Appendices

Appendix A The SI System of Units

In this appendix we review briefly the basic equations of electromagnetism when written in the SI (System International, or rationalized mksa) system of units. Conversion between the SI and gaussian systems of units is summarized in an additional appendix. The intent of this appendix is to establish notation and not to present a rigorous exposition of electromagnetic theory. More complete treatments of electromagnetism can be found for example in Jackson (1998), Marion and Heald (1994), Purcell and Morin (2013), and Stratton (2008).

In the SI system, mechanical properties are measured in mks units, that is, distance is measured in meters (m), mass in kilograms (kg), and time in seconds (s). The unit of force is thus the kg m/sec^2, known as a newton (N), and the unit of energy is the kg m^2/sec^2, known as the joule (J). The fundamental electrical unit is a unit of charge, known as the coulomb (C). It is defined such that the force between two charged point particles, each containing 1 coulomb of charge and separated by a distance of 1 meter, is 9.0 giganewton, which is the numerical value of $(4\pi\epsilon_0)^{-1}$. More generally, the force between two charged particles of charges q_1 and q_2 separated by the directed distance $\mathbf{r} = r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$ direction, is given by

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{A.1}$$

This result is known as Coulomb's law. The parameter ϵ_0 that appears in this equation is known as the permittivity of free space and has the value $\epsilon_0 = 8.85 \times 10^{-12}$ F/m. Here F is the abbreviation for the farad, which is defined as 1 coulomb/volt. The unit of electrical current is the ampere (A), which is 1 coulomb/sec. The unit of electrical potential (i.e., potential energy per unit charge) is the volt, which is 1 joule/coulomb. Note that the units of ϵ_0 can thus be expressed as $[\mathrm{F/m}] = \mathrm{A^2\,s^4\,kg^{-1}\,m^{-3}} = \mathrm{C^2\,N^{-1}\,m^{-2}} = \mathrm{C\,V^{-1}\,m^{-1}}$.

In the SI system, Maxwell's equations have the form*

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{A.2a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{A.2b}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{A.2c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{A.2d}$$

The units and names of the field vectors are as follows:

$$[\mathbf{E}] = \mathrm{V/m} \quad \text{(electric field)}, \tag{A.3a}$$

$$[\mathbf{D}] = \mathrm{C/m^2} \quad \text{(electric displacement)}, \tag{A.3b}$$

$$[\mathbf{B}] = \mathrm{T} \quad \text{(magnetic field, or magnetic induction)}, \tag{A.3c}$$

$$[\mathbf{H}] = \mathrm{A/m} \quad \text{(magnetic intensity)}, \tag{A.3d}$$

$$[\mathbf{P}] = \mathrm{C/m^2} \quad \text{(polarization)}, \tag{A.3e}$$

$$[\mathbf{M}] = \mathrm{A/m} \quad \text{(magnetization)}. \tag{A.3f}$$

In Eq. (A.3c), T notes the tesla, the unit of magnetic field strength. The tesla is equivalent to the Wb/m^2, where Wb denotes the weber, the unit of magnetic flux, which is equivalent to 1 joule/ampere or to 1 volt second.

The vectors $\mathbf{P}$ and $\mathbf{M}$ are known as the polarization and magnetization, respectively. The polarization $\mathbf{P}$ represents the electric dipole moment per unit volume, which may be present in a material. The magnetization $\mathbf{M}$ denotes the magnetic dipole moment per unit volume, which may be present in the material. These quantities are discussed further in the discussion given below.

The two additional quantities appearing in Maxwell's equations are the free charge density ρ, measured in units of coulombs/m^3, and the free current density $\mathbf{J}$, measured in units of amperes/m^2. Under many circumstances, $\mathbf{J}$ is given by the expression

$$\mathbf{J} = \sigma \mathbf{E}, \tag{A.4}$$

which can be considered to be a microscopic form of Ohm's law. Here σ is the electrical conductivity, whose units are ohm^{-1} m^{-1}. The ohm is the unit of electrical resistance and has units of volt/ampere.

The relationships that exist among the four electromagnetic field vectors because of purely material properties are known as the constitutive relations. These relations, even in the presence

* In this appendix, we dispense with our usual notation of using a tilde to denote time-varying quantities.

of nonlinearities, have the form

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{A.5a}$$

$$\mathbf{H} = \mu_0^{-1} \mathbf{B} - \mathbf{M}. \tag{A.5b}$$

Here μ_0 is the magnetic permeability of free space, which has the value $\mu_0 = 1.26 \times 10^{-6}$ H/m. Here H is the abbreviation for the henry, which is defined as 1 weber/ampere or as 1 volt second/ampere. Note that the units of μ_0 are thus given by any of the relations $\mathrm{H/m} = \mathrm{NA}^{-2} = \mathrm{TmA}^{-1} = \mathrm{WbA}^{-1}\mathrm{m}^{-1} = \mathrm{VsA}^{-1}\mathrm{m}^{-1}$.

