Toric Varieties

Mathilde Kermorgant, Ameer Muhialdeen, and Yifan Guo

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Our presentations are mainly based on [Cox05, Lec. 1]. The road map of our chain of three talks is as follows:

1. Necessary algebraic background for toric varieties, and toric varieties;
2. The combinatorial object called cone, and how to construct a variety from it;
3. Correspondence between normal affine toric varieties and cones.

1 Algebraic background

This sections aims to introduce the necessary algebraic background to define and understand normal toric varieties.

1.1 Affine varieties and coordinate rings

An important construction we can associate with an affine variety V is its coordinate ring $\mathbb{C}[V]$. We begin from the following observation: let g and g' be two polynomials in $k[x_1, ..., x_n]$. g and g' take the same values on V iff $g(p) - g'(p) = 0$ for all $p \in V$ iff $g - g' \in I(V)$. Therefore, the ring of distinct polynomial function on V is $\mathbb{C}[x_1, ..., x_n]/I(V)$.

Definition 1.1. Let V be an affine variety. Then, its coordinate ring is $\mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]/I(V)$.

Remark. Why is it called a coordinate ring? Let V_1 and V_2 be two varieties. A map of varieties $V_1 \to V_2$ is given by $(x_1, ..., x_n) \mapsto (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$ where $f_i \in \mathbb{C}[V_1]$. This induces a ring morphism from $C[V_2] \to C[V_1]$ mapping the coordinates y_i of $\mathbb{C}[V]$ to f_i in $\mathbb{C}[V_1]$. Therefore, we can say that the polynomials in $\mathbb{C}[V_1]$ "represent the coordinates" of V_1 when we embed it into another variety.

The Nullstellensatz indicates that the maximal ideals of $\mathbb{C}[x_1, ..., x_n]$ are $(x_1 - a_1, ..., x_n - a_n)$, so we can bijectively identify the maximal ideals of $\mathbb{C}[x_1, ..., x_n]$ with the points of $\mathbb{C}^n$. Viewing $\mathbb{C}^n$ as a variety, $I(\mathbb{C}^n) = (0)$ so $\mathbb{C}[\mathbb{C}^n] = \mathbb{C}[x_1, ..., x_n]$. This generalizes to the maximal ideals of $\mathbb{C}[V]$.

Proposition 1.1. *The maximal ideals of* $\mathbb{C}[V]$ *can be bijectively identified with the points of* V*. We write* $\mathrm{Specm}(\mathbb{C}[V]) = V$ *to represent the close relationship between them, where* $\mathrm{Specm}(\mathbb{C}[V])$ *denotes the set of maximal ideals of* $\mathbb{C}[V]$*.*

Proposition 1.2. *Given any finitely generated* $\mathbb{C}$*-algebra* R *with no nilpotents, we may construct a variety* $\mathrm{Specm}(R)$ *with coordinate ring* R*.*

Proof. (Sketch) R is a finitely generated $\mathbb{C}$ algebra so $R \cong \mathbb{C}[x_1, ..., x_n]/I$ for some integer n and some ideal I. I is radical because R has no nilpotent elements, and this is important because in the definition of the coordinate ring, we take the modulus by $I(V)$, which is a radical ideal. Then we can take $\mathrm{Specm}(R) = V(I)$. $\square$

1.2 Irreducible affine varieties

Irreducible affine varieties are the "building blocks" of affine varieties.

Definition 1.2. An affine variety V is irreducible if $V = V_1 \cup V_2$ for subvarieties V_1 and V_2 implies that $V_1 = V$ or $V_2 = V$.

Proposition 1.3. *Let* V *be an affine variety. TFAE:*

1. *V is irreducible*
2. *$I(V)$ is prime*
3. *$\mathbb{C}[V]$ is an integral domain*

Example 1.1. $\mathbb{C}[x, y]/(xy)$ *is not irreducible and* $\mathbb{C}[x, y]/(x)$ *is irreducible.*

Proposition 1.4. *Every affine variety can be written as* $V = V_1 \cup ... \cup V_r$ *where the* V_i *are irreducible and* $V_i \not\subseteq \cup_{i \neq j} V_j$*.*

Proof. (Sketch) In a Noetherian ring, every ideal has an irredundant primary decomposition. If a radical ideal I has an irredundant primary decomposition $p_1 \cap ... \cap p_r$ where $p_i \not\subset \cap_{i \neq j} p_j$ and p_i is prime (because I is radical), then $V(I) = V(p_1 \cap ... \cap p_r) = V = V(p_1) \cup ... \cup V(p_r)$ where the V_i are irreducible and $V_i \not\subseteq \cup_{i \neq j} V_j$! $\square$

