INTEGRATION

6.2 $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda)$

(a)

$$f(w) = \begin{cases} w & \text{for } w = 0, 1, \dots, n \\ 0 & \text{elsewhere} \end{cases}$$

f is a non-negative simple and measurable function, because:

$$f(w) = \sum_{k=0}^{n} k \cdot \mathcal{X}_{\{k\}} + 0 \cdot \mathcal{X}_{\mathbb{R}\setminus\{1,\dots,n\}} \quad \text{therefore}$$

measurable measurable

$$\int_{\mathbb{R}} f \, d\lambda = \int_{\mathbb{R}} \sum_{k=0}^{n} k \cdot \mathcal{X}_{\{k\}} \, d\lambda = \sum_{k=0}^{n} k \cdot \underbrace{\lambda(\{k\})}_{=0} = 0$$

(b)

$$f(w) = \begin{cases} 1 & \text{for } w = \mathbb{Q}^c \cap [0,1] \\ 0 & \text{elsewhere} \end{cases}$$

$\mathbb{Q}^c \cap [0,1]$ represents the irrational numbers in the interval $[0,1]$, and we proved in the example 4.2.1. of the theory that $\lambda(\mathbb{Q}^c \cap [0,1]) = 1$.

Therefore, f is a non-negative simple and measurable function because

$$f = 1 \cdot \mathcal{X}_{\{\mathbb{Q}^c \cap [0,1]\}} + 0 \cdot \mathcal{X}_{\{\text{elsewhere}\}} \quad \text{so} \quad \int_{\mathbb{R}} f \, d\lambda = \int 1 \cdot \mathcal{X}_{\{\mathbb{Q}^c \cap [0,1]\}} \, d\lambda = 1 \cdot \lambda(\mathbb{Q}^c \cap [0,1]) = 1.$$

measurables

(c)

$$f = \begin{cases} n & \text{for } w = \mathbb{Q}^c \cap [0,n] \\ 0 & \text{elsewhere} \end{cases}$$

The measure of $\mathbb{Q}^c \cap [0,n]$ will be n, following the same proof of example 4.2.1.:

The measure of $[0,n]$ is n, the measure of $\mathbb{Q}$ is 0, so $[0,n] = (\mathbb{Q}^c \cap [0,n]) \cup (\mathbb{Q} \cap [0,n])$

so $n = \lambda(\mathbb{Q}^c \cap [0,n]) + \lambda(\mathbb{Q} \cap [0,n]) \Rightarrow n = \lambda(\mathbb{Q}^c \cap [0,n])$. Therefore:

$\lambda(\mathbb{Q}) = 0$

$$\int_{\mathbb{R}} f \, d\lambda = \int_{\mathbb{R}} n \cdot \mathcal{X}_{\{\mathbb{Q}^c \cap [0,n]\}} \, d\lambda = n \cdot \lambda(\mathbb{Q}^c \cap [0,n]) = n \cdot n = n^2$$