

11.3

Exercise 11.3 Borrowed from Wasserman. Let $X_n \sim N(0, \frac{1}{n})$ and let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Does X_n converge to X in distribution? How about in probability? Prove or disprove these statement. R:
Plot the CDF of X_n for $n = 1, 2, 5, 10, 100, 1000$.

Let's study first the convergence in distribution, because if $X_n \xrightarrow{D} X \Rightarrow X_n \xrightarrow{P} X$.

We know that $X_n \sim N(0, 1/n) \Rightarrow X_n = 1/\sqrt{n} \cdot Z$ where $Z \sim N(0, 1)$

Now, let's fix $x \in \mathbb{R}$:

$$F_{X_n}(x) = P(X_n \leq x) = P(Z/\sqrt{n} \leq x) = P(Z \leq \sqrt{n}x) = F_Z(\sqrt{n}x)$$

$$\text{If: } \begin{cases} x > 0 : \sqrt{n}x \xrightarrow{n \rightarrow \infty} \infty & \text{and } \lim_{x \rightarrow \infty} F_Z(x) = 1 \Rightarrow \lim_{n \rightarrow \infty} F_{X_n}(x) = 1 \\ x < 0 : \sqrt{n}x \xrightarrow{n \rightarrow \infty} -\infty & \text{and } \lim_{x \rightarrow -\infty} F_Z(x) = 0 \Rightarrow \lim_{n \rightarrow \infty} F_{X_n}(x) = 0 \end{cases}$$

$$\text{Therefore, } F_{X_n}(x) \xrightarrow{D} \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} = F_X(x) \text{ because we don't need to}$$

check $x=0$, the D-convergence is only in the continuous points of F_X and $x=0$ is not.

Now, before checking the convergence in probability, let's notice that

F_X is describing a degenerate (point-mass) random variable at 0, because

$$P_X(0) = 1 \text{ a.s.} \Rightarrow X = 0 \text{ a.s.}$$

These means that here, convergence in probability is $P(|X_n| > \varepsilon) \xrightarrow{n \rightarrow \infty} 0 \quad \forall \varepsilon > 0$

To prove this:

$$\begin{aligned} P(|X_n| > \varepsilon) &= P(\{X_n > \varepsilon\} \cup \{-X_n < -\varepsilon\}) = \\ &= P(X_n > \varepsilon) + P(-X_n < -\varepsilon) = \\ &= 1 - F_{X_n}(\varepsilon) + F_{X_n}(-\varepsilon) = 1 - \underbrace{F_Z(\sqrt{n}\varepsilon)}_{\xrightarrow{n \rightarrow \infty} 1} + \underbrace{F_Z(-\sqrt{n}\varepsilon)}_{\xrightarrow{n \rightarrow \infty} 0} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

Where we used $F_{X_n}(x) = F_Z(\sqrt{n}x)$ as before.