Projective Transformations as Versors

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1 The Space of Lines $\mathbb{R}^{3,3}$

Take the homogeneous model (or the conformal model) V^4 of 3D Euclidean space and form the lines. Using e_0 for the vector representing the point at the origin, and bold for Eulicdean vectors, a line through $p = e_0 + \mathbf{p}$ in the direction $\mathbf{u}$ is formed as:

$$L = p \wedge \mathbf{u} = e_0 \wedge \mathbf{u} + \mathbf{p} \wedge \mathbf{u}.$$

These are the Plücker coordinates, on a basis $\{e_0\mathbf{e}_1, e_0\mathbf{e}_2, e_0\mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$. This is a basis for bivectors in the homogeneous model, of dimension $\binom{4}{2} = 6$. We then write it as basis 'vectors'

$$\{\ e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\ \}.$$

A bivector A of $\bigwedge^2(V^4)$ is a line iff

$$A \wedge A = 0,$$

which is called the Grassmann-Plücker relation. This effectively evaluates to the purely Euclidean relation $\mathbf{u} \wedge (\mathbf{p} \wedge \mathbf{u}) = 0$. In terms of coordinates l_{ij} of L this relation gives:

$$l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0 \tag{1}$$

This is called the (Grassmann-)Plücker condition of the line. It only depends on the outer product in V^4.

We now switch to the 6-dimensional space of the Plücker coordinates. We use it in a homogeneous fashion, as a 5D projective space: multiples of vectors represent the same geometry (at least, positionally, if we want to represented weighted elements the homogeneous freedom becomes interpretable). In that space, the lines are represented by all vectors l satisfying eq.(1). That 4-D manifold of vectors is called the *Klein quadric* when considered in the 5D projective space.

We are going to turn this 6D space into a metric space, to do geometric algebra. That means we need to define a metric for it, and we let that be inspired by the non-metric nature of V^4, in a clever correspondence (see [1]):

$$A \cdot B \equiv [A \wedge B],$$

with the inner product defined between elements of the 6D space on the left, defined in terms of a bracket taken in the Grassmann algebra $\bigwedge(V^4)$ on the right. That bracket is effectively the dual with the pseudoscalar $e_0\mathbf{I}_3$ of V^4, but since no metric for V^4 is given it just means: find out what part of $A\,B$ is proportional to the pseudoscalar and return the proportionality factor. It therefore only uses ratios of volumes in V^4, and hence volume ratios in actual 3-space, to induce a metric of lines (thus the metric issue of what to make of e_0^2 is avoided).

[1] proves that this metric is nondegenerate: if $A \cdot B = 0$ for all $B \in \bigwedge^2(V^4)$, then $A = 0$. Under this metric, lines are therefore precisely identified with the *null vectors* in the 6D space. Moreover, the 6D space has the metric structure of $\mathbb{R}^{3,3}$.

An orthonormal basis to show off the $\mathbb{R}^{3,3}$ nature is:

$$\{a_+, b_+, c_+, a_-, b_-, c_-\} = \left\{ \frac{e_{01} + e_{23}}{\sqrt{2}}, \frac{e_{02} + e_{31}}{\sqrt{2}}, \frac{e_{03} + e_{12}}{\sqrt{2}}, \frac{e_{01} - e_{23}}{\sqrt{2}}, \frac{e_{02} - e_{31}}{\sqrt{2}}, \frac{e_{03} - e_{12}}{\sqrt{2}} \right\}.$$

We have $a_+^2 = b_+^2 = c_+^2 = 1$ and $a_-^2 = b_-^2 = c_-^2 = -1$. As a practical insight speeding up hand computations: any repeated index in a product leads to a zero contribution to the inner product.

