CHAPTER 1

1.4. $\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ is finite or } A^c \text{ is finite}\}$, $\Omega = \mathbb{N}$

Show $\mathcal{A}$ algebra but not a σ-algebra.

- $\mathcal{A}$ is an algebra:

(i) $\emptyset \in \mathcal{A}$ because is finite

(ii) If $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ for the same definition of $\mathcal{A}$.

(iii) Given $\{A_n\}_{n=1}^{N}$ s.t. $A_n \in \mathcal{A}$ $\forall n \leq N$, we need to have $\bigcup_{n=1}^{N} A_n \in \mathcal{A}$.

Induction:

Base: $n=2 \Rightarrow$ if $\overset{\in \mathcal{A}}{A_1}$ and $\overset{\in \mathcal{A}}{A_2}$ are finite $\Rightarrow A_1 \cup A_2$ also and $A_1 \cup A_2 \in \mathcal{A}$

if $\exists j \in \{1,2\}$ s.t. A_j infinite $\Rightarrow A_j^c$ finite bc. $A_j \in \mathcal{A}$

and $(A_1 \cup A_2)^c = A_1^c \cap A_2^c \in \mathcal{A}$ bc A_1^c or A_2^c finite

We suppose to have that given $\{A_n\}_{n=1}^{N} \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{N} A_n \in \mathcal{A}$

for $\underline{N+1}$ we can divide $\bigcup_{n=1}^{N+1} A_n = \bigcup_{n=1}^{N} A_n \cup A_{N+1}$ and we know that $B := \bigcup_{n=1}^{N} A_n \in \mathcal{A}$ and $A_{N+1} \in \mathcal{A}$ so given two elements of $\mathcal{A}$, we can use the same proof of the base case of the induction.

- $\mathcal{A}$ is not a σ-algebra.

Given the countable sequence of the odd and given numbers in $\mathbb{N}$:

$\{A_n\}_{n=0}^{\infty}$ with $A_n := 2n$ and $\{B_n\}_{n=0}^{\infty}$ with $B_n := 2(n+1)$

we have that $\bigcup_{n=0}^{\infty} A_n \cup \bigcup_{n=0}^{\infty} B_n = \mathbb{N} = \Omega$

and $\bigcup_{n=0}^{\infty} A_n$ infinite and $\left(\bigcup_{n=0}^{\infty} A_n\right)^c = \bigcup_{n=0}^{\infty} B_n$ infinite so $\bigcup_{n=0}^{\infty} A_n \notin \mathcal{A}$.

1.11.

(a) For showing a head turns up sooner or later, we know that

$P(\text{head}) = P(\text{tail}) = 1/2$

The prob. of not having head in the first n tosses is:

$P(\text{no heads in first } n \text{ tosses}) = (1/2)^n \quad \forall n \in \mathbb{N}$

So $(1/2)^n \xrightarrow[n\to\infty]{} 0$, that means the probability of having head in infinite $(n\to\infty)$ tosses is 0.

$P(\text{eventually have head}) = 1 - P(\text{never having head}) = 1 - 0 = 0$.

(b) If we have a pattern s for a block of $k \in \mathbb{N}$ consecutive tosses, the probability that the pattern is equal at this random block is

$$P(\text{pattern } s \text{ occurs in a given block of size } k) = (1/2)^k$$

so, the probability of s not appear in a block of size k is: $1 - \left(\frac{1}{2}\right)^k$

In the same way of (a):

$$P(s \text{ occurs}) = \lim_{n\to\infty} P(s \text{ occurs in first } nk \text{ tosses})$$

The probability that s occurs somewhere in the first nk tosses is at least as large as the prob. that s occurs in one of those n separate disjoint blocks, so:

$$P(s \text{ occurs}) \geq \lim_{n\to\infty} P(s \text{ occurs in first } n \text{ disjoint blocks of size } k) =$$

$$= \lim_{n\to\infty} \left(1 - P(s \text{ doesn't occur in first } n \text{ disjoint blocks of size } k)\right) =$$

$$= \lim_{n\to\infty} 1 - \Big(\underbrace{1 - \frac{1}{2^k}}_{<1}\Big)^n \xrightarrow[n\to\infty]{} 0 \quad = 1.$$

1.15. $(\mathbb{Z}, 2^{\mathbb{Z}})$

(a)

An easy measure could be defined by $P_0 : \mathbb{Z} \to [0,\infty]$ s.t.

$$P_0(\omega) = \begin{cases} 1 & \omega = 1 \\ 0 & \omega \neq 1 \end{cases} \quad \forall \omega \in \mathbb{Z} \quad \text{because} \quad \sum_{\omega\in\mathbb{Z}} P_0(\omega) = 1$$

and we use the proposition 1.4.1.

(b) We will define P_0 s.t. $P_0(\omega) > 0 \quad \forall \omega \in \mathbb{Z}$.

We can define a similar geometric distribution of $\mathbb{N}$ knowing that every number in $n \in \mathbb{N}$ has the negative one in $\mathbb{Z}$.

For each number $z \in \mathbb{Z}\setminus\{0\}$: $P_0(z) = \frac{1}{2}p(1-p)^{|z|}$ and $P_0(0) = p$

$$\sum_{\omega\in\mathbb{Z}} P_0(z) = \sum_{\omega\in\mathbb{Z}\setminus\{0\}} \frac{1}{2}p(1-p)^{|z|} + p = p + \frac{1}{2}p \sum_{k=1}^{\infty} (1-p)^{|z|} = p + \frac{1}{\cancel{2}}\cancel{p}\left(\cancel{2}\cdot\frac{(1-p)}{\cancel{p}}\right) = \boxed{1}$$