M4: Skalär ODE:

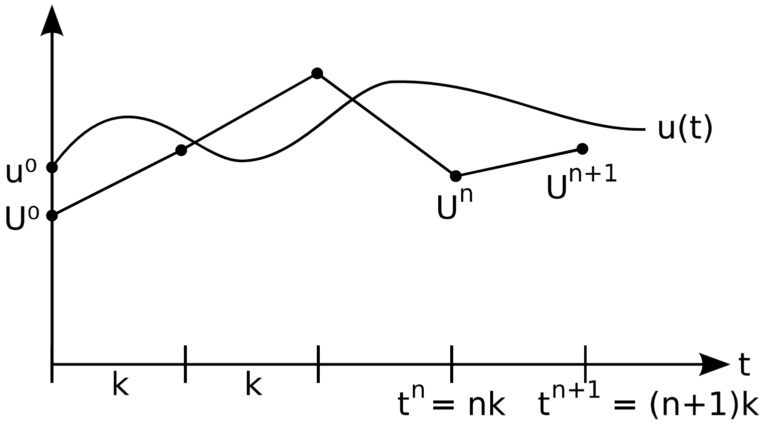
$$\begin{cases} \dot{u}(t)+Au(t)=F(t), \quad t>0 \\ u(0)=u^0 \end{cases}$$

A konstant, F(t) given funktion

Trapetsmetod:

Hitta en styckvis linjär U(t)

$$U^{n+1}+\frac{k}{2}\left(AU^{n+1}+AU^n\right)=U^n+\int_{I_n} F(t)\, dt, \quad n=1, 2, \ldots$$



$U^0 \approx u^0$

Inte en interpolation, men ett försöl att approximera den.

Residual:

$$R(U)=\dot{U}+AU-F \qquad (R(u) = 0)$$

Dualproblem:

$$\begin{cases} -\varphi+A\varphi=0, \quad T>t\geq 0 \\ \varphi(T)=\operatorname{sgn}(e(t)) \end{cases}$$

Fel: $e = u - U$

Kommentar:

$\langle Av;\, w\rangle=\ldots=\langle v;\, A^*\, w\rangle$ där * är en dualoperator

$A=\dot{v}-\Delta u$

$$\langle \dot{v}+Av;\, w\rangle=\int v(-\dot{w}+Aw)\, dx\, dt=\langle v\,;-\dot{w}+Aw\rangle$$

Felrepresentation:

$$|e(T)| = -\int_0^T R(U)\varphi\, dt + e(0)\varphi(0) \qquad (1)$$

Låt $\bar{\varphi}$, medelvärdet över I_n: $\quad \bar{\varphi}(t) = \frac{1}{k}\int_{I_n} \varphi(s)\, ds$

($\bar{\varphi}$ konstant över I_n)

Vi har att $\int_{I_n} R(U)\, dt = 0$

$$\Downarrow$$

$$\int_{I_n} R(U)\bar{\varphi}\, dt = 0 \ \Rightarrow \ \int_0^T R(U)\bar{\varphi}\, dt = 0 \qquad (2)$$

(1) − (2):

$$|e(T)| \ = \ -\int_0^T R(U)(\varphi - \bar{\varphi})\, dt \ + \ e(0)\varphi(0)$$

$$\Downarrow$$

$$|e(T)| \ \leq \ \sum_{n=0}^{N-1} R_n \int_{I_n} |\varphi - \bar{\varphi}|\, dt \ + \ |e(0)| \cdot |\varphi(0)|$$

$$R_n(U) \ \triangleq \ \max_{t \in I_n} |R(U(t))|$$

A posterion-feluppskattning:

$$|u(T) - U(T)| \ \leq \ S_c(t) \max_n kR_n(U) \ + \ S_d(t)\, |u(0) - U(0)|$$

$$\begin{cases} S_d(T) = |\varphi(0)| \\ S_c(T) = \int_0^T |\dot{\varphi}|\, dt \end{cases}$$

