II

Electromagnetic potentials and polarization

IN the preceding chapter the effect of matter on an electromagnetic field was expressed in terms of a number of macroscopic constants. These have only a limited range of validity and are in fact inadequate to describe certain processes, such as the emission, absorption and dispersion of light. A full account of these phenomena would involve an extensive study of the atomistic theory and lies therefore outside the scope of this book.

It is possible, however, to describe the interaction of field and matter by means of a simple model which is entirely adequate for most branches of optics. For this purpose each of the vectors **D** and **B** is expressed as the sum of two terms.[*] Of these one is taken to be the vacuum field and the other is regarded as arising from the influence of matter. Thus one is led to the introduction of two new vectors for describing the effects of matter: the *electric polarization* (**P**) and the *magnetic polarization* or *magnetization* (**M**). Instead of the material equations (10) and (11) in §1.1 connecting **D** and **B** with **E** and **H**, we now have equations connecting **P** and **M** with **E** and **H**. These new equations have a more direct physical meaning and lead to the following conception of the propagation of an electromagnetic field in matter:

An electromagnetic field produces at a given volume element certain amounts of polarization **P** and **M** which, in the first approximation, are proportional to the field, the constant of proportionality being a measure of the reaction of the field. Each volume element then becomes a source of a new secondary or scattered wavelet, whose strength is related in a simple way to **P** and **M**. All the secondary wavelets combine with each other and with the incident field and form the total field, this being the one from which the whole consideration proceeds. By expressing this identity formally, we obtain two integral equations,[†] which may be shown to be equivalent to Maxwell's differential equations, but which describe the propagation of the electromagnetic field in a manner more clearly related to the atomic constitution of matter.

The two main results which will be obtained from the theory are: (1) The Lorentz–Lorenz formula, which relates the macroscopic-optical properties of the medium to the number and the properties of the scattering particles, and (2) the so-called extinction

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[*] According to the remarks on p. 1, it would be more appropriate to write **H** in place of **B**. This departure from the general theory is made here for the sake of the trivial convenience of having the positive signs on the right-hand side of §2.2 (2), as is customary.

[†] In the case of nonmagnetic substances there is only one integral equation. The second integral equation reduces to a relatively simple expression for the magnetic field, and may be evaluated when the solution of the first equation has been determined.

The 'material' integral equations discussed here must be distinguished from the 'geometrical' integral equations used in the treatment of certain diffraction problems (Chapter XI).

theorem of Ewald and Oseen, which shows how an external electromagnetic disturbance travelling with the velocity of the light in vacuum is exactly cancelled out and replaced in the substance by the secondary disturbance travelling with an appropriately smaller velocity.

The theory provides an alternative mathematical approach to the treatment of some problems of electromagnetic theory. This will be illustrated by the derivation of the laws of refraction and reflection and the Fresnel formulae, and will be further illustrated by the treatment of a more complex problem in Chapter XII.

In the derivation of these results some new mathematical apparatus will be required. Accordingly, we shall first discuss the representation of an electromagnetic field in terms of so-called retarded potentials, these being generalizations of the well-known static potentials. The expression of the potentials in terms of the polarization vectors leads to another set of auxiliary quantities, known as Hertz vectors. In §2.1 and §2.2 we discuss these mathematical preliminaries and in §2.3 we briefly explain the underlying physical concepts. The integral equations and the two basic theorems already referred to will be derived in §2.4.

2.1 The electrodynamic potentials in the vacuum

2.1.1 The vector and scalar potentials

Let us consider an electromagnetic field in vacuum due to a given distribution of charges $\rho(\mathbf{r}, t)$ and currents $\mathbf{j}(\mathbf{r}, t)$. It obeys Maxwell's equations §1.1 (1)–(4) which may be written (since in vacuum $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$) in the form:

$$\operatorname{curl} \mathbf{B} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{j}, \tag{1}$$

$$\operatorname{curl} \mathbf{E} + \frac{1}{c}\dot{\mathbf{B}} = 0, \tag{2}$$

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \tag{3}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{4}$$

Since the divergence of the curl of any vector is zero, (4) will be satisfied if we set

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \tag{5}$$

$\mathbf{A}$ being an arbitrary vector function of position and time. If we substitute from (5) into Maxwell's second equation, we have

$$\operatorname{curl}\left(\mathbf{E} + \frac{1}{c}\dot{\mathbf{A}}\right) = 0. \tag{6}$$

Eq. (6) will hold if

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\phi, \tag{7}$$

ϕ being an arbitrary scalar function. $\mathbf{A}$ and ϕ must now be determined in such a way as to satisfy the remaining Maxwell equations.

Substituting from (5) and (7) into (1) and (3), and using the identities $\operatorname{curl}\operatorname{curl} \equiv \operatorname{grad}\operatorname{div} - \nabla^2$ and $\operatorname{div}\operatorname{grad} \equiv \nabla^2$, we obtain

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \ddot{\mathbf{A}} - \text{grad} \left(\text{div} \, \mathbf{A} + \frac{1}{c} \dot{\phi} \right) = -\frac{4\pi}{c} \mathbf{j}, \tag{8}$$

and

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} + \frac{1}{c} \frac{\partial}{\partial t} \left(\text{div} \, \mathbf{A} + \frac{1}{c} \dot{\phi} \right) = -4\pi\rho. \tag{9}$$

If we prescribe between $\mathbf{A}$ and ϕ the relation

$$\text{div} \, \mathbf{A} + \frac{1}{c} \dot{\phi} = 0, \tag{10}$$

then (8) and (9) reduce to the inhomogeneous wave equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \ddot{\mathbf{A}} = -\frac{4\pi}{c} \mathbf{j}, \tag{11}$$

and

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} = -4\pi\rho. \tag{12}$$

The two functions $\mathbf{A}$ and ϕ, from which $\mathbf{B}$ and $\mathbf{E}$ may be determined by means of the relations (5) and (7), are known as the *magnetic vector potential* and the *electric scalar potential* respectively. The relation (10), which couples the two potentials, is called the *Lorentz condition*. We note that it is consistent with the continuity equation §1.1 (5)

$$\dot{\rho} + \text{div} \, \mathbf{j} = 0. \tag{13}$$

It is to be observed that (11), (12) and (10) do not define the two potentials uniquely. For if we add to $\mathbf{A}$ the vector $\text{grad} \, \chi$, where χ is arbitrary, $\mathbf{B}$ will remain unchanged, and if, in addition, ϕ is replaced by $\phi - \dot{\chi}/c$, $\mathbf{E}$ also will remain unchanged. In other words $\mathbf{B}$ and $\mathbf{E}$ are invariant under the transformation

$$\mathbf{A}' = \mathbf{A} + \text{grad} \, \chi, \tag{14a}$$

$$\phi' = \phi - \frac{1}{c} \dot{\chi}. \tag{14b}$$

Now from (10) and (14),

$$\text{div} \, \mathbf{A}' + \frac{1}{c} \dot{\phi}' - \left(\nabla^2 \chi - \frac{1}{c^2} \ddot{\chi} \right) = 0. \tag{15}$$

Hence $\mathbf{A}'$ and ϕ' will satisfy the Lorentz relation, if one imposes on χ the condition

$$\nabla^2 \chi - \frac{1}{c^2} \ddot{\chi} = 0. \tag{16}$$

Eqs. (14), subject to the condition (16), express the so-called *gauge transformation*. The gauge transformation may be used to simplify the expressions for the field vectors. For example in a region where the charge density ρ is zero, ϕ satisfies the homogeneous wave equation

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} = 0 \tag{17}$$

and χ may then be so chosen, that the scalar potential vanishes. According to (14b) and (16) it is only necessary to take

$$\chi = c \int \phi \, \mathrm{d}t. \tag{18}$$

The field can then be derived from the vector potential alone[*] by means of the relations (dropping the prime on **A**)

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \qquad \mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}}, \tag{19}$$

whilst the Lorentz condition reduces to

$$\operatorname{div} \mathbf{A} = 0. \tag{20}$$

2.1.2 Retarded potentials

We now consider the solution [subject to the relation (10)] of the inhomogeneous wave equations (11) and (12) for the vector and scalar potentials, and show first that these equations are satisfied by the following functions:

$$\mathbf{A}(\mathbf{r},\, t) = \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}',\, t - R/c)}{R}\, \mathrm{d}V', \tag{21}$$

$$\phi(\mathbf{r},\, t) = \int \frac{\rho(\mathbf{r}',\, t - R/c)}{R}\, \mathrm{d}V'. \tag{22}$$

Here

$$R = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{23}$$

is the distance from the point $\mathbf{r}(x,\, y,\, z)$ to the volume element $\mathrm{d}V'$ at $\mathbf{r}'(x',\, y',\, z')$, the integration being carried throughout the whole space.

To verify that (22) satisfies the inhomogeneous wave equation for the scalar potential, we imagine that the point $\mathbf{r}$ is surrounded by a sphere of radius a, centred on that point, and divide (22) into two parts:

$$\phi = \phi_1 + \phi_2, \tag{24}$$

where ϕ_1 represents the contribution to the integral from the interior of the sphere and ϕ_2 the contribution from the rest of space. Since $R \neq 0$ for every point $\mathbf{r}'(x',\, y',\, z')$ outside the sphere it will be permissible to differentiate ϕ_2 under the integral sign. We can then verify by a straightforward calculation that ϕ_2 satisfies the homogeneous wave equation

$$\nabla^2 \phi_2 - \frac{1}{c^2}\ddot{\phi}_2 = 0, \tag{25}$$

consisting, simply as it does, of a set of superposed spherical waves [see §1.3 (12)]. In

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[*] Eq. (19) expresses the field in a charge-free region in vacuum in terms of three scalar wave functions (Cartesian components of **A**). On account of (20) they are, however, not independent. It may be shown that in such a region the field can in fact be derived from two real scalar wave functions alone (see, for example, S. A. Schelkunoff, *Electromagnetic Waves* (New York, Van Nostrand, 1943) p. 382; H. S. Green and E. Wolf, *Proc. Phys. Soc.*, A, **66** (1953), 1129; A. Nisbet, *Proc. Roy. Soc.*, A, **231** (1955), 251).

the case of ϕ_1 we must, however, proceed differently, since the integrand has a singularity at the centre $R = 0$.

We note that by making the radius of the sphere sufficiently small, we can secure (assuming ρ to be a continuous function of $\mathbf{r}$ and t) that for all points $\mathbf{r}'$ inside the sphere $\rho(\mathbf{r}', t - R/c)$ differs from $\rho(\mathbf{r}, t)$ by an amount smaller than any prescribed value. Hence, as the radius a tends to zero, $\nabla^2 \phi_1$ will approach more and more closely the value of the electrostatic potential of a homogeneously charged sphere, with charge density ρ; i.e. for a sufficiently small a,

$$\nabla^2 \phi_1 = -4\pi\rho(\mathbf{r}, t). \tag{26}$$

Also, as $a \to 0$, $\ddot{\phi}_1 \to 0$. For if a is sufficiently small we may write

$$\ddot{\phi}_1 = \ddot{\rho}(\mathbf{r}, t) \int_{R \leqslant a} \frac{\mathrm{d}V'}{R}$$

$$= 4\pi\ddot{\rho} \int_0^a R\,\mathrm{d}r$$

$$= 2\pi a^2 \ddot{\rho}, \tag{27}$$

and this tends to zero with a. Hence it follows from (24)–(26) that as $a \to 0$,

$$\nabla^2 \phi - \frac{1}{c^2}\ddot{\phi} = \nabla^2(\phi_1 + \phi_2) - \frac{1}{c^2}(\ddot{\phi}_1 + \ddot{\phi}_2) \to -4\pi\rho(\mathbf{r}, t), \tag{28}$$

so that (22) satisfies the inhomogeneous wave equation for the scalar potential. In a strictly analogous manner we can show that each Cartesian component of (21) is a solution of the corresponding scalar wave equation, whose inhomogeneous term contains, in place of ρ, the appropriate component of $\mathbf{j}/c$. Hence (21) satisfies the inhomogeneous wave equation for the vector potential. Moreover, on account of the continuity relation (13), these solutions also satisfy the Lorentz relation (10).

