XV

Optics of crystals

15.1 The dielectric tensor of an anisotropic medium

It will be remembered that our optical theory is based on two quite separate foundations; on the one hand on Maxwell's equations §1.1 (1) and §1.1 (2), on the other hand on the material equations which in the case of an isotropic medium were given by the formulae §1.1 (9)–(11). In dealing with crystals we must generalize these latter equations so as to take account of anisotropy. In the greater part of this chapter we assume that the medium is homogeneous, nonconducting ($\sigma = 0$) and magnetically isotropic,* but allow *electrical anisotropy*, i.e. we consider substances whose electrical excitations depend on the direction of the electric field. In general the vector **D** will then no longer be in the direction of the vector **E**. In place of §1.1 (10) we assume the relation between **D** and **E** to have the simplest form which can account for anisotropic behaviour, namely one in which each component of **D** is linearly related to the components of **E**:

$$\left.\begin{aligned} D_x &= \varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z, \\ D_y &= \varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z, \\ D_z &= \varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z. \end{aligned}\right\} \tag{1}$$

The nine quantities ε_{xx}, ε_{yy}, ... are constants of the medium, and constitute the *dielectric tensor*; the vector **D** is thus the product of this tensor with **E**.

We shall write (1) in shorter form as

$$D_k = \sum_l \varepsilon_{kl} E_l, \tag{2}$$

where k stands for one of the three indices x, y and z, and l stands for each of x, y and z in turn in the summation. The summation sign would be omitted in formal tensor notation, the occurrence of the index l in two places in the product being understood as an instruction to sum over all l's. We shall, however, retain the summation sign, as this will help to avoid any ambiguities for readers unfamiliar with tensor calculus.

* There are also magnetic crystals, but as the effect of magnetization on optical phenomena (rapid oscillations) is small, the magnetic anisotropy may be neglected. The magnetic permeability will, however, be retained, and will be represented by a scalar μ, in order to preserve some symmetry in the formulae and to include weakly magnetic crystals; moreover, by retaining it, we facilitate the expression of the equations in systems of units in which μ is not equal to unity in free space.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

The expressions §1.1 (31) for electric and magnetic energy densities retain their validity; thus

$$w_e = \frac{1}{8\pi}\mathbf{E}\cdot\mathbf{D} = \frac{1}{8\pi}\sum_{kl} E_k \varepsilon_{kl} E_l. \tag{3}$$

and

$$w_m = \frac{1}{8\pi}\mathbf{B}\cdot\mathbf{H} = \frac{1}{8\pi}\mu H^2. \tag{4}$$

We retain also the definition §1.1 (38) of the Poynting vector, or the 'ray-vector'

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) \tag{5}$$

and investigate whether these definitions are consistent with the principle of the conservation of energy.

We have, as in §1.1.4, by multiplying the first Maxwell equation by $\mathbf{E}$ and the second by $\mathbf{H}$ and using the vector identity §1.1 (27),

$$\begin{aligned} -c\,\mathrm{div}(\mathbf{E}\times\mathbf{H}) &= \mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}} \\ &= \sum_{kl} E_k\varepsilon_{kl}\dot{E}_l + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mu H^2). \end{aligned} \tag{6}$$

If we divide both sides of this equation by 4π, the second term on the right represents the rate of change of the magnetic energy per unit volume, but the first term does not represent the rate of change of the electric energy density unless

$$\frac{1}{4\pi}\sum_{kl} E_k\varepsilon_{kl}\dot{E}_l = \frac{\mathrm{d}w_e}{\mathrm{d}t} = \frac{1}{8\pi}\sum_{kl}\varepsilon_{kl}(E_k\dot{E}_l + E_l\dot{E}_k) \tag{7}$$

that is, unless

$$\sum_{kl}\varepsilon_{kl}(E_k\dot{E}_l - \dot{E}_k E_l) = 0.$$

The suffixes k and l are dummy suffixes; both run over the same values (x, y, z). Hence the expression is not altered if we interchange k and l in the second term. This leads to

$$\sum_{kl}(\varepsilon_{kl} - \varepsilon_{lk})E_{k}\dot{E}_{l} = 0.$$

As this equation must hold whatever the value of the field, it follows that

$$\varepsilon_{kl} = \varepsilon_{lk}. \tag{8}$$

This means that the *dielectric tensor must be symmetric*; it has only six instead of nine independent components. Conversely, condition (8) is sufficient to ensure the validity of (7), and we obtain the energy theorem in differential form [the 'hydrodynamical continuity equation' §1.1 (43)]

$$-\mathrm{div}\,\mathbf{S} = \frac{\partial w}{\partial t}, \qquad (w = w_e + w_m). \tag{9}$$

The symmetry of the tensor ε makes it possible to reduce the expression for the

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

electric energy w_e to a form in which only the squares of the field components, and not their products, enter. Consider in a space x, y, z the surface of the second degree

$$\varepsilon_{xx}x^2 + \varepsilon_{yy}y^2 + \varepsilon_{zz}z^2 + 2\varepsilon_{yz}yz + 2\varepsilon_{xx}xz + 2\varepsilon_{xy}xy = \text{constant}. \tag{10}$$

The left-hand side of (10) must be a positive definite quadratic form, because if x, y, and z are replaced by the components of **E** the expression becomes equal to $8\pi w_e$, and the energy w_e must be positive for any value of the field vector. Therefore (10) represents an ellipsoid. The ellipsoid can always be transformed to its principal axes; thus there exists a coordinate system fixed in the crystal such that the equation of the ellipsoid is

$$\varepsilon_x x^2 + \varepsilon_y y^2 + \varepsilon_z z^2 = \text{constant}. \tag{11}$$

In this system of *principal dielectric axes* the material equations and the expression for the electrical energy take the simple forms

$$D_x = \varepsilon_x E_x, \qquad D_y = \varepsilon_y E_y, \qquad D_z = \varepsilon_z E_z, \tag{12}$$

$$\begin{aligned} w_e &= \frac{1}{8\pi}(\varepsilon_x E_x^2 + \varepsilon_y E_y^2 + \varepsilon_z E_z^2) \\ &= \frac{1}{8\pi}\left(\frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z}\right). \end{aligned} \tag{13}$$

ε_x, ε_y, ε_z are called the *principal dielectric constants* (or *principal permittivities*). It may be seen immediately from these formulae that **D** and **E** will have different directions, unless **E** coincides in direction with one of the principal axes, or the principal dielectric constants are all equal; in the latter case ($\varepsilon_x = \varepsilon_y = \varepsilon_z$) the ellipsoid degenerates into a sphere.

A note must be added here on the effect of dispersion. Just as, in the case of isotropic substances, the dielectric constant is not a constant of the material but depends on the frequency, so in an anisotropic medium the six components ε_{kl} of the dielectric tensor will also vary with frequency. As a result not only the values of the principal dielectric constants ε_x, ε_y, ε_z will vary but also the directions of the principal axes. This phenomenon is known as *dispersion of the axes*. It can arise, however, only in crystals in which the symmetry of the structure does not determine a preferential orthogonal triplet of directions, i.e. it can be observed only in monoclinic and triclinic systems (see §15.3.1).*

If we restrict ourselves to monochromatic waves we may disregard dispersion; the quantities ε_{kl} are then constants depending only on the medium.

15.2 The structure of a monochromatic plane wave in an anisotropic medium

15.2.1 The phase velocity and the ray velocity

In a monochromatic plane wave of angular frequency $\omega = 2\pi\nu$ propagated with velocity c/n in the direction of the unit wave-normal **s**, the vectors **E**, **D**, **H** and **B** are

* This dispersion phenomenon is particularly conspicuous in the infrared: cf. Th. Liebisch and H. Rubens, *Sitzber. preuss. Akad. Wiss., Phys.-math. Kl.* (1919), 198, 876. H. Rubens, *ibid.*, 976.

in complex notation proportional to $\exp\{\mathrm{i}\omega[(n/c)(\mathbf{r}\cdot\mathbf{s})-t]\}$. It may be mentioned at once that in addition to the *phase* (or *wave-normal*) *velocity* c/n, we shall later have to introduce a *ray* (or *energy*) *velocity*, since, as will be seen, in an anisotropic medium the energy is in general propagated with a different velocity and in a direction different from that of the wave normal.

In such an oscillatory field the operation $\partial/\partial t$ is always equivalent to multiplication by $-\mathrm{i}\omega$, while the operation $\partial/\partial x$ is equivalent to multiplication by $\mathrm{i}\omega n s_x/c$. In particular,

$$\dot{\mathbf{E}} = -\mathrm{i}\omega\mathbf{E}, \qquad \operatorname{curl}\mathbf{E} = \mathrm{i}\omega\frac{n}{c}\mathbf{s}\times\mathbf{E}. \tag{1}$$

Maxwell's equations, for a region which does not contain currents, viz.

$$\operatorname{curl}\mathbf{H} - \frac{1}{c}\dot{\mathbf{D}} = 0, \qquad \operatorname{curl}\mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0, \tag{2}$$

become

$$n\mathbf{s}\times\mathbf{H} = -\mathbf{D}, \qquad n\mathbf{s}\times\mathbf{E} = \mu\mathbf{H}, \tag{3}$$

where the relation $\mathbf{B} = \mu\mathbf{H}$ has been used. Eliminating $\mathbf{H}$ between (3) and using a well-known vector identity, we obtain

$$\mathbf{D} = -\frac{n^2}{\mu}\mathbf{s}\times(\mathbf{s}\times\mathbf{E}) = \frac{n^2}{\mu}[\mathbf{E}-\mathbf{s}(\mathbf{s}\cdot\mathbf{E})] = \frac{n^2}{\mu}\mathbf{E}_\perp, \tag{4}$$

where $\mathbf{E}_\perp$ denotes the vector components of $\mathbf{E}$ perpendicular to $\mathbf{s}$ in the plane of $\mathbf{E}$ and $\mathbf{s}$ (see Fig. 15.1).

We see from (3) that the vector $\mathbf{H}$ (and hence also $\mathbf{B}$) is at right angles to $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{s}$, which must therefore be *coplanar*. Further $\mathbf{D}$ is seen to be orthogonal to $\mathbf{s}$. Thus $\mathbf{H}$ and $\mathbf{D}$ are transversal to the direction of propagation $\mathbf{s}$, as before, but $\mathbf{E}$ is not. Fig. 15.1 shows the relative directions of these vectors, and in addition the unit vector in the direction of the ray-vector $\mathbf{S}$ to be denoted $\mathbf{t}$, which is perpendicular to $\mathbf{E}$ and $\mathbf{H}$. The angle between $\mathbf{E}$ and $\mathbf{D}$ is the same as the angle between $\mathbf{s}$ and $\mathbf{t}$, and will be denoted by α. We see that $\mathbf{D}$, $\mathbf{H}$ *and* $\mathbf{s}$, *on the one hand, and* $\mathbf{E}$, $\mathbf{H}$ *and* $\mathbf{t}$, *on the other hand, form orthogonal vector triplets, with the common vector* $\mathbf{H}$, rotated relatively to one another through the angle α. An important conclusion is that *in a crystal the energy is not in general propagated in the direction of the wave normal*. On the other hand the

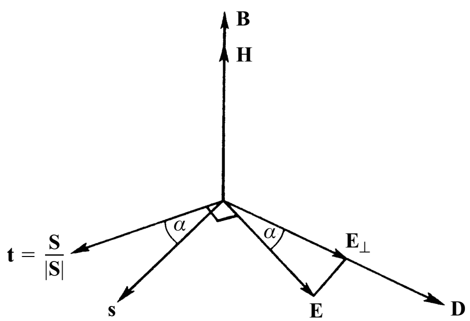

Fig. 15.1 Directions of the wave normal, of the field vectors and of the energy flow in an electrically anisotropic medium.

theorem of equal electric and magnetic energy densities retains its validity. This follows from (3) since*

$$\left.\begin{aligned} w_e &= \frac{1}{8\pi}\mathbf{E}\cdot\mathbf{D} = -\frac{n}{8\pi}\mathbf{E}\cdot(\mathbf{s}\times\mathbf{H}), \\ w_m &= \frac{1}{8\pi}\mathbf{B}\cdot\mathbf{H} = \quad \frac{n}{8\pi}(\mathbf{s}\times\mathbf{E})\cdot\mathbf{H}, \end{aligned}\right\} \tag{5}$$

and by a well-known property of the scalar triple product the right-hand sides of these equations are equal to each other. Moreover, they are both equal to $n(\mathbf{E}\times\mathbf{H})\cdot\mathbf{s}/8\pi$, so that the total energy density $w = w_e + w_m$ is

$$w = \frac{n}{c}\mathbf{S}\cdot\mathbf{s}. \tag{6}$$

We must distinguish between the phase velocity and the velocity of energy transport. The former, *the phase velocity*, is in the direction of the unit vector **s** and its magnitude is

$$v_p = \frac{c}{n}. \tag{7}$$

The latter, *the ray velocity*, is in the same direction as the Poynting vector **S**, i.e. in the direction of the unit vector **t**. Its magnitude v_r is equal to the energy that crosses in unit time an area perpendicular to the flow direction, divided by the energy per unit volume. In accordance with the energy theorem §15.1 (9) this is given by

$$v_r = \frac{S}{w}. \tag{8}$$

From (6), (7) and (8)

$$v_p = v_r\mathbf{t}\cdot\mathbf{s} = v_r\cos\alpha, \tag{9}$$

i.e. *the phase velocity is the projection of the ray velocity onto the direction of the wave normal*.

It may be noted that the ray velocity, being derived from the Poynting vector shares with it (see §1.1, p. 10) a certain degree of arbitrariness. It is nevertheless a useful concept although, like the phase velocity, it has no directly verifiable physical significance.

If **E** and **D** are known [as for instance when **E** is prescribed and **D** is determined from it by §15.1 (1)], the refractive index n and the wave normal **s** are thereby determined. For in the first place, since $\mathbf{E}_\perp$ is the vector component of **E** in the direction of **D**,

$$\mathbf{E}_\perp = \left(\mathbf{E}\cdot\frac{\mathbf{D}}{D}\right)\frac{\mathbf{D}}{D}, \tag{10}$$

so that (4) gives

$$n^2 = \frac{\mu\mathbf{D}}{\mathbf{E}_\perp} = \frac{\mu\mathbf{D}^2}{(\mathbf{E}\cdot\mathbf{D})}. \tag{11}$$

* As we are now dealing with quadratic functions of the field variables, it is the real field vectors and not the associated complex vectors that enter (5) and (6) (see §1.3, p. 18).

Further, since the unit vector $\mathbf{s}$ is perpendicular to $\mathbf{D}$ and coplanar with $\mathbf{D}$ and $\mathbf{E}$, it may be expressed in the form

$$\mathbf{s} = \frac{\mathbf{E} - \mathbf{E}_\perp}{|\mathbf{E} - \mathbf{E}_\perp|} = \frac{\mathbf{E} - \dfrac{(\mathbf{E}\cdot\mathbf{D})\mathbf{D}}{\mathbf{D}^2}}{\sqrt{\mathbf{E}^2 - \dfrac{(\mathbf{E}\cdot\mathbf{D})^2}{\mathbf{D}^2}}} = \frac{\mathbf{D}^2\mathbf{E} - (\mathbf{E}\cdot\mathbf{D})\mathbf{D}}{\sqrt{\mathbf{D}^2[\mathbf{E}^2\mathbf{D}^2 - (\mathbf{E}\cdot\mathbf{D})^2]}}. \tag{12}$$

By analogy with the refractive index n we may also define a *ray index* or *energy index* n_r by means of the formula

$$n_r = \frac{c}{v_r}. \tag{13}$$

By (7) and (9) we have

$$n_r = n\cos\alpha. \tag{14}$$

We shall now show that the ray index n_r and the unit vector $\mathbf{t}$ in the direction of the propagation of energy are given by formulae analogous to (11) and (12). We have from (14), (11) and the relation $\mathbf{E}\cdot\mathbf{D} = ED\cos\alpha$,

$$n_r^2 = \frac{\mu(\mathbf{E}\cdot\mathbf{D})}{\mathbf{E}^2}. \tag{15}$$

The unit vector $\mathbf{t}$ is perpendicular to $\mathbf{E}$ and coplanar with $\mathbf{E}$ and $\mathbf{D}$ and so must (apart perhaps from the sign) be given by the formula that results from interchanging $\mathbf{E}$ and $\mathbf{D}$ in (12). Hence

$$-\mathbf{t} = \frac{\mathbf{E}^2\mathbf{D} - (\mathbf{E}\cdot\mathbf{D})\mathbf{E}}{\sqrt{\mathbf{E}^2[\mathbf{E}^2\mathbf{D}^2 - (\mathbf{E}\cdot\mathbf{D})^2]}}. \tag{16}$$

The negative sign on the left ensures that $\mathbf{s}$ and $\mathbf{t}$ point towards the same side of $\mathbf{E}$ and $\mathbf{D}$ as in Fig. 15.1.

Both (12) and (16) reduce to $0/0$ when $\mathbf{E}$ and $\mathbf{D}$ coincide in direction, i.e. when $\mathbf{E}$ is in the direction of one of the principal axes of the crystal. This is to be expected since the directions of $\mathbf{s}$ and $\mathbf{t}$ are then undetermined apart from the fact that they must be perpendicular to $\mathbf{E}$.

We may also express the magnitude of the Poynting vector in terms of $\mathbf{E}$ and $\mathbf{D}$. According to (8), (13) and (15), remembering also that for a plane wave $w = 2w_e$

$$S = v_r w = \frac{c}{n_r}\frac{\mathbf{E}\cdot\mathbf{D}}{4\pi} = \frac{c}{4\pi\sqrt{\mu}}E\sqrt{\mathbf{E}\cdot\mathbf{D}}, \tag{17}$$

a formula that, in the case of an isotropic medium, is seen to be in agreement with §1.4 (8) and §1.4 (9).

15.2.2 Fresnel's formulae for the propagation of light in crystals

The formulae derived in §15.2.1 are consequences of Maxwell's equations alone and therefore independent of the properties of the medium. We shall now combine these with the material equation §15.1 (1).

We shall use a system of coordinate axes coincident with the principal dielectric

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

axes. Relations §15.1 (1) then reduce to the simpler form §15.1 (12), and substitution for **D** into (4) gives

$$\mu\varepsilon_k E_k = n^2[E_k - s_k(\mathbf{E}\cdot\mathbf{s})], \qquad (k = x,\ y,\ z). \tag{18}$$

Eqs. (18), being three homogeneous linear equations in E_x, E_y and E_z, can be satisfied by nonzero values of these components only if the associated determinant vanishes. This implies that a certain relation must be satisfied by the refractive index n, the vector $\mathbf{s}(s_x,\ s_y,\ s_z)$ and the principal dielectric constants ε_x, ε_y, ε_z. This relation may be derived by writing (18) in the form

$$E_k = \frac{n^2 s_k(\mathbf{E}\cdot\mathbf{s})}{n^2 - \mu\varepsilon_k}, \tag{19}$$

multiplying it by s_k and adding the resulting three equations; dividing the expression which then results by the common factor $\mathbf{E}\cdot\mathbf{s}$, we obtain

$$\frac{s_x^2}{n^2 - \mu\varepsilon_x} + \frac{s_y^2}{n^2 - \mu\varepsilon_y} + \frac{s_z^2}{n^2 - \mu\varepsilon_z} = \frac{1}{n^2}. \tag{20}$$

This formula may be expressed in a slightly different form. We multiply both sides of (20) by n^2 and subtract $s_x^2 + s_y^2 + s_z^2 = 1$. Next we multiply the resulting expression by $-n^2$ and find that

$$\frac{s_x^2}{\dfrac{1}{n^2} - \dfrac{1}{\mu\varepsilon_x}} + \frac{s_y^2}{\dfrac{1}{n^2} - \dfrac{1}{\mu\varepsilon_y}} + \frac{s_z^2}{\dfrac{1}{n^2} - \dfrac{1}{\mu\varepsilon_z}} = 0. \tag{21}$$

We define three *principal velocities of propagation* by the formulae*

$$v_x = \frac{c}{\sqrt{\mu\varepsilon_x}}, \qquad v_y = \frac{c}{\sqrt{\mu\varepsilon_y}}, \qquad v_z = \frac{c}{\sqrt{\mu\varepsilon_z}}. \tag{22}$$

When the expression (7) is used for the phase velocity v_p, (19) and (21) take the form

$$E_k = \frac{v_k^2}{v_k^2 - v_p^2} s_k(\mathbf{E}\cdot\mathbf{s}), \qquad (k = x,\ y,\ z), \tag{23}$$

$$\frac{s_x^2}{v_p^2 - v_x^2} + \frac{s_y^2}{v_p^2 - v_y^2} + \frac{s_z^2}{v_p^2 - v_z^2} = 0. \tag{24}$$

Eqs. (20), (21) and (24) are equivalent forms of *Fresnel's equation of wave normals*. This is a quadratic equation in v_p^2, as can be seen by multiplying (24) by the product of the denominators. Thus to every direction **s** there correspond two phase velocities v_p. (The two values $\pm v_p$ corresponding to any value v_p^2 are counted as one, since the negative value evidently belongs to the opposite direction of propagation $-\mathbf{s}$.) With each of the two values of v_p, (23) may then be solved for the ratios E_x:E_y:E_z; the corresponding ratios involving the **D** vector may be subsequently obtained from §15.1 (12). Since these ratios are real, the **E** and **D** fields are *linearly polarized*. Thus we have the important result that the *structure of an anisotropic medium permits two*

* Note that v_x, v_y, v_z are not components of a vector and are defined only with reference to the principal axes.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

monochromatic plane waves with two different linear polarizations and two different velocities to propagate in any given direction. It will be shown later that the two directions of the electric displacement vector **D** corresponding to a given direction of propagation **s** are perpendicular to each other.