Recall: the distinguished open at f is $V_f = \{a \in V : f(a) \neq 0\}$

Proposition 1.5. *If* V *is irreducible and* f *is non-zero in* $\mathbb{C}[V]$*,* $\mathrm{Specm}(\mathbb{C}[V]_f)$ *is the distinguished open* V_f

Proof. (Sketch) Indeed, $\mathbb{C}[V]_f = \{g/f^n : g \in \mathbb{C}[V]\}$. The corresponding variety is precisely the points of V on which the elements of $\mathbb{C}[V]_f$ are defined, aka $\mathrm{Specm}(\mathbb{C}[V]_f) = \{a \in V : f(a) \neq 0\} = V_f$. $\square$

1.3 Toric varieties

Finally, we can define toric varieties.

Definition 1.3. A Toric variety is an irreducible variety V such that $(C^d)^*$ is a Zariski open subset of V and the action of $(C^d)^*$ on itself extends to an action of $(C^d)^*$ on V.

Remark. Be careful, the d in $(C^d)^*$ is not the number of variables of $\mathbb{C}[x_1, ..., x_n]$ but the dimension of X.

Remark. A Zariski open subset of V is dense in V. This is what makes the notion of "extending the action of $(\mathbb{C}^*)^d$ on itself to an action on V" palletable.

Example 1.2. *Let $A = \mathbb{C}[x,y]/(y^2 - x^3)$.*
We have an injective map $\mathbb{C}^* \to V(y^2 - x^3)$ such that $t \mapsto (t^2, t^3)$. Notice that (t^2, t^3) is the parametrization of the curve $y^2 - x^3 = 0$ minus the origin. This is precisely what makes $\mathbb{C}^*$ into a Zariski open subset of $V(y^2 - x^3)$: $\{0\}$ is Zariski-closed (because it is the zero locus of the ideal (x, y)), so its complement is Zariski-open
$\mathbb{C}^*$ is a multiplicative group and acts on itself via coordinate-wise multiplication: $c \cdot t = ct$. This action carries over to $V(y^2 - x^3)$ such that $c \cdot (t^2, t^3) = (c^2t^2, c^3t^3)$ since $c \cdot t = ct \mapsto (c^2t^2, c^3t^3)$.
Therefore, $V(y^2 - x^3)$ is a toric variety.

1.4 Normal varieties

Philosophically, a normal variety is one that is "almost smooth". However, it has a very algebraic definition that can be hard to see through.

Definition 1.4. Let V be an irreducible variety. Then V is normal $\iff$ $\mathbb{C}[V]$ is integrally closed in its field of fractions ($\mathbb{C}[V]$ is normal).

Example 1.3. *To understand better the meaning of "normal", we will study the example of a domain that is not normal:*
Let $A = \mathbb{C}[x,y]/(y^2 - x^3)$. It's a domain because $y^2 - x^3$ is irreducible. However it is not normal: $y/x \notin A$ but $y/x \in \mathrm{Frac}(A)$ and $(y/x)^2 - x = 0$.
When we draw the Spec of this, we can see that it has a singularity at 0.

Exercise 1.1. *The intersection of normal domains with the same field if fractions is normal.*
Hint: Show that being normal is a local property

2 Cones, Duals and & Lattices

We take a quick detour into the world of combinatorics, and define combinatorial objects called cones in $\mathbb{R}^n$, with the notions of dual and strong convexity. We then define lattices, and rational cones, and show how we can obtain an affine toric variety from a strongly convex rational cone.

