Appendix VI

Propagation of discontinuities in an electromagnetic field (§3.1.1)

It was mentioned in §3.1.1 that the eikonal equation of geometrical optics is identical with an equation which describes the propagation of discontinuities in an electromagnetic field. More generally, the four equations §3.1 (11a)–(14a) governing the behaviour of the electromagnetic field associated with the geometrical light rays may be shown to be identical with equations which connect the field vectors on a moving discontinuity surface. It is the purpose of this appendix to demonstrate this mathematical equivalence.

1 Relations connecting discontinuous changes in field vectors

In §1.1.3 we considered discontinuities in field vectors which arise from abrupt changes in the material parameters ε and μ, for example at a surface of a lens. Discontinuous fields may also arise from entirely different reasons, namely because a source suddenly begins to radiate. The field then spreads into the space surrounding the source and with increasing time fills a larger and larger region. On the boundary of this region the field has a discontinuity, the field vectors being in general finite inside this region and zero outside it. We shall first establish certain general relations which hold on any surface at which the field is discontinuous. For simplicity we assume that at any instant of time $t > 0$ there is only one such surface; the extension to several discontinuity surfaces (which may arise, for example, from reflections at obstacles present in the medium) is straightforward.

Let $F(x, y, z, t) = 0$ be any surface at which at least one of the field vectors is discontinuous. If this surface is fixed in space, F is, of course, independent of t. The points on either side of the surface may be distinguished by the inequalities $F < 0$ and $F > 0$ respectively. Let $\mathbf{E}$ be the electric vector and let

$$\left.\begin{aligned} \mathbf{E}(x, y, z, t) &= \mathbf{E}^{(1)}(x, y, z, t), \quad \text{when } F(x, y, z, t) < 0, \\ &= \mathbf{E}^{(2)}(x, y, z, t), \quad \text{when } F(x, y, z, t) > 0. \end{aligned}\right\} \tag{1}$$

$\mathbf{E}$ may then be written in the form

$$\mathbf{E} = \mathbf{E}^{(1)} U(-F) + \mathbf{E}^{(2)} U(F), \tag{2}$$

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where U is the Heaviside unit function [see Appendix IV (17)].

Using the representation (2), we derive expressions for the quantities curl $\mathbf{E}$, div $\mathbf{E}$, $\partial\mathbf{E}/\partial t$, etc., which enter Maxwell's equations. To differentiate a sum or a product that contains a discontinuous factor, we apply the ordinary rules of differentiation and use relation (18) of Appendix IV,

$$\frac{\mathrm{d}}{\mathrm{d}x}U(x) = \delta(x), \tag{3}$$

where δ is the Dirac delta function. Thus from (2) we have, for example,

$$\begin{aligned}\operatorname{curl}\mathbf{E} = {} & U(-F)\operatorname{curl}\mathbf{E}^{(1)} + U(F)\operatorname{curl}\mathbf{E}^{(2)} + [\operatorname{grad} U(-F)]\times\mathbf{E}^{(1)} \\ & + [\operatorname{grad} U(F)]\times\mathbf{E}^{(2)}.\end{aligned} \tag{4}$$

Now

$$\operatorname{grad} U(-F) = -\operatorname{grad} U(F) = -\frac{\mathrm{d}U(F)}{\mathrm{d}F}\operatorname{grad} F = -\delta(F)\operatorname{grad} F, \tag{5}$$

and from (4) and (5) it follows that

$$\operatorname{curl}\mathbf{E} = U(-F)\operatorname{curl}\mathbf{E}^{(1)} + U(F)\operatorname{curl}\mathbf{E}^{(2)} + \delta(F)\operatorname{grad} F\times(\Delta\mathbf{E}), \tag{6}$$

where

$$\Delta\mathbf{E} = \mathbf{E}^{(2)} - \mathbf{E}^{(1)}. \tag{7}$$

In a similar way we find

$$\operatorname{div}\mathbf{E} = U(-F)\operatorname{div}\mathbf{E}^{(1)} + U(F)\operatorname{div}\mathbf{E}^{(2)} + \delta(F)\operatorname{grad} F\cdot\Delta\mathbf{E}, \tag{8}$$

$$\frac{\partial\mathbf{E}}{\partial t} = U(-F)\frac{\partial\mathbf{E}^{(1)}}{\partial t} + U(F)\frac{\partial\mathbf{E}^{(2)}}{\partial t} + \delta(F)\frac{\partial F}{\partial t}\Delta\mathbf{E}. \tag{9}$$

