Projective Transformations as Versors

Leo Dorst

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1 The Space of Lines $\mathbb{R}^{3,3}$

Take the homogeneous model (or the conformal model) V^4 of 3D Euclidean space and form the spatial lines. Using e_0 for the vector representing the point at the origin, and bold for Euclidean vectors, a line through $p = e_0 + \mathbf{p}$ in the direction $\mathbf{u}$ is formed as:

$$L = p \wedge \mathbf{u} = e_0 \wedge \mathbf{u} + \mathbf{p} \wedge \mathbf{u}.$$

These are effectively the Plücker coordinates, on a basis $\{e_0\mathbf{e}_1, e_0\mathbf{e}_2, e_0\mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$. This is a basis for bivectors in the homogeneous model, of dimension $\binom{4}{2} = 6$. We then write them as basis 'vectors' of the bivector space $\bigwedge^2 V^4$

$$\{\ e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\ \}. \tag{1}$$

A bivector A of $\bigwedge^2 V^4$ is a line iff

$$A \wedge A = 0,$$

which is called the Grassmann-Plücker relation. This effectively evaluates to the purely Euclidean relation $\mathbf{u} \wedge (\mathbf{p} \wedge \mathbf{u}) = 0$. In terms of coordinates l_{ij} of L this relation gives:

$$\ell_{01}\ell_{23} + \ell_{02}\ell_{31} + \ell_{03}\ell_{12} = 0 \tag{2}$$

This is called the (Grassmann-)Plücker condition of the line. It only depends on the outer product in V^4.

We now switch to the 6-dimensional space of the Plücker coordinates and call it $\mathbb{R}^{3,3}$ for reasons which will become clear below. We use it in a homogeneous fashion, as a 5D projective space: multiples of vectors represent the same geometry (at least, positionally, if we want to represented weighted elements the homogeneous freedom becomes interpretable). In that space, the lines are represented by all vectors ℓ satisfying eq.(2). The embedding of a line L of $\mathbb{R}^3$ represented as a 2-blade of V^4 in this way leads to a vector of $\mathbb{R}^{3,3}$ which we denote $\ell = \mathrm{Em}(L)$. That 4-D manifold of vectors representing lines is called the *Klein quadric* when considered in the 5D projective space. (This mapping with as range the Klein quadric is γ from [4], pg.141, who denotes the quadric as M_2^4.) When considering general elements on the 6D-basis of bivectors in V^4, we can also consider them as vectors in $\mathbb{R}^{3,3}$; [4] denotes this mapping as $\gamma^\star$, but we will simply overload the use of $\mathrm{Em}(\cdot)$ since the star is too reminiscent of duality.

We are going to turn this 6D space into a metric space, to do geometric algebra. That means we need to define a metric for it, and we let that be inspired by the non-metric outer product nature of V^4, in a clever correspondence (see [2]):

$$\mathrm{Em}(A) \cdot \mathrm{Em}(B) \equiv [A \wedge B], \tag{3}$$

with the inner product defined between elements of the 6D space on the left, defined in terms of a bracket taken in the Grassmann algebra $\bigwedge(V^4)$ on the right. That bracket is effectively the dual with the pseudoscalar $e_0\mathbf{I}_3$ of V^4, but since no metric for V^4 is given it just means: find out what part of $A\,B$ is proportional to the pseudoscalar and return the proportionality factor. It therefore only uses ratios of volumes in V^4, and hence volume ratios in actual 3-space, to induce a metric of lines (thus the metric issue of what to make of e_0^2 is avoided), used as a metric for vectors in $\mathbb{R}^{3,3}$. But for those computations which are independent of a metric, we may just as well consider V^4 as the metric space $\mathbb{R}^4$.

We briefly make the geometry of this dot product explicit in V^4. For two finite lines $L = (e_0+\mathbf{p})\wedge\mathbf{u}$ and $M+(e_0+\mathbf{q})\wedge\mathbf{v}$, the value of the dot product is $[(e_0+\mathbf{p})\wedge\mathbf{u}\wedge(e_0+\mathbf{q})\wedge\mathbf{v}] = [e_0 \wedge \mathbf{u} \wedge (\mathbf{q}-\mathbf{p}) \wedge \mathbf{v}]$, which equals $\det[\mathbf{u}\ (\mathbf{q}-\mathbf{p})\ \mathbf{v}]$ relative to the Euclidean unit volume element. It is the (relative) volume spanned by the direction vector $\mathbf{u}$ of L, a relative position vector $(\mathbf{q}-\mathbf{p})$ from a point $\mathbf{p}$ on L to a point $\mathbf{q}$ on M, and the direction vector $\mathbf{v}$ of M. It thus quantifies a combination of dissimilarity in position and direction: for lines with perpendicular signed distance δ and unit direction vectors making a signed angle ϕ, it equals $\delta\,\sin(\phi)$. For a finite line $L = (e_0+\mathbf{p})\wedge\mathbf{u}$ and an ideal line $\mathbf{v}\wedge\mathbf{w}$, we obtain $\det[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$, the volume spanned by their directions (for unit directions, this is equal to the sine of the angle that $\mathbf{u}$ makes with the $\mathbf{v}\wedge\mathbf{w}$-plane). Two ideal lines in 3D have dot product zero - they always intersect in a common 1-direction.

[2] proves that this metric defined by eq.(3) is nondegenerate: if $a\cdot b = 0$ for all $b \in \bigwedge^2(V^4)$, then $a = 0$. Under this metric, lines are therefore precisely identified with the *null vectors* in the 6D space. Moreover, the 6D space has the metric structure of $\mathbb{R}^{3,3}$.

An orthonormal basis to expose explicitly the $\mathbb{R}^{3,3}$ nature is:

$$\{a_+, b_+, c_+, a_-, b_-, c_-\} = \left\{ \frac{e_{01}+e_{23}}{\sqrt{2}}, \frac{e_{02}+e_{31}}{\sqrt{2}}, \frac{e_{03}+e_{12}}{\sqrt{2}}, \frac{e_{01}-e_{23}}{\sqrt{2}}, \frac{e_{02}-e_{31}}{\sqrt{2}}, \frac{e_{03}-e_{12}}{\sqrt{2}} \right\}.$$

We have $a_+^2 = b_+^2 = c_+^2 = 1$ and $a_-^2 = b_-^2 = c_-^2 = -1$. As a practical insight speeding up hand computations: any repeated index in a product leads to a zero contribution to the inner product.

The inner product multiplication table on the unit basis is:

$\cdot$	a_+	b_+	c_+	a_-	b_-	c_-
a_+	1	0	0	0	0	0
b_+	0	1	0	0	0	0
c_+	0	0	1	0	0	0
a_-	0	0	0	-1	0	0
b_-	0	0	0	0	-1	0
c_-	0	0	0	0	0	-1

Note that these orthonormal basis vectors for $\mathbb{R}^{3,3}$ do not represent lines - they are clearly not null. The 'standard' null basis of eq.(1)

$$\{\ e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\ \}$$

does consist of representatives of lines: three orthogonal lines through the origin, and three orthogonal lines at infinity (orthogonal great circles on the celestial sphere of V^4). Algebraically, the null basis elements may be grouped into two 3D null subspaces. The 3D subspace span$\{e_{01}, e_{02}, e_{03}\}$ is null: any two vectors in it are orthogonal to each other. The same is true for span$\{e_{23}, e_{31}, e_{12}\}$. Actually performing this grouping (or a similar split) means distinguishing between the finite and the infinite (ideal lines in V^4, and therefore requires an affine structure which is not projectively meaningful.

The multiplication table on the null basis is:

$\cdot$	e_{01}	e_{02}	e_{03}	e_{23}	e_{31}	e_{12}
e_{01}	0	0	0	1	0	0
e_{02}	0	0	0	0	1	0
e_{03}	0	0	0	0	0	1
e_{23}	1	0	0	0	0	0
e_{31}	0	1	0	0	0	0
e_{12}	0	0	1	0	0	0

This can be encoded as a metric matrix $[M]$. It is called K in [4], Eq.(2.28), pg 141. Two lines A, B intersect iff $0 = \mathrm{Em}(A) \cdot \mathrm{Em}(B) = [\mathrm{Em}(A)]^T [M][\mathrm{Em}(B)] = 0$, where the latter form implements the dot product of $\mathbb{R}^{3,3}$ in terms of matrices.

[[[**just said all this, choose where**]]] As we have seen above, geometrically interpreted, the inner product of two finite line representatives is proportional to the volume spanned by their direction vectors and separation vector; the inner product of two infinite lines is zero; the inner product of a finite and infinite line is the volume spanned at infinity by their directions (a 1-direction and a 2-direction). With $\mathrm{Em}(\cdot)$ the embedding function of lines and the 3D volume bracket denoted as $[\cdot]_{3D}$:

$$\mathrm{Em}(p \wedge \mathbf{u}) \cdot \mathrm{Em}(q \wedge \mathbf{v}) = [p \wedge \mathbf{u} \wedge q \wedge \mathbf{v}] = [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge (\mathbf{q} - \mathbf{p})] = [\mathbf{u} \wedge (\mathbf{p} - \mathbf{q}) \wedge \mathbf{v}]_{3D}$$

and

$$\mathrm{Em}(p \wedge \mathbf{u}) \cdot \mathrm{Em}(\mathbf{v} \wedge \mathbf{w}) = [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}] = [\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}]_{3D}.$$

The inner product of two intersecting line representatives, of finite or infinite lines, is zero.