We now show that there is an analogous formula for the ray velocity v_r. This is most easily done by showing first that there is a relation analogous to (4) in which the roles of **D** and **E** and of **s** and **t** are interchanged. It is convenient to introduce a vector $\mathbf{D}_\perp$ defined as the vector component of **D** perpendicular to **t**, in the plane of **D** and **t**. It is evidently given by

$$\mathbf{D}_\perp = \mathbf{D} - \mathbf{t}(\mathbf{D}\cdot\mathbf{t}). \tag{25}$$

Since the electric vector is also perpendicular to **t** and is coplanar with **D** and **t** (see Fig. 15.1), $\mathbf{D}_\perp$ is parallel to **E** and so may also be expressed in the form

$$\mathbf{D}_\perp = \left(\mathbf{D}\cdot\frac{\mathbf{E}}{E}\right)\frac{\mathbf{E}}{E} = \frac{n_r^2}{\mu}\mathbf{E}, \tag{26}$$

where (15) has been used. From (25) and (26) it follows that

$$\mathbf{E} = \frac{\mu}{n_r^2}[\mathbf{D} - \mathbf{t}(\mathbf{D}\cdot\mathbf{t})] = \frac{\mu}{n_r^2}\mathbf{D}_\perp. \tag{27}$$

This equation is analogous to (4) and may be formally obtained from it by interchanging the roles of **E** and **D**, n and $1/n_r$, μ and $1/\mu$ and **s** and $-$**t**. Quite generally the following *rule of duality* follows from the basic equations:

Let the variables be arranged in two rows as follows:

$$\left.\begin{array}{cccccccccccccc} \mathbf{E}, & \mathbf{D}, & \mathbf{s}, & \mathbf{t}, & c, & \mu & v_p, & n, & \varepsilon_x, & \varepsilon_y, & \varepsilon_z, & v_x, & v_y, & v_z, \\ \mathbf{D}, & \mathbf{E}, & -\mathbf{t}, & -\mathbf{s}, & \dfrac{1}{c}, & \dfrac{1}{\mu}, & \dfrac{1}{v_r}, & \dfrac{1}{n_r}, & \dfrac{1}{\varepsilon_x}, & \dfrac{1}{\varepsilon_y}, & \dfrac{1}{\varepsilon_z}, & \dfrac{1}{v_x}, & \dfrac{1}{v_y}, & \dfrac{1}{v_z}. \end{array}\right\} \tag{28}$$

If in any relation which holds between the quantities in one row each quantity is replaced by the corresponding quantity in the other row another valid relation is obtained.

Applying this rule to Fresnel's equation of wave normals (24) we immediately obtain the required *ray equation*

$$\frac{t_x^2}{\dfrac{1}{v_r^2}-\dfrac{1}{v_x^2}} + \frac{t_y^2}{\dfrac{1}{v_r^2}-\dfrac{1}{v_y^2}} + \frac{t_z^2}{\dfrac{1}{v_r^2}-\dfrac{1}{v_z^2}} = 0. \tag{29}$$

We may, of course, express this equation also in forms analogous to (20) and (21). Like (24) this is again a quadratic equation and gives two possible ray velocities v_r for each ray direction $t(t_x, t_y, t_z)$. The corresponding direction of **D** may be obtained on solving with the appropriate value v_r the equations dual to (23), namely

$$D_k = -\frac{v_r^2}{v_k^2 - v_r^2}t_k(\mathbf{D}\cdot\mathbf{t}), \qquad (k = x, y, z). \tag{30}$$

The directions of the two **E** vectors (which as we saw are orthogonal to **t**) may then be obtained by using §1.4.1 (12).

As a rule only one of the vectors **s** or **t** is given and it is, therefore, desirable to derive relations from which the other may be directly calculated. We have from Fig. 15.1

$$\mathbf{E}\cdot\mathbf{s} = E_\perp \tan\alpha, \qquad \mathbf{D}\cdot\mathbf{t} = -D\sin\alpha. \tag{31}$$

But from (4), $D = n^2 E_\perp/\mu$. Hence

$$\mathbf{D}\cdot\mathbf{t} = -\frac{n^2}{\mu}E_\perp \sin\alpha = -\frac{n^2}{\mu}\mathbf{E}\cdot\mathbf{s}\cos\alpha = -\frac{1}{\mu}\frac{c^2}{v_p v_r}\mathbf{E}\cdot\mathbf{s}, \tag{32}$$

where the relations (9) and (7) have been used. Substitution from (32) into (30) gives

$$D_k = \varepsilon_k E_k = \frac{1}{\mu}\frac{c^2 v_r}{v_p(v_k^2 - v_r^2)}t_k(\mathbf{E}\cdot\mathbf{s}). \tag{33}$$

Comparing (33) with (23) and remembering that $\mu\varepsilon_k v_k^2 = c^2$, we obtain

$$\frac{v_p s_k}{v_k^2 - v_p^2} = \frac{v_r t_k}{v_k^2 - v_r^2}. \tag{34}$$

Solving for t_k we have

$$t_k = \frac{v_p}{v_r}\frac{v_k^2 - v_k^2}{v_r^2 - v_p^2}s_k, \tag{35}$$

so that

$$v_r t_k - v_p s_k = v_p s_k \frac{v_r^2 - v_p^2}{v_p^2 - v_k^2}. \tag{36}$$

Squaring and adding the three equations (36) and using the relation §15.2 (9), namely $\mathbf{s}\cdot\mathbf{t} = v_p/v_r$, we obtain

$$v_r^2 - v_p^2 = v_p^2(v_r^2 - v_p^2)^2\left[\left(\frac{s_x}{v_p^2 - v_x^2}\right)^2 + \left(\frac{s_y}{v_p^2 - v_y^2}\right)^2 + \left(\frac{s_z}{v_p^2 - v_z^2}\right)^2\right]. \tag{37}$$

Hence we may define

$$g^2 \equiv v_p^2(v_r^2 - v_p^2) = \frac{1}{\left(\frac{s_x}{v_p^2 - v_x^2}\right)^2 + \left(\frac{s_y}{v_p^2 - v_y^2}\right)^2 + \left(\frac{s_z^2}{v_p^2 - v_z^2}\right)^2}. \tag{38}$$

This relation expresses v_r in terms of **s**, since v_p is already known in terms of **s** from Fresnel's equation (24). With v_r so determined, (35) then gives the unit ray vector **t** as a function of **s**. Using the expression g, (35) may be written as

$$t_k = \frac{s_k}{v_p v_r}\left(v_p^2 + \frac{g^2}{v_p^2 - v_k^2}\right) \qquad (k = x,\ y,\ z). \tag{39}$$

Since to every **s** there correspond in general two phase velocities v_p, there are in general two ray directions* for each wave-normal direction. There are, however, two

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

singular directions in certain crystals (biaxial crystals — cf. §15.3.1) due to vanishing of the denominators in (39), to each of which there corresponds an *infinite* number of rays; there are also two singular ray directions to each of which there corresponds an infinite number of wave normal directions. These special cases give rise to an interesting phenomenon (conical refraction), which will be investigated in §15.3.4.

15.2.3 Geometrical constructions for determining the velocities of propagation and the directions of vibration

Many results concerning the phase and the ray velocities and the directions of vibration may be illustrated by means of certain geometrical constructions.

(a) The ellipsoid of wave normals

By §15.1 (13) the components of the vector **D** at a given energy density $w = 2w_e$ satisfy the relation

$$\frac{D_x^2}{\varepsilon_x} + \frac{D_y^2}{\varepsilon_y} + \frac{D_z^2}{\varepsilon_z} = C \qquad (C = 8\pi w_e = \mathbf{E} \cdot \mathbf{D}). \tag{40}$$

Let us write x, y and z in place of $D_x/\sqrt{C}$, $D_y/\sqrt{C}$, $D_z/\sqrt{C}$, and consider these as Cartesian coordinates in space. Then

$$\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} = 1. \tag{41}$$

This equation represents an ellipsoid, the semiaxes of which are equal to the square roots of the principal dielectric constants and coincide in directions with the principal dielectric axes. We call this ellipsoid the *ellipsoid of wave normals* in preference to the widely used but rather vague term 'optical indicatrix' (also known as the index ellipsoid, or the reciprocal ellipsoid).

With the aid of the ellipsoid of wave normals we can find the two phase velocities v_p and the two directions of vibrations **D** which belong to a given wave-normal direction **s** as follows: We draw a plane through the origin at right angles to **s**. The curve of intersection of this plane with the ellipsoid is an ellipse; the principal semiaxes of this ellipse are proportional to the reciprocals $1/v_p$ of the phase velocities, and their directions coincide with the corresponding directions of vibrations of the vector **D** (Fig. 15.2).

To establish this result consider the two equations that specify the ellipse:

$$xs_x + ys_y + zs_z = 0, \tag{42}$$

$$\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} = 1. \tag{43}$$

Since the principal axes of an ellipse are, by definition, its shortest and longest diameters, we can determine them by finding the extrema of

* The position of the rays that correspond to a given wave normal is investigated in detail for the case of a biaxial crystal in M. Born, *Optik* (Berlin, Springer, 1933; reprinted 1965), pp. 235–237.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

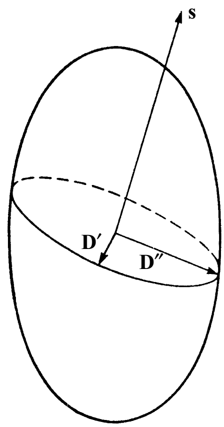

Fig. 15.2 The ellipsoid of wave normals. Construction of the directions of vibrations of the **D** vectors belonging to a wave normal **s**.

$$r^2 = x^2 + y^2 + z^2, \tag{44}$$

subject to the conditions (42) and (43). This we do by means of Lagrange's method of undetermined multipliers.* We introduce two multipliers $2\lambda_1$ and λ_2 and construct the function

$$F = x^2 + y^2 + z^2 + 2\lambda_1(xs_x + ys_y + zs_z) + \lambda_2\left(\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} - 1\right). \tag{45}$$

Our problem is then equivalent to finding the extremum of F subject to no subsidiary conditions. The necessary conditions for the extremum of F is that its derivatives with respect to x, y and z should vanish, i.e. that

$$x + \lambda_1 s_x + \frac{\lambda_2 x}{\varepsilon_x} = 0, \qquad y + \lambda_1 s_y + \frac{\lambda_2 y}{\varepsilon_y} = 0, \qquad z + \lambda_1 s_z + \frac{\lambda_2 z}{\varepsilon_z} = 0. \tag{46}$$

Multiplying these equations by x, y and z respectively and adding, we obtain, because of (42) and (43):

$$r^2 + \lambda_2 = 0. \tag{47}$$

Next we multiply (46) by s_x, s_y, and s_z and add, and again use (42). We then obtain

$$\lambda_1 + \lambda_2\left(\frac{xs_x}{\varepsilon_x} + \frac{ys_y}{\varepsilon_y} + \frac{zs_z}{\varepsilon_z}\right) = 0. \tag{48}$$

Substitution for λ_1 and λ_2 from (47) and (48) into (46) gives

$$x\left(1 - \frac{r^2}{\varepsilon_x}\right) + s_x r^2\left(\frac{xs_x}{\varepsilon_x} + \frac{ys_y}{\varepsilon_y} + \frac{zs_z}{\varepsilon_z}\right) = 0, \tag{49}$$

* For a full account of this method see, for example, R. Courant, *Differential and Integral Calculus*, Vol. II (London, Blackie & Son, Ltd., 1942), pp. 188–199.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

with two similar equations. With a given $\mathbf{s}$, these are three homogeneous equations for x, y and z. They are compatible only if the associated determinant vanishes, a condition that gives an algebraic equation for r^2. Now it is immediately seen that Eqs. (49) differ only in notation from equations (18). For if we replace x by $D_x/\sqrt{C}$, x/ε_x by $E_x/\sqrt{C}$ and r^2 by $\mathbf{D}^2/C = \mathbf{D}^2/\mathbf{E}\cdot\mathbf{D} = n^2/\mu$ [in accordance with (11)], (49) becomes

$$\mu D_x = n^2[E_x - s_x(\mathbf{E}\cdot\mathbf{s})], \tag{50}$$

which, together with the two similar equations, is identical with (18).

Thus we find that the roots of the determinantal equation for $n = c/v_p$ (which as we saw is of the second degree) are proportional to the lengths r of the semiaxes of the elliptical section at right angles to $\mathbf{s}$, and, moreover, that the two possible directions x:y:z of the vector $\mathbf{D}$ coincide with the directions of these axes. Since the axes of an ellipse are perpendicular to each other, we obtain the important result that *the directions of vibrations of the two vectors* $\mathbf{D}$ *corresponding to a given direction of propagation* $\mathbf{s}$ *are perpendicular to each other*. In what follows we shall denote the two $\mathbf{D}$ directions that correspond to a particular wave-normal direction $\mathbf{s}$ by $\mathbf{D}'$ and $\mathbf{D}''$; thus $\mathbf{s}$, $\mathbf{D}'$ and $\mathbf{D}''$ form an orthogonal triplet.

In the special case when the direction of propagation coincides with one of the principal axes of the ellipsoid of wave normals, say the x-axis, the extrema of r are, according to our construction, equal to the lengths of the other two semiaxes, i.e. to $\sqrt{\varepsilon_y}$ and $\sqrt{\varepsilon_z}$. But we saw that the extrema of r are also equal to $n/\sqrt{\mu} = c/v_p\sqrt{\mu}$. Hence *the phase velocities of waves which are propagated in the direction of the principal dielectric axis x are equal to $c/\sqrt{\mu\varepsilon_y}$ and $c/\sqrt{\mu\varepsilon_z}$, i.e. to the principal velocities of propagation v_y and v_z introduced formally by* (22). A corresponding result holds, of course, for propagation in the directions of the other two axes.

There is another construction by which the directions of vibrations can be determined. It is known that an ellipsoid has two circular sections C_1 and C_2 passing through the centre, and that the normals $\mathbf{N}_1$ and $\mathbf{N}_2$ to these sections are coplanar with the longest and shortest principal axes (z and x) of the ellipsoid. These two directions $\mathbf{N}_1$ and $\mathbf{N}_2$ are called the *optic axes*,* and will be considered more fully later (§15.3.3). Since the sections C_1 and C_2 are circular (also of the same radius), the directions $\mathbf{N}_1$, $\mathbf{N}_2$ have the property that there is only one velocity of propagation along them: $\mathbf{D}$ can then take *any* direction perpendicular to $\mathbf{s}$. Let E be the elliptical section through the centre, at right angles to an arbitrary unit normal $\mathbf{s}$. This plane intersects the circles C_1, C_2 in two radial vectors $\mathbf{r}_1$, $\mathbf{r}_2$ which are of equal length and must therefore make equal angles with the principal axes of E (see Figs. 15.3 and 15.4). The required directions of vibration are therefore the bisectors of the directions $\mathbf{r}_1$, $\mathbf{r}_2$. But $\mathbf{r}_1$ is perpendicular to $\mathbf{N}_1$ and $\mathbf{s}$, and is, therefore, perpendicular to the plane containing $\mathbf{N}_1$ and $\mathbf{s}$; similarly $\mathbf{r}_2$ is perpendicular to the plane containing $\mathbf{N}_2$ and $\mathbf{s}$. If these planes intersect the ellipse E in vectors $\mathbf{r}_1'$, $\mathbf{r}_2'$ the principal axes of the ellipse must also bisect the directions of $\mathbf{r}_1'$, $\mathbf{r}_2'$. It follows that *the planes of vibration of the electric displacement*, i.e. the planes containing $\mathbf{s}$ and $\mathbf{D}'$ or $\mathbf{D}''$, *are the internal or external*

* More precisely *the optic axes of wave normals*. The corresponding sections of the ray ellipsoid (see next page) define *the optic ray axes*.

† Throughout this chapter ($\mathbf{a}$, $\mathbf{b}$) denotes the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

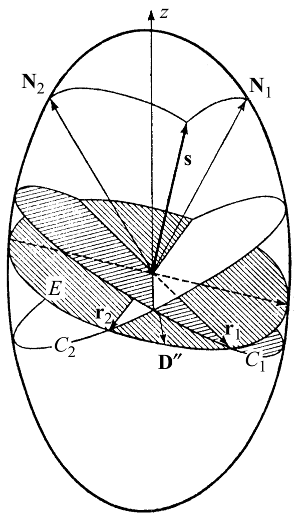

Fig. 15.3 A construction for determining the planes of vibrations ($\mathbf{s}$, $\mathbf{D}'$) and ($\mathbf{s}$, $\mathbf{D}''$).

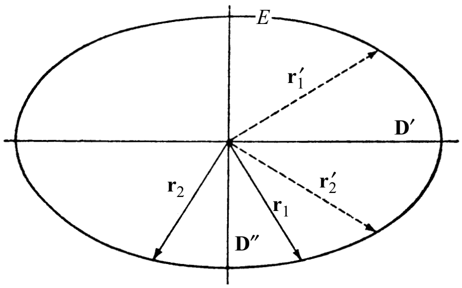

Fig. 15.4 The plane E of Fig. 15.3.

bisectors of the angle between the planes ($\mathbf{N}_1$, $\mathbf{s}$) *and* ($\mathbf{N}_2$, $\mathbf{s}$).† This construction becomes indeterminate when $\mathbf{s}$ coincides in direction with either $\mathbf{N}_1$ or $\mathbf{N}_2$, as we should expect.

(b) The ray ellipsoid

Rays must be treated in the same manner as wave normals if, in accordance with the duality rule (28), one starts from the *ray ellipsoid*

$$\varepsilon_x x^2 + \varepsilon_y y^2 + \varepsilon_z z^2 = 1. \tag{51}$$

In particular, the central section of this ellipsoid perpendicular to a ray direction $\mathbf{t}$ is an ellipse with semiaxes of lengths proportional to the two corresponding ray velocities v_r, oriented in the two permissible directions $\mathbf{E}'$ and $\mathbf{E}''$ of the electric vector. Thus $\mathbf{t}$, $\mathbf{E}'$ and $\mathbf{E}''$ form an orthogonal triplet of vectors.

(c) The normal surface and the ray surface

Imagine that with a fixed point O inside a crystal as origin two vectors are plotted in the same direction $\mathbf{s}$ and of lengths that are proportional to the two corresponding phase velocities. As $\mathbf{s}$ takes on all possible directions the end points give rise to a surface of two shells known as *the wave-normal surface* or, for short, *the normal surface*.

Similarly, the end points of position vectors plotted from a fixed origin in all directions $\mathbf{t}$, and of lengths that are proportional to the corresponding ray velocities will generate a two-sheeted surface, called *the ray surface*.

These two surfaces are more complicated than the ellipsoids that we just considered. The ray surface is of the fourth degree, and the surface of normals of the sixth degree,* as may be verified from the formulae (24) and (29). There exists an important relation between these two surfaces which we shall now derive.

We have seen that if $\mathbf{E}$ or $\mathbf{D}$ is known, the directions of $\mathbf{s}$ and $\mathbf{t}$ as well as the corresponding velocities v_p and v_r, and consequently the corresponding points (P and P' in Fig. 15.5) on the two surfaces are thereby determined. Let $\mathbf{r}$ and $\mathbf{r}'$ be the vectors representing these points:

$$\mathbf{r} = v_r\mathbf{t}, \qquad \mathbf{r}' = v_p\mathbf{s}. \tag{52}$$

We shall show that a small variation in $\mathbf{E}$ or $\mathbf{D}$ produces a change in the vector $\mathbf{r}$ at right angles to $\mathbf{r}'$.

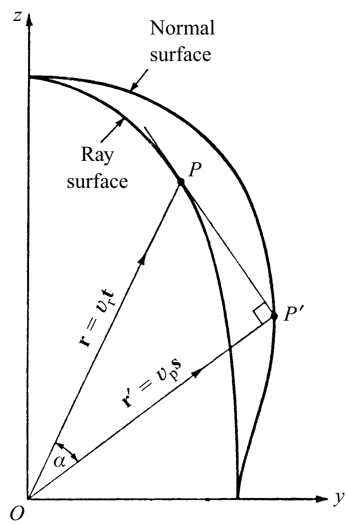

Fig. 15.5 The relation between the normal surface and the ray surface.

* The normal surface and the ray surface cannot be expected to have equations of the same degree, since they are not duals of each other. In order to construct the dual of the normal surface one would have to plot vectors of lengths $1/v_r$ (not v_r), in accordance with the duality rule (28).

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

We start from (27)

$$\frac{1}{\mu}\mathbf{E} = \frac{1}{n_r^2}[\mathbf{D} - \mathbf{t}(\mathbf{D}\cdot\mathbf{t})]. \tag{53}$$

Substitution for $\mathbf{t}$ from the first equation (52) gives, if we also set $n_r = c/v_r$:

$$\frac{c^2}{\mu}\mathbf{E} = \mathbf{r}^2\mathbf{D} - \mathbf{r}(\mathbf{D}\cdot\mathbf{r}). \tag{54}$$

Suppose now that $\mathbf{E}$ is changed by a small amount $\delta\mathbf{E}$. If $\delta\mathbf{D}$ and $\delta\mathbf{r}$ are the corresponding changes in $\mathbf{D}$ and $\mathbf{r}$ respectively, we have, according to (54),

$$\frac{c^2}{\mu}\delta\mathbf{E} = 2(\mathbf{r}\cdot\delta\mathbf{r})\mathbf{D} + \mathbf{r}^2\delta\mathbf{D} - \delta\mathbf{r}(\mathbf{D}\cdot\mathbf{r}) - \mathbf{r}(\delta\mathbf{r}\cdot\mathbf{D}) - \mathbf{r}(\mathbf{r}\cdot\delta\mathbf{D}). \tag{55}$$

If we multiply both sides of this equation scalarly by $\mathbf{D}$ and use the relation

$$\mathbf{D}\cdot\delta\mathbf{E} = \varepsilon_x E_x\delta E_x + \varepsilon_y E_y\delta E_y + \varepsilon_z E_z\delta E_z = \mathbf{E}\cdot\delta\mathbf{D}, \tag{56}$$

we obtain

$$\frac{c^2}{\mu}\mathbf{E}\cdot\delta\mathbf{D} = \delta\mathbf{D}\cdot[\mathbf{r}^2\mathbf{D} - \mathbf{r}(\mathbf{D}\cdot\mathbf{r})] + 2\delta\mathbf{r}\cdot[\mathbf{r}\mathbf{D}^2 - \mathbf{D}(\mathbf{D}\cdot\mathbf{r})]. \tag{57}$$

The terms having $\delta\mathbf{D}$ as a factor cancel because of (54) and the term multiplying $\delta\mathbf{r}$ may be written as $2\delta\mathbf{r}\cdot[(\mathbf{D}\times\mathbf{r})\times\mathbf{D}]$. Hence, since $\mathbf{r} = v_r\mathbf{t}$,

$$\delta\mathbf{r}\cdot[(\mathbf{D}\times\mathbf{t})\times\mathbf{D}] = 0. \tag{58}$$

Now the vector $\mathbf{D}\times\mathbf{t}$ is perpendicular to both $\mathbf{D}$ and $\mathbf{t}$ and so $(\mathbf{D}\times\mathbf{t})\times\mathbf{D}$ is in the plane of $\mathbf{D}$ and $\mathbf{t}$ and at right angles to $\mathbf{D}$; it is therefore parallel to $\mathbf{s}$ (see §15.1). Hence

$$\mathbf{s}\cdot\delta\mathbf{r} = 0, \tag{59}$$

i.e. $\delta\mathbf{r}$ is perpendicular to $\mathbf{s}$, thus proving our statement. This result implies that *the tangent plane of the ray surface is always perpendicular to the corresponding wave normal*. Fig. 15.5 illustrates this relation in plane section. Since the perpendicular distance from the origin to this plane is equal to $v_r\ \mathbf{t}\cdot\mathbf{s} = v_r\cos\alpha = v_p$ by (9), it follows that *the normal surface is the pedal surface of the ray surface* and conversely *the ray surface is the envelope of planes drawn through points on the normal surface, at right angles to the radius vectors from the origin to these points*. If we know the form of either of these surfaces, the other may be determined from this relationship.