We now substitute from (6), (8) and (9) and from similar expressions involving the other field vectors into Maxwell's equations §1.1 (1)–(4), assuming $\mathbf{j} = \rho = 0$. The terms with superscript (1) cancel out, and so do the terms with superscript (2), since the fields on either side of the discontinuity surface satisfy separately Maxwell's equations. The remaining terms give the following *relations connecting the discontinuous changes in the field vectors*

$$\operatorname{grad} F\times\Delta\mathbf{H} - \frac{1}{c}\frac{\partial F}{\partial t}\Delta\mathbf{D} = 0, \tag{10}$$

$$\operatorname{grad} F\times\Delta\mathbf{E} + \frac{1}{c}\frac{\partial F}{\partial t}\Delta\mathbf{B} = 0, \tag{11}$$

$$\operatorname{grad} F\cdot\Delta\mathbf{D} = 0, \tag{12}$$

$$\operatorname{grad} F\cdot\Delta\mathbf{B} = 0. \tag{13}$$

It is of interest to note that these equations may be formally obtained from the source-free Maxwell's equations by replacing the vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$ by the differences $\Delta\mathbf{E}$, $\Delta\mathbf{H}$, $\Delta\mathbf{D}$, $\Delta\mathbf{B}$; and the differential operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, $\partial/\partial t$ by the multiplicative operators $(1/|\operatorname{grad} F|)\partial F/\partial x$, $(1/|\operatorname{grad} F|)\partial F/\partial y$, $(1/|\operatorname{grad} F|)\partial F/\partial z$, $(1/|\operatorname{grad} F|)\partial F/\partial t$.

Let $\mathbf{n}_{12}$ be the unit vector normal to the discontinuity surface and pointing from the region $F < 0$ (suffix 1), into the region $F > 0$ (suffix 2),

$$\mathbf{n}_{12} = \frac{\operatorname{grad} F}{|\operatorname{grad} F|}. \tag{14}$$

We also introduce the speed v with which the discontinuity surface advances. To a small displacement $\delta\mathbf{r}(\delta x, \delta y, \delta z)$ from a point on the discontinuity surface $F(x, y, z, t) = 0$ to a point on a neighbouring discontinuity surface, there corresponds a change δt in time, such that

$$\operatorname{grad} F \cdot \delta\mathbf{r} + \frac{\partial F}{\partial t}\delta t = 0. \tag{15}$$

In particular, for a displacement along the normal, $\delta\mathbf{r} = \delta s\,\mathbf{n}_{12}$, so that the speed v is given by

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial t}. \tag{16}$$

It follows that the relations (10)–(13) may also be written in the form*

$$\mathbf{n}_{12} \times \Delta\mathbf{H} + \frac{v}{c}\Delta\mathbf{D} = 0, \tag{10a}$$

$$\mathbf{n}_{12} \times \Delta\mathbf{E} - \frac{v}{c}\Delta\mathbf{B} = 0, \tag{11a}$$

$$\mathbf{n}_{12} \cdot \Delta\mathbf{D} = 0, \tag{12a}$$

$$\mathbf{n}_{12} \cdot \Delta\mathbf{B} = 0. \tag{13a}$$

If the discontinuities in the field vectors arise from abrupt changes in the material parameters ε and μ on a surface $F(x, y, z) = 0$ which is fixed in space, then $v = 0$ and (10a)–(13a) reduce to §1.1 (25), §1.1 (23), §1.1 (19), and §1.1 (15) (with $\mathbf{j} = \rho = 0$).

