BAYESIAN INFERENCE

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Exercise 17.1 (Poisson-gamma model) Let us assume a Poisson likelihood and a gamma prior on the Poisson mean parameter (this is a conjugate prior).

a. Derive posterior
b. Below we have some data, which represents number of goals in a football match. Choose sensible prior for this data (draw the gamma density if necessary), justify it. Compute the posterior. Compute an interval such that the probability that the true mean is in there is 95%. What is the probability that the true mean is greater than 2.5?
c. Back to theory: Compute prior predictive and posterior predictive. Discuss why the posterior predictive is overdispersed and not Poisson?
d. Draw a histogram of the prior predictive and posterior predictive for the data from (b). Discuss.
e. Generate 10 and 100 random samples from a Poisson distribution and compare the posteriors with a flat prior, and a prior concentrated away from the truth.

(a) We have $y=(y_1,\dots,y_n)$ where $y_i|\lambda \overset{iid}{\sim} \text{Poisson}(\lambda)$ and we assume $\lambda\sim \text{Gamma}(\alpha,\beta)$

The posterior will be:

$$p(\lambda|y) \propto p(y|\lambda)\,p(\lambda) = \left(\prod_{i=1}^n p(y_i|\lambda)\right)p(\lambda) =$$

$$= \left(\prod_{i=1}^n \frac{\lambda^{y_i}}{y_i!}\, e^{-\lambda}\right)\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda} \propto \lambda^{\sum y_i+\alpha-1}\, e^{-(\beta+n)\lambda}$$

and this is a $\text{Gamma}\left(\alpha_0=\alpha+\sum_{i=1}^n y_i\,,\ \beta_0=n+\beta\right)$

(b) We will need to compute the prior predictive distribution:

$\lambda\sim\text{Gamma}$

$$p(y_{new}) = \int_0^{+\infty} p(y_{new},\lambda)\,d\lambda = \int_0^{+\infty} p(y_{new}|\lambda)\,p(\lambda)\,d\lambda =$$

$$= \int_0^{+\infty}\frac{\lambda^{y_{new}}}{y_{new}!}\,e^{-\lambda}\,\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,d\lambda =$$

$\Gamma(y_{new}+1)$

$$= \frac{\beta^\alpha}{\Gamma(y_{new}+1)\,\Gamma(\alpha)}\int_0^{+\infty}\lambda^{y_{new}+\alpha-1}\,e^{-\lambda(\beta+1)}\,d\lambda =$$

$$= \frac{\beta^\alpha}{\Gamma(y_{new}+1)\,\Gamma(\alpha)}\,\frac{\Gamma(y_{new}+\alpha)}{(\beta+1)^{y_{new}+\alpha}} = \frac{\Gamma(y_{new}+\alpha)}{\Gamma(y_{new}+1)\,\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{y_{new}}$$

Thus is a negative binomial distribution $r=\alpha$ and $p=\frac{1}{\beta+1}$

(c) The posterior is done in the textbook, and is analogous calculation.