XI

Rigorous diffraction theory

11.1 Introduction

On the basis of Maxwell's equations, together with standard boundary conditions, the scattering of electromagnetic radiation by an obstacle becomes a well-defined mathematical boundary-value problem. In the present chapter some aspects of the theory of diffraction of monochromatic waves are developed from this point of view, and in particular the rigorous solution to the classical problem of diffraction by a perfectly conducting half-plane is given in detail.

In the early theories of Young, Fresnel, and Kirchhoff, the diffracting obstacle was supposed to be perfectly 'black'; that is to say, all radiation falling on it was assumed to be absorbed, and none reflected. This is an inherent source of ambiguity in that such a concept of absolute 'blackness' cannot legitimately be defined with precision; it is, indeed, incompatible with electromagnetic theory.

Cases in which the diffracting body has a finite dielectric constant and finite conductivity have been examined theoretically, one of the earliest comprehensive treatments of such a case being Mie's discussion in 1908 of scattering by a sphere, which is described in Chapter XIV in connection with the optics of metals. In general, however, the assumption of finite conductivity tends to make the mathematics very complicated, and it is often desirable to accept the concept of a perfectly conducting (and therefore perfectly reflecting) body. This is clearly an idealization, but one which is compatible with electromagnetic theory; furthermore, since the conductivity of some metals (e.g. copper) is very large, it may represent a good approximation if the frequency is not too high, though it should be stressed that the approximation is never entirely adequate at optical frequencies. The simplifying assumption that the diffracting obstacle has infinite conductivity is made in most of the treatments based on a precise mathematical formulation, and the subsequent discussion is confined to this case.

The first rigorous solution of such a diffraction problem was given by Sommerfeld* in 1896, when he treated the two-dimensional case of a plane wave incident on an infinitely thin, perfectly conducting half-plane. The fame of this achievement rests partly on the skill with which the solution was constructed, and partly on the remarkable fact that it could be expressed exactly and simply in terms of the Fresnel integrals which had been such a conspicuous feature of previous approximate theories.

* A. Sommerfeld, *Math. Ann.*, **47** (1896), 317.

Many mathematicians followed Sommerfeld's lead. Early variants of his problem, dealing with line and point sources, and a noteworthy generalization to the treatment of a wedge rather than a half-plane, are associated with the names of Carslaw,* MacDonald,† and Bromwich.‡ Other problems were attacked, and more recently new methods have been introduced, stimulated by the advance in ultra-short-wave radio techniques. Before proceeding to the main body of the chapter the nature of some of these investigations is very briefly indicated.

If there exists an orthogonal coordinate system, u_1, u_2, u_3, say, such that the surface of the diffracting body is identical with one of the surfaces $u_i =$ constant, the classical technique for solving partial differential equations by separation of variables may be appropriate; this, indeed, was Mie's approach in the case of the finitely conducting sphere mentioned above. The solution of the boundary-value problem then appears, in general, as an infinite series, and its utility depends on the ease with which computation of the relevant functions can be carried out and the rapidity with which the series converges. This method has been applied to various cases apart from the sphere, notably to the circular disc or aperture.§ It should be mentioned, however, that some of the work only relates to strictly scalar problems, such as those in the theory of small-amplitude sound waves; as is shown later, two-dimensional problems in electromagnetic theory are essentially of this type, but otherwise the vector nature of the electromagnetic field introduces further complications.

Another approach is based on integral equation formulations, a method apparently first considered by Rayleigh.|| Certain problems, the simplest being that of the half-plane, yield integral equations that can be solved exactly by the method of Wiener and Hopf,¶ and the appreciation of this fact by Copson,** Schwinger and others has led to a number of new solutions in closed form.†† Mention should also be made, in this connection, of powerful, if somewhat complicated, variational procedures which can be used to calculate the power diffracted through an aperture.‡‡

For reasons of space the discussion in this chapter is largely confined to one method.§§ First, some aspects of considerable generality in the theory of the scattering of electromagnetic waves by perfectly conducting structures are developed. Next, a representation of any field as an integral over a spectrum of plane waves is introduced

* H. S. Carslaw, *Proc. Lond. Math. Soc.*, **30** (1899), 121.
† H. M. MacDonald, *Electric Waves* (Cambridge, Cambridge University Press, 1902).
‡ T. J. I'A. Bromwich, *Proc. Lond. Math. Soc.*, **14** (1916), 450.
§ C. J. Bouwkamp, Dissertation, Groningen (1941).
J. Meixner and W. Andrejewski, *Ann. d. Physik*, **7** (1950), 157.
|| Lord Rayleigh, *Phil. Mag.*, **43** (1897), 259.
¶ For a discussion of the Wiener–Hopf method see E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, Clarendon Press, 1937), p. 339.
E. T. Copson, *Quart. J. Maths.*, **17 (1946), 19.
††J. F. Carlson and A. E. Heins, *Quart. Appl. Maths.*, **4** (1947), 313; **5** (1947), 82. A. E. Heins, *Quart. Appl. Maths.*, **6** (1948), 157, 215. H. Levine and J. Schwinger, *Phys. Rev.*, **73** (1948), 383. For a more comprehensive list of references see J. W. Miles, *J. Appl. Phys.*, **20** (1949), 760, and C. J. Bouwkamp, *Rep. Progr. Phys.* (London, Physical Society), **17** (1954), 35.
‡‡H. Levine and J. Schwinger, *Phys. Rev.*, **74** (1948), 958; **75** (1949), 1423.
§§ For general accounts of other methods see the article by G. Wolfsohn in *Handbuch der Physik*, Vol. 20 (Berlin, Springer, 1928), p. 263; B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, Clarendon Press, 1950), Chapters 4 and 5 and an article by H. Hönl, A. W. Maue and K. Westpfahl in *Handbuch d. Physik*, Vol. 25/1 (Berlin, Springer, 1961). The review by Bouwkamp just mentioned gives in outline a most comprehensive selection of methods and formulae.

and is shown to lead to the formulation of certain diffraction problems in terms of 'dual' integral equations.* The Sommerfeld half-plane problem is then tractable†; the solution of this is obtained and examined in some detail, together with a number of ramifications. Several allied problems are discussed.

11.2 Boundary conditions and surface currents

It is well known that an electromagnetic field penetrates but little into a good conductor. The idealization of infinite conductivity, when there is no penetration at all, results in the concept of electric currents existing purely on the surface of the conductor, as the following argument shows.

It is a consequence of Maxwell's curl equations (see §1.1.3) that the tangential component of $\mathbf{E}$ is continuous in crossing an infinitely thin electric current sheet whereas that of $\mathbf{H}$ is discontinuous; more particularly, the discontinuity in $\mathbf{H}$ is in the tangential component normal to the surface current density‡ $\mathbf{J}$, the relative sense of the directions being indicated schematically in Fig. 11.1, and is of amount $4\pi\mathbf{J}/c$. Furthermore, in line with the behaviour of the tangential components of $\mathbf{E}$ and $\mathbf{H}$, the normal component of $\mathbf{H}$ is continuous across the current sheet whereas that of $\mathbf{E}$ is discontinuous, the magnitude of the discontinuity being equal to 4π times the surface charge density. Hence it is clear that the field in free space exterior to a perfectly conducting body is such that on the surface of the conductor

(a) the tangential component of $\mathbf{E}$ is zero;
(b) the tangential component of $\mathbf{H}$ is perpendicular to the surface electric current density $\mathbf{J}$, in the sense indicated above, and of amount $4\pi\mathbf{J}/c$;
(c) the normal component of $\mathbf{H}$ is zero;
(d) the outward normal component of $\mathbf{E}$ is equal to 4π times the surface charge density.

The effect of radiation incident on a perfectly conducting body may be interpreted conveniently in terms of the induced surface currents. If $\mathbf{E}^{(i)}$ is the electric vector of the incident field and $\mathbf{E}^{(s)}$ is that of the 'scattered' field due to induced currents, then the total electric vector, everywhere, is $\mathbf{E}^{(i)} + \mathbf{E}^{(s)}$. The diffraction problem may

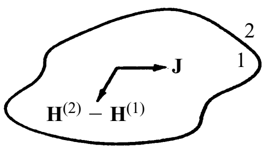

Fig. 11.1 Showing the sense of the discontinuity in $\mathbf{H}$ relative to the surface current density $\mathbf{J}$; $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$ are the respective magnetic fields on sides 1 and 2 of the surface.

* For a discussion of 'dual' integral equations (defined on pp. 642–643 below) see E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, Clarendon Press, 1937), p. 334.

† For an elementary treatment of the Sommerfeld half-plane problem see F. Gori, *Atti. d. Fond. Giorgio Ronchi*, **XXXVIII** (1983), 593. See also F. Gori, *Opt. Commun.*, **48** (1983), 67.

‡ The surface current was denoted by $\mathbf{j}$ in §1.1.3.

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therefore be stated as follows: *given* $\mathbf{E}^{(i)}$, *to find a field* $\mathbf{E}^{(s)}$ *which could arise from a current distribution in the surface of the conductor and which is such that its tangential component on the surface is minus that of* $\mathbf{E}^{(i)}$. It is worth stressing that the boundary condition (a) above is fundamental and sufficient alone to specify the problem uniquely in the form stated.* As regards the other conditions, (b) is of value in relating the field to the induced currents, but (c) and (d) are of no particular interest.

It should be noted that an implication of $\mathbf{E}^{(s)} = -\mathbf{E}^{(i)}$ at interior points of the conductor is the existence of a unique current density, in any closed surface S, which reproduces at all points inside S the field due to sources located outside S. Likewise, from a consideration of the case when the boundary of a perfect conductor on which the radiation falls is a complete infinite plane, it follows that there is a unique current density in any plane which reproduces on one side of the plane the field due to sources situated on the other side of the plane.

In problems of diffraction by perfectly conducting screens it is desirable to make the assumption that the screens are infinitely thin; if this is not done, the mathematical difficulties become very great. Of course, the opacity of the screen is maintained, the concept, in fact, being that of a perfect conductor whose thickness tends to zero in the limit. From what has been said, the effect of such a screen can be interpreted in terms of an electric current sheet, with the difference now that the sheet is no longer a closed surface. Of particular interest is the relatively simple case when the sheet is plane; in this case some important relations satisfied by the field $\mathbf{E}^{(s)}$, $\mathbf{H}^{(s)}$ which it radiates may be deduced immediately and are given below.

Suppose that the current sheet occupies part of the plane $y = 0$. Then, by reason of symmetry, it is clear that

$$\left.\begin{aligned}
E_x^{(s)}(x, y, z) &= E_x^{(s)}(x, -y, z), & H_x^{(s)}(x, y, z) &= -H_x^{(s)}(x, -y, z),\\
E_y^{(s)}(x, y, z) &= -E_y^{(s)}(x, -y, z), & H_y^{(s)}(x, y, z) &= H_y^{(s)}(x, -y, z),\\
E_z^{(s)}(x, y, z) &= E_z^{(s)}(x, -y, z), & H_z^{(s)}(x, y, z) &= -H_z^{(s)}(x, -y, z).
\end{aligned}\right\} \tag{1}$$

Moreover, if the current density in the sheet has components J_x and J_z, evidently, on $y = 0$,

$$H_x^{(s)} = \mp\frac{2\pi}{c}J_z, \qquad H_z^{(s)} = \pm\frac{2\pi}{c}J_x, \tag{2}$$

with the upper or lower sign according as to whether y reaches zero through positive or negative values respectively. As discussed in the next section, the application of these simple relations to the interesting problem of diffraction by a plane screen leads to a useful formulation which, in particular, puts in evidence an exact electromagnetic analogue of Babinet's principle.

11.3 Diffraction by a plane screen: electromagnetic form of Babinet's principle

Suppose that an electromagnetic field $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ is incident on a set of infinitely thin, perfectly conducting laminae lying in the plane $y = 0$. Let M signify those areas of the

* A discussion of proofs of uniqueness, which present some difficulties, is deferred to a later section (§11.9).

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plane occupied by the metal and A the remaining 'apertures', so that M and A together comprise the whole plane. Either M or A, or both M and A, may be of infinite extent.

As previously explained, a scattered field is sought which satisfies a certain boundary condition on M. Now in view of the relations §11.2 (1), it is, in fact, only necessary to consider the scattered field in one of the half-spaces $y \geqslant 0$, $y \leqslant 0$, provided the requirement of continuity across A is explicitly recognized. Hence the problem may be formulated thus: to find, in $y \geqslant 0$ (on in $y \leqslant 0$), an electromagnetic field $\mathbf{E}^{(s)}$, $\mathbf{H}^{(s)}$, which could be generated by currents in $y = 0$, such that

$$\begin{array}{lll} \text{(I)} & E_x^{(i)} + E_x^{(s)} = E_z^{(i)} + E_z^{(s)} = 0 & \text{on } M, \\ \text{(II)} & H_x^{(s)} = H_z^{(s)} = 0 & \text{on } A. \end{array}$$

Here, (I) is the fundamental boundary condition for a perfect conductor, whereas (II), which follows from §11.2 (2), is a convenient way of expressing the fact that there are no induced currents in A. If (II) is satisfied by the scattered field in $y \geqslant 0$, and §11.2 (1) used to deduce the scattered field in $y \leqslant 0$, then continuity across A is achieved.

A form of Babinet's principle for electromagnetic waves and perfectly conducting screens which is exact* may now be easily derived. As in the classical principle (§8.3.2), a relation is established between the respective fields existing in the presence of the screen and the 'complementary' screen obtained by interchanging the conducting laminae and the apertures; the difference lies in the fact that the field incident on the complementary screen is no longer the same as that incident on the original screen, but is derived from it by the transformation $\mathbf{E} \rightarrow \mathbf{H}$.

In the first case, then, let the field (suffix 1) defined by $\mathbf{E}_1^{(i)} = \mathbf{F}^{(i)}$ be incident, in $y > 0$, on the screen described above. From (I) and (II)

$$\begin{array}{llll} \text{(I}'\text{)} & E_{1x}^{(s)} = -F_x^{(i)}, & E_{1z}^{(s)} = -F_z^{(i)} & \text{on } M, \\ \text{(II}'\text{)} & H_{1x}^{(s)} = H_{1z}^{(s)} = 0 & & \text{on } A. \end{array}$$

Secondly, let the field (suffix 2) defined by $\mathbf{H}_2^{(i)} = \mathbf{F}^{(i)}$ be incident on the complementary screen. Then, writing the boundary conditions now in terms of the total field,

$$\begin{array}{llll} \text{(I}''\text{)} & E_{2x} = E_{2z} = 0 & & \text{on } A, \\ \text{(II}''\text{)} & H_{2x} = F_x^{(i)}, & H_{2z} = F_z^{(i)} & \text{on } M. \end{array}$$

Since Maxwell's equations in free space are invariant under the transformation $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{H} \rightarrow -\mathbf{E}$, and since there is a unique surface current density in $y = 0$ which would produce the incident field at all points in $y \leqslant 0$, it is clear from a comparison of (I′), (II′) with (II″), (I″), respectively, that

$$\mathbf{H}_2 = -\mathbf{E}_1^{(s)} \tag{1}$$

in the half-space *behind* the screen. In terms of the total field $\mathbf{E}_1$, (1) gives

$$\mathbf{E}_1 + \mathbf{H}_2 = \mathbf{F}^{(i)}, \tag{2}$$

which is the required electromagnetic form of Babinet's principle.

* H. G. Booker, *J. Instn. Elect. Engrs.*, **93**, Pt. III A (1946), 620. L. G. H. Huxley, *The Principles and Practice of Waveguides* (Cambridge, Cambridge University Press, 1947), p. 284.

11.4 Two-dimensional diffraction by a plane screen

11.4.1 The scalar nature of two-dimensional electromagnetic fields

A problem which is completely independent of one Cartesian coordinate, say z, is said to be two-dimensional. As already remarked, problems of this type in electromagnetic theory are essentially of a scalar nature in that they can straightaway be expressed in terms of a single dependent variable. This will now be shown.