The inner product multiplication table on the unit basis is:

$\cdot$	a_+	b_+	c_+	a_-	b_-	c_-
a_+	1	0	0	0	0	0
b_+	0	1	0	0	0	0
c_+	0	0	1	0	0	0
a_-	0	0	0	-1	0	0
b_-	0	0	0	0	-1	0
c_-	0	0	0	0	0	-1

Note that these orthonormal basis vectors for $\mathbb{R}^{3,3}$ do not represent lines - they are clearly not null. The 'standard' null basis of eq.(1)

$$\{\ e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}\ \}$$

does consist of lines: three orthogonal lines through the origin, and three orthogonal lines at infinity (orthogonal great circles on the celestial sphere). Algebraically, the null basis elements may be grouped into two 3D null subspaces. The 3D subspace span$\{e_{01}, e_{02}, e_{03}\}$ is null: any two vectors in it are orthogonal to each other. The same is true for span$\{e_{23}, e_{31}, e_{12}\}$.

The multiplication table on the null basis is:

$\cdot$	e_{01}	e_{02}	e_{03}	e_{23}	e_{31}	e_{12}
e_{01}	0	0	0	1	0	0
e_{02}	0	0	0	0	1	0
e_{03}	0	0	0	0	0	1
e_{23}	1	0	0	0	0	0
e_{31}	0	1	0	0	0	0
e_{12}	0	0	1	0	0	0

This can be encoded as a metric matrix $[M]$.

Geometrically, the inner product of two finite lines is proportional to the volume spanned by their direction vectors and separation vector; the inner product of two infinite lines is zero; the inner product of a finite and infinite line is the volume spanned at infinity by their directions (a 1-direction and a 2-direction). With $E[\cdot]$ the embedding function of lines and the 3D volume bracket denoted as $[\cdot]_{3D}$:

$$E[p \wedge \mathbf{u}] \cdot E[q \wedge \mathbf{v}] = [p \wedge \mathbf{u} \wedge q \wedge \mathbf{v}] = [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge (\mathbf{q} - \mathbf{p})] = [\mathbf{u} \wedge (\mathbf{p} - \mathbf{q}) \wedge \mathbf{v}]_{3D}$$

and

$$E[p \wedge \mathbf{u}] \cdot E[\mathbf{v} \wedge \mathbf{w}] = [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}] = [\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}]_{3D}.$$

The inner product of two intersecting lines, finite or infinite, is zero.

In V^4, the dual of a line would be produced by switching moment and direction components (for instance [[[**example**]]]). Dualization then amounts to the swapping of the coordinates:

$$l_{01} \leftrightarrow l_{23}, \quad l_{02} \leftrightarrow l_{31}, \quad l_{03} \leftrightarrow l_{12}.$$

(It may be achieved in a metric way in V^4 by taking $e_0^2 = 1$ and $I_4 = e_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, and the defining $A^* = A/I_4$; which is inelegant, but unambiguous.) This also extends naturally to non-lines on the same basis eq.(1), and hence to $\mathbb{R}^{3,3}$ space. One may verify that $(A^*)^* = A$, so that undualization is the same as dualization.

Having thus used the structure of V^4 to define a basis and a metric for $\mathbb{R}^{3,3}$, we need no longer consider it, but can focus fully on the line space $\mathbb{R}^{3,3}$.

2 Interpretation of Null Blades in $\mathbb{R}^{3,3}$

The blades in $\mathbb{R}^{3,3}$ have to contain null vectors to represent collections of lines. (If they do not, we have no geometrical interpretation for them.) If a blade contains a null vector, it is a null blade. Therefore we are interested in null blades in $\mathbb{R}^{3,3}$. [1] makes an inventory, and displays the geometry.

- $\mathbb{R}^{0,0,1}$: a line (possibly ideal, this applies to all below)
- $\mathbb{R}^{1,1,0}$: a pair of skew lines (one of which can be ideal)
- $\mathbb{R}^{0,0,2}$: a 2D pencil of lines through one point (possibly ideal)
- $\mathbb{R}^{2,1,0}$ or $\mathbb{R}^{1,2,0}$: a regulus pencil: a family of lines ruling a hyperboloid of revolution.
- $\mathbb{R}^{1,1,1}$: two pencils with one common line
- $\mathbb{R}^{0,0,3}$: all lines through a common point, or all lines in a common plane