The manner in which the response of a material medium can lead to a nonlinear dependence of **P** upon **E** is of course the subject of this book. For the limiting case of a purely linear response, the relation between **P** and **E** and the relation between **M** and **H** can be expressed (assuming an isotropic medium for notational simplicity) as

$$\mathbf{P} = \epsilon_0 \chi^{(1)} \mathbf{E}, \tag{A.6a}$$

$$\mathbf{M} = \chi_m^{(1)} \mathbf{H}. \tag{A.6b}$$

Note that the linear electric susceptibility $\chi^{(1)}$ and the linear magnetic susceptibility $\chi_m^{(1)}$ are dimensionless quantities. We now introduce the linear dimensionless relative permittivity (also known as the dielectric constant) $\epsilon^{(1)}$ and the linear dimensionless relative magnetic permeability $\mu^{(1)}$, which are defined by

$$\mathbf{D} = \epsilon_0 \epsilon^{(1)} \mathbf{E}, \tag{A.7a}$$

$$\mathbf{B} = \mu_0 \mu_m^{(1)} \mathbf{H}. \tag{A.7b}$$

We then find by consistency of Eqs. (A.5a), (A.6a), and (A.7a) and of (A.5b), (A.6b), and (A.7b) that

$$\epsilon^{(1)} = 1 + \chi^{(1)}, \tag{A.8a}$$

$$\mu^{(1)} = 1 + \chi_m^{(1)}. \tag{A.8b}$$

The fields **E** and **B** (rather than **D** and **H**) are usually taken to constitute the fundamental electromagnetic fields. For example, the force on a particle of charge q moving at velocity **v** through an electromagnetic field is given by the Lorentz force law in the form

$$\mathbf{F} = q\big[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\big]. \tag{A.9}$$

A.1 Energy Relations and Poynting's Theorem

Poynting's theorem can be derived from Maxwell's equations in the following manner. We begin with the vector identity (that is, this equation is true for any vector fields **E** and **H**)

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{A.10}$$

and introduce expressions for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ from the Maxwell equations (A.2a) and (A.2b) to obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \left[\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\right] = -\mathbf{J} \cdot \mathbf{E}. \tag{A.11}$$

Assuming for simplicity the case of a purely linear response, the second term on the left-hand side of this equation can be expressed as $\partial u/\partial t$, where

$$u = \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{A.12}$$

represents the energy density of the electromagnetic field. We also introduce the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{A.13}$$

which gives the rate at which electromagnetic energy passes through a unit area whose normal is in the direction of **S**. Eq. (A.11) can then be written as

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E}, \tag{A.14}$$

where $\mathbf{J} \cdot \mathbf{E}$ gives the rate per unit volume at which energy is lost to the field through Joule heating.

A.2 The Wave Equation

A wave equation for the electric field can be derived from Maxwell's equations, as described in greater detail in Section 2.1. We assume the case of a linear, isotropic, nonmagnetic (i.e., $\mu = 1$) medium that is free of sources (i.e., $\rho = 0$ and $\mathbf{J} = 0$). We first take the curl of the first Maxwell equation (A.2a), reverse the order of differentiation on the right-hand side, replace **B** by $\mu_0\mathbf{H}$, and use the second Maxwell equation (A.2b) to replace $\nabla \times \mathbf{H}$ by $\partial \mathbf{D}/\partial t$ to obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{A.15}$$

On the left-hand side of this equation, we make use of the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{A.16}$$

and drop the first term on the right-hand side because $\nabla \cdot \mathbf{E}$ must vanish whenever ρ vanishes in an isotropic medium because of the Maxwell equation (A.2c). On the right-hand side, we replace $\mathbf{D}$ by $\epsilon_0 \epsilon^{(1)} \mathbf{E}$, and set $\mu_0 \epsilon_0$ equal to $1/c^2$. We thus obtain the wave equation in the form

$$-\nabla^2 \mathbf{E} + \frac{\epsilon^{(1)}}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{A.17}$$

This equation possesses solutions in the form of infinite plane waves—that is,

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.}, \tag{A.18}$$

where $\mathbf{k}$ and ω must be related by

$$k = n\omega/c \quad \text{where} \quad n = \sqrt{\epsilon^{(1)}} \quad \text{and} \quad k = |\mathbf{k}|.$$