In V^4, the dual of a line would be produced by switching moment and direction components (see for instance [1] [[[**example**]]]). Dualization then amounts to the swapping of the coordinates of the representative:

$$\ell_{01} \leftrightarrow \ell_{23}, \quad \ell_{02} \leftrightarrow \ell_{31}, \quad \ell_{03} \leftrightarrow \ell_{12}.$$

This depends on the origin, and is therefore not a geometric operation. [[[**so we should not use it?**]]] (It may be achieved in a metric way in V^4 by considering it as $\mathbb{R}^4$ through taking $e_0^2 = 1$ and $I_4 = e_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, and the defining $A^\star = A/I_4$; which is inelegant, but unambiguous.) This also extends naturally to non-lines on the same basis eq.(1), and hence to the entire $\mathbb{R}^{3,3}$-space. One may verify that $(A^\star)^\star = A$, so that in $\mathbb{R}^4$, undualization is the same as dualization. Note that $[\mathrm{Em}(A^\star)] = -[M][\mathrm{Em}(A)]$.

Having thus used the structure of V^4 to define a basis and a metric for $\mathbb{R}^{3,3}$, we need no longer consider it, but can focus fully on the space $\mathbb{R}^{3,3}$, which contains line representatives on the Klein quadric.

1.1 Correspondence with Pottmann notation

Pottmann & Wallner's *Computational Line Geometry* [4] performs a detailed analysis of the elements of line geometry, in non-GA and non-metric terms, and with its definitions coordinate-based. We will redo this analysis within the coordinate-free GA framework. For precision in the description, we will use a different notation and different terminology, but indicate the correspondence throughout.

- A *correlation* is a projective mapping from a projective space (such as V^4) to its dual. In GA, we would blur the distinction between a space and its dual, endow the space with a metric, and use duality relative to the pseudoscalar. This maps vectors to 3-blades, interpreted as mapping points to hyperplanes. For the space V^4, this duality is somewhat strained since there is no natural choice for e_0^2.

- A *linear line complex* is a set of lines characterized by a linear homogeneous equation in their Plücker coordinates [4] pg.161, i.e. an equation of the form $\ell \cdot c = 0$. We would consider the c that characterizes this complex as the dual of a 5-blade in $\mathbb{R}^{3,3}$, and might prefer to characterize the linear line complex directly by the 5-blade c^{-*} through $x \wedge c^{-*} = 0$.

- If the c of the complex represents a line (so that $c^2 = 0$) the complex is called *singular*, otherwise *regular*. The *axis* of a singular linear complex is the line $\mathrm{Em}^{-1}(c)$ which is met by all lines in the complex.

- [4] appears to characterize the axis in a V^4-dual manner as Mc. The reason behind this is that they have used the 'canonical' dot product for the 6D space in (1.30), not recognizing the necessity for the metric matrix $[M]$ (their K). But their formulation of Lemma 3.1.2 'denote the entries in the matrix C by c_{ij}' is not sufficiently precise to see what is going on.

  Let us make this more precise by translating the projective correlation of (1.30) into GA. A skew-symmetric mapping like the null polarity C can be written as the contraction with a bivector $\mathbf{C}$: $Cx = -x \rfloor \mathbf{C}$ (we introduce a minus to avoid signs later). We rewrite in (1.30), using $\rfloor$ for the inner product in V^4:

$$0 = y \rfloor (Cx) = -y \rfloor (x \rfloor \mathbf{C}) = (x \wedge y) \rfloor \mathbf{C}.$$

  A correlation for points of V^4 is therefore recast in GA into the kernel [[[**huh?**]]] of a mapping of their connecting line. Assuming a Euclidean metric for V^4 to implement this contraction, and setting $\ell = (x \wedge y) = \ell_{01} e_0 \wedge e_1 + \cdots$, this product evaluates as $\ell_{01} C_{01} + \cdots$.

- The correspondence in $Cx = -x \rfloor \mathbf{C}$ in matrix terms, using a sensible labeling for the coefficients of $\mathbf{C}$ as $\mathbf{C} = C_{01} e_0 \wedge e_1 + \cdots$, is on the orthonormal basis $\{e_0, e_1, e_2, e_3\}$:

$$C = \begin{bmatrix} 0 & C_{01} & C_{02} & C_{03} \\ -C_{01} & 0 & C_{12} & -C_{31} \\ -C_{02} & -C_{12} & 0 & C_{23} \\ -C_{03} & C_{31} & -C_{23} & 0 \end{bmatrix}.$$

  Unfortunately, [4] does not give the parametrization of their C by the c_{ij} explicitly, but it can be inferred from their C^{-T} on pg.162:

$$C = \begin{bmatrix} 0 & c_{01} & c_{02} & c_{03} \\ -c_{01} & 0 & c_{12} & -c_{31} \\ -c_{02} & -c_{12} & 0 & c_{23} \\ -c_{03} & c_{31} & -c_{23} & 0 \end{bmatrix}.$$

  Due to the introduction of our minus sign above, the matrices are therefore very similar. However, [4] characterizes this C by the Plücker space vector $c' = (\mathbf{c}, \bar{\mathbf{c}})$ with 'perversion' of the components: $(\mathbf{c}, \bar{\mathbf{c}}) = (c_{23}, c_{31}, c_{12}, c_{01}, c_{02}, c_{03})$ (see (3.1) and Remark 3.1.1.). We see that their 6D vector c' characterizing the matrix C of the linear line complex $\mathcal{C}$ is therefore M times the natural mapping $c = \mathrm{Em}(\mathbf{C})$ of the bivector $\mathbf{C}$ that we would use to characterize the correlation. This does not change the numbers involved in the computation, merely the interface from the characterization of $\mathcal{C}$ by c' or $\mathbf{C}$ to the actual setting up of the matrix C: they have to make $c = Mc'$ and plug in the c-numbers, we have to read off $c = \mathrm{Em}(\mathbf{C})$.

- The reason for this perversion operation is that [4] uses the canonical 'Euclidean' dot product of the 6D space, whereas we can use the natural inner product of $\mathbb{R}^{3,3}$. Denoting the former by a transpose and matrix product, we have:

  $$y^T C x = \mathrm{Em}(x \wedge y)^T c' = \mathrm{Em}(x \wedge y)^T (Mc) = \mathrm{Em}(x \wedge y) \cdot c = \mathrm{Em}(x \wedge y) \cdot \mathrm{Em}(\mathbf{C}).$$

  (We see this for instance in the proof of Lemma 3.1.2.) Thus the use of the Euclidean dot product in 6D forces one to the perverse characterization c' to obtain the same equation, and hence the same geometry, as the $\cdot$ would give for c. (At the top of pg.161, they show that K (our M) could have been incorporated; if only they had.)

- In [4], all mappings are denoted in bracketless postfix notation, but we use prefix with brackets. They use γ for the mapping of actual lines of V^4, which we denote by Em( ). Their $\gamma^\star$ maps a linear complex $\mathcal{C}$ to the characterizing vector which we denote Mc (actually, $Mc\mathbb{R}$ since it is a homogeneous representation). We thus have

  $$M(\mathcal{C}\gamma^\star) = \mathrm{Em}(\mathbf{C}) = c.$$

- But that equation $M(\mathcal{C}\gamma^\star) = \mathrm{Em}(\mathbf{C}) = c$ appears to be in contradiction with the first part of Prop.3.1.10: 'If a linear complex $\mathcal{C}$ is singular with axis A, then $\mathcal{C}\gamma^\star = A\gamma$.' Looking at the actual proof which refers to the proof of Theorem 3.1.3, which refers to (2.26), there is indeed an M missing. For (2.26) refers to the intersection of two lines without perverted coordinates; therefore the equation refers to the intersection of $(\mathbf{l}, \bar{\mathbf{l}})$ with the γ-preimage of $c' = (\mathbf{c}, \bar{\mathbf{c}}) = (c_{23}, \cdots) = Mc$. If the linear complex has a line as characterizing vector c (so that $c^2 = 0$), then its axis A is characterized by $(\mathrm{Em}(A) \cdot x) = 0) \Leftrightarrow (c \cdot x = 0)$, so the axis satisfies $c = \mathrm{Em}(A) = A\gamma$. Therefore $\mathcal{C}\gamma^\star = Mc \neq c = A\gamma$.

- The 5-blade characterizing the linear complex is for us c^*; for [4] it requires invoking the polarity μ_2^4 associated with the Klein quadric. There is a freedom of sign (and actually also of scale), but we should therefore have: $\mathcal{C}\gamma^\star\mu_2^4 = \pm c^* = \pm(\mathrm{Em}(\mathbf{C}))^*$. This could still hold, if the apparent omission of the M, referred to above, is absorbed into the definition of μ_2^4.

- The Klein quadric is denoted M_2^4 in [4]; we will use Q_K.

- They give a specific expression for the null polarity π characterized by its coordinate matrix C or by $c' = (\mathbf{c}, \bar{\mathbf{c}})$ in (3.3) as

  $$((x_0, \mathbf{x})\mathbb{R})\pi = \mathbb{R}(\mathbf{x} \cdot \bar{\mathbf{c}}, -x_0\bar{\mathbf{c}} + \mathbf{x} \times \mathbf{c}).$$

  In our notation but using their characterizing parameters $(\mathbf{c}, \bar{\mathbf{c}})$, the bivector of the polarity would be $\mathbf{C} = e_0 \wedge \bar{\mathbf{c}} - \mathbf{c}^\star$. Then the null polarity for a point $x = x_0 e_0 + \mathbf{x}$ would be computed as:

  $$\pi_{\mathbf{C}}(x) = -x \rfloor \mathbf{C} = (e_0 x_0 + \mathbf{x}) \rfloor (e_0 \wedge \bar{\mathbf{c}} - \mathbf{c}^\star) = e_0 \mathbf{x} \cdot \bar{\mathbf{c}} - x_0 \bar{\mathbf{c}} + \mathbf{x} \rfloor \mathbf{c}^\star = e_0 \mathbf{x} \cdot \bar{\mathbf{c}} - x_0 \bar{\mathbf{c}} + \mathbf{x} \times \mathbf{c},$$

  confirming the result.