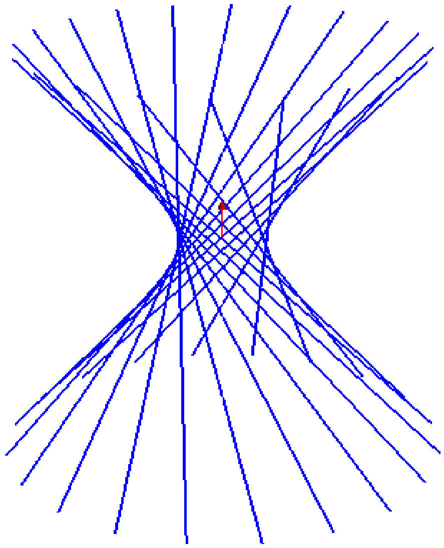
Example of the regulus pencil: Take the blade $P_1 \wedge P_2 \wedge N$, with $1 = P_1 \cdot P_1 = P_2 \cdot P_2 = -N \cdot N$, and the vectors orthogonal. (An example is $(e_{01} + e_{23}) \wedge (e_{02} + e_{31}) \wedge (e_{03} - e_{12})$.) Then a line in this is a linear combination of those components, that is null. So $L = \alpha P_1 + \beta P_2 + \gamma N$, while $0 = \alpha^2 + \beta^2 - \gamma^2$.

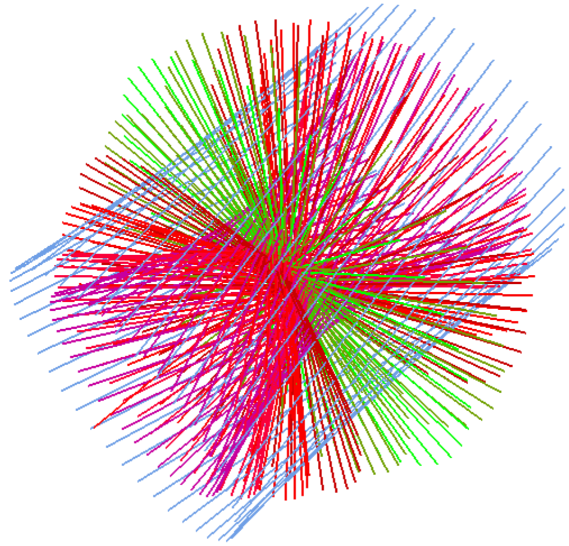
When $\gamma = 0$, then $\alpha^2 = -\beta^2$, so this does not lead to a solution. Setting $\gamma = 1$ to set the weight of L, we find that $\alpha^2 + \beta^2 = 1$. It follows that (modulo weight) there is a 1-parameter family $L = \cos(\phi)P_1 + \sin(\phi)P_2 + N$ that satisfies the equation. Each of these lines has the same non-zero inner product with the axis line $A =$ [[[**? related to direction of** N]]]. In the example, that is the line e_{03}.

2.1 Non-lines containing lines: the unit basis elements

Some impression of the meaning of the non-line elements like $E = e_{03} + e_{12}$ (this is P_3 from the unit basis) can be obtained by probing them with lines. Taking a line $L = (e_0 + \mathbf{d}) \wedge \mathbf{u}$ as a probe, with $\mathbf{u}$ a unit vector and $\mathbf{d} \cdot \mathbf{u} = 0$, we can ask when the dot product of such a line L with E equals zero. We find;

$$\begin{aligned}
0 &= L \cdot E \\
&= [e_0\mathbf{u}\mathbf{e}_1\mathbf{e}_2 + \mathbf{d}\mathbf{u}e_0\mathbf{e}_3] \\
&= \mathbf{u} \cdot \mathbf{e}_3 + (\mathbf{d} \wedge \mathbf{u} \wedge \mathbf{e}_3)^* \\
&= \mathbf{e}_3 \cdot \big(\mathbf{u} + (\mathbf{d} \wedge \mathbf{u})^*\big) \\
&= \mathbf{e}_3 \cdot \big(\mathbf{u}\,(1 - \mathbf{d}^*)\big).
\end{aligned}$$





The operation $(1 - \mathbf{d}^*)$ is rotation and scaling in the plane perpendicular to $\mathbf{d}$, over a specific angle determined by the magnitude of $\mathbf{d}$ (to be precise, its tangent is $\|\mathbf{d}\|$; the scaling is over $\sqrt{1 + \mathbf{d}^2}$). Lines for which this operation applied to $\mathbf{u}$ results a vector without $\mathbf{e}_3$ component are admitted.