With a time factor $\exp(-\mathrm{i}\omega t)$ suppressed and writing $k = \omega/c$, Maxwell's equations in free space are

$$\operatorname{curl}\mathbf{H} = -\mathrm{i}k\mathbf{E}, \qquad \operatorname{curl}\mathbf{E} = \mathrm{i}k\mathbf{H}.$$

Equating to zero all partial derivatives with respect to z, these may be split up into the two independent sets

$$\frac{\partial E_z}{\partial y} = \mathrm{i}kH_x, \qquad \frac{\partial E_z}{\partial x} = \mathrm{i}kH_y, \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -\mathrm{i}kE_z, \tag{1}$$

and

$$\frac{\partial H_z}{\partial y} = -\mathrm{i}kE_x, \qquad \frac{\partial H_z}{\partial x} = \mathrm{i}kE_y, \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \mathrm{i}kH_z. \tag{2}$$

The first group involves only H_x, H_y, E_z, the second only E_x, E_y, H_z. Simplicity can therefore be obtained by separating any solution into a linear combination of the two solutions for which every member of one of the above sets is zero. For the sake of nomenclature we characterize the two types of field as follows:

E-polarization

$$E_x = E_y = H_z = 0,$$

$$H_x = \frac{1}{\mathrm{i}k}\frac{\partial E_z}{\partial y}, \qquad H_y = -\frac{1}{\mathrm{i}k}\frac{\partial E_z}{\partial x},$$

and, as is evident on substituting for H_x and H_y into the third equation of (1),

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = 0.$$

Here the complete field is specified in terms of E_z, which, of course, satisfies the two-dimensional form of the standard wave equation.

H-polarization

$$H_x = H_y = E_z = 0,$$

and

$$E_x = -\frac{1}{\mathrm{i}k}\frac{\partial H_z}{\partial y}, \qquad E_y = \frac{1}{\mathrm{i}k}\frac{\partial H_z}{\partial x}.$$

Here the complete field is specified in terms of H_z.

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11.4.2 An angular spectrum of plane waves

For two-dimensional problems, to which the discussion is now confined, it has been shown that the Cartesian components of **E** and **H** satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0. \tag{3}$$

This equation has to be solved subject to the appropriate boundary conditions.

A fundamental elementary solution of (3) is

$$\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)} = \mathrm{e}^{\mathrm{i}k(x\cos\alpha + y\sin\alpha)}, \tag{4}$$

r, θ ($0 \leqslant \theta \leqslant 2\pi$) being polar coordinates related to x, y by the equations $x = r\cos\theta$, $y = r\sin\theta$. If α is real (4) represents a *homogeneous* plane wave, i.e. one whose equiamplitude and equiphase planes coincide: α is the angle between the direction of propagation and the x-axis (Fig. 11.2(a)). If, on the other hand, α is complex, (4) represents an *inhomogeneous* plane wave, i.e. one whose equiamplitude and equiphase planes do not coincide. In fact, writing $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, where α_1 and α_2 are real, (4) becomes

$$\mathrm{e}^{\mathrm{i}kr\cosh\alpha_2\cos(\theta-\alpha_1)}\,\mathrm{e}^{-kr\sinh\alpha_2\sin(\theta-\alpha_1)}, \tag{5}$$

from which it follows that the equiamplitude and equiphase planes are mutually perpendicular (Fig. 11.2(b)): the direction of phase propagation makes an angle α_1 with the x-axis, the phase velocity being reduced by the factor $\operatorname{sech}\alpha_2$, and there is exponential attenuation, governed by the attenuation factor $k\sinh\alpha_2$, in the direction at right angles.

Now it can be shown* that any solution of (3) can be put in the form of an *angular spectrum of plane waves*†

$$\int f(\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha$$

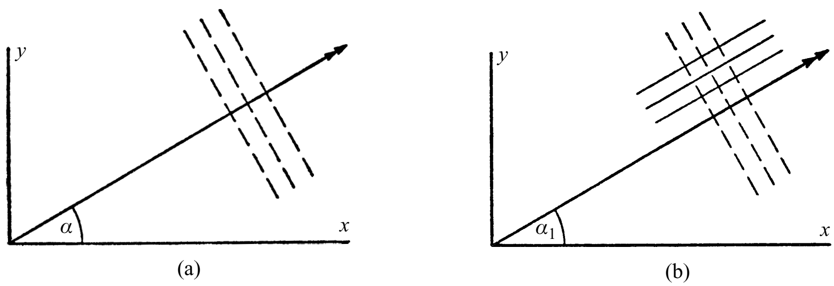

Fig. 11.2(a) The homogeneous plane wave (4) when α is real; the dashed lines denote both equiamplitude and equiphase planes. (b) The inhomogeneous plane wave (4) when α is complex; the full lines denote equiamplitude planes, the dashed lines equiphase planes.

* E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge, Cambridge University Press, 1927), p. 397.

† For another form of the angular spectrum representation see §13.2.1 and the reference cited on p. 712.

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by a suitable choice of the path of integration and the function $f(\alpha)$. Such a representation is closely linked with the representation of an arbitrary function by means of a Fourier integral, and is likewise of great power in application. Without significant loss of generality a certain fixed path of integration can be prescribed, so that any problem becomes a matter of determining the appropriate $f(\alpha)$. We shall first express the electromagnetic field due to a plane current sheet in this way, and then show that the result leads to a formulation of the problem of diffraction by a plane screen in terms of dual integral equations.

Consider a two-dimensional current sheet in $y = 0$. As already pointed out, it is convenient to deal with E-polarization and H-polarization separately. We treat, first, the former case, in which the current density has a z component only, J_z say, and begin by asking what particular distribution will radiate the E-polarized plane wave

$$E = (0,\ 0,\ 1)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}, \qquad H = (\sin\alpha,\ -\cos\alpha,\ 0)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)} \tag{6}$$

into the half-space $y > 0$. From the first relation in §11.2 (2) it is, in fact, immediately seen that

$$J_z(\xi) = -\frac{c}{2\pi}\mathrm{e}^{\mathrm{i}k\xi\cos\alpha}\sin\alpha \tag{7}$$

at the point $(\xi,\ 0)$. This could, of course, be verified by the standard method of Hertz potentials for finding the field generated by a current distribution, though the evaluation of a quite complicated integral is then necessary.

Now, broadly speaking, any current distribution can be built up by the appropriate superposition of expressions (7) for different values of α, and the radiated field will be obtained by the corresponding superposition of the plane waves (6). More precisely, suppose the current density can be written in the form of a Fourier integral as

$$J_z(\xi) = -\frac{c}{2\pi}\int_{-\infty}^{\infty} P(\mu)\mathrm{e}^{\mathrm{i}k\xi\mu}\,\mathrm{d}\mu. \tag{8}$$

The change of variable $\mu = \cos\alpha$ gives

$$J_z(\xi) = -\frac{c}{2\pi}\int_C \sin\alpha P(\cos\alpha)\mathrm{e}^{\mathrm{i}k\xi\cos\alpha}\,\mathrm{d}\alpha, \tag{9}$$

where C is the path in the complex α-plane along which $\cos\alpha$ ranges through real values from ∞ to $-\infty$, as shown in Fig. 11.3. The resulting nonzero field components are therefore

$$E_z^{(s)} = \int_C P(\cos\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{10}$$

$$H_x^{(s)} = \pm\int_C \sin\alpha\, P(\cos\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{11}$$

$$H_y^{(s)} = -\int_C \cos\alpha\, P(\cos\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{12}$$

with the upper sign for $y \geqslant 0$ and the lower sign for $y \leqslant 0$.

Eqs. (10), (11) and (12) represent the field in the form of a plane wave spectrum determined by the function $P(\cos\alpha)$. The individual plane waves corresponding to the section of C along the real axis are homogeneous; they radiate into the regions $y > 0$

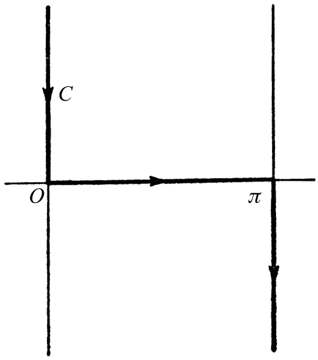

Fig. 11.3 The path C in the complex α plane.

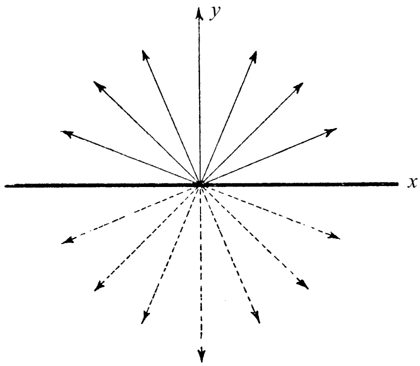

Fig. 11.4 Showing the directions of propagation of the homogeneous waves which radiate into half-space $y > 0$ (full lines) and the half-space $y < 0$ (dashed lines).

and $y < 0$, their directions of propagation embracing a range of angles π in each region, as illustrated diagrammatically in Fig. 11.4. The plane waves corresponding to the two arms of C on which $\alpha = \mathrm{i}\beta$ and $\alpha = \pi - \mathrm{i}\beta$, ($\beta = 0$ to ∞), are inhomogeneous; all their directions of phase propagation are along the positive or negative x-axis, and they are exponentially attenuated in the direction normal to and away from the plane $y = 0$. It can easily be shown, by an examination of the Poynting vector, that on the average no energy is carried away from the plane $y = 0$ into the half-space $y > 0$ by any of these *evanescent* waves. Their presence is necessary in order to take account of structure in the current distribution which is finer than a wavelength.

For the case of H-polarization, the field due to a current density J_x in $y = 0$ would likewise be written as

$$H_z^{(s)} = \pm \int_C P(\cos\alpha) \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{13}$$

$$E_x^{(s)} = -\int_C \sin\alpha\, P(\cos\alpha) \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{14}$$

$$E_y^{(s)} = \pm \int_C \cos\alpha\, P(\cos\alpha) \mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{15}$$

with the upper sign for $y \geqslant 0$, the lower sign for $y \leqslant 0$, where

$$J_x(\xi) = \frac{c}{2\pi}\int_{-\infty}^{\infty} \frac{P(\mu)}{\sqrt{1-\mu^2}} e^{ik\xi\mu}\, d\mu. \tag{16}$$

11.4.3 Formulation in terms of dual integral equations

The two-dimensional problem of diffraction by a plane screen can now be formulated in terms of dual integral equations.

Suppose that an electromagnetic field $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ is incident on a set of infinitely thin, perfectly conducting strips lying in $y = 0$; designate by M the ranges of x within which there is metal, by A those within which there is not. If the scattered field $\mathbf{E}^{(s)}$, $\mathbf{H}^{(s)}$ is represented as an angular spectrum of plane waves, in the form §11.4 (10), §11.4 (11), §11.4 (12) or §11.4 (13), §11.4 (14), §11.4 (15) according to the polarization, the conditions (I) and (II) of §11.3 yield the following integral equations:

E-polarization

$$\int_{-\infty}^{\infty} \frac{P(\mu)}{\sqrt{1-\mu^2}} e^{ikx\mu}\, d\mu = -E_x^{(i)} \qquad \text{on } M, \tag{17}$$

$$\int_{-\infty}^{\infty} P(\mu) e^{ikx\mu}\, d\mu = 0 \qquad \text{on } A. \tag{18}$$

H-polarization

$$\int_{-\infty}^{\infty} P(\mu) e^{ikx\mu}\, d\mu = E_x^{(i)} \qquad \text{on } M, \tag{19}$$

$$\int_{-\infty}^{\infty} \frac{P(\mu)}{\sqrt{1-\mu^2}} e^{ikx\mu}\, d\mu = 0 \qquad \text{on } A. \tag{20}$$

A consideration of the way in which the complex α-plane between $\mathcal{R}\alpha = 0$ and $\mathcal{R}\alpha = \pi$ ($\mathcal{R}$ denoting the real part) maps into the complete complex μ-plane ($\mu = \cos\alpha$) shows that the path of integration along the real axis avoids the possible branch-points at $\mu = \pm 1$ as illustrated diagrammatically in Fig. 11.5. Integral equations of this type, in which a single unknown function $P(\mu)$ satisfies different equations for two distinct ranges of the parameter x, are called 'dual'.*

The formulation used by Copson, Schwinger, and others, mentioned in §11.1, is

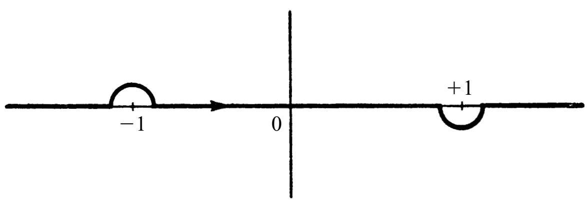

Fig. 11.5 The path of integration from $-\infty$ to ∞ in the complex μ plane.

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, Clarendon Press, 1937), p. 334.

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somewhat different from the one given above in that it only involves a single integral equation. Although it will not be wanted here, its connection with the present method should be pointed out. For the case of E-polarization, say, the solution of (16) obtained by taking its Fourier transform could be written in the form

$$P(\mu) = -\frac{k}{c}\int_M J_z(\xi)e^{-ik\mu\xi}\,d\xi, \tag{21}$$

in agreement, of course, with (8) and the fact that $J_z(\xi) = 0$ on A. Substituting this value of $P(\mu)$ into (17), and carrying out the integration with respect to μ, we have the integral equation

$$\frac{k}{c}\int_M J_z(\xi)H_0^{(1)}(k|x-\xi|)d\xi = E_z^{(i)} \qquad \text{on } M, \tag{22}$$

involving the Hankel function $H_0^{(1)}$ of the first kind and zero order, to be solved for $J_z(\xi)$. Obviously, the left-hand side of (22) could have been derived from the direct expression of the scattered field in terms of the induced current density.

11.5 Two-dimensional diffraction of a plane wave by a half-plane

11.5.1 Solution of the dual integral equations for E-polarization

In the next few pages the diffraction of a plane wave by a semiinfinite plane sheet is treated rigorously by obtaining a simple explicit solution of the appropriate dual integral equations.

Consider, first, the E-polarized plane wave

$$E_z^{(i)} = e^{-ikr\cos(\theta-\alpha_0)} \tag{1}$$

incident on the perfectly conducting half-plane $y = 0$, $x > 0$, where it is assumed, for convenience, that α_0 is real and $0 < \alpha_0 < \pi$ (Fig. 11.6). Eqs. §11.4 (17) and (18) are now

$$\int_{-\infty}^{\infty}\frac{P(\mu)}{\sqrt{1-\mu^2}}e^{ikx\mu}\,d\mu = -e^{-ikx\mu_0} \qquad \text{for } x > 0, \tag{2}$$

$$\int_{-\infty}^{\infty}P(\mu)e^{ikx\mu}\,d\mu = 0 \qquad \text{for } x < 0, \tag{3}$$

where $\mu_0 = \cos\alpha_0$. We proceed to solve these equations by the use of standard techniques in contour integration.

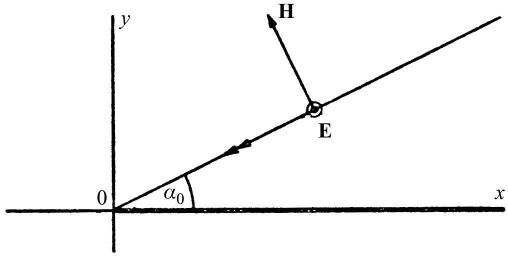

Fig. 11.6 A plane wave incident on a perfectly conducting half-plane.

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In the integral on the left-hand side of (3) x is negative. Hence, by Jordan's lemma,* provided $P(\mu) \to 0$ as $|\mu| \to \infty$ when $0 \geqslant \arg\mu \geqslant -\pi$, we can close the path of integration with an infinite semicircle *below* the real axis without making any additional contribution to the integral. Thus we only require further that $P(\mu)$ should have no singularities in the half-plane below the path of integration for (3) to be satisfied, since the integral is then effectively round a closed contour within which the integrand is regular.