We may interpret this result in more physical terms. Consider, not a single wave, but a group formed by plane waves of the same frequency with slightly different directions of propagation. The wave normals $\mathbf{s}$ of the component waves fill a solid angle around a 'mean wave normal' $\mathbf{s}_0$, and we assume that the amplitudes of only those waves are appreciable whose normals are close to $\mathbf{s}_0$. Suppose that at time $t = 0$ all the waves are in phase at a point O; the disturbance is then a maximum at this point. Let us now examine how this maximum is propagated.

Consider all the wave-fronts which pass through O at time $t = 0$. After a unit time a wave-front W, propagated with velocity v_p in the direction $\mathbf{s}$, will have reached a position W' such that the foot of the perpendicular dropped on to it from O has the position vector $v_p\mathbf{s}$; thus W' is a plane perpendicular to the appropriate radius vector of the normal surface. The amplitude of the group will be largest in a region where the waves reinforce each other, i.e. where this plane intersects the planes with neighbouring wave normals. But this is precisely the region around the envelope of these planes, i.e. in the neighbourhood of the corresponding points $v_r\mathbf{t}$ on the ray surface. This consideration confirms that the energy carried by the group is propagated with velocity v_r in the direction of the unit vector $\mathbf{t}$.

15.3 Optical properties of uniaxial and biaxial crystals

15.3.1 The optical classification of crystals

Transparent crystals fall into three distinct groups as regards their optical properties:

Group I. *Crystals in which three crystallographically-equivalent, mutually-orthogonal directions may be chosen*. These are crystals of the so-called *cubic* system. The equivalent directions evidently coincide with the principal dielectric axes and one has $\varepsilon_x = \varepsilon_y = \varepsilon_z$ ($= \varepsilon$ say); then $\mathbf{D} = \varepsilon\mathbf{E}$ and the crystal is *optically isotropic* and equivalent to an amorphous body.

Group II. *Crystals not belonging to group I in which two or more crystallographically-equivalent directions may be chosen in one plane*. These are crystals of the trigonal, tetragonal and hexagonal systems, the plane containing the equivalent directions being perpendicular to the axis of three-fold, four-fold or six-fold symmetry. One dielectric principal axis must coincide with this distinguished direction, whilst for the other two one may choose any orthogonal line pair perpendicular to it. If the distinguished direction is taken as the z-axis one then has $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$. Such crystals are said to be optically *uniaxial*.

Group III. *Crystals in which no two crystallographically-equivalent directions may be chosen*. These are the crystals belonging to the so-called orthorhombic, monoclinic and triclinic systems. Here $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$ and the directions of the dielectric axes may or may not be determined by symmetry (see Table 15.1) and may therefore be wavelength dependent. Crystals of this group are said to be optically *biaxial*.

That all crystals fall into these three types as regards their optical properties may be clearly seen by considering one of the associated ellipsoids, e.g. the ellipsoid of wave normals. Evidently the ellipsoid must be unchanged by the symmetry operations that leave the crystal structure unaltered.* Now there are only *two steps in the degeneration of an ellipsoid* into a sphere: an ellipsoid has (a) all axes of unequal lengths or (b) two axes equal and one unequal (spheroid, i.e. ellipsoid of revolution), or (c) all axes equal (sphere); these correspond to the three groups [in the order (III), (II) and (I)] which we just considered. The terms uniaxial and biaxial refer to the number of optic axes which the ellipsoid has, i.e. the number of diameters with the property that a plane section at right angles to them through the centre of the ellipsoid is a circle. A general ellipsoid has two such diameters (biaxial crystals), a spheroid has one (uniaxial crystals), and a sphere an infinite number (isotropic crystals).

* For instance, crystals of the monoclinic class are characterized by a two-fold axis parallel to, or a mirror plane perpendicular to, one of the crystal axes, or both. In each case it is clear that the ellipsoid must have one of its axes parallel to this crystal axis, in order to remain unchanged by the symmetry operation.

Of the many accounts of crystal classes and symmetry operations, we may cite C. W. Bunn, *Chemical Crystallography* (Oxford, Clarendon Press, 1945), Chapter 2.

Table 15.1.

Crystal system	Dielectric axes	Ellipsoid of wave normals	Optical classification
Triclinic	CCC 	General ellipsoid	Biaxial
Monoclinic	CCF 	General ellipsoid	Biaxial
Orthorhombic	FFF 	General ellipsoid	Biaxial
Trigonal Tetragonal Hexagonal	FRR 	Spheroid	Uniaxial
Cubic	RRR 	Sphere	Isotropic

C = Axis with colour dispersion; F = Axis fixed in direction; R = Freely rotatable, or indeterminate, axis.

Table 15.1 gives a survey of all the possible cases. Principal dielectric axes which may be colour-dependent are shown by two thin lines at a small angle to each other (signifying positions for two wavelengths), fixed axes by thick lines, and freely rotatable (or indeterminate) axes by broken lines ending on a circle or a sphere.

15.3.2 Light propagation in uniaxial crystals

We start from Fresnel's equation of wave normals §15.2 (24) and write it in the form

$$s_x^2(v_p^2 - v_y^2)(v_p^2 - v_z^2) + s_y^2(v_p^2 - v_z^2)(v_p^2 - v_x^2) + s_z^2(v_p^2 - v_x^2)(v_p^2 - v_y^2) = 0. \qquad (1)$$

For an optically uniaxial crystal with the optic axis in the z direction, $v_x = v_y$. Writing* v_o in place of this common velocity and v_e in place of v_z, (1) reduces to

$$(v_p^2 - v_o^2)[(s_x^2 + s_y^2)(v_p^2 - v_e^2) + s_z^2(v_p^2 - v_o^2)] = 0. \qquad (2)$$

Let ϑ denote the angle which the wave normal s makes with the z-axis; then

$$s_x^2 + s_y^2 = \sin^2\vartheta, \qquad s_z^2 = \cos^2\vartheta,$$

and (2) becomes

$$(v_p^2 - v_o^2)[(v_p^2 - v_e^2)\sin^2\vartheta + (v_p^2 - v_o^2)\cos^2\vartheta] = 0. \qquad (3)$$

The two roots of this equation (v_p' and v_p'' say) are given by

$$\left.\begin{aligned} v_p'^2 &= v_o^2, \\ \text{and} \qquad v_p''^2 &= v_o^2\cos^2\vartheta + v_e^2\sin^2\vartheta. \end{aligned}\right\} \qquad (4)$$

* Suffixes o and e stand here for ordinary and extraordinary, a terminology which will be clarified later.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

Eqs. (4) show that *the two shells of the normal surface are a sphere of radius* $v'_p = v_o$ *and a surface of revolution* (*of the fourth order*), *an ovaloid*. Thus one of the two waves that correspond to any particular wave-normal direction is an *ordinary wave*, with a velocity independent of the direction of propagation, the other an *extraordinary wave* with velocity depending on the angle between the direction of the wave normal and the optic axis. The two velocities are only equal when $\vartheta = 0$, i.e. when the wave normal is in the direction of the optic axis.

When $v_o > v_e$ [see Fig. 15.6(a)], the ordinary wave travels faster than the extraordinary wave (except for $\vartheta = 0$ when they are equal); such a crystal is said to be a *positive* uniaxial crystal (e.g. quartz). If $v_o < v_e$ [see Fig. 15.6(b)], the ordinary wave travels more slowly than the extraordinary wave, and we speak of a *negative* uniaxial crystal (e.g. felspar).

The directions of vibration may conveniently be found with the help of the ellipsoid of wave normals, which now has two of its principal axes equal. The plane containing the wave normal **s** and the optic axis *OZ* is called the *principal plane* (shaded in Fig. 15.7). The elipsoid is symmetrical about this plane. It follows that the elliptical section

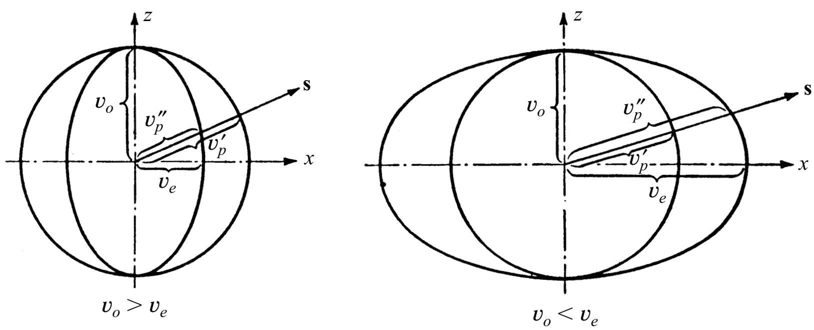

Fig. 15.6 The normal surfaces of a uniaxial crystal: (a) positive uniaxial crystal; (b) negative uniaxial crystal.

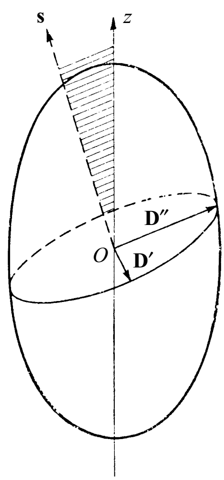

Fig. 15.7 The directions of vibrations in a uniaxial crystal.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

through O by the plane perpendicular to $\mathbf{s}$ is symmetrical about the principal plane, and therefore the principal axes of the ellipse are perpendicular and parallel to the principal plane, as shown in Fig. 15.7. The semiaxis perpendicular to the principal plane is equal to the radius of the equatorial circle of the spheroid, i.e. it is inversely proportional to the velocity v_o of the ordinary wave. We see that *the vector* $\mathbf{D}$ *of the ordinary wave* ($\mathbf{D}'$ in Fig. 15.7) *vibrates at right angles to the principal plane, the vector of the extraordinary wave* ($\mathbf{D}''$) *is in this plane*.

Optical phenomena in uniaxial crystals played a considerable part in the history of optics in connection with the question whether a 'light vector' vibrates at right angles or parallel to the plane of polarization (defined as the plane of incidence for reflection from a plane air–dielectric interface at an angle of incidence at which any incident wave becomes linearly polarized, i.e. the ($\mathbf{H}$,$\mathbf{s}$)-plane in the language of electromagnetic theory; see pp. 29 and 45). Today it is no longer relevant to discuss this question in detail,* since we know that there is no single physical entity that can be identified with a 'light vector'.

15.3.3 Light propagation in biaxial crystals

We shall now investigate the main consequences of the basic equation (1) in the general case of a biaxial crystal. It will help us to visualize the normal surface if we consider first the sections of this surface by the three coordinate planes $x = 0$, $y = 0$ and $z = 0$ of our reference system (principal dielectrical axes). In order to fix the form of the intersection curves uniquely the three axes will be labelled so that

$$\varepsilon_x < \varepsilon_y < \varepsilon_z \qquad (v_x > v_y > v_z). \tag{5}$$

If we set $s_x = 0$ in (1), the equation breaks into two factors, giving

$$\left.\begin{aligned} v_p'^2 &= v_x^2, \\ v_p''^2 &= v_z^2 s_y^2 + v_y^2 s_z^2. \end{aligned}\right\} \tag{6}$$

We set $v_p s_y = y$, $v_p s_z = z$; then $v_p^2 = y^2 + z^2$ and the equations become

$$y^2 + z^2 = v_x^2, \qquad (y^2 + z^2)^2 = v_z^2 y^2 + v_y^2 z^2. \tag{6a}$$

Thus the section of the normal surface by the coordinate plane $x = 0$ is a circle and an oval. The section by each of the other two coordinate planes likewise consists of a circle and an oval, the only difference being in the relative positions of the two curves. With the choice of axes indicated by (5), the circle lies completely outside the oval in the y,z-plane and lies completely inside it on the x,y-plane; in the z,x-plane the circle and the oval intersect in four points (Fig. 15.8). An octant of the normal surface is shown in perspective in Fig. 15.9 where each of the curves $A'B'C'N$ and $A''B''C''N$ is spanned by a smooth surface. In general two surfaces intersect in a curve, but in this case the two surfaces have only four points in common, the point N and the corresponding points in the other quadrants. The two lines joining the origin with each of these points are the two optic axes of wave normals. That there are no other such

* For an account of the historical background see E. T. Whittaker, *History of the Theories of Aether and Electricity*, Vol. I: *The Classical Theories* (London, T. Nelson, 1951), p. 116.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

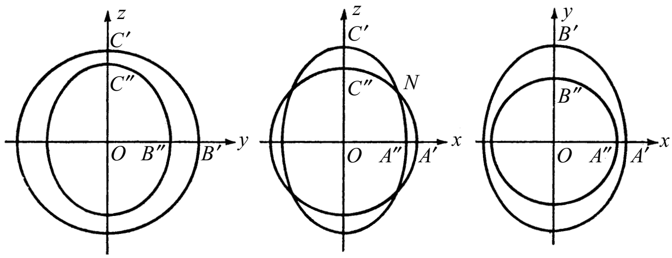

Fig. 15.8 Sections of a normal surface of a biaxial crystal.

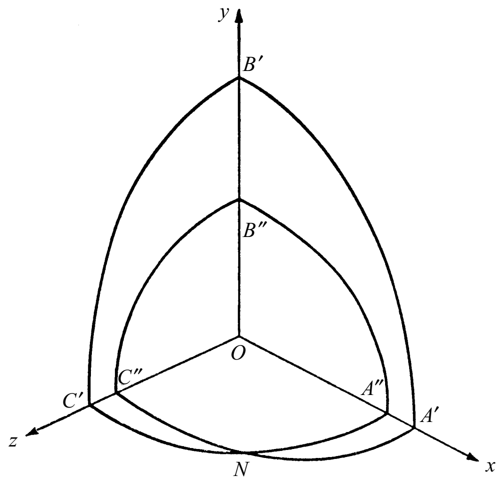

Fig. 15.9 The normal surface of a biaxial crystal.

points and, therefore, no other optic axes of wave normals, follows from the geometrical theorem relating to the number of central circular sections of an ellipsoid, mentioned on p. 801. But we will now confirm this by direct calculations and establish at the same time an inequality that we shall need later.

Let

$$v_x^2 = v_y^2 + q_x, \qquad v_z^2 = v_y^2 - q_z, \qquad v_p^2 = v_y^2 + q, \tag{7}$$

where q_x and q_z are positive because of (5). With this substitution (1) becomes

$$s_x^2 q(q + q_z) + s_y^2(q + q_z)(q - q_x) + s_z^2(q - q_x)q = 0, \tag{8a}$$

or

$$q^2 + [s_x^2 q_z + s_y^2(q_z - q_x) - s_z^2 q_x]q - s_y^2 q_x q_z = 0. \tag{8b}$$

Since the constant term $-s_y^2 q_x q_z$ cannot be positive, the roots of this equation must be real. If we denote them by q' and q'', then

$$q' q'' = -s_y^2 q_x q_z \leqslant 0.$$

Hence q' and q'' must have opposite signs. Let $q' \geqslant 0$ and $q'' \leqslant 0$.

If $q > q_x$ or $q < -q_z$ the terms on the left-hand side of (8a) would be positive. Hence q must lie in the range $-q_z \leqslant q \leqslant q_x$ and so

$$-q_z \leqslant q'' \leqslant 0 \leqslant q' \leqslant q_x. \tag{9}$$

The two roots q' and q'' can therefore only be equal if they both are zero; and for this to be the case, one must, according to (8b), have simultaneously

$$s_y^2 = 0, \qquad s_x^2 q_z = s_z^2 q_x. \tag{10}$$

Thus we obtain two directions **s** for each of which the corresponding two velocities are equal, confirming that there are two optical axes of wave normals. As already noted earlier, these axes lie in the x,z-plane. If β is the angle which one of the axes makes with the z direction, then $s_x = \sin\beta$, $s_z = \cos\beta$ where, according to (10),

$$\tan\beta = \frac{s_x}{s_z} = \pm\sqrt{\frac{q_x}{q_z}} = \pm\sqrt{\frac{v_x^2 - v_y^2}{v_y^2 - v_z^2}}; \tag{11}$$

thus the optic axes are situated symmetrically with respect to the z-axis.

In terms of the velocities, the inequalities (9) become

$$v_z^2 \leqslant v_p''^2 \leqslant v_y^2 \leqslant v_p'^2 \leqslant v_x^2, \tag{12}$$

a relation that we shall need later. We see that the two phase velocities are real whatever the direction of the wave normal; this conclusion is, of course, also evident from the geometrical construction for the phase velocities by means of the ellipsoid of wave normals.

The expressions for the two phase velocities corresponding to a given wave-normal direction **s** takes a very simple form if **s** is specified in terms of the angles ϑ_1 and ϑ_2 which it makes with the two optic axes of wave normals. Since the direction cosines of the optic axes are $\pm\sin\beta$, 0, $\cos\beta$, the angles ϑ_1 and ϑ_2 are given by the equations

$$\left.\begin{aligned} \cos\vartheta_1 &= s_x\sin\beta + s_z\cos\beta, \\ \cos\vartheta_2 &= -s_x\sin\beta + s_z\cos\beta. \end{aligned}\right\} \tag{13}$$

In terms of s_x and s_z, the roots of (8b) are

$$q = -\tfrac{1}{2}P \pm \tfrac{1}{2}\sqrt{\Delta}, \tag{14}$$

where, if we also use the identity $s_x^2 + s_y^2 + s_z^2 = 1$,

$$P = s_x^2 q_x - s_z^2 q_z + q_z - q_x, \tag{15}$$

$$\begin{aligned} \Delta &= P^2 + 4s_y^2 q_x q_z \\ &= (q_x + q_z)^2 - 2(q_x + q_z)(s_x^2 q_x + s_z^2 q_z) + (s_z^2 q_z - s_x^2 q_x)^2. \end{aligned} \tag{16}$$

Now from (13) and (11),

$$\cos\vartheta_1\cos\vartheta_2 = \frac{q_z s_z^2 - q_x s_x^2}{q_x + q_z}, \qquad \cos^2\vartheta_1 + \cos^2\vartheta_2 = 2\frac{q_x s_x^2 + q_z s_z^2}{q_x + q_z}. \tag{17}$$

Hence in terms of ϑ_1 and ϑ_2, (15) and (16) become

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$$P = q_z - q_x - (q_z + q_x)\cos\vartheta_1 \cos\vartheta_2, \tag{15a}$$

$$\Delta = [(q_z + q_x)\sin\vartheta_1 \sin\vartheta_2]^2, \tag{16a}$$

and we finally obtain, on substituting from (15a) and (16a) into (14), and replacing q, q_x and q_z by the expressions (7):

$$v_p^2 = \tfrac{1}{2}[v_x^2 + v_z^2 + (v_x^2 - v_z^2)\cos(\vartheta_1 \pm \vartheta_2)]. \tag{18}$$

Although v_y does not appear in this equation, it is contained implicitly in ϑ_1 and ϑ_2, since these angles depend on β and this is a function of all three principal velocities.

In the special case of a uniaxial crystal ϑ_1 and ϑ_2 are equal; it may readily be verified that (18) then correctly reduces to (4).

Similar analysis applies to the rays. Starting from the ray equation §15.2 (29) we find that the section of the ray surface with the coordinate plane $x = 0$ consists of the two curves

$$v_r'^2 = v_x^2, \qquad \frac{1}{v_r''^2} = \frac{t_y^2}{v_z^2} + \frac{t_z^2}{v_y^2}. \tag{19}$$

If we set $v_r t_y = y$, $v_r t_z = z$ then $v_r^2 = y^2 + z^2$ and (19) becomes

$$y^2 + z^2 = v_x^2, \qquad \frac{y^2}{v_z^2} + \frac{z^2}{v_y^2} = 1, \tag{19a}$$

i.e. the section of the ray surface by the plane $x = 0$ is a circle and an ellipse [not a more general type of an oval as in (6a)]. Each of the sections by the other two coordinate planes is likewise a circle and an ellipse. Because of the inequalities (5), the circle encloses the ellipse in the y,z-plane, whilst in the x,y-plane the ellipse encloses the circle; and in the x,z-plane the circle and the ellipse intersect in four points. These points specify the position of the *optic ray axes* R_1, R_2; their directions are given by the dual of (10):

$$t_y^2 = 0, \qquad t_x^2\left(\frac{1}{v_y^2} - \frac{1}{v_z^2}\right) = t_z^2\left(\frac{1}{v_x^2} - \frac{1}{v_y^2}\right). \tag{20}$$

The angle γ between either of these axes and the z-axis is therefore given by

$$\tan\gamma = \frac{t_x}{t_z} = \pm\frac{v_z}{v_x}\sqrt{\frac{v_x^2 - v_y^2}{v_y^2 - v_z^2}} = \pm\frac{v_z}{v_x}\tan\beta. \tag{21}$$

If $v_z < v_x$ the optic ray axis makes a smaller angle with the z axis than the axis of normals.

15.3.4 Refraction in crystals

(a) Double refraction

Consider a plane wave incident from vacuum on a plane surface Σ of an anisotropic medium. It will give rise to a transmitted and a reflected field. We shall briefly consider the nature of the transmitted field, using substantially the same argument as in the case of an isotropic medium (§1.5.1). We shall confine our attention to finding the direction of propagation of the disturbance within the crystal; expressions for the amplitude ratios (corresponding to the Fresnel formulae) will not be investigated.*

Let $\mathbf{s}$ be the unit wave normal of the incident wave and $\mathbf{s}'$ that of the transmitted wave. We shall see shortly that in general *two* waves are transmitted so that there are two possible values of $\mathbf{s}'$. The field vectors of the incident wave and of the transmitted waves are functions of $(t - \mathbf{r}\cdot\mathbf{s}/c)$ and $(t - \mathbf{r}\cdot\mathbf{s}'/v')$ respectively. The continuity of the field across the boundary demands that for any point $\mathbf{r}$ on the plane Σ and for all times t,

$$t - \frac{\mathbf{r}\cdot\mathbf{s}}{c} = t - \frac{\mathbf{r}\cdot\mathbf{s}'}{v'},$$

i.e.

$$\mathbf{r}\cdot\left(\frac{\mathbf{s}'}{v'} - \frac{\mathbf{s}}{c}\right) = 0. \tag{22}$$

Hence the vector $\mathbf{s}'/v' - \mathbf{s}/c$ must be perpendicular to the boundary.