  The effect of the null polarity on lines is (3.5):

  $$L\pi = (\mathbf{c} \cdot \bar{\mathbf{c}})(\mathbf{l}, \bar{\mathbf{l}}) - (\mathbf{c} \cdot \bar{\mathbf{l}} + \bar{\mathbf{c}} \cdot \mathbf{l})(\mathbf{c}, \bar{\mathbf{c}}).$$

We can compute our expression for this mapping, using the polar definition of 'the transform of a line spanned by two points is the intersection of their polar planes'. Let us denote the mapping by $\bar{\pi}_{\mathbf{C}}$ (for reasons we will explain below).

$$\begin{aligned}
\bar{\pi}_{\mathbf{C}}(L) &= \bar{\pi}_{\mathbf{C}}(x \wedge y) \\
&\equiv \big(\pi_{\mathbf{C}}(x) \wedge \pi_{\mathbf{C}}(y)\big)^\star \\
&= \big((-x \rfloor \mathbf{C}) \wedge (-y \rfloor \mathbf{C})\big)^\star \\
&= (x \rfloor \mathbf{C}) \rfloor (y \wedge \mathbf{C}^\star) \\
&= \big((x \rfloor \mathbf{C}) \rfloor y\big) \wedge \mathbf{C}^\star - y \wedge \big((x \rfloor \mathbf{C}) \rfloor \mathbf{C}^\star\big) \\
&= \big(y \rfloor (x \rfloor \mathbf{C})\big) \wedge \mathbf{C}^\star - y \wedge \big((x \rfloor \mathbf{C}) \wedge \mathbf{C}\big)^\star \\
&= \big((y \wedge x) \rfloor \mathbf{C}\big) \wedge \mathbf{C}^\star - y \wedge x \wedge (\mathbf{C} \rfloor \mathbf{C}^\star)/2 \\
&= -(L \rfloor \mathbf{C})\, \mathbf{C}^\star + L(\mathbf{C} \wedge \mathbf{C})^\star/2.
\end{aligned}$$

The correctness of the derivation has been checked by GAViewer examples. The 'half' derives from the slightly tricky step

$$\begin{aligned}
\big((x \rfloor \mathbf{C}) \wedge \mathbf{C}\big)^\star &= \big((x\mathbf{C} - \mathbf{C}x)\mathbf{C} + \mathbf{C}(x\mathbf{C} - \mathbf{C}x)\big)^\star/4 \\
&= (x\mathbf{C}^2 - \mathbf{C}^2 x)^\star/4 \\
&= (x \rfloor \mathbf{C}^2)^\star/2 \\
&= x \wedge (\mathbf{C}^2)^\star/2 \\
&= x \wedge (\mathbf{C} \rfloor \mathbf{C} + \mathbf{C} \times \mathbf{C} + \mathbf{C} \wedge \mathbf{C})^\star/2 \\
&= 0 + 0 + x \wedge (\mathbf{C} \wedge \mathbf{C})^\star/2.
\end{aligned}$$

Setting $c = \mathrm{Em}(\mathbf{C})$, the quantity $(\mathbf{C} \wedge \mathbf{C})^\star = [\mathbf{C} \wedge \mathbf{C}]$ can be evaluated in $\mathbb{R}^{3,3}$ as $c \cdot c$, by the definition eq.(3) of the inner product. Similarly, $(L \rfloor \mathbf{C}) = \big(L \wedge \mathbf{C}^\star\big)^\star = [L \wedge \mathbf{C}^\star] = \ell \cdot (Mc)$, with $\ell = \mathrm{Em}(L)$. Together the equation maps to

$$\mathrm{Em}(\bar{\pi}_{\mathbf{C}}(L)) = \ell c^2/2 - (\ell \cdot Mc)Mc. \tag{4}$$

Seemingly, our result differs from [4], but their factor for ℓ, which is $\mathbf{c} \cdot \bar{\mathbf{c}} = c_{01}c_{23} + \cdots$, is indeed only half of $c \cdot c = c_{01}c_{23} + c_{23}c_{01} + \cdots$.

Since $c^2 = (Mc)^2$, the mapping for lines can be expressed more simply in terms of Mc; perhaps this is why [4] uses that quantity - for the original definition of the polarity for points, it is somewhat unnatural; but for lines it works better.

- But we should actually characterize the plane dual to a point plane directly, and hence take the dual of the result above as the proper null polarity. Let us denote that by an overbar:

  $$\bar{\pi}_{\mathbf{C}^\star} : V^4 \to \textstyle\bigwedge^3 V^4 : x \mapsto -x \wedge \mathbf{C}^\star$$

  We see that the bivector $\mathbf{C}' \equiv \mathbf{C}^\star$ is the natural bivector to characterize this mapping, and therefore also its $\mathbb{R}^{3,3}$-vector $c' = \mathrm{Em}(\mathbf{C}') = \mathrm{Em}(\mathbf{C}^\star) = Mc$. The polarity for lines is then simply the meet of the polarities for points

  $$\big(\bar{\pi}_{\mathbf{C}^\star}(x \wedge y)\big)^\star = \big(\bar{\pi}_{\mathbf{C}^\star}(x)\big)^\star \wedge \big(\bar{\pi}_{\mathbf{C}^\star}(y)\big)^\star.$$

  Alternatively, we could characterize the polarity for lines dually, and then it is just an outermorphism extension of the polarity for points:

  $$\pi_{\mathbf{C}}(x \wedge y) \equiv \big(\bar{\pi}_{\mathbf{C}^\star}(x \wedge y)\big)^\star = \pi_{\mathbf{C}}(x) \wedge \pi_{\mathbf{C}}(y).$$

This shows up that the use of duality in [4] is not wholly consistent. This is reflected in our use of the overbar for our line polarity, which conveys a more consistent algebraic structure of the operators.

With this notation, we would have

$$\mathrm{Em}(\pi_{\mathbf{C}}(L)) = (M\ell)\, c^2/2 - \big((M\ell)\cdot c\big)\, c$$

instead of eq.(4). So this mapping would work more naturally on the dual line $L^\star$...

- Note that when c is a line, so that $c^2 = 0$, the polarity returns a weighted multiple of Mc, whatever the input line ℓ (as long as ℓ is not a mulitple of c). These weighted multiples are projectively equivalent, so the polarity is rather degenerate.

- So the problem of the correlation description is that the correlation for points involves c, whereas the associated correlation for lines then involves Mc. Since lines are central, it is perhaps more natural to ensure that the line description is simplest. That is effectively what [4]'s use of $c' = Mc$ for the points achieves.

  Now that we know that, we could follow that example, and use as point correlation not $x \mapsto \pi_{\mathbf{C}}(x) = -x\rfloor\mathbf{C}$, but $x \mapsto \pi'_{\mathbf{B}}(x) = -x\rfloor\mathbf{B}^\star = -(x\wedge\mathbf{B})^\star$ for a bivector $\mathbf{B}$. This uses the fact that V^4 is 4-dimensional, so that the dual of a bivector is again a bivector, so that there is an alternative bivector characterization for a skew transformation. Then the corresponding 6D vector characterizing the correlation is $b \equiv \mathrm{Em}(\mathbf{B}) = c' = Mc$.

  The computations essentially proceed as before, with this substitution of $\mathbf{B}^\star$ for $\mathbf{C}$, and we get for the correlation of lines instead of eq.(4):

$$\mathrm{Em}(\bar{\pi}'_{\mathbf{B}}(L)) = \ell\, b^2/2 - (\ell\cdot b)\, b. \tag{5}$$

  The null lines of the null polarity, for which homogeneously $\bar{\pi}'_{\mathbf{B}}(L) = L$, should therefore now satisfy $\ell\cdot b = 0$. That is indeed how we will compute them from a vector b (itself the 6-D dual of a 5-blade).

- Alternatively, we would use the idea behind the correlation, mapping a point to a plane, and use $\bar{\pi}'_{\mathbf{B}}(x) = -x\wedge\mathbf{B}$ (with the minus still to correspond to the convention for c' of [4], which is $c' = \mathrm{Em}(\mathbf{B})$. It is then more natural to use the meet to construct the line correlation, and therefore employ:

- The geometric usage of the polarity $x \mapsto -x\wedge\mathbf{B}$ is that it converts points to planes in V^4. The bivector $\mathbf{B}$ of the polarity determines which plane. Performing this for an arbitrary point x leads to the local 'normal plane' at x of a helical motion of the space V^4, determined by $\mathbf{B}$. If $\mathbf{B}$ is a 2-blade, the helical motion degenerates to a rotation or a translation, and the normal planes contain the 'axis' $\mathbf{B}$.

- The computation of the pitch of a linear complex in Theorem 3.1.9 as $p = \mathbf{c}\cdot\bar{\mathbf{c}}/\mathbf{c}^2$ involves $\mathbf{c}\cdot\bar{\mathbf{c}} = c^2/2$, but also the non-projective quantity $\mathbf{c}$, the square of the ideal part of c (due to the perversion).

2 The Blades of $\mathbb{R}^{3,3}$ Interpreted

Let us make an inventory of the blades in $\mathbb{R}^{3,3}$, whether they are spanned by lines (the cases [2] treats) or not. The full inventory may also be found in [4], in different terms using subspaces in direct and dual ways by studying their 'carrier' versus their 'singular set' ([4], section 3.2, pg.171). The terms used in [4] to describe the various entities are perhaps traditional, but rather non-geometric; we will suggest some friendlier alternatives.

2.1 1-blades

A 1-blade b of $\mathbb{R}^{3,3}$ can represent a line of V^4 (homogeneously unique) if $b^2 = 0$, or a non-line of V^4 if $b^2 \neq 0$. Since the term non-line will come up a lot, we propose to use the term 'kine'.

The line can be ideal (like e_{23}), or finite (like e_{01}). When it is finite, it acts like the axis of a rotation (as we will see when treating 5-blades), when ideal like the 'axis' of a translation (same). A kine(like $e_{01} + e_{23}$) transforms under projective transformations like the lines do, being a sum of lines. It acts very much like a screw axis (as we will see with 5-blades) and could be drawn as such, somehow depicting the rotation plane, its perpendicular translation axis, and the pitch.

