
Spontaneous Light Scattering and Acoustooptics

8.1 Features of Spontaneous Light Scattering

In this chapter, we describe spontaneous light scattering; Chapters 9 and 10 present descriptions of various stimulated light-scattering processes. By spontaneous light scattering, we mean light scattering under conditions such that the optical properties of the material system are unmodified by the presence of the incident light beam. We shall see in the following two chapters that the character of the light-scattering process is profoundly modified whenever the intensity of the incident light is sufficiently large to modify the optical properties of the material system.

Let us first consider the light-scattering experiment illustrated in part (a) of Fig. 8.1.1. Under the most general circumstances, the spectrum of the scattered light has the form shown in part (b) of the figure, in which Raman, Brillouin, Rayleigh, and Rayleigh-wing features are present. By definition, those components of the scattered light that are shifted to lower frequencies are known as Stokes components, and those components that are shifted to higher frequencies are known as anti-Stokes components. Table 8.1.1 lists some of the physical processes that can lead to light scattering of the sort shown in the figure and gives some of the physical parameters that describe these processes.

One of these light-scattering processes is Raman scattering. Raman scattering results from the interaction of light with the vibrational modes of the molecules constituting the scattering medium. Raman scattering can equivalently be described as the scattering of light from optical phonons.

Brillouin scattering is the scattering of light from sound waves—that is, from propagating pressure (and thus density) waves. Brillouin scattering can also be considered to be the scattering of light from acoustic phonons.

Rayleigh scattering (or Rayleigh-center scattering) is the scattering of light from nonpropagating density fluctuations. Formally, it can be described as scattering from entropy fluctuations. It is known as quasielastic scattering because it induces no frequency shift.

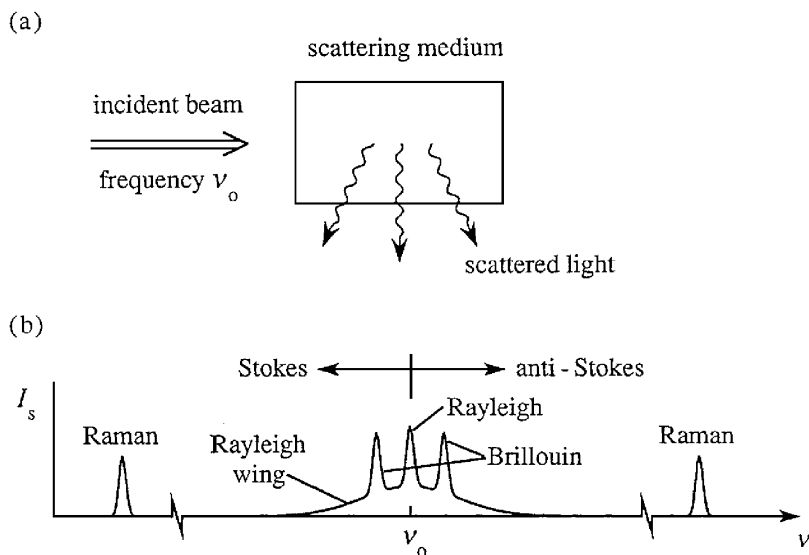


FIGURE 8.1.1: Spontaneous light scattering. (a) Experimental setup. (b) Typical observed spectrum.

Rayleigh-wing scattering (i.e., scattering in the wing of the Rayleigh line) is scattering from fluctuations in the orientation of anisotropic molecules. Since the molecular reorientation process is very rapid, this component is spectrally very broad. Rayleigh-wing scattering does not occur for molecules with an isotropic polarizability tensor.

8.1.1 Fluctuations as the Origin of Light Scattering

Light scattering occurs as a consequence of fluctuations in the optical properties of a material medium; a completely homogeneous material can scatter light only in the forward direction (see, for example, Fabelinskii, 1968). This conclusion can be demonstrated with the aid of

TABLE 8.1.1: Typical values of the parameters describing several light-scattering processes.

Process	Shift (cm^{-1})	Linewidth (cm^{-1})	Relaxation time (sec)	Gain ^a (m/MW)
Raman	1000	5	10^{-12}	5×10^{-5}
Brillouin	0.1	5×10^{-3}	10^{-9}	10^{-4}
Rayleigh	0	5×10^{-4}	10^{-8}	10^{-6}
Rayleigh-wing	0	5	10^{-12}	10^{-5}

^a Gain of the stimulated version of the process.

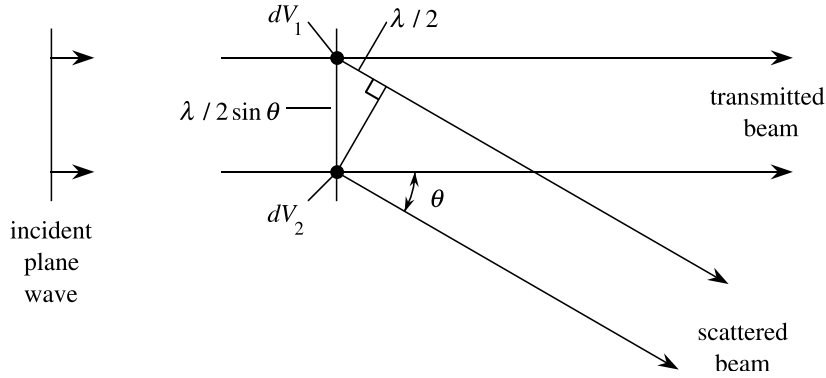


FIGURE 8.1.2: Light scattering cannot occur in a completely homogeneous medium.

Fig. 8.1.2, which shows a completely homogeneous medium being illuminated by a plane wave. We suppose that the volume element dV_1 scatters light into the θ direction. However, for any direction, except the exact forward direction ($\theta = 0$), there must be a nearby volume element (labeled dV_2) whose scattered field interferes destructively with that from dV_1 . Since the same argument can be applied to any volume element in the medium, we conclude that there can be no scattering in any direction except $\theta = 0$. Scattering in the direction $\theta = 0$ is known as coherent forward scattering and is the origin of the index of refraction. (See, for example, the discussion in Section 31 of Feynman et al., 1963.)

Note that the argument that scattering cannot occur (except in the forward direction) requires that the medium be *completely* homogeneous. Scattering can occur as the result of fluctuations in any of the optical properties of the medium. For example, if the density of the medium is nonuniform, then the total number of molecules in the volume element dV_1 may not be equal to the number of molecules in dV_2 , and consequently the destructive interference between the fields scattered by these two elements will not be exact.

Since light scattering results from fluctuations in the optical properties of a material medium, it is useful to represent the dielectric tensor of the medium (which for simplicity we assume to be isotropic in its average properties) as (Landau and Lifshitz, 1960)

$$\epsilon_{ik} = \bar{\epsilon}\delta_{ik} + \Delta\epsilon_{ik}, \quad (8.1.1)$$

where $\bar{\epsilon}$ represents the mean dielectric constant of the medium and where $\Delta\epsilon_{ik}$ represents the (temporally and/or spatially varying) fluctuations in the dielectric tensor that lead to light scattering. It is convenient to decompose the fluctuation $\Delta\epsilon_{ik}$ in the dielectric tensor into the sum of a scalar contribution $\Delta\epsilon\delta_{ik}$ and a (traceless) tensor contribution $\Delta\epsilon_{ik}^{(t)}$ as

$$\Delta\epsilon_{ik} = \Delta\epsilon\delta_{ik} + \Delta\epsilon_{ik}^{(t)}. \quad (8.1.2)$$

The scalar contribution $\Delta\epsilon$ arises from fluctuations in thermodynamic quantities such as the pressure, entropy, density, or temperature. In a chemical solution it also has a contribution from fluctuations in concentration. Scattering that results from $\Delta\epsilon$ is called scalar light scattering; examples of scalar light scattering include Brillouin and Rayleigh scattering.

Scattering that results from $\Delta\epsilon_{ik}^{(t)}$ is called tensor light scattering. The tensor $\Delta\epsilon_{ik}^{(t)}$ has been taken to be traceless (i.e., $\sum_i \Delta\epsilon_{ii}^{(t)} = 0$), since the scalar contribution $\Delta\epsilon$ has been separated out. It is useful to express $\Delta\epsilon_{ik}^{(t)}$ as

$$\Delta\epsilon_{ik}^{(t)} = \Delta\epsilon_{ik}^{(s)} + \Delta\epsilon_{ik}^{(a)}, \quad (8.1.3)$$

where $\Delta\epsilon_{ik}^{(s)}$ is the symmetric part of $\Delta\epsilon_{ik}^{(t)}$ (symmetric in the sense that $\Delta\epsilon_{ik}^{(s)} = \Delta\epsilon_{ki}^{(s)}$) and gives rise to Rayleigh-wing scattering, and where $\Delta\epsilon_{ik}^{(a)}$ is the antisymmetric part of $\Delta\epsilon_{ik}^{(t)}$ (that is, $\Delta\epsilon_{ik}^{(a)} = -\Delta\epsilon_{ki}^{(a)}$) and gives rise to Raman scattering.

It can be shown that the fluctuations $\Delta\epsilon$, $\Delta\epsilon_{ik}^{(s)}$, and $\Delta\epsilon_{ik}^{(a)}$ are statistically independent. Scattering due to $\Delta\epsilon_{ik}^{(t)}$ is called depolarized scattering, because in general the degree of polarization in the scattered light is smaller than that of the incident light.

8.1.2 Scattering Coefficient

A quantity that is used to describe the efficiency of the scattering process is the scattering coefficient R , which is defined in terms of the quantities shown in Fig. 8.1.3. Here a beam of light of intensity I_0 (in units of power per unit area, that is, W m^{-2}) illuminates a scattering region of volume V , and the intensity I_s of the scattered light is measured at a distance L from the interaction region. It is reasonable to assume that the intensity of the scattered light increases linearly with the intensity I_0 of the incident light and with the volume V of the interaction region and that it obeys the inverse square law with respect to the distance L to the point of observation. We can hence represent I_s as

$$I_s = \frac{I_0 R V}{L^2}, \quad (8.1.4)$$

where the constant of proportionality R is known as the scattering coefficient R .

We now assume that the scattered light falls onto a small detector of projected area dA . The power hitting the detector is given by $dP = I_s dA$. Since the detector subtends a solid angle at the scattering region given by $d\Omega = dA/L^2$, the scattered power per unit solid angle is given by $dP/d\Omega = I_s L^2$, or by

$$\frac{dP}{d\Omega} = I_0 R V. \quad (8.1.5)$$

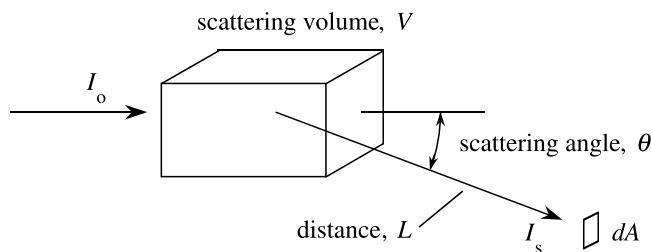


FIGURE 8.1.3: Quantities used to define the scattering coefficient.

Either Eq. (8.1.4) or (8.1.5) can be taken as the definition of the scattering coefficient R . For scattering of visible light through an angle of 90° , R has the value $2 \times 10^{-8} \text{ cm}^{-1}$ for air and $1.4 \times 10^{-4} \text{ m}^{-1}$ for water.

8.1.3 Scattering Cross Section

It is also useful to define the scattering cross section. We consider a beam of intensity I_0 falling onto an individual molecule, as shown in Fig. 8.1.4. We let P denote the total power of the radiation scattered by this molecule. We assume that P increases linearly with I_0 according to

$$P = \sigma I_0, \quad (8.1.6)$$

where the constant of proportionality σ is known as the (total) scattering cross section. Since I_0 has the dimensions of power per unit area, we see that σ has the dimensions of an area, which justifies it being called a cross section. The cross section can be interpreted as the effective geometrical area of the molecule for removing light from the incident beam.

