

5.6.

Exercise 5.6 Let (X, Y) be uniformly distributed on the unit ball $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Let $R = \sqrt{X^2 + Y^2 + Z^2}$. Find the CDF and PDF of R .

Let's define $A := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ the set of points in S^2 . We also define $R = \sqrt{x^2 + y^2 + z^2}$ the distance of the point $(x, y, z) \in A$ from the origin.

First of all, we have to understand the meaning of $R < r$ for $r \in (0, 1]$.

$R < r \equiv x^2 + y^2 + z^2 < r^2 \rightsquigarrow$ event of having the ball of radius r .

Now, since the distribution is uniform \Rightarrow every point is equiprobable in S^2 .

Hence, with the same logic of the uniform in 1 dimension, the CDF represents the length of the r -ball \rightsquigarrow repeat to the total length of S^2 . We are in 3 dimensions, so the length of the r -ball is: $\frac{4}{3}\pi r^3$.

$$F_R(r) = P(R < r) = \frac{\text{Volume of } r\text{-ball}}{\text{Volume of } S^2} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi} = r^3$$

So, the PDF will be $f_R(r) = 3r^2$.