UNCONTABLE PROBABILITY SPACES

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2.4 Prove following subsets of $(0,1]$ are Borel-measurable by finding their measure.

(a) Any countable set:

If A is a countable set $\Rightarrow$ there is a bijection between A and $\mathbb{N}$ $\Rightarrow$

$\Rightarrow A = \bigcup_{i\in\mathbb{N}} \{a_i\}$ and we know that $\lambda(\{a_i\}) = 0$ (measure of a singleton).

Therefore, $\lambda(A) = \lambda\left(\bigcup_{i=1}^{\infty}\{a_i\}\right) = \sum_{i=1}^{\infty} \lambda(\{a_i\}) = 0.$

count. add. and $\{a_i\}\cap\{a_j\} = \emptyset \ \forall i\neq j$

Remark: We know that for a subset $A\subset\mathbb{R}$, if $\lambda(A) = 0 \Rightarrow A$ is Lebesgue measurable (so Borel-measurable) because in the proof of the Caratheodory extension theorem:

- $\mathcal{F}_{\lambda^*} = \mathcal{B}_{\mathbb{R}}$ and $A\subset\mathcal{F}_{\lambda^*} \Leftrightarrow \lambda^*(Y) = \lambda^*(A\cap Y) + \lambda^*(A^c\cap Y)$

if $\lambda(A) = 0 \Rightarrow \lambda^*(A) = 0 \Rightarrow \lambda^*(A\cap Y) \leq \lambda^*(A) = 0$ and:

(i) $\lambda^*(Y) \leq \overset{0}{\lambda^*(A)} + \lambda^*(Y\setminus A) = \lambda^*(Y\setminus A)$

(ii) $\lambda^*(Y\setminus A) \leq \lambda^*(Y)$

$\Rightarrow \lambda^*(Y) = \lambda^*(A\cap Y) + \lambda^*(Y\setminus A)$ ☒

(ii) Set in $(0,1]$ where decimal expansion doesn't contain 7:

If we call A that set, we will try to calculate A^c.

We can see A^c as the set that is the disjoint union between the subset with the first 7 in the decimal exp. in the position $j\in\mathbb{N}$.

(i) $j=1 \Rightarrow$ interval $[0.7, 0.8)$ ⇝ measure 0.1

(ii) $j=2 \Rightarrow [0.07, 0.08)\cup[0.17, 0.18)\cup\cdots =$

$= \bigcup_{k=0}^{9}[0.k7, 0.k8) \setminus [0.77, 0.78)$ (9 intervals of measure 0.01)

. . . . . .

If we continue $\forall j\in\mathbb{N}$, finally we will have:

$\lambda(A^c) = \sum_{k=0}^{\infty}\frac{9^k}{10^{k+1}} = \frac{1}{10}\sum_{j=0}^{\infty}\left(\frac{9}{10}\right)^k = \frac{1}{\cancel{10}}\cdot\cancel{10} = 1 \Rightarrow \lambda(A) = 1 - \lambda(A^c) = 0$

disjoints by construction

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☒

2.3 $\Omega = [0,1]$, $\mathcal{F}_3$ as defined

(a) $\mathcal{F}_3$ is a σ-algebra?

(i) $\emptyset \in \mathcal{F}_3$ because $\{\emptyset\}$ is countable.

(ii) $A \in \mathcal{F}_3 \overset{?}{\Rightarrow} B = A^c \in \mathcal{F}_3$

case 1: A countable $\Rightarrow$ If $B = A^c$, $B^c = A$ countable and $B \in \mathcal{F}_3$

case 2: A uncountable $\Rightarrow$ A^c countable because $A \in \mathcal{F}_3 \Rightarrow B = A^c \in \mathcal{F}_3$ because countable

(iii) $\{A_i\}_{i\in\mathbb{N}}$, $A_i \in \mathcal{F}_3$ $\forall i \in \mathbb{N}$ $\overset{?}{\Rightarrow} B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_3$

case 1: if A_i countable $\forall i \in \mathbb{N} \Rightarrow \bigcup_{i=1}^{\infty} A_i$ countable $\Rightarrow B \in \mathcal{F}_3$

case 2: if $\exists j \in \mathbb{N}$ s.t. A_j uncountable:

$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \subseteq A_j$ countable $\Rightarrow \bigcap_{i=1}^{\infty} A_i^c$ countable $\Rightarrow B^c$ countable $\Rightarrow$

$\Rightarrow B \in \mathcal{F}_3$.

