Appendix IV

The Dirac delta function

THE purpose of this appendix is to summarize the main properties of the delta function,* which we found useful for representing point sources, point charges, etc. This function, which is used extensively in quantum mechanics as well as in classical applied mathematics, may be defined by the equations

$$\left.\begin{aligned} \delta(x) = 0, \quad \text{when } x \neq 0, \quad &\text{(a)} \\ \int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = 1. \quad &\text{(b)} \end{aligned}\right\} \tag{1}$$

Evidently $\delta(x)$ is not a function in the ordinary mathematical sense,† since if a function is zero everywhere except at one point, and the integral of this function exists, the value of the integral is necessarily also equal to zero. It is more appropriate to regard $\delta(x)$ as a quantity with a certain symbolic meaning.

Consider a set of functions $\delta(x, \mu)$ which, with increasing μ, differ appreciably from zero only over a smaller and smaller x-interval around the origin and which are such that for all values of μ

$$\int_{-\infty}^{+\infty} \delta(x, \mu)\mathrm{d}x = 1. \tag{2}$$

Examples are the functions (see Fig. 8)

$$\delta(x, \mu) = \frac{\mu}{\sqrt{\pi}} \mathrm{e}^{-\mu^2 x^2}. \tag{3}$$

* Also known as the *impulse function*. It was brought into prominence by P. Dirac, *The Principles of Quantum Mechanics* (Oxford, Clarendon Press, 1930), but was known to mathematicians and physicists much earlier, chiefly through the writings of O. Heaviside. See B. van der Pol and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge, Cambridge University Press, 1950), pp. 62–66.

† The theory of the delta function may be made mathematically rigorous by using the notion of distributions, as developed by L. Schwartz in his *Théorie des distributions* [Paris, Hermann et Cie., Vol. I (1950), Vol. II (1951)]. A simplified version of Schwartz' theory was developed by G. Temple, *Proc. Roy. Soc.*, A, **228** (1955), 175, and is described fully in M. J. Lighthill, *An Introduction to Fourier Analysis and Generalised Functions* (Cambridge, Cambridge University Press, 1958).

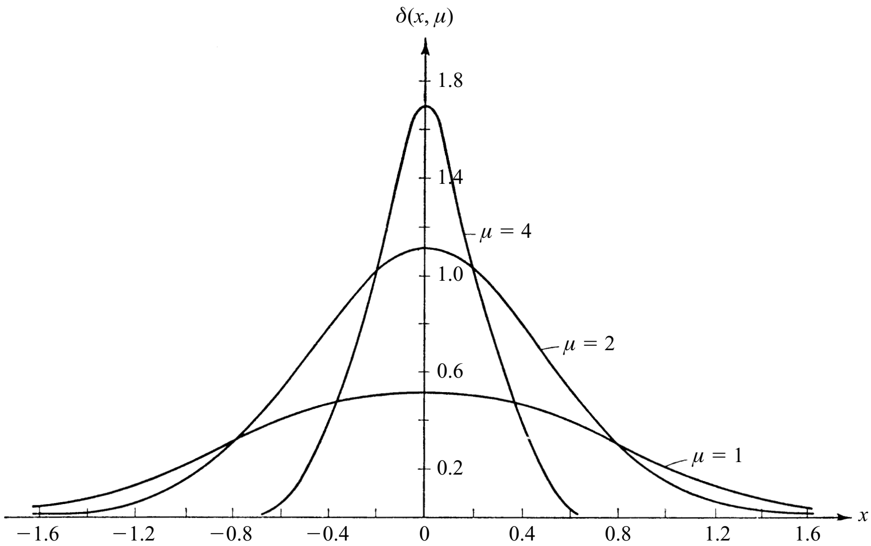

Fig. 8 Illustrating the significance of the Dirac delta function. The functions

$$\delta(x, \mu) = \frac{\mu}{\sqrt{\pi}} e^{-\mu^2 x^2} \quad \text{for } \mu = 1, 2, 4.$$

The total area under each curve is unity.

It is tempting to try to interpret the Dirac delta function as a limit of such a set, for $\mu \to \infty$, but it must be noted that the limit of $\delta(x, \mu)$ need not exist for all x. However,

$$\operatorname*{Lim}_{\mu\to\infty} \int_{-\infty}^{+\infty} \delta(x, \mu) \mathrm{d}x \tag{4}$$

does exist and is equal to unity. *We interpret any operation involving $\delta(x)$ as implying that this operation is to be performed with a function $\delta(x, \mu)$ of a suitable chosen set such as* (3)*, and that the limit $\mu \to \infty$ is taken at the end of the calculation*. With this interpretation (1b) evidently holds. The exact choice of the functions $\delta(x, \mu)$ is not important, provided that their oscillations (if any) near the origin are not too violent.