- For a $\mathbf{d}$ perpendicular to the e_{03} line, this implies a rotation symmetry around the e_{03} axis. At a fixed distance $||\mathbf{d}\|$, a line with a certain angle relative to this axis as allowed, and may then be rotated, giving a regulus. The tilt of $\mathbf{u}$ is zero when $\mathbf{d} = 0$ (giving a pencil of lines through the origin), and gradually increases to $\pi/2$ as $\|\mathbf{d}\| \to \infty$. At infinite distance, the lines of the regulus are parallel to the axis. We therefore obtain at least this nested family of reguli.

- For $\mathbf{d}$ not perpendicular to the e_{03} axis, it is rescaled for the same $\mathbf{u}$ [[[**as what, I do not understand this**]]]. Effectively, the change of $\mathbf{d}$ is precisely correct to translate each of the the just-derived families along the $\mathbf{e}_3$ axis over an arbitrary amount. This implements the obvious translation symmetry of $e_{03} + e_{12}$ into its solutions. (One can derive this easily: translate the support vector $\mathbf{d}$ by $\tau\mathbf{e}_3$ to become a point on the translated line $(e_0 + \mathbf{d} + \tau\mathbf{e}_3) \wedge \mathbf{u} = (e_0 + (\mathbf{d} + \tau(\mathbf{e}_3 \wedge \mathbf{u})/\mathbf{u})\mathbf{u}$. Substiting the new support vector for $\mathbf{d}$ in the equation, we find $\mathbf{e}_3 \cdot (\mathbf{u}(1 + (\mathbf{d} + \tau(\mathbf{e}_3 \wedge \mathbf{u})/\mathbf{u})^*)) = -\tau\mathbf{e}_3 \cdot (\mathbf{e}_3 \wedge \mathbf{u})^* = -\tau(\mathbf{e}_3 \wedge \mathbf{e}_3 \wedge \mathbf{u})^* = 0$, so that the shifted line indeed also satisfies the equation.)

Ultimately, there is therefore a 3-parameter family of lines that has zero dot product with the non-line element $e_{03} + e_{12}$: it can be parametrized by direction $\mathbf{u}$ (giving the distance $\mathbf{d}$), and by rotation around and translation along e_{03}.

We can also ask when the element $E = e_{03} + e_{12}$ is the direct representation of a set of lines. In that case, we need to solve $L \wedge E = 0$ for a line $L = (e_0 + \mathbf{d}) \wedge \mathbf{u}$. This leads to:

$$
\begin{aligned}
0 &= L \wedge E \\
&= \big(u_1 e_{01} + u_2 e_{02} + u_3 e_{03} + (\mathbf{du})_{23} e_{23} + (\mathbf{du})_{31} e_{31} + (\mathbf{du})_{12} e_{12}\big) \wedge \big(e_{03} + e_{12}\big)
\end{aligned}
$$

It is quickly apparent from the generated 2-blades and their coefficients that $u_1 = u_2 = (\mathbf{du})_{23} e_{23} = (\mathbf{du})_{31} e_{31} = 0$, and what remains is:

$$0 = L \wedge E = (u_3 - (\mathbf{du})_{12})\, e_{01} \wedge e_{23} = \mathbf{e}_3 \cdot \big(\mathbf{u}\,(1 + \mathbf{d}^*)\big)\, e_{01} \wedge e_{23}.$$

It follows that the "outer product null space" consists of families of reguli ruled by the 'conjugated' lines relative to those of the "inner product null space", with a particular $\mathbf{u}$ now occurring at $-\mathbf{d}$ rather than at $\mathbf{d}$. Starting from a $\mathbf{d}$ perpendicular to $\mathbf{e}_3$, the 'direct regulus family' and the 'dual regulus family' move in opposite ways along the e_{03} axis as $\mathbf{d}$ varies.