We also define a differential cross section. Rather than describing the total scattered power, this quantity describes the power dP scattered in some particular direction into the element

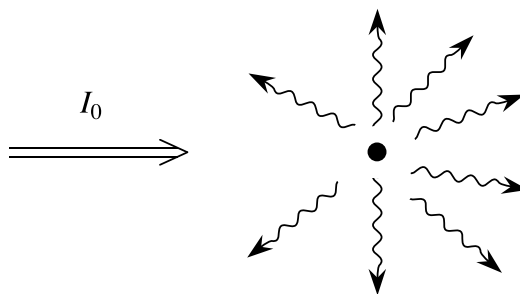


FIGURE 8.1.4: Scattering of light by a molecule.

of solid angle $d\Omega$. We assume that the scattered power per unit solid angle $dP/d\Omega$ increases linearly with the incident intensity according to

$$\frac{dP}{d\Omega} = I_0 \frac{d\sigma}{d\Omega}, \quad (8.1.7)$$

where $d\sigma/d\Omega$ is known as the differential cross section. Clearly, since P is equal to $\int (dP/d\Omega) d\Omega$, it follows from Eqs. (8.1.6) and (8.1.7) that

$$\sigma = \int_{4\pi} \frac{d\sigma}{d\Omega} d\Omega. \quad (8.1.8)$$

Let us next see how to relate the differential scattering cross section $d\sigma/d\Omega$ to the scattering coefficient R . If each of the \mathcal{N} molecules contained in the volume V of Fig. 8.1.3 scatters independently, then the total power per unit solid angle of the scattered light will be \mathcal{N} times larger than the result given in Eq. (8.1.7). Consequently, by comparison with Eq. (8.1.5), we see that the scattering coefficient is given by

$$R = \frac{\mathcal{N}}{V} \frac{d\sigma}{d\Omega}. \quad (8.1.9)$$

One should be wary about taking this equation to constitute a generally valid result. Recall that a completely homogeneous medium does not scatter light at all, which implies that for such a medium R would be equal to zero and not to $(\mathcal{N}/V)(d\sigma/d\Omega)$. In the next section we examine the conditions under which it is valid to assume that each molecule scatters independently. As a general rule, Eq. (8.1.9) is valid for dilute media and is entirely invalid for condensed matter.

8.2 Microscopic Theory of Light Scattering

Let us now consider light scattering in terms of the field scattered by each molecule contained within the interaction region. Such a treatment is particularly well suited to the case of scattering from a dilute gas, where collective effects due to the interaction of the various molecules are relatively unimportant. (Light scattering from condensed matter is more conveniently treated using the thermodynamic formalism presented in the next section.) As illustrated in Fig. 8.2.1, we assume that the optical field

$$\tilde{\mathbf{E}} = \mathbf{E}_0 e^{-i\omega t} + \text{c.c.} \quad (8.2.1)$$

of intensity $I_0 = (2nc\epsilon_0)|E_0|^2$ is incident on a molecule whose linear dimensions are assumed to be much smaller than the wavelength of light. In response to the applied field, the molecule develops the dipole moment

$$\tilde{\mathbf{p}} = \epsilon_0 \alpha(\omega) \mathbf{E}_0 e^{-i\omega t} + \text{c.c.}, \quad (8.2.2)$$

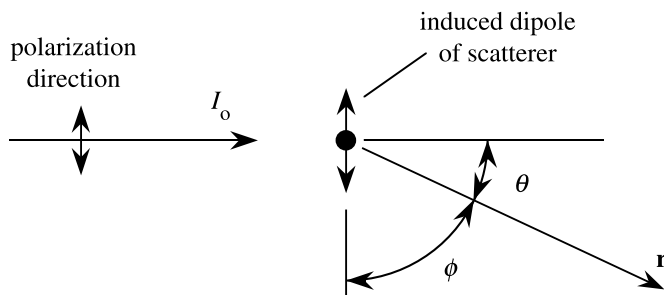


FIGURE 8.2.1: Geometry of light scattering from an individual molecule.

where $\alpha(\omega)$ is the polarizability of the particle. Explicit formulas for $\alpha(\omega)$ for certain types of scatterers are given below, but for reasons of generality we leave the form of $\alpha(\omega)$ unspecified for the present.

As a consequence of the time-varying dipole moment given by Eq. (8.2.2), the particle will radiate. The intensity of this radiation at a distance L from the scatterer is given by the magnitude of the Poynting vector (see, for example, Jackson, 1982, Section 9.2) as

$$I_s = \frac{n \langle \dot{\mathbf{p}}^2 \rangle}{16\pi^2 \epsilon_0 c^3 L^2} \sin^2 \phi = \frac{n \omega^4 \epsilon_0 |\alpha(\omega)|^2 |\mathbf{E}_0|^2}{8\pi^2 c^3 L^2} \sin^2 \phi. \quad (8.2.3)$$

The angular brackets in the first form imply that the time average of the enclosed quantity is to be taken. As shown in Fig. 8.2.1, ϕ is the angle between the induced dipole moment of the particle and the direction \mathbf{r} to the point of observation.

We next use Eq. (8.2.3) to derive an expression for the differential scattering cross section. As in the derivation of Eq. (8.1.5), the scattered power per unit solid angle is given by $dP/d\Omega = I_s L^2$. We introduce the differential cross section of Eq. (8.1.7), $d\sigma/d\Omega = (dP/d\Omega)/I_0 = I_s L^2/I_0$, which through the use of Eq. (8.2.3) becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 \sin^2 \phi. \quad (8.2.4)$$

We note that this expression for the differential cross section $d\sigma/d\Omega$ predicts a $\sin^2 \phi$ dependence for *any* functional form for $\alpha(\omega)$. This result is a consequence of our assumption that the scattering particle is small compared to an optical wavelength and hence that the scattering is due solely to electric dipole and not to higher-order multipole processes. Since the angular dependence of $d\sigma/d\Omega$ is contained entirely in the $\sin^2 \phi$ term, we can immediately obtain an expression for the total scattering cross section by integrating $d\sigma/d\Omega$ over all solid angles, yielding

$$\sigma = \int_{4\pi} d\Omega \frac{d\sigma}{d\Omega} = \frac{8\pi}{3} \frac{1}{16\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 = \frac{1}{6\pi} \frac{\omega^4}{c^4} |\alpha(\omega)|^2. \quad (8.2.5)$$

In deriving Eq. (8.2.4) for the differential scattering cross section, we assumed that the incident light was linearly polarized, and for convenience we took the direction of polarization to lie in the plane of Fig. 8.2.1. For this direction of polarization, the scattering angle θ and the angle ϕ of Eq. (8.2.3) are related by $\theta + \phi = 90^\circ$, and thus for this direction of polarization Eq. (8.2.4) can be expressed in terms of the scattering angle as

$$\left(\frac{d\sigma}{d\Omega}\right)_p = \frac{1}{16\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 \cos^2 \theta. \quad (8.2.6)$$

Other types of polarization can be treated by allowing the incident field to have a component perpendicular to the plane of Fig. 8.2.1. For this component ϕ is equal to 90° for any value of the scattering angle θ , and thus for this component the differential cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_s = \frac{1}{16\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 \quad (8.2.7)$$

for any value of θ . Since unpolarized light consists of equal intensities in the two orthogonal polarization directions, the differential cross section for unpolarized light is obtained by averaging Eqs. (8.2.6) and (8.2.7), giving

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} = \frac{1}{32\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 (1 + \cos^2 \theta). \quad (8.2.8)$$

As an example of the use of these equations, we consider scattering from an atom whose optical properties can be described by the Lorentz model of the atom (that is, we model the atom as a simple harmonic oscillator). According to Eqs. (1.4.17) and (1.4.10) and the relation of $\chi(\omega) = N\alpha(\omega)$, the polarizability of such an atom is given by

$$\alpha(\omega) = \frac{e^2/m\epsilon_0}{\omega_0^2 - \omega^2 - 2i\omega\gamma}, \quad (8.2.9)$$

where ω_0 is the resonance frequency and γ is the dipole damping rate. Through use of this expression, the total scattering cross section given by Eq. (8.2.5) becomes

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\gamma^2}. \quad (8.2.10)$$

The frequency dependence of the scattering cross section predicted by this equation is illustrated in Fig. 8.2.2. Eq. (8.2.10) can be simplified under several different limiting conditions.

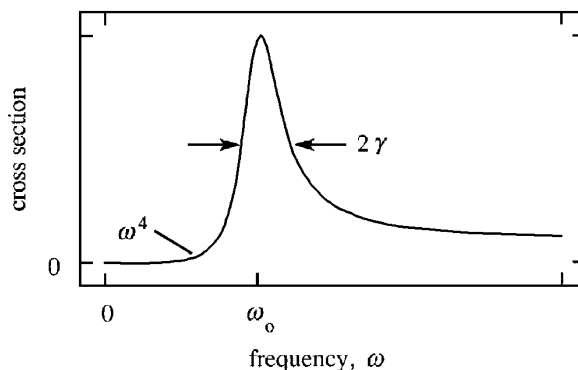


FIGURE 8.2.2: Frequency dependence of the scattering cross section of a Lorentz oscillator.

In particular, we find that

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \frac{\omega^4}{\omega_0^4} \quad \text{for } \omega \ll \omega_0, \quad (8.2.11a)$$

$$\sigma = \frac{2\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \frac{\omega_0^2}{(\omega_0 - \omega)^2 + \gamma^2} \quad \text{for } \omega \simeq \omega_0, \quad (8.2.11b)$$

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \quad \text{for } \omega \gg \omega_0. \quad (8.2.11c)$$

Eq. (8.2.11a) shows that the scattering cross section increases as the fourth power of the optical frequency ω in the limit $\omega \ll \omega_0$. This result leads, for example, to the prediction that the sky is blue, since the shorter wavelengths of sunlight are scattered far more efficiently in the earth's atmosphere than are the longer wavelengths. Scattering in this limit is often known as Rayleigh scattering. Eq. (8.2.11b) shows that near the atomic resonance frequency the dependence of the scattering cross section on the optical frequency has a Lorentzian lineshape. Eq. (8.2.11c) shows that for very large frequencies the scattering cross section approaches a constant value. This value is of the order of the square of the “classical” electron radius, $r_e = e^2/4\pi\epsilon_0 mc^2 = 2.82 \times 10^{-15}$ m. Scattering in this limit is known as Thompson scattering.

As a second example of the application of Eq. (8.2.5), we consider scattering from a collection of small dielectric spheres. We take ϵ_1 to be the dielectric constant of the material within each sphere and ϵ to be that of the surrounding medium. We assume that each sphere is small in the sense that its radius a is much smaller than the wavelength of the incident radiation. We can then calculate the polarizability of each sphere using the laws of electrostatics. It is straightforward to show (see, for example, Stratton, 1941, p. 206; or Jackson, 1982, p. 158) that the

polarizability is given by the expression

$$\alpha = 4\pi\epsilon \frac{\epsilon_1 - \epsilon}{\epsilon_1 + 2\epsilon} a^3. \quad (8.2.12)$$

Note that α depends on frequency only through any possible frequency dependence of ϵ or of ϵ_1 . Through the use of Eq. (8.2.5), we find that the scattering cross section is given by

$$\sigma = \frac{8\pi}{3} \frac{\omega^4}{c^4} a^6 \epsilon^2 \left(\frac{\epsilon_1 - \epsilon}{\epsilon_1 + 2\epsilon} \right)^2. \quad (8.2.13)$$

Note that, as in the low-frequency limit of the Lorentz atom, the cross section scales as the fourth power of the frequency. Note also that the cross section scales as the square of the volume of each particle.