2.1 Cones, Duals & Lattices

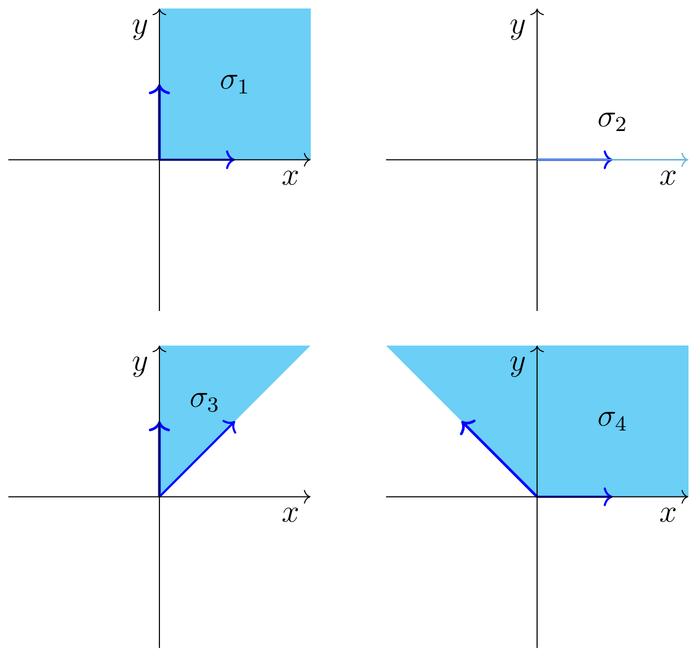
Definition 2.1. Let $S \subseteq \mathbb{R}^n$ be a nonempty finite subset of vectors. A *convex polyhedral cone* σ (or for short, a *cone*) in $\mathbb{R}^n$ is a subset of the form

$$\sigma = \mathrm{Cone}(S) = \{\sum_{v \in S} \lambda_v v : \lambda_v \geq 0\} \subseteq \mathbb{R}^n$$

Note that when $S = \emptyset$, then $\mathrm{Cone}(S) = \{0\}$ since the empty sum is defined to be zero.

We explore this definition through four examples:

Example 2.1. *In $\mathbb{R}^2$, let $S_1 = \{e_1, e_2\}, S_2 = \{e_1\}, S_3 = \{e_2, e_1 + e_2\}, S_4 = \{e_1, -e_1 + e_2\}$, then the cones $\sigma_i = Cone(S_i)$ given by these sets are*



Remark. Convex polyhedral cones are indeed convex sets. For any $x, y \in \mathrm{Cone}(S)$, if we write

$$\begin{cases} x = \sum_{v \in S} \lambda_v v \\ y = \sum_{v \in S} \mu_v v \end{cases}$$

then

$$cx + (1-c)y = \sum_{v \in S} \underbrace{(\lambda_v + (1-c)\mu_v)}_{\geq 0} v \in \mathrm{Cone}(S)$$

for all $c \in [0, 1]$ so $\mathrm{Cone}(S)$ is convex.

We may obtain a cone from a given cone by taking the "dual". We first note that there is a natural pairing $\langle \cdot, \cdot \rangle$ between $v \in \mathbb{R}^n, u \in (\mathbb{R}^n)^*$ given by

$$\langle u, v \rangle = u(v)$$

With this in mind, we give the following definition:

Definition 2.2. Let $\sigma = \text{Cone}(S)$ be a cone in $\mathbb{R}^n$. Then, the dual $\sigma^\vee$ is defined as

$$\sigma^\vee = \{u \in (\mathbb{R}^n)^* : \langle u, v\rangle \geq 0 \text{ for all } v \in S\}$$

or equivalently,

$$\sigma^\vee = \bigcap_{v \in S} \{u \in (\mathbb{R}^n)^* : \langle u, v\rangle \geq 0\}$$

Remark. As a quick exercise, one can verify that for any $v \in \mathbb{R}^n, u \in (\mathbb{R}^n)^*$, if we write

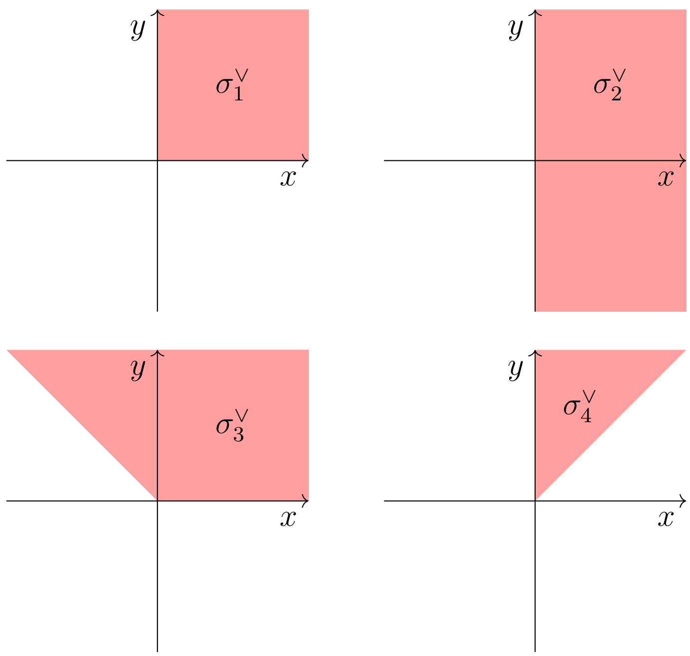
$$\begin{cases} u = \sum_{1 \leq i \leq n} a_i e_i^* \\ v = \sum_{1 \leq i \leq n} b_i e_i \end{cases}$$