The permissible wave normals $\mathbf{s}'$ may be determined as follows: With any point O on Σ as origin we plot vectors in all directions $\mathbf{s}'$, each of the length $1/v'$, where v' is the phase velocity corresponding to each $\mathbf{s}'$ in accordance with Fresnel equation §15.2 (24). The locus of the end points is a two-sheeted surface which differs from the normal surface in that each radius vector is of length $1/v'$ instead of v'. We call this surface the *inverse surface of wave normals*; it is the dual of the ray surface and, therefore, like the ray surface itself, is of the fourth degree. Since the required vector $\mathbf{s}'/v'$ must be such that $\mathbf{s}'/v' - \mathbf{s}/c$ is perpendicular to Σ, its end point Q' must be on the normal to Σ through the end point P of the vector $\mathbf{s}/c$. In general, the normal to Σ cuts the inverse surface in four points, two of which lie on the same side of the boundary as the crystal. Hence there are two such points (Q' and Q'' in Fig. 15.10), and therefore two possible wave-normal directions, so that *in general each incident wave will give rise to two refracted waves*; to each of these waves there corresponds a ray direction and a ray velocity describing the propagation of the energy within the crystal.† This is the phenomenon of *birefringence* or *double refraction*. It is illustrated by the well-known effect that two images are observed when a small object is viewed through a slab of calcite.

Taking the boundary plane Σ as the plane $z = 0$, (22) becomes

$$\frac{xs_x + ys_y}{c} = \frac{xs'_x + ys'_y}{v'} = \frac{xs''_x + ys''_y}{v''}, \tag{23}$$

which must be satisfied for all values of x and y. This implies, firstly, that $s'_x/s_x = s'_y/s_y$ and $s''_x/s_x = s''_y/s_y$, so that *the two refracted rays lie in the plane of*

* These are discussed, for example, in G. Szivessy, *Hdb. d. Phys.*, Vol. 20 (Berlin, Springer, 1928), p. 715.

† The origin of the double refraction may also be illustrated by generalizing Huygens' construction (§3.3.3) to anisotropic media. A proper formulation of this approach is, however, by no means as simple as is usually given in textbooks, since Huygens' construction deals with wavelets proceeding from point sources, and not with independent plane waves, the laws of propagation of which are as a rule taken over without justification. Some of the difficulties inherent in this approach were discussed by M. G. Lamé in his *Leçons sur la Théorie Mathématique de l'Élasticité des Corps Solides* (Paris, Gauthier-Villars, 2nd edition, 1866), and V. Volterra, *Acta Math.*, **16** (1892), 153.

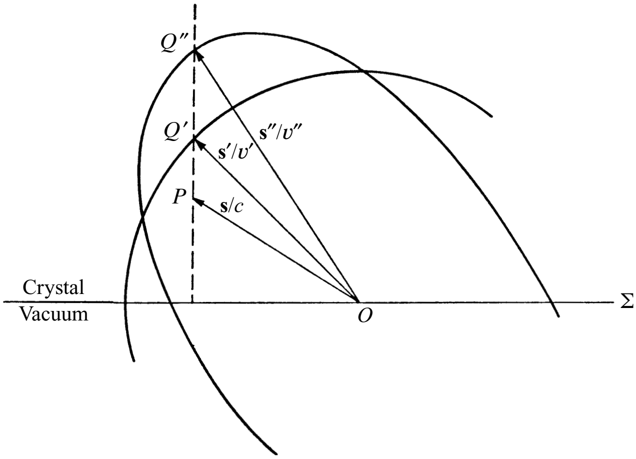

Fig. 15.10 Double refraction: construction for permissible wave normals.

incidence. Further, if θ_i, θ_t' and θ_t'' are the angles which the incident and the two transmitted waves make with the axis, we have from (23)

$$\frac{\sin\theta_i}{\sin\theta_t'} = \frac{c}{v'}, \qquad \frac{\sin\theta_i}{\sin\theta_t''} = \frac{c}{v''}. \tag{24}$$

Thus each of the transmitted waves is seen to obey the same *law of refraction* as in the case of isotropic media. However, the velocity v now depends on θ_t, so that the determination of the direction of propagation in the crystal is more complicated. In a uniaxial crystal one sheet of the inverse wave-normal surface is a sphere, so that the phase velocity of *one* of the transmitted waves is then independent of θ_t, this being an *ordinary wave*.

In the special case of normal incidence ($\theta_i = 0$) one has $\theta_t' = \theta_t'' = 0$, so that the two wave normals in the crystal coincide and are in the direction of the normal to Σ. Another special case, and one of a great theoretical interest, is that of a wave propagated in the direction of one of the optic axes of a biaxial crystal. It gives rise to a phenomenon known as conical refraction, which we shall now consider.

(b) Conical refraction

We mentioned earlier that when **s** coincides with one of the optic axes of the wave normals in a biaxial crystal, the relation between **s** and **t** has a singularity. We must investigate the nature of this singularity before discussing the refraction phenomena associated with propagation in this special direction.

We have shown in §15.2.3 that the electric displacement vectors **D** associated with a wave-normal direction **s** are parallel to the principal axes of the elliptical section through the ellipsoid of wave normals at right angles to **s**. Now in the special case when the vector **s** is in the direction of the optic axis of wave normals, the section is circular, so that all directions of **D** at right angles to **s** are then permissible; consequently an infinity of directions of the electric vector **E** [calculated for each **D** from §15.1 (1)] and an infinity of ray directions **t** (determined as in Fig. 15.1) are possible in this case. We shall now show that all these **t** vectors lie on the surface of a cone.

Let $\mathbf{s}'(s'_x, 0, s'_z)$ be the unit vector along one of the optic axes of wave normals, referred to the principal dielectric axes, subject to the inequality (5). Then s'_x and s'_z are related by (11) and the permissible **D** vectors, being perpendicular to $\mathbf{s}'$, satisfy the relation

$$s'_x D_x + s'_z D_z = 0.$$

In terms of the components of the corresponding **E** vectors, this relation may be written as

$$s'_x \varepsilon_x E_x + s'_z \varepsilon_z E_z = 0. \tag{25}$$

In Fig. 15.11 let Π be any plane perpendicular to $\mathbf{s}'$ and let the line in which the electric vector **E** is localized cut it at P. Since according to (25) all the **E** vectors must lie in a plane Λ perpendicular to the vector with components $(s'_x\varepsilon_x, 0, s'_z\varepsilon_z)$, all the possible points P must lie on the straight line AB in which the planes Π and Λ intersect. Now the ray vector **t** is coplanar with **E** and $\mathbf{s}'$ and perpendicular to **E**. Let **t** and $\mathbf{s}'$ cut the plane Π in the points T and S respectively. Then, by similar triangles,

$$TS \cdot SP = OS^2 = \text{constant}. \tag{26}$$

Since the locus of P is the straight line AB, the locus of T is the inverse of AB, that is,* a circle passing through the centre of inversion S and with tangent at S parallel to AB. Hence *there is an infinity of rays corresponding to the optic axis of wave normals, and these rays form a surface of a cone*. This cone is not a circular one, since the centre of the circle is not the foot of the perpendicular from O to the plane Π.

If **E**, $\mathbf{s}'$ and **t** all are in the x,z-plane, the direction of **t** must be the direction

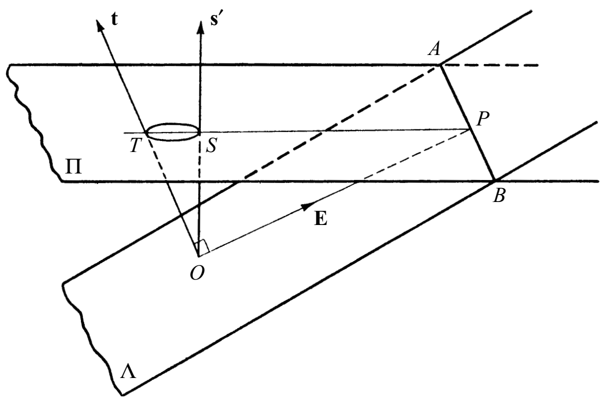

Fig. 15.11 Illustrating the position of rays that correspond to an optic axis of wave normals in a biaxial crystal.

* See, for example, D. M. Y. Sommerville, *Analytical Conics* (London, Bell and Sons, 1941), p. 92.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$(s'_x\varepsilon_x, 0, s'_z\varepsilon_z)$ to which **E** is always perpendicular. If this direction makes an angle ϕ with the z axis, the aperture angle χ of the cone in this plane is given by

$$\tan\chi = \tan(\beta - \phi) = \frac{\dfrac{s'_x}{s'_z} - \dfrac{s'_x\varepsilon_x}{s'_z\varepsilon_z}}{1 + \dfrac{\varepsilon_x s'^2_x}{\varepsilon_z s'^2_z}} = \frac{1}{v_y^2}\sqrt{(v_x^2 - v_y^2)(v_y^2 - v_z^2)}, \tag{27}$$

where (11) has been used. Usually $(v_x^2 - v_y^2)/v_y^2 \ll 1$, $(v_y^2 - v_z^2)/v_y^2 \ll 1$, so that the cone is very nearly circular and of angle χ.

We may show in a strictly similar manner that *there is an infinity of wave normals corresponding to the optic ray axis, and these wave normals form the surface of a cone*. The aperture angle ψ of this cone is given by the dual of (27), i.e. by

$$\tan\psi = \frac{1}{v_x v_z}\sqrt{(v_x^2 - v_z^2)(v_y^2 - v_z^2)} = \frac{v_y^2}{v_x v_z}\tan\chi. \tag{28}$$

The situation may conveniently be illustrated in terms of the normal surface and the ray surface, using the property, derived in §15.2.3, that the normal surface is the pedal surface of the ray surface. The intersection of these surfaces with the x,z-plane is shown in Fig. 15.12. The surface of normals intersects this plane in a circle of radius $v'_p = v_y$, and in an oval with the polar radius v''_p, whilst the ray surface intersects it in the same circle $v'_r = v_y$ and an ellipse v''_r. If the circle and the oval intersect at N, the line ON is in the direction of the optic axis of wave normals, and the plane through N perpendicular to ON must touch the ray surface at all points in which the cone of permitted ray directions cuts this surface. The ray surface thus has the unusual property that certain tangent planes touch it at an infinity of points.*

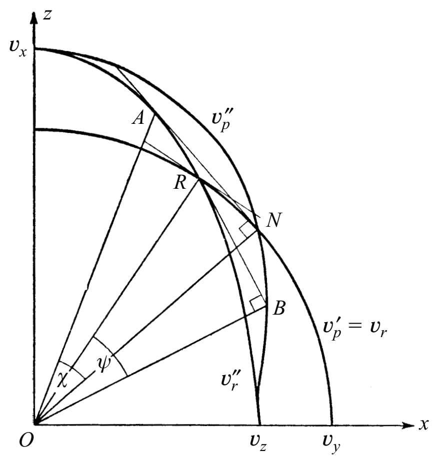

Fig. 15.12 Conical refraction: construction of the cones.

* The properties of this surface are discussed in detail in G. Salmon, *Analytical Geometry of Three Dimensions*. Revised by R. A. P. Rogers, Vol. 2 (London, Longmans, Green & Co., 1915, 5th edition), Chapter IV.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

The ray optic axis is represented by OR, where R is the point of intersection of the two sheets of the ray surface; the two sheets intersect in such a way that there is an infinite number of tangent planes at R, with their normals lying on a cone. The normals from O to these planes form the cone of wave normal directions corresponding to the ray direction OR. The aperture angles χ and ψ of the two cones are also shown in the figure. For reasons that will become evident shortly, the cone belonging to N, i.e. formed by rays such as OA, is called the *cone of internal conical refraction*; the cone belonging to R, i.e. formed by wave normals such as OB, is called *the cone of external conical refraction*.

Consider now a plate of a biaxial crystal, e.g. aragonite, cut so that its two parallel faces are perpendicular to the optic axis of wave normals. If this be illuminated by a narrow beam of monochromatic light incident normally on one of the faces of the plate, the energy will spread out in the plate in a hollow cone, the cone of internal conical refraction, and on emerging at the other side it will form a hollow cylinder, as shown in Fig. 15.13. Thus we should expect to see on a screen parallel to the crystal face, a bright *circular ring*. This remarkable phenomenon was predicted in 1832 by Sir William Rowan Hamilton, and confirmed a year later by Lloyd, who investigated it in aragonite on Hamilton's instigation. The success of this experiment represented one of the most striking confirmations of Fresnel's wave theory of light and contributed greatly to its general acceptance (see Historical introduction, p. xxxii).

In practice the demonstration of the phenomenon of conical refraction is less simple than we have just indicated, for naturally it is impossible to obtain an accurately parallel beam of monochromatic light. In an experiment one always has to use beams of finite angular aperture, and in this case, as Poggendorff* and Haidinger† were the first to show, one observes two bright circles separated by a fine dark circle, as illustrated in Fig. 15.14. In Lloyd's first experiments this structure escaped observation, as the apertures limiting the width of his beam were too large and the two bright circles were therefore blurred into one. It remained unexplained for a long time after its discovery, until Voigt‡ gave it an interpretation which may be summarized as follows:

We have to consider the propagation of waves whose normals are slightly inclined to the optic axis. Each of the wave normals will give rise to two rays inside the crystal,

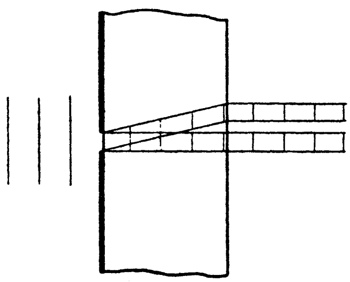

Fig. 15.13 Internal conical refraction.

* J. C. Poggendorff, *Pogg. Ann.*, **48** (1839), 461.
† W. v. Haidinger, *Wiener Ber.* (2), **16** (1854), 129; *Pogg. Ann.* (4), **96** (1855), 486.
‡ W. Voigt, *Phys. Z.*, **6** (1905), 672, 818.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

and we should expect that the directions of these rays will differ only slightly from the directions of the generators of the cone of internal conical refraction. In order to find how the transmitted rays are distributed, we must consider part of the ray surface near the circle of contact with the tangent plane *AN* in Fig. 15.12. This part of the surface may be likened to part of an inflated inner tube of a motor-car tyre, and the tangent plane to a flat board lying on the tyre. Fig. 15.15 shows the section with the x,z-plane. Now the points on the ray surface which represent the directions of the two rays corresponding to the wave-normal direction **s** are the points of contact of the two tangent planes to the ray surface perpendicular to **s** as shown in Fig. 15.5. When the wave normal *ON* is slightly displaced from the optic axis, the tangent plane splits into two parallel planes, one of which rolls over the surface so that its point of contact moves away from the centre of the circle of contact to *F*, while the second (which cannot be represented in our model because it would have to cut through the inner tube) rolls so that its point of contact moves towards the centre to *G*. Fig. 15.15 shows this for a displacement of the wave normal in the x,z-plane, but the same will happen for a displacement in any direction.

From these remarks it is seen that all incident rays with wave normals inclined at a small angle ϕ to the optic axis will give rise to pairs of rays inclined at angles $\frac{1}{2}\chi + a\phi$ and $\frac{1}{2}\chi - a\phi$ to the central axis of the cone of inner conical refraction, where a is some constant. Thus all the energy associated with the angular range ϕ and $\phi + d\phi$ in the incident beam will appear in two cones of angular semiapertures $\frac{1}{2}\chi \pm a\phi$ and angular separation $a\,\mathrm{d}\phi$. But the corresponding energy in the incident beam is proportional to

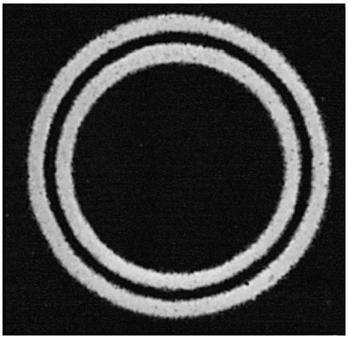

Fig. 15.14 Light distribution arising from conical refraction.

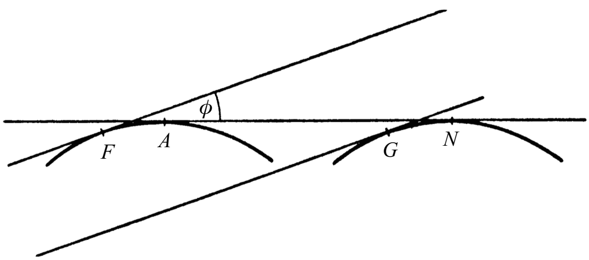

Fig. 15.15 Illustrating the position of rays belonging to wave normals that are inclined at small angles to an optic axis of wave normals.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

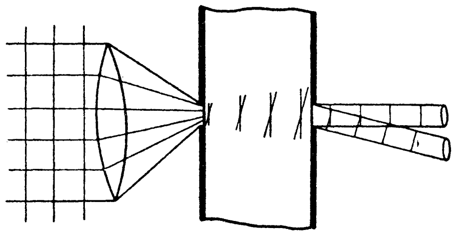

Fig. 15.16 External conical refraction.

$\phi\, \mathrm{d}\phi$, up to the maximum value of ϕ. It follows that, at angles $\frac{1}{2}\chi \pm a\phi$, the intensity in the cone of rays is proportional to ϕ, and in particular that it is zero at $\phi = 0$. Thus we must expect two bright circles to appear, with a dark circle between them, as is observed.

External conical refraction demonstrates the fact established earlier that there is a whole cone of wave-normal directions corresponding to a given ray direction. This phenomenon is observed with a crystal plate cut so that its faces are perpendicular to the optic ray axis. Small apertures are placed on each face exactly opposite to each other, and one of them is illuminated with convergent light, as shown in Fig. 15.16. Only those rays will reach the second aperture which have their directions very close to the direction of the optic ray axis, so that the waves reaching the second aperture all have their normals near the cone of the outer conical refraction. A cone of light will therefore emerge from the crystal. The angular aperture of this cone will be greater than the true angle ψ of the outer conical refraction, because of refraction from emergence from the crystal. Again, two concentric circles of light are observed on a screen parallel to the crystal face, and the explanation of this double circle is similar to that given for the case of internal conical refraction.

15.4 Measurements in crystal optics

In this section we shall describe briefly methods for determining the character of a crystal (i.e. whether uniaxial or biaxial), the position of its optic axes and the values of its principal refractive indices. The optic axes may be located, as we shall see, from observation of interference fringes on crystal plates; the structure of the pattern indicates clearly the intersection of the optic axes with the faces of the plate. The principal refractive indices may be determined with the help of crystal prisms, from measurements of the angles of deviation or of total reflection.

As a preliminary we must discuss the production and analysis of polarized light.

15.4.1 The Nicol prism

One of the most commonly used instruments for the production of linearly polarized light is the *Nicol prism*.* It consists of a natural rhomb of calcite (uniaxial crystal) which is cut into two equal parts along a diagonal plane (represented by *AC* in Fig.

* W. Nicol, *Edinburgh New Philos. Journ.*, **6** (1829), 83.

15.17), and with the two parts cemented together with Canada balsam. The rhomb is about three times as long as it is wide, with the angles at B and D of its principal section equal to 71°; the end faces AD and BC are ground off to reduce this angle to 68°.

A ray of light incident in the direction L parallel to the long edge is split into an ordinary ray and an extraordinary ray. For the first ray the Canada balsam is of lower, for the second of higher, optical density (calcite crystal: $n_0 = 1.66$, $n_e = 1.49$; Canada balsam $n = 1.53$), and it may easily be verified from the formulae of §1.5.4 that on the Canada balsam interfaces conditions for total reflection are satisfied with respect to the ordinary ray; this ray is totally reflected towards the face DC which is blackened and so absorbs it. The extraordinary ray passes through the prism with practically no lateral displacement and is linearly polarized with its **D** vector in the principal section (see §15.3.2). Thus *the Nicol prism produces linearly polarized light, whose direction of vibration is known.*

If the incident ray is inclined to the edge of the rhomb the Nicol prism will still act as a polarizer, provided that for the ordinary ray the angle of incidence on the Canada balsam is not smaller than the critical angle; this limits the angle of the cone of incident rays in air for which the prism is effective to about* 30°.

Fairly pure linearly polarized light may also be obtained by passing natural light through a sheet of an absorbing crystalline material whose absorption coefficients for the two directions of vibrations are appreciably different. We shall consider such 'polaroid sheets' in §15.6.3.

An arrangement such as a Nicol prism, which produces linearly polarized light from light of other states of polarization, is called a *polarizer*. Such an arrangement may also be used as an *analyser*, i.e. as a detector of linearly polarized light and of its direction of vibration. To detect linearly polarized light with a Nicol prism we only have to rotate the prism about its longitudinal axis and note whether there is a position when no light is passed through it. If there is such a position the light is linearly polarized and the direction of vibration of its **D** vector is perpendicular to the principal section.

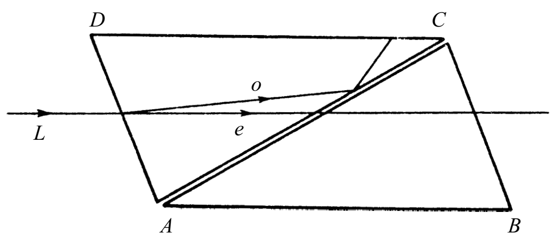

Fig. 15.17 The Nicol prism.

* There are several modifications of the Nicol prism which can be used over a larger angular field. For the description of these see, for example, L. C. Martin, *An Introduction to Applied Optics*, Vol. 1 (London, Pitman, 1930), p. 204, or R. W. Wood, *Physical Optics* (New York, Macmillan, 3rd edition, 1934), pp. 337–338.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

15.4.2 Compensators

Crystalline material may also be used to investigate elliptically polarized light, i.e. the directions of its axes and the ratio of their lengths. For this purpose a crystal plate of suitable material and thickness is employed, and by means of it a path difference is introduced between vibrations in two mutually orthogonal directions. In particular the path difference may be made such as to convert elliptically polarized light into light that is linearly polarized; from the analysis of the linearly polarized light the required information about the elliptically polarized light is then obtained. Such a device is called a *compensator*, as its function is to compensate a phase difference.*

(a) The quarter-wave plate

Consider a plane-parallel crystal plate of thickness h. Let the z-axis be normal to the plate and the x- and y-axes in the directions of the corresponding **D**-vibrations. We may assume that the plate is turned round the normal until the x- and y-axes are parallel to the principal axes of the vibrational ellipse of the incident light. The components of the **D** vector of the incident light are then represented by

$$\left.\begin{aligned} D_x^{(i)} &= a\cos\omega t,\\ D_y^{(i)} &= b\sin\omega t.\end{aligned}\right\} \tag{1}$$

On traversing the plate the two components will suffer different changes in phase because the velocities of the two rays are different. Neglecting reflection losses, the components of the **D** vector of the light emerging from the plate are given by

$$\left.\begin{aligned} D_x^{(t)} &= a\cos(\omega t+\delta'),\\ D_y^{(t)} &= b\sin(\omega t+\delta''),\end{aligned}\right\} \tag{2}$$

where

$$\delta' = \frac{2\pi}{\lambda}n'h, \qquad \delta'' = \frac{2\pi}{\lambda}n''h, \tag{3}$$

λ being the vacuum wavelength.† Hence the phase difference introduced by the plate is

$$\delta''-\delta' = \frac{2\pi}{\lambda}(n''-n')h. \tag{4}$$

In particular, if the emergent light is to be *linearly polarized*, one must have $\delta''-\delta' = \pm\frac{1}{2}\pi$, or more generally $\delta''-\delta' = (2m+1)\pi/2$, where m is any integer, so that the plate must be of thickness

* For a detailed survey of compensators see H. G. Jerrard, *J. Opt. Soc. Amer.*, **38** (1948), 35. A survey of methods available for the analysis of polarized light is given in a paper by M. Richartz and Hsien-Yü Hsü, *ibid.*, **39** (1949), 136.

