RANDOM VARIABLES

4.3

Exercise 4.3 (Convolutions) Convolutions are probability distributions that correspond to sums of independent random variables.

a. Let X and Y be independent discrete variables. Find the PMF of $Z = X + Y$. Hint: Use the law of total probability.

b. Let X and Y be independent continuous variables. Find the PDF of $Z = X + Y$. Hint: Start with the CDF.

(a) For the variables X, Y : $\exists\, B_X, B_Y \in \mathcal{B}(\mathbb{R})$ countable s.t. $P_X(B_X) = P_Y(B_Y) = 1$ and as countable, there is a bijection with $\mathbb{N}$ for every point on both subsets, and we can write $B_X = \bigcup_{i \in \mathbb{N}} \{x_i\}$, $B_Y = \bigcup_{i \in \mathbb{N}} \{y_i\}$ where $x_i \in B_X$, $y_i \in B_Y$.

Now, we can use the law of total probability for $Z = X + Y$:

$$P(Z = z) = P(X + Y = z) = \sum_{i=1}^{\infty} P(X + y_i = z \mid Y = y_i)\, P(Y = y_i) =$$

← X and Y independent

$$= \sum_{i=0}^{\infty} P(X + y_i = z)\, P(Y = y_i) = \sum_{i=0}^{\infty} P(X = (z - y_i))\, P(Y = y_i)$$

(b) Lets define f, g the PDFs of X and Y respectively:

$$F_Z(y) = P(Z < z) = P(X + Y < z) \overset{\text{total probability law}}{=} \int_{\mathbb{R}} P(X + y < z \mid Y = y)\, P(Y = y)\, dy \overset{\text{independents}}{=}$$

by working in dy : $P(y \le Y \le y + dy) = g(y)$

$$= \int_{\mathbb{R}} P(X + y < z)\, P(Y = y)\, dy = \int_{\mathbb{R}} P(X < z - y)\, \underbrace{P(Y = y)}\, dy =$$

$$= \int_{\mathbb{R}} \left(\int_{-\infty}^{z-y} f(x)\, dx \right) g(y)\, dy$$

Now that we have an expression for F_Z, we can derivate to get the PDF:
We will define $h(z, y) = \int_{-\infty}^{z-y} f(x)\, dx$ and by the fundamental theorem of calculus we have $\frac{\partial}{\partial z} h(z, y) = f(z - y) \cdot \frac{\partial}{\partial z}(z - y) = f(z - y)$.
Now, applying Leibniz's rule:

$$\frac{\partial}{\partial z} F_Z(z) = \frac{\partial}{\partial z} \int_{\mathbb{R}} \overbrace{\int_{-\infty}^{z-y} f(x)}^{h(z,y)}\, g(y)\, dy = \int_{\mathbb{R}} \left(\frac{\partial}{\partial z} h(z, y) \right) g(y)\, dy =$$

$$= \boxed{\int_{\mathbb{R}} f(z - y)\, g(y)\, dy}$$