ST-2023-14 to 26

AI24BTECH11017 - MAANYA SRI

1) Consider the probability space $(\Omega, \mathcal{G}, P)$, where $\Omega = [0, 2]$ and $\mathcal{G} = \{\emptyset, \Omega, [0, 1], (1, 2]\}$. Let X and Y be two functions on Ω defined as

$$X(\omega) = \begin{cases} 1 & if \omega \in [0, 1] \\ 2 & if \omega \in (1, 2] \end{cases}$$

and

$$Y(\omega) = \begin{cases} 2 & if \omega \in [0, 1.5] \\ 3 & if \omega \in (1.5, 2]. \end{cases}$$

Then which one of the following statements is true?
   a) X is a random variable with respect to $\mathcal{G}$, but Y is not a random variable with respect to $\mathcal{G}$
   b) Y is a random variable with respect to $\mathcal{G}$, but X is not a random variable with respect to $\mathcal{G}$
   c) Neither X nor Y is a random variable with respect to $\mathcal{G}$
   d) Both X and Y are random variables with respect to $\mathcal{G}$
2) Let $\Phi(\cdot)$ denote the cumulative distribution function of a standard normal random variable. If the random variable X has the cumulative distribution function

$$F(x) = \begin{cases} \Phi(x) & if x < -1 \\ \Phi(x+1) & if x \geq -1, \end{cases}$$

then which one of the following statements is true?
   a) $P(X \leq -1) = \frac{1}{2}$
   b) $P(X = -1) = \frac{1}{2}$
   c) $P(X < -1) = \frac{1}{2}$
   d) $P(X \leq 0) = \frac{1}{2}$
3) Let X be a random variable with probability density function

$$f(x) = \begin{cases} \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}} & if x > 0 \\ 0 & otherwise, \end{cases}$$

where $\alpha > 0$ and $\lambda > 0$. If the median of X is 1 and the third quantile is 2, then (α, λ) equals
   a) $(1, \log_e 2)$
   b) $(1, 1)$
   c) $(2, \log_e 2)$
   d) $(1, \log_e 3)$

4) Let X be a random variable having Poisson distribution with mean $\lambda > 0$. Then $E\left(\frac{1}{X+1} \middle| X > 0\right)$ equals

a) $\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda\left(1-e^{-\lambda}\right)}$
b) $\frac{1-e^{-\lambda}}{\lambda}$
c) $\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda}$
d) $\frac{1-e^{-\lambda}}{\lambda+1}$

5) Suppose that X has the probability density function

$$f(x) = \begin{cases} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & if x > 0 \\ 0 & otherwise, \end{cases}$$

where $\alpha > 0$ and $\lambda > 0$. Which one of the following statements is NOT true?

a) $E(X)$ exists for all $\alpha > 0$ and $\lambda > 0$
b) Variance of X exists for all $\alpha > 0$ and $\lambda > 0$
c) $E\left(\frac{1}{X}\right)$ exists for all $\alpha > 0$ and $\lambda > 0$
d) $E\left(\log_e(1 + X)\right)$ exists for all $\alpha > 0$ and $\lambda > 0$

6) Let (X, Y) have joint probability density function

$$f(x, y) = \begin{cases} 8xy & if 0 < x < y < 1 \\ 0 & otherwise. \end{cases}$$

If $E(X|Y = y_0) = \frac{1}{2}$, then y_0 equals

a) $\frac{3}{4}$
b) $\frac{1}{2}$
c) $\frac{1}{3}$
d) $\frac{2}{3}$

7) Suppose that there are 5 boxes, each containing 3 blue pens, 1 red pen and 2 black pens. One pen is drawn at random from each of these 5 boxes. If the random variable X_1 denotes the total number of blue pens drawn and the random variable X_2 denotes the total number of red pens drawn, then $P(X_1 = 2, X_2 = 1)$ equals

a) $\frac{5}{36}$
b) $\frac{5}{18}$
c) $\frac{5}{12}$
d) $\frac{5}{9}$

8) Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be two sequences of random variables and X and Y be two random variables, all of them defined on the same probability space. Which one of the following statements is true?

