CONDITIONAL PROBABILITY

3.3

Exercise 3.3 A machine reports the true value of a thrown 12-sided die 5 out of 6 times.

a. If the machine reports a 1 has been tossed, what is the probability that it is actually a 1?

b. Now let the machine only report whether a 1 has been tossed or not. Does the probability change?

(a) We will denote the events $T_k \equiv$ The toss has been k $\forall k \in \{1, \dots, 12\}$ and the event $M_k \equiv$ the machine reports k $\forall k \in \{1, \dots, 12\}$

Let's figure out the probabilities:

$P(T_k) = 1/12 \quad \forall k$ because is a 12-sided die

We don't know directly $P(M_k)$, but the conditional probability depending of the real toss: $P(M_k \mid T_{k'})$ $k, k' \in \{1, \dots, 12\}$

- If $k = k' \Rightarrow P(M_k \mid T_k) = 5/6$
- If $k \neq k' \Rightarrow P(M_k \mid T_{k'}) = \frac{1}{6} \cdot \frac{1}{11}$ because $1/6$ is the prob. of giving a false value and $1/11$ is the prob. that the false value is $k \neq k'$ and not other $k_2 \neq k'$.

Now, we can calculate:

$$P(T_1 \mid M_1) = \frac{P(M_1 \mid T_1) P(T_1)}{P(M_1)} = \frac{\overbrace{P(M_1 \mid T_1) P(T_1)}^{5/6 \cdot 1/12}}{\sum_{k=1}^{12} P(M_1 \mid T_k) P(T_k)} =$$

$$= \frac{5/6 \cdot 1/12}{\frac{5}{6} \cdot 1/12 + \sum_{k=2}^{12} P(M_1 \mid T_k) P(T_k)} = \frac{5/6 \cdot 1/\cancel{12}}{5/6 \cdot 1/\cancel{12} + \cancel{11} \cdot \left(\frac{1}{6} \cdot \frac{1}{\cancel{11}} \cdot \frac{1}{\cancel{12}}\right)} = \frac{5/6}{1\,\cancel{6}/\cancel{6}\,2} = \boxed{\frac{5}{6}}$$

(b) In this case, $P(M_k \mid T_{k'})$ is changing, because we don't count the selection of the false value by the machine. Now, $\forall k, k' \in \{1, \dots, 12\}$

- If $k = k' \Rightarrow P(M_k \mid T_k) = 5/6$
- If $k \neq k' \Rightarrow P(M_k \mid T_{k'}) = \frac{1}{6}$ because the machine only reports that k' is not the tossed value.

$$P(T_1 \mid M_1) = \frac{P(M_1 \mid T_1) P(T_1)}{P(M_1)} = \frac{P(M_1 \mid T_1) P(T_1)}{\sum_{k=1}^{12} P(M_1 \mid T_k) P(T_k)} =$$

$$= \frac{5/6 \cdot 1/12}{\frac{5}{6} \cdot 1/12 + \sum_{k=2}^{12} P(M_1 \mid T_k) P(T_k)} = \frac{5/6 \cdot 1/12}{5/6 \cdot 1/12 + 11 \cdot \left(\frac{1}{6} \cdot \frac{1}{12}\right)} = \boxed{\frac{5}{16}} \rightarrow \text{the probability changes!!}$$

3.8

Exercise 3.8 We have two coins of identical appearance. We know that one is a fair coin and the other flips heads 80% of the time. We choose one of the two coins uniformly at random. We discard the coin that was not chosen. We now flip the chosen coin independently 10 times, producing a sequence $Y_1 = y_1$, $Y_2 = y_2$, ..., $Y_{10} = y_{10}$.

a. Intuitively, without doing and computation, are these random variables independent?

b. Compute the probability $P(Y_1 = 1)$.

c. Compute the probabilities $P(Y_2 = 1 | Y_1 = 1)$ and $P(Y_{10} = 1 | Y_1 = 1, \ldots, Y_9 = 1)$.

d. Given your answers to b) and c), would you now change your answer to a)? If so, discuss why your intuition had failed.

(a) Intuitively, should be independent, because every flip doesn't depend of the other ones

(b) For calculating $P(Y_1=1)$, we need to know what coin has been flipped, so we use the law of total probability. If we denote $B = \begin{cases} 0 & \text{fair coin} \\ 1 & \text{biased coin} \end{cases}$

$$P(Y_1 = 1) = \sum_{i=0}^{1} P(Y_1 = 1 | B = i) P(B = i) = \frac{1}{2} \sum_{i=0}^{1} P(Y_1 = 1 | B = i) = \frac{1}{2}\left(\frac{1}{2} + \frac{4}{5}\right) = \boxed{\frac{13}{20}}$$

(c) We will compute $P(Y_2 = 1 | Y_1 = 1)$ first. By the same reasoning of (b), we need to condition it by the coin chosen, because the prob. of $Y_2 = 1$ depends of the coin chosen (and maybe of Y_1).

$$P(Y_2 = 1 | Y_1 = 1) = \sum_{i=0}^{1} \underbrace{P(Y_2 = 1 | B = i, Y_1 = 1)}_{\text{depends only of } B} \underbrace{P(B = i | Y_1 = 1)}_{\text{Bayes theorem}} =$$

$$= \frac{4}{5} \cdot \frac{P(Y_1 = 1 | B = 0) P(B = 0)}{P(Y_1 = 1)} + \frac{1}{2} \frac{P(Y_1 = 1 | B = 1) P(B = 1)}{P(Y_1 = 1)} =$$

$$= \frac{4}{5} \cdot \frac{4/5 \cdot 1/2}{13/20} + \frac{1}{2} \frac{1/2 \cdot 1/2}{13/20} \approx 0.68 \quad \left(\neq 0.65 = \frac{13}{20}\right)$$

We will now define $X = 1 \Leftrightarrow Y_1 = 1, \ldots, Y_9 = 1$

$$P(X = 1) = \sum_{i=0}^{1} P(X = 1 | B = i) P(B_i) = \frac{1}{2}\left[(1/2)^9 + (4/5)^9 \right]$$

so, with this, we can apply the same computations of before:

$$P(Y_2 = 1 | X = 1) = \sum_{i=0}^{1} P(Y_2 = 1 | B = i, X = 1) P(B = i | X = 1) =$$

$$= \frac{4}{5} \cdot \frac{P(X = 1 | B = 0) P(B = 0)}{P(X = 1)} + \frac{1}{2} \frac{P(X = 1 | B = 1) P(B = 1)}{P(X = 1)} =$$

$$= \frac{4}{5} \cdot \frac{4/5 \cdot 1/2}{P(X=1)} + \frac{1}{2} \frac{1/2 \cdot 1/2}{P(X=1)} \approx 0.8 \quad (\neq 0.068 = P(X = 1))$$

(d) I didn't think about how is more probable that the coin is the biased one if we got after a lot of "independent" tosses always head.