Replies to Leo's notes

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1 2-blades

Theorem 1. *Show that* any *2-blade of* $\mathbb{R}^{3,3}$ *contains at least two* Lines

Lemma 1. *A 2-blade of two intersecting* Lines $\ell_1 \wedge \ell_2$ *contains at least two* Lines*.*

See Leo's notes, 2.2, April 19. Also: per definition.

Lemma 2. *A 2-blade of two skew* Lines $\ell_1 \wedge \ell_2$ *contains at least two* Lines*.*

See Leo's notes, 2.2, April 19. Also: per definition.

Lemma 3. *A 2-blade of one* Line *and a non-*Line $\ell \wedge k$ *contains at least two* Lines*.*

We are looking for Lines. As every null vector is a Line, we have the constraint $x^2 = 0$.

$$
\begin{array}{lll}
 & x \wedge (\ell \wedge k) = 0 & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & x = \alpha\ell + \beta k & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & (\alpha\ell + \beta k)^2 = 0 & \\
\Leftrightarrow & \alpha^2\ell^2 + \alpha\beta\,(\ell \cdot k) + \beta^2 k^2 = 0 & \\
\Leftrightarrow & \alpha\beta\,(\ell \cdot k) + \beta^2 k^2 = 0 & \\
\Leftrightarrow & \alpha\,(\ell \cdot k) = -\beta k^2 & \\
\Leftrightarrow & \alpha = -\frac{\beta k^2}{\ell \cdot k} & \text{or} \quad \beta = -\frac{\alpha(\ell \cdot k)}{k^2}
\end{array}
$$

Lemma 4. *A 2-blade of two non-*Lines $k_1 \wedge k_2$ *contains at least two* Lines*.*

Again, x should be a Line. We have the same constraint $x^2 = 0$.

$$
\begin{array}{lll}
 & x \wedge (k_1 \wedge k_2) = 0 & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & x = \alpha k_1 + \beta k_2 & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & (\alpha k_1 + \beta k_2)^2 = 0 & \\
\Leftrightarrow & \alpha^2 k_1^2 + \alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2 = 0 & \\
\Leftrightarrow & \alpha \neq 0 \text{ and } \beta = \frac{\imath\left(\sqrt{3}\alpha k_1 \pm \imath\alpha k_1\right)}{2k_2} & \text{or} \quad \beta \neq 0 \text{ and } \alpha = \frac{\imath\left(\sqrt{3}\beta k_2 \pm \imath\beta k_2\right)}{2k_1}
\end{array}
$$

This is according to Wolfram Alpha... I couldn't get further than this:

$$
\begin{array}{lll}
& x \wedge (k_1 \wedge k_2) = 0 & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & x = \alpha k_1 + \beta k_2 & \text{and} \quad x^2 = 0 \\
\Leftrightarrow & (\alpha k_1 + \beta k_2)^2 = 0 & \\
\Leftrightarrow & \alpha^2 k_1^2 + \alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2 = 0 & \\
\Leftrightarrow & \alpha\beta(k_1 \cdot k_2) = -\alpha^2 k_1^2 - \beta^2 k_2^2 & \\
\Leftrightarrow & \alpha = -\frac{\alpha^2 k_1^2 + \beta^2 k_2^2}{\beta k_1 \cdot k_2} & \text{or} \quad \beta = -\frac{\alpha^2 k_1^2 + \beta^2 k_2^2}{\alpha k_1 \cdot k_2} \\
\Leftrightarrow & \alpha^2 = -\frac{\alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2}{k_1^2} & \text{or} \quad \beta^2 = -\frac{\alpha\beta(k_1 \cdot k_2) + \alpha^2 k_1^2}{k_2^2}
\end{array}
$$