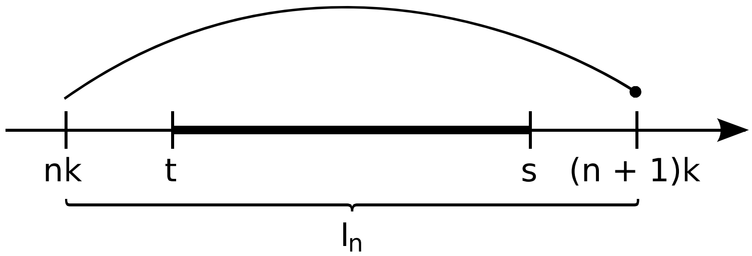
Bevis:

$$\int_{I_n} |\varphi-\bar{\varphi}|\, dt \;\le\; k\int_{I_n} |\dot{\varphi}|\, dt \qquad (3)$$

$$\int_{I_n} |\varphi-\bar{\varphi}|\, dt \;\; ?$$

$$\varphi(t)-\bar{\varphi}\, t \;=\; \frac{1}{k}\int_{I_n} \big(\varphi(t)-\varphi(s)\big)\, ds \qquad (*)$$

$$|\varphi(t)-\varphi(s)| \;=\; \left|\int_s^t \dot{\varphi}(\sigma)\, d\sigma\right| \;\ge\; \int_s^t |\dot{\varphi}(\sigma)\, d\sigma| \qquad (**)$$



nk ≤ t < s ≤ (n+1)k

$$\int_{I_n} |\varphi-\bar{\varphi}|\, dt \underset{(*)}{=} \int_{I_n} \left|\frac{1}{k}\int_{I_n} \big(\varphi(t)-\varphi(s)\big)\, ds\right| dt \;\le$$

$$\le\; \int_{I_n} \frac{1}{k}\int_{I_n} |\varphi(t)-\varphi(s)|\, ds\, dt \underset{(**)}{\le} \int_{I_n} \frac{1}{k}\int_{I_n} \int_{I_n} |\dot{\varphi}(\sigma)|\, d\sigma\, ds\, dt \;=$$

$$=\; k\int_{I_n} |\dot{\varphi}(\sigma)|\, d\sigma \qquad \blacksquare$$

Dual:

$$\begin{cases} -\dot{\varphi}+A\varphi=0 \\ \varphi(T)=\operatorname{sgn}\big(e(t)\big)=\pm 1 \end{cases}$$

Exakt lösning:

$$\varphi(t)=\pm\exp\big(-A(T-t)\big)$$

(Primal):

$$(P)\begin{cases}\dot{u}+Au=F(t)\\ u(0)+u^0\end{cases}$$

Stabilitet hos (P) beror på A.

1) A < 0: $S_d(T)\leq\exp\left(|A|\,T\right)$,

$S_c(T)\leq\exp(|A|\,T)$

2) A ≥ 0: $S_d(T)\leq 1,\ \ S_c(T)\leq 1$

S_c — beräkningsdel
S_d — datafel

Generalisering

1.

$$\begin{cases}\dot{\vec{u}}(t)+\mathbf{A}\vec{u}(t)=\vec{F}(t), \quad t>0\\ \vec{u}(0)=\vec{u}^0\end{cases}, \qquad \vec{u}(t)=\begin{pmatrix}u_1(t)\\ u_2(t)\\ \vdots\\ u_m(t)\end{pmatrix}, \qquad \vec{F}(t)=\begin{pmatrix}F_1(t)\\ F_2(t)\\ \vdots\\ F_m(t)\end{pmatrix}$$

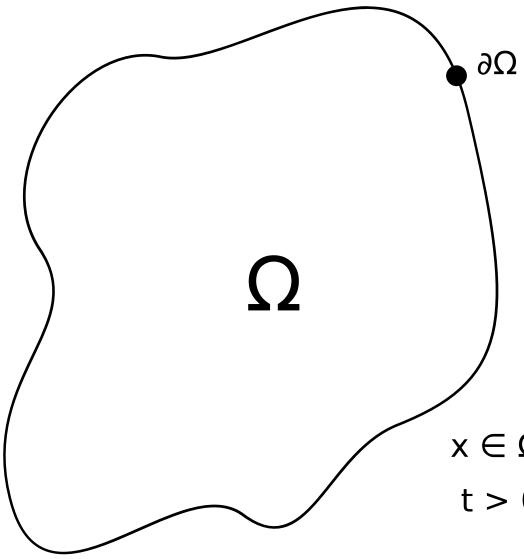
A — m×m-matris

(Modul 4 och 5)

2. Partiella differentialekationer (PDE):

Hitta $\vec{u}(x;t)$:

$$\begin{cases}\dot{\vec{u}}(x;t)+\mathbf{A}\vec{u}(x;t)=\vec{F}(x;t)\\ \vec{u}(x;0)=\vec{u}^0(x)\\ \text{randvillkor}\end{cases}$$



$x\in\Omega$

$t>0$

(Modul 6)