A systematic procedure for the theoretical analysis of fully or partially polarized light is afforded by the coherency matrices or by the Stokes parameters, discussed in §10.9.

† In contrast to §1.3 we write λ rather than λ_0 here, as suffix o refers throughout this section to the ordinary ray.

$$h = \left|\frac{2m+1}{n''-n'}\right|\frac{\lambda}{4}. \tag{5}$$

With this compensator the direction of the linear vibration of the transmitted light is given by

$$\frac{D_y^{(t)}}{D_x^{(t)}} = \pm\frac{b}{a}. \tag{6}$$

A compensator which introduces a phase difference $|\delta'' - \delta'| = \pi/2$, i.e. one for which the difference in the two optical thicknesses is a quarter of a wavelength, is called a *quarter-wave plate**. It may conveniently be made of a foil of mica (biaxial crystal), split to the thickness $h = \lambda/4|n'' - n'|$.

Elliptically polarized light may be analysed with it in the following way: the light is passed through a quarter-wave plate and then through a Nicol prism and both are rotated independently until the field seen through the Nicol prism becomes completely dark. This is the position where the axes of the mica plate are parallel to the axes of the vibrational ellipse of the incident light, and where, according to (6), the Nicol prism is set so as to extinguish linearly polarized light with its $\mathbf{D}$ vector inclined at an angle $\tan^{-1} b/a$ to the x-axis. Thus the orientation of the axes of the ellipse and their ratio are found.

If the incident light is circularly polarized, then $b = +a$ or $b = -a$ and the $\mathbf{D}$ vector of the transmitted light is linearly polarized in a direction which makes an angle of 45° or 135° respectively with the axis OX. The former corresponds to left-handed, the latter to right-handed polarization.

Since the thickness of the quarter-wave plate depends on λ, exact compensation is only possible for monochromatic light of one particular wavelength. In order to achieve compensation for light of any given wavelength a wedge or a combination of wedges must be used in place of a single parallel-sided plate. We shall next consider some compensators of this type.

(b) Babinet's compensator

The compensator due to Babinet† allows the realization of all phase differences (including zero). It consists of two wedges of quartz (positive uniaxial crystal), with equal acute angles. The wedges are placed against each other as shown in Fig. 15.18, and can be displaced along their plane of contact, thus forming a parallel plate of variable thickness. In one wedge the optic axis is parallel, in the other at right angles, to the edge.

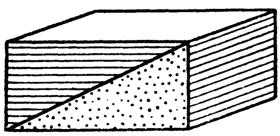

Fig. 15.18 Babinet's compensator.

* G. Airy, *Trans. Camb. Phil. Soc.*, **4** (1833), 313.
† J. Babinet, *C. R. Acad. Sci., Paris*, **29** (1849), 514. J. Jamin, *Ann. Chim. (Phys.)*, (3), **29** (1850), 274.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

Let n_o and n_e be the ordinary and extraordinary refractive indices of quartz and h_1 and h_2 the thicknesses of the two wedges at some particular point. On passing through both wedges, the phase difference between the two rays is

$$\delta = \frac{2\pi}{\lambda}(n_e - n_o)(h_1 - h_2). \tag{7}$$

The two contributions enter here with opposite signs, since the ray whose vector vibrates at right angles to the principal axis (the ordinary ray) is the faster one, and consequently the component vibrating in a direction parallel to the edge will be ahead of the other component in one wedge and lag behind it in the other. Now h_1 and h_2 and consequently also δ change continuously as the point of incidence moves across the plate, and δ is zero in the middle. In consequence there will be a series of lines along which the transmitted light is linearly polarized and can thus be extinguished by a suitably oriented Nicol prism.

Taking the x-axis parallel to the edge, the elliptically polarized incident light may be represented by

$$\left.\begin{aligned} D_x^{(i)} &= a_1 \cos(\omega t + \delta_o'), \\ D_y^{(i)} &= a_2 \cos(\omega t + \delta_o''). \end{aligned}\right\} \tag{8}$$

The light transmitted by the compensator is represented by

$$\left.\begin{aligned} D_x^{(t)} &= a_1 \cos(\omega t + \delta'), \\ D_y^{(t)} &= a_2 \cos(\omega t + \delta''), \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} \delta' &= \delta_0' + \frac{2\pi}{\lambda} n_o(h_1 - h_2), \\ \delta'' &= \delta_0'' + \frac{2\pi}{\lambda} n_e(h_1 - h_2). \end{aligned}\right\} \tag{10}$$

For the emergent light to be linearly polarized one must have $\delta'' - \delta' = m\pi$ ($m = 0$, $\pm 1, \pm 2, \ldots$), i.e.

$$\delta_0'' - \delta_0' = -\frac{2\pi}{\lambda}(n_e - n_o)(h_1 - h_2) + m\pi. \tag{11}$$

The direction of the linear vibrations is given by

$$\frac{D_y^{(t)}}{D_x^{(t)}} = \pm\frac{a_2}{a_1}. \tag{12}$$

Suppose first that a Nicol prism is placed in front of the compensator so that the compensator receives linearly polarized light. Then $\delta_0'' = \delta_0'$, and if the compensator is followed by an analyser that is crossed with the polarizing Nicol prism, dark bands appear running parallel to the edge through points for which the right-hand side of (11) is a multiple of π. These dark bands determine the *zero position*. If next elliptically polarized light is examined by means of the compensator and the analysing prism alone, the displacement of the dark bands from the zero position immediately determines the phase difference $\delta_0'' - \delta_0'$ of the incident light; and the amplitude ratio of the components of $\mathbf{D}^{(i)}$ parallel and perpendicular to the edge can be determined from the orientation of the analyzer, using (12). From these data the position of the principal axes of the vibrational ellipse and their ratio can be found by application of the formulae of §1.4.2.

(c) Soleil's compensator

For some purposes it is necessary to produce a phase difference (positive, negative or zero) which is constant over the whole field of view. This may be achieved by a compensator due to Soleil.* It contains two quartz wedges A and A' which form, as in Babinet's compensator, a plane-parallel plate, but with the difference that the optic axes in *both* wedges are now parallel to the edges. The lower wedge is cemented to a plane-parallel quartz plate B whose optic axis is at right angles to the edge (see Fig. 15.19). The effective path difference which this compensator introduces between the two rays is evidently $h_B - (h_A + h'_A)$. In the zero position, where this difference vanishes, the whole field of view can be obscured by a suitably oriented analyser. The effective path difference can be altered by shifting the upper wedge, but for each position the path difference remains constant over the whole field. The analysis of elliptically polarized light is similar to that with Babinet's compensator.

(d) Berek's compensator

A compensator useful in biological microscopy (e.g. for measuring path differences in threads of birefringent material) is due to Berek.† It consists of a slab of a uniaxial crystalline medium with its optic axes perpendicular to the faces of the slab. Phase differences between the ordinary and extraordinary rays are introduced by tilting the slab. It is mounted so that it can be turned by a graduated wheel which may be calibrated to show directly the path differences introduced.

15.4.3 Interference with crystal plates

As remarked earlier, the directions of optic axes may be determined by means of interference phenomena on crystal plates. The beautiful interference effects which are

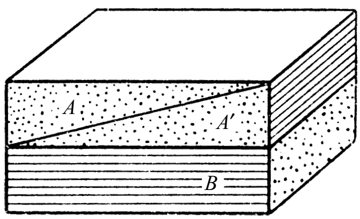

Fig. 15.19 Soleil's compensator.

* H. Soleil, *C. R. Acad. Sci., Paris*, **21** (1845), 426; **24** (1847), 973; **26** (1848), 162. J. Duboscq and H. Soleil, *ibid*., **31** (1850), 248.

† M. Berek, *Zbl. Miner. Geol. Paläont*. (1913), 388, 427, 464, 580.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

then observed, and which give a striking demonstration of the agreement between theory and experiment in crystal optics, deserve to be studied in their own rights.

Consider first a beam of linearly polarized light emerging from a polarizer and incident normally on to a plane-parallel crystal plate of thickness h. On entering the plate, each ray is divided into two rays with different velocities of propagation, and with their **D** vectors vibrating in two mutually orthogonal directions at right angles to the direction of the plate normal. They emerge from the plate with a certain phase difference δ. If an analysing Nicol prism is placed behind the plate, components of the two vibrations in a certain direction are then singled out and may be brought to interference in the focal plane of a lens placed behind the analyser.

In Fig. 15.20, the plane of the drawing is parallel to the plate. $\mathbf{D}'$ and $\mathbf{D}''$ represent the two mutually orthogonal directions of vibrations in the crystal, and OP and OA are the directions of vibrations that are passed by the polarizer and the analyser respectively. Let ϕ be the angle that OP makes with $\mathbf{D}'$ and χ the angle between OA and OP. The amplitude of the light incident on the plate is represented by the vector OE (parallel to OP); its components in the directions of $\mathbf{D}'$ and $\mathbf{D}''$ are

$$OB = E\cos\phi, \qquad OC = E\sin\phi. \tag{13}$$

The analyser transmits only the components parallel to OA which, as may be seen from the figure and from (13), have amplitudes

$$OF = E\cos\phi\cos(\phi-\chi), \qquad OG = E\sin\phi\sin(\phi-\chi). \tag{14}$$

On leaving the plate, the two components differ in phase by the amount

$$\delta = \frac{2\pi}{\lambda}(n''-n')h. \tag{15}$$

According to §7.2 (15), the intensity obtained from the interference of two monochromatic waves with phase difference δ is given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta,$$

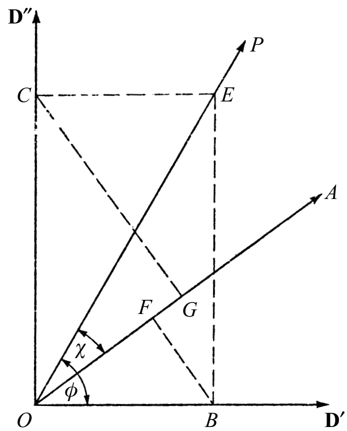

Fig. 15.20 Construction of the vibration components transmitted by a polarizer and an analyser.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

where I_1 and I_2 are the intensities (squared amplitudes) of the two waves. With the amplitudes given by (14) we have

$$I = E^2\left[\cos^2\chi - \sin 2\phi\,\sin 2(\phi-\chi)\,\sin^2\frac{\delta}{2}\right], \tag{16}$$

where the identity $\cos\delta = 1 - 2\sin^2(\delta/2)$ has been used.

If the plate were removed ($\delta = 0$), the intensity would be $I = E^2\cos^2\chi$; thus the second term in (16) represents the effect of the crystal plate.

We now consider two important special cases:

(a) *Analyser and polarizer mutually parallel* ($\chi = 0$). In this case (16) reduces to

$$I_{\parallel} = E^2\left(1 - \sin^2 2\phi\,\sin^2\frac{\delta}{2}\right). \tag{17}$$

There is *maximum transmission* when

$$\phi = 0, \frac{\pi}{2}, \pi, \ldots, \tag{18}$$

i.e. *when the direction of vibrations passed by the analyzer coincides with one of the directions of vibrations in the plate*. The positions (18) are separated by positions of *minima of transmission*, given by $\sin 2\phi = \pm 1$, i.e. by

$$\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots, \tag{19}$$

the minima being

$$I_{\parallel\,\mathrm{min}} = E^2\left(1 - \sin^2\frac{\delta}{2}\right) = E^2\cos^2\frac{\delta}{2}. \tag{20}$$

The minima are not completely dark unless δ is an odd multiple of π, i.e. unless the plate thickness has one of a number of permissible values, depending on the wavelength of the light.

(b) *Analyser and polarizer mutually perpendicular* ($\chi = \pi/2$). In this case (16) gives

$$I_{\perp} = E^2\sin^2 2\phi\,\sin^2\frac{\delta}{2}. \tag{21}$$

Comparison with (17) shows that the interference phenomena are now *complementary*. There are *minima of complete darkness* when ϕ has one of the values given by (18) and relative *maxima* for the intermediate positions (19) whose values are

$$I_{\perp\,\mathrm{max}} = E^2\sin^2\frac{\delta}{2}. \tag{22}$$

These phenomena may be used to produce colours that are exactly complementary; one only has to pass a parallel beam of white light through two Nicol prisms separated by a crystal plate, with the prisms being first parallel ($\chi = 0$) and then crossed ($\chi = \pi/2$). In order to obtain a uniform field it is essential to have the beam sufficiently well collimated to ensure that no significant phase differences are introduced.

If the light incident on the first polarizer originates in an extended incoherent source placed in the focal plane of a lens, each point in the source will give rise to an intensity distribution in the conjugate point, independently of all the other source points. In the image plane one then observes a light distribution which can be described by *curves of equal intensity*, forming a so-called *interference figure* of the crystal plate. To every point in the image plane there corresponds a direction of parallel rays entering and leaving the crystal and we must, therefore, consider the variation of the phase difference δ with this direction. It will be sufficient to consider only the case when the polarizer and analyser are crossed (i.e. $\chi = \pi/2$); for when they are parallel the pattern is complementary, whilst other cases give less marked interference figures.

The intensity corresponding to a given direction of incidence depends on ϕ and δ and it is useful to consider the effects of varying each of these quantities separately. The curves along which ϕ is constant are called *isogyres* (or brushes), those along which δ is constant are called *isochromates*. The isogyres depend on the orientation of the optic axes in the plate and are independent of the thickness of the plate and the wavelength, unless there is dispersion of the axes. The isochromates depend on the direction of the wave-normals and on the thickness of the plate, and are so named because, if white light is used, they are lines of equal colour. The curves of these families along which the intensity is zero are called *principal isogyres* and *principal isochromates* and are according to (21) given by $\sin 2\phi = 0$ and $\sin\frac{1}{2}\delta = 0$ respectively. On these curves the state of polarization of the light is the same as before the passage through the crystal. This is so because on the principal isogyres the direction of vibrations passed by the analyser coincides with one of the directions of vibrations in the crystal, and on the principal isochromates the phase difference between the two emerging beams is an integral multiple of 2π. The two systems of curves are superimposed, but may be studied separately.

Before investigating the form of these curves we must consider how the phase difference δ depends on the angle of incidence. Let SA, AB', AB'' represent the wave normals to the incident and the two refracted waves at A and let θ_1, θ_2', θ_2'' be the angles of incidence and the two angles of refraction respectively (see Fig. 15.21). Further let λ be the wavelength in the first medium (air) and $\lambda' = \lambda/n'$, $\lambda'' = \lambda/n''$ the wavelengths of the two refracted waves. The rays will emerge from the plate parallel to each other and to the incident wave normal (see Fig. 15.21) and with a phase difference

$$\delta = 2\pi\left(\frac{AB''}{\lambda''} + \frac{B''C}{\lambda} - \frac{AB'}{\lambda'}\right), \tag{23}$$

where

$$AB' = \frac{h}{\cos\theta_2'}, \qquad AB'' = \frac{h}{\cos\theta_2''}, \tag{24}$$

and

$$B''C = B''B'\sin\theta_1 = h\sin\theta_1(\tan\theta_2' - \tan\theta_2''). \tag{25}$$

Substituting from (24) and (25) into (23), we obtain

$$\delta = 2\pi h\left[\frac{1}{\cos\theta_2''}\left(\frac{1}{\lambda''} - \frac{\sin\theta_1\sin\theta_2''}{\lambda}\right) - \frac{1}{\cos\theta_2'}\left(\frac{1}{\lambda'} - \frac{\sin\theta_1\sin\theta_2'}{\lambda}\right)\right]. \tag{26}$$

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

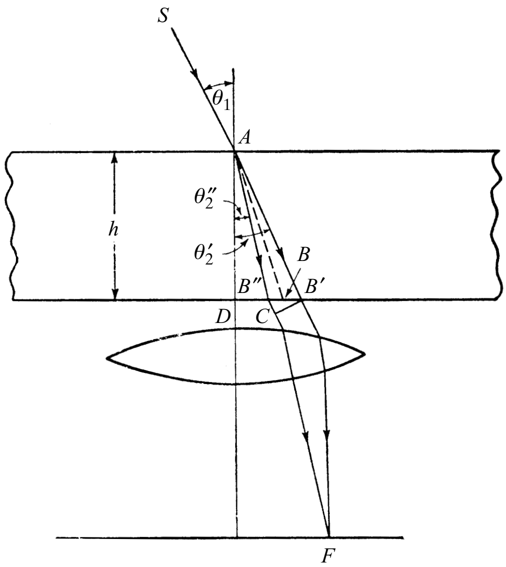

Fig. 15.21 Determination of the phase difference between two waves transmitted by a crystal plate.

Making use of the law of refraction we may replace $\sin\theta_1/\lambda$ by $\sin\theta_2''/\lambda''$ in the first bracket and by $\sin\theta_2'/\lambda'$ in the second bracket, giving

$$\delta = 2\pi h\left[\frac{\cos\theta_2''}{\lambda''} - \frac{\cos\theta_2'}{\lambda'}\right] = \frac{2\pi h}{\lambda}(n''\cos\theta_2'' - n'\cos\theta_2'). \tag{27}$$

As the difference $n'' - n'$ is always small compared with n' and n'', it is permissible to use in place of (27) an approximate expression. We have, to first order,

$$\begin{aligned} n''\cos\theta_2'' - n'\cos\theta_2' &= (n'' - n')\frac{\mathrm{d}}{\mathrm{d}n}(n\cos\theta_2) \\ &= (n'' - n')\left(\cos\theta_2 - n\sin\theta_2\frac{\mathrm{d}\theta_2}{\mathrm{d}n}\right), \end{aligned} \tag{28}$$

where n is an average value of n' and n'' and θ_2 the corresponding average of θ_2' and θ_2''. We also have, on differentiation of the law of refraction $\sin\theta_1 = n\sin\theta_2$, keeping θ_1 fixed,

$$0 = \sin\theta_2 + n\cos\theta_2\frac{\mathrm{d}\theta_2}{\mathrm{d}n}. \tag{29}$$

Hence (28) may be written as

$$n''\cos\theta_2'' - n'\cos\theta_2' = \frac{1}{\cos\theta_2}(n'' - n'), \tag{30}$$

and (27) becomes, on substituting from (30),

$$\delta = \frac{2\pi h}{\lambda\cos\theta_2}(n'' - n'). \tag{31}$$

The quantity $h/\cos\theta_2$ represents the mean geometrical path of the two rays in the plate, and this multiplied by $n'' - n'$ gives the corresponding optical path difference.

Returning to the case of an extended source, one has to consider the transmission of waves with different directions of propagation. It will be assumed that the directions make small angles with the plate normal. Let us represent each of the incident waves by its wave normal through the fixed point A in Fig. 15.21. The points F in which the waves are brought to a focus by the lens are in a one-to-one correspondence with the points B where the transmitted wave normals AB (the average of AB' and AB'') strike the lower face of the plate. Since the inclination of AB to the plate normal AD is small, the points F will form a slightly distorted image — a projection — of the points B. Hence the form of the isochromates is essentially given by the loci of points B for which δ is constant. In particular, for the principal isochromates this constant is an integral multiple of 2π. If we wish to survey the effect of varying the plate thickness, we only have to shift the plane containing the points B parallel to itself.

It follows that all the isochromates may be surveyed by constructing around some point A *the surfaces of constant phase difference* $\delta(h, \theta_2) =$ constant, called also *isochromatic surfaces*, and finding their intersection with the planes $h =$ constant. In determining these surfaces it must be remembered that the refractive indices n' and n'' entering the expression (31) for δ are also functions of θ_2.

We shall consider separately the form of the interference figures obtained from uniaxial and biaxial crystal plates. It will be convenient to specify the points B by the polar distance

$$\rho = AB = \frac{h}{\cos\theta_2}, \tag{32}$$

and by the angle ϑ (or angles ϑ_1 and ϑ_2) which AB makes with the directions of the optic axis (or axes) of wave normals of the crystalline medium (Fig. 15.22).

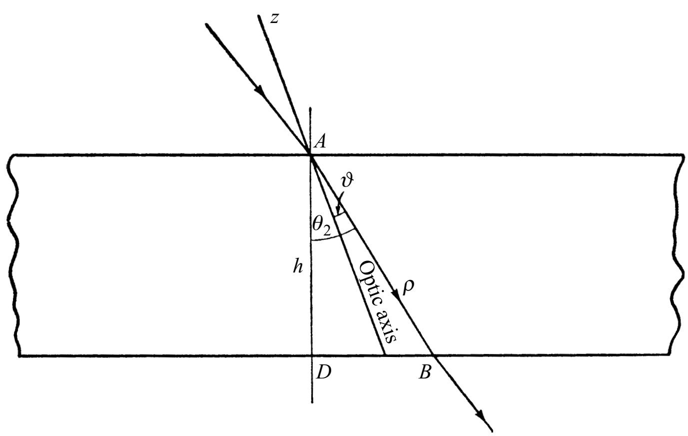

Fig. 15.22 Illustrating the theory of interference figures from crystal plates.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

15.4.4 Interference figures from uniaxial crystal plates

In a uniaxial crystal the phase velocities corresponding to a wave-normal direction that makes an angle ϑ with the optic axis are, according to §15.3 (4), related by

$$v_p'^2 - v_p''^2 = (v_o^2 - v_e^2)\sin^2\vartheta. \tag{33}$$

Since $v_p = c/n$, and similarly for the other velocities, (33) gives

$$\frac{1}{n'^2} - \frac{1}{n''^2} = \left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)\sin^2\vartheta. \tag{34}$$

The difference between these refractive indices is usually small compared with their values, so that (34) may be written in the approximate form

$$n'' - n' = (n_e - n_o)\sin^2\vartheta. \tag{35}$$

Substituting from this relation into (31) and using (32) we obtain

$$\delta = \frac{2\pi\rho}{\lambda}(n_e - n_o)\sin^2\vartheta. \tag{36}$$

Hence the surfaces of constant phase difference are given by

$$\rho\sin^2\vartheta = C, \qquad (C = \text{constant}). \tag{37}$$

To visualize the form of these surfaces we take Cartesian axes with the z-axis along the optic axis. Then

$$\begin{aligned} \rho^2 &= x^2 + y^2 + z^2, \\ \rho^2\sin^2\vartheta &= x^2 + y^2, \end{aligned} \tag{38}$$

and, according to (37), the surfaces of constant phase difference are given by

$$(x^2 + y^2)^2 = C^2(x^2 + y^2 + z^2). \tag{39}$$

These surfaces can be generated by rotating the curves

$$x^4 = C^2(x^2 + z^2), \tag{40}$$

of which a typical one is shown in Fig. 15.23, around the z-axis. At large distances from the origin ($z^2 \gg x^2$) the curve approaches asymptotically the parabolae

$$x^2 = \pm Cz. \tag{41}$$

In the neighbourhood of the x-axis

$$C^2 = \frac{x^4}{x^2 + z^2} = \frac{x^2}{1 + \dfrac{z^2}{x^2}} = x^2\left(1 - \frac{z^2}{x^2} + \cdots\right), \tag{42}$$

so that the curve there is approximated by the hyperbola

$$x^2 - z^2 = C^2. \tag{43}$$

The parabolae and the hyperbola are shown as dashed curves in Fig. 15.23.

