

# Replies to Leo's notes

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## 1 2-blades

**Question 1.** *Show that any 2-blade of  $\mathbb{R}^{3,3}$  contains at least two Lines.*

This can be split up in 4 distinct cases.

1. *A 2-blade of two intersecting Lines  $\ell_1 \wedge \ell_2$  contains at least two Lines.*

See Leo's notes, 2.2, April 19. Also: per definition.

2. *A 2-blade of two skew Lines  $\ell_1 \wedge \ell_2$  contains at least two Lines.*

See Leo's notes, 2.2, April 19. Also: per definition.

3. *A 2-blade of one Line and a non-Line  $\ell \wedge k$  contains at least two Lines.*

We are looking for Lines. As every null vector is a Line, we have the constraint  $x^2 = 0$ .

$$\begin{aligned} x \wedge (\ell \wedge k) &= 0 & \text{and } x^2 &= 0 \\ \Leftrightarrow x &= \alpha\ell + \beta k & \text{and } x^2 &= 0 \\ \Leftrightarrow (\alpha\ell + \beta k)^2 &= 0 \\ \Leftrightarrow \alpha^2\ell^2 + \alpha\beta(\ell \cdot k) + \beta^2k^2 &= 0 \\ \Leftrightarrow \alpha\beta(\ell \cdot k) + \beta^2k^2 &= 0 \\ \Leftrightarrow \alpha(\ell \cdot k) &= -\beta k^2 \\ \Leftrightarrow \alpha &= -\frac{\beta k^2}{\ell \cdot k} \text{ and } \beta \in \mathbb{R} & \text{or } \alpha \in \mathbb{R} \text{ and } \beta &= -\frac{\alpha(\ell \cdot k)}{k^2} \end{aligned}$$

4. *A 2-blade of two non-Lines  $k_1 \wedge k_2$  contains at least two Lines.*

Again,  $x$  should be a Line. We have the same constraint  $x^2 = 0$ .

$$\begin{aligned} x \wedge (k_1 \wedge k_2) &= 0 & \text{and } x^2 &= 0 \\ \Leftrightarrow x &= \alpha k_1 + \beta k_2 & \text{and } x^2 &= 0 \\ \Leftrightarrow (\alpha k_1 + \beta k_2)^2 &= 0 \\ \Leftrightarrow \alpha^2 k_1^2 + \alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2 &= 0 \\ \Leftrightarrow \alpha \neq 0 \text{ and } \beta &= \frac{\imath(\sqrt{3}\alpha k_1 \pm \imath\alpha k_1)}{2k_2} & \text{or } \beta \neq 0 \text{ and } \alpha &= \frac{\imath(\sqrt{3}\beta k_2 \pm \imath\beta k_2)}{2k_1} \end{aligned}$$

This is according to Wolfram Alpha... I couldn't get further than this:

$$\begin{aligned}
& x \wedge (k_1 \wedge k_2) = 0 & \text{and } x^2 = 0 \\
\Leftrightarrow & x = \alpha k_1 + \beta k_2 & \text{and } x^2 = 0 \\
\Leftrightarrow & (\alpha k_1 + \beta k_2)^2 = 0 \\
\Leftrightarrow & \alpha^2 k_1^2 + \alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2 = 0 \\
\Leftrightarrow & \alpha\beta(k_1 \cdot k_2) = -\alpha^2 k_1^2 - \beta^2 k_2^2 \\
\Leftrightarrow & \alpha = -\frac{\alpha^2 k_1^2 + \beta^2 k_2^2}{\beta k_1 \cdot k_2} & \text{or } \beta = -\frac{\alpha^2 k_1^2 + \beta^2 k_2^2}{\alpha k_1 \cdot k_2} \\
\Leftrightarrow & \alpha^2 = -\frac{\alpha\beta(k_1 \cdot k_2) + \beta^2 k_2^2}{k_1^2} & \text{or } \beta^2 = -\frac{\alpha\beta(k_1 \cdot k_2) + \alpha^2 k_1^2}{k_2^2}
\end{aligned}$$

I don't see how to get all  $\alpha$ 's or  $\beta$ 's on one side of the  $=$ ...

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**Question 2.** Show that  $\ell_1 \wedge \ell_2 \wedge \ell_3$ , for relatively skew Lines  $\ell_1, \ell_2, \ell_3$  is a Regulus Pencil.