Visualization of the projective line geometry for geometric algebra

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May 31, 2012

Abstract

This thesis describes a method for computationally recognizing the geometrical interpretation of the elements of the model of projective line geometry in geometric algebra. These different interpretations are not merely recognizable by the grade of the elements, or the absense or presence of some basis vector, as can be done for conformal geometric algebra.

GAViewer, a graphical calculator for the Euclidean and conformal models of geometric algebra, is extended to implement the projective line geometry and this visualization.

Keywords: Geometric algebra, projective geometry, Plücker coordinates, conformal geometry.

Contents

1 Introduction

Just as linear algebra, geometric algebra is an algebra over a given vector space $\mathbb{R}^n$. The difference lies in its operations; whereas linear algebra relies on matrix manipulations which cannot always be inverted, geometric algebra's base operation is the geometric product. From this invertible product, one can deduce the inner product, known from linear algebra, and the outer product. The outer product grants the user access to the exterior algebra $\bigwedge \mathbb{R}^n$. The outer product of $\mathbf{A}, \mathbf{B} \in \bigwedge \mathbb{R}^n : \mathbf{A} \wedge \mathbf{B}$ represents the set of elements that are linear combinations of its operands $\{\alpha\mathbf{A} + \beta\mathbf{B} \mid \alpha, \beta \in \mathbb{R}\}$

There are already several models of geometry in use in the geometric algebra community, of which the conformal model is the most popular and widely known [3]. Although usable in many cases, the transformations of the conformal model are a subset of those of the projective model. The projective transformations are an important class of operations within computer vision and computer graphics.

GAViewer is a visualization and computing tool for the 3-dimensional Euclidean and conformal models of geometric algebra, developed by Daniel Fontijne at the University of Amsterdam [4]. It allows the user to perform calculations in geometric algebra, and shows the results, both numerical as well as graphical. Moreover, it allows the user to rotate, translate and zoom the viewport, as well as to manipulate the displayed elements. The variables in which they are stored are automatically updated, and, if desired by the user, other objects that are parameterized by the manipulated object, may be updated dynamically as well.

Recently, Li and Zhang [5] have found a way to model projective geometry, using a 6 dimensional representation space $\mathbb{R}^{3,3}$ with a special metric, which allows three basis vectors to square to -1. Using Plücker coordinates [5,6], lines in 3-dimensional Euclidean space $\mathbb{E}^3$ are represented in the representation space by vectors $\mathbf{v}^2 = \mathbf{v}\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = 0$. As a consequence, there are also vectors in the representation space $\mathbf{w}^2 \neq 0$ which do not represent lines in $\mathbb{E}^3$. Different geometrical objects are generated by the outer product over lines and non-lines. For example, for two intersecting lines ℓ_1, ℓ_2, the outer product $\ell_1 \wedge \ell_2$ represents a pencil, the collection of all lines that are in the same plane as ℓ_1 and ℓ_2, and pass through the same point. Taking the outer product of this pencil with a third intersecting line that does not lie in the same plane, results in a bundle; the set of all lines intersecting in a certain point. This could be used to represent points in our space of lines.

In their article, Li and Zhang have not discussed what each element in their algebra might represent. Barrau [1,2] and Pottmann and Wallner [6,7] have investigated the objects that can be represented in a 6-dimensional space with Plücker coordinates, both using a different algebra from ours. Pottmann and Wallner use linear algebra, while Barrau's algebra has less expressive power.

1.1 Research question

1.2 Document structure

In section 2 of this thesis, the basics of our mathematical toolkit are explained. This includes a superficial introduction to geometric algebra and its conformal model, as well as a description of the Plücker space and how it is represented in geometric algebra.

Section 3 describes the problem of recognizing the geometrical interpretation of elements of the Plücker space. In subsection 3.2, a method for achieving this classification is given. This method is implemented as a wrapper for GAViewer in section 4.

This documents concludes at section 5 with a discussion of this work as well as possible future work, and a bibliography.

2 Projective geometry and geometric algebra

2.1 Geometric algebra

Paraphrase Moos's work?

2.2 Projective geometry

2.3 Li and Zhang's [5] trick

Needs a different title.

3 Classifying elements of different geometric interpretations

3.1 The problem

In need of better section names.

3.2 The solution

In need of better section names.

4 Implementation

5 Conclusion

References

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