Homework 1

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Exercise 1

a.

$$\begin{aligned}
\frac{1+i}{2-i} \cdot \frac{3-i}{1+i} &= \frac{(1+i)(2+i)}{(2-i)(2+i)} \cdot \frac{(3-i)(1-i)}{(1+i)(1-i)} \\
&= \frac{(2-1)+(1+2)i}{2^2+1} \cdot \frac{(3-1)+(-3-1)i}{2} \\
&= \frac{1+3i}{5} \cdot \frac{2-4i}{2} \\
&= \frac{1+3i}{5} \cdot (1-2i) \\
&= \frac{(1+3i)(1-2i)}{5} \\
&= \frac{(1+6)+(-2+3)i}{5} \\
&= \frac{7+i}{5} = \frac{7}{5} + \frac{1}{5}i
\end{aligned}$$

b.

$$\begin{aligned}
(1+i)^3 + (1-i)^2 &= (1+i)(1+i)(1+i) + (1-i)(1-i) \\
&= (2i)(1+i) - 2i \\
&= -2 + 2i - 2i \\
&= -2
\end{aligned}$$

Exercise 2

$$\begin{aligned}
&z = 3+4i,\ w = 5-12i \\
&z^2 w = (3+4i)(3+4i)(5-12i) \\
&\quad = ((9-16)+(12+12)i)(5-12i) \\
&\quad = (-7+24i)(5-12i) \\
&\quad = (-7\cdot 5) - (24 \cdot -12) + ((-7 \cdot -12) + (24 \cdot 5))i \\
&\quad = -35 + 288 + (84+120)i \\
&\quad = 253 + 204i
\end{aligned}$$

Exercise 3

$$z^2 - \sqrt{3}z + 1 = 0 \text{ to the form } r(\cos(\phi) + i\sin(\phi))$$
$$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$$
$$\phi = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$
$$r(\cos(\phi) + i\sin(\phi)) = 2\left(\cos\left(-\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{3}\right)\right)$$

Exercise 4

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1$ gives $x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$

$z = \dfrac{1+i}{2}$

From this we can say that:

$-b = 1, \sqrt{b^2 - 4c)} = i$

From here, $b = -1$

$\sqrt{b^2 - 4c)} = i$ becomes $\sqrt{(-1)^2 - 4c)} = \sqrt{-1}$

$1 - 4c = -1$

$-4c = -2$

$c = \dfrac{1}{2}, b = -1$

If we plug b and c in our original formula we get:

$x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot \frac{1}{2}}}{2} = \dfrac{1 \pm i}{2}$

The second root is: $\dfrac{1-i}{2}$

Exercise 5

a.

To dertermine if the sequence $a^n = \frac{n}{2^n}, n \geq 1$ is monotonic, we can determine if a^{n+1} is always $\geq$ or $\leq$ then a^n. The sequence a^n looks like this:

$$a^n = \left\{\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{4}{16}, \ldots\right\}$$

You can see that the sequence keeps getting smaller. From this we can conclude that a^n is monotonic, because $a^n \geq a^{n+1}$

limit:

$$\lim_{n\to\infty} \frac{n}{2^n} = \lim_{n\to\infty} \frac{n \,/\, 2^n}{2^n \,/\, 2^n} = \lim_{n\to\infty} \frac{n \,/\, 2^n}{1} = \frac{0}{1} = 0$$

b.

To dertermine if the sequence $b^n = \sqrt{n^2 + n} - n, n \geq 1$ is monotonic, we can determine if b^{n+1} is always $\geq$ or $\leq$ then b^n. The sequence b^n looks like this:

$$a^n = \left\{\sqrt{2} - 1, \sqrt{5} - 2, \sqrt{12} - 3, \ldots\right\}$$

You can see that the sequence keeps getting bigger. From this we can conclude that a^n is monotonic, because $a^n \leq a^{n+1}$

limit:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \frac{\sqrt{n^2+n} - n}{1} \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \\
&= \frac{1}{\sqrt{1+0} + 1} \\
&= \frac{1}{2}
\end{aligned}$$