In [4], a 1-blade is treated dually as a 5-blade. If its square is zero (they would say $\Omega_q(b) = 0$, see pg.161), it is called a *singular linear complex*. The line that is intersected by all lines of this complex is called its *axis* (Theorem 3.1.3). It would be related to the 1-blade as $\mathrm{Em}^{-1}(b)$) If the square is nonzero, the 5-blade is a *regular linear complex*. Such a complex has no axis, e.g. no line that is intersected by all, though there is a kine b for which $\ell \cdot b = 0$ for all line representatives ℓ of the complex.

2.2 2-blades

In [4] Section 3.2.1, the 2-blades are treated dually, as 4-blades, and then called *pencils of linear line complexes*. The *singular set* $S(G)$ of such a complex G would be a representation of the lines in our 2-blade.

A 2-blade can be formed from null vectors or non-null-vectors. We investigate the different possibilities geometrically.

- Study $\ell_1 \wedge \ell_2$ with ℓ_1 and ℓ_2 representatives of intersecting lines L_1 and L_2. They satisfy $\ell_1^2 = \ell_2^2 = \ell_1 \cdot \ell_2 = 0$. In that case, the 2-blade $\ell_1 \wedge \ell_2$ represents a *pencil of lines*, since any linear combination of ℓ_1 and ℓ_2 is again a line: $(\alpha\ell_1 + \beta\ell_2)^2 = 0$. (See also [4], pg.173.)

- One can also make a 2-blade $\ell_1 \wedge \ell_2$ from a pair of *skew* lines. Such a 2-blade contains only the lines ℓ_1 and ℓ_2, for solving $x \wedge (\ell_1 \wedge \ell_2) = 0$ while $x^2 = 0$ leads to $x = \alpha\ell_1 + \beta\ell_2$ while $2\alpha\beta\, \ell_1 \cdot \ell_2 = 0$, which implies $\alpha = 0$ or $\beta = 0$.

  In [4] the dual of this is called a hyperbolic pencil of linear complexes. It contains all lines intersecting the skew line pair.

- Making a 2-blade from non-line vectors a and b may be reducible to the two-line case, but not always. For $x \wedge (a \wedge b) = 0$ gives $x = \alpha\, a + \beta\, b =_H a + \gamma b$. Then $x^2 = 0$ gives a quadratic equation in γ, namey $\gamma^2 b^2 + 2\gamma\, a \cdot b + a^2 = 0$. This solves to $\gamma = \big(-(a \cdot b) \pm \sqrt{(a \wedge b)^2}\,\big)/b^2$.

  - When $(a \wedge b)^2 > 0$, we find two distinct lines in the 2-blade and reduce to one of the cases above.
  - When $(a \wedge b)^2 = 0$, there is only one line in the 2-blade, so it can be written as $\ell \wedge b$ with $\ell^2 = 0$ and $b^2 \neq 0$. In that case, the 2-blade represents a tangent vector to the Klein quadric. For the 2-blade to be null, $\ell \cdot b$ should be zero.

An example of such a 2-blade is obtained by taking the line $\text{Em}(e_0 \wedge \mathbf{e}_1) = e_{01}$ and forming a line pair with the close line $\text{Em}((e_0 + \epsilon\mathbf{e}_2) \wedge (\mathbf{e}_1 + \epsilon\mathbf{e}_3)) = e_{01} + \epsilon(e_{03} + e_{21}) + O(\epsilon^2)$. This gives to first order in ϵ the 2-blade $\epsilon(e_{01} \wedge (e_{03} + e_{21})$, which is of the type studied.

Special cases of the tangent blades are: (a) a pair of coincident lines, then the tangent to the Klein quadric is contained in it (one of the rulers of the rules surface). (b) If the other line is finite, this represents the tangent to a rotational change of the line (with arbitrary fulcrum); (c) if it is ideal, it is a translation (in an arbitary direction). The regular case is that the line is locally perturbable by a screw motion. Such a motion has XXX parameters [[[**Patrick?**]]]. There should therefore maximally be teangent XXX-blades to the Klein quadric. We will meet them below.

– When $(a \wedge b)^2 < 0$, there are no (real) line solutions. There are complex conjugate skew lines satisfying the equation, and in the treatment by [4] pg.175, the 2-blade is now called an *elliptic pencil of linear congruences*. Its carrier is called an *elliptic linear congruence*, and consists ([4] Th.3.2.7) of all real lines which intersect the pair of complex conjugate skew lines. Pottman gives an extensive example 3.2.1 computing with these complex lines. In the end, his 2-blade amounts to $(e_{01} + e_{23}) \wedge (e_{02} + e_{31})$. With the treatment of 5-blades below, we can see that this is the dual of a 4-blade containing, at a point $\mathbf{p}$, the line with direction $\mathbf{u}(\mathbf{p}) = (\mathbf{e}_1 + \mathbf{e}_1 \times \mathbf{p}) \times (\mathbf{e}_2 + \mathbf{e}_2 \times \mathbf{p}) = \mathbf{e}_3 + \mathbf{p} \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2) + (\mathbf{e}_3 \cdot \mathbf{p})\,\mathbf{p}$. There is therefore one line passing through each point (and dually, one line coincident with any plane). This is covariant under rotation around the $\mathbf{e}_3$-axis, so the line set has rotational symmetry. [[[**Patrick, this is nice set to visualize somehow, you can use CGA to do this (for instance `vp(exp(-p ni/2), no ∧ u(p))`. Just pick the points p cleverly to get an interpretable figure.**]]]

2.3 3-blades

A certain way of making a 3-blade that contains lines is to take three lines and construct $\ell_1 \wedge \ell_2 \wedge \ell_3$. To check whether the blade thus constructed contains other lines, we need to solve $x \wedge (\ell_1 \wedge \ell_2 \wedge \ell_3) = 0$ while $x^2 = 0$. This results in

$$\begin{aligned} x &= \alpha\ell_1 + \beta\ell_2 + \gamma\ell_3, \quad \text{while} \\ 0 &= \alpha\beta\,\ell_1 \wedge \ell_2 + \beta\gamma\,\ell_2 \wedge \ell_3 + \gamma\alpha\,\ell_3 \wedge \ell_1. \end{aligned} \tag{6}$$

There are several cases:

- If all lines intersect pairwise, the 3-blade contains only lines. There are two situations in which this can happen:

  – The three lines can lie in a common plane, then the 3-blade represents the *field of lines* in this plane.

  – The three lines can have a point in common, then the 3-blade represents the *bundle (or star) of lines* through this point.

  Note that the 3-blade in $\mathbb{R}^{3,3}$ is now a homogeneous Plücker-plane that is entirely contained in the Klein quadric.

- If the two lines L_1 and L_2 intersect, but L_3 is skew to both, the situation can be standardized to one in which one line intersects each of the others, but those others are skew. Do this by selecting two independent intersecting lines L_1' and L_2' from the pencil of L_1

and L_2, of which L_2' intersects L_3, and the other is skew to L_3. Then we need to solve $\alpha'\gamma\,\ell_1' \wedge \ell_3 = 0$, which results in $\alpha' = 0$ or $\gamma = 0$. So a line in this 3-blade is either of the form $\alpha'\ell_1' + \beta'\ell_2'$, i.e. a member of the pencil represented by $\ell_1 \wedge \ell_2$, or it is a line of the form $\beta'\ell_2' + \gamma\ell_3$, i.e. a line of the pencil represented by $\ell_2' \wedge \ell_3$. This set of lines is called a *coupled wheel pencil* in [2].

- If the lines are all relatively skew, we can test for membership of an arbitrary line x to $\ell_1 \wedge \ell_2 \wedge \ell_3$ by setting $x \wedge (\ell_1 \wedge \ell_2 \wedge \ell_3) = 0$. If we have a line ℓ that intersect all three lines, then $\ell \cdot (\ell_1 \wedge \ell_2 \wedge \ell_3) = 0$. Such a line always exists, the set of such lines forms a regulus, see below. Then $0 = \ell \cdot (x \wedge (\ell_1 \wedge \ell_2 \wedge \ell_3)) = (x \cdot \ell)\,(\ell_1 \wedge \ell_2 \wedge \ell_3)$. Since $\ell_1 \wedge \ell_2 \wedge \ell_3$ is invertible (its square is $-2(\ell_1 \cdot \ell_2)(\ell_2 \cdot \ell_3)(\ell_3 \cdot \ell_1) \neq 0$), this means that $x \cdot \ell = 0$. Therefore, picking three such lines ℓ, we find that any x intersects all three, and so the lines in the 3-blade $\ell_1 \wedge \ell_3 \wedge \ell_3$ forms a *regulus*. [[[**Patrick/Leo here show that this exists**]]]
- The 'dual' 3-blade $(\ell_1 \wedge \ell_2 \wedge \ell_3)^*$ formed from three skew lines is also a regulus. It consists of the lines intersecting all three spanning lines. This is easily verified: $0 = x \wedge (\ell_1 \wedge \ell_2 \wedge \ell_3)^* = (x \cdot (\ell_1 \wedge \ell_2 \wedge \ell_3))^* = \Sigma_{i,j,k}(x \cdot \ell_i)\,(\ell_j \wedge \ell_k)^*$, and from the independence of the 2-blades it follows that $x \cdot \ell_i = 0$.
- [[[**treat the other cases not reducible to previous**]]]

2.3.1 Retrieving Regulus Parameters

We can call the 'standard regulus' R_0 one which has a circular cross section in the e_{12}-plane, a perpendicular axis in the e_3 plane, and lines that slope $\pi/4$ with unit speed. We pick one of the two slants and find:

$$\begin{aligned}
R_0 &= \\
&= \mathrm{Em}((e_0 + \mathbf{e}_1) \wedge (\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}) \wedge \mathrm{Em}((e_0 + \mathbf{e}_2) \wedge (-\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{2}) \wedge \mathrm{Em}((e_0 - \mathbf{e}_1) \wedge (-\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}) \\
&= (e_{02} + e_{12} + e_{03} - e_{31}) \wedge (-e_{01} + e_{12} + e_{03} + e_{23}) \wedge (-e_{02} + e_{12} + e_{03} + e_{31})/(2\sqrt{2}) \\
&= (e_{01} - e_{23}) \wedge (e_{02} - e_{31}) \wedge (e_{03} + e_{12})/(2\sqrt{2}) \\
&= a_- \wedge b_- \wedge c_+.
\end{aligned}$$

The dual regulus to this standard regulus, with the opposite slant, would be: $R_0^* = a_+ \wedge b_+ \wedge c_-$. Both are clearly orthogonal factorizations. The factors correspond to the axes: though they are not axis lines themselves, those can be retrieved from them [[[**specify below**]]].