Let us now consider the rather subtle problem of calculating the total intensity of the light scattered from a collection of molecules. We recall from the discussion of Fig. 8.1.2 that only the fluctuations in the optical properties of the medium can lead to light scattering. As shown in Fig. 8.2.3, we divide the total scattering volume V into a large number of identical small regions of volume V' . We assume that V' is sufficiently small that all of the molecules within V' radiate essentially in phase. The intensity of the light emitted by the atoms in V' in some particular direction can thus be represented as

$$I_{V'} = \nu^2 I_{\text{mol}}, \quad (8.2.14)$$

where ν represents the number of molecules in V' and I_{mol} denotes the intensity of the light scattering by a single molecule.

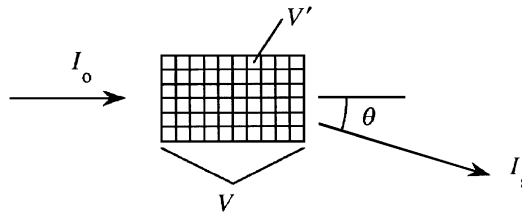


FIGURE 8.2.3: Light scattering from a collection of molecules.

We next calculate the total intensity of the scattered radiation from the entire volume V . We recall from the discussion of Section 8.1 that, for each volume element V' , there will be another element whose radiated field tends to interfere destructively with that from V' . Insofar as each volume element contains exactly the same number of molecules, the cancellation will be complete. However, any deviation of ν from its mean value $\bar{\nu}$ can lead to a net intensity of the scattered radiation. The contribution to the net scattered intensity from volume element

V' is thus given by $\overline{\Delta v^2} I_{\text{mol}}$, where $\overline{\Delta v^2}$ denotes the mean-square fluctuation given by $\overline{\Delta v^2} = (\overline{v} - \bar{v})^2 = \overline{v^2} - \bar{v}^2$. The intensity of the radiation scattered from the total volume V is then given by

$$I_V = I_{\text{mol}} \overline{\Delta v^2} \frac{V}{V'}, \quad (8.2.15)$$

where the last factor V/V' gives the total number of regions of volume V' contained within the interaction volume V . This result shows how the total scattered intensity I_V depends on the fluctuations in the number density of molecules. We also see that the scattered intensity I_V vanishes if the fluctuation Δv^2 vanishes.

For the case of a medium sufficiently dilute that the locations of the individual molecules are uncorrelated (that is, for an ideal gas), we can readily calculate the mean fluctuation $\overline{\Delta v^2}$ in the number of particles. If N denotes the mean number density of particles, then the mean number of particles in the volume element V' is given by

$$\bar{v} = N V', \quad (8.2.16)$$

and the mean-square fluctuation is given by

$$\overline{\Delta v^2} = \overline{v^2} - \bar{v}^2 = \bar{v}, \quad (8.2.17)$$

where the last equality follows from the properties of the Poisson probability distribution, which are obeyed by uncorrelated particles. We hence find from Eqs. (8.2.15) through (8.2.17) that

$$I_V = \bar{v} \frac{V}{V'} I_{\text{mol}} = N V I_{\text{mol}} = \mathcal{N} I_{\text{mol}}. \quad (8.2.18)$$

Hence, for an ideal gas the total intensity is simply the intensity of the light scattered by a single molecule multiplied by the total number of molecules, $\mathcal{N} = NV$. Consequently the scattering coefficient R and differential cross section $d\sigma/d\Omega$ introduced in Section 8.1 are related by Eq. (8.1.9)—that is, by

$$R = N \frac{d\sigma}{d\Omega}. \quad (8.2.19)$$

By introducing Eq. (8.2.4) into this expression, we find that the scattering coefficient is given by

$$R = \frac{N}{16\pi^2} \frac{\omega^4}{c^4} |\alpha(\omega)|^2 \sin^2 \phi. \quad (8.2.20)$$

If the scattering medium is sufficiently dilute that its refractive index can be represented as

$$n = 1 + \frac{1}{2} N \alpha(\omega), \quad (8.2.21)$$

Eq. (8.2.20) can be rewritten as

$$R = \frac{\omega^4}{c^4} \frac{|n-1|^2}{4\pi^2 N} \sin^2 \phi. \quad (8.2.22)$$

This result can be used to determine the number density N of molecules in a gaseous sample in terms of two optical constants: the refractive index n and scattering coefficient R at a fixed angle ϕ . In fact, the first accurate measurement of Loschmidt's number (the number density of molecules at standard temperature and pressure, $N_0 = 2.686 \times 10^{19} \text{ cm}^{-3} = 2.686 \times 10^{25} \text{ m}^{-3}$) was performed through application of Eq. (8.2.22).

8.3 Thermodynamic Theory of Scalar Light Scattering

We next develop a macroscopic description of the light scattering process. We consider the case in which light scattering occurs as the result of fluctuations in the (scalar) dielectric constant and in which these fluctuations are themselves the result of fluctuations in thermodynamic variables, such as the material density and temperature. We assume, as in Fig. 8.2.3 in the preceding section, that the scattering volume V can be divided into a number of smaller volumes V' having the property that all atoms in V' radiate essentially in phase in the θ direction. We let $\Delta\epsilon$ denote the fluctuation of the dielectric constant averaged over the volume V' . Since $\epsilon = 1 + \chi$, the fluctuation in the susceptibility is then given by $\Delta\chi = \Delta\epsilon$. Because of this change in the susceptibility, the volume V' develops the additional polarization

$$\tilde{\mathbf{P}} = \Delta\chi \tilde{\mathbf{E}}_0 = \Delta\epsilon \tilde{\mathbf{E}}_0 \quad (8.3.1)$$

and hence the additional dipole moment

$$\tilde{\mathbf{p}} = \epsilon_0 V' \tilde{\mathbf{P}} = \epsilon_0 \Delta\epsilon V' \tilde{\mathbf{E}}_0. \quad (8.3.2)$$

The intensity $I_s = (nc\epsilon_0) \langle \tilde{\mathbf{E}}_s^2 \rangle$ of the radiation emitted by this oscillating dipole moment is obtained by introducing Eq. (8.3.2) into Eq. (8.2.3), to obtain

$$I_s = I_0 \frac{\omega^4 V'^2 \langle \Delta\epsilon^2 \rangle \sin^2 \phi}{16\pi^2 L^2 c^4}, \quad (8.3.3)$$

where, as before, ϕ is the angle between $\tilde{\mathbf{p}}$ and the direction to the point of observation, and where we have introduced the intensity $I_0 = (nc\epsilon_0) \langle \tilde{\mathbf{E}}_0^2 \rangle$ of the incident light. Eq. (8.3.3) gives the intensity of the light scattered from one cell. The total intensity from all the cells is V/V' times as large, since the fluctuations in the dielectric constant for different cells are uncorrelated.

We next calculate the mean-square fluctuation in the dielectric constant, $\langle \Delta\epsilon^2 \rangle$, for any one cell. We take the density ρ and temperature T as independent thermodynamic variables. We then express the change in the dielectric constant as

$$\Delta\epsilon = \left(\frac{\partial\epsilon}{\partial\rho} \right)_T \Delta\rho + \left(\frac{\partial\epsilon}{\partial T} \right)_\rho \Delta T. \quad (8.3.4)$$

To good accuracy (the error is estimated to be of the order of 2%; see Fabelinskii, 1968), we can usually ignore the second term, since the dielectric constant typically depends much more strongly on density than on temperature.* We thus find

$$\langle \Delta\epsilon^2 \rangle = \left(\frac{\partial\epsilon}{\partial\rho} \right)^2 \langle \Delta\rho^2 \rangle,$$

which can be expressed as

$$\langle \Delta\epsilon^2 \rangle = \gamma_e^2 \frac{\langle \Delta\rho^2 \rangle}{\rho_0^2}, \quad (8.3.5)$$

where ρ_0 denotes the mean density of the material and where we have introduced the electrostrictive constant γ_e , which is defined by[†]

$$\gamma_e = \left(\rho \frac{\partial\epsilon}{\partial\rho} \right)_{\rho=\rho_0}. \quad (8.3.6)$$

The quantity $\langle \Delta\rho^2 \rangle / \rho_0^2$ appearing in Eq. (8.3.5) can be calculated using the laws of statistical mechanics. The result (see, for example, Fabelinskii, 1968, Appendix I, Eq. (I.13); or Landau and Lifshitz, 1969) is

$$\frac{\langle \Delta\rho^2 \rangle}{\rho_0^2} = \frac{kT C_T}{V'} \quad (8.3.7)$$

where

$$C_T = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T \quad (8.3.8)$$

is the isothermal compressibility. Note that the result given by Eq. (8.3.7) (whose proof lies outside the subject area of this book) makes sense: Fluctuations are driven by thermal excitation; the larger the compressibility, the larger will be the resulting excursion; and the smaller the volume under consideration, the easier it is to change its mean density.

* For this reason, it is not crucial that we retain the subscript T on $\partial\epsilon/\partial\rho$.

† The reason why γ_e is called the electrostrictive constant will be described in Section 9.1.

By introducing Eqs. (8.3.5) and (8.3.7) into Eq. (8.3.3) and multiplying the result by the total number of cells, V/V' , we find that the total intensity of the scattered radiation is given by

$$I_s = I_0 \frac{\omega^4 V}{16\pi^2 L^2 c^4} \gamma_e^2 C_T kT \sin^2 \phi. \quad (8.3.9a)$$

We can use this result to find that the scattering coefficient R defined by Eq. (8.1.4) is given by

$$R = \frac{\omega^4}{16\pi^2 c^4} \gamma_e^2 C_T kT \sin^2 \phi. \quad (8.3.9b)$$

8.3.1 Ideal Gas

As an example, let us apply the result given by Eq. (8.3.9a) to light scattering from an ideal gas, for which the equation of state is of the form

$$pV = \mathcal{N}kT, \quad (8.3.10)$$

where \mathcal{N} denotes the total number of molecules in the gas. We then find that $(\partial V/\partial p)_T = -\mathcal{N}kT/p^2$ and thus that the isothermal compressibility is given by

$$C_T = \frac{\mathcal{N}kT}{Vp^2} = \frac{1}{p} = \frac{V}{\mathcal{N}kT}. \quad (8.3.11)$$

We next assume that $\epsilon - 1$ is linearly proportional to ρ , so that we can represent ϵ as $\epsilon = 1 + A\rho$ for some constant A . We hence find that $\partial\epsilon/\partial\rho = A$, or that $\partial\epsilon/\partial\rho = (\epsilon - 1)/\rho$, and that the electrostrictive constant is given by

$$\gamma_e = \epsilon - 1. \quad (8.3.12)$$

If we now introduce Eqs. (8.3.11) and (8.3.12) into Eq. (8.3.9a), we find that the scattered intensity can be expressed as

$$I_s = I_0 \frac{\omega^4 V}{16\pi^2 L^2 c^4} \frac{(\epsilon - 1)^2}{N} \sin^2 \phi, \quad (8.3.13)$$

where we have introduced the mean density of particles $N = \mathcal{N}/V$. Through use of Eq. (8.1.4), we can write this result in terms of the scattering coefficient as

$$R = \frac{(\epsilon - 1)^2 \omega^4 \sin^2 \phi}{16\pi^2 c^4 N}. \quad (8.3.14)$$

Note that, since $\epsilon - 1$ is equal to $2(n - 1)$ for a dilute gas (i.e., for $\epsilon - 1 \ll 1$), this result is in agreement with the prediction of the microscopic model of light scattering for an ideal gas, given by Eq. (8.2.22).