An important property of the Dirac delta function is the so-called *sifting property*, expressed by the relation

$$\int_{-\infty}^{+\infty} f(x)\delta(x - a)\mathrm{d}x = f(a). \tag{5}$$

Here $f(x)$ is any continuous function of x. The validity of (5) is seen immediately if $\delta(x - a)$ is replaced by $\delta(x - a, \mu)$ and the behaviour of the integral is examined for large values of μ. Evidently when μ is large

$$\int_{-\infty}^{+\infty} f(x)\delta(x - a, \mu)\mathrm{d}x \tag{6}$$

depends essentially on the values of $f(x)$ in the immediate neighbourhood of the point $x = a$ only, and the error of replacing $f(x)$ by $f(a)$ may be made negligible by taking

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μ sufficiently large. Using (1b), (5) then follows. This result implies that the process of multiplying a continuous function by $\delta(x-a)$ and integrating over all values of x is equivalent to the process of substituting a for the argument of the function. Actually, for this result to hold, the range of integration need not be taken from $-\infty$ to $+\infty$. It is only necessary that the domain of integration contains the point $x = a$ in its interior. The result is also written symbolically as

$$f(x)\delta(x-a) = f(a)\delta(x-a), \tag{7}$$

the meaning of such a relation being that the two sides give the same results when used as factors in an integral. In particular, with $f(x) = x$, $a = 0$, (7) gives

$$x\delta(x) = 0. \tag{8}$$

With a similar interpretation the following relations may easily be verified*:

$$\delta(-x) = \delta(x), \tag{9}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x), \tag{10}$$

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)], \tag{11}$$

$$\int_{-\infty}^{+\infty} \delta(a-x)\delta(x-b)\mathrm{d}x = \delta(a-b). \tag{12}$$

To verify (10) for example, we compare the integrals of $f(x)\delta(ax)$ and of $f(x)(1/|a|)\delta(x)$. We have

$$\int_{-\infty}^{+\infty} f(x)\delta(ax)\mathrm{d}x = \pm\int_{-\infty}^{+\infty} f\left(\frac{y}{a}\right)\delta(y)\frac{1}{a}\mathrm{d}y = \frac{1}{|a|}f(0),$$

where the upper or lower sign is taken in front of the second integral according as $a \gtrless 0$. We also have

$$\int_{-\infty}^{+\infty} f(x)\frac{1}{|a|}\delta(x)\mathrm{d}x = \frac{1}{|a|}f(0)$$

because of (5). The integrals are seen to be equal and this is the meaning of (10). Similarly (12) implies that if the two sides are multiplied by a continuous function of a or b and are integrated over all values of a or b respectively, an identity is obtained.

Let us next consider what interpretation may be given to the *derivatives* of the delta function. We have, using the 'approximation functions' $\delta(x, \mu)$ and integrating by parts,

$$\int_{-\infty}^{+\infty} f(x)\delta'(x,\mu)\mathrm{d}x = f(\infty)\delta(\infty,\mu) - f(-\infty)\delta(-\infty,\mu) - \int_{-\infty}^{+\infty} f'(x)\delta(x,\mu)\mathrm{d}x.$$

* The formulas (9)–(11) are special cases of the more general relation,

$$\delta[f(x)] = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i),$$

where x_i are the zeros of $f(x)$, i.e. $f(x_i) = 0$.

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Proceeding to the limit as $\mu \to \infty$, the first two terms on the right disappear and we obtain

$$\int_{-\infty}^{+\infty} f(x)\delta'(x)\mathrm{d}x = -f'(0). \tag{13}$$

Repeating this process we find that

$$\int_{-\infty}^{+\infty} f(x)\delta^{(n)}(x)\mathrm{d}x = (-1)^n f^{(n)}(0). \tag{14}$$

The following relations may easily be verified:

$$\delta'(-x) = -\delta'(x), \tag{15}$$

$$x\delta'(x) = -\delta(x). \tag{16}$$

It is often convenient (see, for example, Appendix VI) to express the Dirac delta function in terms of the *Heaviside unit function* (called also the *step function*) $U(x)$, this function being defined as

$$\left.\begin{aligned} U(x) &= 0, \qquad x<0, \\ &= 1, \qquad x>0. \end{aligned}\right\} \tag{17}$$

