LIMIT THEOREMS

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12.1

Exercise 12.1 Show that Monte Carlo integration converges almost surely to the true integral of a bounded function.

Solving it with study of Monte-Carlo method.

We want to compute the integral : $\int_\Omega g(x)\, dx$

where : $- \Omega \subset \mathbb{R}^d$ bounded

$- g$ bounded on Ω

We will sample : $X_1, X_2, \dots, X_n \overset{i.i.d}{\sim} \text{Uniform}(\Omega)$

and if μ is the measure of the measure space $(\Omega, \mathcal{F}, \mu)$, we will also define $V_\Omega = \mu(\Omega)$.

Now, as $X_i \sim \text{Unif}(\Omega)$, the density of every X_i is $f_X(x) = \frac{1}{V_\Omega} \mathbb{I}_\Omega(x)$

We will now define $Y_i = g(X_i)$ and Y_i are also iid.

If we compute the expectation:

$$\mathbb{E}[Y_i] = \mathbb{E}[g(X_i)] = \int_\Omega g(x) f_{X_i}(x)\, dx = \frac{1}{V_\Omega} \int_\Omega g(x)\, dx$$

We can rewrite now the integral as : $\int_\Omega g(x)\, dx = V_\Omega \mathbb{E}[Y_i] = V_\Omega \mathbb{E}[g(X)]$

We can now use the SLLN in Y_i because Y_i iid and the mean is finite because g is bounded:

$$\frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{a.s.} \mathbb{E}[g(X)]$$

And this is the monte-carlo estimator:

$$\frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{a.s.} \mathbb{E}[g(X)] \iff \frac{1}{n} \sum_{i=1}^n g(X_i) \xrightarrow{a.s.} \frac{1}{V_\Omega} \int_\Omega g(x)\, dx \iff$$

$$\iff V_\Omega \frac{1}{n} \sum_{i=1}^n g(X_i) \xrightarrow{a.s.} \int_\Omega g(x)\, dx$$