8.3.2 Spectrum of the Scattered Light

The analysis just presented has led to an explicit prediction (8.3.9a) for the *total* intensity of the light scattered as the result of the fluctuations in the density (and hence the dielectric constant) of a material system in thermal equilibrium. In order to determine the *spectrum* of the scattered light, we have to examine the dynamical behavior of the density fluctuations that give rise to light scattering. As before (see the discussion associated with Eq. (8.3.4)), we represent the fluctuation in the dielectric constant as

$$\Delta\tilde{\epsilon} = \left(\frac{\partial\epsilon}{\partial\rho} \right) \Delta\tilde{\rho}. \quad (8.3.15)$$

We now choose the entropy s and pressure p to be the independent thermodynamic variables. We can then represent the variation in density, $\Delta\tilde{\rho}$, as

$$\Delta\tilde{\rho} = \left(\frac{\partial\rho}{\partial p} \right)_s \Delta\tilde{p} + \left(\frac{\partial\rho}{\partial s} \right)_p \Delta\tilde{s}. \quad (8.3.16)$$

Here the first term describes adiabatic density fluctuations (that is, acoustic waves) and leads to Brillouin scattering. The second term describes isobaric density fluctuations (that is, entropy or temperature fluctuations) and leads to Rayleigh-center scattering. The two contributions to $\Delta\tilde{\rho}$ are quite different in character and lead to very different spectral distributions of the scattered light, because (as we shall see) the equations of motion for $\Delta\tilde{p}$ and $\Delta\tilde{s}$ are very different.

8.3.3 Brillouin Scattering

The equation of motion for a pressure wave is well known from the field of acoustics and is given by (see, e.g., Fabelinskii, 1968, Section 34.9)

$$\frac{\partial^2 \Delta\tilde{p}}{\partial t^2} - \Gamma' \nabla^2 \frac{\partial \Delta\tilde{p}}{\partial t} - v^2 \nabla^2 \Delta\tilde{p} = 0. \quad (8.3.17)$$

Here v denotes the velocity of sound, which is given in terms of thermodynamic variables by

$$v^2 = \left(\frac{\partial p}{\partial \rho} \right)_s. \quad (8.3.18)$$

The equation for the velocity of sound is conveniently expressed in terms of the compressibility C or in terms of its reciprocal, the bulk modulus K , which are defined by

$$C \equiv \frac{1}{K} = -\frac{1}{V} \frac{\partial V}{\partial p} = \frac{1}{\rho} \frac{\partial \rho}{\partial p}. \quad (8.3.19)$$

The compressibility can be measured either at constant temperature or at constant entropy. The two values of the compressibility, denoted respectively as C_T and C_s , are related by

$$\frac{C_T}{C_s} = \frac{c_p}{c_v} \equiv \gamma, \quad (8.3.20)$$

where c_p is the specific heat (i.e., the heat capacity per unit mass, whose units are J/kg K) at constant pressure, c_v is the specific heat at constant volume, and where their ratio γ is known as the adiabatic index. The velocity of sound as defined by Eq. (8.3.18) can thus be written as

$$v^2 = \frac{K_s}{\rho} = \frac{1}{C_s \rho}. \quad (8.3.21)$$

An important special case of the use of this formula is that of an ideal gas, for which the equation of state is given by Eq. (8.3.10) and the isothermal compressibility is given by Eq. (8.3.11). The adiabatic compressibility is thus given by $C_s = C_T/\gamma = 1/\gamma p$. We hence find from Eq. (8.3.21) that the velocity of sound is given by

$$v = \left(\frac{\gamma p}{\rho} \right)^{1/2} = \left(\frac{\gamma \mathcal{N} k T}{\rho V} \right)^{1/2} = \left(\frac{\gamma k T}{\mu} \right)^{1/2}, \quad (8.3.22)$$

where μ denotes the molecular mass. We thus see that the velocity of sound is of the order of the mean thermal velocity of the molecules of the gas. The velocity of sound for some common optical materials is listed in Table 8.3.1.

TABLE 8.3.1: Typical sound velocities.

Material	v (m/sec)
<i>Gases</i>	
Dry air	3.31×10^2
He	9.65×10^2
H ₂	12.84×10^2
Water vapor	4.94×10^2
<i>Liquids</i>	
CS ₂	1.15×10^3
CCl ₄	0.93×10^3
Ethanol	1.21×10^3
Water	1.50×10^3
<i>Solids</i>	
Fused silica	5.97×10^3
Lucite	2.68×10^3

The parameter Γ' appearing in the wave equation (8.3.17) is a damping parameter that can be shown to be expressible as

$$\Gamma' = \frac{1}{\rho} \left[\frac{4}{3} \eta_s + \eta_b + \frac{\kappa}{C_p} (\gamma - 1) \right], \quad (8.3.23)$$

where η_s is the shear viscosity coefficient, η_b is the bulk viscosity coefficient, and κ is the thermal conductivity. For most materials of interest in optics, the last contribution to Γ' is much smaller than the first two. Conventions involving the naming of the viscosity coefficients are discussed briefly in the Appendix to Section 9.6.

As an illustration of the nature of the acoustic wave equation (8.3.17), we consider the propagation of the wave

$$\Delta \tilde{p} = \Delta p e^{i(qz - \Omega t)} + \text{c.c.} \quad (8.3.24)$$

through an acoustic medium. By substituting this form into the acoustic wave equation (Eq. (8.3.17)), we find that q and Ω must be related by a dispersion relation of the form

$$\Omega^2 = q^2 (v^2 - i \Omega \Gamma'). \quad (8.3.25)$$

We rewrite this relation as

$$q^2 = \frac{\Omega^2}{v^2 - i \Omega \Gamma'} = \frac{\Omega^2/v^2}{1 - i \Omega \Gamma'/v^2} \simeq \frac{\Omega^2}{v^2} \left(1 + \frac{i \Omega \Gamma'}{v^2} \right), \quad (8.3.26)$$

which shows that

$$q \simeq \frac{\Omega}{v} + \frac{i \Gamma}{2v}, \quad (8.3.27)$$

where we have introduced the phonon decay rate

$$\Gamma = \Gamma' q^2. \quad (8.3.28)$$

We find by introducing the form for q given by Eq. (8.3.27) into Eq. (8.3.24) that the intensity of the acoustic wave varies spatially as

$$|\Delta p(z)|^2 = |\Delta p(0)|^2 e^{-\alpha_s z}, \quad (8.3.29)$$

where we have introduced the sound absorption coefficient

$$\alpha_s = \frac{q^2 \Gamma'}{v} = \frac{\Gamma}{v}. \quad (8.3.30)$$

It is also useful to define the phonon lifetime as

$$\tau_p = \frac{1}{\Gamma} = \frac{1}{q^2 \Gamma'}. \quad (8.3.31)$$

Next, we calculate the rate at which light is scattered out of a beam of light by these acoustic waves. We assume that the incident optical field is described by

$$\tilde{E}_0(z, t) = E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \text{c.c.} \quad (8.3.32)$$

and that the scattered field obeys the driven wave equation

$$\nabla^2 \tilde{E} - \frac{n^2}{c^2} \frac{\partial^2 \tilde{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2} \frac{\partial^2 \tilde{P}}{\partial t^2}. \quad (8.3.33)$$

We take the polarization \tilde{P} of the medium to be given by Eq. (8.3.1) with the variation $\Delta \tilde{\epsilon}$ in dielectric constant given by Eq. (8.3.15)—that is, we take $\tilde{P} = \epsilon_0 (\partial \epsilon / \partial \rho) \Delta \tilde{\rho} \tilde{E}_0$. We take the variation in density to be given by the first contribution to Eq. (8.3.16)—that is, by $\Delta \tilde{\rho} = (\partial \rho / \partial p) \Delta \tilde{p}$, where $\Delta \tilde{p}$ denotes the incremental pressure. We thus find that

$$\begin{aligned} \tilde{P}(\mathbf{r}, t) &= \epsilon_0 \left(\frac{\partial \epsilon}{\partial \rho} \right) \left(\frac{\partial \rho}{\partial p} \right)_s \Delta \tilde{p}(\mathbf{r}, t) \tilde{E}_0(z, t) \\ &= \epsilon_0 \gamma_e C_s \Delta \tilde{p}(\mathbf{r}, t) \tilde{E}_0(z, t), \end{aligned} \quad (8.3.34)$$

where we have introduced the adiabatic compressibility C_s of Eq. (8.3.19) and the electrostrictive constant of Eq. (8.3.6). We take a typical component of the thermally excited pressure disturbance within the interaction region to be given by

$$\Delta \tilde{p}(\mathbf{r}, t) = \Delta p e^{i(\mathbf{q} \cdot \mathbf{r} - \Omega t)} + \text{c.c.} \quad (8.3.35)$$

By combining Eqs. (8.3.33) through (8.3.35), we find that the scattered field must obey the wave equation

$$\begin{aligned} \nabla^2 \tilde{\mathbf{E}} - \frac{n^2}{c^2} \frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2} &= -\frac{\gamma_e C_s}{c^2} \left[(\omega - \Omega)^2 E_0 \Delta p^* e^{i(\mathbf{k} - \mathbf{q}) \cdot \mathbf{r} - i(\omega - \Omega)t} \right. \\ &\quad \left. + (\omega + \Omega)^2 E_0 \Delta p e^{i(\mathbf{k} + \mathbf{q}) \cdot \mathbf{r} - i(\omega + \Omega)t} + \text{c.c.} \right]. \end{aligned} \quad (8.3.36)$$

The first term in this expression leads to Stokes scattering; the second to anti-Stokes scattering. We study these two contributions in turn.

8.3.4 Stokes Scattering (First Term in Eq. (8.3.36))

The polarization is seen to have a component with wavevector

$$\mathbf{k}' \equiv \mathbf{k} - \mathbf{q} \quad (8.3.37)$$

and frequency

$$\omega' \equiv \omega - \Omega, \quad (8.3.38)$$

where the frequency ω and wavevector \mathbf{k} of the incident optical field are related according to

$$\omega = |\mathbf{k}|c/n, \quad (8.3.39)$$

and where the frequency Ω and wavevector \mathbf{q} of the acoustic wave are related according to

$$\Omega = |\mathbf{q}|v. \quad (8.3.40)$$

This component of the polarization can couple efficiently to the scattered optical wave only if its frequency ω' and wavevector \mathbf{k}' are related by the dispersion relation for optical waves, namely

$$\omega' = |\mathbf{k}'|c/n. \quad (8.3.41)$$

In order for Eqs. (8.3.37) through (8.3.41) to be satisfied simultaneously, the sound-wave frequency and wavevector must each have a particular value for any scattering direction. For the case of scattering at the angle θ , we have the situation illustrated in Fig. 8.3.1. Part (a) of this figure shows the relative orientations of the wavevectors of the incident and scattered fields. Part (b) illustrates Eq. (8.3.37) and shows how the wavevector of the acoustic disturbance is related to those of the incident and scattered optical radiation.

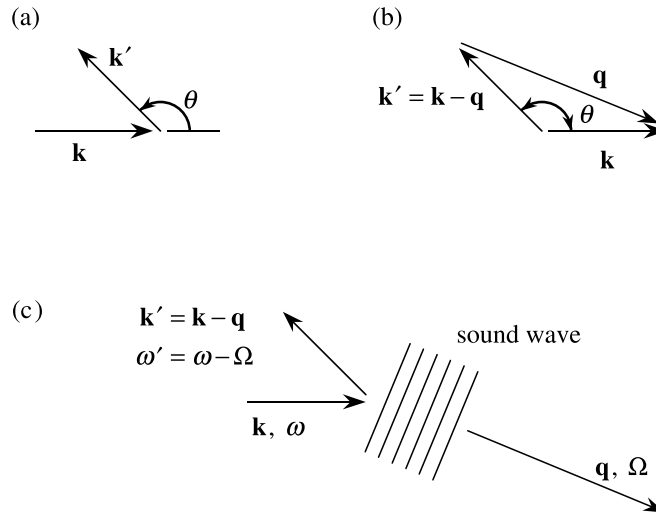


FIGURE 8.3.1: Illustration of Stokes Brillouin scattering.

