Lös värmeledningsekvationen:

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \quad -\infty<x<\infty$$

$$u(x;0)=f(x)=\begin{cases} u_0 & |x|<1 \\ 0 & |x|>1 \end{cases}$$

(Beskriver temperaturen i en oändlig tråd.)

Låt:

$$\hat{u}(\alpha;t)=\mathcal{F}(u(x;t))(\alpha;t)=\int_{-\infty}^{\infty} u(x;t)e^{i\alpha x}\,dx$$

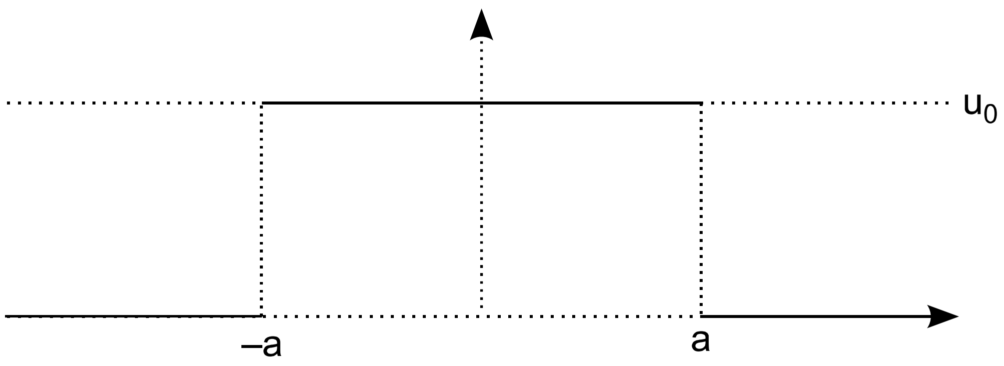
(Med avseende på x, t är fixerad)

(För varje fixerat t söks FT av u(x; t))

Transformera begynnelsevillkor:

Vi har u(x; 0) = f(x).

$$\mathcal{F}(u(x;0))=U(\alpha;0)=\mathcal{F}(f(x))(\alpha)=\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx=\int_{-1}^{1} u_0\cdot e^{i\alpha x}\,dx=$$



$$=u_0\left[\frac{e^{i\alpha x}}{i\alpha}\right]_{x=-1}^{1}=u_0\left(\frac{e^{i\alpha x}-e^{-i\alpha x}}{i\alpha}\right)=2u_0\left(\frac{e^{i\alpha x}-e^{-i\alpha x}}{2i\alpha}\right)=2u_0\cdot\frac{\sin\alpha}{\alpha}$$

Transformera DE:

$$k\mathcal{F}\left(\frac{\partial^2 u(x;t)}{\partial x^2}\right)=k(i\alpha)^2\cdot\mathcal{F}(u(x;t))=k(-\alpha^2)U(\alpha;t)$$

$$\mathcal{F}\left(\frac{\partial u}{\partial t}(x;t)\right)=\int_{-\infty}^{\infty}\frac{\partial u(x;t)}{\partial t}\cdot e^{ixt}\,dx\overset{(*)}{=}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x;t)e^{ixt}\,dx=\frac{\partial}{\partial t}U(\alpha;t)$$

Förklarning av (*):

$$\int \left(\frac{\partial}{\partial t}u(t;\, s)\right)_{t=t_0} ds = \int \lim_{\Delta t \to 0} \frac{u(t_0;\, s)-u(t_0+\Delta t;\, s)}{\Delta t}\, ds =$$

$$= \{\int \text{ är absolut konvergent}\} = \lim_{\Delta t \to 0} \int \frac{u(t_0;\, s)-u(t_0+\Delta t;\, s)}{\Delta t}\, ds =$$

$$= \lim_{\Delta t \to 0} \frac{\int u(t_0;\, s)\, ds - \int u(t_0+\Delta t;\, s)\, ds}{\Delta t} = \frac{\partial}{\partial t} \int (u(t;\, s))_{t=t_0} ds$$

Ekvationen tar formen

$$-k\alpha^2 U(\alpha;\, t) = \frac{\partial}{\partial t} U(\alpha;\, t)$$

För varje fixerat α är ekvationen

$$U_t^{\prime} = -k\alpha^2 U$$

Separabel!