2.2 Regulus pencil

[[[**something like this, order is now confusing**]]] Since each of the elements P_1, P_2 and N are such dual "regulus generators", the lines in $P_1 \wedge P_2 \wedge N$ are dual to the intersection of the generated reguli.

It remains to show that any non-null element of the space can be brought in such a form of 'orthogonal' direction and moment components, so that the computation above computation on $e_{03} + e_{12}$ is prototypical.

3 Projective Transformations as Versors

[1] claims that any special linear transformation of V^4 is isomorhic to a special orthogonal transformation of $\mathbb{R}^{3,3}$, by a reasoning that I do not quite follow. Here's mine:

A linear transformation on V^4 induces an outermorphism. This outermorphism affects its bivectors and therefore the vectors of $\mathbb{R}^{3,3}$ they map to. Then the inner product may be affected. We compute (denoting f for the mapping and the induced mappings):

$$f(A) \cdot f(B) = [f(A) \wedge f(B)] = [f(A \wedge B)] = \det(f)\,[A \wedge B] = \det(f)\, A \cdot B.$$

It follows that when we restrict ourselves to transformations of f for which $\det(f) = 1$, their representation in $\mathbb{R}^{3,3}$ is an orthogonal transformation. The converse is not true: [1] claims that the odd orthogonal transformations of $\mathbb{R}^{3,3}$ swap points and planes whereas the linear transformations of V^4 do not, so that some care is required. Check this, it appears we need to focus on even orthogonal trasnformations of $\mathbb{R}^{3,3}$, but will still get all elements of $SL(V^4)$.

"All special projective transformations can be classified by their spinor generators." [1]

3.1 Translation

The 2-blade $T_1 = e_{12} \wedge e_{31}$ is null. It has the following relationships with the null basis vectors:

$$e_{01}\, T_1 - T_1 e_{01} = 0, \quad e_{02}\, T_1 - T_1\, e_{02} = 2e_{12}, \quad e_{03}\, T_1 - T_1\, e_{03} = -2e_{31},$$
$$e_{23}\, T_1 - T_1 e_{23} = 0, \quad e_{31}\, T_1 = 0, \quad e_{12}\, T_1 = 0.$$

Exponentiating T_1 gives $V_1 = \exp(\tau T_1/2) = 1 + \frac{1}{2}\tau T_1$, and application of this rotor yields:

$$V_1\, e_{02}\, \widetilde{V}_1 = e_{02} + \tau e_{12} \leftrightarrow (e_0 + \tau \mathbf{e}_1) \wedge \mathbf{e}_2, \quad V_1\, e_{03}\, \widetilde{V}_1 = e_{03} - \tau e_{31} \leftrightarrow (e_0 + \tau \mathbf{e}_1) \wedge \mathbf{e}_3,$$

and leaves the other null basis elements invariant. Therefore $V_1 = \exp(\tau T_1/2)$ represents translation over $\tau \mathbf{e}_1$, which should have precisely these properties. Its matrix is:

$$[W_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\tau & 0 & 1 & 0 \\ 0 & \tau & 0 & 0 & 0 & 1 \end{array}\right].$$

3.2 Perspective Transformations

In a similar manner to translations, one can compute perspective transformations. The 2-blade $F_1 = e_{02} \wedge e_{03}$ is null. It has the following relationships with the null basis vectors:

$$e_{23}\,F_1 - F_1 e_{23} = 0, \quad e_{12}\,F_1 - F_1\,e_{12} = 2e_{02}, \quad e_{31}\,F_1 - F_1\,e_{31} = -2e_{03},$$

$$e_{01}\,F_1 - F_1 e_{01} = 0, \quad e_{02}\,F_1 = 0, \quad e_{03}\,F_1 = 0.$$

Exponentiating F_1 gives $W_1 = \exp(fF_1/2) = 1 + \frac{1}{2}fF_1$, and application of this rotor yields:

$$W_1\,e_{31}\,\widetilde{W_1} = e_{31} - fe_{03} \leftrightarrow -(fe_0 + \mathbf{e}_1) \wedge \mathbf{e}_3, \quad W_1\,e_{12}\,\widetilde{W_1} = e_{12} + fe_{02} \leftrightarrow (fe_0 + \mathbf{e}_1) \wedge \mathbf{e}_2,$$

and leaves the other null basis elements invariant. Its matrix is:

$$[W_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & f \\ 0 & 0 & 1 & 0 & -f & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