Likewise, in the integral on the left-hand side of (2) x is positive, and we can close the path of integration with an infinite semicircle *above* the real axis without making any additional contribution to the integral on the assumption that $P(\mu)/\sqrt{1-\mu^2} \to 0$ as $|\mu| \to \infty$ when $\pi \geqslant \arg\mu \geqslant 0$. Then if $U(\mu)$ is any function which is free of singularities in the half-plane above the path of integration and has appropriate behaviour as $|\mu| \to \infty$ therein, (2) is clearly satisfied by

$$\frac{P(\mu)}{\sqrt{1-\mu^2}} = -\frac{1}{2\pi\mathrm{i}}\frac{U(\mu)}{U(-\mu_0)}\frac{1}{(\mu+\mu_0)} \tag{4}$$

if the path of integration be indented below the pole at $\mu = -\mu_0$, as shown diagrammatically in Fig. 11.7. For the only relevant singularity of the function on the right-hand side of (4) is the pole at $\mu = -\mu_0$, with residue $-1/(2\pi\mathrm{i})$, and from Cauchy's residue theorem this will contribute to the integral in (2) precisely the term $-\exp(-\mathrm{i}kx\mu_0)$.

If we now rewrite (4) in the form

$$\frac{P(\mu)}{\sqrt{1-\mu}}(\mu+\mu_0) = -\frac{1}{2\pi\mathrm{i}}\frac{U(\mu)}{U(-\mu_0)}\sqrt{1+\mu} \tag{5}$$

it can be argued that each side of (5) is a constant. For the left-hand side is free of singularities in the half-plane below the path of integration and of algebraic growth at infinity therein, whereas the right-hand side has the same characteristics in the half-plane above the path of integration. The function with which both sides are identical is thus free of singularities and of algebraic growth at infinity over the whole complex μ-plane: it must therefore be a polynomial, and since for some values of $\arg\mu$, $P(\mu) \to 0$ as $|\mu| \to \infty$, this polynomial can contain only a constant term.

The value of the constant is straightaway found by putting $\mu = -\mu_0$ in the right-hand side of (5), whence

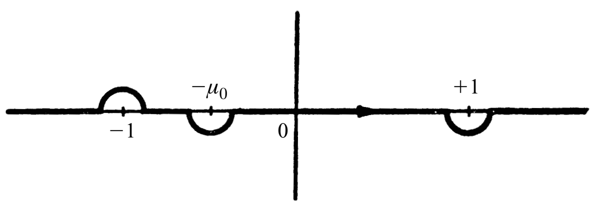

Fig. 11.7 The path of integration in the complex μ plane.

* E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge, Cambridge University Press, 1920), p. 115.

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$$P(\mu) = \frac{\mathrm{i}}{2\pi}\frac{\sqrt{1-\mu_0}\sqrt{1-\mu}}{\mu+\mu_0}, \tag{6}$$

or

$$P(\cos\alpha) = \frac{\mathrm{i}}{\pi}\frac{\sin\frac{1}{2}\alpha_0\sin\frac{1}{2}\alpha}{\cos\alpha+\cos\alpha_0}. \tag{7}$$

The significance of the symmetry of (7) in α and α_0 is mentioned at the end of §11.7.1.

The components of the scattered field follow from §11.4 (10)–(12) with the value (7) for $P(\cos\alpha)$, and hence the total field is given by

$$E_z = \mathrm{e}^{-\mathrm{i}kr\cos(\theta-\alpha_0)} - \frac{1}{\mathrm{i}\pi}\int_C \frac{\sin\frac{1}{2}\alpha_0\sin\frac{1}{2}\alpha}{\cos\alpha+\cos\alpha_0}\mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{8}$$

with the upper sign for $y>0$, and the lower sign for $y<0$. This completes the actual solution, and it only remains to cast it into a more useful form.

11.5.2 Expression of the solution in terms of Fresnel integrals

When kr is large, i.e. for distances greater than a wavelength or so from the origin, the evaluation of integrals of the general type

$$\int P(\cos\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha \tag{9}$$

may be attempted by the method of steepest descent (see Appendix III). The preliminary step in this procedure is to distort the path of integration (making due allowance for the presence of any singularities in the integrand) into that of steepest descents, $S(\theta)$ say, through the saddle-point at $\alpha=\theta$. The path $S(\theta)$ is shown in Fig. 11.8; along it the new variable

$$\tau = \sqrt{2}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\sin\tfrac{1}{2}(\alpha-\theta) \tag{10}$$

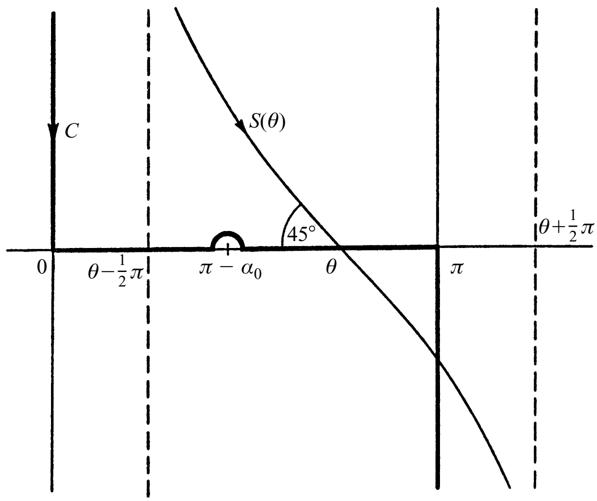

Fig. 11.8 The steepest descent path $S(\theta)$ in the complex α plane.

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goes through real values from $-\infty$ to ∞. The integral (9) then appears in the form

$$\sqrt{2}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\mathrm{e}^{\mathrm{i}kr}\int_{-\infty}^{\infty}\frac{P(\cos\alpha)}{\sqrt{1+\frac{1}{2}\mathrm{i}\tau^2}}\mathrm{e}^{-kr\tau^2}\,\mathrm{d}\tau, \tag{11}$$

from which asymptotic approximations for $kr \gg 1$ can be obtained.

The application of this procedure to the particular integral in (8) in fact leads, without approximation, to its expression in terms of Fresnel integrals. This will now be shown.

Consider, first, the case $0 < \theta < \pi$. Since (7) can be put in the form

$$P(\cos\alpha) = \frac{1}{4\pi\mathrm{i}}[\sec\tfrac{1}{2}(\alpha-\alpha_0) - \sec\tfrac{1}{2}(\alpha+\alpha_0)], \tag{12}$$

it is sufficient to evaluate

$$\int_{S(\theta)}\sec\tfrac{1}{2}(\alpha-\alpha_0)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha, \tag{13}$$

as the contribution from $\sec\frac{1}{2}(\alpha+\alpha_0)$ can subsequently be written down by changing the sign of α_0. Now by simple transformations (13) is

$$\begin{aligned}
&\int_{S(0)}\sec\tfrac{1}{2}(\alpha-\alpha_0+\theta)\mathrm{e}^{\mathrm{i}kr\cos\alpha}\,\mathrm{d}\alpha \\
&\qquad = \tfrac{1}{2}\int_{S(0)}[\sec\tfrac{1}{2}(\alpha-\alpha_0+\theta)+\sec\tfrac{1}{2}(\alpha+\alpha_0-\theta)]\mathrm{e}^{\mathrm{i}kr\cos\alpha}\,\mathrm{d}\alpha \\
&\qquad = 2\int_{S(0)}\frac{\cos\frac{1}{2}(\alpha_0-\theta)\cos\frac{1}{2}\alpha}{\cos\alpha+\cos(\alpha_0-\theta)}\mathrm{e}^{\mathrm{i}kr\cos\alpha}\,\mathrm{d}\alpha,
\end{aligned} \tag{14}$$

and using the substitution

$$\tau = \sqrt{2}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\sin\tfrac{1}{2}\alpha$$

(14) becomes

$$-2\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\,\mathrm{e}^{\mathrm{i}kr}\,\eta\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-kr\tau^2}}{\tau^2-\mathrm{i}\eta^2}\,\mathrm{d}\tau, \tag{15}$$

where

$$\eta = \sqrt{2}\cos\tfrac{1}{2}(\theta-\alpha_0).$$

But

$$\int_{-\infty}^{\infty}\mathrm{e}^{-\xi\tau^2}\,\mathrm{d}\tau = \sqrt{\frac{\pi}{\xi}},$$

whence multiplication by $\exp(\mathrm{i}\eta^2\xi)$ followed by integration over ξ from kr to infinity gives

$$\mathrm{e}^{\mathrm{i}kr\eta^2}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-kr\tau^2}}{\tau^2-\mathrm{i}\eta^2}\,\mathrm{d}\tau = \sqrt{\pi}\int_{kr}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\eta^2\xi}}{\sqrt{\xi}}\,\mathrm{d}\xi = \frac{2\sqrt{\pi}}{|\eta|}\int_{|\eta|\sqrt{kr}}^{\infty}\mathrm{e}^{\mathrm{i}\mu^2}\,\mathrm{d}\mu. \tag{16}$$

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Or, introducing the notation

$$F(a) = \int_a^\infty \mathrm{e}^{\mathrm{i}\mu^2}\,\mathrm{d}\mu \tag{17}$$

for a form of the complex Fresnel integral,*

$$\eta \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-kr\tau^2}}{\tau^2 - \mathrm{i}\eta^2}\,\mathrm{d}\tau = \pm 2\sqrt{\pi}\mathrm{e}^{-\mathrm{i}kr\eta^2} F\{\pm\eta\sqrt{kr}\}, \tag{18}$$

with the upper sign for $\eta > 0$, the lower sign for $\eta < 0$.

Combining these results we finally have, for $y \geqslant 0$,

$$\begin{aligned} -\frac{1}{\mathrm{i}\pi}\int_{S(\theta)} \frac{\sin\frac{1}{2}\alpha_0 \sin\frac{1}{2}\alpha}{\cos\alpha + \cos\alpha_0}\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha = -\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\Big\{&\mathrm{e}^{-\mathrm{i}kr\cos(\theta-\alpha_0)}F[\sqrt{2kr}\cos\tfrac{1}{2}(\theta-\alpha_0)] \\ &\mp \mathrm{e}^{-\mathrm{i}kr\cos(\theta+\alpha_0)}F[\pm\sqrt{2kr}\cos\tfrac{1}{2}(\theta+\alpha_0)]\Big\}, \end{aligned} \tag{19}$$

with the upper sign for $\theta + \alpha_0 < \pi$, the lower sign for $\theta + \alpha_0 > \pi$.

To get the complete field from (8) it only remains to take account of the simple pole at $\alpha = \pi - \alpha_0$. For $0 \leqslant \theta \leqslant \pi$, it is easily verified that in distorting the path C to the path $S(\theta)$ the sectors at infinity make no contribution, and it is clear from Fig. 11.8 that the pole is captured if and only if $\pi - \alpha_0 > \theta$. Its contribution, obtained from the residue theorem, is then

$$-\mathrm{e}^{-\mathrm{i}kr\cos(\theta+\alpha_0)}. \tag{20}$$

In other words, it is the reflected wave of *geometrical optics*, the discontinuity in which across $\theta = \pi - \alpha_0$ exactly counterbalances that in the *diffraction* field (19). In fact, invoking the relation

$$F(a) + F(-a) = \sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}, \tag{21}$$

the complete field (8) can be written in the form

$$\begin{aligned} E_z = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\Big\{&\mathrm{e}^{-\mathrm{i}kr\cos(\theta-\alpha_0)}F[-\sqrt{2kr}\cos\tfrac{1}{2}(\theta-\alpha_0)] \\ &- \mathrm{e}^{-\mathrm{i}kr\cos(\theta+\alpha_0)}F[-\sqrt{2kr}\cos\tfrac{1}{2}(\theta+\alpha_0)]\Big\}, \end{aligned} \tag{22}$$

which is Sommerfeld's famous result.

When $y < 0$ the integral which has to be evaluated is

$$\int_C \frac{\sin\frac{1}{2}\alpha_0 \sin\frac{1}{2}\alpha}{\cos\alpha + \cos\alpha_0}\mathrm{e}^{\mathrm{i}kr\cos(\theta+\alpha)}\,\mathrm{d}\alpha.$$

The appropriate steepest descents path is now $S(2\pi - \theta)$, and the capture of the pole at $\alpha = \pi - \alpha_0$, which occurs only for $\theta > \pi + \alpha_0$, yields minus the incident wave. The complete field is again given by (22).

To obtain the corresponding expressions for the components of **H** is merely a matter

* This form of the Fresnel integral is more convenient here than those defined in §8.7 (12); the change in limits should be noted.

of differentiation, as shown in §11.4.1. Both the Cartesian components H_x, H_y, and the polar components H_r, H_θ are of interest: in view of the fact that (22) is in terms of r, θ, it is convenient to derive the latter first, from the Maxwell equations

$$H_r = \frac{1}{\mathrm{i}kr}\frac{\partial E_z}{\partial \theta}, \qquad H_\theta = -\frac{1}{\mathrm{i}k}\frac{\partial E_z}{\partial r}, \tag{23}$$

and then to deduce the former from the relations

$$H_x = \cos\theta\, H_r - \sin\theta\, H_\theta, \qquad H_y = \sin\theta\, H_r + \cos\theta\, H_\theta.$$

The following notation is introduced in order to make the results more compact:

$$u = -\sqrt{2kr}\cos\tfrac{1}{2}(\theta - \alpha_0), \qquad v = -\sqrt{2kr}\cos\tfrac{1}{2}(\theta + \alpha_0), \tag{24}$$

$$G(a) = \mathrm{e}^{-\mathrm{i}a^2}F(a). \tag{25}$$

Note that

$$\frac{\mathrm{d}G(a)}{\mathrm{d}a} = -1 - 2\mathrm{i}aG(a).$$

The expression (22) then appears in the form

$$E_z = \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kr}\{G(u) - G(v)\}, \tag{26}$$

and it follows that

$$\left.\begin{aligned} H_r &= \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kr}\left\{\sin(\theta-\alpha_0)G(u) - \sin(\theta+\alpha_0)G(v) - \mathrm{i}\sqrt{\frac{2}{kr}}\sin\tfrac{1}{2}\alpha_0\cos\tfrac{1}{2}\theta\right\},\\ H_\theta &= \frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kr}\left\{\cos(\theta-\alpha_0)G(u) - \cos(\theta+\alpha_0)G(v) + \mathrm{i}\sqrt{\frac{2}{kr}}\sin\tfrac{1}{2}\alpha_0\sin\tfrac{1}{2}\theta\right\},\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} H_x &= -\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kr}\left\{\sin\alpha_0[G(u)+G(v)] + \mathrm{i}\sqrt{\frac{2}{kr}}\sin\tfrac{1}{2}\alpha_0\cos\tfrac{1}{2}\theta\right\},\\ H_y &= -\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}kr}\left\{\cos\alpha_0[G(u)-G(v)] - \mathrm{i}\sqrt{\frac{2}{kr}}\sin\tfrac{1}{2}\alpha_0\sin\tfrac{1}{2}\theta\right\}.\end{aligned}\right\} \tag{28}$$

11.5.3 The nature of the solution

We now examine, in some detail, the nature of the results given in §11.5.2. It is evident from their derivation, and can be verified directly, that $G(u)\exp(\mathrm{i}kr)$ is itself a solution of the two-dimensional wave equation, for any value of α_0; the noteworthy point being that it has periodicity 4π, so that $G(u) - G(v)$ vanishes on $\theta = 0$, $\theta = 2\pi$, the two faces of the screen, but does not vanish on $\theta = \pi$; Sommerfeld, indeed, arrived at his result (22) by seeking an appropriate solution of the wave equation of period 4π and combining it with its 'image'.* Incidentally, it follows from (28) that

* A good account of this approach is given in B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, Clarendon Press, 2nd edition, 1950), Chapter 4.

$$\cos\tfrac{1}{2}\theta\,\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}, \qquad \sin\tfrac{1}{2}\theta\,\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}$$

are also solutions of the two-dimensional wave equation, a result which is well known.

The other aspect of (26) which should be examined is its behaviour as $r \to \infty$. This is a straightforward matter and is the main topic of the subsequent discussion in the present section.

