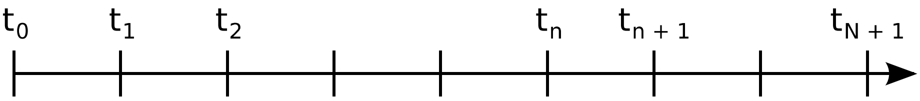
Differentialekvation

$$\begin{cases} \dot{u}(t) = f(t) \\ u(0) = u^0 \end{cases}$$

- tid t, tidssteg dt



$t_n = n\ dt$

$u^n = u(n\ dt)$

$f^n = f(n\ dt)$

- tidsstegning

$$u^{n+1} = u^n + f^n\ dt \qquad \left(\frac{u^{n+1} - u^n}{dt} = f^n \right)$$

$$u^n = u^{n-1} + f^{n-1}\ dt$$

$u^{n-1} = u^{n-2} + f^{n-2}\ dt$

$u^{n+1} = u^0 + (f^0 + f^1 + f^2 + \ldots + f^n)\ dt$

$$u^{n+1} = u^0 + \sum_{i=0}^{n} f^i\ dt$$

$$u^{n+1} - u^0 = \sum_{i=0}^{n} f^i\ dt$$

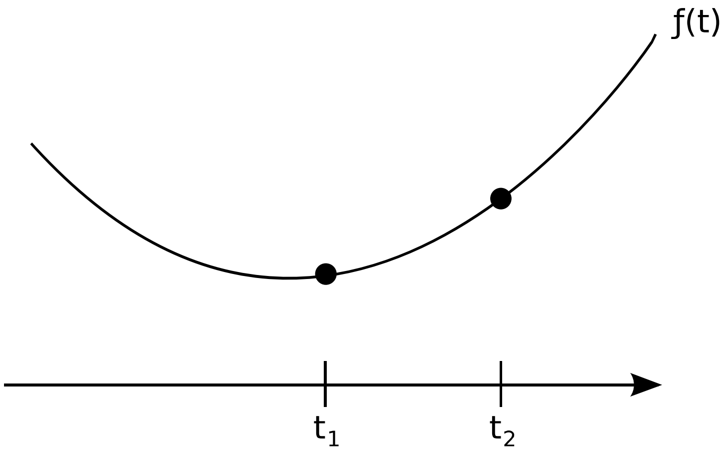
Blir integral

$$\left(u(T) - u(0) = \int_0^T f(t)\ dt \right)$$

Lipschitz-kontinuietet

$f(t)$ är Lipschitz-kontinuerlig om

$$|f(t_2) - f(t_1)| \leq L|t_2 - t_1|$$

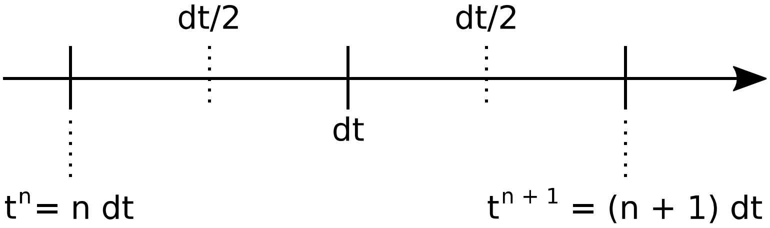


$$\left|\frac{f(t_2) - f(t_1)}{t_2 - t_1}\right| \leq L, \qquad f(t) = t^2$$

$$f(t_2) - f(t_1) = t_2^2 - t_1^2 = (t_2 - t_1)\underbrace{(t_2 + t_1)}_{L = \max} = \underbrace{f'(\xi)}_{L = \max}(t_2 - t_1), \qquad \xi \in [t_1; t_2]$$

$$f(x) - f(y) = f'(\xi)(x - y), \ \xi \in [x; y]$$

Medelvärdessatsen!



Effekt av tidsstegets längd för ett steg

$$u((n+1)dt) = u(n\,dt) + f(n\,dt)\ dt$$

$$\bar{u}((n+1)dt) = u(n\,dt) + f(n\,dt)\,\frac{dt}{2} + f\left(n\,dt + \frac{dt}{2}\right)\frac{dt}{2}$$

$$u\big((n+1)dt\big)-\bar{u}\big((n+1)dt\big)=$$

$$=u(n\,dt)+f(n\,dt)\,dt-\left(u(n\,dt)+f(n\,dt)\,\frac{dt}{2}+f\left(n\,dt+\frac{dt}{2}\right)\,\frac{dt}{2}\right)=$$

$$=f(n\,dt)\,dt-f(n\,dt)\,\frac{dt}{2}-f\left(n\,dt+\frac{dt}{2}\right)\,\frac{dt}{2}=$$

$$=f(n\,dt)\,\frac{dt}{2}-f\left(n\,dt+\frac{dt}{2}\right)\,\frac{dt}{2}=$$

$$=\left[f(n\,dt)-f\left(n\,dt+\frac{dt}{2}\right)\right]\frac{dt}{2}$$

$$\left|f(t_2)-f(t_1)\right| \leq L\left|t_2-t_1\right|$$

där $f(t)$ är Lipschitz-kontinuerlig.

