Appendix IX

Proof of a reciprocity inequality (§10.8.3)

LET $f(\tau)$ and $g(\tau)$ be any two functions, generally complex, of the real variable τ and let λ be a real parameter. Then

$$\int_{-\infty}^{+\infty} |f+\lambda g^\star|^2 \, d\tau = \int_{-\infty}^{+\infty} (f+\lambda g^\star)(f^\star+\lambda g) d\tau \geqslant 0, \tag{1}$$

or

$$\int_{-\infty}^{+\infty} ff^\star \, d\tau + \lambda \int_{-\infty}^{+\infty} (fg+f^\star g^\star) d\tau + \lambda^2 \int_{-\infty}^{+\infty} gg^\star \, d\tau \geqslant 0. \tag{2}$$

The minimum of this quadratic expression in λ is obtained by differentiating:

$$\int_{-\infty}^{+\infty} (fg+f^\star g^\star) d\tau + 2\lambda \int_{-\infty}^{+\infty} gg^\star \, d\tau = 0. \tag{3}$$

The root $\lambda = \lambda_{\min}$ of this expression is

$$\lambda_{\min} = \frac{-\displaystyle\int_{-\infty}^{+\infty} (fg+f^\star g^\star) d\tau}{2\displaystyle\int_{-\infty}^{+\infty} gg^\star \, d\tau}. \tag{4}$$

If this value is substituted into (2) we obtain

$$4\left(\int_{-\infty}^{+\infty} ff^\star \, d\tau\right)\left(\int_{-\infty}^{+\infty} gg^\star \, d\tau\right) \geqslant \left(\int_{-\infty}^{+\infty} (fg+f^\star g^\star) d\tau\right)^2. \tag{5}$$

Let

$$f = \tau\psi(\tau), \qquad g = \frac{d\psi^\star(\tau)}{d\tau}. \tag{6}$$

Then

$$fg + f^\star g^\star = \tau\left(\psi \frac{d\psi^\star}{d\tau} + \psi^\star \frac{d\psi}{d\tau}\right) = \tau \frac{d}{d\tau}(\psi\psi^\star), \tag{7}$$

https://doi.org/10.1017/9781108769914.027 Published online by Cambridge University Press

and (5) becomes, if we integrate by parts on the right and assume that* $\tau\psi\psi^\star \to 0$ as $\tau \to \pm\infty$,

$$4\left(\int_{-\infty}^{+\infty} \tau^2 \psi\psi^\star \,\mathrm{d}\tau\right)\left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\psi}{\mathrm{d}\tau}\frac{\mathrm{d}\psi^\star}{\mathrm{d}\tau}\,\mathrm{d}\tau\right) \geqslant \left(\int_{-\infty}^{+\infty} \psi\psi^\star \,\mathrm{d}\tau\right)^2. \tag{8}$$

This is the required inequality.

The equal sign in (8) can only hold if it holds in (1); this is only possible if $f \equiv -\lambda g^\star$, or using (6), if

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\frac{1}{\lambda}\tau\psi. \tag{9}$$

The general solution of this differential equation is

$$\psi(\tau) = A\,\mathrm{e}^{-\frac{\tau^2}{2\lambda}}, \tag{10}$$

where A is a constant. Only solutions with $\lambda \geqslant 0$ apply, since otherwise $\psi(\tau)$ would not vanish at infinity. Hence (8) *becomes an equality if and only if ψ is a Gaussian function*.

* This condition is in fact satisfied whenever the integrals on the left in (8) converge (see H. Weyl, *The Theory of Groups and Quantum Mechanics*, translated from German (London, Methuen, 1931; also, New York, Dover Publications, Inc.), pp. 393–394).

https://doi.org/10.1017/9781108769914.027 Published online by Cambridge University Press