A very attractive feature of the half-plane problem is that the field can be evaluated at any point from tables of the Fresnel integrals.* Furthermore, in two cases of particular interest, namely $kr \gg 1$ and $kr \ll 1$, simple approximations (mentioned in §8.7.2) to the Fresnel integrals are available. The former condition is, of course, always satisfied in optical experiments, where the point of observation is likely to be millions of wavelengths from the diffracting edge; the latter condition arises in connection with the behaviour of the field in the vicinity of a sharp edge and can be studied at centimetre radio wavelengths (see §11.5.6).

(*a*) $kr \gg 1$. In this case $|u|$ and $|v|$ are large compared to unity except for values of θ sufficiently close to $\pi + \alpha_0$ and $\pi - \alpha_0$, respectively. To be precise, we introduce five regions as shown in Fig. 11.9. The equations of the curves bounding regions II and IV are taken to be $u^2 = 1$ and $v^2 = 1$ respectively, so that the curves are parabolae with foci at the origin and axes $\theta = \pi + \alpha_0$, $\theta = \pi - \alpha_0$. Well within region II (that is to say, inside a parabola $u^2 = \varepsilon$, where $\varepsilon \ll 1$) $|u| \ll 1$; well outside region II (that is to say, outside a parabola $u^2 = \gamma$, where $\gamma \gg 1$) $|u| \gg 1$; and similarly with $|v|$ and region IV. Furthermore, for $0 < \theta < \pi - \alpha_0$, u and v are both negative; for $\pi - \alpha_0 < \theta < \pi + \alpha_0$, u is negative but v is positive; and for $\pi + \alpha_0 < \theta < 2\pi$, u and v are both positive.

Regions I, III and V are obviously closely connected with those which would arise from a geometrical optics treatment according to which the light travels in straight lines; namely, as illustrated in Fig. 11.10, the *shadow* sector behind the screen where there is no field at all, the *illuminated* sector where there is the incident plane wave

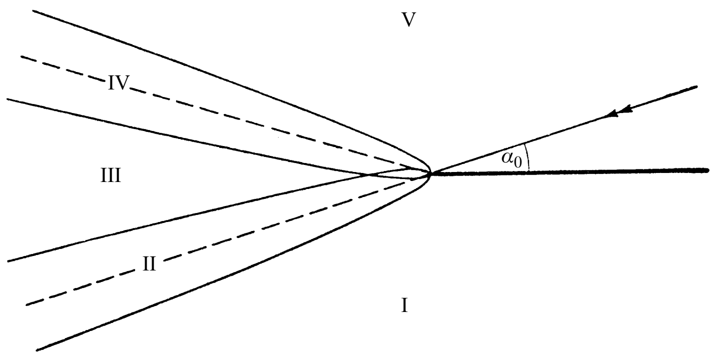

Fig. 11.9 Diffraction of a plane wave by a perfectly conducting half-plane. The five regions in terms of which the behaviour of the field may be described.

* The most convenient for this purpose seem to be those given by R. A. Rankin, *Phil. Trans. Roy. Soc.* A, **241** (1949), 457.

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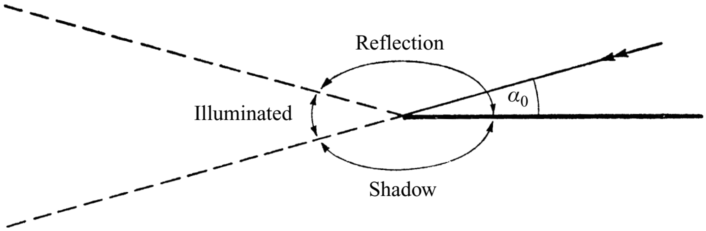

Fig. 11.10 Diffraction of a plane wave by a perfectly conducting half-plane. The three regions of geometrical optics.

only, and the *reflection* sector where there is the incident plane wave together with the reflected plane wave appropriate to reflection at an *infinite* screen. In fact, broadly speaking, regions II and IV are those in which the exact solution effects a smooth transition from the geometrical field in one sector to that in an adjacent sector. To see this in more detail we must pause to derive an asymptotic approximation to the Fresnel integral for large values of the argument.

If a is positive we write

$$G(a) = \mathrm{e}^{-\mathrm{i}a^2} \int_a^\infty \mathrm{d}(\mathrm{e}^{\mathrm{i}\mu^2})/2\mathrm{i}\mu$$

and integrate by parts twice and find that

$$G(a) = \frac{\mathrm{i}}{2a} + \frac{1}{4a^3} - \frac{3}{4}\mathrm{e}^{-\mathrm{i}a^2} \int_a^\infty \frac{\mathrm{e}^{\mathrm{i}\mu^2}}{\mu^4}\,\mathrm{d}\mu. \tag{29}$$

A continuation of this process would, in fact, yield a complete asymptotic expansion of $G(a)$, but for our present purposes we merely observe that the modulus of the integral in the last term of (29) is less than

$$\int_a^\infty \mathrm{d}\mu/\mu^4 = 1/3a^3, \tag{30}$$

and consequently

$$G(a) = \frac{\mathrm{i}}{2a} + O\left(\frac{1}{a^3}\right). \tag{31}$$

It is worth noting that this result could also have been obtained from the general method outlined in §11.5.2, which here consists in expanding the factor $(\tau^2 - \mathrm{i}\eta^2)^{-1}$ in the integrand of (18) as a power series in τ, and then integrating term by term.

If a is negative the left-hand side of (30) diverges, but this case can easily be handled by using (21) in conjunction with the result for a positive argument. Thus

$$G(a) = \sqrt{\pi}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}a^2} + \frac{\mathrm{i}}{2a} + O\left(\frac{1}{a^3}\right). \tag{32}$$

The fact that the asymptotic approximations (31) for a positive and (32) for a negative are different is a particular example of the Stokes phenomenon.*

* G. G. Stokes, *Trans. Camb. Phil. Soc.*, **10** (1864), 105. For a modern discussion of this phenomenon see, for example, R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic Press, New York, 1973), Secs. 1.2 and 21.6.

Now let us write

$$E_z = E_z^{(g)} + E_z^{(d)},$$

where $E_z^{(g)}$ is the geometrical optics field given by

$$E_z^{(g)} = \begin{cases} e^{-ikr\cos(\theta-\alpha_0)} - e^{-ikr\cos(\theta+\alpha_0)} & \text{for } 0 \leqslant \theta < \pi - \alpha_0, \\ e^{-ikr\cos(\theta-\alpha_0)} & \text{for } \pi - \alpha_0 < \theta < \pi + \alpha_0, \\ 0 & \text{for } \pi + \alpha_0 < \theta \leqslant 2\pi, \end{cases} \tag{33}$$

and $E_z^{(d)}$ is the *diffraction* field, which is simply the field which must be added to that of geometrical optics to give the complete field. Then, for $kr \gg 1$, the application of (31) and (32) to (26) gives

$$E_z^{(d)} \sim \sqrt{\frac{2}{\pi}} e^{\frac{1}{4}i\pi} \frac{\sin\frac{1}{2}\alpha_0 \sin\frac{1}{2}\theta}{(\cos\theta + \cos\alpha_0)} \frac{e^{ikr}}{\sqrt{kr}} \tag{34}$$

at points not too close to regions II and IV, in the sense indicated above. It is readily seen, from either (23) or (27), that the components of $\mathbf{H}^{(d)}$ to the same order of approximation as (34) are $H_\theta^{(d)} = -E_z^{(d)}$ and $H_r^{(d)} = 0$. Evidently (34) implies that the diffraction field behaves as though it originates from a line-source situated along the diffracting edge whose 'polar diagram' varies with angle as specified. This accords with the experimental fact that the diffracting edge, when viewed from the shadow sector for the sake of contrast, appears illuminated.

When $\cos\theta + \cos\alpha_0$ approaches zero, the approximation (34) breaks down and appeal must be made to the exact solution. Since

$$G(0) = \int_0^\infty e^{i\mu^2}\, d\mu = \tfrac{1}{2}\sqrt{\pi} e^{\frac{1}{4}i\pi}, \tag{35}$$

we see from (26) that, on $\theta = \pi + \alpha_0$,

$$E_z = \tfrac{1}{2} e^{ikr} + O\left\{\frac{1}{\sqrt{kr}}\right\}, \tag{36}$$

and on $\theta = \pi - \alpha_0$,

$$E_z = e^{ikr\cos(2\alpha_0)} - \tfrac{1}{2} e^{ikr} + O\left\{\frac{1}{\sqrt{kr}}\right\}. \tag{37}$$

Hence, near $\theta = \pi + \alpha_0$ and $\theta = \pi - \alpha_0$ the diffraction field is of the same order as the incident field. In particular, at infinity the transition between the geometrical optics fields in adjacent sectors is via their arithmetic mean.

Interference between the geometrical optics field and the diffraction field in regions where they are comparable gives rise to fringes. These are evident in Fig. 11.11, which is discussed in §11.5.5.

(*b*) $kr \ll 1$. In this case $|u|$ and $|v|$ are small compared to unity and series expansions of the Fresnel integral are useful. Writing

$$F(a) = \int_0^\infty e^{i\mu^2}\, d\mu - \int_0^a e^{i\mu^2}\, d\mu,$$

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and expanding the exponential in the integrand of the second integral, we have

$$F(a) = \tfrac{1}{2}\sqrt{\pi}e^{\frac{1}{4}i\pi} - a + O(a^3). \tag{38}$$

Hence, from (26) and (28), neglecting powers of kr greater than a half,

$$\left.\begin{aligned} E_z &= 2\sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}\sqrt{kr}\sin\tfrac{1}{2}\alpha_0\sin\tfrac{1}{2}\theta, \\ H_x &= -\sin\alpha_0 - \sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}\alpha_0\cos\tfrac{1}{2}\theta\left\{\frac{i}{\sqrt{kr}} + (1+2\cos\alpha_0)\sqrt{kr}\right\}, \\ H_y &= \sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}\alpha_0\sin\tfrac{1}{2}\theta\left\{-\frac{i}{\sqrt{kr}} + (1+2\cos\alpha_0)\sqrt{kr}\right\}. \end{aligned}\right\} \tag{39}$$

It should be noticed that E_z is finite and continuous at $r = 0$, but that H_x and H_y diverge like $r^{-1/2}$, except on $\theta = \pi$ when $H_x = -\sin\alpha_0\exp(ikr\cos\alpha_0)$, and $\theta = 0, 2\pi$ when $H_y = 0$. Such behaviour, peculiar in a physical problem, arises of course from the idealized concept of an infinitely sharp edge. The existence of singularities in the field components in this case must be taken into account in formulating any theorem about the uniqueness of the solution (see §11.9).

We conclude this investigation into the nature of the solution by examining the current density induced in the diffracting screen. This is $-c/4\pi$ times the difference in H_x at $\theta = 0$ and $\theta = 2\pi$; that is, from (28),

$$\frac{2\pi}{c}J_z = \sin\alpha_0\,e^{-ikx\cos\alpha_0} - \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{ikx}\left\{2\sin\alpha_0\,G[\sqrt{2kx}\cos\tfrac{1}{2}\alpha_0] - i\sqrt{\frac{2}{kx}}\sin\tfrac{1}{2}\alpha_0\right\}. \tag{40}$$

For $\sqrt{2kx}\cos\frac{1}{2}\alpha_0 \gg 1$, (40) reads

$$J_z = \frac{c}{2\pi}\sin\alpha_0\,e^{-ikx\cos\alpha_0} + O\{(kx)^{-3/2}\}. \tag{41}$$

This result is of interest in indicating the extent to which it is permissible to assume that the current density is that given by geometrical optics, a standard procedure in problems which cannot be solved exactly. Clearly the assumption is only reasonable for values of α_0 not near π, and benefits from the fact that the 'correction' term in (41) tends to zero as $x \to \infty$ like $x^{-3/2}$ rather than $x^{-1/2}$.

On the other hand, for $\sqrt{2kx}\cos\frac{1}{2}\alpha_0 \ll 1$,

$$J_z = \frac{c}{\pi\sqrt{2\pi}}e^{-\frac{1}{4}i\pi}\sin\tfrac{1}{2}\alpha_0\left\{\frac{i}{\sqrt{kx}} + 4\cos^2\tfrac{1}{2}\alpha_0\sqrt{kx}\right\}e^{ikx}, \tag{42}$$

which diverges at the diffracting edge.

11.5.4 The solution for H-polarization

The case of H-polarization, namely when the incident field is specified by

$$H_z^{(i)} = e^{-ikr\cos(\theta-\alpha_0)}, \tag{43}$$

can be treated in the same way as that of E-polarization, the analysis being, in fact,

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practically identical. Alternatively, the former can be deduced from the latter by invoking the exact electromagnetic form of Babinet's principle, given in §11.3, because the screen complementary to a half-plane is itself a half-plane. It turns out that the complete field is given by

$$H_z = \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}\Big\{e^{-ikr\cos(\theta-\alpha_0)}F[-\sqrt{2kr}\cos\tfrac{1}{2}(\theta-\alpha_0)] \\ +e^{-ikr\cos(\theta+\alpha_0)}F[-\sqrt{2kr}\cos\tfrac{1}{2}(\theta+\alpha_0)]\Big\}. \tag{44}$$

This differs from the corresponding expression for E_z for an E-polarized field, (22), only in the sign of the second term.

Using the notation of §11.5.2 the nonzero field components appear in the form

$$\left.\begin{aligned} H_z &= \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{ikr}\{G(u)+G(v)\}, \\ E_x &= \frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{ikr}\left\{\sin\alpha_0[G(u)-G(v)]+i\sqrt{\frac{2}{kr}}\cos\tfrac{1}{2}\alpha_0\sin\tfrac{1}{2}\theta\right\}, \\ E_y &= -\frac{e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{ikr}\left\{\cos\alpha_0[G(u)+G(v)]+i\sqrt{\frac{2}{kr}}\cos\tfrac{1}{2}\alpha_0\cos\tfrac{1}{2}\theta\right\}. \end{aligned}\right\} \tag{45}$$

Clearly E_x vanishes on $\theta = 0$ and $\theta = 2\pi$, and the behaviour of the field for $kr \gg 1$ can again be interpreted in terms of a diffraction field which appears to originate in a line-source along the diffracting edge for points sufficiently far from $\theta = \pi - \alpha_0$ and $\theta = \pi + \alpha_0$. As $r \to 0$, H_z remains finite and continuous, whereas E_x and E_y diverge like $r^{-1/2}$, except in so far as $E_x = 0$ for $\theta = 0, 2\pi$, and $E_y = -\cos\alpha_0 \exp(ikr\cos\alpha_0)$ for $\theta = \pi$.

The current density is given by

$$\frac{2\pi}{c}J_x = e^{-ikx\cos\alpha_0} - \frac{2e^{-\frac{1}{4}i\pi}}{\sqrt{\pi}}e^{ikx}G\{\sqrt{2kx}\cos\tfrac{1}{2}\alpha_0\}. \tag{46}$$

For $\sqrt{(2kx)}\cos\frac{1}{2}\alpha_0 \gg 1$,

$$\frac{2\pi}{c}J_x = e^{-ikx\cos\alpha_0} - \frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}}\sec\tfrac{1}{2}\alpha_0\frac{e^{ikx}}{\sqrt{kx}}, \tag{47}$$

which approximates less rapidly to the current density of geometrical optics than in the case of E-polarization. For $\sqrt{2kx}\cos\frac{1}{2}\alpha_0 \ll 1$,

$$J_x = \frac{c}{\pi}\sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}\cos\tfrac{1}{2}\alpha_0\sqrt{kx}e^{ikx}. \tag{48}$$

This vanishes at $x = 0$, so that, as might be expected, at the edge itself there is no current normal to the edge.