The expressions (21) and (22) have a simple physical interpretation. They show that one may regard $\mathbf{A}$ and ϕ as arising from contributions from each volume element of space, a typical element $\mathrm{d}V'$ contributing the amounts

$$\frac{1}{c}\frac{\mathbf{j}(\mathbf{r}', t - R/c)}{R} \qquad \text{and} \qquad \frac{\rho(\mathbf{r}', t - R/c)}{R}$$

to $\mathbf{A}$ and ϕ respectively. Now R/c is precisely the time needed for light to travel from the point $\mathbf{r}'$ to $\mathbf{r}$, so that each contribution has to leave the element at such preceding time as to reach the point of observation at the required time t. For this reason (21) and (22) are called *retarded potentials*.[*]

Eqs. (21) and (22) represent a special solution of the wave equations (11) and (12), namely that which arises from the given charges and currents. The general solution is

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[*] It is also possible to construct solutions in the form of *advanced potentials* (with $t + R/c$ in place of $t - R/c$). These represent the effect of incoming spherical waves, whereas the retarded potentials represent the effect of outgoing spherical waves.

   At the cost of some formal asymmetry, we may restrict our attention to solutions in terms of retarded potentials only. When the interaction of radiation and matter is considered thermodynamically, then there is an asymmetry in the physical situation which reinforces this choice (see M. Born, *Natural Philosophy of Cause and Chance* (Oxford, Clarendon Press, 1949; New York, Dover Publications, 1964), especially p. 26).

obtained by adding to this the general solutions (again subject to the Lorentz condition) of the homogeneous wave equations

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \ddot{\mathbf{A}} = 0, \tag{29}$$

$$\nabla^2 \phi - \frac{1}{c^2} \ddot{\phi} = 0. \tag{30}$$

2.2 Polarization and magnetization

2.2.1 The potentials in terms of polarization and magnetization

In Chapter I the field was studied with the help of the material relations $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$; these imply that the fields $\mathbf{E}$ and $\mathbf{H}$ are regarded as producing at each point of the substance certain displacements $\mathbf{D}$ and $\mathbf{B}$ which are proportional to them. Instead of these 'multiplicative relations' we shall now describe the interaction of matter and field by means of 'additive relations',

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}, \tag{1}$$

$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}. \tag{2}$$

Here $\mathbf{P}$ is called the *electric polarization* and $\mathbf{M}$ the *magnetic polarization* or *magnetization*. The physical meaning of these quantities will become clear later; here we only observe that $\mathbf{P}$ and $\mathbf{M}$ both vanish in vacuum so that these quantities represent the influence of matter on the field in a simple intuitive way.

Assume the material to be nonconducting ($\sigma = 0$) and consider the field in a region where the current and charge densities are zero ($\rho = \mathbf{j} = 0$). Eliminating $\mathbf{D}$ and $\mathbf{H}$ from Maxwell's equations §1.1 (1)–(4) by means of (1) and (2), we obtain

$$\operatorname{curl} \mathbf{B} - \frac{1}{c} \dot{\mathbf{E}} = \frac{4\pi}{c} \tilde{\mathbf{j}}, \tag{3}$$

$$\operatorname{curl} \mathbf{E} + \frac{1}{c} \dot{\mathbf{B}} = 0, \tag{4}$$

$$\operatorname{div} \mathbf{E} = 4\pi \tilde{\rho}, \tag{5}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{6}$$

where the *free current density* $\tilde{\mathbf{j}}$ and the *free charge density* $\tilde{\rho}$ are defined by the relations

$$\tilde{\mathbf{j}} = \dot{\mathbf{P}} + c \operatorname{curl} \mathbf{M}, \tag{7}$$

$$\tilde{\rho} = -\operatorname{div} \mathbf{P}. \tag{8}$$

Eqs. (3)–(6) *are formally identical with the equations for the field in vacuum considered* in the previous section. Hence, as in the vacuum case, we may introduce a vector potential $\mathbf{A}$ and a scalar potential ϕ such that

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \tag{9}$$

$$\mathbf{E} = -\frac{1}{c} \dot{\mathbf{A}} - \operatorname{grad} \phi. \tag{10}$$

Moreover, if we impose, as before, the Lorentz condition

$$\operatorname{div}\mathbf{A} + \frac{1}{c}\dot{\phi} = 0, \tag{11}$$

we have, by analogy with §2.1 (11) and §2.1 (12), the following equations for $\mathbf{A}$ and ϕ:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = -\frac{4\pi}{c}\tilde{\mathbf{j}}, \tag{12}$$

$$\nabla^2\phi - \frac{1}{c^2}\ddot{\phi} = -4\pi\tilde{\rho}. \tag{13}$$

Now condition (11) is consistent with (12) and (13) provided that

$$\frac{\partial\tilde{\rho}}{\partial t} + \operatorname{div}\tilde{\mathbf{j}} = 0; \tag{14}$$

this relation is, in fact, satisfied identically, as can be immediately seen from (7) and (8), on using the vector identity $\operatorname{div}\operatorname{curl} \equiv 0$.

In terms of the polarization and magnetization, the solutions of (12) and (13) may, according to §2.1 (21) and §2.1 (22), and by (7) and (8), be expressed in the form

$$\phi = -\int \frac{1}{R}[\operatorname{div}'\mathbf{P}]\,\mathrm{d}V', \tag{15}$$

$$\mathbf{A} = \int \frac{1}{R}\left[\operatorname{curl}'\mathbf{M} + \frac{1}{c}\dot{\mathbf{P}}\right]\mathrm{d}V'. \tag{16}$$

Here the differential operators div' and curl' are taken with respect to the coordinates (x', y', z') of the variable point of integration $\mathbf{r}'$ at which the volume element $\mathrm{d}V'$ is situated, and the square brackets denote *retarded values*, i.e. they imply that the argument t is replaced by $t - R/c$ inside each bracket.

By a straightforward calculation the following identities can be verified:

$$\operatorname{div}'[\mathbf{P}] = [\operatorname{div}'\mathbf{P}] + \frac{1}{cR}\mathbf{R}\cdot[\dot{\mathbf{P}}], \tag{17a}$$

$$\operatorname{curl}'[\mathbf{M}] = [\operatorname{curl}'\mathbf{M}] + \frac{1}{cR}\mathbf{R}\times[\dot{\mathbf{M}}], \tag{17b}$$

where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}'. \tag{18}$$

Hence (15) and (16) may be written as

$$\phi = -\int\left\{\frac{1}{R}\operatorname{div}'[\mathbf{P}] - \frac{1}{cR^2}\mathbf{R}\cdot[\dot{\mathbf{P}}]\right\}\mathrm{d}V', \tag{19}$$

$$\mathbf{A} = \int\left\{\frac{1}{R}\operatorname{curl}'[\mathbf{M}] - \frac{1}{cR^2}\mathbf{R}\times[\dot{\mathbf{M}}] + \frac{1}{cR}[\dot{\mathbf{P}}]\right\}\mathrm{d}V'. \tag{20}$$

From the first term in each of these integrals we can separate an integral over the boundary surface. To do this we use the vector identities

$$\text{div}\left(\frac{1}{R}\mathbf{P}\right) = \frac{1}{R}\,\text{div}\,\mathbf{P} + \mathbf{P}\cdot\text{grad}\,\frac{1}{R}, \tag{21a}$$

$$\text{curl}\left(\frac{1}{R}\mathbf{M}\right) = \frac{1}{R}\,\text{curl}\,\mathbf{M} - \mathbf{M}\times\text{grad}\,\frac{1}{R}. \tag{21b}$$

Integrating these relations throughout an arbitrary finite domain, we obtain, on using the Gauss theorem,

$$\int\frac{1}{R}(\mathbf{n}\cdot\mathbf{P})\,\mathrm{d}S = \int\left\{\frac{1}{R}\,\text{div}\,\mathbf{P} + \mathbf{P}\cdot\text{grad}\,\frac{1}{R}\right\}\mathrm{d}V, \tag{22a}$$

$$\int\frac{1}{R}(\mathbf{n}\times\mathbf{M})\,\mathrm{d}S = \int\left\{\frac{1}{R}\,\text{curl}\,\mathbf{M} - \mathbf{M}\times\text{grad}\,\frac{1}{R}\right\}\mathrm{d}V, \tag{22b}$$

the integrals on the left extending over the surface bounding the domain, $\mathbf{n}$ being the unit outward normal to the surface.

Assume now that the substance (i.e. the region of space where $\mathbf{P}$ and $\mathbf{M}$ differ from zero) remains within a finite closed surface. If then the integrals (19) and (20) are extended over the volume inside this surface, the surface integrals in (22) vanish and (19) and (20) may be expressed in the form

$$\phi = \int\left\{[\mathbf{P}]\cdot\text{grad}'\,\frac{1}{R} + \frac{1}{cR^2}\mathbf{R}\cdot[\dot{\mathbf{P}}]\right\}\mathrm{d}V', \tag{23}$$

$$\mathbf{A} = \int\left\{[\mathbf{M}]\times\text{grad}'\,\frac{1}{R} - \frac{1}{cR^2}\mathbf{R}\times[\dot{\mathbf{M}}] + \frac{1}{cR}[\dot{\mathbf{P}}]\right\}\mathrm{d}V'. \tag{24}$$

From (23) and (24) the physical significance of $\mathbf{P}$ and $\mathbf{M}$ can be appreciated. For this purpose consider the case when $\mathbf{P}$ and $\mathbf{M}$ are zero everywhere except within a vanishingly small volume element at $\mathbf{r}_0(x_0, y_0, z_0)$. We may express this assumption formally with the help of Dirac's δ function (see Appendix IV):

$$\mathbf{P}(\mathbf{r}', t) = \mathbf{p}(t)\delta(\mathbf{r}' - \mathbf{r}_0), \tag{25}$$

$$\mathbf{M}(\mathbf{r}', t) = \mathbf{m}(t)\delta(\mathbf{r}' - \mathbf{r}_0). \tag{26}$$

(23) and (24) then become

$$\phi = [\mathbf{p}]\cdot\text{grad}_0\,\frac{1}{R} + \frac{1}{cR^2}\mathbf{R}\cdot[\dot{\mathbf{p}}], \tag{27}$$

$$\mathbf{A} = [\mathbf{m}]\times\text{grad}_0\,\frac{1}{R} - \frac{1}{cR^2}\mathbf{R}\times[\dot{\mathbf{m}}] + \frac{1}{cR}[\dot{\mathbf{p}}], \tag{28}$$

where now

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0, \qquad R = |\mathbf{r} - \mathbf{r}_0|, \tag{29}$$

and grad_0 implies that the operator is taken with respect to the coordinates x_0, y_0, z_0 of the point $\mathbf{r}_0$.

