Homework 2

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Exercise 1

$$a_1 = 2, a_2 = 1$$
$$a_1 < a_2$$
$$\text{if } a_{n+1} < a_n, \text{then we have}$$
$$a_{n+2} = \frac{1}{3 - a_{n+1}} < \frac{1}{3 - a_n} = a_{n+1}$$
$$\text{So } \{a_n\} \text{ is decreasing.}$$
$$\lim_{a \to \infty} \frac{1}{3 - a} = 0$$

Exercise 2

a.

$$\sum_{n=3}^{\infty} \frac{(-9)^n}{11^n} = \sum_{n=3}^{\infty} \left(\frac{-9}{11}\right)^n$$

$$S = \frac{a}{1-r},\ a = \left(\frac{-9}{11}\right)^3,\ r = \frac{-9}{11}$$

$$S = \frac{\frac{-729}{1331}}{1 - \frac{-9}{11}} = -\frac{729}{2420}$$

b.

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n} + \frac{2}{6^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n + \sum_{n=0}^{\infty} \frac{2}{6^n}$$

$$\text{First part of formula:} \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$$

$$S = \frac{a}{1-r},\ a = 1,\ r = \frac{2}{7}$$

$$\frac{1}{1 - \frac{2}{7}} = \frac{\frac{1}{5}}{7} = \frac{7}{5} = 1\frac{3}{5}$$

$$\text{Second part of formula:} \sum_{n=0}^{\infty} \frac{2}{6^n}$$

$$\frac{2}{6^n} = 2 \cdot \frac{1}{6^n} = 2\left(\frac{1}{6}\right)^n$$

$$S = \frac{a}{1-r},\ a = 2,\ r = \frac{1}{6}$$

$$\frac{2}{1 - \frac{1}{6}} = \frac{\frac{2}{5}}{6} = \frac{12}{5} = 2\frac{2}{5}$$

$$\text{Sum:}$$

$$1\frac{2}{5} + 2\frac{2}{5} = \frac{19}{5} = 3\frac{4}{5}$$

Exercise 3

First we need to rewrite the formula

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

If $n = 0, A = \frac{1}{3}$ and $B = -\frac{1}{3}$

$$\frac{1}{n(n+3)} = \frac{1}{3n} - \frac{1}{3(n+3)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{3n} - \sum_{n=1}^{\infty} \frac{1}{3(n+3)}$$

If we write out this sum, it looks like this:

$$\frac{1}{3}\left\{\frac{1}{1} - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + ... + \frac{1}{n} - \frac{1}{n+3}\right\}$$

You can see that from $\frac{1}{4}$ every term gets cancelled out.

$$= \frac{1}{3}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}$$

Exercise 4

a.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^{n+1}}{7^{n+1}(n+2)} \Big/ \frac{2^n}{7^n(n+1)} &= \lim_{n\to\infty} \frac{2^{n+1}}{7^{n+1}(n+2)} \cdot \frac{7^n(n+1)}{2^n} \\
&= \lim_{n\to\infty} \frac{2^{n+1}7^n(n+1)}{2^n7^{n+1}(n+2)} \\
&= \lim_{n\to\infty} \frac{2(n+1)}{7(n+2)} \\
&= \frac{2}{7} \lim_{n\to\infty} \frac{n+1}{n+2} \\
&= \frac{2}{7} \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \\
&= \frac{2}{7} \cdot \frac{1}{1} \\
&= \frac{2}{7}
\end{aligned}$$

$\frac{2}{7}$ is between 0 and 1, so this sum converges b.

$$\begin{aligned}
\lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \bigg/ \frac{5^n}{n!} &= \lim_{n\to\infty} \frac{5^{n+1}n!}{5^n(n+1)!} \\
&= \lim_{n\to\infty} \frac{5n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{5}{n+1} = 0
\end{aligned}$$

this sum converges

c.

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)2^{n+1}} \bigg/ \frac{3^n}{n2^n} &= \lim_{n\to\infty} \frac{3^{n+1}n2^n}{3^n(n+1)2^{n+1}} \\
&= \lim_{n\to\infty} \frac{3n2^n}{(n+1)2^{n+1}} \\
&= \lim_{n\to\infty} \frac{3n}{2(n+1)} \\
&= \frac{3}{2} \lim_{n\to\infty} \frac{n}{n+1} \\
&= \frac{3}{2} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\
&= \frac{3}{2} \cdot \frac{1}{1} \\
&= 1\frac{1}{2}
\end{aligned}$$

$1\frac{1}{2} > 1,$ so this sum diverges to infinity

d.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)^3+1}{(n+1)!} \bigg/ \frac{n^3+1}{n!} &= \lim_{n\to\infty} \frac{\left((n+1)^3+1\right)n!}{(n^3+1)(n+1)!} \\
&= \lim_{n\to\infty} \frac{n^3+3n^2+3n+2}{(n+1)(n^3+1)} \\
&= \lim_{n\to\infty} \frac{n^3+3n^2+3n+2}{n^4+n^3+n+1} \\
&= \lim_{n\to\infty} \frac{\frac{n^3}{n^4}+3\frac{n^2}{n^4}+3\frac{n}{n^4}+\frac{2}{n^4}}{1+\frac{n^3}{n^4}+\frac{n}{n^4}+\frac{1}{n^4}} \\
&= \frac{0+0+0+0}{1+0+0+0} \\
&= 0
\end{aligned}$$

$0 < 1$, so this sum converges
