Appendix VII

The circle polynomials of Zernike (§9.2.1)

In this appendix, the circle polynomials discussed briefly in §9.2.1 will be considered more fully. These polynomials were introduced and first investigated by Zernike,* in his important paper on the knife edge test and the phase contrast method, and were studied further by him and Brinkman,† and by Nijboer.‡ They were later derived from the requirement of orthogonality and invariance alone by Bhatia and Wolf;§ we shall follow substantially their treatment.

1 Some general considerations

It is not difficult to show that there exists an infinity of complete sets of polynomials in two real variables x, y which are orthogonal for the interior of the unit circle, i.e. which satisfy the orthogonality condition

$$\iint\limits_{x^2+y^2\leqslant 1} V^{\star}_{(\alpha)}(x,\ y)V_{(\beta)}(x,\ y)\mathrm{d}x\,\mathrm{d}y = A_{\alpha\beta}\,\delta_{\alpha\beta}. \tag{1}$$

Here $V_{(\alpha)}$ and $V_{(\beta)}$ denote two typical polynomials of the set, the asterisk denotes complex conjugate, δ is the Kronecker symbol, and $A_{\alpha\beta}$ are normalization constants, to be chosen later. The circle polynomials of Zernike are distinguished from the other sets by certain simple invariance properties which can best be explained from group theoretical considerations. It is, however, possible to avoid the abstract formalism of group theory with the help of a kind of normalization. One considers first of all such sets as are 'invariant in form' with respect to rotations of axes about the origin. By such invariance we mean that when any rotation

$$\left.\begin{aligned} x' &= x\cos\phi + y\sin\phi,\\ y' &= -x\sin\phi + y\cos\phi \end{aligned}\right\} \tag{2}$$

* F. Zernike, *Physica*, **1** (1934), 689.
† F. Zernike and H. C. Brinkman, *Verh. Akad. Wet. Amst.* (Proc. Sec. Sci.), **38** (1935), 11.
‡ B. R. A. Nijboer, Thesis (University of Groningen, 1942).
§ A. B. Bhatia and E. Wolf, *Proc. Camb. Phil. Soc.*, **50** (1954), 40.

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is applied, each polynomial $V(x, y)$ is transformed into a polynomial of the same form, i.e. V satisfies the following relation under the transformation (2):

$$V(x, y) = G(\phi)V(x', y'), \tag{3}$$

where $G(\phi)$ is a continuous function with period 2π of the angle of rotation ϕ, and $G(0) = 1$.

Now the application of two successive rotations through angles ϕ_1 and ϕ_2 is equivalent to a single rotation through an angle $\phi_1 + \phi_2$. Hence it follows from (3) that G must satisfy the functional equation

$$G(\phi_1)G(\phi_2) = G(\phi_1 + \phi_2). \tag{4}$$

The general solution with the period 2π of this equation is well known, and is*

$$G(\phi) = \mathrm{e}^{\mathrm{i}l\phi}. \tag{5}$$

Here l is any integer, positive, negative or zero. On substituting from (5) into (3), setting $x' = \rho$, $y' = 0$, and using (2) it follows that V must be of the form

$$V(\rho\cos\phi, \rho\sin\phi) = R(\rho)\mathrm{e}^{\mathrm{i}l\phi}, \tag{6}$$

where $R(\rho) = V(\rho, 0)$ is a function of ρ alone. Next we expand $\mathrm{e}^{\mathrm{i}l\phi}$ in powers of $\cos\phi$ and $\sin\phi$. Suppose that V is a polynomial of degree n in the variables $x = \rho\cos\phi$, $y = \rho\sin\phi$; it then follows from (6) that $R(\rho)$ is a polynomial in ρ of degree n and contains no power of ρ of degree lower than $|l|$. Moreover, $R(\rho)$ is evidently an even or an odd polynomial according as l is even or odd. The set of *the Zernike circle polynomials* is distinguished from all other such sets by the property that it *contains a polynomial for each pair of the permissible values of n* (*degree*) *and l* (*angular dependence*), i.e. for integral values of n and l, such that $n \geqslant 0$, $l \gtreqless 0$, $n \geqslant |l|$, and $n - |l|$ is even. We denote a typical polynomial of this set by

$$V_n^l(\rho\cos\phi, \rho\sin\phi) = R_n^l(\rho)\mathrm{e}^{\mathrm{i}l\phi}. \tag{7}$$