We are interested to know what happens to the axes when a conformal transformation takes place. This transforms the basis vectors a_-, b_-, c_+ by an orthogonal transformation and hence keeps them orthogonal. From the transformed regulus R, the new axis indicators can be retrieved by orthogonal factorization; but it is perhaps simpler to do an eigenvector analysis of a linear operator related to the regulus.

Because of the orthogonal factorization, a vector like a_- commutes with R_0:

$$a_-\,R_0 = a_- \rfloor R_0 + a_- \wedge R_0 = a_- \rfloor R_0 = R_0 \lfloor a_- = R_0\,a_-,$$

whereas its associated [[[**introduce that term**]]] vector a_+ anticommutes:

$$a_+\,R_0 = a_+ \rfloor R_0 + a_+ \wedge R_0 = a_+ \wedge R_0 = -R_0 \wedge a_+ = -R_0\,a_+.$$

R_0 is invertible, its square is $R_0^2 = (a_- b_- c_+)^2 = -a_-^2 b_-^2 c_+^2 = -1$, so that $R^{-1} = -R$. Therefore the commutation relations can be expressed as eigenequations of a linear operator $\underline{R_0}$ associated with R_0:

$$\underline{R_0} : x \mapsto\ -R_0\,x R_0.$$

Vectors 'in' the factorization commute are eigenvectors of eigenvalue 1, their associates are eigenvectors of eigenvalue -1 (and actually eigenvectors of eigenvalue +1 for the dual regulus (since $-R_0^* x R_0^* = -R_0 I_6^{-1} x I_6 R_0 = R_0 I_6^{-1} I_6 x R_0 = R_0 x R_0$, $\underline{R_0} = -\underline{R_0^*}$). The operator is an orthogonal transformation, and it is symmetric, so its eigenbasis is complete and orthogonal. The eigenvectors come in pairs, and the 'axis' e_{03} can be reconstructed as being proportional to $c_+ + c_-$. [[[**discuss how to match the pairs, or just take the finite part of a result as the axis.**]]]

Such relationships transform covariantly as the standard regulus R_0 undergoes a conformal transformation to become a general regulus R. Therefore the recipe to retrieve the axes of a general regulus R are:

1. Determine the eigenvectors of the operator $\underline{R}$ of eigenvalue 1.
2. Depending on the sign of the slant (left-regulus or right-regulus), there are two positive or negatively squared eigenvectors. The other eigenvector denotes the main axis.
3. The line of an axis can be found by [[[**spell this out**]]]

The length of the axis after transformation is also established in this manner: for instance, if a scaling by e^γ had taken place, the resulting vectors transform as:

$$u'_{01} = (\frac{e^\gamma e_{01} + e^{-\gamma} e_{23}}{\sqrt{2}} + \frac{e^\gamma e_{01} - e^{-\gamma} e_{23}}{\sqrt{2}})/\sqrt{2} = e^\gamma e_{01}.$$

The unit vectors remain unit vectors, but their nullvector components change length appropriately. [[[**add details on more general case, how to match the associates?**]]]

The mapping $\underline{R}$ van be represented by a 6×6 matrix, depending on the 20 components of R. However, since $\underline{R}$ should be symmetric and orthogonal, there are constraints on its matrix $[R]$, namely $[M][R]^T[M] = [R]$ for symmetry, and $[M][R]^T[M][R] = I$ for orthogonality. There are only $\binom{6}{2=15}$ dofs in orthogonal transformations, and symmetry would impose imposes 15; but detailed analysis shows that there are 9 overlaps, so that the total dofs are $36-(21+15-9) = 9$ dofs. And there are indeed 9 degrees of freedom in a regulus: 3 for its location, 2 for the orientation of its symmetry axis, 1 for the orientation of its major ellipse axis in the orthogonal plane, and 3 more for the scalings along the axes.

2.4 4-blades

[[[**treat this as dual to 2-blade once we have completed that characterization**]]] [[[**or perhaps more easily tractable as the intersection op normal pencils at helix points**]]] [[[**have done some special cases that look interesting**]]]

If the 4-blade is the dual of a 2-blade that is factorizable as two intersecting lines, it consists of the set of lines that intersect all lines of a pencil. This set is the *union of a bundle and a field* of lines: the bundle through the vertex and a field containing the two lines. See [4], pg.173.

2.5 5-Blades

In [4] it is shown that (in our terms) all regular duals of 5-blades are projectively equivalent (the dual is regular if it does not lie on the Klein quadric). To compute the interpretation of such a 5-blade, we can therefore use a specific case such as $(e_{03} + e_{12})^*$ (this is the dual of c_+ from the unit basis). Taking a line $L = (e_0 + \mathbf{d}) \wedge \mathbf{u}$ as a probe, with $\mathbf{u}$ a unit vector and

$\mathbf{d} \cdot \mathbf{u} = 0$, we can ask when the dot product of the representative $\ell = \mathrm{Em}(L)$ with c_+ equals zero. We find for the demand $0 = \ell \wedge c_*^*$:

$$\begin{aligned} 0 &= \ell \cdot c_+ \\ &= [e_0 \wedge \mathbf{u} \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{d} \wedge \mathbf{u} \wedge e_0 \wedge \mathbf{e}_3] \\ &= \mathbf{u} \cdot \mathbf{e}_3 + (\mathbf{d} \wedge \mathbf{u} \wedge \mathbf{e}_3)^\star \\ &= \mathbf{e}_3 \cdot \big(\mathbf{u} + (\mathbf{d} \wedge \mathbf{u})^\star\big) \\ &= \mathbf{e}_3 \cdot \big(\mathbf{u}\,(1 - \mathbf{d}^\star)\big). \end{aligned}$$

The operation $(1 - \mathbf{d}^\star)$ is rotation and scaling in the plane perpendicular to $\mathbf{d}$, over a specific angle determined by the magnitude of $\mathbf{d}$ (to be precise, its tangent is $\|\mathbf{d}\|$; the scaling is over $\sqrt{1 + \mathbf{d}^2}$). Lines for which this operation applied to $\mathbf{u}$ results a vector without $\mathbf{e}_3$ component are admitted.

- For a $\mathbf{d}$ perpendicular to the e_{03} line, this implies a rotation symmetry around the e_{03} axis. At a fixed distance $||\mathbf{d}\|$, a line with a certain angle relative to this axis as allowed, and may then be rotated, giving a regulus. The tilt of $\mathbf{u}$ is zero when $\mathbf{d} = 0$ (giving a pencil of lines through the origin), and gradually increases to $\pi/2$ as $\|\mathbf{d}\| \to \infty$. At infinite distance, the lines of the regulus are parallel to the axis. We therefore obtain at least this nested family of reguli. [[[**seems opposite behavior of what it should be, check this**]]]

- For $\mathbf{d}$ not perpendicular to the e_{03} axis, it is rescaled for the same $\mathbf{u}$ [[[**as what, I do not understand this**]]]. Effectively, the change of $\mathbf{d}$ is precisely correct to translate each of the the just-derived families along the $\mathbf{e}_3$ axis over an arbitrary amount. This implements the obvious translation symmetry of $e_{03} + e_{12}$ into its solutions. (One can derive this easily: translate the support vector $\mathbf{d}$ by $\tau\mathbf{e}_3$ to become a point on the translated line $(e_0 + \mathbf{d} + \tau\mathbf{e}_3) \wedge \mathbf{u} = (e_0 + (\mathbf{d} + \tau(\mathbf{e}_3 \wedge \mathbf{u})/\mathbf{u})\mathbf{u}$. Substiting the new support vector for $\mathbf{d}$ in the equation, we find $\mathbf{e}_3 \cdot (\mathbf{u}(1 + (\mathbf{d} + \tau(\mathbf{e}_3 \wedge \mathbf{u})/\mathbf{u})^\star)) = -\tau\mathbf{e}_3 \cdot (\mathbf{e}_3 \wedge \mathbf{u})^\star = -\tau(\mathbf{e}_3 \wedge \mathbf{e}_3 \wedge \mathbf{u})^\star = 0$, so that the shifted line indeed also satisfies the equation.)

Ultimately, there is therefore a 3-parameter family of lines that has zero dot product with the non-line element $e_{03} + e_{12}$: it can be parametrized by direction $\mathbf{u}$ (giving the distance $\mathbf{d}$), and by rotation around and translation along e_{03}.

An alternative way of inspecting the collection of lines is to ask which of the family pass through a fixed point p with position vector $\mathbf{p}$. Then we can compute $\mathbf{d} = (\mathbf{p} \wedge \mathbf{u})/\mathbf{u}$, and find that $0 = \mathbf{e}_3 \cdot (\mathbf{u} + (\mathbf{p} \wedge \mathbf{u})^\star) = \mathbf{e}_3 \cdot (\mathbf{u} - \mathbf{u} \cdot (\mathbf{p})^\star) = \mathbf{u} \cdot (\mathbf{e}_3 + \mathbf{e}_3 \cdot (\mathbf{p})^\star = \mathbf{u} \cdot (\mathbf{e}_3 + (\mathbf{e}_3 \wedge \mathbf{p})^\star$. Therefore the lines form a pencil through the point p, with the normal vector of the pencil plane equal to $\mathbf{n} = \mathbf{e}_3 + (\mathbf{e}_3 \wedge \mathbf{p})^\star$.

