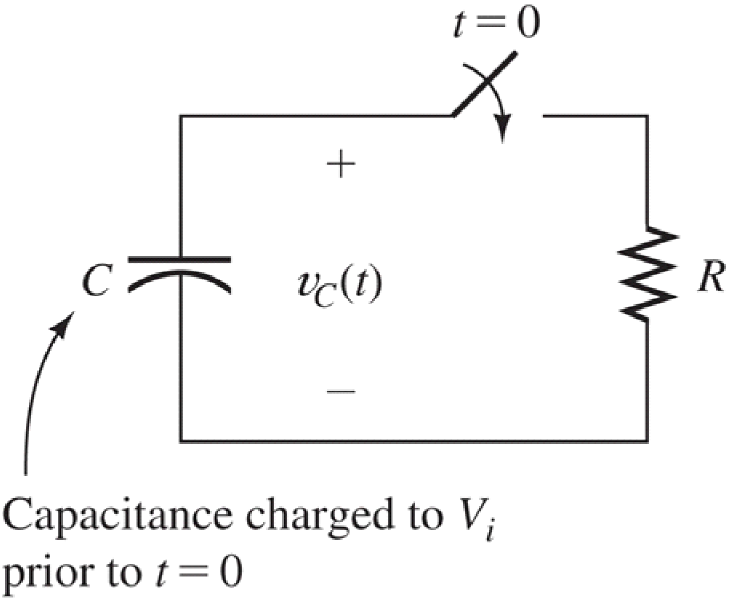
P4.2

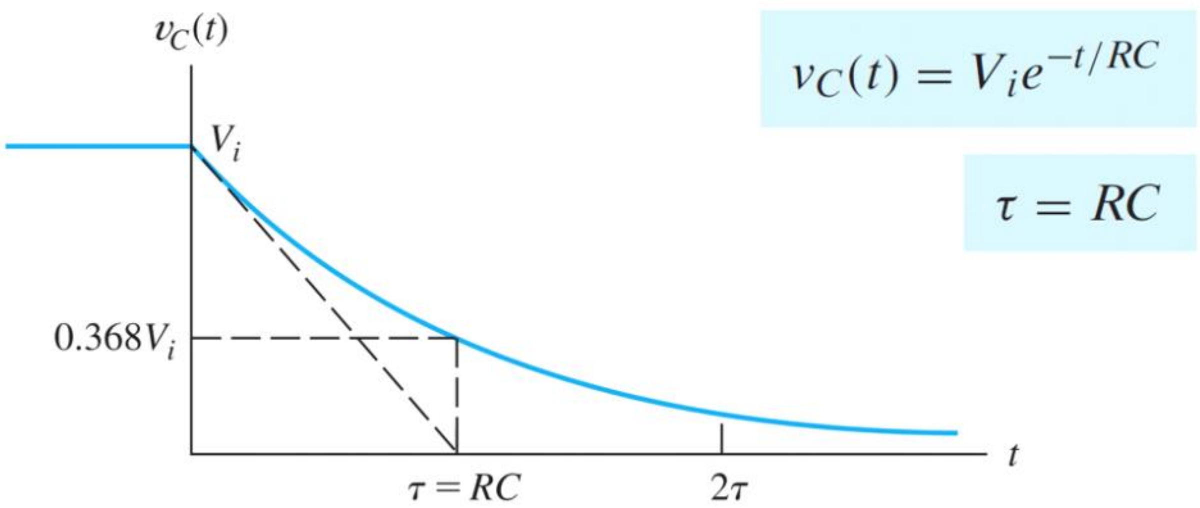
***P4.2.** The dielectric materials used in real capacitors are not perfect insulators. A resistance called a leakage resistance in parallel with the capacitance can model this imperfection. A 100-μF capacitor is initially charged to 100 V. We want 90 percent of the initial energy to remain after one minute. What is the limit on the leakage resistance for this capacitor?

Actually, if we think of this situation, we can draw a circuit that is very much like what we've seen:



Now the switch is not actually present, but we will use it to signify the time (t=0 s) at which we stop charging the capacitor.

Since we know the circuit, we can now draw the voltage v_C(t) (look at Slide 11 in the NET4 Week A slideset):



For the parameters, we were given the values:
V_i = 100 V
C = 100 μF

Furthermore, there's talk of the **energy** stored in a Capacitor. What's the formula for that again?

Aha - Slide 8 in the Week A slideset:

Energy	$W(t) = \frac{1}{2}Cv^2(t)$
Energy stored in:	Electric field

Looking at this formula, and also at the voltage curve above, we realize that stored energy is also decaying from t=0 s, since it is the square of the decaying v_C(t).

