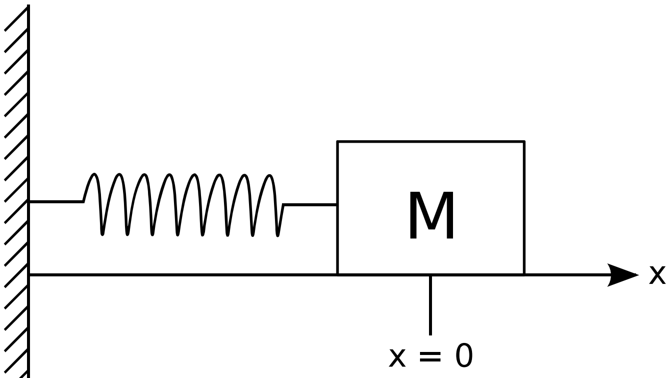
Modul 3: ODE



x(t) — position
v(t) — hastighet

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = -x \end{cases}$$

M = 1

$$\begin{matrix} (1) \\ (2) \end{matrix} \begin{cases} x^{n+1} = x^n + kv^n \\ v^{n+1} = v^n - kx^n \end{cases} \qquad \begin{cases} x^{n+1} = x^n + kv^{n+1} \\ v^{n+1} = v^n - kx^{n+1} \end{cases}$$

Trapetsregel:

$$\begin{cases} x^{n+1} = x^n + \dfrac{k}{2}(v^n + v^{n+1}) \\ v^{n+1} = v^n - \dfrac{k}{2}(x^n + x^{n+1}) \end{cases}$$

Total Energy, E (M = 1)

$$E(t) = \underbrace{\frac{1}{2}x^2(t)}_{\text{Potentiell energi}} + \underbrace{\frac{1}{2}v^2(t)}_{\text{Kinetisk energi}}$$

$$\frac{dE}{dt} = \dot{x}\,x + \dot{v}\,v = \underset{(1)}{vx} - \underset{(2)}{xv} = 0$$

Alternativt:

Multiplicera (1) med x och (2) med v.
Adderar sedan resultaten.

$$\begin{cases} \dot{x}\,x = vx \\ \dot{v}\,v = -xv \end{cases} \Rightarrow \dot{x}\,x + \dot{v}\,v = vx - xv = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}x^2\right) + \frac{d}{dt}\left(\frac{1}{2}v^2\right) = 0 \Rightarrow \frac{d}{dt}E = 0$$

$$\begin{cases} x^{n+1} - x^n = \dfrac{k}{2}(v^n + v^{n+1}) \\ v^{n+1} - v^n = -\dfrac{k}{2}(x^n + x^{n+1}) \end{cases}$$

Multiplicera med

$$\begin{cases} x^{n+1} + x^n \\ v^{n+1} + v^n \end{cases}$$

$$\Downarrow$$

$$\begin{cases} (x^{n+1} - x^n)(x^{n+1} + x^n) = \dfrac{k}{2}\overbrace{(v^n + v^{n+1})(x^n + x^{n+1})}^{\alpha} \\ (v^{n+1} - v^n)(v^{n+1} + v^n) = -\dfrac{k}{2}\underbrace{(x^n + x^{n+1})(v^n + v^{n+1})}_{\alpha} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} (x^{n+1})^2 - (x^n)^2 = \dfrac{k}{2}\alpha \\ (v^{n+1})^2 - (v^n)^2 = -\dfrac{k}{2}\alpha \end{cases}$$

$$\Downarrow$$

$$(x^{n+1})^2 - (x^n)^2 + (v^{n+1})^2 - (v^n)^2 = \frac{k}{2}\alpha - \frac{k}{2}\alpha$$

$$\Updownarrow$$

$$(x^{n+1})^2 - (x^n)^2 + (v^{n+1})^2 - (v^n)^2 = 0$$

$$\Updownarrow$$

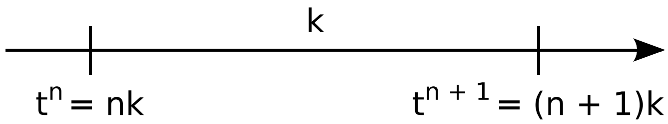
$$(x^{n+1})^2 + (v^{n+1})^2 = (x^n)^2 + (v^n)^2$$

