Appendix X

Evaluation of two integrals (§12.2.2)*

In this appendix we shall evaluate the two integrals of §12.2 (8), §12.2 (9)

$$\mathcal{J}_1 = \frac{1}{4\pi}\iiint\limits_{V_1}\left[e^{i(px_1+qy_1)}\frac{\partial^2}{\partial z_1^2}\left(\frac{e^{i\omega R/c}}{R}\right)\right] dx_1\,dy_1\,dz_1, \tag{1}$$

$$\mathcal{J}_2 = \frac{1}{4\pi}\frac{\omega^2}{c^2}\iiint\limits_{V_1}\left[e^{i(px_1+qy_1)}\left(\frac{e^{i\omega R/c}}{R}\right)\right] dx_1\,dy_1\,dz_1, \tag{2}$$

where

$$R = +\sqrt{x_1^2 + y_1^2 + z_1^2} \tag{3}$$

and $\omega^2 > c^2p^2$. We are interested in two cases, namely:

(a) $\underline{0 < y < d}$, where y and d are constants and the volume V_1 of integration is the slab $-\infty < x_1 < \infty$, $-y \leqslant y_1 \leqslant d - y$, $-\infty < z_1 < \infty$, except for a small sphere of vanishingly small radius around the origin $x_1 = y_1 = z_1 = 0$.

(b) $\underline{y > d \text{ or } y < 0}$, the volume V_1 of integration now being the full region $-\infty < x_1 < \infty$, $-y \leqslant y_1 \leqslant d - y$, $-\infty < z_1 < \infty$.

To evaluate $\mathcal{J}_1$ we apply Gauss's theorem, in the form

$$\iiint\limits_{V_1}\operatorname{div}\mathbf{G}\,dV_1 = \iint\limits_{S_1}\mathbf{G}\cdot\mathbf{n}\,dS_1, \tag{4}$$

where $\mathbf{G}$ is an arbitrary vector function of position and $\mathbf{n}(n_x, n_y, n_z)$ is the unit outward normal to the surface S_1 bounding the volume V_1. We take

$$G_{x_1} = G_{y_1} = 0, \qquad G_{z_1} = \frac{1}{4\pi}\frac{\partial}{\partial z_1}\left(\frac{e^{i(px_1+qy_1+\omega R/c)}}{R}\right) \tag{5}$$

and obtain from (1) and (4)

* After C. G. Darwin, *Trans. Camb. Phil. Soc.*, **23** (1924), §6 and §8.

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$$\mathcal{J}_1 = \frac{1}{4\pi} \iint_{S_1} n_{z_1} \frac{\partial}{\partial z_1} \left(\frac{e^{i(px_1+qy_1+\omega R/c)}}{R} \right) dS_1. \tag{6}$$

Since n_{z_1} is zero at each face ($y_1 = -y$ and $y_1 = d - y$) of the slab, the integral $\mathcal{J}_1$ is zero in cases (b).* In case (a), we must also include the contribution from the small sphere σ_1 around the origin. If a is the radius of the sphere, we have $n_z = -z_1/a$ and the contribution is

$$\frac{1}{4\pi} \iint_{\sigma_1} \left(\frac{z_1}{a} \right)^2 e^{i(px_1+qy_1+\omega a/c)} \left(\frac{1}{a^2} - \frac{i\omega}{ac} \right) d\sigma_1 \to \tfrac{1}{3} \quad \text{as } a \to 0. \tag{7}$$

Hence we have in all

$$\left. \begin{aligned} \mathcal{J}_1 &= \tfrac{1}{3} \quad \text{when} \quad 0 < y < d, \\ &= 0 \quad \text{when} \quad y < 0 \text{ or } y > d. \end{aligned} \right\} \tag{8}$$

Before calculating $\mathcal{J}_2$ we note that because the integrand contains only a singularity of order $1/R$, integration throughout the vanishingly small sphere around the origin will not give a contribution. Hence in case (a), just as in case (b) we may now integrate throughout the full volume bounded by the slab:

$$\mathcal{J}_2 = \frac{\omega^2}{4\pi c^2} \int_{-y}^{d-y} e^{iqy_1} \mathcal{L}(y_1) dy_1, \tag{9}$$

where

$$\mathcal{L}(y_1) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{e^{i(px_1+\omega R/c)}}{R} \, dx_1 \, dz_1. \tag{10}$$