Since $|\mathbf{k}|$ is very nearly equal to $|\mathbf{k}'|$ (because Ω is much smaller than ω), diagram (b) shows that

$$|\mathbf{q}| = 2|\mathbf{k}| \sin(\theta/2). \quad (8.3.42)$$

The dispersion relation (8.3.40) then shows that the acoustic frequency is given by

$$\Omega = 2|\mathbf{k}|v \sin(\theta/2) = 2n\omega \frac{v}{c} \sin(\theta/2). \quad (8.3.43)$$

We note that the Stokes shift Ω is equal to zero for forward scattering and is maximum for backscattering (i.e., for $\theta = 180^\circ$). The maximum frequency shift is thus given by

$$\Omega_{\max} = 2n \frac{v}{c} \omega. \quad (8.3.44)$$

For $\omega/2\pi = 3 \times 10^{14}$ Hz (i.e., at $\lambda = 1 \mu\text{m}$), $v = 1 \times 10^3$ m/sec (a typical value), and $n = 1.5$, we obtain $\Omega_{\max}/2\pi = 3 \times 10^9$ Hz.

Stokes scattering can be visualized as the scattering of light from a retreating acoustic wave, as illustrated in part (c) of Fig. 8.3.1.

8.3.5 Anti-Stokes Scattering (Second Term in Eq. (8.3.36))

The analysis here is analogous to that for Stokes scattering. The polarization is seen to have a component with wavevector

$$\mathbf{k}' \equiv \mathbf{k} + \mathbf{q} \quad (8.3.45)$$

and frequency

$$\omega' \equiv \omega + \Omega, \quad (8.3.46)$$

where, as before, $\omega = |\mathbf{k}|c/n$ and $\Omega = |\mathbf{q}|v$. This component of the polarization can couple efficiently to an electromagnetic wave only if ω' and $|\mathbf{k}'|$ are related by $\omega' = |\mathbf{k}'|c/n$. We again assume that θ denotes the scattering angle, as illustrated in Fig. 8.3.2. The condition (8.3.45) is illustrated as part (b) of the figure. Since (as before) $|\mathbf{k}|$ is very nearly equal to $|\mathbf{k}'|$, the length of the acoustic wavevector is given by

$$|\mathbf{q}| = 2|\mathbf{k}| \sin(\theta/2). \quad (8.3.47)$$

Hence, by Eq. (8.3.40), the acoustic frequency is given by

$$\Omega = 2n\omega \frac{v}{c} \sin(\theta/2). \quad (8.3.48)$$

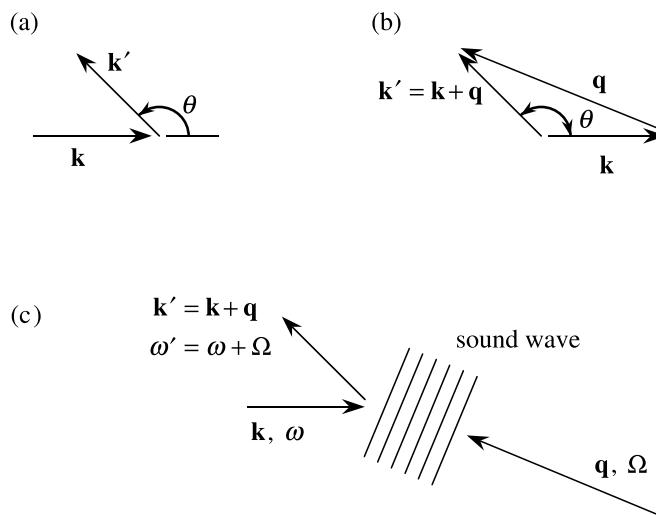


FIGURE 8.3.2: Illustration of anti-Stokes Brillouin scattering.

Anti-Stokes scattering can be visualized as scattering from an oncoming sound wave, as shown in part (c) of Fig. 8.3.2.

We have thus far ignored attenuation of the acoustic wave in our analysis. If we include this effect, we find that the light scattered into direction θ is not monochromatic but has a spread in angular frequency whose width (FWHM) is given by

$$\delta\omega = 1/\tau_p = \Gamma' q^2, \quad (8.3.49)$$

which becomes, through use of Eq. (8.3.42),

$$\delta\omega = 4\Gamma' |\mathbf{k}|^2 \sin^2(\theta/2) = 4n^2 \Gamma' \frac{\omega^2}{c^2} \sin^2(\theta/2). \quad (8.3.50)$$

For the case of backscattering ($\theta = 180^\circ$), $\delta\omega/2\pi$ is typically of the order of 100 MHz for organic liquids. Since the acoustic frequency is given by Eq. (8.3.43), we see that the ratio of the linewidth to the Brillouin frequency shift is given by

$$\frac{\delta\omega}{\Omega} = \frac{2n\Gamma'\omega}{vc} \sin(\theta/2). \quad (8.3.51)$$

We see that the linewidth normalized in this manner is largest for back scattering ($\theta = 180^\circ$) and becomes vanishingly small for forward scattering ($\theta = 0$). The spectrum of the scattered light has the form shown in Fig. 8.3.3.

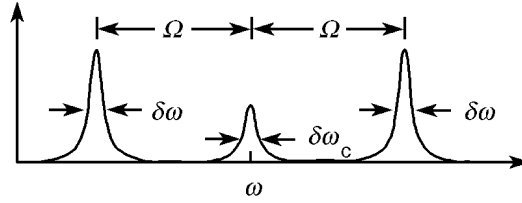


FIGURE 8.3.3: Spectrum showing Brillouin and Rayleigh scattering.

8.3.6 Rayleigh Center Scattering

We now consider the contribution to $\Delta\tilde{\rho}$ (and hence to $\Delta\tilde{\epsilon}$) resulting from isobaric density fluctuations, which are described by the second term in Eq. (8.3.16) and are proportional to the entropy fluctuation $\Delta\tilde{s}$. Entropy fluctuations are described by the same equation as that describing temperature variations:

$$\rho c_p \frac{\partial \Delta\tilde{s}}{\partial t} - \kappa \nabla^2 \Delta\tilde{s} = 0, \quad (8.3.52)$$

where, as before, c_p denotes the specific heat at constant pressure, and where κ denotes the thermal conductivity. Note that these fluctuations obey a diffusion equation and not a wave equation. A solution to the diffusion equation (8.3.52) is

$$\Delta\tilde{s} = \Delta s_0 e^{-\delta t} e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad (8.3.53)$$

where the damping rate of the entropy disturbance is given by

$$\delta = \frac{\kappa}{\rho c_p} q^2. \quad (8.3.54)$$

We see that, unlike pressure waves, entropy waves do not propagate. As a result, the nonlinear polarization proportional to Δs can give rise only to an unshifted component of the scattered light. The width (FWHM) of this component is given by $\delta\omega_c = \delta$, that is, by

$$\delta\omega_c = \frac{4\kappa}{\rho c_p} |\mathbf{k}|^2 \sin^2(\theta/2). \quad (8.3.55)$$

As a representative case, for liquid water $\kappa = 6$ mW/cm K, $\rho = 1$ g/cm³, $c_p = 4.2$ J/g K, and the predicted width of the central component for backscattering ($\theta = 180^\circ$) of radiation at 500 nm is $\delta\omega_c/2\pi = 1.4 \times 10^7$ Hz.

It can be shown (Fabelinskii, 1968, Eq. 5.39) that the relative intensities of the Brillouin and Rayleigh center components are given by

$$\frac{I_c}{2I_B} = \frac{c_p - c_v}{c_v} = \gamma - 1. \quad (8.3.56)$$

Here I_c denotes the integrated intensity of the central component, and I_B that of either of the Brillouin components. This result is known as the Landau–Placzek relation.

8.4 Acoustooptics

The analysis just presented of the scattering of light from a sound wave can be applied to the situation in which the sound wave is applied to the interaction region externally by means of a transducer. Such acoustooptic devices are useful as intensity or frequency modulators for laser beams or as beam deflectors.

Acoustooptic devices are commonly classified as falling into one of two regimes, each of which will be discussed in greater detail below. These regimes are as follows:

Bragg scattering. This type of scattering occurs for the case of interaction lengths that are sufficiently long that phase-matching considerations become important. Bragg scattering leads to a single diffracted beam. The name is given by analogy to the scattering of X-rays from the atomic planes in a crystal. Bragg scattering can lead to an appreciable scattering efficiency ($>50\%$).

Raman–Nath scattering. This type of scattering occurs in cells with a short interaction length. Phase-matching considerations are not important, and several scattered orders are usually present.

We shall first consider the case of Bragg scattering of light waves; a more precise statement of the conditions under which each type of scattering occurs is given below in connection with the discussion of Raman–Nath scattering.

8.4.1 Bragg Scattering of Light by Sound Waves

The operation of a typical Bragg scattering cell is shown schematically in Fig. 8.4.1. A traveling acoustic wave of frequency Ω and wavelength $\Lambda = 2\pi v/\Omega$ (where v denotes the velocity of sound) is established in the scattering medium. The density variation associated with this acoustic wave produces a variation in the dielectric constant of the medium, and the incident optical wave scatters from this variation. Although the amplitude of the wave scattered from each acoustic wavefront is typically rather small, the total scattered field can become quite intense if the various contributions add in phase to produce constructive interference. The condition for constructive interference to occur is obtained with the help of the construction shown in Fig. 8.4.2 and is given by the relation

$$\lambda = 2\Lambda \sin \theta, \quad (8.4.1)$$

where λ is the wavelength of light in the medium. This condition is known as the Bragg condition. It ensures that the path length difference between rays that reflect from successive acoustic

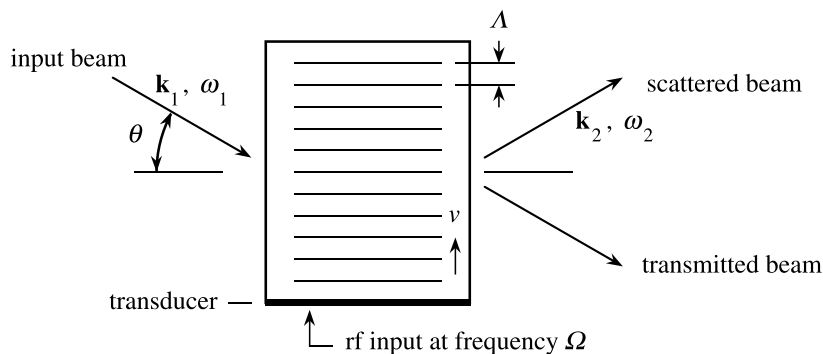


FIGURE 8.4.1: Bragg-type acoustooptic modulator.

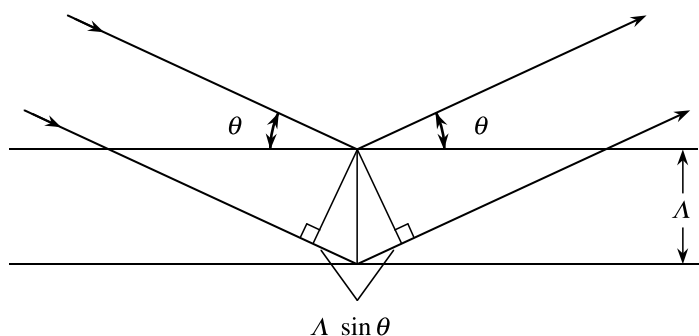


FIGURE 8.4.2: The Bragg condition for acoustooptic scattering.

maxima is equal to an optical wavelength. In a typical acoustooptic device, relevant parameters might be $v = 1.5 \times 10^5$ cm/sec and $\Omega/2\pi = 200$ MHz, which imply that the acoustic wavelength is equal to $\Lambda = 2\pi v/\Omega = 7.5$ μm . If the optical wavelength is 0.5 μm , we see from Eq. (8.4.1) that $\sin \theta = 1/30$ and thus the deflection angle is given by $2\theta = 4^\circ$.