It follows from (1) and (7) that the *radial polynomials* $R_n{}^l(\rho)$ satisfy the relation

$$\int_0^1 R_n^l(\rho)R_{n'}^l(\rho)\rho\,\mathrm{d}\rho = a_n^l\delta_{nn'}, \tag{8}$$

where

$$\frac{a_n = A_n^l}{2\pi.} \tag{9}$$

For any given value l, the lower index n can only take the values $|l|$, $|l| + 2$, $|l| + 4, \ldots$. The corresponding sequence $R_{|l|}^l(\rho)$, $R_{|l|+2}^l(\rho)$, $R_{|l|+4}^l(\rho), \ldots$ may be obtained by orthogonalizing the powers

$$\rho^{|l|}, \rho^{|l|+2}, \rho^{|l|+4}, \ldots \tag{10}$$

with the weighting factor ρ over the interval $0 \leqslant \rho \leqslant 1$. Moreover, since only the absolute values of l occur in (10),

* See, for example, M. Born, *Natural Philosophy of Cause and Chance* (Oxford, Clarendon Press, 1949; Dover Publications, New York, 1964), p. 153.

$$R_n^{-l}(\rho) = \beta_n^l R_n^l(\rho), \tag{11}$$

where β_n^l is a constant depending only on the normalization of the two polynomials R_n^{-l} and R_n^l. In particular we may normalize in such a way that $\beta_n^l = 1$ for all l and n, and then

$$V_n^{\pm m}(\rho\cos\phi,\ \rho\sin\phi) = R_n^m(\rho)\mathrm{e}^{\pm \mathrm{i}m\phi}, \tag{12}$$

where $m = |l|$ is a nonnegative integer.

The set of the circle polynomials contains $\frac{1}{2}(n+1)(n+2)$ linearly independent polynomials of degree $\leqslant n$. Hence every monomial $x^i y^j$ ($i \geqslant 0$, $j \geqslant 0$ integers) and, consequently every polynomial in x, y may be expressed as a linear combination of a finite number of the circle polynomials V_n^l. By Weierstrass' approximation theorem* it then follows that the set is *complete*.

2 Explicit expressions for the radial polynomials $R_n^{\pm m}(\rho)$

Since $R_n^{\pm m}(\rho)$ is a polynomial in ρ of degree n and contains no power of ρ lower than m and is an even or odd polynomial according as n is even or odd, it follows that R may be expressed in the form

$$R_n^{\pm m}(\rho) = t^{\frac{m}{2}} Q_{\frac{n-m}{2}}(t), \tag{13}$$

where $t = \rho^2$ and $Q_{(n-m)/2}(t)$ is a polynomial in t of degree $\frac{1}{2}(n-m)$. According to (8), the Q polynomials must satisfy the relations

$$\left.\begin{aligned} &\tfrac{1}{2}\int_0^1 t^m Q_k(t) Q_{k'}(t)\mathrm{d}t = a_n^{\pm m}\delta_{kk'}, \\ \text{where}\qquad & \\ &k = \tfrac{1}{2}(n-m), \qquad k' = \tfrac{1}{2}(n'-m). \end{aligned}\right\} \tag{14}$$

It follows that the polynomials $Q_0(t)$, $Q_1(t)$, ..., $Q_k(t)$, ... may be obtained by orthogonalizing the sequence of natural powers

$$1,\ t,\ t^2,\ \ldots,\ t^k,\ \ldots \tag{15}$$

with the weighting factor $w(t) = t^m$ over the range $0 \leqslant t \leqslant 1$. Now the well-known Jacobi (or hypergeometric) polynomials†

$$G_k(p,\ q,\ t) = \frac{(q-1)!}{(q+k-1)!} t^{1-q}(1-t)^{q-p} \frac{\mathrm{d}^k}{\mathrm{d}t^k}[t^{q-1+k}(1-t)^{p-q+k}] \tag{16}$$

$$= \frac{k!(q-1)!}{(p+k-1)!} \sum_{s=0}^{k} (-1)^s \frac{(p+k+s-1)!}{(k-s)!s!(q+s-1)!} t^s, \tag{17}$$

($k \geqslant 0$, $q > 0$, $p - q > -1$) may be defined as functions obtained by orthogonalizing (15) with the more general weighting function

* See, for example, R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1 (New York, Interscience Publishers, 1st English edition, 1953), p. 65.