Eqs. (27) and (28) have a simple interpretation. Consider the electrostatic potential of two fixed electric charges $-e$ and $+e$ situated at points whose position vectors with respect to a fixed point $Q_0(\mathbf{r}_0)$ are $-\frac{1}{2}\mathbf{a}$ and $\frac{1}{2}\mathbf{a}$ respectively. The Coulomb potential $\phi(\mathbf{R})$ due to the charges is (see Fig. 2.1):

Fig. 2.1 Notation used in calculating the potentials of an electric dipole.

$$\phi = \frac{e}{R_2} - \frac{e}{R_1}. \tag{30}$$

If a is sufficiently small, we may expand $1/R_2$ in powers of the components of a and obtain

$$\frac{1}{R_2} = \frac{1}{R_1} + \mathbf{a} \cdot \mathrm{grad}\, \frac{1}{R_1} + \cdots, \tag{31}$$

where the operator grad is taken with respect to the coordinates of the point at which the charge $-e$ is situated. If we neglect the higher-order terms in (31), (30) becomes

$$\phi = e\mathbf{a} \cdot \mathrm{grad}\, \frac{1}{R_1} = e\frac{\mathbf{a} \cdot \mathbf{R}_1}{R_1{}^3}. \tag{32}$$

Suppose now that a is gradually decreased whilst e is increased without limits in such a way that $e\mathbf{a}$ approaches a finite value $\mathbf{p}$:

$$\lim_{a \to 0} e\mathbf{a} = \mathbf{p}. \tag{33}$$

Then R_1 approaches R and (32) becomes in the limit

$$\phi = \mathbf{p} \cdot \mathrm{grad}_0 \frac{1}{R}. \tag{34}$$

This expression is identical with (27) in the special case when $\mathbf{p}$ is independent of time.

If the charges depend on time but at each instant of time differ only in sign, we have, in place of (30),

$$\phi = \frac{e(t - R_2/c)}{R_2} - \frac{e(t - R_1/c)}{R_1} \tag{35}$$

and obtain by the same limiting process

$$\phi = [\mathbf{p}] \cdot \mathrm{grad}_0 \frac{1}{R} + \frac{1}{cR^2} \mathbf{R} \cdot [\dot{\mathbf{p}}],$$

which is (27). Thus (23) may be interpreted as the scalar potential of a *distribution of electric dipoles of moment* $\mathbf{P}$ *per unit volume*. The last term in (28) may be shown to be the magnetic potential arising from these dipoles.

In a similar way (28) may be shown to be the vector potential of a magnetic dipole of moment $\mathbf{m}(t)$, such a dipole being equivalent, of course, to an infinitesimal closed

circuit of area A normal to $\mathbf{m}$, carrying the current* $c\mathbf{m}/A$. Hence the first two terms in (24) may be interpreted as the vector potential of a *distribution of magnetic dipoles of moment* $\mathbf{M}$ *per unit volume*.

2.2.2 Hertz vectors

Instead of using the potentials $\mathbf{A}$ and ϕ one can represent the field in terms of another pair of potential functions $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ which depend on $\mathbf{P}$ and $\mathbf{M}$ in a much simpler way. They are known as the *Hertz vectors* or *polarization potentials*† and may be introduced by means of the relations

$$\mathbf{A} = \frac{1}{c}\dot{\mathbf{\Pi}}_e + \mathrm{curl}\,\mathbf{\Pi}_m, \tag{36}$$

$$\phi = -\mathrm{div}\,\mathbf{\Pi}_e. \tag{37}$$

$\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ are seen to bear the same mathematical relationship to $\mathbf{A}$ and ϕ as the polarizations $\mathbf{P}$ and $\mathbf{M}$ bear to $\tilde{j}/c$ and $\tilde{\rho}$.

The Lorentz condition is now automatically satisfied and (12) and (13) for $\mathbf{A}$ and ϕ will also be satisfied if $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ are solutions of the inhomogeneous wave equations

$$\nabla^2\mathbf{\Pi}_e - \frac{1}{c^2}\ddot{\mathbf{\Pi}}_e = -4\pi\mathbf{P}, \tag{38}$$

$$\nabla^2\mathbf{\Pi}_m - \frac{1}{c^2}\ddot{\mathbf{\Pi}}_m = -4\pi\mathbf{M}. \tag{39}$$

According to §2.1, particular solutions of these equations may be expressed in the form of retarded potentials

$$\mathbf{\Pi}_e = \int \frac{[\mathbf{P}]}{R}\,\mathrm{d}V', \tag{40}$$

$$\mathbf{\Pi}_m = \int \frac{[\mathbf{M}]}{R}\,\mathrm{d}V'. \tag{41}$$

The field vectors may be expressed directly in terms of $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ by simple differentiation. We have from (9) and (36),

$$\mathbf{B} = \mathrm{curl}\left(\frac{1}{c}\dot{\mathbf{\Pi}}_e + \mathrm{curl}\,\mathbf{\Pi}_m\right), \tag{42}$$

and from (10), (36) and (37),

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* For a discussion of Ampère's infinitesimal currents see any standard textbook on electricity, or Max Born, *Optik* (Berlin, Springer, 1933, reprinted 1965), pp. 305−306.
† $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$ have similar transformation properties as $\mathbf{E}$ and $\mathbf{B}$ (or $\mathbf{D}$ and $\mathbf{H}$), i.e. they form a *six vector* (antisymmetric tensor of the second rank). They are generalizations of a certain potential function introduced for the electromagnetic field of an oscillating dipole by H. Hertz, *Ann. d. Physik.*, **36** (1889), 1. The vector character of the Hertz potential was noted by A. Righi, *Nuovo Cimento*, **2** (1901), 104, who introduced the corresponding magnetic potential. A general theory of the Hertz vectors and the associated gauge transformation is due to F. Kottler, *Ann. d. Physik* (4), **71** (1923), 462 and in *Progress in Optics*, Vol. 6, ed. E. Wolf (Amsterdam, North-Holland Publishing Company, 1967), p. 331 (Appendix B). See also A. Nisbet, *Proc. Roy. Soc.*, A **231** (1955), 250, and W. H. McCrea, *ibid.*, **240** (1957), 447.

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\dot{\mathbf{\Pi}}_e + \operatorname{curl}\mathbf{\Pi}_m\right) + \operatorname{grad}\operatorname{div}\mathbf{\Pi}_e$$

$$= \operatorname{curl}\left(-\frac{1}{c}\dot{\mathbf{\Pi}}_m + \operatorname{curl}\mathbf{\Pi}_e\right) + \left(\nabla^2\mathbf{\Pi}_e - \frac{1}{c^2}\ddot{\mathbf{\Pi}}_e\right), \tag{43}$$

where the identity $\operatorname{grad}\operatorname{div} \equiv \operatorname{curl}\operatorname{curl} + \nabla^2$ has been used. According to (38) the term in the second bracket is equal to $-4\pi\mathbf{P}$; hence, using (1),

$$\mathbf{D} = \operatorname{curl}\left(-\frac{1}{c}\dot{\mathbf{\Pi}}_m + \operatorname{curl}\mathbf{\Pi}_e\right). \tag{44}$$

Finally, from (2), (42) and (39)

$$\mathbf{H} = \operatorname{curl}\left(\frac{1}{c}\dot{\mathbf{\Pi}}_e + \operatorname{curl}\mathbf{\Pi}_m\right) + \left(\nabla^2\mathbf{\Pi}_m - \frac{1}{c^2}\ddot{\mathbf{\Pi}}_m\right)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\dot{\mathbf{\Pi}}_m + \operatorname{curl}\mathbf{\Pi}_e\right) + \operatorname{grad}\operatorname{div}\mathbf{\Pi}_m. \tag{45}$$

It was shown in §2.1 that the potentials $\mathbf{A}$ and ϕ associated with a given field are not unique; any other pair $(\mathbf{A}', \phi')$ which can be derived from a possible $(\mathbf{A}, \phi)$ by means of a gauge transformation §2.1 (14) will represent the same field. The Hertz vectors too are not unique. They may be subjected to the following transformation, which leaves the field vectors invariant:

$$\left.\begin{aligned} \mathbf{\Pi}'_e &= \mathbf{\Pi}_e + \operatorname{curl}\mathbf{F} - \operatorname{grad}G, \\ \mathbf{\Pi}'_m &= \mathbf{\Pi}_m - \frac{1}{c}\dot{\mathbf{F}}, \end{aligned}\right\} \tag{46}$$

where the vector function $\mathbf{F}$ and the scalar function G are any solutions of the homogeneous wave equations

$$\left.\begin{aligned} \nabla^2\mathbf{F} - \frac{1}{c^2}\ddot{\mathbf{F}} &= 0, \\ \nabla^2 G - \frac{1}{c^2}\ddot{G} &= 0. \end{aligned}\right\} \tag{47}$$

The invariance under this transformation is readily verified by substituting from (46) into (36) and (37). $\mathbf{A}$ and ϕ are then transformed into $\mathbf{A}'$ and ϕ' according to §2.1 (14), with

$$\chi = -\frac{1}{c}\dot{G}.$$

Moreover, substituting from (46) and (38) and (39) shows that $\mathbf{\Pi}'_e$ and $\mathbf{\Pi}'_m$ also satisfy the inhomogeneous wave equations for the Hertz vectors.

2.2.3 The field of a linear electric dipole

For the purpose of later applications it will be useful to write down the full explicit solutions of the field equations for a field in vacuum produced by a linear electric

dipole, situated at a point $\mathbf{r}_0$ and vibrating in a fixed direction specified by a unit vector $\mathbf{n}$. Such a dipole is characterized by an electric polarization

$$\mathbf{P}(\mathbf{r},\ t) = p(t)\delta(\mathbf{r} - \mathbf{r}_0)\mathbf{n}, \tag{48}$$

where p is a function of time and δ the Dirac delta function. A general dipole whose orientation $\mathbf{n}$ is also a function of time is equivalent to three linear dipoles, with their moment vectors oriented along three mutually perpendicular directions.

According to (40) the electric Hertz vector associated with (48) is

$$\mathbf{\Pi}_e = \frac{p(t - R/c)}{R}\mathbf{n}, \tag{49}$$

where R is the distance of the point $\mathbf{r}$ from the point $\mathbf{r}_0$.