We can now determine all the isochromates by simply taking the sections of the surface (39) with planes at different distances h from the origin, these planes

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

representing the exit face of the crystal plate. It is evident from Fig. 15.23 that isochromates of very different forms can arise, depending on the orientation of this face of the crystal relative to the optic axis. If the face of the plate is perpendicular to the optic axis, the isochromates are evidently circular; if the normal to the face makes a small angle with the optic axis, the curves are closed, approximating to ellipses; but if the normal makes a large angle with the optic axis the curves approximate to hyperbolae.

The *principal isogyres* (curves on which $\sin 2\phi = 0$) appear as a dark blurred cross whose arms are parallel to the directions of the polarizer and the analyser, and whose centre corresponds to a wave-normal direction parallel to the optic axis. This follows from the fact that for any given direction of the wave normal in the crystal the directions of vibration are parallel to, and perpendicular to, the principal plane containing the wave normal and the optic axis. Thus there will be darkness at all points of the field of view for which the principal plane is parallel to the direction of vibration passed by either the polarizing or the analysing Nicol prism.

Fig. 15.24 shows a typical interference figure from a uniaxial crystal; the principal isogyres and isochromates can be clearly seen.

Interference figures from plane-parallel crystal plates have an important practical application in connection with *polariscopes*; these are instruments used for the detection of small proportions of polarized light in a beam which is largely unpolarized. An example is Savart's plate.* It consists of two quartz plates, each cut with its

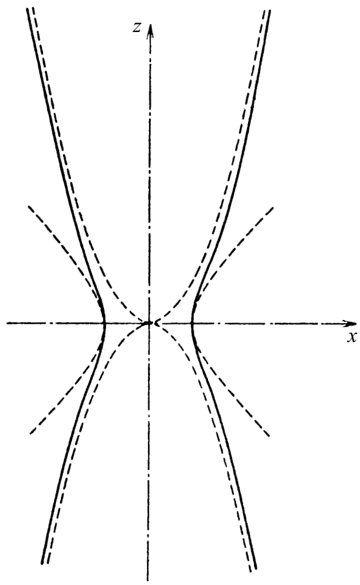

Fig. 15.23 Meridional curve of a surface of constant phase difference for an optically uniaxial crystal.

* See *Ann. d. Physik*, **49** (1840), 292.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

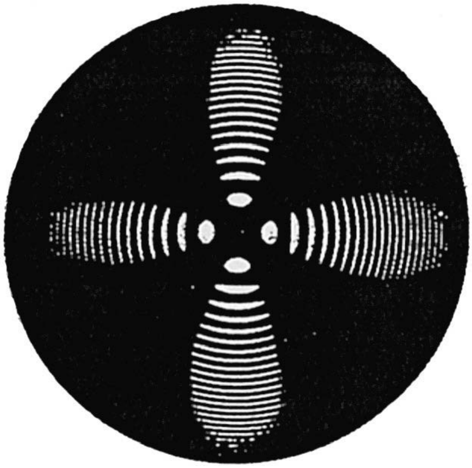

Fig. 15.24 Interference figure from fluorspar cut perpendicular to the optic axis, between crossed Nicol prisms.

optic axis at 45° to the plate normal, and cemented together so that the planes containing the plate normals and the optic axes are perpendicular to each other. Thus the direction of ordinary vibrations in one plate is the direction of extraordinary vibration in the other, and the phase differences have opposite signs; they cancel each other exactly when the wave passes normally through the plates. This arrangement ensures that the phase difference varies very little with wavelength for nearly normal incidence, so that white light fringes may be obtained. If Savart's plate is placed between crossed Nicol prisms with its principal planes at 45° to the directions of the Nicol prisms, the interference pattern consists of light and black, almost straight, fringes. If the polarizing Nicol prism is removed, and partially polarized light passed through the system, the polarized component will give rise to fringes that are superimposed on a uniform background produced by the unpolarized component, but the fringes can be detected even when the proportion of the polarized light is very low. The fringes have the greatest contrast when the plane of vibration of the incident light is at 45° to the principal planes of the Savart plate.

15.4.5 Interference figures from biaxial crystal plates

For a plane-parallel biaxial crystal plate we have, in place of (33), the more general relation

$$v_p'^2 - v_p''^2 = (v_x^2 - v_z^2)\sin\vartheta_1 \sin\vartheta_2, \tag{44}$$

which follows from §15.3 (18). Here ϑ_1 and ϑ_2 are the angles which the wave normal direction AB makes with the two optic axes of wave normals. Since $v_p = c/n$, $v_x = c/n_x$, etc., (44) gives

$$\left(\frac{1}{n'^2} - \frac{1}{n''^2}\right) = \left(\frac{1}{n_x^2} - \frac{1}{n_z^2}\right)\sin\vartheta_1 \sin\vartheta_2, \tag{45}$$

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

or approximately, since the differences between the refractive indices are small compared with their values,

$$n'' - n' = (n_z - n_x)\sin\vartheta_1 \sin\vartheta_2. \tag{46}$$

Substituting from (46) into (31) and again setting $h/\cos\vartheta_2 = \rho$, we obtain for the phase difference δ the expression

$$\delta = \frac{2\pi\rho}{\lambda}(n_z - n_x)\sin\vartheta_1 \sin\vartheta_2. \tag{47}$$

We see that the surfaces of constant phase difference are now given by

$$\rho\sin\vartheta_1 \sin\vartheta_2 = C \qquad (C = \text{constant}). \tag{48}$$

In the direction of each optic axis ($\vartheta_1 = 0$ or $\vartheta_2 = 0$) ρ tends to infinity, so that the surfaces approximate asymptotically to cylinders surrounding the optic axes. When ϑ_1 is small, ϑ_2 is approximately equal to the angle 2β between the two optic axes, and (48) then becomes

$$\rho\sin\vartheta_1 = \frac{C}{\sin 2\beta}. \tag{49}$$

But $\rho\sin\vartheta_1$ is the distance of a point on the surface from the optic axis $\vartheta_1 = 0$; similarly for $\rho\sin\vartheta_2$. Hence the ‘asymptotic cylinders’ are circular. The general form of a surface of constant phase difference is shown in Fig. 15.25. It is evident that near the optic axes the isochromates are closed curves, approximating to ellipses, surrounding the two points in the focal plane that correspond to the optic axes.

The *principal isogyres* are obtained, as before, by finding all directions of the wave normal such that the directions of vibrations in the crystal coincide with the directions passed by the Nicol prisms. We may use the construction of §15.2.3 (a), where it was shown that the planes of vibration bisect the angles between the planes ($\mathbf{N}_1$, $\mathbf{s}$) and ($\mathbf{N}_2$, $\mathbf{s}$), $\mathbf{N}_1$ and $\mathbf{N}_2$ being the directions of the optic axes. Thus if the crystal has its z-axis vertical and the directions of vibrations which are transmitted by the Nicol prisms

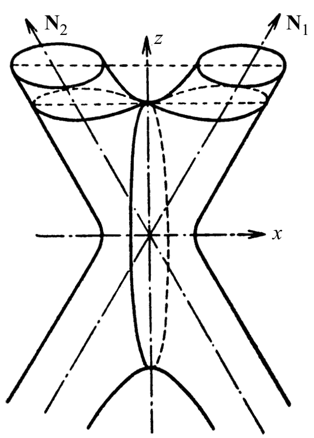

Fig. 15.25 Surface of constant phase difference for an optically biaxial crystal.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

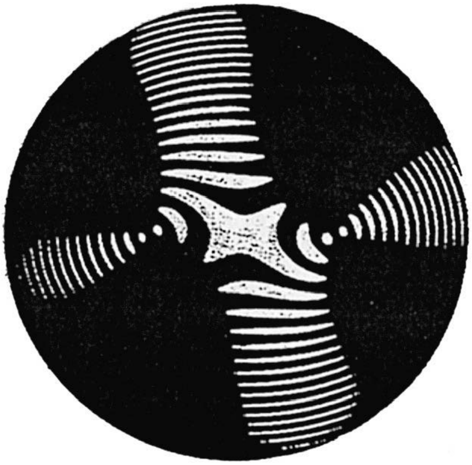

Fig. 15.26 Interference figure from Brazil topaz.

are parallel to the x- and y-axes, the principal isogyres will lie in the x,z-plane and the y,z-plane, and the interference figure will thus show a dark cross with one pair of arms passing through the points corresponding to the optic axes.

More generally, if the crystal is in any other orientation relative to the directions of the Nicol prisms, the principal isogyres will have the form of rectangular hyperbolae with asymptotes in the directions of vibrations transmitted by the Nicol prisms, and with arms passing through the points corresponding to the optic axes. If, with both the Nicol prisms fixed, the crystal plate is rotated in its plane, the pattern of isogyres will change, but the isochromates will, apart from rotation, remain the same; for the isochromates are defined by conditions which do not depend on the directions of the Nicol prisms. A typical interference figure from biaxial crystal plate is shown in Fig. 15.26.

15.4.6 Location of optic axes and determination of the principal refractive indices of a crystalline medium

Since the isochromates form closed curves around the optic axis (or axes), the observation of interference figures affords a ready method of determining whether a crystal is uniaxial or biaxial, and of locating the axes. The interference figures may be observed in a microscope fitted with two Nicol prisms (called a *polarizing microscope*), either by removing the eyepiece and focusing the eye on the back focal plane of the objective lens (thus reproducing the conditions of Fig. 15.21) or by inserting an additional lens in the body of the microscope so that the back focal plane of the objective can be observed through the eyepiece. The second method has the advantage that a larger image of the interference figure is seen, and measurements may be made on the interference figure by the use of a calibrated scale in the eyepiece. Thus the angle between the optic axes of a biaxial crystal may be measured (naturally one must take into account the fact that the light is refracted as it leaves the crystal). Even very small crystal fragments, such as may occur in thin mineralogical sections, are

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

sufficient for the location of the optic axes and measurements of their inclination by these means.

The principal refractive indices, n_x, n_y, n_z of a crystal may be determined either from measurements of the angle of deviation or of total reflection in a prism, or by immersion in a series of liquids of graded refractive index.

The prism method is more convenient for uniaxial crystals than for biaxial crystals. A prism is cut with its refracting edge parallel to the optic axis of wave normal. Then the ordinary and extraordinary waves have their **D** vectors respectively perpendicular and parallel to this edge. The two refractive indices can be found from the deviations of the two rays which emerge from the prism when an unpolarized beam is incident on one of its faces. By means of a Nicol prism one can distinguish between the ordinary and extraordinary rays.

The immersion method depends on the fact that a transparent body is not visible when immersed in a liquid of equal refractive index. Since a crystal has two refractive indices for any given direction of propagation, it will be visible in any liquid if it is observed with unpolarized light. However, if polarized light is used with its **D** vector in one of the directions of vibrations in the crystal, the crystal will be invisible in a liquid of the appropriate refractive index, n' or n''. If the directions of the principal dielectric axes are known, the crystal can be oriented so that light travels parallel to each axis in turn, and n' and n'' can be made equal to pairs of n_x, n_y and n_z in turn.

If the directions of the principal dielectric axes are unknown, one can obtain a fair estimate of the refractive indices by immersing a large number of crystals, of the type under consideration, with random orientations, in a series of liquids of graded refractive indices. Each crystal will become invisible at two different values of the refractive index, for two directions of vibrations of the incident light. If these two refractive indices are n' and n'', we have, by the inequalities §15.3 (12)

$$n_x \leqslant n' \leqslant n_y \leqslant n'' \leqslant n_z. \tag{50}$$

Thus n_x equals the lower limit of the values of n', n_z equals the upper limit of the values of n'', and n_y is equal both to the upper limit of n' and to the lower limit of n''; these limits should coincide if a sufficient number of measurements have been taken.

If the crystals are uniaxial, then every crystal will give n_o for one of its refractive indices, and the other will range between n_o and n_e.

15.5 Stress birefringence and form birefringence

15.5.1 Stress birefringence

When a transparent isotropic material is subject to mechanical stresses it may become optically anisotropic. This phenomenon, known as *stress birefringence* or the *photo-elastic effect* was first noted by Brewster* and finds useful practical applications. We shall only briefly indicate how optical methods may be used to obtain information about the state of stress in an initially isotropic material. As a preliminary we must consider the relations that exist between the elastic and the optical constants of matter.

The state of stress and the state of strain in an elastic solid body are characterized by

* D. Brewster, *Phil. Trans.* (1815), 60; (1816), 156. *Trans. Roy. Soc. Ed.*, **8** (1818), 369.

second-order tensors, the stress tensor P_{kl} and the strain tensor r_{kl}, the components of which are linearly related to each other. These two tensors are always symmetric, but their principal axes are generally different from those of the dielectric tensor, which, as we saw in §15.1, determines the optical properties of the body.

When a stress is applied to the body, the dielectric tensor is modified, and it may be assumed in the first approximation that the *changes* in the components of the dielectric tensor are linearly related to the six stress components, and hence also to the six strain components. Thus we are led to the introduction of two new sets of coefficients, *the stress-optical constants* and *the strain-optical constants* which characterize these relationships.

If we refer coordinates to the principal dielectric axes of the unstressed material, the ellipsoid of wave normals has the equation

$$\frac{x^2}{\varepsilon_x}+\frac{y^2}{\varepsilon_y}+\frac{z^2}{\varepsilon_z}=1. \tag{1}$$

On applying a stress whose components are P_{xx}, P_{xy}, ..., this ellipsoid is changed into another one whose equation may be written as

$$a_{xx}x^2+a_{yy}y^2+a_{zz}z^2+a_{yz}yz+a_{zx}zx+a_{xy}xy=1. \tag{2}$$

By our assumptions, each coefficient a_{kl} differs from the corresponding coefficient in (1) by a linear function in the P's. We thus have six relations, of which two typical ones are

$$\left.\begin{aligned} a_{xx}-\frac{1}{\varepsilon_x}&=q_{11}P_{xx}+q_{12}P_{yy}+q_{13}P_{zz}+q_{14}P_{yz}+q_{15}P_{zx}+q_{16}P_{xy},\\ a_{yz}&=q_{41}P_{xx}+q_{42}P_{yy}+q_{43}P_{zz}+q_{44}P_{yz}+q_{45}P_{zx}+q_{46}P_{xy}.\end{aligned}\right\} \tag{3}$$

In this notation each numeral 1–6 in the subscripts refers to a pair of axes, thus: $1=xx$, $2=yy$, $3=zz$, $4=yz$, $5=zx$, $6=xy$.

There is a similar set of equations relating the coefficients a_{kl} to the strain components. The optical effect of the stress may also be described in terms of the deformation of the ray ellipsoid, and thus two further sets of linear equations with 36 coefficients are obtained. These coefficients are related to those of the ellipsoid of wave normals, since the two ellipsoids always have their principal axes in the same directions, and the semiaxes of one are reciprocals of the other.

Relations (3) take a simpler form if there are symmetry elements present in the structure. For crystals of the *cubic system* the three principal axes x, y and z are equivalent, and in consequence the following relations hold between the stress-optical coefficients*

$$\left.\begin{aligned} &q_{11}=q_{22}=q_{33},\\ &q_{12}=q_{21}=q_{23}=q_{32}=q_{31}=q_{13},\\ &q_{44}=q_{55}=q_{66},\end{aligned}\right\} \tag{4}$$

* The relations which exist between the stress-optical coefficients of each of the crystal systems are discussed by F. Pockels, *Ann. d. Physik*, **37** (1889), 158. Also his *Lehrbuch der Kristalloptik*, (Leipzig, 1906), pp. 469–474. They are summarized in G. Szivessy, *Handbuch der Physik*, Vol. 21 (Berlin, Springer, 1929), p. 840.

all the remaining coefficients being zero.

For *isotropic materials*, relations (3) must remain unaltered for any change of axes. This is only possible if the stress-optical constants satisfy the conditions for cubic symmetry, and in addition the relation

$$2q_{44} = q_{11} - q_{12} \tag{5}$$

holds. Thus in this case there are only two independent constants. Since all systems of axes are now equivalent, we may use any set of axes and, in particular, the principal axes of the stress tensor; then $P_{yz} = P_{zy} = P_{xy} = 0$ and we have in place of (3) the simpler relations

$$\left.\begin{aligned} a_{xx} - \frac{1}{\varepsilon} &= q_{11}P_{xx} + q_{12}P_{yy} + q_{12}P_{zz}, \\ a_{yy} - \frac{1}{\varepsilon} &= q_{12}P_{xx} + q_{11}P_{yy} + q_{12}P_{zz}, \\ a_{zz} - \frac{1}{\varepsilon} &= q_{12}P_{xx} + q_{12}P_{yy} + q_{11}P_{zz}, \\ a_{yz} = a_{zx} &= a_{xy} = 0. \end{aligned}\right\} \tag{6}$$

Thus the principal axes of the stress tensor and those of the ellipsoid of wave normals are the same in this case, as one would expect from symmetry considerations.

Although cubic crystals such as rock-salt are optically isotropic when unstrained, they nevertheless behave differently, when strained, from truly isotropic materials like glass. The effect of stress may be conveniently observed by viewing the body between crossed Nicol prisms (or other polarizing devices such as polaroid sheets — see §15.6.3). Consider a sheet of material of thickness h and let light be incident normally on it. Suppose that the sheet is stressed so that two of the principal axes, say x and y, of the stress tensor and, therefore, of the dielectric tensor, lie in the plane of the sheet and make angles ϕ and $\phi + \pi/2$ with the directions of the polarizer and the analyser, as in §15.4.3. The section of the ellipsoid of normals by the x,y-plane is the ellipse

$$a_{xx}x^2 + a_{yy}y^2 = 1, \tag{7}$$

where a_{xx} and a_{yy} are given by (6). The refractive indices n' and n'' for the two waves propagated in the sheet are given by

$$n' = \frac{1}{\sqrt{a_{xx}}}, \qquad n'' = \frac{1}{\sqrt{a_{yy}}}. \tag{8}$$

Hence

$$\begin{aligned} \frac{1}{n'^2} - \frac{1}{n''^2} &= a_{xx} - a_{yy} \\ &= (q_{11} - q_{12})(P_{xx} - P_{yy}). \end{aligned} \tag{9}$$

Now n' and n'' will in practice usually differ only slightly from n_x, so that, approximately

$$n'' - n' = \tfrac{1}{2}n_x^3(q_{11} - q_{12})(P_{xx} - P_{yy}). \tag{10}$$

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

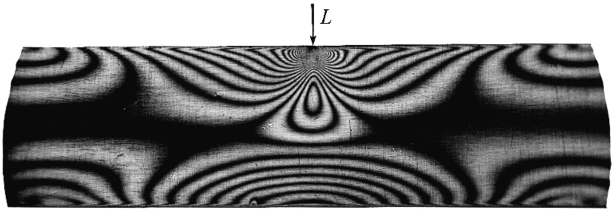

Fig. 15.27 The stress pattern of a beam under the action of a concentrated load. The position of the load is indicated by L; the ends (not shown) are clamped.

Substitution into §15.4 (31) gives the phase difference δ between the two waves emerging from the sheet:

$$\delta = \frac{\pi h}{\lambda} n_x^3 (q_{11} - q_{12})(P_{xx} - P_{yy}). \tag{11}$$

Thus the phase difference is proportional to $P_{xx} - P_{yy}$, which represents twice the shearing stress over planes inclined at 45° to the x and y directions. There is in this case only *one* relevant stress-optical constant, namely $q_{11} - q_{12}$.

It follows that if a stressed sheet of glass or transparent plastic is observed between crossed Nicol prisms, bright and dark fringes will be seen, and these fringes are contours of equal shearing stress. Such a 'stress pattern' is shown in Fig. 15.27. The fringes are seen at maximum intensity in any given region of the pattern only if the principal axes of the stress system in this region are at 45° to the directions of vibrations transmitted by the Nicol prisms. When the principal axes of the stress are parallel to the directions transmitted by the Nicol prisms, the fringes disappear and the field of view becomes black; thus the directions of the axes of the stress system can be determined by rotating the crossed Nicol prisms, while the magnitude of the shearing stress can be obtained from the order of the fringes. This method is used to investigate stresses in engineering structures; a model of the structure is made from a suitable plastic material and the effect of the stresses is directly observed in the manner just described. In this way lengthy calculations are often avoided.*

15.5.2 Form birefringence

The birefringent properties of crystals may be explained in terms of the anisotropic electrical properties of molecules of which the crystals are composed. Birefringence may, however, arise from anisotropy on a scale much larger than molecular, namely when there is an ordered arrangement of similar particles of optically isotropic material whose size is large compared with the dimensions of molecules, but small compared with the wavelength of light. We then speak of *form birefringence*.

From optical measurements information may often be obtained about the submicroscopic particles that give rise to form birefringence. We shall explain the principle of

* For a full account of this method see for instance E. G. Coker and L. N. G. Filon, *A Treatise on Photoelasticity* (Cambridge, Cambridge University Press, 1931), or M. M. Frocht, *Photoelasticity*, Vol. I (1941), Vol. II (1948), (New York, John Wiley).

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

the method by considering the somewhat idealized case of a regular assembly of particles that have the form of thin parallel plates. Let t_1 be the thickness of each plate and t_2 the widths of the spaces between them (Fig. 15.28). Further, let ε_1 be the dielectric constant of each plate and ε_2 the dielectric constant of the medium in which they are immersed.