This normal vector is a tangent to a helix generated by a unit pitch (one meter per radian) screw around $\mathbf{e}_3$ (just rewrite in the classical form $\mathbf{n} = \mathbf{e}_3 + \mathbf{e}_3 \times \mathbf{p}$. Differentiating the helical motion $\mathbf{p}(t) = t\mathbf{e}_3 + R_{\mathbf{e}_3\mathbf{I}_3 t}[\mathbf{p}_0]$ gives $\dot{\mathbf{p}} = \mathbf{n}$).

2.6 Overview

2.6.1 Factorization

[[[**how to factorize blades into lines?**]]]

2.6.2 Testing Equivalence of Intersections

In [2] it is stated that the difference between a field (lines with a common plane) and a bundle (lines with a common point) cannot be made on projective grounds only, but that one would need to introduce an affine structure. This would seem strange, since the difference is one of multiplicity of intersection, and intersections are projectively covariant.

The intersection point between two 3D lines in general position (not in a plane through the origin) is the homogeneous point

$$(e_0\mathbf{v} + \mathbf{V}) \cap (e_0\mathbf{u} + \mathbf{U}) = (\mathbf{V} \wedge \mathbf{u})^\star e_0 + \mathbf{U}^\star \cdot \mathbf{V},$$

with the duals being Euclidean. If it has already been established that three lines are not pairwise skew, it is therefore sufficient to establish whether they have a common intersection by testing for the proportionaliy of the outcome of the cross product of their moment vectors. This uses:

$$\mathbf{U}^\star \cdot \mathbf{V} = \mathbf{U}^\star \times \mathbf{V}^\star.$$

This equation also works for ideal lines; but if the two lines happen to lie in a plane through the origin, the result is 0 and gives no indication of the intersection point. In that case, one should use:

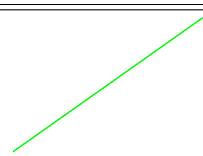
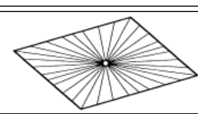
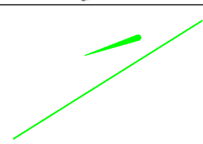
$$(e_0\mathbf{v} + \mathbf{V}) \cap (e_0\mathbf{u} + \mathbf{U}) = (\mathbf{u} \wedge \mathbf{v})^\star e_0 + \mathbf{V}^\star\,\mathbf{u} - \mathbf{U}^\star\,\mathbf{v},$$

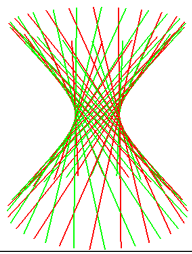
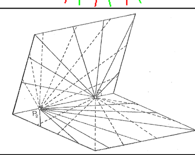
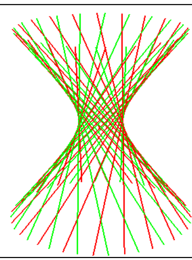
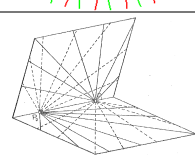
with the dual now taken relative to the 2-blade of the common plane, which is of course proportional to $(\mathbf{u}\wedge\mathbf{v})$. For comparison of equivalence, it is again sufficient to use the positional part only; that can be simplified to

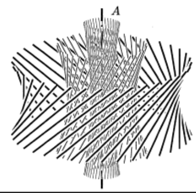
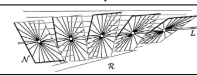
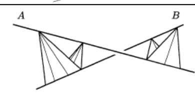
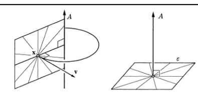
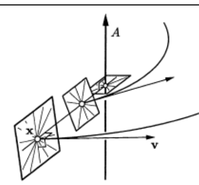
$$\mathbf{V}^\star\,\mathbf{u} - \mathbf{U}^\star\,\mathbf{v} = (\mathbf{u} \cdot \mathbf{V} - \mathbf{v} \cdot \mathbf{U})^\star,$$

and one could then even drop the dual when testing for equivalence only, after establishing that the lines are in a common plane (through the origin). Note that when pairwise testing a triple of lines, proportionality of the three computed vectors is then enough to investigate their incidence relationships. And one needs to test *all* three in this case, for if a line would pass through the origin, all of its intersections with other lines would be proportional anyway.

Given a factorization of a null 3-blade which can represent either a field or a bundle, there is therefore a simple test to establish which it is.

g	s	form	carrier	S	description
0	> 0				
1	0	ℓ			line, or ideal line [axis of a singular linear complex]
1	$\neq 0$	k			not-a-line, 'kine'
2	0	$\ell \wedge \ell$			pencil of lines
2	$\neq 0$	$\ell \wedge \ell$			skew line pair
2	$\neq 0$	$\ell \wedge k$			'line tangent' ?
2	< 0	$k \wedge k$			'dual regulus pencil' ?

g	s	form	carrier	S	description
3	0	$\ell \wedge \ell \wedge \ell$, pt com			bundle of lines (point)
3	$\neq 0$	$\ell \wedge \ell \wedge \ell$, pt coin			field of lines (plane)
3	0	$\ell \wedge \ell \wedge \ell$			bundle of lines
3					[[[**more cases**]]]
3	$\neq 0$	$\ell \wedge \ell \wedge \ell$, pt skew			regulus
3	$\neq 0$	$\ell \wedge \ell \wedge \ell$, 2 pw coin			double wheel pencil
3	0	$(\ell \wedge \ell \wedge \ell)^*$ pt com			bundle of lines (point)
3	$\neq 0$	$(\ell \wedge \ell \wedge \ell)^*$, pw coin			field of lines (plane)
3	0	$(\ell \wedge \ell \wedge \ell)^*$			bundle of lines
3	$\neq 0$	$(\ell \wedge \ell \wedge \ell)^*$ pw skew			regulus
3	$\neq 0$	$(\ell \wedge \ell \wedge \ell)^*$ 2 pw coin			double wheel pencil
3					[[[**more cases**]]]

g	s	form	carrier	S	description
4	< 0	$(k \wedge k)^*$			[elliptic linear congruence] regulus pencil ?
4	$= 0$	$(\ell \wedge \ell)^*$			[parabolic linear congruence]
4	> 0	$(\ell \wedge \ell)^*$ skew			[hyperbolic linear congruence]
4	> 0	$(\ell \wedge \ell)^*$ pt com			[hyperbolic linear congruence], bundle $\cup$ field
4					[[[**more cases ?**]]]
5	$= 0$	ℓ^*			[singular linear complex], all lines hitting line
5	$\neq 0$	k^*			[regular linear complex], regulus bundle?

2.7 Interpretation of Null Blades in $\mathbb{R}^{3,3}$

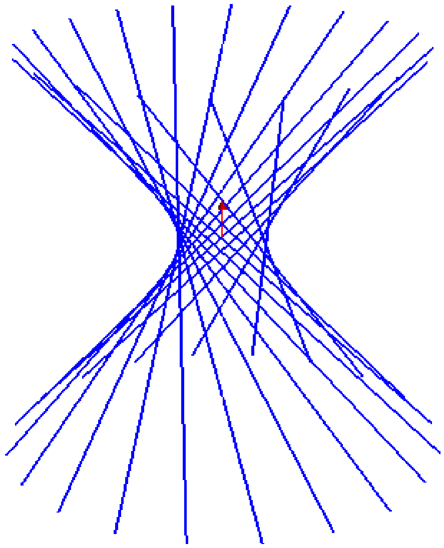
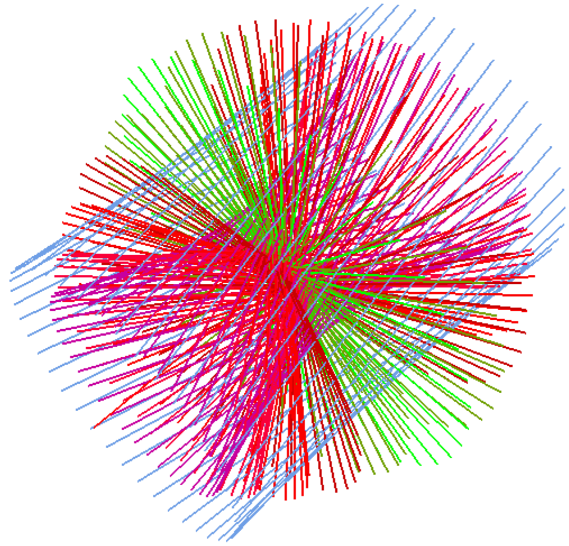
[[[**this section now old, but we will get back to metric issues**]]] The blades in $\mathbb{R}^{3,3}$ have to contain null vectors to represent collections of lines. (If they do not, we have no geometrical interpretation for them.) If a blade contains a null vector, it is a null blade. Therefore we are interested in null blades in $\mathbb{R}^{3,3}$. [2] makes an inventory, and displays the geometry.

- $\mathbb{R}^{0,0,1}$: a line (possibly ideal, this applies to all below)
- $\mathbb{R}^{1,1,0}$: a line-pair, i.e., a pair of skew lines (one of which can be ideal)
- $\mathbb{R}^{0,0,2}$: a 2D pencil of lines through one point (possibly ideal)
- $\mathbb{R}^{2,1,0}$ or $\mathbb{R}^{1,2,0}$: a regulus: a family of lines ruling a hyperboloid of revolution.
- $\mathbb{R}^{1,1,1}$: two pencils with one common line
- $\mathbb{R}^{0,0,3}$: all lines through a common point, or all lines in a common plane

2.7.1 Regulus

[[[**old material based on Li's metric characterization, possibly true but we would rephrase this**]]] Example of the regulus pencil: Take the blade $p_1 \wedge p_2 \wedge n$, with $1 = p_1 \cdot p_1 = p_2 \cdot p_2 = -n \cdot n$, and the vectors orthogonal. (An example is $(e_{01}+e_{23}) \wedge (e_{02}+e_{31}) \wedge (e_{03}-e_{12})$.) Then a line in this is a linear combination of those components, that is null. So $L = \alpha p_1 + \beta p_2 + \gamma n$, while $0 = \alpha^2 + \beta^2 - \gamma^2$.

