Funktionsmängden

$$\{1\} \cup \left(\coprod_{i=1}^{n} \cos \frac{i\pi x}{p} \right) \cup \left(\coprod_{i=1}^{n} \sin \frac{i\pi x}{p} \right)$$

är ortogonal på intervallet [–p; p] med den inre produkten

$$(f_1; f_2) = \int_{-p}^{p} f_1(x) \cdot f_2(x) dx$$

Fourierserien till en funktion ƒ definierad på intervallet]–p; p[ges av:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx \qquad a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx$$

$$n \neq 0 \qquad b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx$$

Fourierserien för en jämn funktion på intervallet]–p; p[:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} \right)$$

$$a_0 = \frac{2}{p} \int_{0}^{p} f(x)\, dx \qquad a_n = \frac{2}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx$$

Fourierserien för en udda funktion på intervallet]–p; p[:

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{p} \right) \qquad b_n = \frac{2}{p} \int_{0}^{p} f(x) \sin \frac{n\pi x}{p}\, dx$$

Konvergensvillkor:

Låt ƒ och ƒ' vara styckvis kontinuerliga på intervallet]–p; p[.

Då konvergerar ƒ:s Fourierserie mot $\frac{f(x^-) + f(x^+)}{2}$.

[z.c.11.2.7.]

$f(x) = x + \pi, \qquad -\pi < x < \pi$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx=\frac{1}{\pi}(0+\pi 2\pi)=2\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx=\frac{1}{\pi}\left(\left[(x+\pi)\frac{\sin nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{\sin nx}{n}dx\right)=$$

$$=-\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\frac{\sin nx}{n}}_{\text{udda}}dx=0, \qquad (n\neq 0)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx\,dx=\frac{1}{\pi}\left(\left[(x+\pi)\frac{-\cos nx}{n}\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{\cos nx}{n}dx\right)=$$

$$=\frac{1}{\pi}(2\pi)\frac{-\cos nx}{n}+\frac{1}{\pi}\left[\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}=-\frac{2\pi}{\pi n}\cos nx+0=\frac{2(-1)^{n+1}}{n}$$

$$f\sim\pi+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

[z.c.12.4.1.]

$$u(x;t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{an\pi t}{L}+b_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

Det återstår nu att bestämma konstanterna a_n och b_n.

Begynnelsevillkoret ger oss:

$$\frac{\partial}{\partial t}u(x;t)=\sum_{n=1}^{\infty}\frac{an\pi}{L}\left(b_n\cos\frac{an\pi t}{L}-a_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

Fourierserien för en udda funktion på intervallet]–p; p[:

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{p}, \quad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi x}{p}\,dx$$

$$u(x;0)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}=\underbrace{\frac{1}{4}x(L-x)}_{\text{Givet}}$$

$$\frac{\partial}{\partial t}u(x;0)=\sum_{n=1}^{\infty}\frac{an\pi}{L}b_n\sin\frac{n\pi x}{L}=\{\text{Givet}\}=0 \quad , \quad b_n=0$$

$$a_n=\frac{2}{L}\int_0^L\frac{1}{4}x(L-x)\sin\frac{n\pi x}{L}\,dx=\{\text{slut}\ldots\}$$

[z.c.12.3.3.]

Värmeledesekvation.

Find the temperature $u(x;t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\quad,\qquad 0 < x < L,\quad t > 0$$

Randvillkor: $\begin{cases}\frac{\partial}{\partial x}u(0;t)=0\\ \frac{\partial}{\partial x}u(L;t)=0\end{cases}\quad,\quad t > 0$

Begynnelsevillkor: $u(x;0) = f(x)$, $\quad 0 < x < L$

Separera variablerna: $u(x;t) = X(x)T(t)$

$kX''(x)T(t) = X(x)T'(t)$

Dividera med $kX(x)T(t)$

$$\frac{X''(x)}{X(x)}=\frac{T'(t)}{kT(t)}=\text{“konstant”}=\lambda$$

$$\begin{cases}X''(x)-\lambda X(x)=0\\ T'(t)-\lambda kT(t)=0\end{cases}$$

$\lambda > 0$, $\lambda = \mu^2$, $\mu \in \mathbb{R}$:

$X''(x) - \mu^2 X(x) = 0$

Lösningarna ges av

$X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$

$\lambda = 0$:

$X''(x) = 0$

$X(x) = A_2 x + B_2$

$\lambda < 0$, $\lambda = -\mu^2$, $\mu \in \mathbb{R}$:

$$X''(x) + \mu^2 X(x) = 0$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

Substitutionen ger att randvilloren kan skrivas

$$0 = \frac{\partial}{\partial x} u(0; t) = X'(0) T(t)$$

$$0 = \frac{\partial}{\partial x} u(L; t) = X'(L) T(t)$$

Dessa samband skall stämma för alla t.

Detta innebär att: $0 = X'(0)$, $0 = X'(L)$.

$\lambda > 0$:

$$X'(x) = \mu \cdot (A_1 e^{\mu x} - B_1 e^{-\mu x})$$

$$\begin{cases} 0 = X'(0) = \mu \cdot (A_1 - B_1) \\ 0 = X'(L) = \mu \cdot (A_1 e^{L\mu} - B_1 e^{-L\mu}) \end{cases}$$

$$A_1 = B_1 = 0$$

Endast den triviala lösningarna $\mu = 0$

$\lambda = 0$:

$$X'(x) = A_2$$

$$\begin{cases} 0 = X'(0) = \mu \cdot (B_3) \\ 0 = X'(L) = \mu \cdot (-A_3 \sin \mu L + B_3 \cos \mu L) \end{cases}$$

$$X(x) = B_2$$

$$T(t) = C_2$$

$\lambda < 0$:

$X'(x) = \mu \cdot (-A_3 \sin \mu x - B_3 \cos \mu x)$

$$\begin{cases} 0 = X'(0) = \mu \cdot (B_3) \\ 0 = X'(L) = \mu \cdot (-A_3 \sin \mu L + B_3 \cos \mu L) \end{cases}$$

$B_3 = 0$

$A_3 \sin \mu L = 0$

Icke-triviala lösningar erhålles då $\mu L = n\pi$, $n \in \mathbb{Z}$

$$X(x) = A_3 \cos \frac{n\pi x}{L}$$

$$T(t) = C_3 e^{\overbrace{-\left(\frac{n\pi x}{L}\right)^2}^{\lambda = -\mu^2} kt}$$

$$u(x;t) = B_2 C_2 + \sum_{n=1}^{\infty} (A_3 C_3)_n \cos \frac{n\pi x}{L} \; e^{-\left(\frac{n\pi}{L}\right)^2 kt} =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \; e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

Begynnelsevillkoret ger:

$$f(x) = u(x;0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$