Visualization of the projective geometry for geometric algebra

Final Presentation

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Project description

Extend a graphical calculator of geometric algebra to model the projective geometry of lines.

- Much like linear algebra, but has structure preserving operations
- Outer product $\wedge$ to create subspaces
- Implementations are fast [Fontijne and Dorst(2003)]

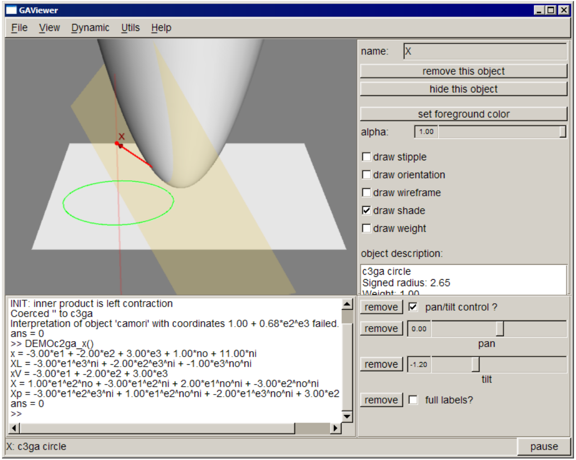
Project description

Extend a graphical calculator of geometric algebra to model the projective geometry of lines.

- Useful for computer vision, graphics, robotics...
- Plücker coordinates: 3D lines are 6D null vectors
  - Representation is homogeneous
- Recently found an operational model for geometric algebra [Li and Zhang(2011)]

Project description

Extend a graphical calculator of geometric algebra to model the projective geometry of lines.



http://geometricalgebra.net → Downloads → GAViewer

Approach

1. Understand Plücker model for GA
2. Explore part of the representation space
   - Geometric interpretation of subspaces
   - Compute location, stance, weight, orientation
3. Extend GAViewer
   - Recognise geometric type of input
   - Implement drawing routines

Plücker model

Really new to GA community

- Original article defines metric and vector space $\mathbb{R}^{3,3}$
- No further connection with model in other algebras shown
- No further articles on Plücker model for GA

. . . but well known for other algebras

- Big difference in vocabulary
- Less focus on subspace nature
- Some GA concepts not well defined in LA

But first. . .

Recall homogeneous model

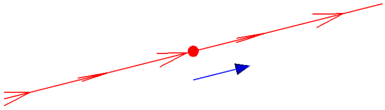
- Extends Euclidean directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with point e_0
- Point $p = \mathbf{p} + \lambda e_0$ has location $\frac{p}{e_0 \cdot p}$ and weight λ
- Directions have weight $\lambda = 0$, so location is at infinity
- Average of two points gives new point

Lines defined

Define line L by a point p and direction $\mathbf{d}$:

$$L = p \wedge \mathbf{d}$$

λL only differs in weight and orientation

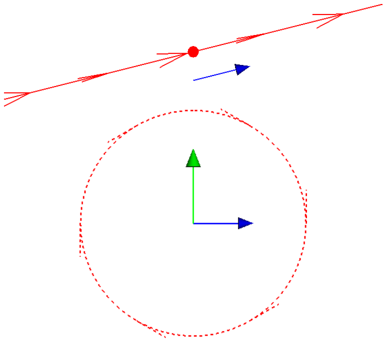


Plücker space

Basis of our vector space $\mathbb{R}^{3,3}$:

$$\left.\begin{aligned} e_0 \wedge \mathbf{e}_1 &= e_{01} \\ e_0 \wedge \mathbf{e}_2 &= e_{02} \\ e_0 \wedge \mathbf{e}_3 &= e_{03} \end{aligned}\right\} \text{Finite lines}$$

$$\left.\begin{aligned} \mathbf{e}_2 \wedge \mathbf{e}_3 &= e_{23} \\ \mathbf{e}_3 \wedge \mathbf{e}_1 &= e_{31} \\ \mathbf{e}_1 \wedge \mathbf{e}_2 &= e_{12} \end{aligned}\right\} \text{Lines at infinity}$$

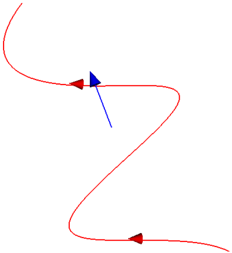


Lines are screws

Screws are general basic geometric entity

- Finite lines only rotate
- Lines at infinity only translate

. . . but we only look at subspaces of lines

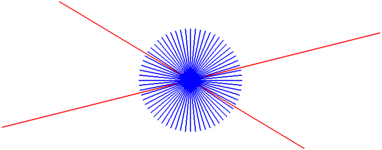


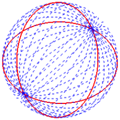
Combining lines

Geometrical different cases

1. 2 intersecting and parallel lines
2. 2 skew lines
3. 3 lines intersect in 1 point
4. 3 lines intersect in different points
5. 1 line intersects 2 other lines
6. 3 skew lines

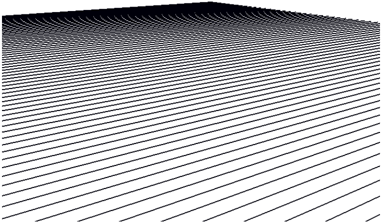
Pencils of lines





Two lines ℓ_1, ℓ_2 intersect

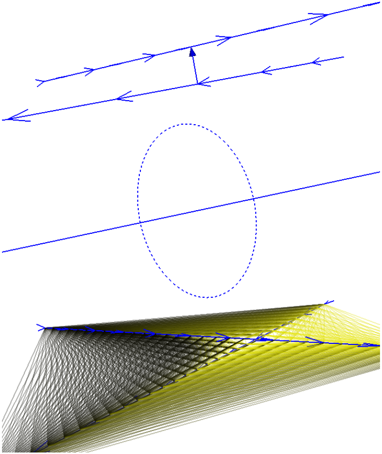
Parallel lines intersect in point at infinity!



