Question 2

The equilibrium states of structural and mechanical systems are characterized by the stationary points of the total potential energy of the system. This is known as the principle of stationary potential energy. If at a stationary point the potential energy actually has a minimum value, the equilibrium state is called stable.

Taken from : https://www.sciencedirect.com/topics/engineering/equilibrium-state

As the above-mentioned statement states, we are looking for a stationary point of the potential function to make the whole system stable.

As we are placing the (n) new charges on a surface(2D surface) ,each point is defined as p(xi,yi) and the potential of two pairwise charges P and Q is defined as follow:

$$E(P,Q) \quad = K_e \frac{e_P * e_Q}{distance(P,Q)}$$

Where e_P denotes the charge of a given electric charge. As all the charges (m existing charges on the surface and the ones we add later on) have the same magnitude, we consider them as constant, and they do not affect our problem. Also K_e is a constant multiplier. So we can neglect that as well.

As the problem mentions that the new charges are place on a circle, we have a constrained problem with the following equality constraint:

$(X_i)^2$+$(Y_i)^2$-1=0

Which means the new charges are placed on a circle around origin.

Additionally, as each pint has 2 input variables(x,y) in 2D plane, the size of the input will be 2*n. we can simplify our problem by using polar coordinate where each point on 2D space is represented by R and Θ. For the sake of simplicity, we may assume that the radius of the circle which new charges are added is one. So, the number of the input is reduced to 1 and we now have n input variables for the new charges.

As the problem mentions, the m previously charges are immobile and static on the surface. Consequently, we do not consider them as a variable, and we consider them as a parameter to the problem. As we changed our coordinate system to polar coordinate, we also change the coordinate od the immobile charges to polar coordinate so that we can compute the distance of the immobile charges and new charges correctly. The only fact we must consider is that the immobile charges are not on a unit circle and they have their own radius. The variables are the positions of the n additional charges which we denote by polar coordinate. As the immobile charges are fixed, we calculate the potential of the system before putting new charges and add it to our new equation.

Additionally, the old immobile charges also have interactoin with the new charges. Each pair of the charges(immobile,mobile) have a potential that we should consider as well.

The formalization of the problem is as follows:

$E_{system} = (\sum_{immobile\ cahrges} \frac{1}{distance(pairwise)}) + (\sum_{mobile\ charges} \frac{1}{distance(pairwise)}) +$
$(\sum_{immobile\ cahrges\ and\ mobile\ charges} \frac{1}{distance(immobile-mobile\ pair)})$

Note: the distance measured in both cases is the pairwise distance and we have to keep in mind that we should not calculate a pair two times. We calculate a pair only once.

Note: the first summation is a constant number as the immobile charges are fixed and they are just parameters to our problem. So we can exclude this term from the objective function.

The objective function is as follow:

$E_{whole\ system} =$

$$\sum_{mobile\ charges} \frac{1}{distance(pairwise)} + \sum_{immobile\ cahrges\ and\ mobile\ charges} \frac{1}{distance(pairwise)}$$

Also in polar coordinate, we have a hidden constraint. The angle Θ should be in the interval [-π,π] for each given point. But as we know that the cosine function used in distance function is a periodic function(described in the below section), we can solve our problem regardless of these constraints.

Calculating the distance:

We calculate the distance of a pair charges in polar coordinates which can be done with the following formula:

distance$(p_i, p_j) = \sqrt{r_i^2 + r_j^2 + 2 * r_i * r_j * \cos(\Theta_i - \Theta_j)}$

for mobile charges, the radius of all charges is unit length. So the distance function for mobile charges is:

$distance_{mobile\ charges}(p_i, p_j) = \sqrt{1^2 + 1^2 + 2 * 1 * 1 * \cos(\Theta_i - \Theta_j)} = \sqrt{2 + 2 * \cos(\Theta_i - \Theta_j)}$

While for calculating the potential of a mobile and immobile charge, the radius of mobile charge is unit length, but the radius of immobile charge is not. So, the distance function for the pair of a mobile charge and an immobile charge is:

$distance_{mobile\ and\ immobile\ charges}(p_i, p_j) = \sqrt{r_{immobile\ charge}^2 + 1^2 + 2 * r_{immobile} * 1 * \cos(\Theta_i - \Theta_j)}$
$= \sqrt{r_{immobile\ charge}^2 + 1 + 2 * r_{immobile} * \cos(\Theta_i - \Theta_j)}$

so our minimization objective function is as follows:

Min($E_{whole\ system}$) =min (E(mobile and immobile pair)+E($\Theta_1, \Theta_2, \dots\dots, \Theta_n$))

Which method should we apply for the problem?

Since this problem (like Thomson problem) is a very complicated and calculation intensive problem, we have to choose an efficient algorithm which terminates within finite steps.

Generally, we can choose between zero order, first order and second order algorithms.

Zero order methods are very inefficient in this case as so many function evaluations are required in each step which leads to a very high runtime of the program, as well as a high demand of calculation.

The second order methods also perform many function evaluations, and they require the first and the second order derivative which is not efficient in our case study.

The method I think would perform good in this scenario is the bisection method. It only uses the first order derivative of the function and it has a good convergence rate of $\frac{1}{2}$ which leads to finding the minimum in finite steps.