When $\gamma = 0$, then $\alpha^2 = -\beta^2$, so this does not lead to a solution. Setting $\gamma = 1$ to set the weight of L, we find that $\alpha^2 + \beta^2 = 1$. It follows that (modulo weight) there is a 1-parameter family $L = \cos(\phi) p_1 + \sin(\phi) p_2 + n$ that satisfies the equation. Each of these lines has the same

non-zero inner product with the axis line $A =$ [[[**? related to direction of** n]]]. In the example, that is the line e_{03}.

3 Projective Transformations as Versors

[3] claims that any special linear transformation of V^4 is isomorhic to a special orthogonal transformation of $\mathbb{R}^{3,3}$, by a reasoning that I do not quite follow. Here's mine:

A linear transformation f on V^4 induces an outermorphism. This outermorphism (also denoted f) affects its bivectors and therefore the vectors of $\mathbb{R}^{3,3}$ they map to. Then the inner product may be affected. We compute:

$$\mathrm{Em}(f(A)) \cdot \mathrm{Em}(f(B)) = [f(A) \wedge f(B)] = [f(A \wedge B)] = \det(f)\,[A \wedge B] = \det(f)\,\mathrm{Em}(A) \cdot \mathrm{Em}(B).$$

It follows that when we restrict ourselves to transformations of f for which $\det(f) = 1$, their representation in $\mathbb{R}^{3,3}$ is an orthogonal transformation. The determinant constraint is simply a scaling freedom of the homogeneous matrices representing projective transformations, and therefore not a restriction to the geometry that can be represented. The converse is not true: [3] claims that the odd orthogonal transformations of $\mathbb{R}^{3,3}$ swap points and planes whereas the linear transformations of V^4 do not, so that some care is required. [[[**Check this, it appears we need to focus on even orthogonal trasnformations of $\mathbb{R}^{3,3}$, but will still get all elements of $SL(V^4)$.**]]]

"All special projective transformations can be classified by their spinor generators." [3]

We will also give the matrix representations of the transformation, on the null basis and on the unit basis (denoted with prime). The transformation between the two is by means of the matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$. This transforms a matrix like:

$$QAQ = \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A+B+C+D & A-B+C-D \\ A+B-C-D & A-B-C+D \end{bmatrix}.$$

The matrices on the bull basis are generally more sparse, as we will see.

3.1 Translation

The 2-blade $T_1 = e_{12} \wedge e_{31}$ is null. It has the following relationships with the null basis vectors:

$$e_{01}\,T_1 - T_1 e_{01} = 0, \quad e_{02}\,T_1 - T_1\,e_{02} = 2e_{12}, \quad e_{03}\,T_1 - T_1\,e_{03} = -2e_{31},$$

$$e_{23}\,T_1 - T_1 e_{23} = 0, \quad e_{31}\,T_1 = 0, \quad e_{12}\,T_1 = 0.$$

Exponentiating T_1 gives $V_1 = \exp(\tau T_1/2) = 1 + \frac{1}{2}\tau T_1$, and application of this rotor yields:

$$V_1\,e_{02}\,\widetilde{V}_1 = e_{02} + \tau e_{12} \leftrightarrow (e_0 + \tau \mathbf{e}_1) \wedge \mathbf{e}_2, \quad V_1\,e_{03}\,\widetilde{V}_1 = e_{03} - \tau e_{31} \leftrightarrow (e_0 + \tau \mathbf{e}_1) \wedge \mathbf{e}_3,$$

and leaves the other null basis elements invariant. Therefore $V_1 = \exp(\tau T_1/2)$ represents translation over $\tau \mathbf{e}_1$, which should have precisely these properties. Its matrix is: [[[**changed into transpose relative to Patrick's thesis**]]]

$$[V_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \tau \\ 0 & 0 & 1 & 0 & -\tau & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad [V_1'] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \tau/2 & 0 & 0 & -\tau/2 \\ 0 & -\tau/2 & 1 & 0 & \tau/2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \tau/2 & 0 & 1 & -\tau/2 \\ 0 & -\tau/2 & 0 & 0 & \tau/2 & 1 \end{array}\right].$$

3.2 Perspective Transformations

In a similar manner to translations, one can compute perspective transformations. The 2-blade $F_1 = e_{02} \wedge e_{03}$ is null. It has the following relationships with the null basis vectors:

$$e_{23}\,F_1 - F_1 e_{23} = 0, \quad e_{12}\,F_1 - F_1\,e_{12} = 2e_{02}, \quad e_{31}\,F_1 - F_1\,e_{31} = -2e_{03},$$

$$e_{01}\,F_1 - F_1 e_{01} = 0, \quad e_{02}\,F_1 = 0, \quad e_{03}\,F_1 = 0.$$

Exponentiating F_1 gives $W_1 = \exp(fF_1/2) = 1 + \frac{1}{2} f F_1$, and application of this rotor yields:

$$W_1\,e_{31}\,\widetilde{W_1} = e_{31} - f e_{03} \leftrightarrow -(f e_0 + \mathbf{e}_1) \wedge \mathbf{e}_3, \quad W_1\,e_{12}\,\widetilde{W_1} = e_{12} + f e_{02} \leftrightarrow (f e_0 + \mathbf{e}_1) \wedge \mathbf{e}_2,$$

and leaves the other null basis elements invariant. Its matrix is: [[[**changed into transpose relative to Patrick's thesis**]]]

$$[W_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -f & 0 & 1 & 0 \\ 0 & f & 0 & 0 & 0 & 1 \end{array}\right] \qquad [W_1'] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -f/2 & 0 & 0 & -f/2 \\ 0 & f/2 & 1 & 0 & f/2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f/2 & 0 & 1 & f/2 \\ 0 & -f/2 & 0 & 0 & -f/2 & 1 \end{array}\right].$$

3.3 Directional Scaling

Directional scaling bivectors should square to a non-zero scalar, say 1. For a scaling in the $\mathbf{e}_1$ direction, use $S_1 = e_{01} \wedge e_{23}$. It commutes with all null basis vectors except:

$$e_{01} \rfloor S_1 = -e_{01}, \quad e_{23} \rfloor S_1 = e_{23}.$$

Applying the versor $V_1^S = \exp(\gamma\,S_1/2)$ with $S_1^2 = 1$ to a vector v gives $v - 2\sinh(\gamma/2) V_1^S (v \rfloor S_1)$. It leaves all null basis elements unchanged except:

$$e_{01} \mapsto e^{\gamma} e_{01}, \quad e_{23} \mapsto e^{-\gamma} e_{23}.$$

The e_{01} scaling is as expected, and the e_{23} scaling is to keep the inner product $e_{01} \cdot e_{23}$ invariant. Its matrix is:

$$[S_1] = \left[\begin{array}{ccc|ccc} e^{\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & e^{-\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad [S_1'] = \left[\begin{array}{ccc|ccc} ch & 0 & 0 & sh & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline sh & 0 & 0 & ch & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

with $ch = \cosh(\gamma)$ and $sh = \sinh(\gamma)$.

3.4 Skewing

The null bivector $B = e_{02} \wedge e_{12}$ produces a skewing transformation. It commutes with all null basis vectors except:

$$e_{03} \rfloor B = -e_{02}, \quad e_{13} \rfloor B = -e_{12}.$$

Since $B^2 = 0$, the action of the versor $V = \exp(-\sigma B/2)$ on a vector $v \in \mathbb{R}^{3,3}$ is $v \mapsto v + \sigma\, v \rfloor B$. Computing that happens to the null basis of $\mathbb{R}^{3,3}$, we find that its matrix is:

$$[S_{32}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\sigma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \sigma & 1 \end{array}\right] \qquad [S'_{32}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\sigma/2 & 0 & 0 & -\sigma/2 \\ 0 & \sigma/2 & 1 & 0 & -\sigma/2 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sigma/2 & 0 & 1 & -\sigma/2 \\ 0 & -\sigma/2 & 0 & 0 & \sigma/2 & 1 \end{array}\right].$$

This is interpretable as a skewing in V^4, namely the one in which $\mathbf{e}_3 \mapsto \mathbf{e}_3 - \sigma\, \mathbf{e}_2$, and all other vectors are left invariant.