Since $\mathbf{\Pi}_m = 0$ and $\mathbf{\Pi}_e$ satisfies (except at the origin) the homogeneous wave equation for vacuum, (42)–(45) become

$$\mathbf{E} = \mathbf{D} = \operatorname{curl}\operatorname{curl}\mathbf{\Pi}_e, \tag{50}$$

$$\mathbf{B} = \mathbf{H} = \frac{1}{c}\operatorname{curl}\dot{\mathbf{\Pi}}_e. \tag{51}$$

Using the identity $\operatorname{curl}\operatorname{curl} \equiv \operatorname{grad}\operatorname{div} - \nabla^2$ and the relation $\nabla^2\mathbf{\Pi}_e = (1/c^2)\ddot{\mathbf{\Pi}}_e$, Eq. (50) may be re-written in the form

$$\mathbf{E} = \mathbf{D} = \operatorname{grad}\operatorname{div}\mathbf{\Pi}_e - \frac{1}{c^2}\ddot{\mathbf{\Pi}}_e. \tag{52}$$

From (49) we obtain, after a simple calculation,

$$\operatorname{div}\mathbf{\Pi}_e = -\left\{\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2}\right\}(\mathbf{n}\cdot\mathbf{R}),$$

$$\operatorname{grad}\operatorname{div}\mathbf{\Pi}_e = \left\{\frac{3[p]}{R^5} + \frac{3[\dot{p}]}{cR^4} + \frac{[\ddot{p}]}{c^2R^3}\right\}(\mathbf{n}\cdot\mathbf{R})\mathbf{R} - \left\{\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2}\right\}\mathbf{n},$$

$$\operatorname{curl}\mathbf{\Pi}_e = \left\{\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2}\right\}(\mathbf{n}\times\mathbf{R}),$$

where, as before, the square brackets denote retarded values. Substituting into (52) and (51) we obtain the required formulae for the field vectors:

$$\mathbf{E} = \mathbf{D} = \left\{\frac{3[p]}{R^5} + \frac{3[\dot{p}]}{cR^4} + \frac{[\ddot{p}]}{c^2R^3}\right\}(\mathbf{n}\cdot\mathbf{R})\mathbf{R} - \left\{\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2} + \frac{[\ddot{p}]}{c^2R}\right\}\mathbf{n}, \tag{53}$$

$$\mathbf{B} = \mathbf{H} = \left\{\frac{[\dot{p}]}{cR^3} + \frac{[\ddot{p}]}{c^2R^2}\right\}(\mathbf{n}\times\mathbf{R}). \tag{54}$$

We shall also later need expressions for $\mathbf{E}$ and $\mathbf{H}$ in terms of spherical polar coordinates. Taking the $\mathbf{n}$ direction as the z-axis (see Fig. 2.2), and denoting by $\mathbf{i}_R$, $\mathbf{i}_\theta$ and $\mathbf{i}_\psi$ the unit vectors in the direction of increasing R, θ and ψ we have

$$\mathbf{R} = R\mathbf{i}_R, \qquad \mathbf{n} = (\cos\theta)\mathbf{i}_R - (\sin\theta)\mathbf{i}_\theta,$$

$$(\mathbf{n}\cdot\mathbf{R}) = R\cos\theta, \qquad (\mathbf{n}\times\mathbf{R}) = (R\sin\theta)\mathbf{i}_\psi, \tag{55}$$
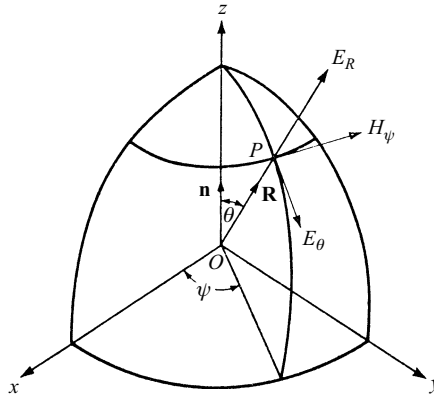
and (53) and (54) give

Fig. 2.2 Calculation of a linear electric dipole with momentum vector along the *z*-axis.

$$\mathbf{E} = E_R \mathbf{i}_R + E_\theta \mathbf{i}_\theta, \qquad \mathbf{H} = H_\psi \mathbf{i}_\psi, \tag{56}$$

where

$$
\left.
\begin{aligned}
E_R &= 2\left(\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2}\right)\cos\theta, \\[2mm]
E_\theta &= \left(\frac{[p]}{R^3} + \frac{[\dot{p}]}{cR^2} + \frac{[\ddot{p}]}{c^2 R}\right)\sin\theta, \\[2mm]
H_\psi &= \left(\frac{[\dot{p}]}{cR^2} + \frac{[\ddot{p}]}{c^2 R}\right)\sin\theta.
\end{aligned}
\right\} \tag{57}
$$

Thus the electric vector lies in the meridional plane through the axis of the dipole, and the magnetic vector is perpendicular to this plane.

Of particular interest is the field in regions which are so far away from the dipole that all terms but those in $1/R$ may be neglected in the above expressions. This *wave-* (or *radiation-*)*zone* is characterized by

$$R \gg c\left|\frac{p}{\dot{p}}\right|, \qquad R \gg c\left|\frac{\dot{p}}{\ddot{p}}\right|. \tag{58}$$

In this region

$$E_\theta \sim H_\psi \sim \frac{[\ddot{p}]}{c^2 R}\sin\theta, \tag{59}$$

and the other components are negligible. Hence, in the wave zone, **E** and **H** are of equal magnitude and perpendicular to each other and to the radius vector **R**, which now coincides with the direction of the Poynting vector. In this zone the structure of the field of a linear electric dipole is therefore similar to that of a plane wave. However, on each 'wave surface' (sphere centred on $\mathbf{r}_0$) the field vectors now vary from point to point, diminishing in magnitude from the equator to the poles, being zero on the axis

of the oscillator. The dipole therefore does not radiate energy in the direction of its axis.

Let us now calculate the amount of energy radiated per second across each of the spherical wave surfaces. It is given by the integral of the magnitude S of the Poynting vector taken over this surface. In the wave zone,

$$S = \frac{c}{4\pi} |\mathbf{E} \times \mathbf{H}| = \frac{c}{4\pi} |E_\theta H_\psi| = \frac{[\ddot{p}]^2}{4\pi c^3 R^2} \sin^2 \theta. \tag{60}$$

Hence the total amount of energy which is radiated across the spherical surface per second is ($d\sigma$ denoting element of the surface)

$$\int S \, d\sigma = \frac{[\ddot{p}]^2}{4\pi c^3 R^2} \int_0^\pi \sin^2 \theta \cdot 2\pi R^2 \sin \theta \, d\theta$$

$$= \frac{2[\ddot{p}]^2}{3c^3}. \tag{61}$$

Consider the special case when $p(t)$ is periodic with angular frequency ω:

$$p(t) = p_0 \, e^{-i\omega t}, \tag{62}$$

where p_0 is a complex constant, and the real part of (62) is understood to represent p. The two conditions (58) now reduce to the single condition

$$R \gg \lambda/2\pi \qquad (\lambda = 2\pi c/\omega) \tag{63}$$

and (59) gives, for the nonvanishing components of the field in the wave zone,

$$E_\theta = H_\psi \sim -\left(\frac{\omega}{c}\right)^2 p_0 \sin \theta \frac{e^{-i\omega(t-R/c)}}{R}, \tag{64}$$

it being understood that the real part of the expression on the right-hand side of (64) is taken. The amount of energy crossing per second a unit area of the spherical surface in the wave zone is

$$S = \frac{c}{4\pi} \left(\frac{\omega}{c}\right)^4 \frac{|p_0|^2}{R^2} \sin^2 \theta \cos^2[\omega(t - R/c) - \alpha], \tag{65}$$

α being the phase of p_0.

On taking the time average over an interval of time which is large compared with the period $T = 2\pi/\omega$, (65) reduces to

$$\langle S \rangle = \frac{|p_0|^2}{8\pi R^2} \frac{\omega^4}{c^3} \sin^2 \theta. \tag{66}$$

The (time averaged) energy crossing the whole surface per second is therefore

$$\int \langle S \rangle \, d\sigma = \frac{1}{3} \frac{|p_0|^2 \omega^4}{c^3}. \tag{67}$$

2.3 The Lorentz–Lorenz formula and elementary dispersion theory[*]

2.3.1 The dielectric and magnetic susceptibilities

The interpretation of the potentials in terms of polarization and magnetization is of fundamental importance in the atomic theory of matter. From the standpoint of such a theory the additive relations §2.2 (1) and (2) connecting **E** and **D** and **P** and **H** with **B** and **M** have a more direct physical meaning than the multiplicative relations [the material equations §1.1 (10) and §1.1 (11)].

In the atomic theory one regards matter as composed of interacting particles (atoms and molecules) embedded in the vacuum. These entities produce a field which has large local variations in the interior of the matter. This internal field is modified by any field which is applied externally; the properties of the matter are then derived by averaging over the total field within it. As long as the region over which the average is taken is large compared with the linear dimensions of the particles, the electromagnetic properties of each can be simply described by an electric and a magnetic dipole; the secondary field is then the field due to these dipoles (with retardation). This is, in fact, exactly what we have just described by regarding matter as a continuous distribution interacting with the field: it corresponds to the first approximation (for slow variation in space) of the atomic theory. In this approximation for sufficiently weak fields† one can assume **P** and **M** to be proportional to **E** and **H** respectively:

$$\mathbf{P} = \eta\mathbf{E}, \qquad \mathbf{M} = \chi\mathbf{H}. \tag{1}$$

The factor η is called the *dielectric susceptibility* and χ the *magnetic susceptibility*, the latter being a concept current in practical magnetism. From (1) and from §2.2 (1) and §2.2 (2) it follows, on comparison with the material equations §1.1 (10) and §1.1 (11), that the dielectric constant ε and the magnetic permeability μ are related to the dielectric and magnetic susceptibilities by the formulae

$$\varepsilon = 1 + 4\pi\eta, \qquad \mu = 1 + 4\pi\chi. \tag{2}$$

By means of (2) the complete formal equivalence of the 'multiplicative' and 'additive' treatments is secured. But such formal comparison of the constants makes no use of any of the consequences of the atomistic structure. While a detailed discussion involving the results of the atomistic theory is outside the scope of this book, it will be useful to introduce some of the relevant concepts and formulae. With their help we shall obtain a clearer understanding of the physical content of the integral equations which will be introduced in §2.4 in place of the usual differential equations of Maxwell's theory.

The simplest assumption which leads one step into the atomistic theory is to regard

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[*] In this section we depart from our deductive treatment in order to obtain a clearer understanding of the phenomena before formulating the integral equations which will be introduced in §2.4 in place of Maxwell's differential equations. The considerations of this section are more illustrative than rigorous, since a rigorous treatment would involve most of the atomistic theory of matter and take us far beyond the reasonable bounds of this book.

† A field which is considered strong on the atomic level is one in which the electrostatic energy of an electron changes by an appreciable fraction of the ionization potential of the atom in a distance equal to the atomic diameter, i.e. a field of millions of volts per centimetre. Electrical breakdowns in real materials actually occur at much smaller fields.

matter as being made up of certain physical entities — the molecules — which are polarizable, so that under the influence of an external field they show electric and magnetic moments. In the first approximation it may be assumed that the components of these moments are linear functions of the components of the field; in general the direction of the moment vector does not coincide with the direction of the field. This assumption has many consequences, which can only be briefly studied here. We shall confine our attention to isotropic nonmagnetic substances, and examine first the dependence of the electric constants on the density in a substance composed of similar molecules. We shall also study the dependence of the refractive index on frequency. Only somewhat oversimplified arguments can be put forward here, but a rigorous, though a more formal, derivation of the main result (the Lorentz–Lorenz formula) will be given in §2.4.

2.3.2　The effective field

To begin with we must distinguish between the *effective field* $\mathbf{E}'$, $\mathbf{H}'$ acting on a molecule and the *mean* or *observed* field $\mathbf{E}$, $\mathbf{H}$ obtained by averaging over a region which contains a great number of molecules. The difference between the two fields is due to the gaps between the molecules and depends on the number of molecules per unit volume (their number density N).