then

$$\langle u, v\rangle = (a_1, ..., a_n) \cdot (b_1, ..., b_n)$$

where $\cdot$ denotes the dot product.

Example 2.2. *The last characterization of $\sigma^\vee$ in Definition 6.2 gives us a quick way of computing duals. We compute $\{u \in (\mathbb{R}^n)^* : \langle u, v\rangle \geq 0\}$ for every $s \in S_i$ in Example 6.1, and compute their intersection (shaded in red) to find $\sigma_i^\vee$.*



We note that all $\sigma_i^\vee$ above are two dimensional in $\mathbb{R}^2$, but this is not always the case. For example, if $\text{Cone}(\{\pm e_1, \pm e_2\})^\vee = \{0\}$. We formulate this observation into the following definition.

Definition 2.3. Let $\sigma = \text{Cone}(S)$ in $\mathbb{R}^n$. We say that σ is strongly convex if and only if $\dim(\sigma^\vee) = n$.

Definition 2.4. We define a lattice N a free abelian group of rank n. We immediately get that $N \cong \mathbb{Z}^n$. We define $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ to be the dual lattice. Since $N \cong \mathbb{Z}^n$, we have that $M \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n$.

Definition 2.5. Let N be a lattice of rank n with dual M. We define the associated $\mathbb{R}$-vector space $N_{\mathbb{R}}$ of N as

$$N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$$

Similarly, we define the associated $\mathbb{R}$-vector space $M_{\mathbb{R}}$ of M as

$$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} = (\mathbb{R}^n)^*$$

Remark. For a given lattice N and its dual M, there is a natural pairing $\langle \cdot, \cdot \rangle$ between $v \in N_{\mathbb{R}}, u \in M_{\mathbb{R}}$ given by

$$\langle u, v \rangle = u(v)$$

similar to $\mathbb{R}^n, (\mathbb{R}^n)^*$.

The discussion above cones carries over to lattices with $N_{\mathbb{R}}, M_{\mathbb{R}}$ acting as $\mathbb{R}^n, (\mathbb{R}^n)^*$, respectively. Moreover, we define a new kind of cones that respects the structure of lattices.

Definition 2.6. Let N be a lattice with its dual M. Let $\sigma = \mathrm{Cone}(S)$ be a cone in $N_{\mathbb{R}}$ where S is finite. We say that σ is rational if and only if $S \subseteq N$.

2.2 Affine Toric Varieties From Strongly Convex Cones

For the rest of this section, let N be a lattice with its dual M, and let σ be a strongly convex rational cone $\sigma = \mathrm{Cone}(S)$.

Definition 2.7. The semigroup S_σ associated to σ is defined as

$$S_\sigma = M \cap \sigma^\vee$$

One can verify that is indeed a semigroup under addition with $0 \in S_\sigma$ acting as the additive identity.

Definition 2.8. The semigroup algebra $\mathbb{C}[S_\sigma]$ is the $\mathbb{C}$-algebra generated by elements m of S_σ, denoted symbolically as χ^m. Elements of $\mathbb{C}[S_\sigma]$ are all formal linear combinations $\sigma_{m \in S_\sigma} a_m \chi^m$ where all but finitely many a_m are zero. Products in $\mathbb{C}[S_\sigma]$ are given by the distributive law and the exponential rule:

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}$$

Thus, $\chi^0 = 1$ is the multiplicative identity, and $\chi^m \in \mathbb{C}[S_\sigma]$ has an inverse if and only if $-m \in S_\sigma$.

There is a special property of semigroups obtained via cones, which is captured in Gordan's lemma.