$$u\big((n+1)dt\big)-\bar{u}\big((n+1)dt\big)=$$

$$=\left(f(n\,dt)-f\left(n\,dt+\frac{dt}{2}\right)\right)\frac{dt}{2}\leq$$

$$\leq L\left|n\,dt-\left(n\,dt+\frac{dt}{2}\right)\right|\,\frac{dt}{2}=$$

$$=L\left|-\frac{dt}{2}\right|\,\frac{dt}{2}=$$

$$=\frac{L}{4}\,dt^2$$

Skillnad över [0; T] $\qquad T = (N + 1)\,dt$

$$\left|u(T)-\bar{u}(T)\right| \;\leq\; \frac{L}{4}\,dt^2 \times (N+1) \;=\; \frac{LT}{4}\,dt$$

❑ Samma sak för $\bar{u}$ motsvarande $\frac{dt}{4}$

— Låt $u_{dt}(T)$ motsvara lösning med tidssteg dt

$$|u_{dt}(T) - u_{dt/4}(T)| =$$

$$= |u_{dt}(T) - u_{dt/2}(T) + u_{dt/2}(T) - u_{dt/4}(T)| \le$$

$$\le |u_{dt}(T) - u_{dt/2}| + |u_{dt/2}(T) - u_{dt/4}| \le$$

$$\le \frac{LT}{4}\,dt + \frac{LT}{4}\,\frac{dt}{2}$$

❑ Upprepa för $\frac{dt}{8}$, $\frac{dt}{16}$, ...

❑ Låt $\bar{u}$ vara lösning med godtyckligt litet dt

$$|u_{dt}(T) - \bar{u}(T)| \le$$

$$\le \frac{LT}{4}\,dt + \frac{LT}{4}\,\frac{dt}{2} + \frac{LT}{4}\,\frac{dt}{4} + \frac{LT}{4}\,\frac{dt}{8} + \ldots =$$

$$\le \frac{LT}{2}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) <$$

$$< \left\{\sum_{i=1}^{N} \frac{1}{2^n} \cong 1\right\} < \frac{LT}{2}$$

Fundamentalsatsen

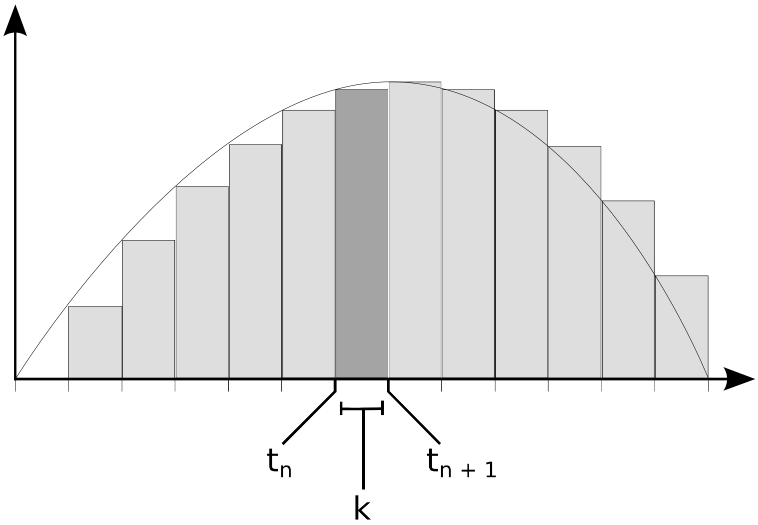
Om $f : [0; T] \to \mathbb{R}$ är Lipschitz-kontinuerlig så är funktionen $u(t) = \int_0^t f(s)\,ds$

(som definieras genom tidstegning med försvinnande kort tidssteg,

motsvarande $\bar{u}$) lösning till ekvationen

$$\begin{cases} \dot{u}(t) = f(t) \\ u(0) = u^0 \end{cases} \quad \text{för } t \in [0; T]$$

Riemann-summa



$$u(T) = \int_0^t f(t)\,dt \approx \sum_{n=0}^{N} f(nk)k$$