

# Matrix Theory Assignment 1

AI24BTECH11017-Maanya Sri

- 16) Using mathematical induction, prove that  $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \dots + \tan^{-1}\left\{\frac{1}{(n^2+n+1)}\right\} = \tan^{-1}\left\{\frac{n}{(n+2)}\right\}$  (1993-5 Marks)
- 17) Prove that  $\sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2n-1} = 0$ , where  $k = \frac{(3n)}{2}$  and  $n$  is an even positive integer. (1993-5 Marks)
- 18) If  $x$  is not an integral multiple of  $2\pi$  use mathematical induction to prove that :  $\cos x + \cos 2x + \dots + \cos nx = \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$  (1994-4 Marks)
- 19) Let  $n$  be a positive integer and  $(1+x+x^2)^n = a_0 + a_1x + \dots + a_{2n}x^{2n}$ . Show that  $a_0^2 - a_1^2 + a_2^2 - \dots + a_{2n}^2 = a_n$  (1994-5 Marks)
- 20) Using mathematical induction prove that for every integer  $n \geq 1$ ,  $(3^{2n} - 1)$  is divisible by  $2^{n+2}$  but not by  $2^{n+3}$ . (1996-3 Marks)
- 21) Let  $0 < A_i < \pi$  for  $i = 1, 2, \dots, n$ . Use mathematical induction to prove that  $\sin A_1 + \sin A_2 + \dots + \sin A_n \leq n \sin\left(\frac{A_1 + A_2 + \dots + A_n}{n}\right)$  where  $\geq 1$  is a natural number. {You may use the fact that  $p \sin x + (1-p) \sin y \leq \sin [px + (1-p)y]$ , where  $0 \leq p \leq 1$  and  $0 \leq x, y \leq \pi$ .} (1997-5 Marks)
- 22) Let  $p$  be a prime number and  $m$  a positive integer. By mathematical induction on  $m$ , or otherwise, prove that whenever  $r$  is an integer such that  $p$  does not divide  $r$ ,  $p$  divides  ${}^mP_r$  [Hint: You may use the fact that  $(1+x)^{(m+1)p} = (1+x)^p (1+x)^{mp}$ ] (1998-8 Marks)
- 23) Let  $n$  be any positive integer. Prove that  $\sum_{k=0}^m \frac{\binom{2n-k}{k}}{\binom{2n-1}{n}} \cdot \frac{2n-4k+1}{2n-2k+1} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m}$  for each non-negative integer  $m \leq n$ . (Here  $\binom{p}{q} = {}^pC_q$ ). (1999-10 Marks)
- 24) For any positive integer  $m, n$  (with  $n \geq m$ ), let  $\binom{n}{m} = {}^nC_m$ . Prove that  $\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m} = \binom{n+1}{m+2}$ . Hence or otherwise, prove that
- $$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \dots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2} \quad (24.1)$$
- (2000-6 Marks)
- 25) For every positive integer  $n$ , prove that  $\sqrt{4n+1} < \sqrt{n} + \sqrt{n+1} < \sqrt{4n+2}$ . Hence or otherwise, prove that  $\lceil \sqrt{n} + \sqrt{n+1} \rceil = \lceil \sqrt{4n+1} \rceil$ , where  $\lceil x \rceil$  denotes the greatest integer not exceeding  $x$ . (2000-6 Marks)
- 26) Let  $a, b, c$  be positive real numbers such that  $b^2 - 4ac > 0$  and let  $\alpha_1 = c$ . Prove by induction that  $\alpha_{n+1} = \frac{aa_n^2}{(b^2 - 2a(\alpha_1 + \alpha_2 + \dots + \alpha_n))}$  is well-defined and  $\alpha_{n+1} < \frac{\alpha_n}{2}$  for all  $n = 1, 2, \dots$  (Here, 'well-defined' means that the denominator in the expression for  $\alpha_{n+1}$  is not zero.) (2001-5 Marks)
- 27) Use mathematical induction to show that  $(25)^{n+1} - 24n + 5735$  is divisible by  $(24)^2$  for all  $n = 1, 2, \dots$  (2002-5 Marks)

28) Prove that

$$2^k \binom{n}{0} \binom{n}{k} - 2^{k-1} \binom{n}{1} \binom{n-1}{k-1} + 2^{k-2} \binom{n-2}{k-2} - \dots (-1)^k \binom{n}{k} \binom{n-k}{0} = \binom{n}{k} \quad (28.1)$$

(2003-2 Marks)

- 29) A coin has probability  $p$  of showing head when tossed. It is tossed  $n$  times. Let  $p_n$  denote the probability that no two (or more) consecutive heads occur. Prove that  $p_1 = 1, p_2 = 1 - p^2$  and  $p_n = (1 - p) \cdot p_{n-1} + p(1 - p)p_{n-2}$  for all  $n \geq 3$ . Prove by induction on  $n$ , that  $p_n = A\alpha^n + B\beta^n$  for all  $n \geq 1$ , where  $\alpha$  and  $\beta$  are the roots of quadratic equation  $x^2 - (1 - p)x - p(1 - p) = 0$  and  $A = \frac{p^2 + \beta - 1}{\alpha\beta - \alpha^2}$ ,  $B = \frac{p^2 + \alpha - 1}{\alpha\beta - \beta^2}$ . (2000-5 Marks)