3.3 Directional Scaling

Directional scaling bivectors should square to a non-zero scalar, say 1. For a scaling in the $\mathbf{e}_1$ direction, use $S_1 = e_{01} \wedge e_{23}$. It commutes with all null basis vectors except:

$$e_{01} \rfloor S_1 = -e_{01}, \quad e_{23} \rfloor S_1 = e_{23}.$$

Applying the versor $V_1^S = \exp(\gamma\,S_1/2)$ with $S_1^2 = 1$ to a vector v gives $v - 2\sinh(\gamma/2) V_1^S (v \rfloor S_1)$. It leaves all null basis elements unchanged except:

$$e_{01} \mapsto e^{\gamma} e_{01}, \quad e_{23} \mapsto e^{-\gamma} e_{23}.$$

The e_{01} scaling is as expected, and the e_{23} scaling is to keep the inner product $e_{01} \cdot e_{23}$ invariant. Its matrix is:

$$[L] = \left[\begin{array}{ccc|ccc} e^{\gamma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & e^{-\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

3.4 Rotation

The rotation bivector is a bit more involved. The bivector B for rotation in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B = \tfrac{1}{2}(e_{02} \wedge e_{12} - e_{03} \wedge e_{31})$$

This is not a blade, and its square is not a scalar, but of the form 'scalar plus quadvector':

$$B^2 = -\tfrac{1}{2}(1 + e_{02} \wedge e_{12} \wedge e_{03} \wedge e_{31}).$$

One may verify that $B^3 = -B$. The algebraic properties of B therefore determine a rotation-like versor, since they allow for a grouping into trigonometric functions:

$$\begin{aligned} e^{-\phi B} &= 1 - \tfrac{1}{1!}\phi B + \tfrac{1}{2!}\phi^2 B^2 - \tfrac{1}{3!}\phi^3 B^3 + \tfrac{1}{4!}\phi^4 B^4 + \cdots \\ &= 1 - \tfrac{1}{1!}\phi B + \tfrac{1}{2!}\phi^2 B^2 + \tfrac{1}{3!}\phi^3 B - \tfrac{1}{4!}\phi^4 B^2 + \cdots \\ &= 1 + B^2 - (\tfrac{1}{1!}\phi - \tfrac{1}{3!}\phi^3 + \cdots)\, B - (1 - \tfrac{1}{2!}\phi^2 + \tfrac{1}{4!}\phi^4 + \cdots)\, B^2 \\ &= 1 - \sin(\phi)\, B + \big(1 - \cos(\phi)\big)\, B^2 \end{aligned}$$

Let us appy this to a vector b 'in' B (such as e_{02} or e_{03}). One may verify that:

$$Bb - bB = -b^\perp, \qquad B^2 b + bB^2 = -b, \qquad BbB = 0.$$

with $b^\perp$ achieved from b by the regular 90-degree rotation sending $\mathbf{e}_2$ to $\mathbf{e}_3$, with as a consequence for the null basis:

$$e_{02}^\perp = e_{03}, \quad e_{03}^\perp = -e_{02}, \quad e_{31}^\perp = -e_{13}^\perp = e_{12}, \quad .e_{12}^\perp = e_{13} = -e_{31}.$$

With that:

$$\begin{aligned} &e^{-\phi B}\, b\, e^{\phi B} \\ &= (1 - sB + (1-c)B^2)\, b\, (1 + sB + (1-c)B^2) \\ &= b - s(Bb - bB) + (1-c)(B^2 b + bB^2) - s^2 BbB + s(1-c)(BbB^2 - B^2 bB) + (1-c)^2 B^2 b B^2 \\ &= \cos(\phi)\, b + \sin(\phi)\, b^\perp. \end{aligned}$$

So $\exp(-\phi B)$ is the versor for a rotation over ϕ, leaving the vectors commuting with B invariant.