The magnetic intensity associated with this wave has the form

$$\mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.} \tag{A.19}$$

Note that, in accordance with the convention followed in this book, factors of $\frac{1}{2}$ are not included in these expressions. From Maxwell's equations, one can deduce that $\mathbf{E}_0$, $\mathbf{H}_0$, and $\mathbf{k}$ are mutually orthogonal and that the magnitudes of $\mathbf{E}_0$ and $\mathbf{H}_0$ are related by

$$n|\mathbf{E}_0| = \sqrt{\mu_0/\epsilon_0}\,|\mathbf{H}_0|. \tag{A.20}$$

The quantity $\sqrt{\mu_0/\epsilon_0}$ is known as the impedance of free space, is often represented as Z_0, and has the value 377 ohms. Since $\epsilon_0\mu_0 = 1/c^2$, the impedance of free space can alternatively be written as $Z_0 = \sqrt{\mu_0/\epsilon_0} = 1/\epsilon_0 c$. For completeness, we also note that for a plane wave the field amplitudes $\mathbf{D_0}$ and $\mathbf{B_0}$ are related to $\mathbf{E_0}$ and $\mathbf{H_0}$ through

$$|\mathbf{E}_0| = (1/\epsilon_0)|\mathbf{D}_0| = (c/n)|\mathbf{B}_0| = (1/n\epsilon_0 c)|\mathbf{H}_0|, \tag{A.21a}$$

$$(c/n)|\mathbf{B}_0| = (c\mu_0/n)|\mathbf{H}_0| = \sqrt{(\mu_0/\epsilon_0)}|\mathbf{H}_0|/n = (1/\epsilon_0 c)|\mathbf{H}_0|. \tag{A.21b}$$

In considerations of the energy relations associated with a time-varying field, it is useful to introduce a time-averaged (cycle-averaged) Poynting vector $\langle \mathbf{S} \rangle$ and a time-averaged energy density $\langle u \rangle$. Through use of Eqs. (A.18)–(A.20) and the defining relations (A.12) and (A.13), we find that these quantities are given by

$$\langle \mathbf{S} \rangle = 2n\sqrt{\epsilon_0/\mu_0}\,|E_0|^2 \hat{\mathbf{k}} = 2n\epsilon_0 c |E_0|^2 \hat{\mathbf{k}}, \tag{A.22a}$$

$$\langle u \rangle = 2n^2 \epsilon_0 |E_0|^2, \tag{A.22b}$$

where $\hat{\mathbf{k}}$ is a unit vector in the $\mathbf{k}$ direction. In this book the magnitude of the time-averaged Poynting vector is called the intensity $I = |\langle \mathbf{S} \rangle|$ and is given by

$$I = 2n\sqrt{\epsilon_0/\mu_0}\,|E_0|^2 = 2n\epsilon_0 c |E_0|^2. \tag{A.23}$$

A.3 Boundary Conditions

There are many situations in electromagnetic theory in which one needs to calculate the fields in the vicinity of a boundary between two regions of space with different optical properties. The way in which the fields are related on the opposite sides of the boundary constitutes the topic of boundary conditions.

To treat this topic, we first express the Maxwell equations in their integral rather than differential forms. We recall the divergence theorem, which states that, for any well-behaved vector field $\mathbf{A}$, the following identity holds:

$$\int_V \nabla \cdot \mathbf{A}\, dV = \int_S \mathbf{A} \cdot \mathbf{n}\, da. \tag{A.24}$$

The integral on the left-hand side is to be performed over any closed three-dimensional volume V and the integral on the right-hand side is to be performed over the surface S that encloses this volume. The quantity $\mathbf{n}$ represents a unit vector pointing in the outward normal direction. If the divergence theorem is applied to Maxwell's equations (A.2c) and (A.2d), one obtains

$$\int_S \mathbf{D} \cdot \mathbf{n}\, da = \int_V \rho\, dV, \tag{A.25}$$

$$\int_S \mathbf{B} \cdot \mathbf{n}\, da = 0. \tag{A.26}$$

The first of these equations expresses Gauss's law, and the second the absence of magnetic monopoles.

We can similarly express the two "curl" Maxwell equations in integral form through use of Stokes's theorem, which states that for any well-behaved vector field $\mathbf{A}$

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \int_C \mathbf{A} \cdot d\mathbf{l}. \tag{A.27}$$

Here S is any open surface, C is a curve that bounds it, and $d\mathbf{l}$ is a directed line element along this curve. When this theorem is applied to Maxwell's equations (A.2a) and (A.2b), one obtains

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, da, \tag{A.28}$$

$$\int_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot \mathbf{n}\, da. \tag{A.29}$$

The first of these equations expresses Faraday's law, and the second expresses Ampere's law with the inclusion of Maxwell's displacement current.