Lemma 2.1. *(Gordan's lemma) The semigroup S_σ is finitely generated.*

This means that we may find a finite set of elements $m_1, ..., m_r$ that generate S_σ, and so $\chi^{m_1}, ..., \chi^{m_t}$ generate $\mathbb{C}[S_\sigma]$ as a $\mathbb{C}$-algebra. By choosing a $\mathbb{Z}$-basis for N, we may write every element of S_σ as a $\mathbb{Z}$-linear combination of $m_1, ..., m_r$. This allows us to consider the following inclusion

$$\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$$

given by

$$\chi^{\sum a_i m_i} \to t_1^{a_1} \cdots t_n^{a_n}$$

Example 2.3. *Let $N = \mathbb{Z}^n$ with the dual $M = \mathbb{Z}^n$. The cone σ_2 as defined in Example 6.1 is indeed rational since $\{e_1\} \subseteq N$ so we may look at its associated semigroup and semigroup algebra. We already determined that $\sigma_2^\vee$ is generated by the set $\{e_1, \pm e_2\}$. Thus, the semigroup $S_{\sigma_2} = \sigma_2^\vee \cap M$ has generators $e_1, \pm e_2$, and so $\chi^{e_1}, \chi^{e_2}, \chi^{-e_2}$ generate $\mathbb{C}[S_\sigma]$ as a $\mathbb{C}$-algebra. We may identify $\chi^{e_1} \mapsto t_1, \chi^{e_2} \mapsto t_2, \chi^{-e_2} \mapsto t_2^{-1}$ so $\mathbb{C}[S_\sigma] \cong \mathbb{C}[t_1, t_2^{\pm 1}]$.*

This example is a special case from a more general case that will be useful to us. In $\mathbb{R}^n$, $Cone(\{e_1, ..., e_d\})^\vee = Cone(\{e_1, ..., e_d, \pm e_{d+1}, ..., \pm e_n\}))$ for any $1 \leq d < n$, and so S_σ has generators $e_1, ..., e_d, \pm e_{d+1}, ..., \pm e_n$. This means that $\mathbb{C}[S_\sigma]$ is generated by elements $\chi^{e_1}, ..., \chi^{e_d}, \chi^{\pm e_{d+1}}, ..., \chi^{\pm e_n}$ so $\mathbb{C}[S_\sigma] \cong \mathbb{C}[t_1, ..., t_d, t_{d+1}^{\pm 1}, ..., t_n^{\pm 1}]$ with a similar identification as above.

Example 2.4. *Let $N = \mathbb{Z}^n$ with the dual $M = \mathbb{Z}^n$ again, and consider the cone σ_3 in Example 6.1 which is rational since $\{e_2, e_1 + e_2\} \subseteq N$. We computed already that the dual $\sigma_3^\vee$ is generated by the set $\{e_1, -e_1 + e_2\}$ so $S_\sigma = \sigma_3^\vee \cap M$ has generators $e_1, -e_1 + e_2$. This means that $\chi^{e_1}, \chi^{-e_1+e_2} = \chi^{-e_1}\chi^{e_2}$ generate $\mathbb{C}[S_\sigma]$ as a $\mathbb{C}$-algebra. Making the idenification $\chi^{e_1} \mapsto t_1, \chi^{e_2} \mapsto t_2$, we get that $\mathbb{C}[S_\sigma] \cong \mathbb{C}[t_1, t_1^{-1}t_2]$.*

Since $\mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]$ is an integral domain, then $\mathbb{C}[S_\sigma]$ is also an integral domain that is finitely generated over $\mathbb{C}$. This allows us to make the following fundamental definition:

Definition 2.9. For a strongly convex rational cone σ, we define the affine toric variety V_σ associated with σ as

$$V_\sigma = \mathrm{Specm}(\mathbb{C}[S_\sigma])$$

We explore this definition further in the next section, and show that indeed affine toric varieties obtained through this way are indeed all possible affine toric varieties.

3 Correspondence between Cones and Toric Varieties

We have previously defined the variety $V_\sigma := \mathrm{Specm}(\mathbb{C}[S_\sigma])$. We have been referring to it as an 'affine toric variety'. We now justify this terminology with the following theorem.

Theorem 3.1. *Let $\sigma \in N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then, $V_\sigma :=$ $\mathrm{Specm}(\mathbb{C}[S_\sigma])$ is an affine toric variety.*

Proof. Let $\sigma \in N_\mathbb{R} \simeq \mathbb{R}^n$ be a strongly convex rational polyhedral cone, with dual lattice $M \simeq \mathbb{Z}^n$. We wish to show:

1. $(\mathbb{C}^*)^n$ is a Zariski open subset of V_σ.