Suppose that a plane monochromatic wave is incident on the assembly and assume first that its electric vector is perpendicular to the plates. If the linear dimensions of the faces of the plates are assumed to be large but the thicknesses t_1 and t_2 small compared to the wavelength, the field in the plates and in the spaces may be considered to be uniform. Further, according to §1.1.3, the normal component of the electric displacement must be continuous across a surface at which the properties of the medium change abruptly. Hence the electric displacement must have the same value $\mathbf{D}$ inside the plates and in the spaces. If $\mathbf{E}_1$ and $\mathbf{E}_2$ are the corresponding electric fields,

$$\mathbf{E}_1 = \frac{\mathbf{D}}{\varepsilon_1}, \qquad \mathbf{E}_2 = \frac{\mathbf{D}}{\varepsilon_2}, \tag{12}$$

and the mean field $\mathbf{E}$ averaged over the total volume is

$$\mathbf{E} = \frac{t_1 \dfrac{\mathbf{D}}{\varepsilon_1} + t_2 \dfrac{\mathbf{D}}{\varepsilon_2}}{t_1 + t_2}. \tag{13}$$

The effective dielectric constant $\varepsilon_\perp$ is, therefore,

$$\varepsilon_\perp = \frac{\mathbf{D}}{\mathbf{E}} = \frac{(t_1 + t_2)\varepsilon_1\varepsilon_2}{t_1\varepsilon_2 + t_2\varepsilon_1} = \frac{\varepsilon_1\varepsilon_2}{f_1\varepsilon_2 + f_2\varepsilon_1}, \tag{14}$$

where $f_1 = t_1/(t_1 + t_2)$, $f_2 = t_2/(t_1 + t_2) = 1 - f_1$ are the fractions of the total volume occupied by the plates and by the surrounding medium respectively.

Suppose next that the incident field has its electric vector parallel to the plates. According to §1.1.3, the tangential component of the electric vector is continuous across a discontinuity surface, so that in this case the electric field will have the same value $\mathbf{E}$ inside the plates and in the spaces. The electric displacements in the two regions are

$$\mathbf{D}_1 = \varepsilon_1\mathbf{E}, \qquad \mathbf{D}_2 = \varepsilon_2\mathbf{E}, \tag{15}$$

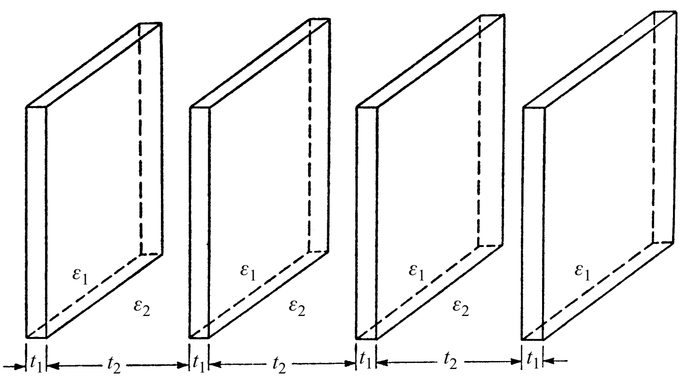

Fig. 15.28 A regular assembly of thin parallel plates.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

so that the mean electric displacement **D** is

$$\mathbf{D} = \frac{t_1 \varepsilon_1 \mathbf{E} + t_2 \varepsilon_2 \mathbf{E}}{t_1 + t_2}. \tag{16}$$

Hence the effective dielectric constant is now given by

$$\varepsilon_{\|} = \frac{\mathbf{D}}{\mathbf{E}} = \frac{t_1 \varepsilon_1 + t_2 \varepsilon_2}{t_1 + t_2} = f_1 \varepsilon_1 + f_2 \varepsilon_2. \tag{17}$$

Since the effective dielectric constant is the same for all directions parallel to the plates, but different for directions normal to the plates, the assembly behaves as a uniaxial crystal with its optic axis perpendicular to the plane of the plates. The difference $\varepsilon_{\|} - \varepsilon_{\perp}$ is always positive, since according to (14) and (17),

$$\varepsilon_{\|} - \varepsilon_{\perp} = \frac{f_1 f_2 (\varepsilon_1 - \varepsilon_2)^2}{f_1 \varepsilon_2 + f_2 \varepsilon_1} \geqslant 0. \tag{18}$$

The ordinary wave has its electric vector perpendicular to the optic axis, i.e. parallel to the plane of the plates. Eq. (18) implies that the assembly always behaves like a *negative uniaxial crystal* (see §15.3.2). In terms of refractive indices the last equation may be written as

$$n_e^2 - n_o^2 = -\frac{f_1 f_2 (n_1^2 - n_2^2)^2}{f_1 n_2^2 + f_2 n_1^2}. \tag{19}$$

For assemblies of particles of less idealized forms the calculations are naturally more complicated.*

A case of considerable practical interest is that of an assembly of parallel and similar thin cylindrical rods. It was shown by Wiener† that if the rods occupy a small fraction of the total volume ($f_1 \ll 1$), one has, in place of (19),

$$n_e^2 - n_o^2 = \frac{f_1 f_2 (n_1^2 - n_2^2)^2}{(1 + f_1) n_2^2 + f_2 n_1^2}. \tag{20}$$

Such an assembly therefore behaves as a *positive uniaxial crystal*, with its optic axis parallel to the axes of the rods.

Observations on form birefringence are useful in biological microscopy. The sign of the observed difference indicates whether the shape of the particles is nearer to that of a rod or a plate, and if n_1 and n_2 are known, it may be possible to estimate from (19) or (20) the fraction of the volume occupied by the particles. To distinguish between form birefringence and intrinsic birefringence in the material of the particles, the refractive index n_2 of the medium is varied; the form birefringence will disappear when $n_2 = n_1$, while intrinsic birefringence will be unaffected by variation of n_2. If

* For a general treatment see O. Wiener, *Abh. Sächs. Ges. Akad. Wiss., Math.-Phys. Kl. No.* 6, **32** (1912), 575. Formulae relating to ellipsoidal particles are also given by W. L. Bragg and A. B. Pippard, *Acta Cryst.*, **6** (1953), 865.

† Weiner, *loc. cit.* p. 581. The principal dielectric constants of a rectangular array of parallel cylinders were also calculated by Lord Rayleigh, *Phil. Mag.*, (5), **34** (1892), 481, and (20) is in agreement with his results even for values of f_1 that are not very small compared to unity, provided that the difference between the refractive indices n_1 and n_2 is small.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

both forms of birefringence are present, a graph of $|n_e^2 - n_o^2|$ plotted against n_2 will show a minimum, but not a zero, at $n_2 = n_1$.

The ordering of the particles that give rise to form birefringence may be permanent or semipermanent as, for example, in tobacco-mosaic virus crystals,* or it may be a temporary ordering of similar particles suspended in a liquid. A suspension of similar particles in a liquid appears optically isotropic if the particles are randomly oriented, as will usually be the case if the liquid is stationary; if, however, the liquid is made to flow, there will be a tendency for the particles to align themselves in a particular direction, and the assembly will then behave as a crystal. This effect may be observed by placing the suspension between two coaxial cylinders, one rotating and the other fixed, and by viewing the liquid between crossed Nicol prisms, with light propagated in a direction parallel to the axes of the cylinders.† This phenomenon, first observed by Maxwell,‡ is of assistance in investigations of the flow of liquids past obstacles, giving information about the direction and magnitude of the velocity gradient.

15.6 Absorbing crystals

15.6.1 Light propagation in an absorbing anisotropic medium

The crystalline media which we have so far considered have been characterized, with regard to their optical propeties, by the dielectric tensor ε_{kl}. To describe media which are not only anisotropic, but are also absorbing, we must in addition introduce the *conductivity tensor* σ_{kl}. The directions of the principal axes of the two tensors will not be the same in general and, in consequence, the theory of propagation of light in such media is rather complicated. However, the principal axes of the two tensors coincide in direction for crystals of the higher symmetry classes (of at least orthorhombic symmetry), and we shall confine our attention to crystals of this type, as they illustrate all the essential features of the general theory. We then only have to replace the real dielectric constants ε_x, ε_y, and ε_z, by complex ones, $\hat{\varepsilon}_x$, $\hat{\varepsilon}_y$ and $\hat{\varepsilon}_z$. We shall see that all the earlier formulae of optics of crystals are formally retained, provided that all quantities which depend on $\hat{\varepsilon}_x$, $\hat{\varepsilon}_y$, and $\hat{\varepsilon}_z$ are assumed to be complex.

We start from Maxwell's equations for a conducting medium, viz.

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\dot{\mathbf{D}} + \frac{4\pi}{c}\mathbf{j}, \qquad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}, \tag{1}$$

and consider the propagation of a plane, damped wave. In the complex notation the vectors **E**, **D**, **B**, **H** and **j** are then each proportional to

$$\exp\left[\mathrm{i}\omega\left(\frac{\hat{n}}{c}\mathbf{r}\cdot\mathbf{s} - t\right)\right].$$

In analogy to §15.2 (3) we obtain the equations

* See M. H. F. Wilkins, A. R. Stokes, W. E. Seeds and G. Oster, *Nature*, **166** (1950), 127–129.

† The determination of the orientation of the particles from such measurements is discussed by P. Boeder, *Z. Phys.*, **75** (1932), 258, and by J. T. Edsall in *Advances in Colloidal Science*, Vol. I, ed. E. O. Kraemer (New York, Interscience Publishers Inc., 1942), p. 269.

‡ J. C. Maxwell, *Proc. Roy. Soc.*, **22** (1873), 46. Also his *Collected Papers*, Vol. 2 (Cambridge, Cambridge University Press, 1890), p. 379.

$$\hat{n}\mathbf{s} \times \mathbf{H} = -\mathbf{D} + \frac{4\pi}{\mathrm{i}\omega}\mathbf{j}, \qquad \hat{n}\mathbf{s} \times \mathbf{E} = \mathbf{B}. \tag{2}$$

Setting $\mathbf{B} = \mu\mathbf{H}$ and eliminating $\mathbf{H}$ between these two equations, we find

$$\mu\left[\mathbf{D} + \frac{4\pi\mathrm{i}}{\omega}\mathbf{j}\right] = \hat{n}^2[\mathbf{E} - \mathbf{s}(\mathbf{s}\cdot\mathbf{E})]. \tag{3}$$

If we take the coordinate axes in the direction of the principal dielectric tensor (which by our assumption coincide with those of the conductivity tensor), we have

$$\left.\begin{aligned} D_x &= \varepsilon_x E_x, \quad D_y = \varepsilon_y E_y, \quad D_z = \varepsilon_z E_z, \\ j_x &= \sigma_x E_x, \quad j_y = \sigma_y E_y, \quad j_z = \sigma_z E_z. \end{aligned}\right\} \tag{4}$$

Substituting from (4) into (3) and introducing the complex dielectric constants

$$\hat{\varepsilon}_k = \varepsilon_k + \frac{4\pi\mathrm{i}}{\omega}\sigma_k \qquad (k = x, y, z), \tag{5}$$

(3) becomes

$$\mu\hat{\varepsilon}_k E_k = \hat{n}^2[E_k - s_k(\mathbf{E}\cdot\mathbf{s})]. \tag{6}$$

This relation is formally identical with the relation §15.2 (18), the real constants ε_k and n being replaced by complex constants $\hat{\varepsilon}_k$ and $\hat{n}_k$. Re-writing (6) in the form

$$E_k = \frac{\hat{n}^2 s_k(\mathbf{E}\cdot\mathbf{s})}{\hat{n}^2 - \mu\hat{\varepsilon}_k}, \tag{7}$$

we obtain, by the same argument as in §15.2 (or simply by using the formal substitution $\varepsilon_k \to \hat{\varepsilon}_k$, $n \to \hat{n}$), *Fresnel's equation*

$$\frac{s_x^2}{\dfrac{1}{\hat{n}^2} - \dfrac{1}{\mu\hat{\varepsilon}_x}} + \frac{s_y^2}{\dfrac{1}{\hat{n}^2} - \dfrac{1}{\mu\hat{\varepsilon}_y}} + \frac{s_z^2}{\dfrac{1}{\hat{n}^2} - \dfrac{1}{\mu\hat{\varepsilon}_z}} = 0. \tag{8}$$

Introducing the complex velocities

$$\hat{v}_p = \frac{c}{\sqrt{\mu\hat{\varepsilon}}} = \frac{c}{\hat{n}}, \qquad \hat{v}_x = \frac{c}{\sqrt{\mu\hat{\varepsilon}_x}} = \frac{c}{\hat{n}_x}, \tag{9}$$

etc., we may write Fresnel's equation again in the form §15.2 (24)

$$\frac{s_x^2}{\hat{v}_p^2 - \hat{v}_x^2} + \frac{s_y^2}{\hat{v}_p^2 - \hat{v}_y^2} + \frac{s_z^2}{\hat{v}_p^2 - \hat{v}_z^2} = 0. \tag{10}$$

These relations are strictly analogous to those relating to nonabsorbing crystals; their physical interpretation is, however, somewhat different. From (8) or (10) we again obtain a quadratic equation for $\hat{n}^2(\mathbf{s})$, i.e. we find two refractive indices and two principal vibrations $\mathbf{D}'$ and $\mathbf{D}''$ corresponding to each direction of propagation $\mathbf{s}$. From (7) we see that the ratios $D_x : D_y : D_z$ are complex, so that the principal vibrations are in general no longer linear, but elliptical. A further difference is that the electric displacement vectors are no longer perpendicular to the wave normal $\mathbf{s}$. For we have from the first equation (2), on scalar multiplication by $\mathbf{s}$,

$$\mathbf{s}\cdot\mathbf{D} = \frac{4\pi}{\mathrm{i}\omega}\mathbf{s}\cdot\mathbf{j} = \frac{4\pi}{\mathrm{i}\omega}\left[\frac{\sigma_x}{\varepsilon_x}s_xD_x + \frac{\sigma_y}{\varepsilon_y}s_yD_y + \frac{\sigma_z}{\varepsilon_z}s_zD_z\right], \tag{11}$$

and the right-hand side of (11) will in general not be zero. However, if the ratios $4\pi\sigma_k/\omega\varepsilon_k$ are small compared with unity, the component of **D** in the direction of **s** will be small compared with **D** itself.

Further analysis is considerably simplified if the absorption is assumed to be small, i.e. if metals are excluded and only substances that are to some extent transparent are considered. We shall restrict our discussion to this case. Formally, weak absorption implies that the second power of the attenuation index κ may be neglected in comparison with unity. We may therefore write, on using (9),

$$\left.\begin{aligned} \hat{n} &= n(1+\mathrm{i}\kappa), & \hat{n}^2 &= n^2(1+2\mathrm{i}\kappa),\\ \hat{v}_p &= \frac{c}{n(1+\mathrm{i}\kappa)} = v_p(1-\mathrm{i}\kappa), & \hat{v}_p^2 &= v_p^2(1-2\mathrm{i}\kappa), \end{aligned}\right\} \tag{12}$$

with similar expressions with subscripts x, y and z, e.g.

$$\hat{n}_x = n_x(1+\mathrm{i}\kappa_x), \qquad \hat{v}_x = c/\hat{n}_x = v_x(1-\mathrm{i}\kappa_x) \quad \text{etc.}$$

Also

$$\mu\hat{\varepsilon}_k = \hat{n}_k^2 = n_k^2(1+2\mathrm{i}\kappa_k) = \mu\varepsilon_k(1+2\mathrm{i}\kappa_k), \qquad (k = x,\ y,\ z) \tag{13}$$

and comparison with (5) gives

$$\kappa_k = \frac{2\pi}{\omega}\frac{\sigma_k}{\varepsilon_k}. \tag{14}$$

Returning to Fresnel's equation (10) we separate the real and imaginary parts. A typical term of (10) is

$$\frac{s_x^2}{\hat{v}_p^2 - \hat{v}_x^2} = \frac{s_x^2}{v_p^2 - v_x^2 - 2\mathrm{i}(\kappa v_p^2 - \kappa_x v_x^2)} = \frac{s_x^2}{v_p^2 - v_x^2}\left(1 + 2\mathrm{i}\,\frac{\kappa v_p^2 - \kappa_x v_x^2}{v_p^2 - v_x^2}\right), \tag{15}$$

and it follows that the real part of (10) is Fresnel's equation in the old form §15.2 (24). The imaginary part gives the equation

$$\begin{aligned} \kappa v_p^2&\left[\frac{s_x^2}{(v_p^2 - v_x^2)^2} + \frac{s_y^2}{(v_p^2 - v_y^2)^2} + \frac{s_z^2}{(v_p^2 - v_z^2)^2}\right]\\ &= \frac{\kappa_x v_x^2 s_x^2}{(v_p^2 - v_x^2)^2} + \frac{\kappa_y v_y^2 s_y^2}{(v_p^2 - v_y^2)^2} + \frac{\kappa_z v_z^2 s_z^2}{(v_p^2 - v_z^2)^2}. \end{aligned} \tag{16}$$

For any given wave-normal direction **s**, Fresnel's equation in general gives two phase velocities v_p as before. Some of the energy carried by the two waves is now absorbed, and (16) gives approximate values for the two attenuation indices.

We may express the formula for κ in a different form. From (7), (4), (9) and (13),

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$$\begin{aligned} D_k &= \frac{c^2}{\mu}\frac{\varepsilon_k}{\hat{\varepsilon}_k}\frac{s_k(\mathbf{E}\cdot\mathbf{s})}{\hat{v}_k^2-\hat{v}_p^2} \\ &= -\frac{c^2}{\mu}\frac{s_k(\mathbf{E}\cdot\mathbf{s})}{v_p^2-v_k^2}\left[1+2\mathrm{i}\frac{(\kappa-\kappa_k)v_p^2}{v_p^2-v_k^2}\right], \qquad (k=x,\,y,\,z), \end{aligned} \tag{17}$$

where again only terms up to the first power in the attenuation indices have been retained. Now the imaginary term in (17) involves the *difference* in the κ's and may in many cases be neglected. This means that one *neglects the ellipticity of the vibrations*. In this approximation the directions of the two **D** vectors belonging to any given wave normal **s** are the same as for a nonabsorbing crystal which has the same (real) principal dielectric constants. We then have

$$\frac{s_k^2}{(v_p^2-v_k^2)^2} = \left(\frac{\mu}{c^2}\right)^2\frac{D_k^2}{(\mathbf{s}\cdot\mathbf{E})^2}, \tag{18}$$

and (16) may be written as

$$\kappa v_p^2 = \frac{\kappa_x v_x^2 D_x^2 + \kappa_y v_y^2 D_y^2 + \kappa_z v_z^2 D_z^2}{\mathbf{D}^2}. \tag{19}$$

This formula may evidently break down as v_p approaches one of the principal velocities v_k, since the imaginary part in (17), which we neglected, involves the difference $v_p^2 - v_k^2$ in the denominator (see remarks at the end of §15.6.3).

Since the two coefficients κ' and κ'' belonging to a given wave-normal direction **s** are in general different, the two waves are absorbed with different strengths. These two coefficients may be frequency-dependent and vary in different ways with the frequency, so that if white light is incident upon the crystal, the crystal will in general appear coloured, and the colour will depend on the direction of vibration of the incident light. This phenomenon is known as *pleochroism*; in the case of a uniaxial crystal one also speaks of *dichroism*; in the case of a biaxial crystal one speaks of *trichroism*.*

For a *uniaxial crystal* ($\hat{\varepsilon}_x = \hat{\varepsilon}_y = \hat{\varepsilon}_o$, $\hat{\varepsilon}_z = \hat{\varepsilon}_e$; also $\kappa_x = \kappa_y = \kappa_o$, $\kappa_z = \kappa_e$) the relations take a simpler form. As in the case of a nonabsorbing uniaxial crystal, Fresnel's equation breaks into two factors (see §15.3.2), giving

$$\left.\begin{aligned} \hat{v}_p'^2 &= \hat{v}_o^2 \\ \hat{v}_p''^2 &= \hat{v}_o^2\cos^2\vartheta + \hat{v}_e^2\sin^2\vartheta, \end{aligned}\right\} \tag{20}$$

where, as before, ϑ is the angle which the wave normal **s** makes with the optic axis. The first equation gives, on separating real and imaginary parts,

$$v_p' = v_o, \qquad \kappa_p' = \kappa_o, \tag{21a}$$

and the second gives

$$v_p''^2 = v_o^2\cos^2\vartheta + v_e^2\sin^2\vartheta, \qquad \kappa_p''^2 v_p''^2 = \kappa_o v_o^2\cos^2\vartheta + \kappa_e v_e^2\sin^2\vartheta, \tag{21b}$$

* This terminology derives from the fact that for a uniaxial crystal there are two characteristic colours and for a biaxial crystal there are three. Some authors, however, mean by dichroic material, any material whose absorption coefficient depends on the state of polarization of the incident light.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

where again we have neglected the terms involving the second power of the attenuation indices. We see that the absorption of the ordinary wave is the same for all directions of propagation.

For a *biaxial crystal* the relations are much more complicated and we shall confine our attention to special cases of interest. As in §15.3.3 we consider first those directions of propagation for which $s_x = 0$. Fresnel's equation (10) then gives, in analogy with §15.3 (6), the equations

$$\left.\begin{aligned} \hat{v}_p'^2 &= \hat{v}_x^2, \\ \hat{v}_p''^2 &= \hat{v}_z^2 s_y^2 + \hat{v}_y^2 s_z^2. \end{aligned}\right\} \tag{22}$$

Separating real and imaginary parts, we obtain the two velocities and the two attenuation indices corresponding to the wave-normal direction $\mathbf{s}(0,\ s_y,\ s_z)$:

$$v_p'^2 = v_x^2, \qquad \kappa_p' = \kappa_x, \tag{23a}$$

and

$$v_p''^2 = v_z^2 s_y^2 + v_y^2 s_z^2, \qquad \kappa_p'' v_p''^2 = \kappa_z v_z^2 s_y^2 + \kappa_y v_y^2 s_z^2. \tag{23b}$$

Similar relations hold, of course, for directions of propagation at right angles to the y and z directions.

In general there are no real values s_y and s_z for which the two roots $\hat{v}_p'$ and $\hat{v}_p''$ are equal. Directions may be found for which the real phase velocities v_p' and v_p'' are equal, but the corresponding attenuation indices (κ' and κ'') will in general be different.