† R. Courant and D. Hilbert, *loc. cit.*, Vol. I, p. 90.

$$w(t) = t^{q-1}(1-t)^{p-q}$$

over the range $0 \leqslant t \leqslant 1$. Their orthogonality and normalization properties are given by*

$$\int_0^1 t^{q-1}(1-t)^{p-q} G_k(p, q, t) G_{k'}(p, q, t) \mathrm{d}t = b_k(p, q)\delta_{kk'}, \tag{18}$$

where

$$b_k(p, q) = \frac{k![(q-1)!]^2[p-q+k]!}{[q-1+k]![p-1+k]![p+2k]}. \tag{19}$$

(With this choice of b_k, $G_k(p, q, 0) = 1$ for all k.) On comparing (18) and (14) it follows that†

$$Q_k(t) = \sqrt{\frac{2a_n^{\pm m}}{b_k(m+1, m+1)}} G_k(m+1, m+1, t). \tag{20}$$

From (13) and (20) we obtain the following expression for the radial polynomials in terms of Jacobi polynomials:

$$R_n^{\pm m}(\rho) = \sqrt{\frac{2a_n^{\pm m}}{b_k(m+1, m+1)}} \rho^m G_k(m+1, m+1, \rho^2), \qquad [k = \tfrac{1}{2}(n-m)]. \tag{21}$$

Following Zernike, we choose the normalization so that for all n and m

$$R_n^{\pm m}(1) = 1. \tag{22}$$

Then from (21) and (22)

$$\sqrt{\frac{b_k(m+1, m+1)}{2a_n^{\pm m}}} = G_k(m+1, m+1, 1). \tag{23}$$

The value of $G_k(m+1, m+1, 1)$ can be obtained from the generating function for the Jacobi polynomials.‡ We have

$$\frac{\left[z - 1 + \sqrt{1 - 2z(1-2\rho^2) + z^2}\right]^m}{(2z\rho^2)^m \sqrt{1 - 2z(1-2\rho^2) + z^2}} = \sum_{s=0}^{\infty} \binom{m+s}{s} G_s(m+1, m+1, \rho^2) z^s. \tag{24}$$

For $\rho = 1$, the left-hand side reduces to $(1+z)^{-1}$ and we obtain on expanding it in a power series and on comparing it with the right-hand side:

$$G_s(m+1, m+1, 1) = \frac{(-1)^s}{\binom{m+s}{s}}. \tag{25}$$

From (25) and (23) it follows that

* E. Kemble, *The Fundamental Principles of Quantum Mechanics* (New York, McGraw-Hill, 1937), p. 594.
† The sign of the square root on the right-hand side of (20) is determined from (26) below.
‡ See R. Courant and D. Hilbert, *loc. cit.*, Vol. I, p. 91.

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$$\sqrt{\frac{2a_n^{\pm m}}{b_k(m+1,\, m+1)}} = (-1)^{\frac{n-m}{2}} \binom{\frac{1}{2}(n+m)}{\frac{1}{2}(n-m)}, \tag{26}$$

and using (16), (17) and (26) we finally obtain from (21) the following expressions for the radial polynomials:

$$R_n^{\pm m}(\rho) = \frac{1}{\left(\dfrac{n-m}{2}\right)!\rho^m} \left[\frac{\mathrm{d}}{\mathrm{d}(\rho^2)}\right]^{\frac{n-m}{2}} \left[(\rho^2)^{\frac{n+m}{2}}(\rho^2-1)^{\frac{n-m}{2}}\right] \tag{27}$$

$$= \sum_{s=0}^{\frac{1}{2}(n-m)} (-1)^s \frac{(n-s)!}{s!\left(\dfrac{n+m}{2}-s\right)!\left(\dfrac{n-m}{2}-s\right)!} \rho^{n-2s}. \tag{28}$$

Explicit expressions for the first few polynomials are given in Table 9.1 on p. 524.