2. The $(\mathbb{C}^*)^n$-action extends to acting on V_σ.

By Gordan's Lemma (Lemma 6.1), say $S_\sigma = \sigma^\vee \cap M$ is finitely generated by $m_1, \dots, m_r \in M$. Then, the corresponding semigroup algebra is

$$\mathbb{C}[S_\sigma] = \mathbb{C}[t^{m_1}, \dots, t^{m_r}] \subset \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}].$$

We obtain the following diagram of $\mathbb{C}$-algebra homomorphisms.

$$\begin{tikzcd}
\mathbb{C}[x_1, \dots, x_r] \arrow[rr, "x_i \mapsto t^{m_i}", "\alpha"'] \arrow[dr, two heads] & & \mathbb{C}[S_\sigma] \arrow[r, hook, "\beta"'] & \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \\
& \mathbb{C}[x_1, \dots, x_r]/I \arrow[ur, "\sim"'] & &
\end{tikzcd}$$

Since $\mathbb{C}[S_\sigma]$ is generated by r elements, we obtain the surjection α. Let $I = \mathrm{Ker}(\alpha)$. Here, β is the inclusion map. Now, apply the $\mathrm{Specm}(-)$ functor.

$$\begin{tikzcd}
\mathrm{Specm}(\mathbb{C}[x_1, \dots, x_r]) \supset V(I) & V_\sigma \arrow[l, "\alpha^*", "\sim"'] & \mathrm{Specm}(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]) \arrow[l, "\beta^*"'] \\
\alpha^{-1}\beta^{-1}(\mathfrak{p}) \arrow[d, equal] & \beta^{-1}(\mathfrak{p}) \arrow[l, mapsto] \arrow[d, equal] & \mathfrak{p} \arrow[l, mapsto] \arrow[d, equal] \\
(x_1 - a^{m_1}, \dots, x_r - a^{m_r}) & (t^{m_1} - a^{m_1}, \dots, t^{m_r} - a^{m_r}) & (t_1 - a_1, \dots, t_n - a_n)
\end{tikzcd}$$

The notation a^{m_i} means $a_1^{m_i^1} \cdots a_n^{m_i^n}$ for $m_i = (m_i^1, \dots, m_i^n) \in \mathbb{Z}^n$ and $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. Identify $\mathrm{Specm}(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]) = \{(t_1 - a_1, \dots, t_n - a_n) \mid a_i \in \mathbb{C}^*\}$ with $(\mathbb{C}^*)^n$ and $\mathrm{Specm}(\mathbb{C}[x_1, \dots, x_r]) = \{(x_1 - b_1, \dots, x_r - b_r) \mid b_i \in \mathbb{C}\}$ with $\mathbb{C}^r$ to obtain another version of this diagram of morphisms.

$$\begin{tikzcd}
\mathbb{C}^r \supset V(I) & V_\sigma \arrow[l, "\alpha^*", "\sim"'] & (\mathbb{C}^*)^n \arrow[l, "\beta^*"'] \\
(a^{m_1}, \dots, a^{m_r}) & & (a_1, \dots, a_n) \arrow[ll, mapsto]
\end{tikzcd} \tag{1}$$

<u>Step 1: Show that $(\mathbb{C}^*)^n$ is a Zariski open subset of V_σ.</u>

Before proving the general case, let us explore the key idea through an example.

Example 3.1. *Recall the cone $\sigma = Cone(e_2, e_1 + e_2)$ in a previous example, with $\sigma^\vee$ generated by*

$$m_1 = (1, 0) \quad \textit{and} \quad m_2 = (-1, 1).$$

Hence, $\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, t_1^{-1} t_2] \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$.

The trick of this proof is to choose any $m_0 \in M \simeq \mathbb{Z}^2$ in the interior of $\sigma^\vee$ and use m_0 to 'pull' any arbitrary lattice point of M into $\sigma^\vee$. This corresponds to multiplying by a sufficient power of t^{m_0} to 'correct' any Laurent monomial so that it is in $\mathbb{C}[S_\sigma]$.
In this example, setting $m_0 = (0,1)$ could work. Then, the corresponding monomial is $t^{m_0} = t_2$ and $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] \simeq \mathbb{C}[S_\sigma]_{t_2}$, since all generators of the Laurent polynomial ring are in the localization as shown below.

$$
\begin{aligned}
&t_1^{-1} = \frac{t_1^{-1}t_2}{t_2} \in \mathbb{C}[S_\sigma]_{t_2}\\
&t_2^{-1} = \frac{1}{t_2} \in \mathbb{C}[S_\sigma]_{t_2}\\
&t_1,\ t_2 = t_1 \cdot t_1^{-1}t_2 \in \mathbb{C}[S_\sigma]
\end{aligned}
$$

Hence, $(\mathbb{C}^*)^2 = \operatorname{Specm} \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}] = \operatorname{Specm} \mathbb{C}[S_\sigma]_{t_2} = (V_\sigma)_{t_2}$, which is a Zariski open set of V_σ.

