

10.3

**Exercise 10.3** From Jagannathan. Let  $X_i, i = 1, \dots, n$ , be a random sample of size  $n$  of a random variable  $X$ . Let  $X$  have mean  $\mu$  and variance  $\sigma^2$ . Find the size of the sample  $n$  required so that the probability that the difference between sample mean and true mean is smaller than  $\frac{\sigma}{10}$  is at least 0.95.  
Hint: Derive a version of the Chebyshev inequality for  $P(|X - \mu| \geq a)$  using Markov inequality.

Let's define the sample mean as:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i. \quad \text{We know that } E[\bar{X}] = \mu \text{ and } V[\bar{X}] = \frac{\sigma^2}{n}.$$

First of all, we will derive a version of the Chebyshev inequality:

$$P(|X - \mu| \geq a) = P((X - \mu)^2 \geq a^2) \leq \frac{E[(X - \mu)^2]}{a^2} = \frac{Var[X]}{a^2}$$

We will apply this inequality to,  $X = \bar{X}$  and  $a = \frac{\sigma}{10}$ :

$$P(|\bar{X} - \mu| \geq a) \leq \frac{Var[\bar{X}]}{\sigma^2} \cdot 100 = \frac{\sigma^2 \cdot 100}{n \sigma^2} = \frac{100}{n}$$

Now, with this last result we can finally solve the problem:

$$P(|\bar{X} - \mu| < a) = 1 - P(|\bar{X} - \mu| \geq a) \geq 1 - \frac{100}{n} \stackrel{a}{\geq} 0.95$$

$$-\frac{100}{n} \geq -0.05 \iff n \geq \frac{100}{0.05} = \underline{\underline{2000}}$$