The Bragg condition given by Eq. (8.4.1) can alternatively be understood as a phase-matching condition. If \mathbf{k}_1 denotes the wavevector of the incident optical wave, \mathbf{k}_2 that of the diffracted optical wave, and \mathbf{q} that of the acoustic wave, the Bragg condition can be seen with the help of Fig. 8.4.3(a) to be a statement that

$$\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{q}. \quad (8.4.2)$$

By comparison with the analysis of Section 8.3 for spontaneous Brillouin scattering (and as shown explicitly below), we see that the frequency of the scattered beam is shifted upward to

$$\omega_2 = \omega_1 + \Omega. \quad (8.4.3)$$

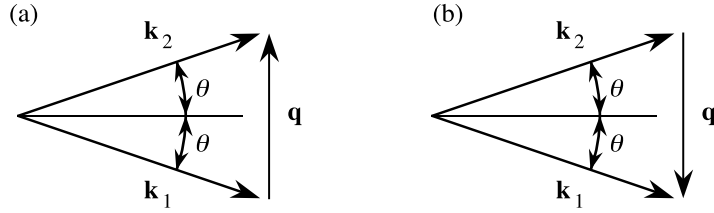


FIGURE 8.4.3: The Bragg condition described as a phase-matching relation.

Since Ω is much smaller than ω_1 , we see that ω_2 is approximately equal to ω_1 , and hence that $|\mathbf{k}_2| \simeq |\mathbf{k}_1|$. The configuration shown in Fig. 8.4.1 shows the case in which the acoustic wave is advancing toward the incident optical wave. For the case of a sound wave propagating in the opposite direction, Eqs. (8.4.2) and (8.4.3) must be replaced by

$$\mathbf{k}_2 = \mathbf{k}_1 - \mathbf{q}, \quad (8.4.4a)$$

$$\omega_2 = \omega_1 - \Omega. \quad (8.4.4b)$$

Figs. 8.4.1 and 8.4.2 are unchanged except for the reversal of the direction of the sound velocity vector, although Fig. 8.4.3(a) must be replaced by Fig. 8.4.3(b).

Bragg scattering of light by sound waves can be treated theoretically by considering the time-varying change $\Delta\tilde{\epsilon}$ in the dielectric constant induced by the acoustic density variation $\Delta\tilde{\rho}$. It is usually adequate to assume that $\Delta\tilde{\epsilon}$ scales linearly with $\Delta\tilde{\rho}$, so that

$$\Delta\tilde{\epsilon} = \frac{\partial\epsilon}{\partial\rho}\Delta\tilde{\rho} = \gamma_e \frac{\Delta\tilde{\rho}}{\rho_0}. \quad (8.4.5)$$

Here ρ_0 denotes the mean density of the material, and γ_e denotes the electrostrictive constant defined by Eq. (8.3.6). Eq. (8.4.5) applies rigorously to the case of liquids, and it predicts the correct qualitative behavior for other materials. For the case of anisotropic materials, the change in the optical properties is described more precisely by means of a tensor relation, which conventionally is given by

$$[\Delta(\epsilon^{-1})]_{ij} = \sum_{kl} p_{ijkl} S_{kl}, \quad (8.4.6)$$

where the quantity p_{ijkl} is known as the strain-optic tensor and where

$$S_{kl} = \frac{1}{2} \left(\frac{\partial d_k}{\partial x_l} + \frac{\partial d_l}{\partial x_k} \right) \quad (8.4.7)$$

is the strain tensor, in which d_k is the k component of the displacement of a particle from its equilibrium position. Whenever the change in the inverse of the dielectric tensor $(\epsilon^{-1})_{ij}$ given

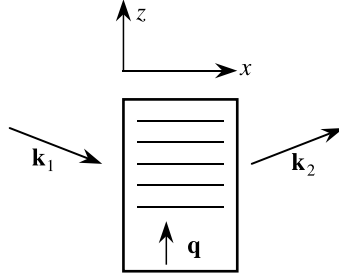


FIGURE 8.4.4: Geometry of a Bragg-type acoustooptic modulator.

by the right-hand side of Eq. (8.4.6) is small, the change in the dielectric tensor ϵ_{ij} is given by

$$(\Delta\epsilon)_{il} = - \sum_{jk} \epsilon_{ij} [\Delta(\epsilon^{-1})]_{jk} \epsilon_{kl}. \quad (8.4.8)$$

Our theoretical treatment of Bragg scattering assumes the geometry shown in Fig. 8.4.4. The interaction of the incident field

$$\tilde{E}_1 = A_1 e^{i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)} + \text{c.c.} \quad (8.4.9)$$

with the acoustic wave of wavevector \mathbf{q} produces the diffracted wave

$$\tilde{E}_2 = A_2 e^{i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)} + \text{c.c.} \quad (8.4.10)$$

with $\omega_2 = \omega_1 + \Omega$. The interaction is assumed to be nearly Bragg-matched (i.e., phase-matched) in the sense that

$$\mathbf{k}_2 \simeq \mathbf{k}_1 + \mathbf{q}. \quad (8.4.11)$$

The variation of the dielectric constant induced by the acoustic wave is represented as

$$\Delta\tilde{\epsilon} = \Delta\epsilon e^{i(\mathbf{q} \cdot \mathbf{r} - \Omega t)} + \text{c.c.}, \quad (8.4.12)$$

where the complex amplitude $\Delta\epsilon$ is given by $\Delta\epsilon = \gamma_e \Delta\rho / \rho_0$ under those conditions where the change in dielectric constant is accurately predicted by Eq. (8.4.5). More generally, for anisotropic interactions, $\Delta\epsilon$ is the amplitude of the appropriate tensor component of $\Delta\tilde{\epsilon}_{ij}$ given by Eq. (8.4.6). The total optical field $\tilde{E} = \tilde{E}_1 + \tilde{E}_2$ is required to satisfy the wave equation

$$\nabla^2 \tilde{E} - \frac{n^2 + \Delta\tilde{\epsilon}}{c^2} \frac{\partial^2 \tilde{E}}{\partial t^2} = 0, \quad (8.4.13)$$

where n denotes the refractive index of the material in the absence of the acoustic wave. Since according to Eq. (8.4.12) $\Delta\tilde{\epsilon}$ oscillates at frequency Ω , it couples the optical waves of frequencies ω_1 and $\omega_2 = \omega_1 + \Omega$.

We first consider the portion of Eq. (8.4.13) that oscillates at frequency ω_1 . This part is given by

$$\begin{aligned} \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial z^2} + 2ik_{1x} \frac{\partial A_1}{\partial x} + 2ik_{1z} \frac{\partial A_1}{\partial z} - (k_{1x}^2 + k_{1z}^2) A_1 \\ + \frac{n^2 \omega_1^2}{c^2} A_1 + \frac{\omega_2^2}{c^2} A_2 \Delta \epsilon^* e^{i(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q}) \cdot \mathbf{r}} = 0. \end{aligned} \quad (8.4.14)$$

This equation can be simplified in the following manner: (1) We introduce the slowly varying amplitude approximation, which entails ignoring the second-order derivatives; (2) we note that A_1 depends only on x and not on z , since the interaction is invariant to a translation in the z direction, and so we set $\partial A_1 / \partial z$ equal to 0; and (3) we note that $k_{1x}^2 + k_{1z}^2 = n^2 \omega_1^2 / c^2$. Eq. (8.4.14) thus becomes

$$2ik_{1x} \frac{dA_1}{dx} = -\frac{\omega_2^2}{c^2} A_2 \Delta \epsilon^* e^{i(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q}) \cdot \mathbf{r}}. \quad (8.4.15)$$

Next, we note that the propagation vector mismatch $\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q} \equiv -\Delta \mathbf{k}$ can have a nonzero component only in the x direction, because the geometry we are considering has infinite extent in the z direction, and the z component of the \mathbf{k} wavevector mismatch must therefore vanish. We thus see that

$$(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{q}) \cdot \mathbf{r} \equiv -\Delta kx, \quad (8.4.16)$$

and hence that Eq. (8.4.15) can be written as

$$\frac{dA_1}{dx} = \frac{i\omega_2^2 \Delta \epsilon^*}{2k_{1x} c^2} A_2 e^{-i\Delta kx}. \quad (8.4.17)$$

By a completely analogous derivation, we find that the portion of the wave equation (8.4.13) that describes a wave at frequency ω_2 is given by

$$\frac{dA_2}{dx} = \frac{i\omega_1^2 \Delta \epsilon}{2k_{2x} c^2} A_1 e^{i\Delta kx}. \quad (8.4.18)$$

Finally, we note that since $\omega_1 \simeq \omega_2 \equiv \omega$ and $k_{1x} \simeq k_{2x} \equiv k_x$, the coupled equations (8.4.17) and (8.4.18) can be written as

$$\frac{dA_1}{dx} = i\kappa A_2 e^{-i\Delta kx}, \quad (8.4.19a)$$

$$\frac{dA_2}{dx} = i\kappa^* A_1 e^{i\Delta kx}, \quad (8.4.19b)$$

where we have introduced the coupling constant

$$\kappa = \frac{\omega^2 \Delta \epsilon^*}{2k_x c^2}. \quad (8.4.20)$$

The solution to these coupled-amplitude equations is particularly simple for the case in which \tilde{E}_1 is incident at the Bragg angle. In this case, the interaction is perfectly phase-matched, so that $\Delta k = 0$, and thus Eq. (8.4.19b) reduces to the set

$$\frac{dA_1}{dx} = i\kappa A_2, \quad \frac{dA_2}{dx} = i\kappa^* A_1. \quad (8.4.21)$$

These equations are easily solved using methods similar to those introduced in Chapter 2. The solution appropriate to the boundary conditions illustrated in Fig. 8.4.4 is

$$A_1(x) = A_1(0) \cos(|\kappa|x), \quad (8.4.22a)$$

$$A_2(x) = \frac{i\kappa^*}{|\kappa|} A_1(0) \sin(|\kappa|x). \quad (8.4.22b)$$

Note that these solutions obey the relation

$$|A_1(x)|^2 + |A_2(x)|^2 = |A_1(0)|^2, \quad (8.4.23)$$

which shows that the energy of the optical field is conserved in the Bragg scattering process (since we have assumed that $\Omega \ll \omega$). We define the diffraction efficiency of the Bragg scattering process to be the ratio of the output intensity of the ω_2 wave to the input intensity of the ω_1 wave, and we find that the diffraction efficiency is given by

$$\eta \equiv \frac{|A_2(L)|^2}{|A_1(0)|^2} = \sin^2(|\kappa|L). \quad (8.4.24)$$

For practical purposes, it is useful to express the coupling constant κ defined by Eq. (8.4.20) in terms of the intensity (i.e., power per unit area) of the acoustic wave. The intensity of a sound wave is given by the relation

$$I = Kv \frac{\langle \Delta \tilde{\rho}^2 \rangle}{\rho_0^2} = 2Kv \left| \frac{\Delta \rho}{\rho_0} \right|^2, \quad (8.4.25)$$

where, as before, $K = 1/C$ is the bulk modulus, v is the sound velocity, and $\Delta \rho$ is the complex amplitude of the density disturbance associated with the acoustic wave. It follows from Eq. (8.4.5) that $\Delta \epsilon$ is equal to $\gamma_e \Delta \rho / \rho$, and thus the acoustic intensity can be written as $I = 2Kv |\Delta \epsilon|^2 / \gamma_e^2$. The coupling constant $|\kappa|$ (see Eq. (8.4.20)) can thus be expressed as

$$|\kappa| = \frac{\omega \gamma_e}{2nc \cos \theta} \left(\frac{I}{2Kv} \right)^{1/2}, \quad (8.4.26)$$

where we have replaced k_x by $n(\omega/c) \cos \theta$.