11.5.5 Some numerical calculations

A typical theoretical curve, obtained from (26), is shown in Fig. 11.11. It is for a normally incident E-polarized plane wave of amplitude unity, and is a plot of the

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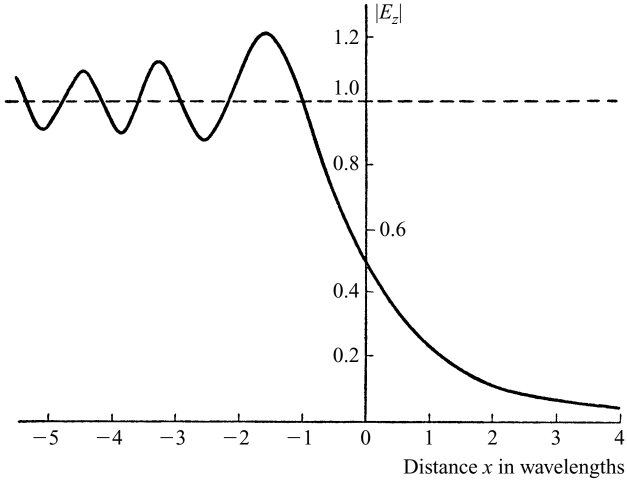

Fig. 11.11 Diffraction of a normally incident E-polarized plane wave of amplitude unity by a perfectly conducting half-plane. The variation of $|E_z|$ with x at a distance of three wavelengths behind the screen.

amplitude of E_z against x at a distance of three wavelengths behind the screen ($ky = -6\pi$). It puts in evidence the diffraction fringes in the illuminated region, and the monotonic decay with deeper penetration into the shadow region.

Some interesting calculations have been made by Braunbek and Laukien.* For a normally incident H-polarized plane wave of amplitude unity they give contours of equal amplitude (Fig. 11.12) and equal phase (Fig. 11.13) of H_z for the region within about a wavelength of the diffracting edge. They also give the lines of average energy flow (Fig. 11.14) which are orthogonal to the phase contours. That this is the case for *any* two-dimensional H-polarized field is easily proved: write $H_z = h\mathrm{e}^{\mathrm{i}\phi}$, where h and ϕ are real; then using the relations

$$E_x = -\frac{1}{\mathrm{i}k}\frac{\partial H_z}{\partial y}, \qquad E_y = \frac{1}{\mathrm{i}k}\frac{\partial H_z}{\partial x},$$

the averaged Poynting vector (§1.4 (56))

$$\frac{c}{8\pi}\mathcal{R}(\mathbf{E} \times \mathbf{H}^\star) = \frac{c}{8\pi}\mathcal{R}(E_y H_z^\star, -E_x H_z^\star, 0)$$

is seen to be

$$\frac{c}{8\pi}\frac{h^2}{k}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, 0\right), \tag{49}$$

which is orthogonal to the surfaces $\phi =$ constant. A corresponding result holds for any E-polarized field.

* W. Braunbek and G. Laukien, *Optik*, **9** (1952), 174.

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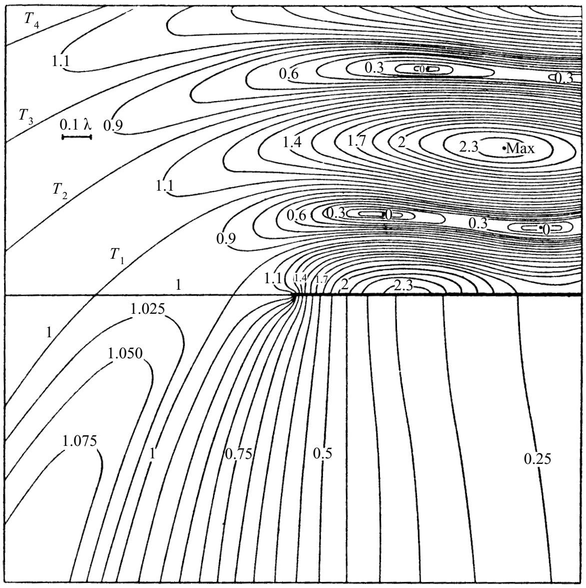

Fig. 11.12 Amplitude contours of H_z (amplitude of incident wave is taken as unity) for diffraction of a normally incident *H*-polarized plane wave by a perfectly conducting half-plane. (After W. Braunbek and G. Laukien, *Optik*, **9** (1952), 174.)

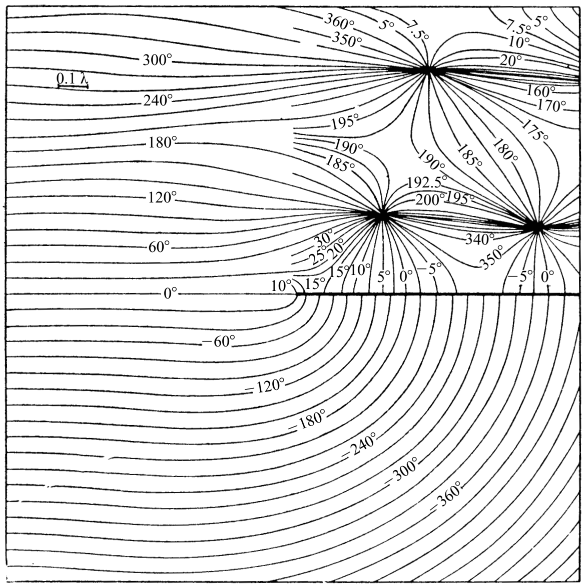

Fig. 11.13 Phase contours of H_z for diffraction of a normally incident *H*-polarized plane wave by a perfectly conducting half-plane. (After W. Braunbek and G. Laukien, *Optik*, **9** (1952), 174.)

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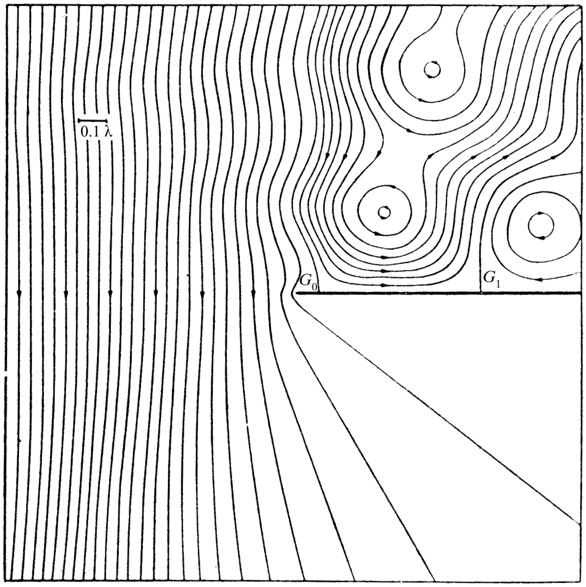

Fig. 11.14 Lines of average energy flow for diffraction of a normally incident H-polarized plane wave by a perfectly conducting half-plane. (After W. Braunbek and G. Laukien, *Optik*, **9** (1952), 174.)

11.5.6 Comparison with approximate theory and with experimental results

For points at a great distance from the diffracting edge in the illuminated part of region II (Fig. 11.9), where the fringes appear, the second term in each of the solutions (22) and (44) can be neglected. The intensity for both E- and H-polarization, and hence also for unpolarized light, is therefore

$$\tfrac{1}{2}\left[\tfrac{1}{2}+\mathcal{C}\left(2\sqrt{\frac{2r}{\lambda}}\cos\tfrac{1}{2}(\theta-\alpha_0)\right)\right]^2+\tfrac{1}{2}\left[\tfrac{1}{2}+\mathcal{S}\left(2\sqrt{\frac{2r}{\lambda}}\cos\tfrac{1}{2}(\theta-\alpha_0)\right)\right]^2, \tag{50}$$

where λ is the wavelength and $\mathcal{C}$, $\mathcal{S}$ are the Fresnel 'cosine' and 'sine' integrals defined by §8.7 (12). This should be compared with the analogous result §8.7 (28) for a black half-plane on the Fresnel–Kirchhoff theory. It has indeed been suggested* that the first term of the exact solution for the perfectly conducting half-plane could be regarded as giving the solution for a black half-plane.

Well into region I, the shadow region, the E-polarization field is given by (34), namely

$$E_z=\sqrt{\frac{2}{\pi}}\,e^{\frac{1}{4}i\pi}\,\frac{\sin\frac{1}{2}\alpha_0\sin\frac{1}{2}\theta}{(\cos\alpha_0+\cos\theta)}\,\frac{e^{ikr}}{\sqrt{kr}}. \tag{51}$$

* See the account in B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, Clarendon Press, 2nd edition, 1950), p. 149 *et seq*.

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It can likewise be shown that the H-polarization field there is

$$H_z = -\sqrt{\frac{2}{\pi}}\, e^{\frac{1}{4}i\pi} \frac{\cos\frac{1}{2}\alpha_0 \cos\frac{1}{2}\theta}{(\cos\alpha_0 + \cos\theta)} \frac{e^{ikr}}{\sqrt{kr}}. \tag{52}$$

The corresponding ratio of the field strengths is therefore

$$\frac{E\text{-polarization}}{H\text{-polarization}} = -\tan\tfrac{1}{2}\alpha_0 \tan\tfrac{1}{2}\theta, \tag{53}$$

and unpolarized incident light will accordingly become partly polarized on diffraction. These results are in broad agreement with optical experiments.*

Developments in microwave radio techniques provide excellent opportunities for the experimental study of the diffraction of electromagnetic waves. In particular, a diffracting screen can be used which is much nearer to the idealization of a perfectly conducting half-plane than can be realized in optical measurements, and the field in the neighbourhood of the diffracting edge can be examined. A number of measurements have been made, mainly on a wavelength of about 3 cm, which show good agreement between theory and experiment.†

11.6 Three-dimensional diffraction of a plane wave by a half-plane

In §11.5 we solved, in effect, the problem of diffraction by a half-plane of a plane wave which was arbitrary except for the restriction that its direction of propagation was normal to the diffracting edge. It will now be shown that, by a simple device, the previous results can be extended to yield the solution for a completely arbitrary incident plane wave.

Let the incident plane wave be characterized by the phase factor

$$e^{-ikS} = e^{-ik(x\cos\alpha\cos\beta + y\sin\alpha\cos\beta + z\sin\beta)}, \tag{1}$$

where, as before, the perfectly conducting screen occupies $y = 0$, $x > 0$. The angles α and β, which specify the direction of propagation, are shown in Fig. 11.15.

Now we note that (1) is obtained from the two-dimensional form corresponding to $\beta = 0$ on first replacing k by $k\cos\beta$ and then multiplying by $\exp(-ikz\sin\beta)$. In fact, this procedure applied to any two-dimensional solution of the wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0 \tag{2}$$

clearly yields a solution of the three-dimensional wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + k^2 V = 0 \tag{3}$$

in which z enters only via the factor $\exp(-ikz\sin\beta)$. Moreover, if U, say, is such a

* See Wolfsohn's article in *Handbuch der Physik*, Vol. 20 (Berlin, Springer, 1928), p. 275, and J. Savornin, *Ann. de Physique*, **11** (1939), 129.

† C. W. Horton and R. B. Watson, *J. Appl. Phys.*, **21** (1950), 16; B. N. Harden, *Proc. Inst. Elec. Engrs.*, **99**, Pt. III (1952), 229; R. D. Kodis, *J. Appl. Phys.*, **23** (1952), 249; R. V. Row, *J. Appl. Phys.*, **24** (1953), 1448.

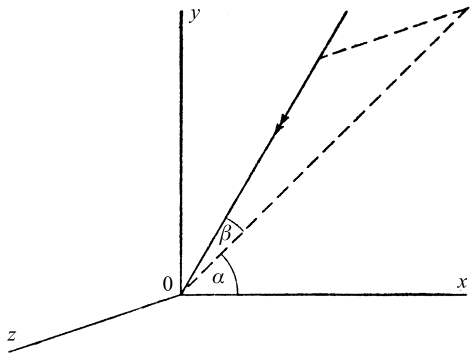

Fig. 11.15 The direction of propagation of the incident plane wave.

solution of (3), then it is easily verified that two electromagnetic fields satisfying Maxwell's equations are

$$\left.\begin{aligned}\mathbf{E} &= \left(-\frac{\mathrm{i}\sin\beta}{k}\frac{\partial U}{\partial x}, -\frac{\mathrm{i}\sin\beta}{k}\frac{\partial U}{\partial y}, \cos^2\beta U\right),\\ \mathbf{H} &= \left(-\frac{\mathrm{i}}{k}\frac{\partial U}{\partial y}, \frac{\mathrm{i}}{k}\frac{\partial U}{\partial x}, 0\right),\end{aligned}\right\}\tag{4}$$

and

$$\left.\begin{aligned}\mathbf{E} &= \left(\frac{\mathrm{i}}{k}\frac{\partial U}{\partial y}, -\frac{\mathrm{i}}{k}\frac{\partial U}{\partial x}, 0\right),\\ \mathbf{H} &= \left(-\frac{\mathrm{i}\sin\beta}{k}\frac{\partial U}{\partial x}, -\frac{\mathrm{i}\sin\beta}{k}\frac{\partial U}{\partial y}, \cos^2\beta U\right).\end{aligned}\right\}\tag{5}$$

When $\beta = 0$, (4) gives a two-dimensional field which is E-polarized, (5) one which is H-polarized.

If we take expression (1) for U, (4) and (5) yield respectively the two plane waves

$$\left.\begin{aligned}\mathbf{E} &= (-\cos\alpha\sin\beta, -\sin\alpha\sin\beta, \cos\beta)\mathrm{e}^{-\mathrm{i}kS},\\ \mathbf{H} &= (-\sin\alpha, \cos\alpha, 0)\mathrm{e}^{-\mathrm{i}kS},\end{aligned}\right\}\tag{6}$$

and

$$\left.\begin{aligned}\mathbf{E} &= (\sin\alpha, -\cos\alpha, 0)\mathrm{e}^{-\mathrm{i}kS},\\ \mathbf{H} &= (-\cos\alpha\sin\beta, -\sin\alpha\sin\beta, \cos\beta)\mathrm{e}^{-\mathrm{i}kS},\end{aligned}\right\}\tag{7}$$

where a factor $\cos\beta$ has been removed throughout. Now *any* plane wave with space variation (1) is specified by two components of $\mathbf{E}$ (or $\mathbf{H}$), because the third component would follow from $\operatorname{div}\mathbf{E} = 0$ (or $\operatorname{div}\mathbf{H} = 0$). Hence *any* plane wave can be formed by suitable superposition of (6) and (7); with the consequence that, in the diffraction problem, attenuation can be confined, without loss of generality, to the two cases in which they are the respective incident fields.

It should now be clear that the solution to the diffraction problem with (6) as the incident wave is given by (4) with U obtained from the known expression for $E_z \sec\beta$ in the two-dimensional case on first replacing k by $k\cos\beta$ and secondly multiplying

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by $\exp(-\mathrm{i}kz\sin\beta)$: for the fact that $U=0$ on $y=0$, $x>0$ implies also that $\partial U/\partial x=0$ there, whence from (4) $E_x=E_z=0$ on the screen, as required. Explicitly, we have, from §11.5 (24) and §11.5 (26)

$$U=\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\sec\beta\,\mathrm{e}^{\mathrm{i}k(r\cos\beta-z\sin\beta)}\{G(p)-G(q)\}, \tag{8}$$

where

$$p=-\sqrt{2kr\cos\beta}\cos\tfrac{1}{2}(\theta-\alpha), \qquad q=-\sqrt{2kr\cos\beta}\cos\tfrac{1}{2}(\theta+\alpha). \tag{9}$$

Thus, from (4),

$$\left.\begin{aligned}
E_z&=\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\cos\beta\mathrm{e}^{\mathrm{i}k(r\cos\beta-z\sin\beta)}\{G(p)-G(q)\},\\
H_x&=-\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}k(r\cos\beta-z\sin\beta)}\left\{\sin\alpha[G(p)+G(q)]+\mathrm{i}\sqrt{\frac{2}{kr\cos\beta}}\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\theta\right\},\\
H_y&=\frac{\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}}{\sqrt{\pi}}\mathrm{e}^{\mathrm{i}k(r\cos\beta-z\sin\beta)}\left\{\cos\alpha[G(p)-G(q)]-\mathrm{i}\sqrt{\frac{2}{kr\cos\beta}}\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\theta\right\},\\
E_x&=-H_y\sin\beta, \qquad E_y=H_x\sin\beta, \qquad H_z=0.
\end{aligned}\right\} \tag{10}$$

When $\beta=0$ the expressions in (10) reduce at once to the corresponding expressions in §11.5 (26) and §11.5 (28).