3.5 Rotation

The rotation bivector is a bit more involved. It is effectively two related skews in the same plane, with the same direction, with the orthogonality conditions automatically introducing a proper scaling. The bivector B for rotation in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B = \tfrac{1}{2}(e_{02} \wedge e_{12} - e_{03} \wedge e_{31})$$

This is not a blade, and its square is not a scalar, but of the form 'scalar plus quadvector':

$$B^2 = -\tfrac{1}{2}(1 + e_{02} \wedge e_{12} \wedge e_{03} \wedge e_{31}).$$

One may verify that $B^3 = -B$. The algebraic properties of B therefore determine a rotation-like versor, since they allow for a grouping into trigonometric functions:

$$\begin{aligned} e^{-\phi B} &= 1 - \tfrac{1}{1!}\phi B + \tfrac{1}{2!}\phi^2 B^2 - \tfrac{1}{3!}\phi^3 B^3 + \tfrac{1}{4!}\phi^4 B^4 + \cdots \\ &= 1 - \tfrac{1}{1!}\phi B + \tfrac{1}{2!}\phi^2 B^2 + \tfrac{1}{3!}\phi^3 B - \tfrac{1}{4!}\phi^4 B^2 + \cdots \\ &= 1 + B^2 - (\tfrac{1}{1!}\phi - \tfrac{1}{3!}\phi^3 + \cdots)\, B - (1 - \tfrac{1}{2!}\phi^2 + \tfrac{1}{4!}\phi^4 + \cdots)\, B^2 \\ &= 1 - \sin(\phi)\, B + \big(1 - \cos(\phi)\big)\, B^2 \end{aligned}$$

A vector that commutes with B is unchanged. Applying the versor to a vector b 'in' B (such as e_{02} or e_{03}), one may verify that:

$$Bb - bB = -b^\perp, \qquad B^2 b + bB^2 = -b, \qquad BbB = 0.$$

with $b^\perp$ achieved from b by the regular 90-degree rotation sending $\mathbf{e}_2$ to $\mathbf{e}_3$, with as a consequence for the null basis:

$$e_{02}^\perp = e_{03}, \quad e_{03}^\perp = -e_{02}, \quad e_{31}^\perp = -e_{13}^\perp = e_{12}, \quad .e_{12}^\perp = e_{13} = -e_{31}.$$

With that:

$$\begin{aligned} &e^{-\phi B}\, b\, e^{\phi B} \\ &= (1 - sB + (1-c)B^2)\, b\, (1 + sB + (1-c)B^2) \\ &= b - s(Bb - bB) + (1-c)(B^2 b + bB^2) - s^2 BbB + s(1-c)(BbB^2 - B^2 bB) + (1-c)^2 B^2 b B^2 \\ &= \cos(\phi)\, b + \sin(\phi)\, b^\perp. \end{aligned}$$

So $\exp(-\phi B)$ is the versor for a rotation over ϕ, leaving the vectors commuting with B invariant.

The matrix of the $\mathbf{e}_2 \wedge \mathbf{e}_3$ rotation is:

$$[R_{23}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -s & 0 & 0 & 0 \\ 0 & s & c & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{array}\right] \qquad [R'_{23}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -s & 0 & 0 & 0 \\ 0 & s & c & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{array}\right].$$

where $c = \cos(\phi)$ and $s = \sin\phi)$. As a consequence, $[R_{23}]^T\,[M]\,[R_{23}] = [M]$, so this is indeed an orthogonal matrix in the $[M]$-metric.

3.6 Squeeze (Lorentz Transformation)

The constituent 2-blade parts of the rotator bivector are pure skews; a rotation combines them an $\mathbf{e}_2$ axis skew and an $\mathbf{e}_3$-axis skew in the same direction; changing the sign of their combination gives a volume preserving 'squeezing' operation.

The bivector B for squeeze in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B = \tfrac{1}{2}(e_{02} \wedge e_{12} + e_{03} \wedge e_{31})$$

This is not a blade, and its square is not a scalar, but of the form 'scalar plus quadvector':

$$B^2 = \tfrac{1}{2}(1 + e_{02} \wedge e_{12} \wedge e_{03} \wedge e_{31}).$$

One may verify that $B^3 = B$. The algebraic properties of B therefore determine a rotation-like versor, since they allow for a grouping into hyperbolic functions:

$$e^{\gamma B} = 1 + \sinh(\gamma)\,B + \big(\cosh(\gamma) - 1\big)\,B^2.$$

Again vectors commuting with B are unchanged. But applying the versor to a vector b 'in' B (such as e_{02} or e_{03}), one may verify that:

$$Bb - bB = b^\perp, \qquad B^2 b + bB^2 = -b, \qquad BbB = 0.$$

with $b^\perp$ achieved from b as

$$e_{02}^\perp = e_{03}, \quad e_{03}^\perp = e_{02}, \quad e_{31}^\perp = -e_{12}, \quad e_{12}^\perp = -e_{31}.$$

With that, for a vector b 'in'B:

$$\begin{aligned} & e^{\gamma B}\,b\,e^{-\gamma B} \\ &= (1 + sB + (c-1)B^2)\,b\,(1 - sB + (c-1)B^2) \\ &= b + s(Bb - bB) + (c-1)(B^2 b + bB^2) - s^2 BbB - s(c-1)(BbB^2 - B^2 bB) + (c-1)^2 B^2 bB^2 \\ &= \cosh(\gamma)\,b + \sinh(\gamma)\,b^\perp. \end{aligned}$$

So $\exp(\gamma B)$ is the versor for a squeeze parametrized by γ, leaving the vectors commuting with B invariant.

This keeps volumes invariant, and is orthogonal due to the simultaneous effect on finite and infinite lines. The matrix of the $\mathbf{e}_2 \wedge \mathbf{e}_3$ squeeze shows this clearly:

$$[L_{23}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & ch & -sh & 0 & 0 & 0 \\ 0 & sh & ch & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ch & sh \\ 0 & 0 & 0 & 0 & -sh & ch \end{array}\right] \qquad [L'_{23}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & ch & -sh & 0 & 0 & 0 \\ 0 & sh & ch & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ch & sh \\ 0 & 0 & 0 & 0 & -sh & ch \end{array}\right].$$

where $ch = \cosh(\gamma)$ and $sh = \sinh(\gamma)$. As a consequence, $[L_{23}]^T\,[M]\,[L_{23}] = [M]$, so this is indeed an orthogonal matrix in the $[M]$-metric.

3.7 Orthogonality Conditions for Matrices

Writing in block form, we can see when a matrix $[X]$ represents an orthogonal transformation, since it should satisfy $[M]\,[X]^T\,[M]\,[X] = [I]$:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} B^TC + D^TA & B^TD + D^TB \\ A^TC + C^TA & A^TD + C^TB \end{bmatrix}.$$

Therefore A^TC and B^TD must be skew-symmetric matrices, and $B^TC + D^TA$ should be the identity.

For unit-based matrices, the metric matrix is different, so we get:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A'^T & C'^T \\ B'^T & D'^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A'^TA' - C'^TC' & A'^TB' - C'^TD' \\ -B'^TA' + D'^TC' & -B'^TB' + D'^TD' \end{bmatrix}$$

Therefore $A'^TA' - C'^TC' = I$, $-B'^TB' + D'^TD' = I$, and $B'^TA' = D'^TC'$ for this representation.

4 2D Projective Geometry

When concentrating on a single plane in space, and considering the transformations that keep the component of the volume perpendicular to it invariant, we can establish an orthogonal representation of the projective transformations in 2D. There are then 8 elementary fundamental transformations: 2 translations, 2 scalings, 2 perspective transformations, 1 rotation and 1 squeeze. Their matrices are, on the null basis: [[[**also changed April 2013**]]]

$$[T] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -\tau_2 \\ 0 & 1 & 0 & 0 & 0 & \tau_1 \\ 0 & 0 & 1 & \tau_2 & -\tau_1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad [P] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \delta_2 & 1 & 0 & 0 \\ 0 & 0 & -\delta_1 & 0 & 1 & 0 \\ -\delta_2 & \delta_1 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$[S] = \left[\begin{array}{ccc|ccc} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$[R] = \left[\begin{array}{ccc|ccc} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad [L] = \left[\begin{array}{ccc|ccc} ch & -sh & 0 & 0 & 0 & 0 \\ sh & ch & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & ch & sh & 0 \\ 0 & 0 & 0 & -sh & ch & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

4.1 Cross Ratio

On a line, a cross ratio is preserved under a projective transformation. Here is a way to embed that in the line geometry. On a line L with direction vector $\mathbf{u}$, consider four points P_1, P_2, P_3, P_4. They can be considered as the intersection of L with other lines L_i. Pick those lines to have specific direction vectors, and allow the points to be represented with non-unit weights (since projective transformations can change weights): $L_1 = \alpha_1 P_1 \wedge \mathbf{v}$, $L_2 = \alpha_2 P_2 \wedge \mathbf{w}$, $L_3 = \alpha_3 P_3 \wedge \mathbf{v}$, $L_4 = \alpha_4 P_4 \wedge \mathbf{w}$. Now transfer these to the line space $\mathbb{R}^{3,3}$, labeling them ℓ_i. For the dot product between two lines with both odd or both even labels, we get zero. For the others, we introduce coordinates π_i to label the points P_i along L from some unit weight point P, in amounts of $\mathbf{u}$. Then we compute:

$$\begin{aligned}
\ell_i \cdot \ell_j &= [\alpha_i P_i \wedge \mathbf{v} \wedge \alpha_j P_j \wedge \mathbf{w}] \\
&= \alpha_i \alpha_j [(P + \pi_i \mathbf{u}) \wedge \mathbf{v} \wedge (P + \pi_j \mathbf{u}) \wedge \mathbf{w}] \\
&= \alpha_i \alpha_j [(\pi_i - \pi_j) \mathbf{u} \wedge P \wedge \mathbf{v} \wedge \mathbf{w}] \\
&= (\pi_j - \pi_i) \alpha_i \alpha_j [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}]
\end{aligned}$$

This is therefore proportional to the signed distance between the points, by the arbitrary point weight, and a rather arbitrary volume-based proportionality constant that is independent of the points.

To achieve an invariance measure involving the distance, we need to divide out the embedding constants. The cross ratio C in V^4 is constructed in line space as a ratio of dot products of bivectors:

$$\begin{aligned}
C &= \frac{(\ell_1 \wedge \ell_4) \cdot (\ell_2 \wedge \ell_3)}{(\ell_1 \wedge \ell_2) \cdot (\ell_3 \wedge \ell_4)} \\
&= -\frac{(\ell_4 \cdot \ell_3)(\ell_1 \cdot \ell_2)}{(\ell_2 \cdot \ell_3)(\ell_1 \cdot \ell_4)} \\
&= \frac{(\pi_2 - \pi_1)(\pi_4 - \pi_3)}{(\pi_3 - \pi_2)(\pi_1 - \pi_4)}
\end{aligned}$$

Since C is defined as a ratio of dot products, each projectively invariant, it is invariant. And since it can be expressed completely in terms of distances in V^4, it is a measurable quantity, independent of the embedding.

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