$$E^{n+1} = \frac{1}{2}(x^{n+1})^2 + \frac{1}{2}(v^{n+1})^2 = \frac{1}{2}(x^n)^2 + \frac{1}{2}(v^n)^2 = E^n$$

$$\Updownarrow$$

$$E^{n+1} = E^n$$

$$\begin{cases} \dot{u}(t) = f(u(t)) \\ u(0) = u^0 \end{cases}$$

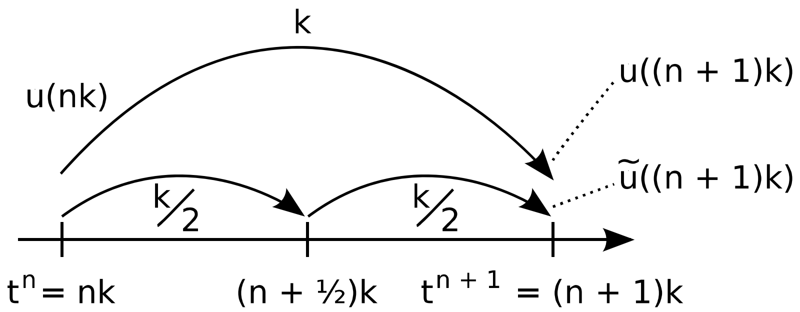


Tidsstegning med Euler framåt:

$$u\big((n+1)k\big) \;=\; u(nk) \;+\; f\big(u(nk)\big)\cdot k$$

f, Lipschitz-kontinuerlig

Skillnad mellan att ta ett tidssteg med längden k,
jämfört med tidssteg av längden $\frac{k}{2}$?



1. Antag att u och ũ startar från samma u(nk):

$$\left| u\big((n+1)k\big) - \tilde{u}\big((n+1)k\big) \right| \;=\; \{ S \triangleq u(nk) \} \;=$$

$$= \left| S + kf(S) - \left[S + \frac{k}{2} f(S) + \frac{k}{2} f\left(S + \frac{k}{2} f(S) \right) \right] \right| \;=$$

$$= \left| S + kf(S) - S - \frac{k}{2} f(S) - \frac{k}{2} f\left(S + \frac{k}{2} f(S) \right) \right| \;=$$

$$= \left| \frac{k}{2} f(S) - \frac{k}{2} f\left(S + \frac{k}{2} f(S) \right) \right| \;=$$

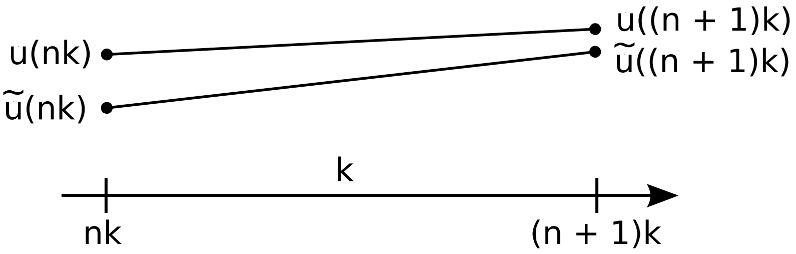
$$= \frac{k}{2} \left| f(S) - f\left(S + \frac{k}{2} f(S) \right) \right| \;\leq$$

$$\leq \frac{k}{2} L \left| S - \left(S + \frac{k}{2} f(S) \right) \right| \;=$$

$$= \frac{k^2}{4} L \left| f\big(u(nk)\big) \right|$$

2. Uppskatta skillnaden efter ett tidssteg med olika startvärden

$$u(nk) - \tilde{u}(nk)$$



$$\left| u\left((n+1)k\right) - \tilde{u}\left((n+1)k\right) \right| =$$

$$= \left| u(nk) + kf\left(u(nk)\right) - \tilde{u}(nk) + kf\left(\tilde{u}(nk)\right) \right| \le$$

$$= \left| u(nk) - \tilde{u}(nk) \right| + k \left| f\left(u(nk)\right) - f\left(\tilde{u}(nk)\right) \right| \le$$

$$= (1 + kL) \left| u(nk) - \tilde{u}(nk) \right|$$

1. + 2.