Line pair

When ℓ_1, ℓ_2 don't intersect, $\ell_1 \wedge \ell_2$ only contains those two lines

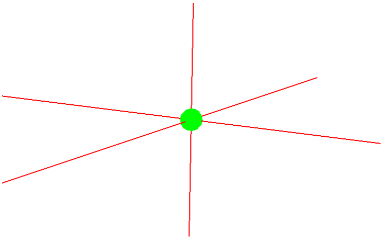
Dual of $\ell_1 \wedge \ell_2$ contains all lines intersecting both



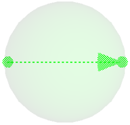
Finite and infinite points

Two interpretations:

1. All lines intersecting in a point
2. The point of intersection

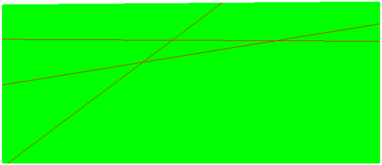


Three parallel lines intersect in a pair of points

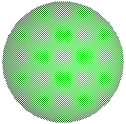


Planes

If 3 lines intersect in 3 points, the linear combination is their common plane Only 1 infinite

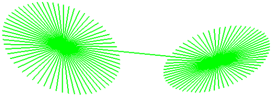


plane, corresponds to Euclidean space



Pair of pencils

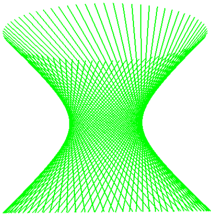
Combines the set of lines of
$\ell_1 \wedge \ell_2$, $\ell_2 \wedge \ell_3$ and $\ell_3 \wedge \ell_1$

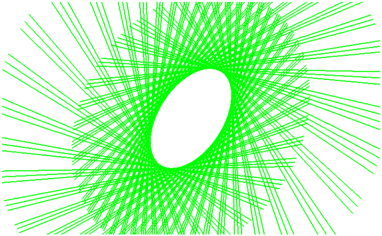


Regulus

Three skew lines

Not implemented; unclear how to parameterize the set of lines





Extending GAViewer

Easily done:

- Generate GA code for Plücker model
- Only need to implement classification, visualization, what to do on mouse drag

. . . if you know how.

- Little documentation on where to put new code

Results

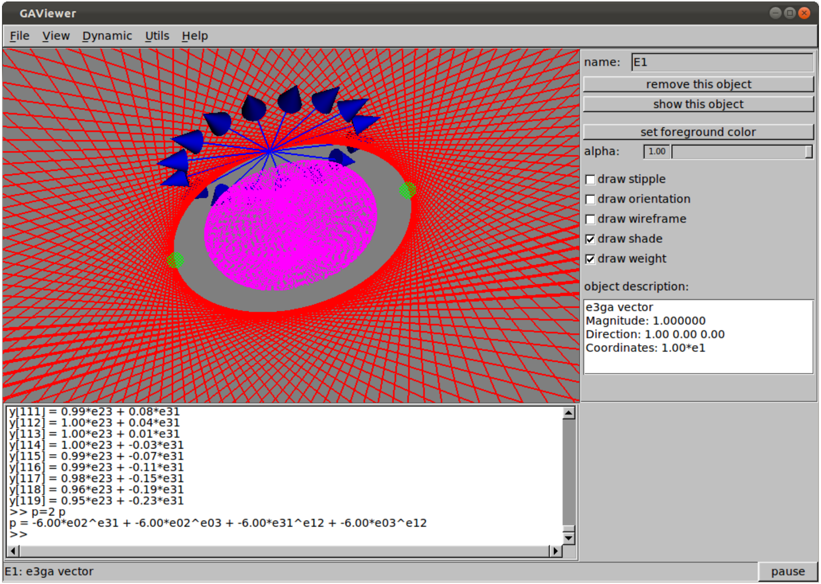
Much is done

1. Show correspondencies between Plücker model for LA and GA
2. Computed 4 geometrically important features of almost all line-subspaces
3. Implemented classifications and visualizations

But much more can be done

1. Compute features of reguli
2. Compute combinations of screws
3. Connections with other models?
4. Ellipse representation?
5. Plücker model for $\mathbb{R}^2$?

Questions


GAViewer
File View Dynamic Utils Help
name: E1
remove this object
show this object
set foreground color
alpha: 1.00
draw stipple
draw orientation
draw wireframe
draw shade
draw weight
object description:
e3ga vector
Magnitude: 1.000000
Direction: 1.00 0.00 0.00
Coordinates: 1.00*e1
y[111] = 0.99*e23 + 0.08*e31
y[112] = 1.00*e23 + 0.04*e31
y[113] = 1.00*e23 + 0.01*e31
y[114] = 1.00*e23 + -0.03*e31
y[115] = 0.99*e23 + -0.07*e31
y[116] = 0.99*e23 + -0.11*e31
y[117] = 0.98*e23 + -0.15*e31
y[118] = 0.96*e23 + -0.19*e31
y[119] = 0.95*e23 + -0.23*e31
>> p=2 p
p = -6.00*e02^e31 + -6.00*e02^e03 + -6.00*e31^e12 + -6.00*e03^e12
>>
E1: e3ga vector
pause

Bibliography

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Line geometry in terms of the null geometric algebra over $\mathbb{R}^{3,3}$, and application to the inverse singularity analysis of generalized stewart platforms.
In Leo Dorst and Joan Lasenby, editors, *Guide to Geometric Algebra in Practice*, pages 253–272. Springer London, 2011.
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`http://dx.doi.org/10.1007/978-0-85729-811-9_13`.