To estimate the difference $\mathbf{E}' - \mathbf{E}$, consider a particular molecule and imagine it to be surrounded by a small sphere whose radius is nevertheless large compared with its linear dimensions. We shall consider separately the effects on the central molecule produced by the matter outside and inside this sphere.

In determining the effect of the matter *outside* the sphere we may clearly neglect the molecular structure and treat the substance as continuous. We may then assume that, outside the sphere, the polarization $\mathbf{P}$ produced by the mean electric field is constant. Concerning the effect of the molecules *inside* the sphere we shall assume, as may be shown for a number of important special cases, including that of random distribution, that they do not produce any resulting field at the central molecule. Hence we may regard the molecule as being situated in a spherical region, inside which there is vacuum and outside which there is a homogeneously polarized medium. We must then determine the potential ϕ of this configuration, i.e. of the free charges on the spherical discontinuity surface at which $\mathbf{P}$ changes from the interior value zero to a constant exterior value.

For this purpose consider the potential $\tilde{\phi}$ of the 'complementary' configuration, namely of a homogeneously polarized sphere surrounded by vacuum. The superposition of these two configurations is a homogeneously polarized substance with no boundary. Hence the potential due to the boundary is zero and we have

$$\phi + \tilde{\phi} = 0. \tag{3}$$

Now the potential $\tilde{\phi}$ can be immediately obtained from §2.2 (23) by taking $\mathbf{P}$ to be constant. This gives

$$\phi = -\tilde{\phi} = -\mathbf{P} \cdot \int \operatorname{grad}' \frac{1}{R} \, dV'. \tag{4}$$

Since

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

we may replace grad$'$ by $-$grad in (4), and write

$$\phi = -\tilde{\phi} = \mathbf{P} \cdot \mathrm{grad} \int \frac{\mathrm{d}V'}{R} = -\mathbf{P} \cdot \mathrm{grad}\, \phi_0, \tag{5}$$

where

$$\phi_0 = -\int \frac{\mathrm{d}V'}{R}. \tag{6}$$

Eq. (6) may be interpreted as the potential of a uniformly charged sphere of charge density -1. Hence it satisfies the Poisson equation

$$\nabla^2 \phi_0 = 4\pi. \tag{7}$$

Now the components of the field strengths associated with the potential (5) are

$$-\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \left(P_x \frac{\partial \phi_0}{\partial x} + P_y \frac{\partial \phi_0}{\partial y} + P_z \frac{\partial \phi_0}{\partial z} \right)$$

$$= P_x \frac{\partial^2 \phi_0}{\partial x^2} + P_y \frac{\partial^2 \phi_0}{\partial x \partial y} + P_z \frac{\partial^2 \phi_0}{\partial x \partial z} \tag{8}$$

etc. At the centre of the sphere we have, by symmetry,

$$\frac{\partial^2 \phi_0}{\partial x \partial y} = \frac{\partial^2 \phi_0}{\partial y \partial z} = \frac{\partial^2 \phi_0}{\partial z \partial x} = 0, \tag{9}$$

and

$$\frac{\partial^2 \phi_0}{\partial x^2} = \frac{\partial^2 \phi_0}{\partial y^2} = \frac{\partial^2 \phi_0}{\partial z^2}. \tag{10}$$

Using (7) it follows that each term in (10) is equal to $4\pi/3$; (8) then shows that the required contribution to the effective field strength is

$$-\nabla \phi = \frac{4\pi}{3} \mathbf{P}. \tag{11}$$

The total field inside the sphere, which is the effective field acting on the central molecule, is obtained by adding to this the mean field $\mathbf{E}$, giving

$$\mathbf{E}' = \mathbf{E} + \frac{4\pi}{3} \mathbf{P}. \tag{12}$$

For magnetizable substances there is a corresponding relation between $\mathbf{H}'$, $\mathbf{H}$ and $\mathbf{M}$. As we shall, however, treat only nonmagnetic substances we shall always have $\mathbf{H}' = \mathbf{H}$.

Next we must connect the polarization $\mathbf{P}$ with the density.

2.3.3 The mean polarizability: the Lorentz–Lorenz formula

As already explained, it will be assumed that, for each molecule, the electric dipole moment **p** established under the influence of the field is proportional to the effective field* **E**′

$$\mathbf{p} = \alpha \mathbf{E}'. \tag{13}$$

The molecule is here assumed to be isotropic;† but as we are interested only in the average effect over all possible orientations of the molecule, it will not be necessary to assume that each *individual* molecule is isotropic. α will then be regarded as representing the *mean polarizability*, and since **p** is of dimensions $[el]$ and **E**′ of dimensions $[el^{-2}]$ (e = charge, l = length), α is seen to have the dimension $[l^3]$, i.e. those of a volume.

If, as before, N is the number of molecules per unit volume, the total *electric moment* **P** *per unit volume* is given by

$$\mathbf{P} = N\mathbf{p} = N\alpha \mathbf{E}'. \tag{14}$$

On eliminating **E**′ between (12) and (14) we obtain the first formulae (1), with the following explicit expression for the dielectric susceptibility:

$$\eta = \frac{N\alpha}{1 - \dfrac{4\pi}{3}N\alpha}. \tag{15}$$

If now we substitute for η into the first formula in (2) we obtain the following expression for the dielectric constant:

$$\varepsilon = \frac{1 + \dfrac{8\pi}{3}N\alpha}{1 - \dfrac{4\pi}{3}N\alpha}. \tag{16}$$

Conversely (16) gives information about the dependence of the mean polarizability on ε and N, or (if Maxwell's relation $\varepsilon = n^2$ is used) on n and N:

$$\alpha = \frac{3}{4\pi N}\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{3}{4\pi N}\frac{n^2 - 1}{n^2 + 2}. \tag{17}$$

By a remarkable coincidence, the relation (17) was discovered independently and practically at the same time by two scientists of almost identical names,‡ Lorentz and Lorenz, and is accordingly called the *Lorentz–Lorenz formula*. It is seen to be a bridge

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* We are restricting our discussion to so-called nonpolar molecules, having no permanent dipole moment in the absence of the field.
† In general the polarizability α is a tensor of the second rank.
‡ H. A. Lorentz, *Wiedem. Ann.*, **9** (1880), 641; L. Lorenz, *Wiedem. Ann.*, **11** (1881), 70.
    An analogous formula for statical fields was derived earlier by R. Clausius, *Mechanische Wärmetheorie*, Vol. 2 (Braunschweig, 2nd ed. 1879), p. 62, and by O. F. Mossotti, *Mem. Soc. Sci. Modena*, **14** (1850), 49. These authors tried to explain the dielectric properties of insulators on the assumption that the atoms are small conducting spheres whose mutual distances are large compared to their diameters, and derived the following expression for the number density g of the spheres in terms of the dielectric constant:

$$g = \frac{\varepsilon - 1}{\varepsilon + 2}.$$

Table 2.1. *The molar refractivity A of air at different*
pressures and at about 14.5 °C, for sodium D light.

Pressure in atm	n	A
1.00	1.0002929	4.606
42.13	1.01241	4.661
96.16	1.02842	4.713
136.21	1.04027	4.743
176.27	1.05213	4.772

which connects Maxwell's phenomenological theory with the atomistic theory of matter.

Instead of the mean polarizability α one often uses another quantity A, called *molar refractivity*. (In the case of monatomic substances it is also called *atomic refractivity*.) This is essentially the total polarizability of a mole of the substance,[*] being defined as

$$A = \frac{4\pi}{3} N_m \alpha. \tag{18}$$

Here $N_m = 6.02 \times 10^{23}$ is the Avogadro number,[†] namely the number of molecules in a mole. If W is the molecular weight, ρ the density, p the pressure and T the absolute temperature, then the molar volume is

$$\frac{N_m}{N} = \frac{W}{\rho} = \frac{RT}{p}, \tag{19}$$

where Boyle's law was used. Hence the molar refractivity may be written in the alternative forms

$$A = \frac{W}{\rho} \frac{n^2 - 1}{n^2 + 2} = \frac{RT}{p} \frac{n^2 - 1}{n^2 + 2}. \tag{20}$$

Eq. (20) shows that A has the dimensions and the order of magnitude of a molar volume.

For a gas, n^2 is nearly unity, so that

$$\alpha \sim \frac{n^2 - 1}{4\pi N}, \qquad A \sim \frac{W}{\rho} \frac{n^2 - 1}{3} = \frac{RT}{p} \frac{n^2 - 1}{3}. \tag{21}$$

The formula (20) gives the explicit dependence of the refractive index, for light of any one particular colour, on the density, and it should hold when the density is changed provided that isotropy is preserved. In the case of *gases* we actually find that $n^2 - 1$ and ρ are very nearly proportional to each other as demanded by the second formula in (21). But even at high pressure, when n will differ appreciably from unity, the molar refractivity remains substantially constant, as illustrated in Table 2.1. The

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[*] A mole or a gram-molecule is that amount of the substance whose weight, in grams, is equal to the molecular weight of the substance.

[†] This is the name internationally accepted; however, in the German literature the term *Loschmidt's number* is used, after the physicist who first gave a numerical estimate for it from experimental data.

Table 2.2. *Molar refractivity A of different compounds for sodium D light; in the columns giving n and A the first entry refers to liquid state, the second to vapour. The densities of the vapours are not stated but were calculated assuming perfect gas law from the known density of hydrogen.*

Substance	Formula	W	n	ρ	A
Oxygen	O_2	32	1.221	1.124	4.00
			1.000271		4.05
Hydrochloric acid	HCl	36.5	1.245	0.95	5.95
			1.000447		6.68
Water	H_2O	18	1.334	1.00	3.71
			1.000249		3.72
Carbon disulphide	CS_2	76	1.628	1.264	21.33
			1.00147		21.99
Acetone	C_3H_6O	58	1.3589	0.791	16.14
			1.00108		16.16

Table 2.3. *Molar refractivity for sodium D light of mixtures of water and sulphuric acid at 15 °C.*

Weight per cent	$\dfrac{N_1}{N_1 + N_2}$	W	n	ρ	A (interpolated from (22))	A (calculated from (20))
0	0	18.00	1.3336	0.9991	3.72	3.72
19.98	0.044	21.52	1.3578	1.1381	4.19	4.15
39.76	0.109	26.72	1.3817	1.2936	4.80	4.81
59.98	0.216	35.28	1.4065	1.4803	5.86	5.86
80.10	0.425	52.00	1.4308	1.6955	7.95	7.93
100	1	98	1.4277	1.8417	13.68	13.68

molar refractivity is also found to remain practically constant when the gas is condensed into a liquid (Table 2.2).

From (20) it can be shown that, to a good approximation, the molar refractivity of a mixture of two substances is equal to the sum of the contributions due to each substance. Thus, if two liquids of refractivities A_1 and A_2 are mixed, and if a unit volume of the first liquid contains N_1 molecules and of the second N_2 molecules, then the molar refractivity of the mixture will be

$$A = \frac{N_1 A_1 + N_2 A_2}{N_1 + N_2}. \tag{22}$$

This result is also found to be in reasonably good agreement with experiment (see Table 2.3).