$\Downarrow$

Total felet $\left| u(T) - \tilde{u}(T) \right| \le k\left(\exp(LT)\right)$

L = 10
T = 30

$\exp(LT) \gg 10^{100}$

Analys av felet och stabilitet

ODE: Hitta u(t) så att

$$\begin{cases} \dot{u}(t) + Au(t) = F(t), & t > 0 \\ u(0) = u^0 \end{cases}$$

A — konstant

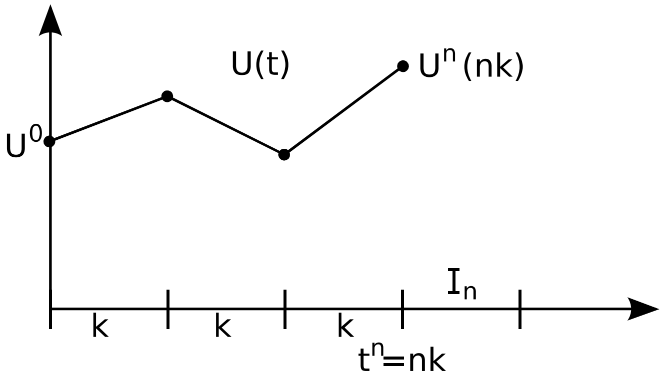
F(t) — given funktion

$$\left(\dot{u}(t) = f\left(t; u(t)\right) = F(t) - Au(t) \right)$$

Beräkna approximativt U(t) med trapetsmetoden.

Hitta $U^n = U(nk)$ så att $U^{n+1} + \frac{k}{2}\left(AU^{n+1} + AU^n\right) = U^n + \int_{I_n} F(t)\,dt, \quad n = 0, 1, 2, \ldots$

$U^0 \approx u^0$



$\int_{I_n} F(t)\,dt$ kan beräknas exakt.

$\int_{I_n} \dot{U}(t)\,dt = U^{n+1} - U^n, \quad \int_{I_n} AU(t)\,dt = \frac{k}{2}\left(AU^{n+1} + AU^n\right)$

Trapetsmetoden uppfyller:

$$\int_{I_n} \left(\dot{U} + AU - F\right) dt = 0, \quad n = 0, 1, 2, \ldots$$

Medelvärdet av residualen är noll:

$$\boxed{R(u) \equiv \dot{U} + AU - F}$$

över varje I_n

Beroende på A har ekvationen $\dot{u} + Au = F$ olika stabilitetsegenskaper:

1. konstant icke-negativ: $A \geq 0$ (stabil)
2. konstant imaginär: $A = i$
3. konstant negativ: $A < 0$ (instabil)
4. oscillerande positiv–negativ: $A(t) = \sin(t)$

Uppskatta felet till tiden T = Nk

Uppskatta e(t) ≜ u(t) - U(t)

Inför dualproblem för att mäta känslighet

$$-\dot{\varphi}(t)+A\varphi(t)=0,\quad T>t\geq 0$$ (baklänges i tiden)

$$\varphi(T)=\pm 1=\mathrm{sgn}(e(t))$$

(Vi räknar med signum(0) = 0)

Felrepresentation:

$$0=\int_0^T e(t)\underbrace{(-\dot{\varphi}(t)+A\varphi(t))}_{0}\,dt=\{\text{Partialintegration}\}=$$

$$=\int_0^T (\dot{e}+Ae)\varphi(t)\,dt-e(T)\varphi(T)+e(0)\varphi(0)$$

$$\Updownarrow$$

$$|e(T)|=-\int_0^T R(U)\varphi(t)\,dt+(u^0-U^0)\varphi(0)$$

$$|u(T)-U(T)|\ \leq\ S_c(T)\max_n\ kR_n(U)\ +\ S_d\,|u(0)-U(0)|$$

Stabilitetsfaktorer:

$$\begin{cases} S_c(T)=\int_0^T |\dot{\varphi}(s)|\,ds \\ S_d(T)=|\varphi(0)| \end{cases}$$