As an example, we evaluate Eq. (8.4.26) for the case of Bragg scattering in water, which is characterized by the following physical constants: $n = 1.33$, $\gamma_e = 0.82$, $v = 1.5 \times 10^3$ m/sec, and $K = 2.19 \times 10^{11}$ N m⁻². We assume that $\cos \theta \simeq 1$, as is usually the case; that the vacuum optical wavelength is $0.5 \mu\text{m}$, so that $\omega = 3.8 \times 10^{15}$ rad/sec; and that the acoustic intensity is 1.0 W/cm^2 (as might be obtained using 1 W of acoustic power and an acoustic beam diameter of approximately 1 cm). Under these conditions, Eq. (8.4.26) gives the value $|\kappa| = 1.5 \text{ cm}^{-1}$. According to Eq. (8.4.24), 100% conversion of the incident beam into the diffracted beam is predicted for $|\kappa|L = \pi/2$, or under the present conditions for a path length through the acoustic beam of $L = 1.0 \text{ cm}$.

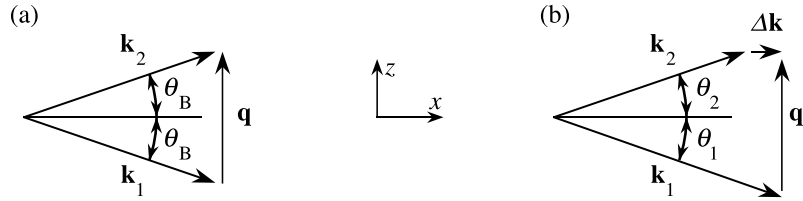


FIGURE 8.4.5: Wavevector diagrams for (a) incidence at the Bragg angle, so that $\Delta k = 0$, and (b) non-Bragg-angle incidence, so that $\Delta k \neq 0$.

For the case in which the incident beam does not intercept the acoustic wavefronts at the Bragg angle, the theoretical analysis is more complicated because the wavevector mismatch Δk does not vanish. The phase-matching diagrams for the cases of Bragg-angle and non-Bragg-angle incidence are contrasted in Fig. 8.4.5. As discussed in connection with Eq. (8.4.16), the wavevector mismatch can have a component only in the x direction, since the medium is assumed to have infinite extent in the z direction.

We first determine the relationship between the wavevector mismatch Δk and the angle of incidence θ_1 . We note that the x and z components of the vectors of diagram (b) obey the relations

$$k \cos \theta_1 - k \cos \theta_2 = \Delta k, \quad (8.4.27a)$$

$$k \sin \theta_1 + k \sin \theta_2 = q, \quad (8.4.27b)$$

where we have let $k_1 \simeq k_2 = k$. We note that if the angle of incidence θ_1 is equal to the Bragg angle

$$\theta_B = \sin^{-1} \frac{q}{2k} = \sin^{-1} \frac{\lambda}{2\Lambda}, \quad (8.4.28)$$

then Eqs. (8.4.27b) imply that the diffraction angle θ_2 is also equal to θ_B and that $\Delta k = 0$. For the case in which the light is not incident at the Bragg angle, we set

$$\theta_1 = \theta_B + \Delta\theta, \quad (8.4.29a)$$

where we assume that $\Delta\theta \ll 1$. We note that Eq. (8.4.27b) will be satisfied so long as

$$\theta_2 = \theta_B - \Delta\theta. \quad (8.4.29b)$$

These values of θ_1 and θ_2 are now introduced into Eq. (8.4.27a). The cosine functions are expanded to lowest order in $\Delta\theta$ as

$$\cos(\theta_B \pm \Delta\theta) = \cos\theta_B \mp (\sin\theta_B)\Delta\theta,$$

and we obtain $(2k \sin\theta_B)\Delta\theta = \Delta k$, which through use of Eq. (8.4.28) shows that the wavevector mismatch Δk that occurs as the result of an angular misalignment $\Delta\theta$ is given by

$$\Delta k = -\Delta\theta q. \quad (8.4.30)$$

We next solve Eqs. (8.4.19b) for arbitrary values of Δk . The solution for the case in which no field at frequency ω_2 is applied externally is

$$A_1(x) = e^{-i(1/2)\Delta k x} A_1(0) \left(\cos sx + i \frac{\Delta k}{2s} \sin sx \right), \quad (8.4.31a)$$

$$A_2(x) = i e^{i(1/2)\Delta k x} A_1(0) \frac{\kappa^*}{s} \sin sx, \quad (8.4.31b)$$

where

$$s^2 = |\kappa|^2 + \left(\frac{1}{2}\Delta k\right)^2. \quad (8.4.32)$$

The diffraction efficiency for arbitrary Δk is now given by

$$\eta(\Delta k) \equiv \frac{|A_2(L)|^2}{|A_1(0)|^2} = \frac{|\kappa|^2}{|\kappa|^2 + (\frac{1}{2}\Delta k)^2} \sin^2 \left\{ \left[|\kappa|^2 + \left(\frac{1}{2}\Delta k\right)^2 \right]^{1/2} L \right\}. \quad (8.4.33)$$

We see that for $\Delta k \neq 0$ the maximum efficiency is always less than 100%. Let us examine the rate at which the efficiency decreases as the phase mismatch Δk is increased. We expand $\eta(\Delta k)$ as a power series in Δk as

$$\eta(\Delta k) = \eta(0) + \Delta k \left. \frac{d\eta}{d(\Delta k)} \right|_{\Delta k=0} + \frac{1}{2} (\Delta k)^2 \left. \frac{d^2\eta}{d(\Delta k)^2} \right|_{\Delta k=0} + \dots \quad (8.4.34)$$

By calculating these derivatives, we find that, correct to second order in Δk , the efficiency is given by

$$\eta(\Delta k) = \eta(0) \left[1 - \frac{(\Delta k)^2}{4|\kappa|^2} \left(1 - \frac{|\kappa|K \cos(|\kappa|L)}{\sin(|\kappa|L)} \right) \right], \quad (8.4.35a)$$

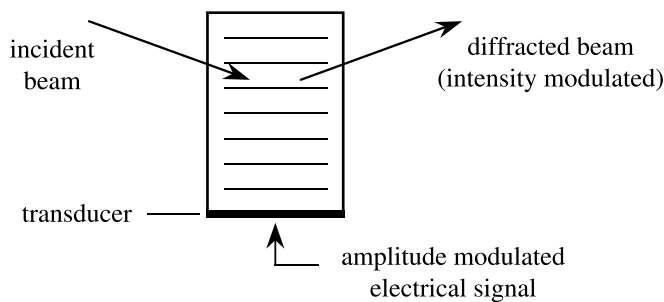
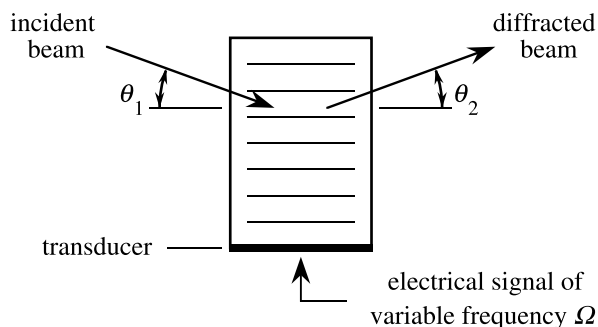


FIGURE 8.4.6: Acoustooptic amplitude modulator.


 FIGURE 8.4.7: Acoustooptic beam deflector. The angle θ_2 depends on the frequency of Ω of the electrical signal.

where

$$\eta(0) = \sin^2(|\kappa|L). \quad (8.4.35b)$$

One common use of the Bragg acoustooptic effect is to produce an amplitude-modulated laser beam, as illustrated in Fig. 8.4.6. In such a device, the frequency of the electrical signal that is fed to the acoustic transducer is held fixed, but the amplitude of this wave is modulated. As a result, the depth of modulation of the acoustic grating is varied, leading to a modulation of the intensity of the scattered wave.

Another application of Bragg acoustooptic scattering is to produce a beam deflector (Fig. 8.4.7). In such a device, the frequency Ω of the electrical signal that is fed to the acoustic transducer is allowed to vary. As a result, the acoustic wavelength Λ varies, and thus the diffraction angle θ_2 given by Eq. (8.4.29b) can be controlled. It should be noted that the diffraction efficiency given by Eq. (8.4.33) decreases for diffraction at angles different from the Bragg angle, and this effect places limitations on the range of deflection angles that are achievable by means of this technique.

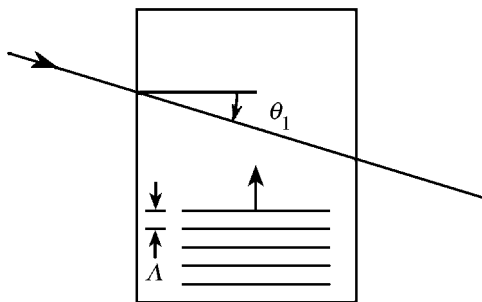


FIGURE 8.4.8: Illustration of the condition under which Bragg scattering occurs.

8.4.2 Raman–Nath Effect

The description of Bragg scattering given in the preceding subsection implicitly assumed that the width L of the interaction region was sufficiently large that an incident ray of light would interact with a large number of acoustic wavefronts. As illustrated in Fig. 8.4.8, this condition requires that

$$L \tan \theta_1 \gg \Lambda, \quad (8.4.36)$$

where Λ is the acoustic wavelength. However, the angle of incidence θ_1 must satisfy the Bragg condition

$$\sin \theta_1 = \frac{\lambda}{2\Lambda} \quad (8.4.37)$$

if efficient scattering is to occur. In most cases of interest, θ_1 is much smaller than unity, and hence $\tan \theta_1 \simeq \theta_1$. Eq. (8.4.37) can then be used to eliminate θ_1 from Eq. (8.4.36), which becomes

$$\frac{\lambda L}{\Lambda^2} \gg 1. \quad (8.4.38)$$

If this condition is satisfied, Bragg scattering can occur. Scattering in the opposite limit is known as Raman–Nath scattering.

Raman–Nath scattering can be understood in terms of the diagram shown in Fig. 8.4.9. A beam of light falls onto the scattering cell, typically at near-normal incidence. Because of the presence of the acoustic wave, whose wavelength is denoted Λ , the refractive index of the medium varies spatially with period Λ . The incident light diffracts off this index grating; the characteristic angular spread of the diffracted light is

$$\delta\theta = \frac{\lambda}{\Lambda}. \quad (8.4.39)$$

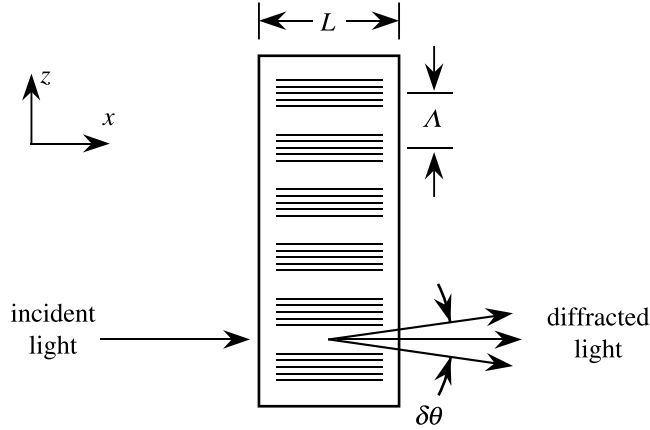


FIGURE 8.4.9: Raman–Nath diffraction.