Finally, we consider the dependence of the molar refractivity A of a compound on the atomic refractivities of its constituent atoms. If the molecule consists of $\mathcal{N}_1$ atoms of refractivity A_1 and of $\mathcal{N}_2$ atoms of refractivity A_2, etc., then obviously

Table 2.4. *Atomic refractivities of various
elements for sodium D light.*

Element	Formula	Atomic weight	A
Hydrogen	H	1	1.02
Carbon	C	12	2.11
Oxygen	O	16	2.01
Sulphur	S	32	8.23
Chlorine	Cl	35.5	5.72

Table 2.5. *The molar refractivities of various compounds for
sodium D light.*

Substance	Formula	A (calculated from (20))	A (calculated from (23))
Hydrochloric acid	HCl	6.68	6.74
Water	H_2O	3.72	4.05
Carbon disulphide	CS_2	21.97	18.57
Acetone	C_3H_6O	16.14	14.46

$$A = \mathcal{N}_1 A_1 + \mathcal{N}_2 A_2. \tag{23}$$

This formula also agrees fairly well with experiment, as is seen from Table 2.5. The refractivities of the constituent atoms are given in Table 2.4.

The good agreement obtained in this case is really better than one could reasonably expect, on the basis of our simple considerations. Refinements are in fact necessary in some cases; for example in the case of nitrogen one has to attribute different atomic polarizabilities to the nitrogen atom according to the compounds which it enters. With the necessary modifications, the polarizability α is quite a well-defined atomic parameter which may be used also in the study of other phenomena, such as the van der Waals cohesion forces, adsorption phenomena on surfaces, etc. These can only be satisfactorily treated, however, with the help of quantum mechanics.

2.3.4 Elementary theory of dispersion

In §1.2 it was pointed out that the phase velocity and consequently the refractive index cannot be constants of the medium as suggested by our earlier formal considerations, but that these quantities must in fact depend on frequency. The variation of the refractive index with frequency constitutes the phenomenon of *dispersion*. For an adequate treatment of dispersion, it would be necessary to go deeply into the atomic theory of matter, but it is possible to give a simplified model of a dispersing medium (essentially due to H. A. Lorentz) by making use of one or two basic results concerning the structure of molecules.

A molecule consists of a number of heavy particles (the atomic nuclei of the atoms

forming the molecule) around which light particles (electrons) revolve. The electrons carry negative and the nuclei positive charges. In neutral molecules the charges of the electrons just compensate those of the nuclei. However, the centres of the positive (nuclear) charges and that of the negative (electronic) charges may not coincide; this system has then an electric dipole and is called *polar*. For the sake of simplicity we shall here exclude polar molecules, although these play a great part in many physical and chemical phenomena.

If a nonpolar molecule is subjected to an electric field, the electrons and nuclei are displaced and a dipole moment is generated. The vectorial sum of all the dipole moments of the molecules in the unit of volume is essentially the polarization vector **P** introduced formally in the preceding section.

In order to determine the dependence of the polarization and of the refractive index on the frequency of the field we must first find the displacement **r** of each charged particle from its equilibrium position. We may assume that each electron is acted on by the Lorentz force **F** [see §1.1 (34)]:

$$\mathbf{F} = e\left(\mathbf{E}' + \frac{\mathbf{v}}{c} \times \mathbf{B}'\right),$$

where e is the charge of the electron and **v** its velocity. It will be assumed that the velocity of the electron is small in comparison with the velocity c of light in vacuum, so that the contribution of the magnetic field may be neglected in the expression for the Lorentz force. The rigorous determination of the effective displacement of the nuclei and electrons under the action of the electric force is a complicated problem of quantum mechanics. It is, however, plausible, and actually confirmed by the rigorous theory, that the electron behaves, to a good approximation, as if it were bound to an equilibrium position by a quasi-elastic restoring force

$$\mathbf{Q} = -q\mathbf{r}.$$

Hence if m denotes the mass of the electron, its equation of motion is

$$m\ddot{\mathbf{r}} + q\mathbf{r} = e\mathbf{E}'. \tag{24}$$

Let ω be the angular frequency of the incident field,

$$\mathbf{E}' = \mathbf{E}_0'\, e^{-i\omega t}, \tag{25}$$

and take as trial solution of (24) the expression

$$\mathbf{r} = \mathbf{r}_0\, e^{-i\omega t}. \tag{26}$$

(24) then gives the required stationary solution

$$\mathbf{r} = \frac{e\mathbf{E}'}{m(\omega_0{}^2 - \omega^2)}, \tag{27}$$

where

$$\omega_0 = \sqrt{\frac{q}{m}} \tag{28}$$

is called the *resonance* (or *absorption*) *frequency*. According to (27) the electron oscillates with the frequency of the incident field.

Each electron contributes to the polarization a moment $\mathbf{p} = e\mathbf{r}$. There will also be contributions from the nuclei; but since the nuclear masses are heavy in comparison with the masses of the electrons, their contributions may in the first approximation be neglected. Also, assuming for the moment that there is only one effective electron in a molecule with resonance frequency ω_0, we obtain for the total polarization $\mathbf{P}$ the expression

$$\mathbf{P} = N\mathbf{p} = Ne\mathbf{r} = N\frac{e^2}{m}\frac{\mathbf{E}'}{(\omega_0^2 - \omega^2)}. \tag{29}$$

Comparison of (29) with (14) gives

$$N\alpha = N\frac{e^2}{m(\omega_0^2 - \omega^2)}, \tag{30}$$

expressing the 'density of polarizability' in terms of atomic parameters. Thus it turns out that the quantity α is not a constant, as it ought to be if the ε in (17) denotes the static dielectric constant. It is convenient to introduce the notion of a frequency-dependent dielectric constant $\varepsilon(\omega)$ defined by Maxwell's relation $\varepsilon = n^2$, where n is the refractive index, a function $n(\omega)$ of ω. The static dielectric constant is then the value $\varepsilon(0) = n^2(0)$; it corresponds, according to (17), to the limiting value $N\alpha(0)$ which is found from (30) to be

$$N\alpha(0) = \frac{Ne^2}{m\omega_0^2}; $$

if this is introduced into (16), one obtains the static constant $\varepsilon(0)$.

For $\omega \neq 0$, the function $N\alpha(\omega)$ is, according to (30), monotonically increasing with ω, but has an infinity (resonance point) for $\omega = \omega_0$; for $\omega > \omega_0$, it approaches with increasing ω the value 0 from the negative side. Substituting from (30) into (17), we find the explicit dependence of the refractive index on frequency:

$$\frac{n^2 - 1}{n^2 + 2} = \frac{4\pi}{3}\frac{Ne^2}{m(\omega_0^2 - \omega^2)}. \tag{31}$$

For a gas, n is close to unity so that we may set $n^2 + 2 \sim 3$ in the denominator on the left-hand side and obtain

$$n^2 - 1 \sim 4\pi N\alpha = \frac{4\pi Ne^2}{m(\omega_0^2 - \omega^2)}. \tag{32}$$

It is seen that n is an increasing function of the frequency. The dispersion is then said to be *normal*. Further $n \gtrless 1$ according as $\omega \lessgtr \omega_0$, and n approaches unity with increasing ω (see Fig. 2.3).

At the resonance frequency ($\omega = \omega_0$) n and α are actually not infinite as our formula suggests. The singularity arises only formally, because we neglected the effect of damping. Damping is in fact an essential factor in the whole process as the vibrating electrons emit electromagnetic waves which carry away energy. But there are also other reasons for the dissipation of energy (e.g. because of collisions between the atoms). Formally damping may be taken into account by adding in the equation of motion (24) a term $g\dot{\mathbf{r}}$ representing a resisting force:
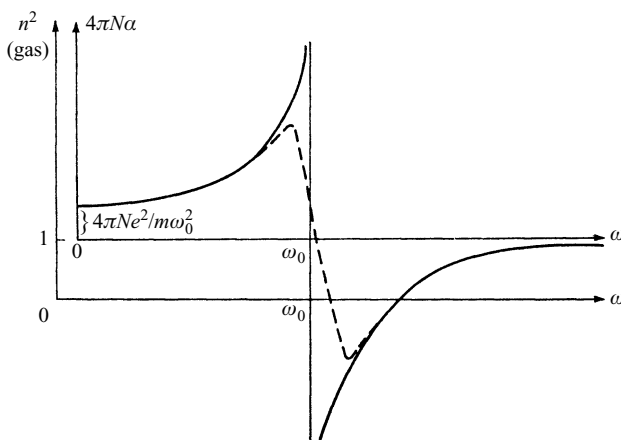
Fig. 2.3 Dispersion curves: full curve – effect of damping neglected; dashed curve – effect of damping taken into account. (The ordinates represent the real parts of n^2 and of $4\pi N\alpha$.)

$$m\ddot{\mathbf{r}} + g\dot{\mathbf{r}} + q\mathbf{r} = e\mathbf{E}'. \tag{33}$$

Instead of (27) we now have

$$\mathbf{r} = \frac{e\mathbf{E}'}{m(\omega_0^2 - \omega^2) - i\omega g}. \tag{34}$$

The polarization, and hence also $N\alpha$, becomes a complex quantity. It can be shown (cf. the similar situation in the theory of metals where a complex refractive index is used — Chapter XIV) that the modulus of this complex function is — apart from a small factor, due to absorption — the actual 'density of polarizability' and this is shown by the dashed curve in Fig. 2.3. The curve has a sharp maximum at a value of ω which is slightly smaller than ω_0, and a sharp minimum at a value slightly greater than ω_0. Between the maximum and minimum the function decreases with increasing frequency and we speak of a region of *anomalous dispersion*. The rays of shorter wavelength are then refracted less than those of longer wavelength, and this results in the reversal of the usual sequence of prismatic colours. For our purposes the region of anomalous dispersion is of little importance, since the absorption frequencies of free atoms lie almost exclusively in the ultra-violet region of the spectrum. The refractive index for visible light is then always greater than unity.

So far we have assumed that the system has only one resonance frequency. In general there will be many such frequencies, even in systems with the same kind of molecules; (31) and (32) must then be replaced by more general expressions. Neglecting for the moment again the motion of the nuclei, one has, in place of (31),

$$\frac{4\pi}{3}N\alpha = \frac{n^2 - 1}{n^2 + 2} = \frac{4\pi}{3}N\frac{e^2}{m}\sum_k \frac{f_k}{\omega_k^2 - \omega^2}, \tag{35}$$

where Nf_k is the number of electrons corresponding to the resonance frequency ω_k. For gases ($n \sim 1$) we may rewrite (35) in the form

$$n^2 - 1 = 4\pi N\alpha = \sum_k \frac{\rho_k}{\nu_k^2 - \nu^2} = \sum_k \frac{\rho_k}{c^2} \frac{\lambda^2 \lambda_k^2}{\lambda^2 - \lambda_k^2}, \tag{36}$$

where

$$\left.\begin{aligned} \rho_k &= N \frac{e^2}{\pi m} f_k, \\ \nu_k &= \frac{\omega_k}{2\pi} = \frac{c}{\lambda_k}, \qquad \nu = \frac{\omega}{2\pi} = \frac{c}{\lambda}. \end{aligned}\right\} \tag{37}$$

Using the identity

$$\frac{\lambda^2}{\lambda^2 - \lambda_k^2} = 1 + \frac{\lambda_k^2}{\lambda^2 - \lambda_k^2},$$

(36) becomes

$$n^2 - 1 = a + \sum_k \frac{b_k}{\lambda^2 - \lambda_k^2}, \tag{38}$$

with

$$\left.\begin{aligned} a &= \sum_k \frac{\rho_k}{\nu_k^2} = \frac{1}{c^2} \sum_k \rho_k \lambda_k^2, \\ b_k &= \frac{c^2 \rho_k}{\nu_k^4} = \frac{1}{c^2} \rho_k \lambda_k^4. \end{aligned}\right\} \tag{39}$$

For the representation of the refractive index over the whole visible range it is usually sufficient to take account of only one or two resonance frequencies in the ultraviolet region. Koch,[*] for example, found that for the range from $\lambda = 0.436\,\mu$m to $\lambda = 8.68\,\mu$m, the following formula holds for hydrogen, oxygen and air:

$$n^2 - 1 = a + \frac{b}{\lambda^2 - \lambda_0^2}, \tag{40}$$

where the constants, a, b and λ_0 are shown in Table 2.6.

