EXPECTED VALUE

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7.7.

Exercise 7.7 (Exponential) Let $X \sim \text{Exp}(\lambda)$.

a. Find $E[X]$. Hint: $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$.

b. Find $Var[X]$.

(a)

$t = \lambda x$

$1/\lambda \, dt = dx$

$$\mathbb{E}[X] = \int_0^{+\infty} x \lambda e^{-\lambda x} \, dx = \lambda \int_0^{+\infty} x e^{-\lambda x} \, dx = \lambda \int_0^{+\infty} \frac{t}{\lambda} e^{-t} \frac{dt}{\lambda} =$$

$$= 1/\lambda \int_0^{+\infty} t e^{-t} \, dt = \frac{\Gamma(2)}{\lambda} = \frac{1}{\lambda}$$

$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt$ and $\Gamma(2) = 1 \cdot \Gamma(1) = 1$

(b)

same substitution

$$\mathbb{E}[X^2] = \int_0^{+\infty} \lambda x^2 e^{-\lambda x} \, dx = \lambda \int_0^{+\infty} \frac{t^2}{\lambda^2} e^{-\lambda x} \, dt/\lambda = \frac{1}{\lambda^2} \int_0^{+\infty} t^2 e^{-t} \, dt =$$

$$= \frac{\Gamma(3)}{\lambda^2} = \frac{2\,\Gamma(2)}{\lambda^2} = 2/\lambda^2$$

$$V[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$$

7.9

Exercise 7.9 (Expectation of transformations) Let X follow a normal distribution with mean μ and variance σ^2.

a. Find $E[2X+4]$.

b. Find $E[X^2]$.

c. Find $E[\exp(X)]$. Hint: Use the error function $\text{erf}(x) = \frac{1}{\sqrt{(\pi)}} \int_{-x}^{x} e^{-t^2} dt$. Also, $\text{erf}(\infty) = 1$.

(a)

$$\mathbb{E}[2X+4] \overset{\text{linearity}}{=} \mathbb{E}[2X] + \mathbb{E}[4] = 2\,\mathbb{E}[X] + 4 \overset{X \sim N(\mu,\sigma^2)}{=} 2\mu + 4$$

(b)

$$V[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \Rightarrow \mathbb{E}[X^2] = V[X] + (\mathbb{E}[X])^2 = \sigma^2 + \mu^2$$

(c)

$$\mathbb{E}[\exp(X)] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{x}\, dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2} + x}\, dx =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2 + 2x\sigma^2}{2\sigma^2}} \, dx \overset{(*)}{=}$$

★ $$-(x-\mu)^2 + 2x\sigma^2 = -x^2 - \mu^2 + 2x\mu + 2x\sigma^2 = -x^2 - \mu^2 + 2x(\mu+\sigma^2) \overset{\pm(\mu+\sigma^2)^2}{=}$$

$$= -(x-(\mu+\sigma^2))^2 - \mu^2 + (\mu+\sigma^2)^2$$

$$(*) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2))^2 + \mu^2 - (\mu+\sigma^2)^2}{2\sigma^2}} \, dx =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mu^2 - (\mu+\sigma^2)^2}{2\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{(x-(\mu+\sigma^2))^2}{2\sigma^2}} \, dx \overset{t = \frac{x-(\mu+\sigma^2)}{\sqrt{2}\sigma}}{=}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-\mu^2 + (\mu+\sigma^2)^2}{2\sigma^2}} \int_{-\infty}^{+\infty} \sqrt{2}\sigma\, e^{-t^2} \, dt =$$

$$= \frac{1}{\cancel{\sqrt{2\pi\sigma^2}}} e^{\frac{-\mu^2 + (\mu+\sigma^2)^2}{2\sigma^2}} \cancel{\sqrt{2}}\,\cancel{\sigma}\,\cancel{\sqrt{\pi}}\, \overset{1}{\overset{\parallel}{\mathrm{erf}(\infty)}} = e^{\frac{-\mu^2 + (\mu+\sigma^2)^2}{2\sigma^2}} = e^{\frac{2\mu+\sigma^2}{2}}$$

$\boxed{7.14.}$

Exercise 7.14 Let $X \sim \mathrm{Uniform}(0,1)$ and $Y|X = x \sim \mathrm{Uniform}(0,x)$.

a. Find the covariance of X and Y.

b. Find the correlation of X and Y.

(a) We know that $p_X(x) = 1$ and $p_{Y|X}(y|x) = 1/x$, so we can compute the joint distribution $p_{X,Y}(x,y) = p_X(x)\, p_{X|Y}(x|y) = 1/x$

now we can compute expectations in this distribution

$$\mathbb{E}[XY] = \int_0^1 \int_0^x xy \, {}^1\!/_x \, dy\, dx = \int_0^1 \int_0^x y \, dy\, dx = \int_0^1 \frac{x^2}{2} dx = \frac{x^3}{6}\Big|_0^1 = \frac{1}{6}$$

$Y \sim U(0,x)$

$X \sim U(0,1)$ so $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[Y|X] = \frac{X}{2}$ so now we can compute:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}\left[\frac{X}{2}\right] = \frac{1}{2}\mathbb{E}[X] = \frac{1}{4}$$

Now, we can combine everything and

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{6} - \frac{1}{2}\frac{1}{4} = \frac{1}{24}$$ //

(b) To calculate the correlation, we need to compute first $V(X)$ and $V(Y)$.

$$\mathbb{E}[X^2] = \int_0^1 x^2\,dx = 1/3 \implies V[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\mathbb{E}[Y^2] = \mathbb{E}\big[\,\mathbb{E}[Y^2 \mid X]\,\big] = \mathbb{E}\left[\frac{X^2}{3}\right] = \frac{1}{3}\,\mathbb{E}[X^2] = \frac{1}{9}$$

$$\hookrightarrow \int_0^x \frac{1}{x} y^2\,dy = \frac{1}{x}\frac{x^3}{3} = \frac{x^2}{3}$$

$$\implies V[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

Finally, combining all:

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V[X]\,V[Y]}} = \frac{1/24}{\sqrt{\frac{1}{12}\cdot\frac{7}{144}}} = 0.65$$ //