We now assume that the cell is sufficiently thin that multiple scattering *cannot* occur. This condition can be stated as

$$\delta\theta L < \Lambda. \quad (8.4.40)$$

If $\delta\theta$ is eliminated from this inequality through use of Eq. (8.4.39), we find that

$$\frac{\lambda L}{\Lambda^2} < 1. \quad (8.4.41)$$

We note that this condition is the opposite of the inequality (8.4.38) for the occurrence of Bragg scattering.

We now present a mathematical analysis of Raman–Nath scattering. We assume that the acoustic wave within the scattering cell can be represented as the density variation

$$\Lambda \tilde{\rho} = \Delta \rho e^{i(qz - \Omega t)} + \text{c.c.} \quad (8.4.42)$$

A refractive index variation

$$\Lambda \tilde{n} = \Delta n e^{i(qz - \Omega t)} + \text{c.c.} \quad (8.4.43)$$

is associated with this acoustic wave. We relate the complex amplitude Δn of the refractive index disturbance to the amplitude $\Delta \rho$ of the acoustic wave as follows: We let $\tilde{n} = n_0 + \Delta \tilde{n}$, where $\tilde{n} = \tilde{\epsilon}^{1/2}$ with $\tilde{\epsilon} = \epsilon_b + \Delta \tilde{\epsilon}$. We thus find that $n_0 = \epsilon_b^{1/2}$ and that $\Delta \tilde{n} = \Delta \tilde{\epsilon} / 2n_0$. We now represent $\Delta \tilde{\epsilon}$ as $\Delta \tilde{\epsilon} = (\partial \epsilon / \partial \rho) \Delta \tilde{\rho} = \gamma_e \Delta \tilde{\rho} / \rho_0$ and find that $\Delta \tilde{n} = \gamma_e \Delta \tilde{\rho} / 2n_0 \rho_0$, and thus that

$$\Delta n = \frac{\gamma_e \Delta \rho}{2n_0 \rho_0}. \quad (8.4.44)$$

The ensuing analysis is simplified by representing $\Delta\tilde{n}$ using real quantities; we assume that the phase conventions are chosen such that

$$\Delta\tilde{n}(z, t) = 2\Delta n \sin(qz - \Omega t). \quad (8.4.45)$$

The electric field of the incident optical wave is represented as

$$\tilde{E}(\mathbf{r}, t) = A e^{i(kx - \omega t)} + \text{c.c.} \quad (8.4.46)$$

After passing through the acoustic wave, the optical field will have experienced a phase shift

$$\phi = \Delta\tilde{n} \frac{\omega}{c} L = 2\Delta n \frac{\omega}{c} L \sin(qz - \Omega t) \equiv \delta \sin(qz - \Omega t), \quad (8.4.47)$$

where the quantity

$$\delta = 2\Delta n \omega L / c \quad (8.4.48)$$

is known as the modulation index. The transmitted field can hence be represented as $\tilde{E}(\mathbf{r}, t) = A \exp[i(kx - \omega t + \phi)] + \text{c.c.}$, or as

$$\tilde{E}(\mathbf{r}, t) = A e^{i[kx - \omega t + \delta \sin(qz - \Omega t)]} + \text{c.c.} \quad (8.4.49)$$

We see that the transmitted field is phase-modulated in time. To determine the consequences of this form of modulation, we note that Eq. (8.4.49) can be transformed through use of the Bessel function identity

$$e^{i\delta \sin y} = \sum_{l=-\infty}^{\infty} J_l(\delta) e^{ily} \quad (8.4.50)$$

so that the transmitted field can be expressed as

$$\tilde{E}(\mathbf{r}, t) = A \sum_{l=-\infty}^{\infty} J_l(\delta) e^{i[(kx + lqz) - (\omega + l\Omega)t]} + \text{c.c.} \quad (8.4.51)$$

We see that the transmitted field is a linear superposition of plane wave components with frequencies $\omega + l\Omega$ and wavevectors $\mathbf{k} + l\mathbf{q}$. As shown in Fig. 8.4.10 (for the case $l = 2$), the l th-order diffracted wave is emitted at angle

$$\theta_l = \tan^{-1}\left(\frac{lq}{k}\right) \simeq \frac{lq}{k} = \frac{l\lambda}{\Lambda}. \quad (8.4.52)$$

The intensity of the light in this diffraction order is

$$I_l = |A|^2 J_l(\delta)^2, \quad (8.4.53)$$

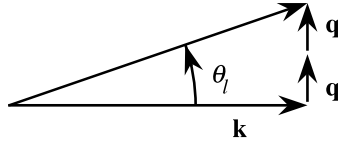


FIGURE 8.4.10: Illustration of the $l = 2$ contribution to the transmitted field of Eq. (8.4.51) for Raman–Nath scattering.

where, as before, $\delta \equiv 2\Delta n(\omega/c)L$. Eqs. (8.4.48) and (8.4.53) constitute the Raman–Nath equations.

It is instructive to repeat this analysis for the case of a standing sound wave. For convenience, we take the resulting modulation of the refractive index to be of the form

$$\tilde{n}(z, t) = 2\Delta n \cos \Omega t \sin qz. \quad (8.4.54)$$

The phase shift induced in the optical wave is then given by

$$\begin{aligned} \phi &= 2\Delta n \frac{\omega}{c} L \cos \Omega t \sin qz \\ &\equiv \delta \cos \Omega t \sin qz, \end{aligned} \quad (8.4.55)$$

and the transmitted optical field is given by

$$\tilde{E}(\mathbf{r}, t) = A e^{i(kx - \omega t + \delta \cos \Omega t \sin qz)} + \text{c.c.} \quad (8.4.56)$$

We now use the Bessel function identity (8.4.50) to transform the factor $\sin qz$ that appears in the exponent of this expression. We find that

$$\tilde{E}(\mathbf{r}, t) = A \sum_{l=-\infty}^{\infty} J_l(\delta \cos \Omega t) \exp[i(kx + lqz) - i\omega t] + \text{c.c.} \quad (8.4.57)$$

We see that once again the transmitted field is composed of plane wave components; the l th diffracted order makes an angle

$$\theta_l \simeq \frac{lq}{k} = \frac{l\lambda}{\Lambda} \quad (8.4.58)$$

with the forward direction. The intensity of the l th order is now given by

$$I_l = |A|^2 J_l^2(\delta \cos \Omega t). \quad (8.4.59)$$

We see that in this case each component is amplitude-modulated.

Problems

1. *Light scattering in air.* Estimate numerically, using the Lorentz model of the atom, the value of the scattering cross section for molecular nitrogen (N_2) for visible light at a wavelength of 500 nm. Use this result to estimate the value of the scattering coefficient R for air at STP. Compare this value with that obtained using Eq. (8.2.22) and the known refractive index of air. (The measured value for 90-degree scattering of unpolarized light, R_{90}^u , is approximately $2 \times 10^{-8} \text{ cm}^{-1}$.) Also, estimate numerically the attenuation distance of light in air, that is, the propagation distance through which the intensity falls to $1/e$ of its initial value due to scattering losses.
[Ans.: $\alpha = (16\pi/3)R_{90}^u = 3 \times 10^{-7} \text{ cm}^{-1} = (33 \text{ km})^{-1}$.]
2. *Light scattering in water.* Through use of Eq. (8.3.9b) and handbook values of γ_e and C_T , estimate numerically the value of the scattering coefficient R for liquid water at room temperature for 90-degree scattering of visible light at a wavelength of 500 nm. Use this result to estimate the attenuation distance of light in water.
[Ans.: Using the values $C_T = 4.5 \times 10^{-11} \text{ cm}^2/\text{dyne}$, $n = 1.33$, $\gamma_e = (n^2 - 1)(n^2 + 2)/3 = 0.98$, and $T = 300 \text{ K}$, we find that $R = 2.8 \times 10^{-6} \sin^2 \phi \text{ cm}^{-1}$, and hence that for 90-degree scattering of unpolarized light $R_{90}^u = 1.4 \times 10^{-6} \text{ cm}^{-1}$. Thus the attenuation constant is given by $\alpha = (16\pi/3)R_{90}^u = 2.34 \times 10^{-5} \text{ cm}^{-1} = (426 \text{ m})^{-1}$.]
3. *Polarizability of a dielectric sphere.* Verify Eq. (8.2.12).
4. *Acoustic attenuation in water.* Estimate numerically the value of the acoustic absorption coefficient α_s for propagation through water at frequencies of 10^3 , 10^6 , and 10^9 Hz .
[Ans.: The low-frequency shear viscosity coefficient of water is $\eta_s = 0.01 \text{ dyne sec/cm}^2$, and the Stokes relation tells us that $\eta_d = -(2/3)\eta_s$. We find that $\Gamma' = 0.66 \times 10^{-2} \text{ cm}^2/\text{sec}$. Since $\alpha_s = q^2 \Gamma' / v$ and $q = \Omega / v$, where $v = 1.5 \times 10^6 \text{ cm/sec}$, we find that $\alpha_s = 7.7 \times 10^{-14} \text{ cm}^{-1}$ at 1 kHz and $\alpha_s = 7.7 \times 10^{-2} \text{ cm}^{-1}$ at 1 GHz.]
5. *Inverse dielectric tensor.* Verify Eq. (8.4.8).
6. *Solution of the Bragg acoustooptics equations.* Verify Eqs. (8.4.31a) through (8.4.35b).
7. *Acoustooptic beam deflector.* Consider an acoustooptic beam deflector. The incidence angle θ_1 remains fixed while the acoustic frequency Ω is varied to control the deflection angle θ_2 . Derive a formula that predicts the maximum useful deflection angle, defined arbitrarily to be that deflection angle for which the diffraction efficiency drops to 50% of its maximum value. Evaluate this formula numerically for the case treated following Eq. (8.4.26), where $|\kappa|L = \pi/2$, $L = 1.1 \text{ cm}$, and $\Lambda = 30 \mu\text{m}$.
[Ans.: Starting with Eq. (8.4.25), and the readily derived relation $\Delta k = -\frac{1}{2}q \delta\theta$, we find that the efficiency drops by 50% when the incidence angle is increased by an amount

$$\delta\theta = \frac{2\sqrt{2}|\kappa|}{q} \left[1 - \frac{|\kappa|L \cos |\kappa|L}{\sin |\kappa|L} \right]^{1/2}.$$

For the case $|\kappa|L = \pi/2$, where the efficiency for $\Delta k = 0$ is 100%, this result simplifies to $\delta\theta = 2\sqrt{2}|\kappa|/q$. For the numerical example, $2\delta\theta = 0.22^\circ$.]

References

- Fabelinskii, I.L., 1968. *Molecular Scattering of Light*. Plenum Press, New York.
- Feynman, R.P., Leighton, R.B., Sands, M., 1963. *The Feynman Lectures on Physics*, vol. I. Addison-Wesley, Reading, MA.
- Jackson, J.D., 1982. *Classical Electrodynamics*. Wiley, New York.
- Landau, L.D., Lifshitz, E.M., 1960. *Electrodynamics of Continuous Media*. Addison-Wesley, Reading, MA; see especially Chapter 14.
- Landau, L.D., Lifshitz, E.M., 1969. *Statistical Physics*. Addison-Wesley, Reading, MA.
- Stratton, J.A., 1941. *Electromagnetic Theory*. McGraw-Hill, New York.
- Recommended Further Reading on Light Scattering and Acoustooptics
- Fabelinskii, I.L., 1997. In: Wolf, E. (Ed.), *Progress in Optics*, vol. XXXVII. Elsevier, Amsterdam.
- Yariv, A., 1975. *Quantum Electronics*. Wiley, New York.
- Yariv, A., Yeh, P., 1984. *Optical Waves in Crystals*. Wiley, New York.