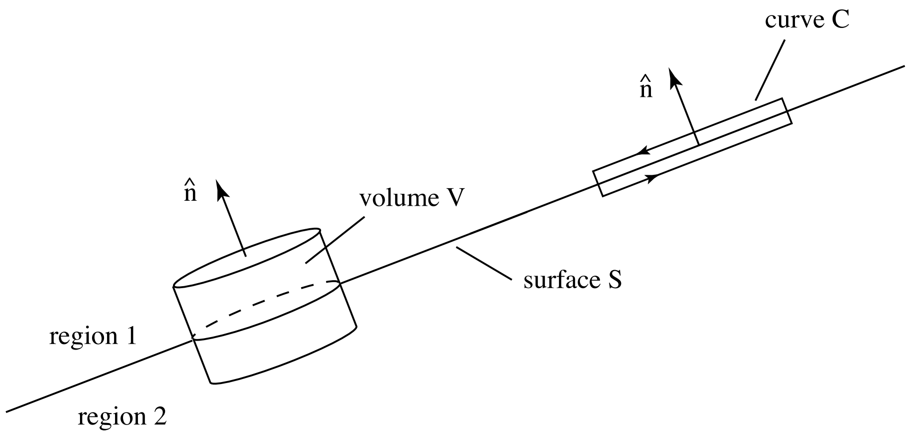

FIGURE A.1: Constructions used to determine the boundary conditions of the electromagnetic fields at the interface (surface S) between regions 1 and 2.

We are now in a position to determine the nature of the boundary conditions on the electromagnetic fields. We refer to Fig. A.1, which shows the interface between regions 1 and 2. We first imagine placing a small cylindrical pill box near the interface so that one circular side extends into region 1 and the other into region 2. We apply Eq. (A.24) to this situation. We next imagine shrinking the height of the pill box while keeping the areas of the two surfaces fixed. Through such a limiting procedure, we are assured that the value of the surface integral is dominated by the fields on the two circular surfaces. We further assume that the circular surfaces are sufficiently small that the fields are essentially constant over these surfaces. Even though the surface integrals then remain appreciable, the volume integral will vanish so long as ρ remains finite, because the volume over which the integration is performed will tend to zero as the height of the pill box is shrunk. The only situation in which the volume integral can be nonvanishing is that in which ρ diverges somewhere within the region of integration, for example, if there is charge located on the surface separating regions 1 and 2. If we let Σ denote the surface charge density—that is, the charge per unit area located on the surface, we find that the boundary condition on **D** is given by

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \Sigma. \tag{A.30}$$

The boundary condition on **B** is found more simply. Since the right-hand side of Eq. (A.26) vanishes, we find immediately that

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0. \tag{A.31}$$

Eq. (A.31) tells us that the normal component of the **B** field must be continuous at the boundary. Eq. (A.30) tells us that the normal component of the **D** field can be discontinuous but only by an amount equal to the charge density accumulated on the surface. This free-charge density can be appreciable for the case of metallic surfaces. However, the surface charge density vanishes at the interface between two dielectric materials.

The boundary conditions for $\mathbf{E}$ and $\mathbf{H}$ can be determined by considering the path integral shown at right-hand side of Fig. A.1. We assume that the long sides of the path lie parallel to the surface, one on each side of the interface. We further assume the limiting situation in which the short sides and very much shorter than the long sides. In this situation the line integrals are dominated by the long sides of the paths, and the surface integrals tend to vanish because the area of the region of integration tends to zero. The surface integrals of $\partial\mathbf{B}/\partial t$ and $\partial\mathbf{D}/\partial t$ always vanish for this reason. However, the surface integral of $\mathbf{J}$ can be nonvanishing if $\mathbf{J}$ diverges anywhere within the region of integration. This can occur if there is a surface current density $\mathbf{j}_s$, of units A/m, at the boundary between the two materials. As a consequence of these considerations, Eqs. (A.27) and (A.28) become

$$(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n} = 0, \tag{A.32}$$

$$(\mathbf{H}_2 - \mathbf{H}_1) \times \mathbf{n} = \mathbf{j}_s. \tag{A.33}$$

The first of these equations states that the tangential component of $\mathbf{E}$ is always continuous at an interface, whereas the second states that the tangential components of $\mathbf{H}$ is discontinuous by an amount equal to the surface current density $\mathbf{j}_s$. Again, the surface current density must vanish for the interface between two dielectric media.

Appendix B The Gaussian System of Units

In this appendix we review briefly the basic equations of electromagnetism when written in the gaussian system of units. Our treatment is a bit more abbreviated than that of Appendix A on the SI system.

In the gaussian system, mechanical properties are measured in cgs units, that is, distance is measured in centimeters (cm), mass in grams (g), and time in seconds (s). The unit of force is thus the g cm/sec^2, known as a dyne, and the unit of energy is the g cm^2/sec^2, known as the erg. The fundamental electrical unit is a unit of charge, known either as the statcoulomb or simply as the electrostatic unit of charge. It is defined such that the force between two charged point particles, each containing 1 statcoulomb of charge and separated by 1 centimeter, is 1 dyne. More generally, the force between two charged particles of charges q_1 and q_2 separated by the directed distance $\mathbf{r} = r\hat{\mathbf{r}}$ where $\hat{\mathbf{r}}$ is a unit vector in the $\mathbf{r}$ direction is given by

$$\mathbf{F} = \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}. \tag{B.1}$$

The unit of current is thus the statcoulomb/sec, which is known as the statampere, or simply as the electrostatic unit of current. The unit of electrical potential (i.e., potential energy per unit charge) is the erg/statcoulomb, also known as the statvolt.