We can write the stored energies at t=0 s and at t=60 s as follows:

- t=0 s: $W_0 = \frac{C}{2}v_C^2(0) = \frac{C}{2}V_i^2$ = 0.5 J
- t=60 s: $W_{60} = \frac{C}{2}v_C^2(60) = \frac{C}{2}(V_i e^{-60/RC})^2$

The only unknown here is R. We can now solve for R, since we know that $W_{60} = 0.9W_0$:

$\frac{C}{2}(V_i e^{-60/RC})^2 =$ 0.45 J

$\frac{C}{2}V_i^2 e^{-120/RC} =$ 0.45 J

$e^{-120/RC} =$ 0.9

$\frac{-120}{RC}$ = ln(0.9)

R = -120/[C*ln(0.9)] = 11.4 MΩ

P4.5

***P4.5.** At $t = 0$, a charged 20-μF capacitance is connected to a voltmeter, as shown in Figure P4.5. The meter can be modeled as a resistance. At $t = 0$, the meter reads 50 V. At $t = 30$ s, the reading is 25 V. Find the resistance of the voltmeter.

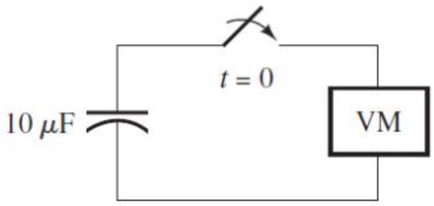


Figure P4.5

The circuit here is actually much the same as that in P4.2. And so are the formulas :-)
The main formula is

$v_C(t) = V_S e^{-t/\tau}$

We have voltage values at t=0 s and at t=30 s, so we can write:

- using the formula at t=0 s: V_S = 50 V
- using the formula at t=30 s:
  25 = $50e^{-30/RC}$
  0.5 = $e^{-30/RC}$
  30/RC = -ln(0.5) = 0.69
  R = 30/(0.69*C) = 2.17 MΩ

P4.7

P4.7. Given an initially charged capacitance that begins to discharge through a resistance at $t = 0$, what percentage of the initial voltage remains at two time constants? What percentage of the initial stored energy remains?

For the discharging capacitor, we know the voltage formula by now:

$v_C(t) = V_S e^{-t/\tau}$

and remember the stored-energy formula:

W = $\frac{C}{2} v_C^2(t)$

We are not given any R and C values, but as you would have guessed, Hambley is not asking for the impossible. Let's see what we can do.

The problem text wants us to compare the times t=0 s and t=2τ s:

- at t=0 s
  - $v_C = V_S$, and as a consequence:
  - W = $\frac{C}{2} V_S^2$
- at t=2τ s
  - $v_C = V_S(e^{-2}) = 0.135V_S$, and as a consequence:
  - W = $\frac{C}{2} 0.135^2 V_S^2$

Now I think you will be able to conclude about the percentages at t=2τ s:

- for voltage v_C, 13.5% remains
- for stored energy W, 0.135^2 = 1.83% remains

P4.10

P4.10. We know that a 50-μF capacitance is charged to an unknown voltage V_i at $t = 0$. The capacitance is in parallel with a 3-kΩ resistance. At $t = 200$ ms, the voltage across the capacitance is 5 V. Determine the value of V_i.

Looking at the text, we identify this as (again) a discharging capacitor. We therefore know the voltage-versus-time formula:

$v_C(t) = V_S e^{-t/\tau}$

with

$\tau = RC$ = 3000*0.00005=0.15 s

and we were told

v_C(200 ms) = 5 V

Inserting all these values into the formula, we get

$5 = V_i e^{-0.2/0.15}$

$V_i = 5/e^{-0.2/0.15} = 5*e^{0.2/0.15}$= 18.97 V

P4.11

P4.11. We know that the capacitor shown in Figure P4.11 is charged to a voltage of 20 V prior to

$t = 0$. **a.** Find expressions for the voltage across the capacitor $v_C(t)$ and the voltage across the resistor $v_R(t)$ for all time. **b.** Find an expression for the power delivered to the resistor. **c.** Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered. **d.** Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