$$U(\alpha;\, t) = Ce^{-k\alpha^2 t}$$

Begynnelsevillkor (BV) ger:

$$2u_0 \frac{\sin\alpha}{\alpha} \overset{BV}{=} U(\alpha;\, 0) = Ce^{-k\alpha^2 \cdot 0} = C$$

$$C = 2u_0 \frac{\sin\alpha}{\alpha}$$

Alltså:

$$U(\alpha;\, t) = \frac{2u_0 \sin\alpha}{\alpha} e^{-kx^2 t}$$

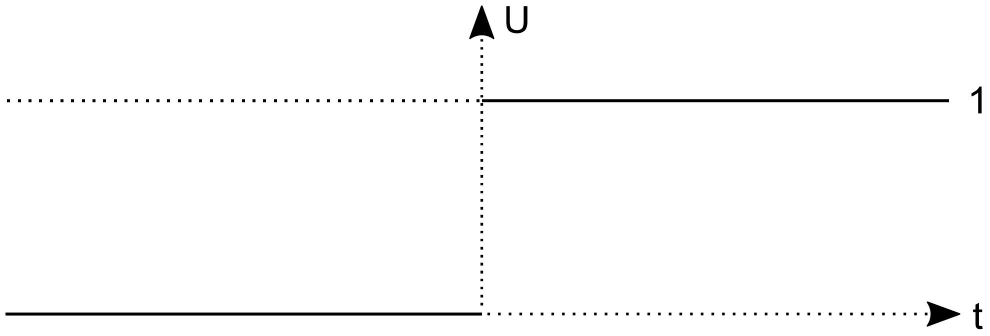
Tillbaka-transformation:

$$u(x;\, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2u_0 \frac{\sin\alpha}{\alpha} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha = \{\text{Se boken sida 506}\} =$$

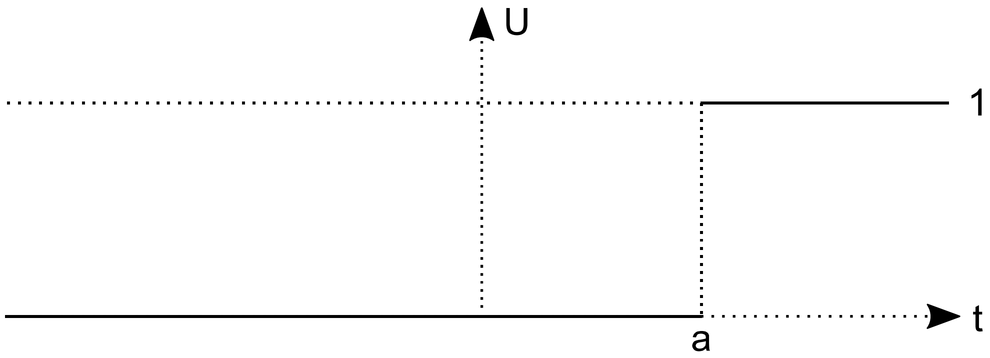
$$\frac{u_0}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\sin\alpha\ \cos\alpha}{\alpha} \cdot e^{-k\alpha^2 t}\, d\alpha$$

Heaviside funktionen:

$$U(t) \stackrel{\text{def}}{=} \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

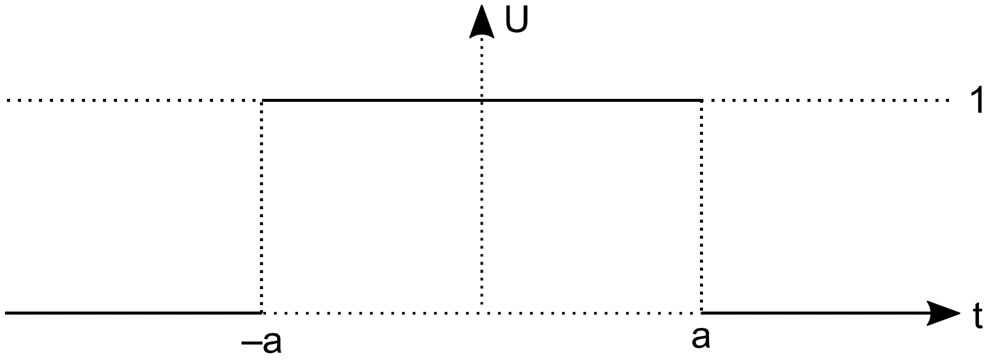


U(t – a), a > 0



Exempel:

U(t + a) – U(t – a)