2 The field on a moving discontinuity surface

Consider a moving discontinuity surface which arises from the presence of a source which suddenly begins to radiate. We represent the surface in the form

$$F(x, y, z, t) \equiv \mathcal{S}(x, y, z) - ct = 0, \tag{17}$$

where c is the vacuum velocity of light. The field vectors on this discontinuity surface will be denoted by small letters,

* These equations are a generalization of a set of discontinuity relations which appear to have been known already to O. Heaviside. See also A. E. H. Love, *Proc. Lond. Math. Soc.*, **1** (1904), 56; H. Bateman, *Partial Differential Equations of Mathematical Physics* (Cambridge, Cambridge University Press, 1932), p. 196; T. Levi-Civita, *Caractéristiques des Systèmes Différentiels et Propagation des Ondes* (Paris, Librairie Félix Alcan, 1932), §10; R. K. Luneburg, *Mathematical Theory of Optics*, (University of California Press, Berkeley and Los Angeles, 1964), §6 and §7; M. Kline, *Comm. Pure and Appl. Math.*, **4** (1951), 239; A. Rubinowicz, *Acta Phys. Polonica*, **14** (1955), 209; M. Kline and I. W. Kay, *Electromagnetic Theory and Geometrical Optics* (New York, Interscience Publishers, 1965), pp. 37–51. The method of derivation used here is due to H. Bremmer, *Comm. Pure and Appl. Math.*, **4** (1951), 419.

$$\mathbf{e}(x, y, z) = \mathbf{E}\left[x, y, z, \frac{1}{c}\mathcal{S}(x, y, z)\right], \tag{18}$$

with similar expressions for the other field vectors. Now in the region outside the moving surface [say $F(x, y, z, t) > 0$] there is no field, so that according to (1) and (7) $\Delta\mathbf{E} = -\mathbf{E}^{(1)} = -\mathbf{e}$, etc. The material equations §1.1 (10), §1.1 (11) give* $\mathbf{d} = \varepsilon\mathbf{e}$, $\mathbf{b} = \mu\mathbf{h}$ and if we use these relations, (10)–(13) become

$$\operatorname{grad}\mathcal{S} \times \mathbf{h} + \varepsilon\mathbf{e} = 0, \tag{19}$$

$$\operatorname{grad}\mathcal{S} \times \mathbf{e} - \mu\mathbf{h} = 0, \tag{20}$$

$$\operatorname{grad}\mathcal{S} \cdot \mathbf{e} = 0, \tag{21}$$

$$\operatorname{grad}\mathcal{S} \cdot \mathbf{h} = 0. \tag{22}$$

These equations are formally identical with the basic equations [§3.1 (11a)–(14a)] of geometrical optics. Hence *the field vectors on a moving discontinuity surface obey the same equations as the field vectors associated with the geometrical optics approximation to a time-harmonic field, the moving discontinuity surface corresponding to the geometrical wave-fronts*.†

Evidently the moving discontinuity surface must obey the eikonal equation

$$\left(\frac{\partial\mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial y}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial z}\right)^2 = n^2, \tag{23}$$

where $n^2 = \varepsilon\mu$. This equation follows as before. It is the consistency condition between (19) and (20) and follows from them on eliminating $\mathbf{h}$ or $\mathbf{e}$ and using (21) or (22). According to (16), (17) and (23), the discontinuity surface is propagated with the velocity $v = c/n$.

* The assumptions $\mathbf{d} = \varepsilon\mathbf{e}$ and $\mathbf{b} = \mu\mathbf{h}$ are approximations that are reasonable, when the effective frequency range of the light interacting with the medium is sufficiently narrow. This may not be the case when the field is discontinuous. Hence (19)–(23) are not rigorously valid, except for propagation in free space.

† It may also be shown that $\mathbf{e}$ and $\mathbf{h}$ obey the same transport equations [§3.1 (41) and (42)] as the complex amplitude vectors of the geometrical optics field. This result was first established by R. K. Luneburg, *loc. cit.*, §11, §12. See also E. T. Copson, *Comm. Pure and Appl. Math.*, **4** (1951), 427; M. Kline and I. W. Kay, *loc. cit.*, p. 162.