The matrix of the $\mathbf{e}_2 \wedge \mathbf{e}_3$ rotation is:

$$[R] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -s & 0 & 0 & 0 \\ 0 & s & c & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{array}\right],$$

where $c = \cos(\phi)$ and $s = \sin \phi)$. As a consequence, $[R]^T\, [M]\, [R] = [M]$, so this is indeed an orthogonal matrix in the $[M]$-metric.

3.5 Squeeze (Lorentz Transformation)

The constituent 2-blade parts of the rotator bivector are pure skews; a rotation combines them an $\mathbf{e}_2$ axis skew and an $\mathbf{e}_3$-axis skew in the same direction; changing the sign of their combination gives a volume preserving 'squeezing' operation.

The bivector B for squeeze in the $\mathbf{e}_2 \wedge \mathbf{e}_3$ plane is:

$$B = \tfrac{1}{2}(e_{02} \wedge e_{12} + e_{03} \wedge e_{31})$$

This is not a blade, and its square is not a scalar, but of the form 'scalar plus quadvector':

$$B^2 = \tfrac{1}{2}(1 + e_{02} \wedge e_{12} \wedge e_{03} \wedge e_{31}).$$

One may verify that $B^3 = B$. The algebraic properties of B therefore determine a rotation-like versor, since they allow for a grouping into hyperbolic functions:

$$e^{\gamma B} = 1 + \sinh(\gamma)\, B + \big(\cosh(\gamma) - 1\big)\, B^2.$$

Let us appy this to a vector b 'in' B (such as e_{02} or e_{03}). One may verify that:

$$Bb - bB = b^{\perp}, \qquad B^2 b + bB^2 = -b, \qquad BbB = 0.$$

with $b^{\perp}$ achieved from b as

$$e_{02}^{\perp} = e_{03}, \quad e_{03}^{\perp} = e_{02}, \quad e_{31}^{\perp} = -e_{12}, \quad e_{12}^{\perp} = -e_{31}.$$

With that, for a vector b 'in'B:

$$\begin{aligned}
&e^{\gamma B}\, b\, e^{-\gamma B} \\
&\quad = (1 + sB + (c-1)B^2)\, b\, (1 - sB + (c-1)B^2) \\
&\quad = b + s(Bb - bB) + (c-1)(B^2 b + bB^2) - s^2 BbB - s(c-1)(BbB^2 - B^2 bB) + (c-1)^2 B^2 bB^2 \\
&\quad = \cosh(\gamma)\, b + \sinh(\gamma)\, b^{\perp}.
\end{aligned}$$

So $\exp(\gamma B)$ is the versor for a squeeze parametrized by γ, leaving the vectors commuting with B invariant.

This keeps volumes invariant, and is orthogonal due to the simultaneous effect on finite and infinite lines. The matrix of the $\mathbf{e}_2 \wedge \mathbf{e}_3$ squeeze shows this clearly:

$$[L] = \left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & ch & sh & 0 & 0 & 0 \\
0 & sh & ch & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & ch & -sh \\
0 & 0 & 0 & 0 & -sh & ch
\end{array}\right],$$

where $ch = \cosh(\gamma)$ and $sh = \sinh(\gamma)$. As a consequence, $[L]^T\,[M]\,[L] = [M]$, so this is indeed an orthogonal matrix in the $[M]$-metric.

3.6 Orthogonality Conditions for Matrices

Writing in block form, we can see when a matrix $[X]$ represents and orthogonal transformation, since it should satisfy $[M]\,[X]^T\,[M]\,[X] = [I]$:

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} B^T C + D^T A & B^T D + D^T B \\ A^T C + C^T A & A^T D + C^T B \end{bmatrix}.$$

Therefore $A^T C$ and $B^T D$ must be skew-symmetric matrices, and $B^T C + D^T A$ should be the identity.