Next we consider propagation in directions that are not very different from that of an optic axis of wave normals. In order to apply (19) we must determine the directions of $\mathbf{D}'$ and $\mathbf{D}''$. This may be done by using the result established in §15.2.3, according to which the two planes of vibrations ($\mathbf{D}'$, $\mathbf{s}$) and ($\mathbf{D}''$, $\mathbf{s}$) bisect the angles between the planes ($\mathbf{N}_1$, $\mathbf{s}$) and ($\mathbf{N}_2$, $\mathbf{s}$), $\mathbf{N}_1$ and $\mathbf{N}_2$ being the axes of wave normals. Let ψ be the angle between the plane ($\mathbf{N}_1$, $\mathbf{s}$) and the x,z-plane (which contains both the optic axes). Since the plane ($\mathbf{N}_2$, $\mathbf{s}$) is nearly parallel to the x,z-plane, it follows from the theorem just quoted that the angle between $\mathbf{D}'$ and the x,z-plane is nearly $\psi/2$ (see Fig. 15.29). The component of $\mathbf{D}'$ in the x,z-plane is therefore $D' \cos(\psi/2)$. To obtain the x-component we must project this vector on to the x-axis. Since, approximately, $\mathbf{s}$ and $\mathbf{N}_1$ coincide, the projection angle is nearly equal to the angle between the optic axis $\mathbf{N}_1$ and the z-axis and consequently (see Fig. 15.30) $D_x' = D' \cos\beta \cos(\psi/2)$. In a similar way we obtain the other components. Hence

$$\left.\begin{aligned} D_x' &= D' \cos\frac{\psi}{2} \cos\beta, \\ D_y' &= D' \sin\frac{\psi}{2}, \\ D_z' &= -D' \cos\frac{\psi}{2} \sin\beta. \end{aligned}\right\} \tag{24a}$$

The vector $\mathbf{D}''$ is orthogonal to $\mathbf{s}$ and $\mathbf{D}'$; its components may be immediately obtained by replacing $\psi/2$ by $\psi/2 + \pi/2$ in (24a), giving

$$D_x'' = -D'' \sin\frac{\psi}{2} \cos\beta, \qquad D_y'' = D'' \cos\frac{\psi}{2}, \qquad D_z'' = D'' \sin\frac{\psi}{2} \sin\beta. \tag{24b}$$

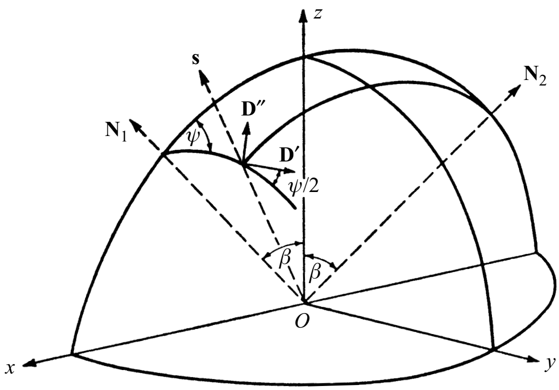

Fig. 15.29 Illustrating the theory of absorbing crystals.

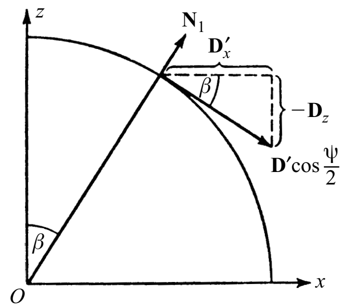

Fig. 15.30 Illustrating the theory of absorbing crystals.

We substitute from (24a) and (24b) into (19), and also use the approximation $\upsilon_p' = \upsilon_p'' = \upsilon_y$ ($\upsilon_x > \upsilon_y > \upsilon_z$), which is justified since we are restricting ourselves to directions not too far from that of the optic axis. We then obtain the required attenuation indices κ' and κ'':

$$\left.\begin{aligned}\kappa' \upsilon_y^2 &= (\kappa_x \upsilon_x^2 \cos^2\beta + \kappa_z \upsilon_z^2 \sin^2\beta)\cos^2\frac{\psi}{2} + \kappa_y \upsilon_y^2 \sin^2\frac{\psi}{2},\\ \kappa'' \upsilon_y^2 &= (\kappa_x \upsilon_x^2 \cos^2\beta + \kappa_z \upsilon_z^2 \sin^2\beta)\sin^2\frac{\psi}{2} + \kappa_y \upsilon_y^2 \cos^2\frac{\psi}{2}.\end{aligned}\right\} \tag{25}$$

In view of the approximation involved in (19) these formulae cannot be expected to remain valid when **s** is in the immediate neighbourhood of the optic axis. In the limiting case when the wave normal is along the optic axis, the angle ψ is undetermined; to obtain the corresponding attenuation indices we return to Fresnel's equation. For any **s** direction in the x,z-plane ($s_y = 0$) we obtain, as in equations (23), by equating real and imaginary parts

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$$\left.\begin{aligned} v_p'^2 &= v_y^2, \qquad & \kappa_p' &= \kappa_y, \\ v_p''^2 &= v_x^2 s_z^2 + v_z^2 s_x^2, \qquad & \kappa_p'' v_p''^2 &= \kappa_x v_x^2 s_z^2 + \kappa_z v_z^2 s_x^2. \end{aligned}\right\} \tag{26}$$

In particular, for the optic axis $s_x = \sin\beta$, $s_z = \cos\beta$, where β is the angle given by §15.3 (11), between either of the optic axes and the z-axis. Also $v_p' = v_p'' = v_y$ and the equations on the right of (26) become, if we also write $\kappa_\perp$ in place of κ' and $\kappa_\parallel$ in place of κ'':

$$\left.\begin{aligned} \kappa_\perp &= \kappa_y, \\ \kappa_\parallel v_y^2 &= \kappa_x v_x^2 \cos^2\beta + \kappa_z v_z^2 \sin^2\beta. \end{aligned}\right\} \tag{27}$$

$\kappa_\perp$ is the attenuation index for a **D** wave polarized at right angles to the plane of the optic axes, and $\kappa_\parallel$ is the index for a **D** wave polarized in the plane of the axes. Thus the absorption of a wave propagated in the direction of the optic axis depends on its direction of vibration.

It is convenient to express κ' and κ'' in terms of the indices $\kappa_\parallel$ and $\kappa_\perp$ and the azimuthal polarization angle ψ, by substituting from (27) into (25). This gives

$$\left.\begin{aligned} \kappa' &= \kappa_\parallel \cos^2\frac{\psi}{2} + \kappa_\perp \sin^2\frac{\psi}{2} = \frac{\kappa_\parallel + \kappa_\perp}{2} + \frac{\kappa_\parallel - \kappa_\perp}{2}\cos\psi, \\ \kappa'' &= \kappa_\parallel \sin^2\frac{\psi}{2} + \kappa_\perp \cos^2\frac{\psi}{2} = \frac{\kappa_\parallel + \kappa_\perp}{2} - \frac{\kappa_\parallel - \kappa_\perp}{2}\cos\psi. \end{aligned}\right\} \tag{28}$$

15.6.2 Interference figures from absorbing crystal plates

We shall now briefly consider interference effects with absorbing crystal plates which are cut at right angles to an optic axis of wave normals. The theory is not very different from that relating to nonabsorbing crystal plates; the only important distinction is that the two interfering rays are absorbed with different strengths, and as a result the visibility of the fringes is decreased. Other conclusions, and in particular the expression for the phase difference, remain unchanged in our approximation, since the geometrical laws of propagation are the same as before.

On travelling a distance l in an absorbing medium, the amplitude of a plane wave is, according to §14.1 (19), reduced by a factor $\exp\{-\omega n\kappa l/c\}$. Hence with the same arrangement and notation as in §15.4.3 (see Fig. 15.20) the amplitudes of the principal vibrations on emergence from the plate are given by [§15.4 (13)]

$$OB = E\mathrm{e}^{-\frac{\omega\kappa'}{v'}l}\cos\phi, \qquad OC = E\mathrm{e}^{-\frac{\omega\kappa''}{v''}l}\sin\phi. \tag{29}$$

Here $l = h/\cos\theta_2$, h being the thickness of the plate and θ_2 the angle which the wave normal in the plate makes with the axis, it being assumed that the two paths in the plate are the same for both waves; this is approximately so, if we restrict ourselves to wave-normal directions close to the optic axis. In the same approximation we may take $v' = v'' = v_y$ in (29). It is convenient to set

$$u = \frac{\omega l}{v_y} \sim \frac{\omega l}{v'} \sim \frac{\omega l}{v''}. \tag{30}$$

With this substitution,

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$$OB = Ee^{-\kappa' u} \cos\phi, \qquad OC = Ee^{-\kappa'' u} \sin\phi. \tag{31}$$

In place of §15.4 (14) we now obtain for the amplitudes of the waves, after passing through the polarizer and the analyser, the expressions (see Fig. 15.20)

$$OF = Ee^{-\kappa' u} \cos\phi \cos(\phi - \chi), \qquad OG = Ee^{-\kappa'' u} \sin\phi \sin(\phi - \chi). \tag{32}$$

The total intensity of the light brought to interference is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\delta, \tag{33}$$

where, apart from unessential proportional factors, $I_1 = OF^2$, $I_2 = OG^2$ and the phase difference δ is calculated as before.

We now examine some special cases of interest.

(a) Uniaxial crystals

In this case we have

$$\left.\begin{aligned} v_x = v_y = v_o, \quad & v_z = v_e, \\ \kappa_x = \kappa_y = \kappa_o, \quad & \kappa_z = \kappa_e. \end{aligned}\right\} \tag{34}$$

The direction of vibration of the extraordinary ray is in the principal plane, i.e. in the plane containing the wave normal and the optic axis. We may, therefore, identify the angle between the **D** vector of the extraordinary wave and the direction *OP* of the polarizer with the angle ϕ in (32); to retain agreement with (21a) and (21b) we must, however, interchange κ' and κ''. With the Nicol prisms *crossed* ($\chi = \pi/2$) we have from (32)

$$OF = Ee^{-\kappa' u} \cos\phi \sin\phi, \qquad OG = -Ee^{-\kappa'' u} \sin\phi \cos\phi, \tag{35}$$

and (33) gives

$$I = \frac{E^2}{4} \sin^2 2\phi [e^{-2\kappa' u} + e^{-2\kappa'' u} - 2\, e^{-(\kappa' + \kappa'')u} \cos\delta]. \tag{36a}$$

For the optic axis, $\kappa' = \kappa''$ and $\delta = 0$, and (36a) reduces to

$$I_0 = 0; \tag{36b}$$

hence at the centre of the pattern (the point corresponding to the optic axis) there is darkness. The field of view is crossed by a dark isogyre, given by $\sin 2\phi = 0$, i.e. by $\phi = 0$ and $\phi = \pi/2$. Thus the isogyre has the form of a cross with its arms parallel to the directions of the polarizer and the analyser (see §15.4.4). The minima and maxima of intensity are

$$\left.\begin{aligned} I_{\min} &= \frac{E^2}{4} \sin^2 2\phi [e^{-2\kappa' u} + e^{-2\kappa'' u} - 2e^{-(\kappa' + \kappa'')u}], \\ I_{\max} &= \frac{E^2}{4} \sin^2 2\phi [e^{-2\kappa' u} + e^{-2\kappa'' u} + 2e^{-(\kappa' + \kappa'')u}], \end{aligned}\right\} \tag{37}$$

so that the visibility $\mathcal{V}$ of the fringes is

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2e^{-(\kappa' + \kappa'')u}}{e^{-2\kappa' u} + e^{-2\kappa'' u}} = \frac{1}{\cosh[(\kappa' - \kappa'')u]}. \tag{38}$$

Thus the visibility is the greater the smaller is the difference between κ' and κ''. Since for wave-normal directions near the optic axis (ϑ small) κ' and κ'' are nearly equal, the fringes should be clearly visible in this region, provided that sufficient light is transmitted by the plate. If κ_o is small compared with κ_e (e.g. for magnesium platinum cyanide) there will be relatively little absorption near the optic axis and the central fringes will be bright. If κ_o is large compared to κ_e (e.g. for tourmaline), the absorption will be relatively weakest near the optic axis and the central fringes will be dark. In both cases the visibility decreases from the centre towards the edge of the field. If κ_o and κ_e are nearly equal, the interference figure will be similar to that of a nonabsorbing crystal, apart from a decrease in visibility with increasing distance from the centre.

(b) Biaxial crystals

We again restrict ourselves to the case of crossed Nicol prisms ($\chi = \pi/2$) and assume that the faces of the plate are perpendicular to one of the optic axes (N_1). Only waves whose directions of propagation are close to this axis will be considered.

Let N_1, N_2 and Q be the points of intersection of the two optic axes and of the wave normal with the exit face of the plate. Further let ψ be the angle which the line N_1Q makes with N_1N_2 and α the angle between the plane of the optic axes and the plane of vibrations transmitted by the polarizer P.

If $N_1Q \ll N_1N_2$ the angle which the direction of vibration $\mathbf{D}'$ makes with N_1N_2 is approximately $\psi/2$ so that the angle ϕ between $\mathbf{D}'$ and the direction OP is approximately (see Fig. 15.31)

$$\phi = \alpha - \frac{\psi}{2}. \tag{39}$$

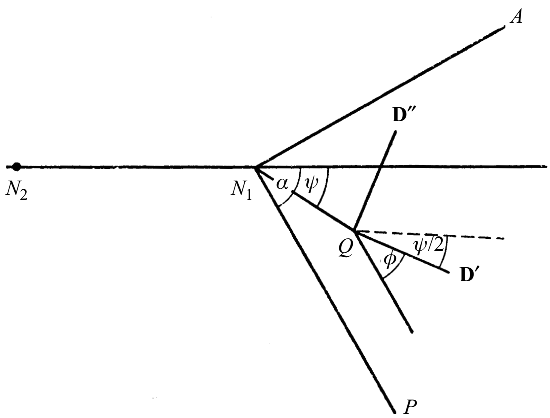

Fig. 15.31 Illustrating the theory of interference on absorbing biaxial crystal plates.

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

The amplitudes (32) of the waves brought to interference, therefore, are

$$OF = E\cos\left(\alpha - \frac{\psi}{2}\right)\sin\left(\alpha - \frac{\psi}{2}\right)\mathrm{e}^{-\kappa' u}, \quad OG = -E\sin\left(\alpha - \frac{\psi}{2}\right)\cos\left(\alpha - \frac{\psi}{2}\right)\mathrm{e}^{-\kappa'' u}. \tag{40}$$

Hence the intensity is

$$I = \frac{E^2}{4}\sin^2(2\alpha - \psi)(\mathrm{e}^{-2\kappa' u} + \mathrm{e}^{-2\kappa'' u} - 2\mathrm{e}^{-(\kappa' + \kappa'')u}\cos\delta). \tag{41a}$$

For a wave propagated in the direction of the optic axis, ψ is undetermined. In this special case we resolve the vibrations into components parallel and perpendicular to the plane containing the optic axes, i.e. we set $\psi = 0$ and write $\kappa_{\parallel}$ and $\kappa_{\perp}$ in place of κ' and κ'' in (41a) [see (28)]. Also $\delta = 0$, so that we obtain, in place of (41a),

$$I_0 = \frac{E^2}{4}\sin^2 2\alpha\,(\mathrm{e}^{-\kappa_{\parallel} u} - \mathrm{e}^{-\kappa_{\perp} u})^2. \tag{41b}$$

Eq. (41a) shows that the intensity is zero along a line on which $\sin(2\alpha - \psi) = 0$, this being a principal isogyre. Whilst in the case of a nonabsorbing crystal the isogyre passed through the point corresponding to the optic axis, now this will no longer be the case, unless, as seen from (41b), $\alpha = 0$ or $\alpha = \pi/2$, i.e. unless the plane of the polarizer is parallel or at right angles to the plane containing the two optic axes.

The dark isochromates, given by $\cos\delta = 1$, i.e. by $\delta = 0, \pm 2\pi, \pm 4\pi, \ldots$ are rings surrounding the point which corresponds to the optic axis. The visibility of the fringes, again given by (38), is appreciable only when κ' and κ'' are nearly equal. According to (28) this is so when ψ is nearly $\pi/2$ or $-\pi/2$.

Let us consider how the intensity changes with ψ when δ is kept constant. If we substitute from (28), (41a) may be written in the form

$$I = \frac{E^2}{2}\mathrm{e}^{-(\kappa_{\parallel} + \kappa_{\perp})u}\sin^2(2\alpha - \psi)\{\cosh[(\kappa_{\parallel} - \kappa_{\perp})u\cos\psi] - \cos\delta\}. \tag{42}$$

The term in the brace brackets is greatest when $|\cos\psi|$ is greatest and least when $|\cos\psi|$ is least. Hence the intensity is maximum when $\psi = 0$ and $\psi = \pi$, and minimum when $\psi = \pi/2$ and $-\pi/2$. Thus in addition to the dark isogyre $\psi = 2\alpha$, the field of view is also crossed by a dark 'brush' with its arms perpendicular to the plane containing the optic axes.*

15.6.3 Dichroic polarizers

In the preceding section we have studied the propagation of polarized light through an absorbing crystal plate. Here we shall consider the effect of such a plate on natural (unpolarized) light.

* From (44) which, by (28), may be written as

$$I = 2E^2\mathrm{e}^{-(\kappa_{\parallel} + \kappa_{\perp})u}\cosh[(\kappa_{\parallel} - \kappa_{\perp})u\cos\psi],$$

it is seen that the dark brush also appears in the absence of a polarizer and analyser, when natural light is passed through the plate.

We again assume that the faces of the plate are perpendicular to the optic axis (or to one such axis if it is biaxial) of wave normals, and consider propagation in a direction close to that of the optic axis. A parallel beam of natural light may be regarded as consisting of two mutually incoherent beams of equal amplitudes, polarized in any two mutually orthogonal directions perpendicular to the direction of propagation [see §10.9 (56)]. Let us choose as the directions of vibration of the two partial beams the directions of the principal vibrations in the crystal. If E is the amplitude of either of the partial beams on entering the plate, their amplitudes after travelling a distance l in the plates are

$$E' = E\mathrm{e}^{-\kappa' u}, \qquad E'' = E\mathrm{e}^{-\kappa'' u}, \tag{43}$$

where as before $u = \omega l / v_y$, it being assumed that the light is quasi-monochromatic and of mean frequency ω. The total intensity is then given by

$$I = I' + I'', \tag{44}$$

where

$$I' = I_0\mathrm{e}^{-2\kappa' u}, \qquad I'' = I_0\mathrm{e}^{-2\kappa'' u}, \tag{45}$$

and $I_0 = E^2$. For the direction of the optic axes, κ' and κ'' must be replaced by $\kappa_{\|}$ and $\kappa_{\perp}$.

We see that after the light has travelled a distance l in the medium, the amplitudes of the two components are in the ratio $\exp[-(\kappa' - \kappa'')u]$, so that the light has become partially polarized. If the two attenuation indices are very different, a relatively thin sheet of the material will be sufficient to transform the incident unpolarized beam into one that is nearly linearly polarized; i.e. the plate acts as a *polarizer*. An example of a natural crystal which will act as a polarizer of this type is tourmaline, which suppresses the ordinary ray much more strongly than the extraordinary ray. However, for most wavelengths it also absorbs a considerable part of the extraordinary ray (see Fig. 15.32), so that it is not suitable for practical applications. It is possible, chiefly as the result of researches of Land (*c*. 1932) and his coworkers, to produce synthetic dichroic materials which act as excellent polarizers. These materials, known commercially as *polaroids*, are not single crystals, but are sheets of organic polymers with long-chain molecules, brought into almost complete alignment by stretching or by some other treatment,* and sometimes dyed. A polymer which is particularly suitable for this purpose is the synthetic product polyvinyl alcohol $(\text{—}CH_2\text{—}CHOH\text{—})_x$. In Fig. 15.33, the dichroism of iodine on oriented polyvinyl alcohol is shown as a function of the wavelength.

The ratio of the absorption coefficients of a good dichroic polarizer may be as high as 100:1. It may transmit about 80 per cent of light polarized in one direction and less than 1 per cent of light polarized in the direction at right angles to it. Large sheets of such polarizing materials are obtainable; machines giving pieces 30 in wide and indefinitely long are in use in commercial production. In this respect these 'polaroid sheets' are superior to Nicol prisms, the dimensions of which are limited by the scarcity of large pieces of calcite of good optical quality.

* For a fuller discussion of dichroic polarizers see the article referred to in the captions to Figs. 15.32 and 15.33 or E. H. Land, *J. Opt. Soc. Amer.*, **41** (1951), 957.

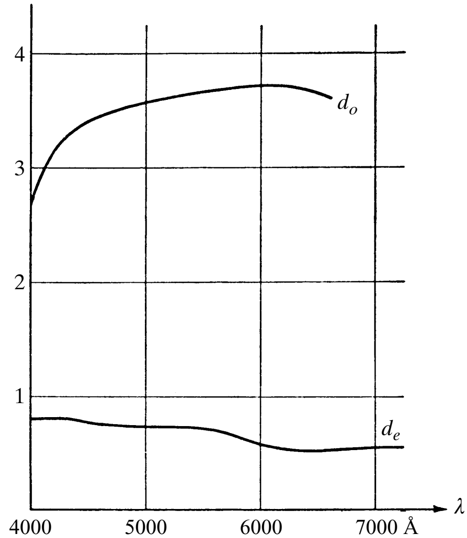

Fig. 15.32 Dichroism of black tourmaline plate about 0.2 mm thick, cut 24° from parallelism with the optic axis, for light incident at right angles to the optic axis.

$$d = 4\pi(\log_{10} e)(n\kappa l/\lambda) \sim 5.5n\kappa l/\lambda.$$

(After E. H. Land and C. D. West, contribution in *Colloid Chemistry*, Vol 6, ed. J. Alexander (New York, Reinhold Publishing Corporation, 1946), p. 167.)

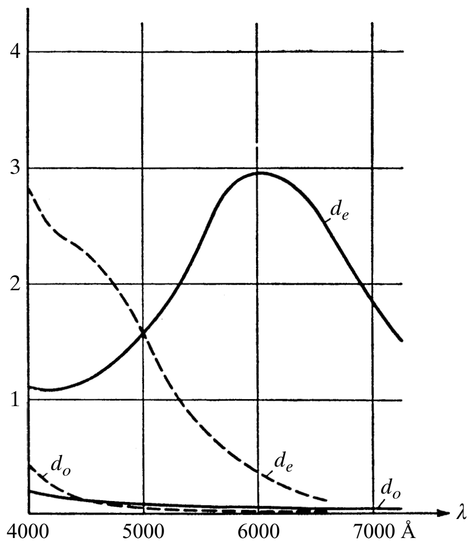

Fig. 15.33 Dichroism of iodine on oriented polyvinyl alcohol, blue (full line) and brown (broken line).

$$d = 4\pi(\log_{10} e)(n\kappa l/\lambda) \sim 5.5n\kappa l/\lambda.$$

(After E. H. Land and C. D. West, contribution in *Colloid Chemistry*, Vol. 6, ed. J. Alexander (New York, Reinhold Publishing Corporation, 1946, p. 177.)

https://doi.org/10.1017/9781108769914.018 Published online by Cambridge University Press

Finally, let us recall that in the main part of this chapter we have assumed that the ellipticity of the principal light vibrations in the crystal may be neglected. This assumption is certainly not always justified; it may be shown, for example, that in every absorbing crystal there exist four directions, lying in pairs near the optic axes, for which the polarization is circular. However, with weaker and weaker absorption the regions of appreciable ellipticity are more and more restricted to the neighbourhood of these special directions, which themselves tend to coincidence with the optic axis. Taking this into account, the general character of the phenomenon is retained and we, therefore, do not need to enter into a more detailed discussion. Naturally our analysis does not reveal some of the finer aspects of the optical theory of absorbing crystals; for their discussion we must refer elsewhere.*

* See, for example, G. Szivessy, *Handbuch der Physik*, Vol. 20 (Berlin, Springer, 1928), pp. 861–904.