ and determine whether $\mathbf{x}$ is unique.

3. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & -9 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ -7 \\ -9 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -4 & 5 \\ 0 & 1 & 1 \\ -3 & 5 & -4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 3 \\ -6 \end{bmatrix}$

7. Let A be a 6×5 matrix. What must a and b by in order to define $T : \mathbb{R}^a \to \mathbb{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

8. How many rows and how many columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^4$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

Find all $\mathbf{x}$ in $\mathbb{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

9. $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$

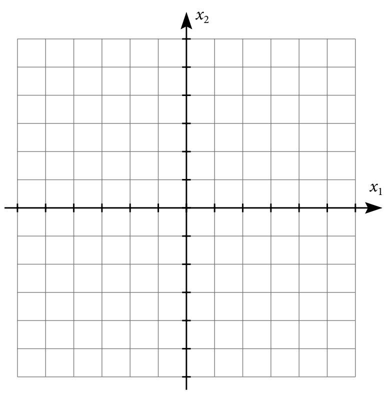
10. $A = \begin{bmatrix} 1 & 3 & 9 & 2 \\ 1 & 0 & 3 & -4 \\ 0 & 1 & 2 & 3 \\ -2 & 3 & 0 & 5 \end{bmatrix}$

11. Let $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and let A be the matrix in Exercise 9. Is $\mathbf{v}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

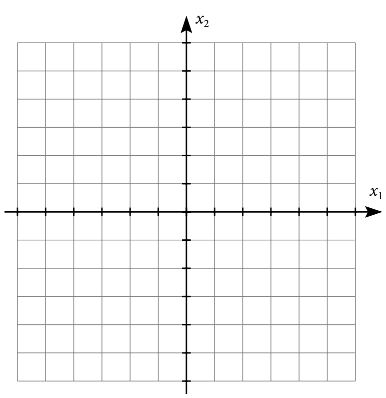
12. Let $\mathbf{b} = \begin{bmatrix} -1 & 3 & -1 & 4 \end{bmatrix}$, and let A be the matrix in Exercise 10. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, and their images under the given transformation T. Describe geoemetrically what T does to each vector in $\mathbb{R}^2$.

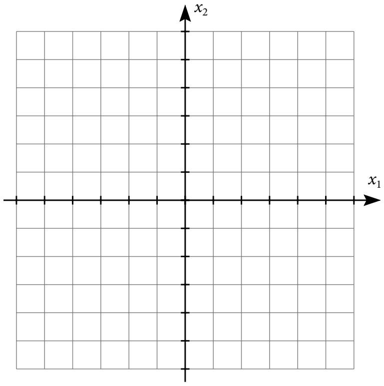
13. $T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



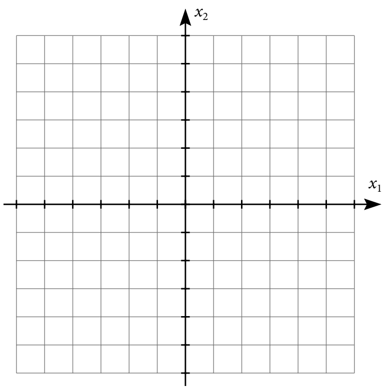
14. $T(\mathbf{x}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



15. $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

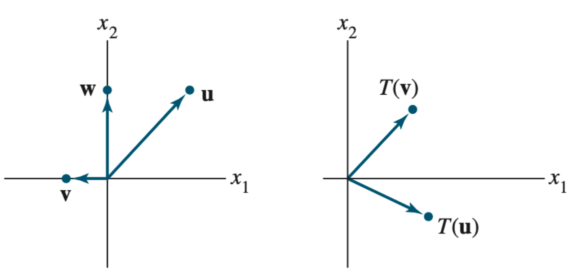


16. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 & 3 \end{bmatrix}$. Use the fact that T is a *linear* transformation to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u} + 2\mathbf{v}$.

18. The figure shows vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, along with the images $T(\mathbf{u})$ and $T(\mathbf{v})$ under the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Copy this figure carefully, and draw the image $T(\mathbf{w})$ as accurately as possible. *Hint:* First, write $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.



19. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

20. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{v}_1 + x_2\mathbf{x}_2$. Find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each $\mathbf{x}$.

Mark each statement True or False (**T**/**F**). Justify each answer.

21. (**T**/**F**) A linear transformation is a special type of function.

22. (**T**/**F**) Every matrix transformation is a linear transformation.

23. (**T**/**F**) If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.

24. (**T**/**F**) The codomain of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the set of all linear combinations of the columns of A.

25. (**T**/**F**) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and if $\mathbf{c}$ is in $\mathbb{R}^m$, then a uniqueness question is "Is $\mathbf{c}$ in the range of T?"

26. (**T**/**F**) Every linear transformation is a matrix transformation.

27. (**T**/**F**) A linear transformation preserves the operations of vetor addition and scalar multiplciation.

28. (**T**/**F**) A linear transformation preserves the operations of vector addition and scalar multiplication.

29. (**T**/**F**) A transformation T is linear if and only if $T\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

30. (**T**/**F**) The superposition principle is a physical description of a linear transformation.

31. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflecs each point through the x_1-axis. (See Practice Problem 2.). Make two sketches similar to Figure 6 that illustrates properties (i) and (ii) of a linear transformation.

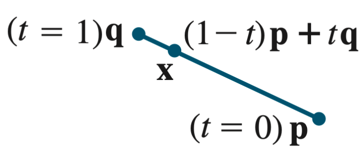
32. Suppose vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, \dots, p$. Show that T is a zero transformation. That is, show that if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x}) = \mathbf{0}$.

33. Given $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p}$ in $\mathbb{R}^n$, the line through $\mathbf{p}$ in the directin of $\mathbf{v}$ has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Show that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ maps this line onto another line or onto a single point (a *degenerate line*).

34. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^3$, and let P be the plane through $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbb{R}$). Show that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ maps P onto a plane through $\mathbf{0}$, or onto a line through $\mathbf{0}$, or onto just the origin in $\mathbb{R}^3$. What must be true about $T(\mathbf{u})$ and $T(\mathbf{v})$ in order fo rthe image of the plane P to be a plane?

35. (a) Show that the line through vectors $\mathbf{p}$ to $\mathbf{q}$ in $\mathbb{R}^n$ may be writte in the parametric form $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$.

    (b) The line spegment from $\mathbf{p}$ to $\mathbf{q}$ is the set of points of the form $(1-t)\mathbf{p} + t\mathbf{q}$ for $0 \leq t \leq 1$ (as shown in the figure below). Show that the linear transformation T maps this line segment onto a line segment or onto a single point.



36. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. It can be shown that thet set P of all points in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \leq q \leq 1$, $0 \leq b \leq 1$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Explain why the image of a point in P under the transformation T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

37. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = mx + b$.
    (a) Show that f is a linear transformation when $b = 0$.
    (b) Find a property of a linear transformation that is violated when $b \neq 0$.
    (c) Why is f called a linear function?

38. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(x) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $b \neq 0$. (Affine transformations are important in computer graphics.)

39. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ by a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.

In Excercises 40-44, column vectors are written as rows, such as $\mathbf{v} = (x_1, v_2)$, and $T(\mathbf{x})$ is written as $T(x_1, x_2)$.

40. Show that the transformation T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

41. Show that the transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is not linear.

42. Let $T :\ mathbbR^n \to \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vetors onto a linearly dependent set, then the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution. *Hint:* Suppose $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are linearly independent and yet $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly depndent. Then $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ for some weights c_1 and c_2, not both zero. Use this equation.

43. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that reflects each vector $\mathbf{x} = (x_1, x_2, x_3)$ through the plane $x_3 = 0$ onto $T(\mathbf{x}) = (x_1, x_2, -x_3)$. Show that T is a linear transformation.

44. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation that projects each vector $\mathbf{x} = (x_1, x_2, x_3)$ onto the plane $x_2 = 0$, so $T(\mathbf{x}) = (x_1, 0, x_3)$. Show that T is a linear transformation.