4 2D Projective Geometry

When concentrating on a single plane in space, and considering the transformations that keep the component of the volume perpendicular to it invariant, we can establish an orthogonal representation of the projective transformations in 2D. There are then 8 elementary fundamental transformations: 2 translations, 2 scalings, 2 perspective transformations, 1 rotation and 1 squeeze. Their matrices are:

$$[T] = \left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & \tau_2 & 1 & 0 & 0 \\
0 & 0 & -\tau_1 & 0 & 1 & 0 \\
-\tau_2 & \tau_1 & 0 & 0 & 0 & 1
\end{array}\right], \qquad [P] = \left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & -\delta_2 \\
0 & 1 & 0 & 0 & 0 & \delta_1 \\
0 & 0 & 1 & \delta_2 & -\delta_1 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right],$$

$$[S] = \left[\begin{array}{ccc|ccc} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/\sigma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$[R] = \left[\begin{array}{ccc|ccc} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad [L] = \left[\begin{array}{ccc|ccc} ch & sh & 0 & 0 & 0 & 0 \\ sh & ch & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & ch & -sh & 0 \\ 0 & 0 & 0 & -sh & ch & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

4.1 Cross Ratio

On a line, a cross ratio is preserved under a projective transformation. Here is a way to embed that in the line geometry. On a line L with direction vector $\mathbf{u}$, consider four points p_1, p_2, p_3, p_4. They can be considered as the intersection of L with other lines L_i. Pick those lines to have specific direction vectors, and allow the points to be represented with non-unit weights (since projective transformations can change weights): $L_1 = \alpha_1\, p_1 \wedge \mathbf{v}$, $L_2 = \alpha_2\, p_2 \wedge \mathbf{w}$, $L_3 = \alpha_3\, p_3 \wedge \mathbf{v}$, $L_4 = \alpha_4\, p_4 \wedge \mathbf{w}$. Now transfer these to the line space, labeling them ℓ_i. For the dot product between two lines with both odd or both even labels, we get zero. For the others, we introduce coordinates π_i to label the points p_i along L from some unit weight point p, in amounts of $\mathbf{u}$. Then we compute:

$$\begin{aligned} \ell_i \cdot \ell_j &= [\alpha_i\, p_i \wedge \mathbf{v} \wedge \alpha_j\, p_j \wedge \mathbf{w}] \\ &= \alpha_i \alpha_j\, [(p + \pi_i \mathbf{u}) \wedge \mathbf{v} \wedge (p + \pi_j \mathbf{u}) \wedge \mathbf{w}] \\ &= \alpha_i \alpha_j\, [(\pi_i - \pi_j)\, \mathbf{u} \wedge p \wedge \mathbf{v} \wedge \mathbf{w}] \\ &= (\pi_j - \pi_i)\, \alpha_i \alpha_j\, [e_0 \wedge \mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}] \end{aligned}$$

This is therefore proportional to the signed distance between the points, by the arbitrary point weight, and a rather arbitrary volume-based proportionality constant that is independent of the points.

To achieve an invariance measure involving the distance, we need to divide out the embedding constants. The cross ratio C in V^4 is constructed in line space as:

$$\begin{aligned} C &= \frac{(\ell_1 \wedge \ell_4) \cdot (\ell_2 \wedge \ell_3)}{(\ell_1 \wedge \ell_2) \cdot (\ell_3 \wedge \ell_4)} \\ &= -\frac{(\ell_4 \cdot \ell_3)\,(\ell_1 \cdot \ell_2)}{(\ell_2 \cdot \ell_3)\,(\ell_1 \cdot \ell_4)} \\ &= \frac{(\pi_2 - \pi_1)\,(\pi_4 - \pi_3)}{(\pi_3 - \pi_2)\,(\pi_1 - \pi_4)} \end{aligned}$$

Since C is defined as a ratio of dot products, it is invariant. And since it can be expressed completely in terms of distances in V^4, it is a measurable quantity, independent of the embedding.

References

[1] Zhang and Li, in Guide to Geometric Algebra in Practice, Springer 2011. [[[**supply**]]]