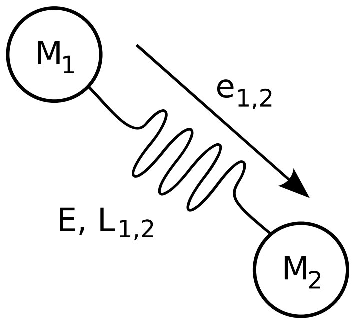
$$\mathbf{A}\vec{u}=\Delta\vec{u}$$

$$\mathbf{A}\vec{u}=\beta\cdot\nabla\vec{u}-\varepsilon\Delta\vec{u}$$

Dual problem:

$$\begin{cases} \text{ODE-system:} & -\dot{\varphi} + A^T \cdot \varphi = 0 \\ \text{PDE:} & -\dot{\varphi} + A^* \cdot \varphi = 0 \end{cases}$$

Exempel: Mass-fjäder-system:



Koordinater: x^1 (M_1)
x^2 (M_2)

Hastigheter: v^1 (M_1)
v^2 (M_2)

Kraft på M_1 från M_2:

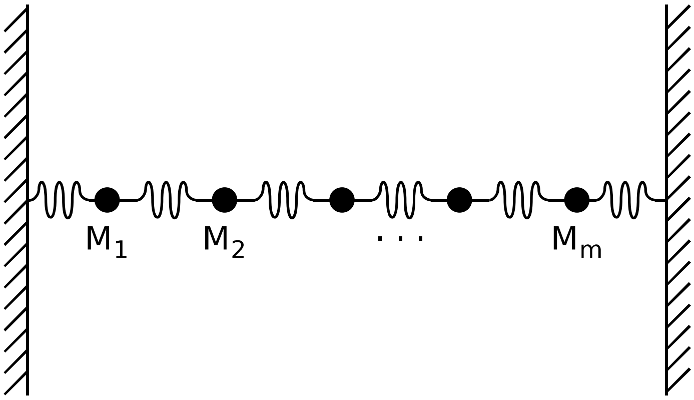
$$F_{1,2} = \begin{Bmatrix} r_{1,2} = |x^1 - x^2| \\ e_{1,2} = \dfrac{x^2 - x^1}{r_{1,2}} \end{Bmatrix} = E(r_{1,2} - L_{1,2}) e_{1,2}$$

M_2 från M_1:

$$F_{2,1} = -F_{1,2} \qquad \text{(Newtons 3:e lag)}$$

Rörelseekvationer:

$$\left\{ \begin{array}{c|c} \dot{x}^1 = v^1 & \dot{x}^2 = v^2 \\ \dot{v}^1 = \dfrac{F_{1,2}}{M_1} & \dot{v}^2 = \dfrac{F_{2,1}}{M_2} \end{array} \right.$$



På varje massa M_i, $i \neq 1, m$ agerar 2 fjäderkrafter:

$$\begin{cases} F_{i,i-1} = E\left(r_{i,i-1} - L_{i,i-1}\right) e_{i,i-1} \\ F_{i,i+1} = E\left(r_{i,i+1} - L_{i,i+1}\right) e_{i,i+1} \end{cases}$$

Rörelseekvation för M_i:

$$\dot{x}^i = v^i$$

$$\dot{v}^i = \frac{\overbrace{E\left(r_{i,i-1} - L_{i,i-1}\right) e_{i,i-1}}^{F_{i,i-1}}}{M_i} + \frac{\overbrace{E\left(r_{i,i+1} - L_{i,i+1}\right) e_{i,i+1}}^{F_{i,i+1}}}{M_i}$$

Antag $E = 1$, $M_i = 1$, $L_{i,j} = 0$

$\Downarrow$

$$\dot{v}^i = \frac{1 \cdot r_{i,i-1}\, e_{i,i-1}}{1} + \frac{1 \cdot r_{i,i+1}\, e_{i,i+1}}{1}$$

$$e_{i,j} = \frac{x_j - x_i}{r_{i,j}} \quad \Rightarrow \quad \dot{v}^i = \left(x^{i-1} - x^i\right) + \left(x^{i+1} - x^i\right) = x^{x-1} - 2x^i + x^{x+1}$$

$$\dot{\vec{u}} = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \\ \hline v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \\ \hline -x^1 + x^2 \\ x^1 - 2x^2 + x^3 \\ \vdots \\ x^{m-1} - x^n \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0} & \begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & 1 & & & 0 \\ & -12 & -1 & & \\ & & -12 & -1 & \\ 0 & & & & \ddots \end{matrix} & \mathbf{0} \end{array}\right] \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^m \\ \hline v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix}$$

(Detta är skumt, men vi kommer gå i på det senare.)