a) If $\{X_n\}_{n\geq 1}$ converges in distribution to a real constant c, then $\{X_n\}_{n\geq 1}$ converges in probability to c
b) If $\{X_n\}_{n\geq 1}$ converges in probability to X, then $\{X_n\}_{n\geq 1}$ converges in 3^{rd} mean to X
c) If $\{X_n\}_{n\geq 1}$ converges in distribution to X and $\{Y_n\}_{n\geq 1}$ converges in distribution to Y, then $\{X_n + Y_n\}_{n\geq 1}$ converges in distribution to $X + Y$
d) If $\{E(X_n)\}_{n\geq 1}$ converges to $E(X)$, then $\{X_n\}_{n\geq 1}$ converges in 1^{st} mean to X

9) Let X be a random variable with probability density function

$$f(x;\lambda) = \begin{cases} \frac{1}{\lambda}e^{-\frac{x}{\lambda}} & if x > 0 \\ 0 & otherwise, \end{cases}$$

where $\lambda > 0$ is an unknown parameter. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from a population having the same distribution as X^2.
If $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, then which one of the following statements is true?

a) $\sqrt{\frac{\bar{Y}}{2}}$ is a method of moments estimator of λ
b) $\sqrt{\bar{Y}}$ is a method of moments estimator of λ
c) $\frac{1}{2}\sqrt{\bar{Y}}$ is a method of moments estimator of λ
d) $2\sqrt{\bar{Y}}$ is a method of moments estimator of λ

10) Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n (\geq 2)$ from a population having probability density function

$$f(x;\theta) = \begin{cases} \frac{2}{\theta x}(-log_e x)\, e^{-\frac{(log_e x)^2}{\theta}} & if 0 < x < 1 \\ 0 & otherwise, \end{cases}$$

where $\theta > 0$ is an unknown parameter. Then which one of the following statements is true?

a) $\frac{1}{n}\sum_{i=1}^{n}(log_e X_i)^2$ is the maximum likelihood estimator of θ
b) $\frac{1}{n-1}\sum_{i=1}^{n}(log_e X_i)^2$ is the maximum likelihood estimator of θ
c) $\frac{1}{n}\sum_{i=1}^{n} log_e X_i$ is the maximum likelihood estimator of θ
d) $\frac{1}{n-1}\sum_{i=1}^{n} log_e X_i$ is the maximum likelihood estimator of θ

11) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population having uniform distribution over the interval $\left(\frac{1}{3}, \theta\right)$,where $\theta > \frac{1}{3}$ is an unknown parameter. If $Y = max\{X_1, X_2, ..., X_n\}$, then which one of the following statements is true?

a) $\left(\frac{n+1}{n}\right)\left(Y - \frac{1}{3}\right) + \frac{1}{3}$ is an unbiased estimator of θ
b) $\left(\frac{n}{n+1}\right)\left(Y - \frac{1}{3}\right) + \frac{1}{3}$ is an unbiased estimator of θ
c) $\left(\frac{n+1}{n}\right)\left(Y + \frac{1}{3}\right) - \frac{1}{3}$ is an unbiased estimator of θ
d) Y is an unbiased estimator of θ

12) Suppose that $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random vectors each having $N_p(\mu, \Sigma)$ distribution, where Σ is non-singular, $p > 1$ and $n > 1$. If $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, then which one of the following statements is true?

a) There exists $c > 0$ such that $c(\bar{X}-\mu)^T\Sigma^{-1}(\bar{X}-\mu)$ has χ^2-distribution with p degrees of freedom
b) There exists a $c > 0$ such that $c(\bar{X}-\bar{Y})^T\Sigma^{-1}(\bar{X}-\bar{Y})$ has χ^2-distribution with $(p-1)$ degrees of freedom.
c) There exists $c > 0$ such that $c\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^T\Sigma^{-1}\left(X_i - \bar{X}\right)$ has χ^2-distribution with p degrees of freedom
d) There exists $c > 0$ such that $c\sum_{i=1}^{n}\left(X_i - Y_i - \bar{X} + \bar{Y}\right)^T\Sigma^{-1}\left(X_i - Y_i - \bar{X} + \bar{Y}\right)$ has χ^2-distribution with p degrees of freedom

13) Consider the following regression model

$$y_k = \alpha_0 + \alpha_1 \log_e k + \epsilon_k, \quad k = 1, 2, \ldots, n,$$

where the ϵ_k's are independent and identically distributed random variables each having probability density function $f(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}$. Then which of the following statements is true?

a) The maximum likelihood estimator of α_0 does not exist
b) The maximum likelihood estimator of α_1 does not exist
c) The least squares estimator of α_0 exists and is unique
d) The least squares estimator of α_1 exists, but it is not unique