In the gaussian system, Maxwell's equations have the form

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{B.2a}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{B.2b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{B.2c}$$

$$\nabla \cdot \mathbf{D} = 4\pi\rho. \tag{B.2d}$$

A remarkable feature of the gaussian system is that the four primary field vectors (i.e., the electric field **E**, the electric displacement field **D**, the magnetic induction **B**, and the magnetic intensity **H**, as well as the polarization vector **P** and the magnetization vector **M**, which will be introduced shortly) all have the same dimensions—that is,

$$\begin{aligned}[\mathbf{E}] &= [\mathbf{D}] = [\mathbf{B}] = [\mathbf{H}] = [\mathbf{D}] = [\mathbf{M}] \\ &= \frac{\text{statvolt}}{\text{cm}} = \frac{\text{statcoulomb}}{\text{cm}^2} = \text{gauss} = \text{oersted} = \left(\frac{\text{erg}}{\text{cm}^3}\right)^{1/2}.\end{aligned} \tag{B.3}$$

By convention the name gauss is used only in reference to the field **B** and oersted only with the field **H**. The two additional quantities appearing in Maxwell's equations are the free charge density ρ, measured in units of statcoulomb/cm^3, and the free current density **J**, measured in units of statampere/cm^2. Under many circumstances **J** is given by the expression

$$\mathbf{J} = \sigma\mathbf{E}, \tag{B.4}$$

which can be considered to be a microscopic form of Ohm's law, where σ is the electrical conductivity, whose units are inverse seconds.

The relationships among the four electromagnetic field vectors are known as the constitutive relations. These relations, even in the presence of nonlinearities, have the form

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{B.5a}$$

$$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}. \tag{B.5b}$$

The manner in which the response of a material medium can lead to a nonlinear dependence of **P** upon **E** is of course the subject of this book. For the limiting case of a purely linear response, the relationships can be expressed (assuming an isotropic medium for notational simplicity) as

$$\mathbf{P} = \chi^{(1)}\mathbf{E}, \tag{B.6a}$$

$$\mathbf{M} = \chi_m^{(1)}\mathbf{H}. \tag{B.6b}$$

Note that the linear electric susceptibility $\chi^{(1)}$ and the linear magnetic susceptibility $\chi_m^{(1)}$ are dimensionless quantities. If we now introduce the linear dielectric constant $\epsilon^{(1)}$ (also known as the dielectric permittivity) and the linear magnetic permeability $\mu^{(1)}$, both of which are dimensionless and which are defined by

$$\mathbf{D} = \epsilon^{(1)}\mathbf{E}, \tag{B.7a}$$

$$\mathbf{B} = \mu_m^{(1)}\mathbf{H}, \tag{B.7b}$$

we find by consistency of Eqs. (B.5a)–(B.7a) and (B.5b)–(B.7b) that

$$\epsilon^{(1)} = 1 + 4\pi\chi^{(1)}, \tag{B.8a}$$

$$\mu^{(1)} = 1 + 4\pi\chi_m^{(1)}. \tag{B.8b}$$

The fields **E** and **B** (rather than **D** and **H**) are usually taken to constitute the fundamental electromagnetic fields. For example, the force on a particle of charge q moving at velocity $\mathbf{v}$ through an electromagnetic field is given by the Lorentz force law

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right). \tag{B.9}$$

Poynting's theorem can be derived from Maxwell's equations in the following manner. We begin with the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{B.10}$$

and introduce expressions for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ from the Maxwell equations (B.2a) and (B.2b), to obtain

$$\frac{c}{4\pi}\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \frac{1}{4\pi}\left[\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\right] = -\mathbf{J} \cdot \mathbf{E}. \tag{B.11}$$

Assuming for simplicity the case of a purely linear response, the second term on the left-hand side of this equation can be expressed as $\partial u/\partial t$, where

$$u = \frac{1}{8\pi}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{B.12}$$

represents the energy density of the electromagnetic field. We also introduce the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}, \tag{B.13}$$

which gives the rate at which electromagnetic energy passes through a unit area whose normal is in the direction of $\mathbf{S}$. Eq. (B.11) can then be written as

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\mathbf{J} \cdot \mathbf{E}, \tag{B.14}$$

where $\mathbf{J} \cdot \mathbf{E}$ gives the rate per unit volume at which energy is lost to the field through Joule heating.