The same idea readily generalizes for any given σ. Since σ is assumed to be strongly convex, $\dim \sigma^\vee = n$. Hence, there exists $m_0 \in M$ in the interior of $\sigma^\vee$. Then for any $m \in M$, we obtain $m + lm_0 \in \sigma^\vee$ for sufficiently large $l \in \mathbb{Z}_{\geq 0}$, and therefore the corresponding monomial in the Laurent polynomial ring satisfies:

$$t^m = \frac{t^{m+lm_0}}{(t^{m_0})^l} \in \mathbb{C}[S_\sigma]_{t^{m_0}}.$$

We now conclude that $(\mathbb{C}^*)^n = \operatorname{Specm} \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] = \operatorname{Specm} \mathbb{C}[S_\sigma]_{t^{m_0}} = (V_\sigma)_{t^{m_0}}$ is a distinguished open of V_σ, which is Zariski open.

Step 2: Show that $(\mathbb{C}^*)^n$-action extends to acting on V_σ.

Recall the map (1) and apply the $(\mathbb{C}^*)^n$-action by $c = (c_1, \dots, c_n) \in (\mathbb{C}^*)^n$ to the right hand side:

$$
\begin{tikzcd}
\mathbb{C}^r & (\mathbb{C}^*)^n \arrow[l] \\
(a^{m_1}, \dots, a^{m_r}) & (a_1, \dots, a_n) \arrow[l, mapsto] \arrow[d, bend left, "c=(c_1,\dots,c_n)\cdot"] \\
(c^{m_1}a^{m_1}, \dots, c^{m_r}a^{m_r}) & (c_1a_1, \dots, c_na_n) \arrow[l, mapsto]
\end{tikzcd}
$$

This induces an $(\mathbb{C}^*)^n$-action on $\mathbb{C}^r$ given by:

$$c \cdot (b_1, \dots, b_r) = (c^{m_1}b_1, \dots, c^{m_r}b_r),$$

for any $(b_1, \dots, b_r) \in \mathbb{C}^r$. We now check that V_σ is closed under this $(\mathbb{C}^*)^n$-action in $\mathbb{C}^r$. Since we have already shown that $(\mathbb{C}^*)^n = (V_\sigma)_f$ for some $f \in \mathbb{C}[S\sigma]$, it suffices to show the following two straightforward exercises.

Exercise 3.1. *The Zariski closure of a non-empty distinguished open* $\overline{V_f} = V$ *for an affine variety* V *and polynomial* f. *(Hint:* $\overline{S} = V(I(S))$ *for any subset* S *of the affine space. It may also be useful to recall a polynomial with infinitely many roots is the zero polynomial.)*

Exercise 3.2. *If subset* $S \subset \mathbb{C}^r$ *is closed under* $(\mathbb{C}^*)^n$*-action, then so is* $\overline{S}$.

□

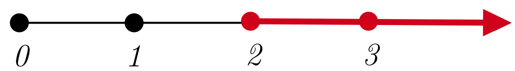
A natural question to ask next is — does the converse of Theorem 3.1 hold? Not necessarily.

Example 3.2. *Recall the toric variety* $V(y^2 - x^3) \subset \mathbb{C}^2$ *in Example 1.2. One can show for its coordinate ring that:*

Exercise 3.3.

$$\frac{\mathbb{C}[x, y]}{(y^2 - x^3)} \simeq \mathbb{C}[t^2, t^3].$$

This ring can also be written as the algebra $\mathbb{C}[S]$ *for the semigroup* $S \subset \mathbb{Z}$ *generated by integers* 2 *and* 3.



We can see that S *cannot correspond to any cone here. Hence, an affine toric variety does not necessarily have the form* V_σ *for some cone* σ. *We also recall from Example 1.3 that* $V(y^2 - x^3)$ *is not normal.*

Luckily, we can fix this! The key issue lies in 'normality'. We refine Theorem 3.1 as follows.