Similar results can likewise be obtained for the case when the incident wave is given by (7), the appropriate two-dimensional solution being that for H-polarization, namely, the expression for H_z in §11.5 (44). As already pointed out, the solution for a quite arbitrary incident plane wave can therefore be derived. Furthermore, a simple generalization of the argument of §11.5.2 shows that the field radiated by *any* source distribution can be represented as a spectrum of plane waves. Thus, in principle, the solution of the diffraction problem for any source distribution can be built up from the solutions for the individual plane waves. Two cases of interest are a line-source parallel to the diffracting edge (a two-dimensional problem) and a point-source: these are treated in the next section.

11.7 Diffraction of a field due to a localized source by a half-plane

11.7.1 A line-current parallel to the diffracting edge

We shall consider a line-source situated at T, the point (r_0,θ_0), where $0\leqslant\theta_0\leqslant\pi$, which would, in free-space, radiate the E-polarized cylindrical wave

$$E_z^{(i)}=\sqrt{\frac{\pi}{2}}\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}H_0^{(1)}(kR)\sim\frac{\mathrm{e}^{\mathrm{i}kR}}{\sqrt{kR}}, \tag{1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zero-order and R is the distance measured from T (see Fig. 11.16).

The source is, in fact, an electric current through T flowing parallel to the z-axis and

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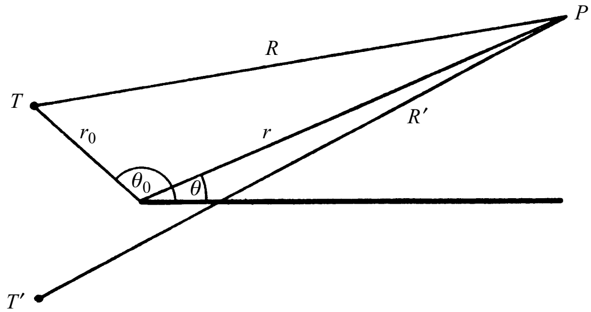

Fig. 11.16 The configuration for a line-source T in the presence of a diffracting half-plane.

oscillating everywhere with the same phase. It is well known that (1) is the fundamental solution of the two-dimensional wave equation representing a diverging wave which depends only on radial distance.

In order to write (1) as an angular spectrum of plane waves we adopt, in essence, Sommerfeld's integral representation of the Hankel function.* Since we require the individual plane waves of the spectrum to be incident on the screen, we consider the representation in the half-space $y < r_0 \sin\theta_0$, namely

$$H_0^{(1)}(kR) = \frac{1}{\pi}\int_{S(\frac{1}{4}\pi)} e^{ikr_0\cos(\theta_0-\alpha)} e^{-ikr\cos(\theta-\alpha)}\, d\alpha. \tag{2}$$

The particular path $S(\frac{1}{2}\pi)$ is chosen here because (apart from the fact that it is valid for all points in $y < r_0 \sin\theta_0$) it is parallel to the path of steepest descents, which proves a convenience in the subsequent analysis. The phase factor $\exp[ikr_0\cos(\theta_0-\alpha)]$ in the integrand of (2) appears because the plane waves of the spectrum must all have zero phase at T.

It is apparent, therefore, that the solution of the diffraction problem for the incident field (1) is obtained from that for an incident field $\exp[-ikr\cos(\theta-\alpha)]$ on multiplying by the factor

$$\frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}} e^{ikr_0\cos(\theta_0-\alpha)} \tag{3}$$

and integrating with respect to α over the path $S(\frac{1}{2}\pi)$.

Now the solution for the incident field $\exp[-ikr\cos(\theta-\alpha)]$, as was shown in §11.5, could be written

$$E_z^{(p)} = E_z^{(pg)} + E_z^{(pd)} \tag{4}$$

(the affix p being introduced here to indicate that the incident wave is plane), where $E_z{}^{(pg)}$ is given by §11.5 (33) and [see §11.5 (19)]

$$E_z^{(pd)} = \frac{i}{\pi}\int_{S(0)} \frac{\sin\frac{1}{2}\alpha\sin\frac{1}{2}(\theta+\beta)}{\cos\alpha+\cos(\theta+\beta)} e^{ikr\cos\beta}\, d\beta. \tag{5}$$

* See, for example, J. A. Stratton, *Electromagnetic Theory* (New York, McGraw-Hill, 1941), p. 367.

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The use of this form for $E_z^{(pd)}$ (rather than that in terms of Fresnel integrals) is adopted because, as will shortly be seen, it brings out the symmetry between α and β. The solution for the incident field (1) is thus

$$E_z = \frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}} \int_{S(\frac{1}{2}\pi)} E_z^{(pg)} e^{ikr_0 \cos(\theta_0 - \alpha)} \, d\alpha + \frac{e^{\frac{1}{4}i\pi}}{\sqrt{2\pi}} \int_{S(\frac{1}{2}\pi)} E_z^{(pd)} e^{ikr_0 \cos(\theta_0 - \alpha)} \, d\alpha. \qquad (6)$$

In order to separate (6) into geometrical optics and diffraction terms, the path of integration for α in the second integral must be displaced to $S(\theta_0)$. In this transposition of the path the poles of $E_z^{(pd)}$, regarded as a function of α, must be taken into account: from (5) they occur where $\cos\alpha = -\cos(\theta + \beta)$ for all β on $S(0)$, and it is easy to verify that the contribution of their residues combines with the first expression in (6) to yield the geometrical optics terms

$$E_z^{(g)} = \begin{cases} \sqrt{(\frac{1}{2}\pi)} e^{\frac{1}{4}i\pi} [H_0^{(1)}(kR) - H_0^{(1)}(kR')] & \text{for } 0 \leqslant \theta \leqslant \pi - \theta_0, \\ \sqrt{(\frac{1}{2}\pi)} e^{\frac{1}{4}i\pi} H_0^{(1)}(kR) & \text{for } \pi - \theta_0 < \theta < \pi + \theta_0, \\ 0 & \text{for } \pi + \theta_0 < \theta \leqslant 2\pi, \end{cases} \qquad (7)$$

where R' is distance measured from T', the image of T in the plane $y = 0$ (Fig. 11.16). The diffraction term may be written

$$E_z^{(d)} = \frac{e^{-\frac{1}{4}i\pi}}{\pi\sqrt{2\pi}} \int_{S(0)} \int_{S(0)} \frac{\sin\frac{1}{2}(\alpha + \theta_0) \sin\frac{1}{2}(\beta + \theta)}{\cos(\alpha + \theta_0) + \cos(\beta + \theta)} e^{ik(r_0 \cos\alpha + r\cos\beta)} \, d\alpha \, d\beta. \qquad (8)$$

It is convenient to express (8) as

$$E_z^{(d)} = \mathcal{J}(\theta_0) - \mathcal{J}(-\theta_0), \qquad (9)$$

where

$$\mathcal{J}(\theta_0) = -\frac{e^{-\frac{1}{4}i\pi}}{4\pi\sqrt{2\pi}} \int_{S(0)} \int_{S(0)} \frac{e^{ik(r_0\cos\alpha + r\cos\beta)}}{\cos\frac{1}{2}(\alpha + \beta + \theta_0 + \theta)} \, d\alpha \, d\beta. \qquad (10)$$

The final step is to reduce $\mathcal{J}(\theta_0)$ to a single integral. Multiplying the top and bottom of the integrand in (10) by $4\cos\frac{1}{2}(\alpha - \beta + \theta_0 + \theta)$ and discarding that part which is an odd function of β, we have

$$\mathcal{J}(\theta_0) = -\frac{e^{-\frac{1}{4}i\pi}}{8\pi\sqrt{2\pi}} \int_{S(0)} \int_{S(0)} \left[\frac{1}{\cos\frac{1}{2}(\alpha + \theta_0 + \theta) - \sin\frac{1}{2}\beta} + \frac{1}{\cos\frac{1}{2}(\alpha + \theta_0 + \theta) + \sin\frac{1}{2}\beta} \right]$$
$$\times \cos\tfrac{1}{2}\beta \, e^{ik(r_0\cos\alpha + r\cos\beta)} \, d\alpha \, d\beta. \qquad (11)$$

In the second term of the integrand in (11) change α to $-\alpha$ and then recombine the two terms. This gives

$$\mathcal{J}(\theta_0) = -\frac{e^{-\frac{1}{4}i\pi}}{2\pi\sqrt{2\pi}}$$
$$\times \int_{S(0)} \int_{S(0)} \frac{1}{N} [\cos\tfrac{1}{2}(\theta_0 + \theta) \cos\tfrac{1}{2}\alpha \cos\tfrac{1}{2}\beta \, e^{ik(r_0\cos\alpha + r\cos\beta)}] d\alpha \, d\beta, \qquad (12)$$

where

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$$N = (\cos\alpha - 1) + (\cos\beta - 1) - 4\sin\tfrac{1}{2}(\theta_0 + \theta)\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta + 2\cos^2\tfrac{1}{2}(\theta_0 + \theta).$$

In (12) make the 'steepest descent' substitutions

$$\xi = \sqrt{2}e^{\frac{1}{4}i\pi}\sin\tfrac{1}{2}\alpha, \qquad \eta = \sqrt{2}e^{\frac{1}{4}i\pi}\sin\tfrac{1}{2}\beta,$$

and write $R_1 = r_0 + r$, to get

$$\mathcal{J}(\theta_0) = \frac{e^{-\frac{1}{4}i\pi}}{\pi\sqrt{2\pi}}e^{ikR_1}\cos\tfrac{1}{2}(\theta_0 + \theta) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-k(r_0\xi^2 + r\eta^2)}}{\xi^2 + \eta^2 + 2\sin\frac{1}{2}(\theta_0 + \theta)\xi\eta - 2i\cos^2\frac{1}{2}(\theta_0 + \theta)}\,d\xi\,d\eta. \tag{13}$$

Next, make the polar substitutions

$$\xi = \sqrt{R_1/r_0}\rho\cos\phi, \qquad \eta = \sqrt{R_1/r}\rho\sin\phi.$$

Then

$$\mathcal{J}(\theta_0) = \frac{e^{-\frac{1}{4}i\pi}}{\pi\sqrt{2\pi}}e^{ikR_1}\cos\tfrac{1}{2}(\theta_0 + \theta)\int_0^{\infty}\rho K(\rho)e^{-kR_1\rho^2}\,d\rho, \tag{14}$$

where

$$K(\rho) = \int_0^{2\pi}\left\{\rho^2\left[\sqrt{\frac{r}{r_0}}\cos^2\phi + \sqrt{\frac{r_0}{r}}\sin^2\phi + 2\sin\tfrac{1}{2}(\theta_0 + \theta)\sin\phi\cos\phi\right] - 2i\frac{\sqrt{rr_0}}{R_1}\cos^2\tfrac{1}{2}(\theta_0 + \theta)\right\}^{-1}d\phi. \tag{15}$$

Now $K(\rho)$ can be evaluated by the standard technique of putting $z = \exp(i\phi)$, when the path of integration becomes the unit circle and it is only necessary to calculate the residues of the enclosed poles. It is found that

$$K(\rho) = 2\pi|\sec\tfrac{1}{2}(\theta_0 + \theta)|\left[\rho^4 - 2i\rho^2 - \frac{4rr_0}{R_1{}^2}\cos^2\tfrac{1}{2}(\theta_0 + \theta)\right]^{-1/2}, \tag{16}$$

where the branch of the square root is that which lies (for real values of ρ) in the fourth quadrant of the complex plane. Hence (10) becomes

$$\mathcal{J}(\theta_0) = \pm\sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}e^{ikR_1}\int_0^{\infty}\frac{\rho e^{-kR_1\rho^2}}{\sqrt{[\rho^2 - i(R_1 - R')/R_1][\rho^2 - i(R_1 + R')/R_1]}}\,d\rho, \tag{17}$$

with the $\begin{smallmatrix}\text{upper}\\\text{lower}\end{smallmatrix}$ sign for $\cos\frac{1}{2}(\theta_0 + \theta) \gtrless 0$. Finally the substitution

$$\mu^2 = ikR_1\rho^2 + k(R_1 - R')$$

gives the required result, namely

$$\mathcal{J}(\theta_0) = \pm\sqrt{\frac{2}{\pi}}e^{-\frac{1}{4}i\pi}e^{ikR'}\int_{\sqrt{k(R_1 - R')}}^{\infty}\frac{e^{i\mu^2}}{\sqrt{\mu^2 + 2kR'}}\,d\mu, \tag{18}$$

with the $\begin{smallmatrix}\text{upper}\\\text{lower}\end{smallmatrix}$ sign for $\cos\frac{1}{2}(\theta_0 + \theta) \gtrless 0$.

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Since the incident field (1) can be written in the form

$$\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\,\mathrm{e}^{\mathrm{i}kR}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu^2}}{\sqrt{\mu^2+2kR}}\,\mathrm{d}\mu, \tag{19}$$

the geometrical optics term (7) combines with the diffraction term (9) to give the total field

$$E_z=\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\left\{\mathrm{e}^{\mathrm{i}kR}\int_{m}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu^2}}{\sqrt{\mu^2+2kR}}\,\mathrm{d}\mu-\mathrm{e}^{\mathrm{i}kR'}\int_{m'}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu^2}}{\sqrt{\mu^2+2kR'}}\,\mathrm{d}\mu\right\}, \tag{20}$$

where

$$\left.\begin{aligned} m&=-2\sqrt{\frac{krr_0}{R_1+R}}\cos\tfrac{1}{2}(\theta_0-\theta)=\mp\sqrt{k(R_1-R)}, &&\mp\text{ for }\cos\tfrac{1}{2}(\theta_0-\theta)\gtrless 0,\\ m'&=-2\sqrt{\frac{krr_0}{R_1+R'}}\cos\tfrac{1}{2}(\theta_0+\theta)=\mp\sqrt{k(R_1-R')}, &&\mp\text{ for }\cos\tfrac{1}{2}(\theta_0+\theta)\gtrless 0.\end{aligned}\right\} \tag{21}$$

The solution was first given in essentially the form (20) by MacDonald,* who obtained it via a transformation of an earlier solution due to Carslaw.† It is very similar in type to that of Sommerfeld for an incident plane wave which, indeed, is immediately recovered on multiplying by $\sqrt{kr_0}\exp(-\mathrm{i}kr_0)$ and letting $r_0\to\infty$. The solution for H-polarization differs only in that the two terms in (20) are added instead of being subtracted.

If $kR_1\gg 1$, μ may be replaced by its lower-limit value in the nonexponential factor in the integrand of (18) to give the approximate result

$$\mathcal{J}(\theta_0)=\pm\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi}\frac{\mathrm{e}^{\mathrm{i}kR'}}{\sqrt{k(R_1+R')}}F[\sqrt{k(R_1-R')}]; \tag{22}$$

thus, using also the corresponding approximation for $\mathcal{J}(-\theta_0)$, the diffraction field is expressed in terms of Fresnel integrals to a degree of accuracy which is only inadequate if both the source and the point of observation are well within a wavelength of the diffracting edge.

Furthermore, if $k(R_1-R')\gg 1$, the asymptotic approximation §11.5 (31) may be applied to (22) to give

$$\mathcal{J}(\theta_0)=\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{2\sqrt{2\pi}}\sec\tfrac{1}{2}(\theta_0+\theta)\frac{\mathrm{e}^{\mathrm{i}kr_0}}{\sqrt{kr_0}}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}; \tag{23}$$

and similarly, if $k(R_1-R)\gg 1$,

$$\mathcal{J}(-\theta_0)=\frac{\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi}}{2\sqrt{2\pi}}\sec\tfrac{1}{2}(\theta_0-\theta)\frac{\mathrm{e}^{\mathrm{i}kr_0}}{\sqrt{kr_0}}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{kr}}. \tag{24}$$

The diffraction field therefore resembles the field of a certain line-source located at the

* H. M. MacDonald, *Proc. Lond. Math. Soc.*, **14** (1915), 410.
† H. S. Carslaw, *Proc. Lond. Math. Soc.*, **30** (1899), 121.

diffracting edge for all points well outside the two *hyperbolae* $k(R_1 - R') = 1$ and $k(R_1 - R) = 1$, the axes of which are $\theta + \theta_0 = \pi$ and $\theta - \theta_0 = \pi$ respectively. These hyperbolae are the counterparts, for an incident cylindrical wave, of the parabolae discussed in §11.5.3 in connection with the solution for an incident plane wave.