If as before, prime denotes the derivative with respect to the argument x, we obtain formally, on integrating by parts (with $x_1 > 0$, $x_2 > 0$),

$$\begin{aligned} \int_{-x_1}^{x_2} f(x)U'(x)\mathrm{d}x &= [f(x)U(x)]_{-x_1}^{x_2} - \int_{-x_1}^{x_2} f'(x)U(x)\mathrm{d}x \\ &= f(x_2) - \int_0^{x_2} f'(x)\mathrm{d}x \\ &= f(x_2) - f(x_2) + f(0) \\ &= f(0). \end{aligned}$$

If we set $x = y - a$, $f(x) = f(y-a) = F(y)$ and proceed to the limit $x_1 \to \infty$, $x_2 \to \infty$, this becomes

$$\int_{-\infty}^{+\infty} F(y)U'(y-a)\mathrm{d}y = F(a),$$

so that U' has the sifting property. In particular, with $F \equiv 1$, $a = 0$ this relation becomes

$$\int_{-\infty}^{+\infty} U'(y)\mathrm{d}y = 1,$$

and shows that U' satisfies a relation of the form (1b). Moreover, $U'(x) = 0$ when $x \neq 0$. Hence we may identify the derivative of the unit function with the delta function:

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}U(x). \tag{18}$$

The delta function may also be introduced with the help of the Fourier integral theorem

$$f(a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}k(x-a)}\,\mathrm{d}x. \tag{19}$$

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If we set

$$K(x-a,\,\mu)=\frac{1}{2\pi}\int_{-\mu}^{\mu}\mathrm{e}^{-\mathrm{i}k(x-a)}\,\mathrm{d}k=\frac{\sin\mu(x-a)}{\pi(x-a)}\tag{20}$$

and invert the order of integration, then (19) may be formally written as

$$f(a)=\int_{-\infty}^{+\infty}f(x)K(x-a)\mathrm{d}x,\tag{21}$$

where $K(x-a)$ is regarded as the limit of $K(x-a,\,\mu)$ when $\mu\to\infty$. Strictly, this limit does not exist in the ordinary sense when $x-a\neq 0$,* but (21) has a similar symbolic meaning as the integrals discussed before, i.e. it should be interpreted as meaning that

$$f(a)=\lim_{\mu\to\infty}\int_{-\infty}^{+\infty}f(x)K(x-a,\,\mu)\mathrm{d}x.\tag{22}$$

Thus K has the sifting property. If we set $f(x)=1$ in (21) we see that the integral of $K(x)$ taken over all x is equal to unity. We thus have another representation of the Dirac delta function, viz.

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k,\tag{23}$$

i.e. $\delta(x)$ may be regarded as the Fourier transform of unity. There is a reciprocal relation, which follows from (21) on setting $f(x)=\mathrm{e}^{\mathrm{i}kx}$, $a=0$,

$$\int_{-\infty}^{+\infty}\delta(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x=1.\tag{24}$$

So far we have considered a space of one dimension only, but the definition may easily be extended to spaces of several dimensions. In particular, consider a space of three dimensions. Then the function

$$\delta(x,\,y,\,z)=\delta(x)\delta(y)\delta(z),\tag{25}$$

often also denoted by $\delta(\mathbf{r})$, where $\mathbf{r}$ is the vector with components x, y, z, evidently satisfies relations analogous to (1), viz.

$$\left.\begin{aligned}&\delta(x,\,y,\,z)=0,\ \text{when } x\neq 0,\ y\neq 0,\ z\neq 0 &&\text{(a)}\\ &\iiint_{-\infty}^{+\infty}\delta(x,\,y,\,z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=1. &&\text{(b)}\end{aligned}\right\}\tag{26}$$

The sifting property is now expressed by the relation

$$\iiint_{-\infty}^{+\infty}f(x,\,y,\,z)\delta(x-a,\,y-b,\,z-c)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=f(a,\,b,\,c),\tag{27}$$

* The limit exists and has the value zero if interpreted in the sense of a Cesàro limit (see B. van der Pol and H. Bremmer, *loc. cit.*, pp. 100–104).

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and $\delta(x, y, z)$ satisfies the Fourier reciprocity relations

$$\delta(x, y, z) = \frac{1}{(2\pi)^3} \iiint\limits_{-\infty}^{+\infty} e^{-i(k_x x + k_y y + k_z z)} \, dk_x \, dk_y \, dk_z, \tag{28}$$

$$\iiint\limits_{-\infty}^{+\infty} \delta(x, y, z) e^{i(k_x x + k_y y + k_z z)} \, dx \, dy \, dz = 1. \tag{29}$$

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