In a spectral range which contains no resonance frequencies, (38) may with a good accuracy be replaced by a simpler formula. For all substances which appear transparent to the eye, the *visible* domain of the spectrum is such a range. Denoting by ν_v the absorption frequencies which are on the short wavelength side (violet), and by ν_r those on the long wavelength side (red), the dispersion formula (36) becomes, on expanding into power series with respect to ν and λ respectively:

$$\left.\begin{aligned} n^2 - 1 &= A + B\nu^2 + C\nu^4 + \cdots - \frac{B'}{\nu^2} - \frac{C'}{\nu^4} - \cdots \\ &= A + \frac{Bc^2}{\lambda^2} + \frac{Cc^4}{\lambda^4} + \cdots - \frac{B'\lambda^2}{c^2} - \frac{C'\lambda^4}{c^4} - \cdots, \end{aligned}\right\} \tag{41}$$

where

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[*] J. Koch, *Nov. Act. Soc. Ups.* (4), **2** (1909), 61.

Table 2.6. *Dispersion constants for oxygen, hydrogen and air between $\lambda = 0.436\ \mu m$ and $\lambda = 8.68\ \mu m$ at 0 °C and 760 mm Hg.*

Gas	$a \times 10^8$	$b \times 10^8$	$\lambda_0{}^2$ in 10^{-8} cm^2	ν_0 in 10^{15} s^{-1}
Hydrogen	27, 216	211.2	0.007760	3.40
Oxygen	52, 842	369.9	0.007000	3.55
Air	57, 642	327.7	0.005685	3.98

$$\left.\begin{array}{ll} A = \sum_v \dfrac{\rho_v}{\nu_v^2}, & B = \sum_v \dfrac{\rho_v}{\nu_v^4}, \quad C = \sum_v \dfrac{\rho_v}{\nu_v^6}, \ \cdots \\[2ex] B' = \sum_r \rho_r, & C' = \sum_r \rho_r \nu_r^2, \ \cdots. \end{array}\right\} \tag{42}$$

In such an absorption-free range the value of n for gases differs so little from unity that one may replace $n^2 - 1$ by $2(n - 1)$. Moreover, the terms B', C', ... arising from the ultra-violet resonance frequencies are usually without appreciable influence. Then, if only terms down to $1/\lambda^2$ are retained, (41) reduces to *Cauchy's formula**

$$n - 1 = A_1 \left(1 + \frac{B_1}{\lambda^2} \right), \tag{43}$$

where

$$A_1 = \frac{A}{2}, \qquad B_1 = \frac{Bc^2}{A}. \tag{44}$$

In Table 2.7 the values of A_1 and B_1 for the more important gases are given, with λ specified in centimetres. Also, to illustrate the accuracy of the formula, the values of the refractive index for air as given by Cauchy's formula and the observed values are compared with each other in Table 2.8.

In the case of substances of high density, i.e. liquids or solids, it is no longer permissible to replace n by unity in the denominator of the second term in (35). Nevertheless, it is possible to reduce (35) to the same form as before. Since

$$\frac{n^2 - 1}{n^2 + 2} = \frac{4\pi}{3} N\alpha,$$

it follows that

$$n^2 - 1 = \frac{12\pi N\alpha}{3 - 4\pi N\alpha}, \tag{45}$$

where

$$4\pi N\alpha = \frac{Ne^2}{\pi m} \sum_k \frac{f_k}{\nu_k^2 - \nu^2} = \sum_k \frac{\rho_k}{\nu_k^2 - \nu^2}. \tag{46}$$

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* L. Cauchy, *Bull. des. sc. math.*, **14** (1830), 9; Sur la dispersion de la lumière (*Nouv. exerc. de math.*, 1836).

Table 2.7. *The constants A_1 and B_1 of Cauchy's dispersion formula for various gases.*

Gas	$A_1 \times 10^5$	$B_1 \times 10^{11}$ cm^{-2}
Argon	27.92	5.6
Nitrogen	29.19	7.7
Helium	3.48	2.3
Hydrogen	13.6	7.7
Oxygen	26.63	5.07
Air	28.79	5.67
Ethane	73.65	9.08
Methane	42.6	14.41

Table 2.8. *Observed values of the refractive index for air and the values given by Cauchy's dispersion formula (43).*

$\lambda \times 10^5$ cm	$(n-1) \times 10^4$ observed	$(n-1) \times 10^4$ calculated	Difference
7.594	2.905	2.907	0.002
6.563	2.916	2.917	0.001
5.896	2.926	2.926	0.000
5.378	2.935	2.935	0.000
5.184	2.940	2.940	0.000
4.861	2.948	2.948	0.000
4.677	2.951	2.954	0.003
4.308	2.966	2.967	0.001
3.969	2.983	2.983	0.000
3.728	2.995	2.996	0.001
3.441	3.016	3.017	0.001
3.180	3.040	3.041	0.001
3.021	3.056	3.058	0.002
2.948	3.065	3.067	0.002

It will, as a rule, be sufficient to take into account only a finite number of the absorption frequencies, and it follows that $n^2 - 1$, given by (45), represents a rational function in ν^2, and may therefore be decomposed into partial fractions. For this purpose one must find the zero values of the denominator, i.e. the roots of the equation

$$3 - 4\pi N\alpha = 3 - \sum_k \frac{\rho_k}{\nu_k^2 - \nu^2} = 0. \tag{47}$$

If the roots are denoted by $\bar{\nu}_k$, then (45) may be reduced to the form

$$n^2 - 1 = \sum_k \frac{\bar{\rho}_k}{\bar{\nu}_k^2 - \nu^2}, \tag{48}$$

which is identical in form with the formula (36) for gases.

For example if there is only a single absorption frequency, $\bar{\nu}_1$, this frequency will be the root of the equation

$$3 - \frac{\rho_1}{\nu_1^2 - \nu^2} = 0, \tag{49}$$

whence

$$\left.\begin{array}{l} \bar{\nu}_1^2 = \nu_1^2 - \tfrac{1}{3}\rho_1, \\[2mm] \bar{\rho}_1 = \rho_1. \end{array}\right\} \tag{50}$$

Eq. (48), which may also be written in the form (38), is known as *Sellmeir's dispersion formula*.

So far the effect of the motion of the nuclei has been neglected. It is, in fact, only of importance at very long wavelengths (infra-red). The reason for this can easily be understood. The electric moment and the mean polarizability may be approximately divided into two parts, one associated with the electrons and the other with the nuclei,

$$\mathbf{p} = \mathbf{p}_e + \mathbf{p}_n, \tag{51}$$

where

$$\mathbf{p}_e = \alpha_e \mathbf{E}', \qquad \mathbf{p}_n = \alpha_n \mathbf{E}'. \tag{52}$$

The electrons will follow the field almost instantly, up to rather high frequencies, including in many cases those of the whole visible spectrum. The nuclear masses, on the other hand, are so heavy that they cannot follow the field in the high-frequency region, i.e. $\alpha_n \sim 0$ for visible light. This can also be seen from the dispersion formula (41) where $n^2 - 1$ is represented by two sets of terms, one due to the high-frequency (violet) vibration ν_v, the other due to the low-frequency (red) vibrations ν_r. It is reasonable to assume, and it follows rigorously from quantum mechanics, that the quasi-elastic forces binding the nuclei and electrons are all of the same order of magnitude; hence the order of magnitude of the frequencies will be related by

$$m\nu_e^2 \sim M\nu_n^2, \tag{53}$$

where m is the electronic and M the nuclear mass, and ν_e and ν_n are the frequencies of vibrations of the electrons and nuclei respectively. In order to extend the formulae (41) and (42) to take into account also the motion of the nuclei, we may simply assume that the violet vibrations are due to the electrons, and the red vibrations due to the nuclei. Then we have also to distinguish two types of factors ρ_k as given by (37), those due to the electronic and those due to the nuclear vibrations, and we have the order of magnitude relation

$$m\rho_e \sim M\rho_n. \tag{54}$$

The coefficients (42) of the dispersion formulae are therefore of the following orders of magnitude:

$$B \sim \frac{A}{\nu_e^2}, \qquad C \sim \frac{A}{\nu_e^4}, \left.\vphantom{\begin{array}{c} \\ \\ \end{array}}\right\}$$

$$B' \sim \frac{m}{M} A \nu_e^2, \quad C' \sim \frac{m}{M} A \nu_e^4, \tag{55}$$

and we have

$$n^2 - 1 \sim A \left[1 + a\frac{\nu^2}{\nu_e^2} + b\frac{\nu^4}{\nu_e^4} + \cdots - \frac{m}{M}\left(a'\frac{\nu_e^2}{\nu^2} + b'\frac{\nu_e^4}{\nu^4} + \cdots \right) \right], \tag{56}$$

where $a, b, \ldots, a', b', \ldots$ are numerical constants of the order of unity. The terms in the first line correspond to the electronic part α_e of the polarizability, in the second line to its nuclear part α_n; since m/M is small ($\sim 1/1840$) we see that the latter terms will be negligible, provided that the frequency ν is not a very small fraction of the violet resonance frequencies ν_e.

Hence for an optical frequency (i.e. one corresponding to visible light), the polarization is given essentially by α_e alone (as assumed in our preceding calculations), while for a static field it is given by $\alpha_e + \alpha_n$.

The considerations of the present section were entirely based on classical mechanics. When similar calculations are carried on the basis of quantum mechanics, the interaction of the field and matter may still be described in terms of virtual oscillators, but the number of the oscillators, even when only a single electron is present, is found to be infinite. Eq. (35) still applies, but the strength factors f_k no longer specify the number of electrons of a particular type, but rather the number of virtual oscillators belonging to one electron or to a set of electrons. In most cases only a finite number of the f_k factors have an appreciable value, the others may be neglected. In fact the whole formal theory is hardly affected by the introduction of quantum mechanics, but with its help one can calculate the f_k factors for a given electronic system.