In (10) we take new variables ρ and χ defined by

$$px_1 + \frac{\omega}{c} R = \rho \sqrt{\left(\frac{\omega}{c} \right)^2 - p^2}, \tag{11a}$$

$$z_1 = \sqrt{\rho^2 - y_1^2} \sin \chi, \tag{11b}$$

in each case the positive square root being taken. On the x_1, z_1-plane, the curves of constant ρ are ellipses, with χ as the eccentric angle. For $\rho = |y_1|$ the ellipse degenerates into a point. Thus the whole x_1, z_1-plane is covered if ρ and χ run through the values $|y_1| \leqslant \rho \leqslant \infty$, $0 \leqslant \chi \leqslant 2\pi$. Hence

$$\mathcal{L}(y_1) = \int_0^{2\pi} \int_{|y_1|}^{\infty} \frac{e^{i\rho\sqrt{\frac{\omega^2}{c^2} - p^2}}}{R} \frac{\partial(x_1, z_1)}{\partial(\rho, \chi)} \, d\rho \, d\chi, \tag{12}$$

where

* Strictly, the surface S_1 should be closed, so that we should also consider the contributions from the remote edges. These contributions may, however, be rejected on physical grounds, since they would require infinitely long time to reach the point where the effect is being considered.

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$$\frac{\partial(x_1, z_1)}{\partial(\rho, \chi)} = \frac{\partial x_1}{\partial \rho}\frac{\partial z_1}{\partial \chi} - \frac{\partial x_1}{\partial \chi}\frac{\partial z_1}{\partial \rho} \tag{13}$$

is the Jacobian of the transformation. From (11a) and (3)

$$\frac{\partial x_1}{\partial \rho} = \frac{\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}}{p + \frac{\omega}{c}\frac{x_1}{R}}, \qquad \frac{\partial x_1}{\partial \chi} = 0, \tag{14a}$$

and from (11b)

$$\frac{\partial z_1}{\partial \rho} = \frac{\rho}{\sqrt{\rho^2 - y_1^2}} \sin\chi, \qquad \frac{\partial z_1}{\partial \chi} = \sqrt{\rho^2 - y_1^2}\cos\chi = \sqrt{\rho^2 - y_1^2 - z_1^2}. \tag{14b}$$

Now we also have the identity

$$\left(pR + \frac{\omega}{c}x_1\right)^2 - \left(px_1 + \frac{\omega}{c}R\right)^2 = \left[\left(\frac{\omega}{c}\right)^2 - \rho^2\right][x_1^2 - R^2],$$

or using (3) and (11a),

$$\left(pR + \frac{\omega}{c}x_1\right) = \sqrt{\rho^2 - y_1^2 - z_1^2}\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}. \tag{15}$$

From (13), (14) and (15) it follows that

$$\frac{\partial(x_1, z_1)}{\partial(\rho, \chi)} = R. \tag{16}$$

We substitute from (16) into (12). The integration with respect to χ gives immediately the value 2π. The integration with respect to ρ is also straightforward and we obtain

$$\mathcal{L}(y) = \frac{2\pi \mathrm{i}\, \mathrm{e}^{\mathrm{i}\left[|y_1|\sqrt{(\omega/c)^2 - p^2}\right]}}{\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}}, \tag{17}$$

where, with the same physical justification as before, we have rejected an oscillatory contribution from infinity.

We now substitute from (17) into (9) and evaluate the resulting integral. This gives:

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$$\left.\begin{aligned}
\mathcal{J}_2 &= \left(\frac{\omega}{c}\right)^2 \frac{e^{-igy}}{2g\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}}(e^{igd} - 1) && \text{when } y > d,\\
&= \left(\frac{\omega}{c}\right)^2 \frac{e^{-ihy}}{2h\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}}(e^{ihd} - 1) && \text{when } y < 0,\\
&= \left(\frac{\omega}{c}\right)^2 \left[\frac{1}{gh} - \frac{e^{-igy}}{2g\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}} + \frac{e^{-ih(y-d)}}{2h\sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}}\right] && \text{when } 0 < y < d,
\end{aligned}\right\} \tag{18}$$

where

$$g = q - \sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}, \qquad h = q + \sqrt{\left(\frac{\omega}{c}\right)^2 - p^2}. \tag{19}$$

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