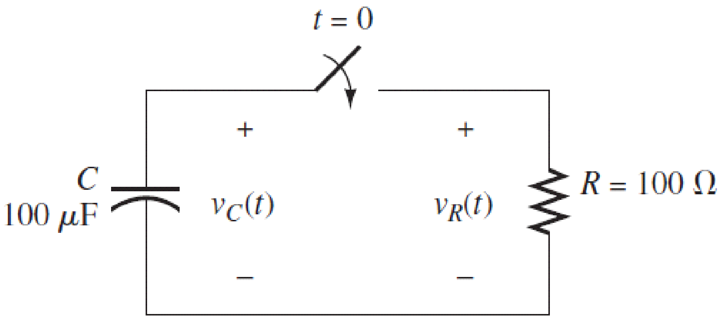


Figure P4.11

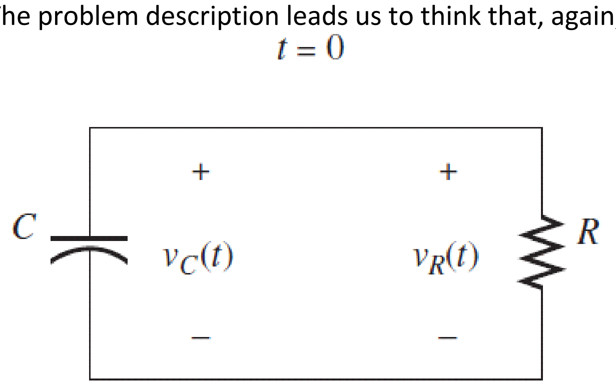
a. You probably know by now:
   $v_C(t) = V_S e^{-t/\tau}$
   with
   $\tau = RC$ = 100*0.0001=0.01 s
   V_S = 20 V
   This is for $t \geq 0$.
   For $t \leq 0$, $v_C(t) = V_S$ = 20 V
   And what about v_R?
   Well...
   For $t \leq 0$, with the switch in open position, no current can flow through R, and therefore $v_R(t)$ = 0 V
   For $t \geq 0$, with the switch closed, $v_R(t) = v_C(t) = 20e^{-100t}$ (C and R are in parallel)
b. Power $p(t) = v(t)*i(t)$ but also $p(t) = v^2(t)/R$ which is easier in this case (since we do not need to derive an expression for $i(t)$).
   So, again before and after the switch is closed:
   For $t \leq 0$, $v_R(t)$ = 0 V, so $p_R(t)$ = 0 W
   For $t \geq 0$, $v_R(t) = 20e^{-100t}$, so $p_R(t) = [20e^{-100t}]^2 / 100 = 4e^{-200t}$
c. We are asked to integrate the power $p_R(t)$ for $t \geq 0$. Mathematically, this means we need to compute the integral
   W = $\int_0^{\infty} 4e^{-200t}$
   For this we are again helped by the special characteristic of the exponential function: it is its own derivative and therefore also its own integral. In our case:
   $$\int 4e^{-200t} = 4\int e^{-200t} = \frac{4}{-200}e^{-200t}$$
   So we can compute the integral of power as the difference of the values at $t = \infty$ and $t = 0$:
   a. at $t = \infty$, $\frac{4}{-200}e^{-200t}$ = 0
   b. at $t = 0$, $\frac{4}{-200}e^{-200t}$ = -4/200 = -0.02
   c. The difference is therefore W = 0 - (-0.02) = 0.02 J (fortunately, this is a positive number, since dissipated energy in a resistor cannot be negative)
d. There's 2 ways of showing this:

a. by simply stating that from t = 0 s, all power is supplied by (the current coming from) the capacitor, which means that the power dissipated in R must be equal to the power supplied by C, which is therefore 0.02 J
b. by computing the energy stored in C at t=0 s, using (see P4.2 above)
W = $\frac{C}{2}v_C^2(0)$ = $\frac{C}{2}V_S^2$ = 50*10^-6*20^2=0.02 J

P4.15

P4.15. Suppose we have a capacitance C that is charged to an initial voltage V_i. Then at $t = 0$, a resistance R is connected across the capacitance. Write an expression for the current. Then, integrate the current from $t = 0$ to $t = \infty$, and show that the result is equal to the initial charge stored on the capacitance.

The problem description leads us to think that, again, this is our well-known circuit:



We know the voltage expression:

$v_C(t) = V_i e^{-t/\tau}$

with

τ = RC

Since the current will flow in the R we can use Ohm's law to get the current:

$i_R(t) = \frac{V_i}{R} e^{-t/RC}$

Now we realize that this is *current*, which is "charge per second", and if we integrate, we are actually summing the charge flow across time - which will result in the total charge that was available at t = 0 s.

If we write the integration mathematically, we have

Charge = $\int_0^\infty \frac{V_i}{R} e^{-t/RC}$

Once again (like in P4.11c), we use the fact that "the exponential function is its own integral", which means we can write:

$$\int e^{-t/RC} = -RCe^{-t/RC}$$

Now the charge integral

$$\int_0^\infty \frac{V_i}{R} e^{-t/RC}$$

reduces to computing the difference of the values of the following function at t = ∞ and t = 0:

$$-\frac{V_i}{R} RCe^{-t/RC}$$

a. at t = ∞, the value is 0
b. at t = 0, the value is $-V_i C$

c. the difference is

   Charge = [0 - $-V_i C$] = $V_i C$

So, the total charge that has flowed is equal to $V_i C$.

Now, for a capacitor we learned a long time ago (NET2 week A) that the stored charge q is equal to the (DC) voltage across the capacitor:

$$q = Cv$$

So, at t = 0 s, the stored charge is $C*V_i$ which equals the total charge that has flowed above. Wow!