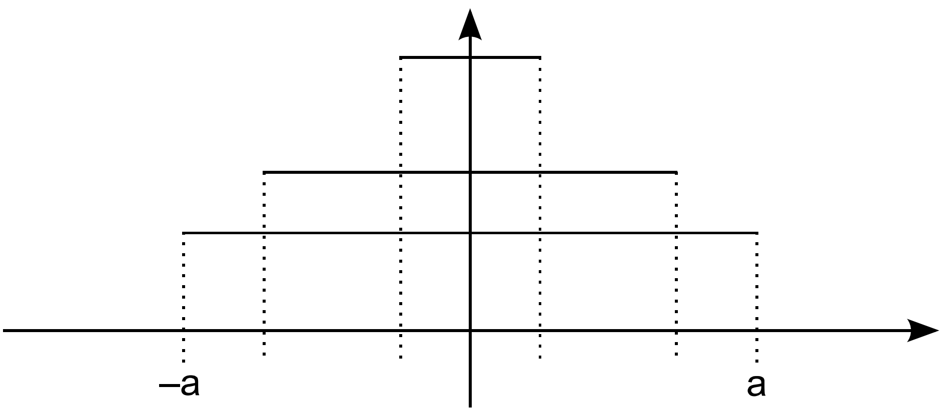
Exempel: |t| med Heavisides funktion:

$$
\begin{aligned}
|t| &= (2\,U(t) - 1)t = \\
&= 2t\,U(t) - t = \\
&= (U(t) - U(-t))t = \\
&= t\,U(t) - t\,U(-t) = \\
&= \mathbf{t\,U(t) + (-t)\,U(-t)}
\end{aligned}
$$

Dirac pulser:

Betrakta gränsvärdet

$$\lim_{a \to 0^+} \frac{1}{2a}\left(U(t+a) - U(t-a)\right) \coloneqq \delta(t)$$



I vanlig mening konvergerar det inte.

Men

$$\int_{-\infty}^{\infty} \frac{1}{2a}\left(U(t+a) - U(t-a)\right) dt = 1 \quad \text{för alla a.}$$

För en glatt funktion ƒ:

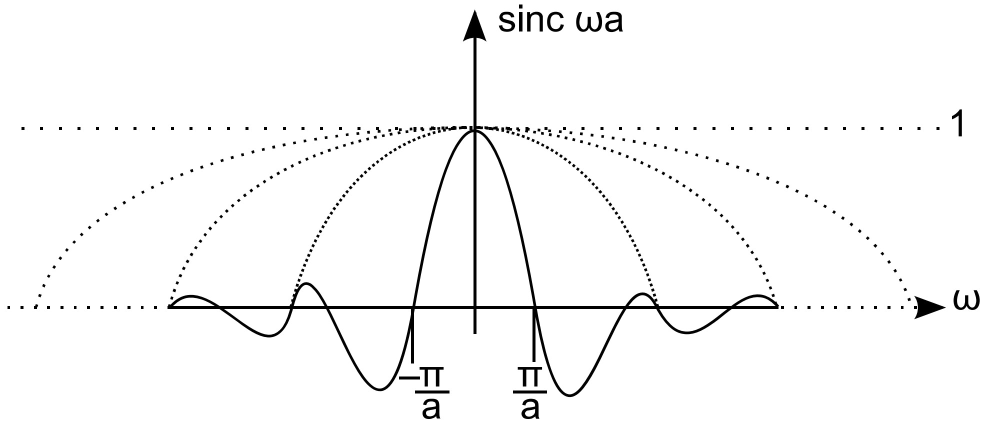
$$\int_{-\infty}^{\infty} f(t) \cdot \frac{1}{2a}\left(U(t+a) - U(t-a)\right) dt \underset{a \to 0^+}{\simeq} f(0) \cdot \frac{1}{2a} \cdot 2a = f(0)$$

Definiera δ(t) som en sådan funtion att

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

För varje oändligt deriverbar $f(t)$.

Vi såg att $\mathcal{F}\left(\frac{1}{2a}\left(U(t+a)-U(t-a)\right)\right)=\frac{2\sin\omega a}{2a\omega}=\underbrace{\frac{\sin\omega a}{\omega a}}_{\text{sinc }\omega a}$



Breddar ut sig med a.

$\operatorname{sinc}\omega a \simeq 1,\ \ a \to \infty$ (sinc används inte av läraren)

Faktiskt:

$$\mathcal{F}(\delta(t))(\omega)=1$$

$$\because \qquad \mathcal{F}(\delta(t))(\omega)=\int_{-\infty}^{\infty}\delta(t)\cdot e^{i\omega t}\,dt=e^{i\omega\cdot 0}=e^{0}=1$$