Theorem 3.2. *A variety can be written as* V_σ *for some strongly convex rational polyhedral cone* $\sigma \in N_{\mathbb{R}}$ *if and only if it is a* ***normal*** *affine toric variety.*

Proof. ($\Longrightarrow$) For the forward direction, it remains to show that V_σ is normal, which is equivalent to $\mathbb{C}[S\sigma]$ being integrally closed by definition. Let $\{v_1, \ldots, v_s\}$ be a minimal set of generators for the cone σ. Recall from previous example, if $\{v_1, \ldots, v_s\}$ happens to be a subset of a $\mathbb{Z}$-basis of $N = \mathbb{Z}^n$, then $\mathbb{C}[S_\sigma] = \mathbb{C}[t_1, \ldots, t_s, t_{s+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$, which is a unique factorization domain and is therefore integrally closed. However, $\{v_1, \ldots, v_s\}$ is not necessarily able to be completed to a $\mathbb{Z}$-basis of N. To resolve this issue, the following exercise is the key.

Exercise 3.4.

$$\mathbb{C}[S_\sigma] = \cap_{i=1}^s \mathbb{C}[S_{cone(v_i)}]$$

(Hint: Use $\sigma^\vee = \cap_{i=1}^s H_{v_i}^+$ *where* $H_v^+ := \{u \in \mathbb{R}^n \mid \langle v, u \rangle \geq 0\}$.*)*

A single elememt v_i always be completed to a $\mathbb{Z}$-basis of N. Now applying the aforementioned example, we know $\mathbb{C}[S_{\text{cone}(v_i)}] = \mathbb{C}[t_1, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$, which is indeed integrally closed. The result then follows from Exercise 1.1 — intersection of integrally closed domains with the same fraction field is still integrally closed.
($\Longleftarrow$) For the reverse direction, we first note the fact from [CLS11, Thm. 1.1.17] that any affine toric variety can be written in the form $\operatorname{Specm} \mathbb{C}[S]$ for some finitely generated semigroup S in some lattice M. Then, the result follows from the following exercise.

Exercise 3.5. $\mathbb{C}[S]$ *is integrally closed* $\implies$ S *is saturated (i.e. If* $km \in S$*, then* $m \in S$*, for any* $k \in \mathbb{Z}_{>0}$ *and* $m \in M$*.)* $\implies$ $S = \sigma^\vee \cap M$ *for some rational polyhedral cone* σ*.(Note: Theorem 3.1 implies that these are in fact three equivalent conditions.)*

$\square$

We have now shown that strongly convex rational polyhedral cones not only provide a construction for normal affine toric varieties, but in fact generates all of them.

Solution of Exercise 3.5

Let $S \subset M \simeq \mathbb{Z}^n$ be minimally generated by $\{m_1, \ldots, m_r\}$.
$\mathbb{C}[S]$ **integrally closed** $\implies$ S **saturated.**
If $km = \sum a_i m_i \in S$, then

$$(t^m)^k - \prod (t^{m_i})^{a_i} = 0.$$

By normality, $t^m \in \mathbb{C}[S]$ and hence $m \in S$.
S **saturated** $\implies$ $S = \sigma^\vee \cap M$ **for some rational polyhedral cone** σ**,**
Since S is saturated, we may assume generators m_i's have relatively prime coordinates with respect to some $\mathbb{Z}$-basis of M. Define $\sigma = \cap H^+_{m_i}$ with $\sigma^\vee = \mathrm{Cone}(m_1, \ldots, m_r)$. For any $x = \sum \lambda_i m_i \in \sigma^\vee \cap M$, the irrational part of each coordinate of x must cancel. Hence, we may assume $\lambda_i \in \mathbb{Q}_{\geq 0}$. Multiplying by a minimal $k \in \mathbb{Z}_{>0}$ such that $k\lambda_i \in \mathbb{Z}$ for all i and thus $kx \in S$. Since S is saturated, $x \in S$ and we conclude that $S = \sigma^\vee \cap M$.

References

[Cox05] David Cox. *Lectures on toric varieties.* CIMPA School on Commutative Algebra in Hanoi. 2005. URL: `https://dacox.people.amherst.edu/lectures/coxcimpa.pdf`.

[CLS11] David Cox, John Little, and Hal Schenck. *Toric varieties.* Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, 2011. ISBN: 978-0-8218-4819-7.