A wave equation for the electric field can be derived from Maxwell's equations, as described in Section 2.1, and for a linear, isotropic nonmagnetic (i.e., $\mu = 1$) medium that is free of sources has the form

$$-\nabla^2 \mathbf{E} + \frac{\epsilon^{(1)}}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{B.15}$$

This equation possesses solutions in the form of infinite plane waves—that is,

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.}, \tag{B.16}$$

where $\mathbf{k}$ and ω must be related by

$$k = n\omega/c, \quad \text{where} \quad n = \sqrt{\epsilon^{(1)}} \text{ and } k = |\mathbf{k}|.$$

The magnetic field associated with this wave has the form

$$\mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.} \tag{B.17}$$

Note that, in accordance with the convention followed in the book, factors of $\frac{1}{2}$ are not included in these expressions. From Maxwell's equations, one can deduce that $\mathbf{E}_0$, $\mathbf{B}_0$, and $\mathbf{k}$ are mutually orthogonal and that the magnitudes of $\mathbf{E}_0$ and $\mathbf{B}_0$ are related by

$$n|\mathbf{E}_0| = |\mathbf{B}_0|. \tag{B.18}$$

In considering the energy relations associated with a time-varying field, it is useful to introduce a time-averaged Poynting vector $\langle \mathbf{S} \rangle$ and a time-averaged energy density $\langle u \rangle$. Through use of Eqs. (B.16)–(B.18), we find that these quantities are given by

$$\langle \mathbf{S} \rangle = \frac{nc}{2\pi} |E_0|^2 \hat{\mathbf{k}}, \tag{B.19a}$$

$$\langle u \rangle = \frac{n^2}{2\pi} |E_0|^2, \tag{B.19b}$$

where $\hat{\mathbf{k}}$ is a unit vector in the $\mathbf{k}$ direction. In this book the magnitude of the time-averaged Poynting vector is called the intensity $I = |\langle \mathbf{S} \rangle|$ and is given by $I = (nc/2\pi)|E_0|^2$.

Appendix C Systems of Units in Nonlinear Optics

There are several different systems of units that are commonly used in nonlinear optics. In this appendix we describe these different systems and show how to convert among them. For simplicity we restrict the discussion to a medium with instantaneous response so that the nonlinear susceptibilities can be taken to be dispersionless. Clearly the rules derived here for among the systems of units are the same for a dispersive medium.

In the gaussian system of units, the polarization $\tilde{P}(t)$ is related to the field strength $\tilde{E}(t)$ by the equation

$$\tilde{P}(t) = \chi^{(1)}\tilde{E}(t) + \chi^{(2)}\tilde{E}^2(t) + \chi^{(3)}\tilde{E}^3(t) + \cdots. \tag{C.1}$$

In the gaussian system, all of the fields $\tilde{E}$, $\tilde{P}$, $\tilde{D}$, $\tilde{B}$, $\tilde{H}$, and $\tilde{M}$ have the same units; in particular, the units of $\tilde{P}$ and $\tilde{E}$ are given by

$$[\tilde{P}] = [\tilde{E}] = \frac{\text{statvolt}}{\text{cm}} = \frac{\text{statcoulomb}}{\text{cm}^2} = \left(\frac{\text{erg}}{\text{cm}^3}\right)^{1/2}. \tag{C.2}$$

Consequently, we see from Eq. (C.1) that the dimensions of the susceptibilities are as follows:

$$\chi^{(1)} \text{ is dimensionless,} \tag{C.3a}$$

$$[\chi^{(2)}] = \left[\frac{1}{\tilde{E}}\right] = \frac{\text{cm}}{\text{statvolt}} = \left(\frac{\text{erg}}{\text{cm}^3}\right)^{-1/2}, \tag{C.3b}$$

$$[\chi^{(3)}] = \left[\frac{1}{\tilde{E}^2}\right] = \frac{\text{cm}^2}{\text{statvolt}^2} = \left(\frac{\text{erg}}{\text{cm}^3}\right)^{-1}. \tag{C.3c}$$

The units of the nonlinear susceptibilities are often not stated explicitly in the gaussian system of units; one rather simply states that the value is given in electrostatic units (esu).

While there are various conventions in use regarding the units of the susceptibilities in the SI system, by far the most common convention is to replace Eq. (C.1) by

$$\tilde{P}(t) = \epsilon_0\left[\chi^{(1)}\tilde{E}(t) + \chi^{(2)}\tilde{E}^2(t) + \chi^{(3)}\tilde{E}^3(t) + \cdots\right], \tag{C.4}$$

where

$$\epsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} \tag{C.5}$$

denotes the permittivity of free space. Since the units of $\tilde{P}$ and $\tilde{E}$ in the SI system are

$$[\tilde{P}] = \frac{\mathrm{C}}{\mathrm{m}^2}, \tag{C.6a}$$

$$[\tilde{E}] = \frac{\mathrm{V}}{\mathrm{m}}, \tag{C.6b}$$

and since 1 farad is equal to 1 coulomb per volt, it follows that the units of the susceptibilities are as follows:

$$\chi^{(1)} \text{ is dimensionless}, \tag{C.7a}$$

$$[\chi^{(2)}] = \left[\frac{1}{\tilde{E}}\right] = \frac{\mathrm{m}}{\mathrm{V}}, \tag{C.7b}$$

$$[\chi^{(3)}] = \left[\frac{1}{\tilde{E}^2}\right] = \frac{\mathrm{m}^2}{\mathrm{V}^2}. \tag{C.7c}$$