Finally, attention should be drawn to the fact that the solution (20) is *reciprocal* in the sense that it is unaltered by the respective interchange of r_0, θ_0 and r, θ. This, of course, is a particular example of the general theorem on reciprocity* implicit in Maxwell's equations: the present analysis shows that it is here associated with the fact that the spectrum function §11.5 (7) is symmetrical in α and α_0.

11.7.2 A dipole

The most elementary source of electromagnetic waves which is localized at a point is a dipole, electric or magnetic. The problem of a dipole in the presence of a half-plane can be solved by representing the undisturbed dipole field as a (three-dimensional) spectrum of plane waves and applying the results of §11.6 to each plane wave. The case of an electric dipole with its axis normal to the diffracting sheet has been treated in this way by Senior,† and the analysis is sketched here.

The configuration is shown in Fig. 11.17 with, as before, Cartesian coordinates x, y, z and cylindrical polar coordinates r, θ, z, where the diffracting sheet occupies $y = 0$, $x > 0$. The dipole at T, (x_0, y_0, z_0) or (r_0, θ_0, z_0), is taken to be parallel to the y-axis; T' is the image of T in the plane $y = 0$, and R, R' are the distances of the point of observation P from T and T' respectively.

With a convenient choice of the magnitude of the dipole moment the undisturbed field of the dipole (see §2.2) may be taken as

$$\mathbf{E} = \left(\frac{\partial^2 \Pi}{\partial x \partial y}, \frac{\partial^2 \Pi}{\partial y^2} + k^2 \Pi, \frac{\partial^2 \Pi}{\partial y \partial z} \right), \qquad \mathbf{H} = \mathrm{i}k \left(\frac{\partial \Pi}{\partial z}, 0, -\frac{\partial \Pi}{\partial x} \right), \tag{25}$$

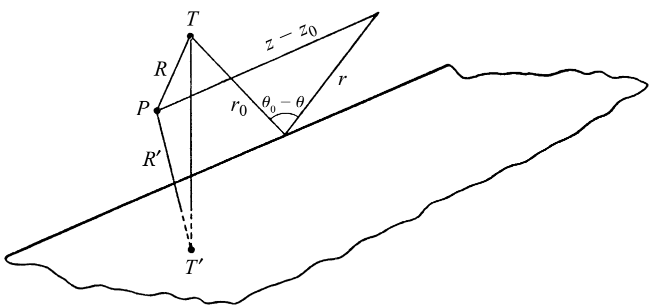

Fig. 11.17 The configuration for a dipole at T in the presence of a diffracting half-plane.

* L. G. H. Huxley, *The Principles and Practice of Waveguides* (Cambridge, Cambridge University Press, 1947), §7.17.

† T. B. A. Senior, *Quart. J. Mech. Appl. Maths.*, **6** (1953), 101. For other methods see A. E. Heins, *Trans. Inst. Radio Engrs.*, AP-**4** (1956), 294; B. D. Woods, *Quart. J. Mech. Appl. Maths.*, **10** (1957), 90; W. E. Williams, *Quart. J. Mech. Appl. Maths.*, **10** (1957), 210.

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where

$$\Pi = \frac{e^{ikR}}{kR}. \tag{26}$$

The required resolution of (25) into plane waves is then obtained from the formula*

$$\frac{e^{ikR}}{kR} = \frac{i}{2\pi}\int_{S(\frac{1}{2}\pi)}\int_{S(\frac{1}{2}\pi)} \cos\beta\, e^{-ik[(x-x_0)\cos\alpha\cos\beta+(y-y_0)\sin\alpha\cos\beta-(z-z_0)\sin\beta]}\, d\alpha\, d\beta. \tag{27}$$

The procedure, therefore, is to write down the total field in the presence of the diffracting sheet arising from the incidence on it of each plane wave implicit in the integrand of (27), and then to carry out the integration.

In view of the subsequent integration it is convenient to use [see §11.7 (5)] the basic solutions for incident plane waves in the form §11.5 (8) (for E-polarization) with the corresponding form for H-polarization. These expressions are modified to yield the three-dimensional solutions corresponding to the incident plane waves implicit in (27) in the manner explained in §11.6. The component of **E** parallel to the dipole then appears in the form

$$E_y = E_y^{(g)} + E_y^{(d)}, \tag{28}$$

where $E_y^{(g)}$ is the field of geometrical optics and

$$\begin{aligned} E_y^{(d)} =& \frac{ik}{2\pi}[A(\theta_0) + A(-\theta_0)] - \frac{i}{2\pi k}\left(\frac{\partial^2}{\partial y\partial y_0} - k^2\right)B(\theta_0) \\ &+ \frac{i}{2\pi k}\left(\frac{\partial^2}{\partial y\partial y_0} + k^2\right)B(-\theta_0), \end{aligned} \tag{29}$$

where

$$\begin{aligned} A(\theta_0) =& \frac{ik}{2\pi}\int_{S(0)}\int_{S(\frac{1}{2}\pi)}\int_{S(0)} \cos\beta\cos\tfrac{1}{2}(\alpha+\gamma+\theta-\theta_0) \\ &\times e^{ik[r\cos\gamma\cos\beta+r_0\cos\alpha\cos\beta+(z-z_0)\sin\beta]}\, d\alpha\, d\beta\, d\gamma, \end{aligned} \tag{30}$$

$$\begin{aligned} B(\theta_0) =& \frac{ik}{2\pi}\int_{S(0)}\int_{S(\frac{1}{2}\pi)}\int_{S(0)} \cos\beta\sec\tfrac{1}{2}(\alpha+\gamma+\theta-\theta_0) \\ &\times e^{ik[r\cos\gamma\cos\beta+r_0\cos\alpha\cos\beta+(z-z_0)\sin\beta]}\, d\alpha\, d\beta\, d\gamma. \end{aligned} \tag{31}$$

By means of an analysis similar to that of Carslaw† and MacDonald‡ it can be shown that

$$A(\theta_0) = \frac{\pi}{\sqrt{rr_0}}\cos\tfrac{1}{2}(\theta-\theta_0)H_0^{(1)}(kR_1), \tag{32}$$

* This is a straightforward modification of a formula given by H. Weyl, *Ann. d. Physik*, **60** (1919), 481. Another form of the Weyl formula is given by §13.2 (2).

† H. S. Carslaw, *Proc. Lond. Math. Soc.*, **30** (1899), 121.

‡ H. M. MacDonald, *Proc. Lond. Math. Soc.*, **14** (1915), 410.

where

$$R_1^2 = (r + r_0)^2 + (z - z_0)^2; \tag{33}$$

$$B(\theta_0) = \pm\pi k \int_{\pm m}^{\infty} H_1^{(1)}(kR\cosh\mu)\mathrm{d}\mu, \qquad \pm \text{ for } \theta - \theta_0 \lessgtr \pi, \tag{34}$$

$$B(-\theta_0) = \pm\pi k \int_{\pm m'}^{\infty} H_1^{(1)}(kR'\cosh\mu)\mathrm{d}\mu, \qquad \pm \text{ for } \theta + \theta_0 \lessgtr \pi, \tag{35}$$

where

$$m = \sinh^{-1}\left[2\frac{\sqrt{rr_0}}{R}\cos\tfrac{1}{2}(\theta - \theta_0)\right], \qquad m' = \sinh^{-1}\left[2\frac{\sqrt{rr_0}}{R'}\cos\tfrac{1}{2}(\theta + \theta_0)\right]. \tag{36}$$

These results, together with the formula

$$\tfrac{1}{2}\mathrm{i}\int_{-\infty}^{\infty} H_1^{(1)}(kR\cosh\mu)\mathrm{d}\mu = \frac{\mathrm{e}^{\mathrm{i}kR}}{kR}, \tag{37}$$

enable (28) to be written

$$E_y = \frac{\mathrm{i}k}{\sqrt{rr_0}}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta_0 H_0^{(1)}(kR_1) - \tfrac{1}{2}\mathrm{i}\left(\frac{\partial^2}{\partial y \partial y_0} - k^2\right)\mathcal{J} + \tfrac{1}{2}\mathrm{i}\left(\frac{\partial^2}{\partial y \partial y_0} + k^2\right)\mathcal{J}', \tag{38}$$

where

$$\mathcal{J} = \int_{-m}^{\infty} H_1^{(1)}(kR\cosh\mu)\mathrm{d}\mu, \qquad \mathcal{J}' = \int_{-m'}^{\infty} H_1^{(1)}(kR'\cosh\mu)\mathrm{d}\mu. \tag{39}$$

The remaining field components can likewise be expressed, exactly, in terms of $\mathcal{J}$ and $\mathcal{J}'$ as follows

$$E_x = -\frac{\mathrm{i}k}{\sqrt{rr_0}}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta_0 H_0^{(1)}(kR_1) - \tfrac{1}{2}\mathrm{i}\frac{\partial^2\mathcal{J}}{\partial x \partial y_0} + \tfrac{1}{2}\mathrm{i}\frac{\partial^2\mathcal{J}'}{\partial x \partial y_0}, \tag{40}$$

$$E_z = -\tfrac{1}{2}\mathrm{i}\frac{\partial^2\mathcal{J}}{\partial z \partial y_0} + \tfrac{1}{2}\mathrm{i}\frac{\partial^2\mathcal{J}'}{\partial z \partial y_0}, \tag{41}$$

$$H_x = \frac{k(z - z_0)}{R_1\sqrt{rr_0}}\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta_0 H_1^{(1)}(kR_1) + \tfrac{1}{2}k\frac{\partial\mathcal{J}}{\partial z_0} + \tfrac{1}{2}k\frac{\partial\mathcal{J}'}{\partial z_0}, \tag{42}$$

$$H_y = \frac{k(z - z_0)}{R_1\sqrt{rr_0}}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta_0 H_1^{(1)}(kR_1), \tag{43}$$

$$H_z = -\tfrac{1}{2}k\frac{\partial\mathcal{J}}{\partial x_0} - \tfrac{1}{2}k\frac{\partial\mathcal{J}'}{\partial x_0}. \tag{44}$$

It is interesting to note that in the complete field there is a nonzero component of **H** parallel to the dipole, so that the analysis cannot be formulated in terms of a single component Hertz vector.

It is again possible, under a relatively trivial restriction, to express the solution in terms of Fresnel integrals. If $kR_1 \gg 1$, it is not difficult to establish the asymptotic approximation

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$$\mathcal{J} = -\frac{2e^{\frac{1}{4}i\pi}}{k}\sqrt{\frac{2}{\pi R_1(R+R_1)}}e^{ikR}F\left[-2\sqrt{\frac{krr_0}{R+R_1}}\cos\tfrac{1}{2}(\theta-\theta_0)\right], \tag{45}$$

with a similar result for $\mathcal{J}'$.

11.8 Other problems

In this section several other diffraction problems are briefly reviewed.

11.8.1 Two parallel half-planes

The problem of diffraction by two parallel half-planes which are perpendicular to the common plane through their edges is tractable by the method used in this chapter for the single half-plane.*

Consider E-polarization, with the problem precisely as in §11.5.1 except that the diffracting obstacle now consists of *two* sheets; one (sheet 1) occupying $y=0$, $x>0$; the other (sheet 2), $y=-a$, $x>0$. It is convenient to introduce the additional coordinates r', θ', measured from the edge $(0, -a)$ of sheet 2.

The scattered fields due to the induced currents in sheets 1 and 2 may be written

$$E_z^{(s1)} = \int_C P_1(\cos\alpha)e^{ikr\cos(\theta\mp\alpha)}\,d\alpha, \qquad \mp \text{ for } y \gtrless 0, \tag{1}$$

$$E_z^{(s2)} = \int_C P_2(\cos\alpha)e^{ikr'\cos(\theta'\mp\alpha)}\,d\alpha, \qquad \mp \text{ for } y \gtrless -a \tag{2}$$

respectively. The continuity of $H_x{}^{(s1)}$ and $H_x{}^{(s2)}$ across the regions $y=0$, $x<0$, and $y=-a$, $x<0$, respectively, is ensured by taking $P_1(\mu)$ and $P_2(\mu)$ to be free of singularities below the path of integration, which is that shown in Fig. 11.7. Furthermore, the boundary condition that E_z should vanish on the two sheets leads to the integral equations

$$\int_{-\infty}^{\infty}\frac{P_1(\mu)}{\sqrt{1-\mu^2}}e^{ikx\mu}\,d\mu + \int_{-\infty}^{\infty}\frac{P_2(\mu)}{\sqrt{1-\mu^2}}e^{ika\sqrt{1-\mu^2}}\,e^{ikx\mu}\,d\mu = -e^{-ikx\mu_0}, \tag{3}$$

$$\int_{-\infty}^{\infty}\frac{P_1(\mu)}{\sqrt{1-\mu^2}}e^{ika\sqrt{1-\mu^2}}\,e^{ikx\mu}\,d\mu + \int_{-\infty}^{\infty}\frac{P_2(\mu)}{\sqrt{1-\mu^2}}e^{ikx\mu}\,d\mu = -e^{ika\sqrt{1-\mu_0{}^2}}\,e^{-ikx\mu_0}, \tag{4}$$

which must hold for $x>0$.

If we write

$$P_1(\mu)+P_2(\mu) = Q_1(\mu), \qquad P_1(\mu)-P_2(\mu) = Q_2(\mu), \tag{5}$$

addition and subtraction, respectively, of (3) and (4) give

$$\int_{-\infty}^{\infty}\frac{Q_1(\mu)}{\sqrt{1-\mu^2}}(1+e^{ika\sqrt{1-\mu^2}})e^{ikx\mu}\,d\mu = -(1+e^{ika\sqrt{1-\mu_0{}^2}})e^{-ikx\mu_0}, \tag{6}$$

* Solutions for an incident plane wave were first given, using the method introduced by Schwinger (mentioned in §11.1), by A. E. Heins, *Quart. Appl. Maths.*, **6** (1948), 157, and L. A. Vainstein, *Izvestiya Akad. Nauk SSSR, Ser. Fiz.* (*Bull. Acad. Sci. USSR*), **12** (1948), 114 and 166.

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$$\int_{-\infty}^{\infty} \frac{Q_2(\mu)}{\sqrt{1-\mu^2}}(1 - \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu^2}})\mathrm{e}^{\mathrm{i}kx\mu}\,\mathrm{d}\mu = -(1 - \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu_0{}^2}})\mathrm{e}^{-\mathrm{i}kx\mu_0}, \tag{7}$$

for $x > 0$.

The form of each of the equations (6) and (7) is similar to §11.5 (2), and to obtain solutions analogous to §11.5 (4) the paths of integration must be closed by infinite semicircles above the real axis. To this end, that branch of $\sqrt{1-\mu^2}$ is chosen which has a positive imaginary part. The required solution of (6) is then

$$\frac{Q_1(\mu)}{\sqrt{1-\mu^2}}(1 + \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu^2}}) = \frac{\mathrm{i}}{2\pi}(1 + \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu_0{}^2}})\frac{U(\mu)}{U(-\mu_0)}\frac{1}{\mu+\mu_0}, \tag{8}$$

where $U(\mu)$ is any function of μ which is free of singularities in the half-plane above the path of integration and tends to zero as $|\mu| \to \infty$ therein.

It remains to be seen how $Q_1(\mu)$ and $U(\mu)$ can be chosen to satisfy (8), remembering that $Q_1(\mu)$ has no singularities below the path of integration. The procedure is to factorize the coefficient of $Q_1(\mu)$ in (8) in the form

$$\frac{1 + \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu^2}}}{\sqrt{1-\mu^2}} = U_1(\mu)L_1(\mu), \tag{9}$$

where $U_1(\mu)$ is free of singularities and zeros in the half-plane above the path of integration, and is of algebraic growth at infinity therein, while $L_1(\mu)$ has similar characteristics in the half-plane below the path of integration. That such factorization is possible is known from the general theory of Wiener and Hopf,* and explicit expressions for $U_1(\mu)$ and $L_1(\mu)$ are given by Heins.† Then clearly

$$Q_1(\mu) = \frac{\mathrm{i}}{2\pi}\sqrt{1-\mu_0{}^2}\,U_1(\mu_0)\frac{1}{L_1(\mu)(\mu+\mu_0)}, \tag{10}$$

where the relation $U_1(\mu) = L_1(-\mu)$, which is implicit in (9) (apart from arbitrary constant factors), has been invoked.

