

10.3

Exercise 10.3 From Jagannathan. Let $X_i, i = 1, \dots, n$, be a random sample of size n of a random variable X . Let X have mean μ and variance σ^2 . Find the size of the sample n required so that the probability that the difference between sample mean and true mean is smaller than $\frac{\sigma}{10}$ is at least 0.95. Hint: Derive a version of the Chebyshev inequality for $P(|X - \mu| \geq a)$ using Markov inequality.

Lets define the sample mean as:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i. \text{ We know that } E[\bar{X}] = \mu \text{ and } V[\bar{X}] = \frac{\sigma^2}{n}.$$

First of all, we will derive a version of the Chebyshev inequality:

$$P(|X - \mu| \geq a) = P((X - \mu)^2 \geq a^2) \leq \frac{E[(X - \mu)^2]}{a^2} = \frac{\text{Var}[X]}{a^2}$$

We will apply this inequality to, $X = \bar{X}$ and $a = \frac{\sigma}{10}$:

$$P(|\bar{X} - \mu| \geq a) \leq \frac{\text{Var}[\bar{X}]}{\sigma^2} \cdot 100 = \frac{\sigma^2 \cdot 100}{n \sigma^2} = \frac{100}{n}$$

Now, with this last result we can finally solve the problem:

$$P(|\bar{X} - \mu| < a) = 1 - P(|\bar{X} - \mu| \geq a) \geq 1 - \frac{100}{n} \geq 0.95$$

$$-\frac{100}{n} \geq -0.05 \Leftrightarrow n \geq \frac{100}{0.05} = \underline{\underline{2000}}$$