C.1 Conversion between the Systems

In order to facilitate conversion between the two systems just introduced, we express the two defining relations (C.1) and (C.4) in the following forms:

$$\tilde{P}(t) = \chi^{(1)}\tilde{E}(t)\left[1 + \frac{\chi^{(2)}\tilde{E}(t)}{\chi^{(1)}} + \frac{\chi^{(3)}\tilde{E}^2(t)}{\chi^{(1)}} + \cdots\right] \text{(gaussian)}, \tag{C.1$'$}$$

$$\tilde{P}(t) = \epsilon_0\chi^{(1)}\tilde{E}(t)\left[1 + \frac{\chi^{(2)}\tilde{E}(t)}{\chi^{(1)}} + \frac{\chi^{(3)}\tilde{E}^2(t)}{\chi^{(1)}} + \cdots\right] \text{(SI)}. \tag{C.4$'$}$$

The power series shown in square brackets must be identical in each of these equations. However, the values of $\tilde{E}$, $\chi^{(1)}$, $\chi^{(2)}$, and $\chi^{(3)}$ are different in different systems. In particular, from Eqs. (C.2) and (C.6b) and the fact that 1 statvolt = 300 V, we find that

$$\tilde{E}\ (\text{SI}) = 3 \times 10^4 \tilde{E}\ (\text{gaussian}). \tag{C.8}$$

To determine how the linear susceptibilities in the gaussian and SI systems are related, we make use of the fact that for a linear medium the displacement is given in the gaussian system by

$$\tilde{D} = \tilde{E} + 4\pi\tilde{P} = \tilde{E}\big(1 + 4\pi\chi^{(1)}\big), \tag{C.9a}$$

and in the SI system by

$$\tilde{D} = \epsilon_0\tilde{E} + \tilde{P} = \epsilon_0\tilde{E}\big(1 + \chi^{(1)}\big). \tag{C.9b}$$

We thus find that

$$\chi^{(1)}\,(\mathrm{SI}) = 4\pi\,\chi^{(1)}\,(\text{gaussian}). \tag{C.10}$$

Using Eqs. (C.8) and (C.9a)–(C.9b), and requiring that the power series of Eqs. (C.1′) and (C.4′) be identical, we find that the nonlinear susceptibilities in the two systems of unit are related by

$$\begin{aligned}\chi^{(2)}\,(\mathrm{SI}) &= \frac{4\pi}{3\times 10^4}\,\chi^{(2)}\,(\text{gaussian})\\ &= 4.189\times 10^{-4}\,\chi^{(2)}\,(\text{gaussian}),\end{aligned} \tag{C.11}$$

$$\begin{aligned}\chi^{(3)}\,(\mathrm{SI}) &= \frac{4\pi}{(3\times 10^4)^2}\,\chi^{(3)}\,(\text{gaussian})\\ &= 1.40\times 10^{-8}\,\chi^{(3)}\,(\text{gaussian}).\end{aligned} \tag{C.12}$$

Appendix D Relationship between Intensity and Field Strength

In the gaussian system of units, the intensity associated with the field

$$\tilde{E}(t) = E e^{-i\omega t} + \text{c.c.} \tag{D.1}$$

is

$$I = \frac{nc}{2\pi}|E|^2, \tag{D.2}$$

where n is the refractive index, $c = 3\times 10^{10}$ cm/sec is the speed of light in vacuum, I is measured in erg/cm^2 sec, and E is measured in statvolts/cm.

In the SI system, the intensity of the field described by Eq. (D.1) is given by

$$I = 2n\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}|E|^2 = \frac{2n}{Z_0}|E|^2 = 2n\epsilon_0 c|E|^2, \tag{D.3}$$

where $\epsilon_0 = 8.85\times 10^{-12}$ F/m, $\mu_0 = 4\pi\times 10^{-7}$ H/m, and $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$. I is measured in W/m^2, and E is measured in V/m. Using these relations we can obtain the results shown in Table D.1. As a numerical example, a pulsed laser of modest energy might produce a pulse energy of $Q = 1$ mJ with a pulse duration of $T = 10$ nsec. The peak laser power would then be of the order of $P = Q/T = 100$ kW. If this beam is focused to a spot size of $w_0 = 100$ μm, the maximum pulse intensity will be $I = P/\pi w_0^2 \simeq 0.3$ GW/cm^2 and the maximum electric field strength will be 790 statvolt/cm or 2.39×10^7 V/m.