Similarly, if

$$\frac{1 - \mathrm{e}^{\mathrm{i}ka\sqrt{1-\mu^2}}}{\sqrt{1-\mu^2}} = U_2(\mu)L_2(\mu) \tag{11}$$

we have

$$Q_2(\mu) = \frac{\mathrm{i}}{2\pi}\sqrt{1-\mu_0{}^2}\,U_2(\mu_0)\frac{1}{L_2(\mu)(\mu+\mu_0)}. \tag{12}$$

The total scattered field is obtained by adding (1) and (2). In $y > 0$, for instance, it is therefore

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, Clarendon Press, 1937), p. 339.
† A. E. Heins, *Quart. Appl. Maths.*, **6** (1948), 157.

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$$
\begin{aligned}
E_z^{(s)} &= \int_C [P_1(\cos\alpha) + P_2(\cos\alpha)\mathrm{e}^{\mathrm{i}ka\sin\alpha}]\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha \\
&= \tfrac{1}{2}\int_C [Q_1(\cos\alpha)(1+\mathrm{e}^{\mathrm{i}ka\sin\alpha}) + Q_2(\cos\alpha)(1-\mathrm{e}^{\mathrm{i}ka\sin\alpha})]\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha \\
&= \int_C P(\cos\alpha)\mathrm{e}^{\mathrm{i}kr\cos(\theta-\alpha)}\,\mathrm{d}\alpha, \qquad \text{say,}
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
P(\mu) &= \tfrac{1}{2}\sqrt{1-\mu^2}[Q_1(\mu)U_1(\mu)L_1(\mu) + Q_2(\mu)U_2(\mu)L_2(\mu)] \\
&= \frac{\mathrm{i}}{4\pi}\frac{\sqrt{1-\mu_0{}^2}\sqrt{1-\mu^2}}{\mu+\mu_0}[U_1(\mu_0)U_1(\mu) + U_2(\mu_0)U_2(\mu)].
\end{aligned}
\tag{14}
$$

Again the symmetry of $P(\mu)$ in μ and μ_0 should be noted. Also that, when $a = 0$, $U_1(\mu) = \sqrt{2/(1+\mu)}$, $U_2(\mu) = 0$, so that (14) reduces, as it should, to §11.5 (6).

11.8.2 An infinite stack of parallel, staggered half-planes

In this problem* we have an infinite set of diffracting sheets, with the nth sheet occupying $y = na$, $x > nb$, where $n = 0, \pm 1, \pm 2, \ldots$.

Then, for an E-polarized incident plane wave as before, the scattered field due to currents induced in the mth sheet may be written

$$
E_z^{(sm)} = \int_C P_m(\cos\alpha)\mathrm{e}^{-\mathrm{i}km(a\cos\alpha\pm b\sin\alpha)}\,\mathrm{e}^{\mathrm{i}kr\cos(\theta\mp\alpha)}\,\mathrm{d}\alpha, \tag{15}
$$

with the upper sign for $y > ma$, and the lower sign for $y < ma$. All the $P_m(\mu)$ must be free of singularities below the path of integration in the $\mu = \cos\alpha$ plane.

The boundary conditions on the nth sheet yield the integral equation

$$
\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P_m(\mu)}{\sqrt{1-\mu^2}}\mathrm{e}^{\mathrm{i}ka|n-m|\sqrt{1-\mu^2}}\,\mathrm{e}^{\mathrm{i}k(x-mb)\mu}\,\mathrm{d}\mu = -\mathrm{e}^{-\mathrm{i}kx\mu_0}\,\mathrm{e}^{-\mathrm{i}kna\sqrt{1-\mu_0{}^2}}, \tag{16}
$$

which must hold for $x > nb$.

From the periodicity of the problem it is clear that

$$
P_m(\mu) = P_0(\mu)\mathrm{e}^{-\mathrm{i}km(b\mu_0+a\sqrt{1-\mu_0{}^2})} \tag{17}
$$

and on putting $n - m = q$, (16) therefore becomes

$$
\sum_{q=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{P_0(\mu)}{\sqrt{1-\mu^2}}\mathrm{e}^{\mathrm{i}kq(b\mu_0+a\sqrt{1-\mu_0{}^2})}\,\mathrm{e}^{\mathrm{i}ka|q|\sqrt{1-\mu^2}}\,\mathrm{e}^{\mathrm{i}k(qb+x)\mu}\,\mathrm{d}\mu = -\mathrm{e}^{-\mathrm{i}kx\mu_0} \tag{18}
$$

for $x > 0$.

The infinite sum over q can be put in closed form, again leaving an integral equation which can be solved by the use of Cauchy's residue theorem. As in the previous problem, it is necessary to 'split' a certain function into a pair of factors, one of which is free of singularities and zeros in the upper half-plane and of algebraic growth at

* J. F. Carlson and A. E. Heins, *Quart. Appl. Maths.*, **4** (1947), 313 and **5** (1947), 82.

infinity therein, the other having similar characteristics in the lower half-plane: the factors are given explicitly, together with further details, in the papers of Carlson and Heins to which reference has already been made.

11.8.3 A strip

Another problem of intrinsic interest is the deceptively simple one in which the diffracting obstacle is a perfectly conducting plane strip, infinitely long with parallel edges; or the corresponding one with the complementary 'screen', a slit in an infinite plane. Various methods of solution have been given,* including a solution as a series of Matthieu functions†. The dual integral equation approach has been used‡ to obtain the first two terms of a series solution in powers of ka, where $2a$ is the width of the strip, for a normally incident plane wave, in the following manner.

For a strip occupying $y=0$, $|x|<a$, and a normally incident H-polarized plane wave, the integral equations §11.4 (19) and §11.4 (20) are

$$\int_{-\infty}^{\infty} P(\mu)\mathrm{e}^{\mathrm{i}kx\mu}\,\mathrm{d}\mu = 1 \qquad \text{for } |x|<a, \tag{19}$$

$$\int_{-\infty}^{\infty} \frac{P(\mu)}{\sqrt{1-\mu^2}}\mathrm{e}^{\mathrm{i}kx\mu}\,\mathrm{d}\mu = 0 \qquad \text{for } |x|>a; \tag{20}$$

or, since the symmetry of the problem implies $P(\mu)=P(-\mu)$,

$$\int_0^{\infty} P(\mu)\cos(kx\mu)\mathrm{d}\mu = \tfrac{1}{2} \qquad \text{for } |x|<a, \tag{21}$$

$$\int_0^{\infty} \frac{P(\mu)}{\sqrt{1-\mu^2}}\cos(kx\mu)\mathrm{d}\mu = 0 \qquad \text{for } |x|>a. \tag{22}$$

A solution is sought of the form

$$\frac{P(\mu)}{\sqrt{1-\mu^2}} = \sum_{m=0}^{\infty} c_m \frac{J_{2m+1}(ka\mu)}{\mu}, \tag{23}$$

since (22) is satisfied by each term of the series. Substitution into (21) shows that the c_m must be found such that

* Lord Rayleigh, *Phil. Mag.*, **43** (1897), 259 (reprinted in *Scientific Papers by John William Strutt, Baron Rayleigh* Vol. 4 (Cambridge, Cambridge University Press, 1899–1920), p. 283); K. Schwarzschild, *Math. Ann.*, **55** (1902), 177; P. M. Morse and P. J. Rubenstein, *Phys. Rev.*, **54** (1938), 895; A. Sommerfeld, *Optics* (New York, Academic Press, 1954), p. 273; B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygens' Principle* (Oxford, Clarendon Press, 2nd edition, 1950), p. 177 *et seq*.; S. Skavlem, *Arch. Math. Naturvid.*, **51** (1951), 61; E. B. Moullin and F. M. Phillips, *Proc. Inst. Elec. Engrs.*, **99**, Pt. IV (1952), 137; R. Müller and K. Westpfahl, *Z. Phys.*, **134** (1953), 245; P. C. Clemmow, *Trans. Inst. Radio Engrs.*, AP-**4** (1956), 282; S. N. Karp and A. Russek, *J. Appl. Phys.*, **27** (1956), 886; R. F. Millar, *Proc. Camb. Phil. Soc.*, **54** (1958), 479, 497.

† B. Sieger, *Ann. der Phys.*, **27** (1908), 626; see also P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (New York, McGraw Hill, Part II, 1953), p. 1428 *et seq.*

‡ E. Groschwitz and H. Hönl, *Z. Phys.*, **131** (1952), 305; H. Hönl and E. Zimmer, *Z. Phys.*, **135** (1953), 196; C. J. Tranter, *Quart. J. Mech. Appl. Maths.*, **7** (1954), 317.

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$$\sum_{m=0}^{\infty} c_m \Phi_m = \tfrac{1}{2} \qquad \text{for } |x| < a, \tag{24}$$

where

$$\Phi_m = \int_0^\infty \frac{\sqrt{1-\mu^2}}{\mu} J_{2m+1}(ka\mu)\cos(kx\mu)\,\mathrm{d}\mu. \tag{25}$$

Now it can be shown that, for $|ka| \ll 1$,

$$\Phi_m = \mathrm{i}\int_0^\infty J_{2m+1}(ka\mu)\cos(kx\mu)\,\mathrm{d}\mu + O(ka) = \frac{\mathrm{i}\cos\left[(2m+1)\sin^{-1}\dfrac{x}{a}\right]}{ka\sqrt{1-x^2/a^2}} + O(ka); \tag{26}$$

whence, to the first order, (24) is

$$\frac{\mathrm{i}}{\sqrt{1-x^2/a^2}} \sum_{m=0}^{\infty} c_m \cos\left[(2m+1)\sin^{-1}\frac{x}{a}\right] = \tfrac{1}{2}ka, \tag{27}$$

giving

$$c_0 = \frac{ka}{2\mathrm{i}}, \qquad c_m = 0 \qquad \text{for } m = 1, 2, 3, \ldots. \tag{28}$$

Thus, to this approximation,

$$\frac{P(\mu)}{\sqrt{1-\mu^2}} = \frac{ka}{2\mathrm{i}} \frac{J_1(ka\mu)}{\mu}, \tag{29}$$

and the current density, derived from §11.4 (16), is

$$J_x(x) = \frac{cka}{2\pi\mathrm{i}} \int_0^\infty \frac{J_1(ka\mu)}{\mu} \cos(kx\mu)\,\mathrm{d}\mu = \frac{cka}{2\pi\mathrm{i}}\sqrt{1-x^2/a^2}. \tag{30}$$

The next approximation, involving terms in $(ka)^3$, is considerably more complicated. Identical expressions for it have been obtained independently by several authors.*

11.8.4 Further problems

There are a number of other interesting problems, allied to those which we have discussed, which are capable of solution, but we cannot do more than mention them here.

Diffraction by a two-dimensional wedge, reducing to a half-plane when the exterior wedge angle is 2π, was solved many years ago.† It involves angular spectra of period $2\pi/n$, where π/n is the exterior wedge angle.

The problem of a half-plane in the plane interface between two different homogeneous media was first considered by Hanson.‡ It is tractable by the method of this chapter and has been applied§ in the theory of the propagation of radio waves over the surface of the earth.

* R. Müller and K. Westpfahl, *Z. Phys.*, **134** (1953), 245; C. J. Tranter, *Quart. J. Mech. Appl. Maths.*, **7** (1954), 317. See also C. J. Bouwkamp, *Rep. Progr. Phys.* (London, Physical Society), **17** (1954), 73.

† H. M. MacDonald, *Electric Waves* (Cambridge, Cambridge University Press, 1902); H. S. Carslaw, *Proc. Lond. Math. Soc.*, **18** (1919), 291.

‡ E. T. Hanson, *Phil. Trans. Roy. Soc.* A, **237** (1938), 35.

§ P. C. Clemmow, *Phil. Trans. Roy. Soc.* A, **246** (1953), 1.

Two investigations have been made of the effect of a half-plane under less idealized conditions. In the first,* finite though large conductivity is introduced, which necessitates the use of approximate boundary conditions; in the second,† the plate is assumed to be perfectly conducting but to have a finite, though small, thickness.

11.9 Uniqueness of solution

In §11.2 we saw that the general diffraction problem with which we were concerned could be stated in the following form: given a field $\mathbf{E}^{(i)}$ incident on a perfectly conducting surface S, to find a field $\mathbf{E}^{(s)}$ which could arise from an electric current distribution in S and which is such that its tangential component on S is minus that of $\mathbf{E}^{(i)}$.

It is, of course, vital that the formulation should yield a unique solution,‡ but the demonstration that there cannot, in fact, be more than one field $\mathbf{E}^{(s)}$ satisfying the stated conditions is by no means straightforward, particularly when the possibilities of S being infinite and the field containing plane waves are taken into account. Only comparatively recently, it appears, has the result been satisfactorily established,§ although it has long been tacitly accepted.

A further difficulty arises in the special, though commonest, type of diffraction problem in which the obstacle may be assumed to have an infinitely sharp edge and to which the discussion in this chapter has, indeed, been confined. The reason for the additional complication is that the solution then contains, as we have seen, a singularity at the edge, thus violating an assumption necessary to the uniqueness proof just mentioned.

It can easily be seen that, if an arbitrary edge singularity is permitted, an infinite sequence of solutions can be obtained by a process of differentiation.|| For example, differentiation of the E-polarization solution of the half-plane problem §11.5 (22) with respect to x gives an essentially new expression which also satisfies the wave equation and vanishes on the screen; or, again, differentiation of §11.5 (22) with respect to y yields an expression which would apparently suffice for the H-polarization solution, but which differs from §11.5 (44).

Each differentiation introduces a singularity of higher order at the diffracting edge. Evidently, in order to ensure uniqueness, some restriction on the nature of the singularity must be specified. The appropriate restriction, and the various ways in which it may be formulated, have been the subject of a number of papers.¶ The reader must be referred to these for details, but broadly speaking it may be said that the

* T. B. A. Senior, *Proc. Roy. Soc.* A, **213** (1952), 436.

† D. S. Jones, *Proc. Roy. Soc.* A, **217** (1953), 153.

‡ To establish the *existence* of a solution is perhaps less important, since this question is settled when one is found in any particular case. But see C. Müller, *Math. Ann.*, **123** (1951), 345, and W. K. Saunders, *Proc. Nat. Acad. Sci. U.S.A.*, **38** (1952), 342.

§ F. Rellich, *Jahr. Deut. Math. Ver.*, **53** (1943), 57. See the account in A. Sommerfeld, *Partial Differential Equations in Physics* (New York, Academic Press, 1949), §28. W. K. Saunders, *Proc. Nat. Acad. Sci. USA*, **38** (1952), 342.

|| C. J. Bouwkamp, *Physica*, **12** (1946), 467.

¶ J. Meixner, *Ann. d. Physik*, **6** (1949), 1; A. W. Maue, *Z. Phys.*, **126** (1949), 601; E. T. Copson, *Proc. Roy. Soc.* A, **202** (1950), 277; D. S. Jones, *Quart. J. Mech. Appl. Maths.*, **3** (1950), 420; A. E. Heins and S. Silver, *Proc. Camb. Phil. Soc.*, **51** (1955), 149; *ibid.*, **54** (1958), 131.

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solution involving the singularity of lowest possible order is to be taken as representing the answer to the physical problem, and that this, in fact, rules out any singularities of order greater than $r^{-1/2}$ as $r \to 0$ at the diffracting edge. In particular, it can be established that a solution is unique which implies for the induced current its integrability over the diffracting surface and the vanishing at the edge of its component normal to the edge. The behaviour of the components of **E** and **H** near the edge can be deduced from these conditions.

Finally, the possibility of an infinite number of 'solutions' having been seen, it may be asked why the method used in this chapter apparently yields a unique solution, which is, moreover, the correct one judged by the above-mentioned criteria. The answer is because of the assumption that the field components in the plane of the diffracting screen can be expressed as convergent Fourier integrals: this precludes them from having singularities of too high an order.