The normalization constant $a_n^{\pm m}$ is obtained from (26) and (19):

$$a_n^{\pm m} = \frac{1}{2n+2}. \tag{29}$$

To obtain the generating function for the radial polynomials, we write s in place of $k = (n-m)/2$ and $m + 2s$ in place of n in (21) and (26) and substitute into (24). We then obtain

$$\frac{\left[1 + z - \sqrt{1 + 2z(1-2\rho^2) + z^2}\right]^m}{(2z\rho)^m \sqrt{1 + 2z(1-2\rho^2) + z^2}} = \sum_{s=0}^{\infty} z^s R_{m+2s}^{\pm m}(\rho). \tag{30}$$

Finally we shall evaluate the integral

$$\int_0^1 R_n^m(\rho) J_m(v\rho)\rho\, \mathrm{d}\rho$$

which, as we saw in Chapter IX, plays an important part in the Zernike–Nijboer diffraction theory of aberrations. We substitute for $R_n^m(\rho)$ the expression (27) and for the Bessel function J_m its series expansion.* The resulting expression may be written in the form

$$\int_0^1 R_n^m(\rho) J_m(v\rho)\rho\, \mathrm{d}\rho$$

$$= \frac{1}{2\left(\dfrac{n-m}{2}\right)!} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+m)!} \left(\frac{v}{2}\right)^{m+2s} f\left(s, \frac{n-m}{2}, \frac{n+m}{2}, \frac{n-m}{2}\right), \tag{31}$$

where

$$f(s,\, p,\, q,\, r) = \int_0^1 u^s \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^p [u^q(u-1)^r]\mathrm{d}u, \tag{32}$$

* R. Courant and D. Hilbert, *loc. cit.*, Vol. I, p. 484.

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p, q, r, s being nonnegative integers. Integrating (32) by parts it follows that

$$f(s,\, p,\, q,\, r) = \left\{ u^s \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{p-1} [u^q(u-1)^r] \right\}_0^1 - s\int_0^1 u^{s-1}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{p-1} [u^q(u-1)^r]\mathrm{d}u. \tag{33}$$

Now if $r \geqslant p$ and $s + q - p \geqslant 0$, the first term on the right vanishes, so that

$$f(s,\, p,\, q,\, r) = -sf(s-1,\, p-1,\, q,\, r). \tag{34}$$

We consider separately the case $s \geqslant p$ and $s < p$.

When $s \geqslant p$, we have, on applying (34) p times,

$$\begin{aligned} f(s,\, p,\, q,\, r) &= (-1)^p s(s-1)(s-2)\cdots(s-p+1)f(s-p,\, 0,\, q,\, r) \\ &= \frac{(-1)^{p+r}s!}{(s-p)!}\int_0^1 u^{s+q-p}(1-u)^r\, \mathrm{d}u. \end{aligned} \tag{35}$$

The integral in (35) is the Euler integral of the first kind (beta function) and its value is* $(s+q-p)!r!/(s+q+r-p+1)!$ Hence, for $s \geqslant p$,

$$f(s,\, p,\, q,\, r) = (-1)^{p+r}\frac{s!(s+q-p)!r!}{(s-p)!(s+q+r-p+1)!}. \tag{36}$$

Next consider the case $s < p$. Applying (34) s times it follows that

$$\begin{aligned} f(s,\, p,\, q,\, r) &= (-1)^s s(s-1)\cdots f(0,\, p-s,\, q,\, r) \\ &= (-1)^s s! \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{p-s-1} [u^q(u-1)^r] \right\}_0^1 \\ &= 0. \end{aligned} \tag{37}$$

We now substitute from (36) and (37) into (31) and introduce a new variable l such that $s = \frac{1}{2}(n-m) + l$. We then obtain

$$\int_0^1 R_n^m(\rho)J_m(v\rho)\rho\, \mathrm{d}\rho = \frac{(-1)^{\frac{3(n-m)}{2}}}{v}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!(n+l+1)!}\left(\frac{v}{2}\right)^{n+2l+1}. \tag{38}$$

The series on the right will be recognized as the expansion of $J_{n+1}(v)$. Since $n-m$ is even the factor $(-1)^{3(n-m)/2}$ may be replaced by $(-1)^{(n-m)/2}$, and we finally obtain

$$\int_0^1 R_n^m(\rho)J_m(v\rho)\rho\, \mathrm{d}\rho = (-1)^{\frac{n-m}{2}}\frac{J_{n+1}(v)}{v}. \tag{39}$$

* R. Courant and D. Hilbert, *loc. cit.*, Vol. I, p. 483.

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