TABLE D.1: Relation between field strength and intensity.[a]

Conventional		gaussian (cgs)		SI (mks)	
I		I (erg/cm^2 sec)	E (statvolt/cm)	I (W/m^2)	E (V/m)
1 kW/m^2		10^6	0.0145	10^3	4.34×10^2
	1 W/cm^2	10^7	0.0458	10^4	1.37×10^3
1 MW/m^2		10^9	0.458	10^6	1.37×10^4
	1 kW/cm^2	10^{10}	1.45	10^7	4.34×10^4
1 GW/m^2		10^{12}	1.45×10	10^9	4.34×10^5
	1 MW/cm^2	10^{13}	45.8	10^{10}	1.37×10^6
1 TW/m^2		10^{15}	4.58×10^2	10^{12}	1.37×10^7
	1 GW/cm^2	10^{16}	1.45×10^3	10^{13}	4.34×10^7
1 ZW/m^2		10^{18}	1.45×10^4	10^{15}	4.34×10^8
	1 TW/cm^2	10^{19}	4.85×10^4	10^{16}	1.37×10^9

[a] Note that the peak field strength is twice the field strength reported in this table.

Appendix E Physical Constants

This appendix provides tables of physical constants in both the Gaussian (cgs) and SI (mks) systems.

TABLE E.1: Physical constants in the cgs and SI systems.

Constant	Symbol	Value	Gaussian (cgs)[a]	SI (mks)[a]
Speed of light in vacuum	c	2.998	10^{10} cm/sec	10^8 m/sec
Elementary charge	e	4.803	10^{-10} esu	
		1.602		10^{-19} C
Avogadro number	N_A	6.023	10^{23} mol	10^{23} mol
Electron rest mass	$m = m_e$	9.109	10^{-28} g	10^{-31} kg
Proton rest mass	m_p	1.673	10^{-24} g	10^{-27} kg
Planck constant	h	6.626	10^{-27} erg sec	10^{-34} J sec
	$\hbar = h/2\pi$	1.054	10^{-27} erg sec	10^{-34} J sec
Fine structure constant[b]	$\alpha = e^2/4\pi\epsilon_0\hbar c$	1/137	–	–
Compton wavelength of electron	$\lambda_C = h/mc$	2.426	10^{-10} cm	10^{-12} m
Rydberg constant[b]	$R_\infty = me^4/32\pi^2\epsilon_0^2\hbar^2$	13.6	eV	eV
Bohr radius[b]	$a_0 = 4\pi\epsilon_0\hbar^2/me^2$	5.292	10^{-9} cm	10^{-11} m
Electron radius[b]	$r_e = e^2/4\pi\epsilon_0 mc^2$	2.818	10^{-13} cm	10^{-15} m
Bohr magneton[b]	$\mu_S = eh/2m_e$	9.273	10^{-21} erg/G	10^{-24} J/T
		$\Rightarrow$	1.4 MHz/G	

continued on next page

TABLE E.1: (*continued.*)

Constant	Symbol	Value	Gaussian (cgs)[a]	SI (mks)[a]
Nuclear magneton[b]	$\mu_N = e\hbar/2m_p$	5.051	10^{-24} erg/G	10^{-27} J/T
Gas constant	R	8.314	10^{7} erg/K m	10^{0} J/K mole
Volume, mole of ideal gas	V_0	2.241	10^{4} cm^3	10^{-2} m^3
Boltzmann constant	k_B	1.381	10^{-16} erg/K	10^{-23} J/K
Stefan–Boltzmann constant	σ	5.670	10^{-5} erg/cm^2 sec K^4	10^{-8} W/m^2 K^4
Gravitational constant	G	6.670	10^{-8} dyne cm^2/g^2	10^{-11} N m^2/kg^2
Electron volt	eV	1.602	10^{-12} erg	10^{-19} J

[a] Abbreviations: C, coulombs; mol, molecules; g, grams; J, joules; N, newtons; G, gauss; T, teslas.
[b] Defining equation is shown in the SI system of units.

TABLE E.2: Physical constants specific to the SI system.

Constant	Symbol[a]	Value[a]
Permittivity of free space	ϵ_0	8.85×10^{-12} F/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ H/m
Velocity of light in free space	$(\epsilon_0\mu_0)^{-1/2} = c$	2.997×10^{8} m/sec
Impedance of free space	$(\mu_0/\epsilon_0)^{1/2} = Z_0 = \epsilon_0 c$	377 Ω

[a] Abbreviations: F, farad = coulomb/volt, H, henry = weber/ampere.

TABLE E.3: Conversion between the systems.

1 m	=	100 cm
1 kg	=	1000 g
1 newton	=	10^5 dynes
1 joule	=	10^7 erg
1 coulomb	=	2.998×10^9 statcoulomb
1 volt	=	1/299.8 statvolt
1 ohm	=	1.139×10^{-12} sec/cm
1 tesla	=	10^4 gauss[a]
1 farad	=	0.899×10^{12} cm
1 henry	=	1.113×10^{-12} sec^2/cm
1 eV	=	1.6×10^{-19} J $= 1.6 \times 10^{-12}$ erg

[a] Here, 1 tesla = 1 weber/m^2; 1 gauss = 1 oersted.

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