(b) Is $(\Omega, \mathcal{F}_3, P)$ a measurable space?

We have to prove that P is a probability measure.

P is a measure:

(i) $P(\emptyset) = 0$ because $\emptyset$ is countable

(ii) We have to check that $\{A_i\}_{i\in\mathbb{N}}$ disjoints $\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

case 1: A_i countable $\forall i \in \mathbb{N} \Rightarrow P(A_i) = 0$ $\forall i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} A_i \overset{\in \mathcal{F}_3}{}$ countable so

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$ and $\sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} 0 = 0$

case 2: $\exists! j \in \mathbb{N}$ s.t. $A_j \in \mathcal{F}_3$ uncountable. In this case, A_j^c countable because $A_j \in \mathcal{F}_3$

If $B = \bigcup_{i=1}^{\infty} A_i$ we have that $B^c = \bigcap_{i=1}^{\infty} A_i^c \subseteq A_j^c$ countable $\Rightarrow$

$\Rightarrow B^c$ countable $\Rightarrow P(B^c) = 0 \Rightarrow P(B) = 1 - P(B^c) = 1$

and $\sum_{i=1}^{\infty} P(A_i) = \underbrace{\sum_{i\neq j}^{\infty} P(A_i)}_{\text{countable}} + 1 = 0 + 1 = 1$

case 3: There is no more case, because if $\exists j_1, j_2 \in \mathbb{N}$ s.t. $j_1 \neq j_2$ and

$A_{j_1}, A_{j_2} \overset{\in \mathcal{F}_3}{}$ are uncountable $\Rightarrow A_1 \cap A_2 \neq \emptyset$ (!) (for construction of $\{A_i\}_{i\in\mathbb{N}}$).

Infact, if A_{j_1} and A_{j_2} are uncountable $\Rightarrow A_{j_1}^c, A_{j_2}^c$ countable (because

$A_{j_1}, A_{j_2} \in \mathcal{F}_3$) $\Rightarrow A_{j_1} \cap A_{j_2} = \big(\underbrace{A_{j_1}^c \cup A_{j_2}^c}_{\to \text{countable set}}\big)^c$ and

So, if $A_{j_1} \cap A_{j_2} = \emptyset \Rightarrow (A_{j_1}^c \cup A_{j_2}^c)^c = \emptyset \Rightarrow A_{j_1}^c \cup A_{j_2}^c = \Omega$ (¡!)
because Ω is uncountable.

As a consequence, $\Omega = A \cup A^c$ where one is countable and the other one uncountable, and $P(\Omega) = P(A) + P(A^c) = 1$. ■

Exercise 2.6 (Theory) - PROPOSITION 2.4.1.

If $C = \{(-\infty, a) \mid a \in \mathbb{R}\}$ we have to prove $\sigma(C) = \mathcal{B}_{\mathbb{R}}$.

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Let's take an $(-\infty, a) \in C$.

We have that $(-\infty, a] = \underbrace{\bigcup_{n=1}^{\infty} (-n, a - \frac{1}{n}]}_{\subseteq \mathcal{B}_{\mathbb{R}}} \cup \underbrace{\{a\}}_{\subseteq \mathcal{B}_{\mathbb{R}}}$ so

as $\sigma(C)$ is generated by C and $C \subseteq \mathcal{B}_{\mathbb{R}} \Rightarrow \sigma(C) \subseteq \mathcal{B}_{\mathbb{R}}$ by being a generated σ-algebra

$\supseteq)$

Let's consider (a,b) open interval. We have $(a,b) = (-\infty, b) \setminus \overbrace{(-\infty, a]}^{\in \sigma(C)}$

But $(-\infty, b) = \bigcup_{n=1}^{\infty} \underbrace{(-\infty, b - \frac{1}{n}]}_{\in C}$ so $(a,b) \in \sigma(C) \Rightarrow \mathcal{B}_{\mathbb{R}} \subseteq \sigma(C)$.