2.4 Propagation of electromagnetic waves treated by integral equations[*]

At the beginning of this chapter a method was outlined for the treatment of propagation of electromagnetic waves in terms of integral equations, and in the sections which followed, auxiliary quantities needed for this purpose were introduced. We shall now formulate these integral equations and discuss some of their consequences. This may be done entirely within the framework of the macroscopic theory as dealt with throughout this book. Within this framework the integral equations are equivalent to Maxwell's equations and represent the mathematical description of the electromagnetic phenomena in terms of mutual interactions over finite distances (which of course need

[*] The theory here described was developed chiefly by W. Esmarch, *Ann. d. Physik*, **42** (1913), 1257; C. W. Oseen, *ibid.*, **48** (1915), 1; and W. Boethe, Beiträge zur Theorie der Brechung und Reflexion (Dissertation, Berlin, 1914), and *Ann. d. Physik*, **64** (1921), 693. A full account of the theory was given by R. Lundblad, *Univ. Årsskrift, Upsala* (1920). See also C. G. Darwin, *Trans. Camb. Phil. Soc.*, **23** (1924), 137, and H. Hoek, Algemeene theorie der optische activiteit van isotrope media (Thesis, Leiden, 1939). A comprehensive review of this work and of more recent developments was given by J. van Kranendonk and J. E. Sipe in *Progress in Optics* Vol. 15, ed. E. Wolf (Amsterdam, North Holland Publishing Company, 1977), p. 245.
Some generalizations of this theory which take into account nonlinear behaviour of the medium (arising when the fields are exceptionally strong) were discussed by N. Bloembergen and P. S. Pershan, *Phys. Rev.*, **128** (1962), 619.

time for transfer). A definite advantage of this treatment, which in some cases is more powerful than the more customary treatment based on differential equations, is that it connects macroscopic phenomena with the molecular ones, discussed in the previous section.

As already explained, a substance may be considered to consist of molecules which react to the incident waves like dipoles. All the emitted dipole waves then give rise to the effective force acting on any other dipole and also determine the average measurable field. It will be assumed that the dipoles are evenly distributed throughout the medium, and the average value **P** of their electric moment per unit volume will be considered as the basic quantity. Actually the molecules are never quite evenly distributed throughout a medium (i.e. there are density fluctuations) and consequently the electric moment of the individual particles exhibits fluctuations about the average value. The resulting phenomena may be accounted for in the present theory by carrying the calculations one step further, i.e. one calculates not only the averages but also their mean square deviations. Such calculations are of importance in some problems, e.g. for the explanation of the blue colour of the sky, as first given by Rayleigh.* But this extension of the theory cannot be given here.†

2.4.1 The basic integral equation

We consider the propagation of an electromagnetic wave in an homogeneous, isotropic nonmagnetic medium. The electric and magnetic fields $\mathbf{E}'_j$ and $\mathbf{H}'_j$ which act on the jth dipole in the interior of the medium can be divided into the incident fields $\mathbf{E}^{(i)}$, $\mathbf{H}^{(i)}$ (propagated with the vacuum velocity c) and the contribution arising from all the dipoles:

$$\left.\begin{aligned}
\mathbf{E}'_j &= \mathbf{E}^{(i)} + \sum_l{}' \mathbf{E}_{jl}, \\
\mathbf{H}'_j &= \mathbf{H}^{(i)} + \sum_l{}' \mathbf{H}_{jl},
\end{aligned}\right\} \tag{1}$$

the summation extending over all the dipoles except the jth one. At the point $\mathbf{r}_j$ where the jth dipole is situated, the field of the lth dipole is obtained from §2.2 (49)–(51):

$$\left.\begin{aligned}
\mathbf{E}_{jl} &= \operatorname{curl}\operatorname{curl} \frac{\mathbf{p}_l(t - R_{jl}/c)}{R_{jl}}, \\
\mathbf{H}_{jl} &= \frac{1}{c} \operatorname{curl} \frac{\dot{\mathbf{p}}_l(t - R_{jl}/c)}{R_{jl}},
\end{aligned}\right\} \tag{2}$$

$\mathbf{p}_l(t)$ denoting the moment of the lth dipole, $R_{jl} = |\mathbf{r}_j - \mathbf{r}_l|$, and the operation curl is taken with respect to the coordinates x_v, y_j, z, of the jth dipole.

As already explained, the distribution may to a good approximation be treated as

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* We shall refer to Rayleigh's theory again in a later section (§14.5) which deals with the scattering of spherical particles situated in vacuum. If the particles are small, the dependence of the intensity of the scattered radiation on the wavelength is the same as in the present case of spontaneous fluctuations of density in a homogeneous medium (namely inversely proportional to the fourth power of the wavelength). The fluctuation theory, however, gives in addition to this result the dependence of the scattered intensity on the density fluctuation.
† It is discussed, for example, in M. Born, *Optik* (Berlin, Springer, 1933, reprinted 1965), §81.

continuous, i.e. the moment of the dipoles may be considered to be a continuous function of position (and of time): $\mathbf{p} = \mathbf{p}(\mathbf{r}, t)$. Likewise the number density N will be assumed to be a continuous function $N(\mathbf{r})$ of position. The total electric dipole $\mathbf{P}$ per unit volume is then given by §2.3 (14):

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}'. \tag{3}$$

For reasons explained in §2.3, the contribution of the magnetic force has been neglected in (3). Since the material is also assumed to be nonmagnetic (i.e. $\mathbf{M} = 0$), the effective field $\mathbf{H}'$ will not enter into the conditions for the dynamical equilibrium to be established.

If we substitute from (2) into (1) and go over to a continuous distribution, using (3), we obtain*

$$\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}^{(i)} + \int \operatorname{curl} \operatorname{curl} N\alpha \frac{\mathbf{E}'(\mathbf{r}', t - R/c)}{R} \, dV', \tag{4}$$

$$\mathbf{H}'(\mathbf{r}, t) = \mathbf{H}^{(i)} + \frac{1}{c} \int \operatorname{curl} N\alpha \frac{\dot{\mathbf{E}}'(\mathbf{r}', t - R/c)}{R} \, dV', \tag{5}$$

where

$$R = |\mathbf{r} - \mathbf{r}'|.$$

If the point of observation $\mathbf{r}$ is outside the medium, the integral is taken throughout the whole medium. If it is inside the medium, a small domain occupied by the atom must be first excluded; we consider it to be a small sphere σ of radius a. Eventually we will proceed to the limit $a \to 0$ in the usual way.

Eq. (4) is an integro-differential equation for $\mathbf{E}'$. When it is solved, $\mathbf{H}'$ may be obtained from (5). The two equations are essentially equivalent to Maxwell's equations for isotropic nonmagnetic substances. Generalization to magnetic substances may be obtained with the help of the second Hertz vector.

2.4.2 The Ewald–Oseen extinction theorem and a rigorous derivation of the Lorentz–Lorenz formula

Eq. (4) connects the effective electric field with the incident electric field in a rather complicated manner and only in special cases can this equation be solved explicitly. Nevertheless it is possible to derive from it some basic results, such as the Lorentz–Lorenz formula, the laws of refraction and reflection, and the Fresnel formulae. Before showing this we shall establish an important general consequence of the solution.

Let Σ denote the boundary of the medium. For points of observation inside the medium the basic equation (4) may be written in the form

$$\mathbf{P} = N\alpha(\mathbf{E}^{(i)} + \mathbf{E}^{(d)}), \tag{6}$$

---

* One must sum the fields and not the potentials, i.e. in (4) the operator curl curl must be under the integration sign and not outside it; the field acting on the molecule as expressed by (3) is obtained by summing all the individual fields. The singularities of the fields at the sources lead to a difference between the sum of the fields and the field associated with the total potential (sum of potentials associated with the individual fields). See (15) below.

where $\mathbf{E}^{(d)}$ denotes the contribution from the dipoles:

$$\mathbf{E}^{(d)} = \int_\sigma^\Sigma \operatorname{curl} \operatorname{curl} \frac{\mathbf{P}(\mathbf{r}', \, t - R/c)}{R} \, \mathrm{d}V'. \tag{7}$$

Here we have indicated explicitly the boundary of the volume over which the integration is carried out.

The incident field $\mathbf{E}^{(i)}$ will be assumed to be monochromatic, with angular frequency ω,

$$\mathbf{E}^{(i)} = \mathbf{A}^{(i)}(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{8}$$

and we take, as a trial solution for $\mathbf{P}$, a wave which is also monochromatic and of the same frequency, but which has a different velocity of propagation (say c/n),

$$\mathbf{P} = (n^2 - 1)k_0^2 \mathbf{Q}(\mathbf{r}) \, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{9}$$

where, as before, $k_0 = \omega/c$ and

$$\nabla^2 \mathbf{Q} + n^2 k_0^2 \mathbf{Q} = 0. \tag{10}$$

In (9) the constant factor $(n^2 - 1)k_0^2$ has been introduced to simplify later formulae. The constant n must be regarded as an unknown quantity, its determination being one of the chief aims of the present analysis.

We also assume that $\mathbf{Q}$ has no sources in the medium, i.e. that inside the medium

$$\operatorname{div} \mathbf{Q} = 0. \tag{11}$$

The possibility of solving the basic integro-differential equation (6) in the form represented by our trial solution may appear somewhat strange at first sight, since $\mathbf{E}^{(i)}$ represents a wave propagated with the vacuum velocity c, whereas $\mathbf{P}$ is assumed to be propagated with the velocity c/n. It will, however, be shown that the dipole field $\mathbf{E}^{(d)}$ may be expressed as the sum of two terms, one of which obeys the wave equation in the vacuum and cancels out exactly the incident wave, whereas the other satisfies the wave equation for propagation with velocity c/n. The incident wave may therefore be regarded as extinguished at any point within the medium by interference with the dipole field and replaced by another wave with a different velocity (and generally also with a different direction) of propagation. This result is known as the *extinction theorem* and was established first for crystalline media by Ewald[*] and for isotropic media by Oseen.[†] In more recent times it has been shown that an analogous extinction theory can also be formulated within the frameworks of Maxwell's electromagnetic theory,[‡] the scalar wave theory [§] and non-relativistic quantum mechanics.[‖]

[*] P. P. Ewald (Dissertation, München, 1912); *Ann. d. Physik*, **49** (1916), 1.

[†] C. W. Oseen, *Ann. d. Physik*, **48** (1915), 1.

[‡] J. J. Sein, *Opt. Comm.*, **2** (1970), 170; J. de Goede and P. Mazur, *Physica*, **58** (1972), 568; D. N. Pattanayak and E. Wolf, *Opt. Comm.*, **6** (1972), 217; E. Wolf in *Coherence and Quantum Optics*, eds. L. Mandel and E. Wolf (New York, Plenum Press, 1973), p. 339. See also H. Fearn, D. F. V. James and P. Milonni, *Am. J. Phys.*, **64** (1996), 986.

   The extinction theorem has also been formulated within the framework of nonrelativistic quantum mechanics by D. N. Pattanayak and E. Wolf, *Phys. Rev. D*, **13** (1976), 913; *ibid.*, **13** (1976), 2287.

[§] T. D. Visser, P. S. Carney and E. Wolf, *Phys. Letts.* A **249** (1998), 243; see also T. D. Visser and E. Wolf, *Phys. Rev.* E **59** (1999) 2355, Sec. IV.

[‖] D. N. Pattanayak and E. Wolf, *Phys. Rev.* **D13** (1976), 913; *ibid.* **13** (1976), 2287. See also E. Wolf, *loc. cit.*

To establish the extinction theorem we rewrite (6) in its time-free form

$$\mathbf{Q} = N\alpha \left[\frac{1}{(n^2 - 1)k_0^2} \mathbf{A}^{(i)} + \mathbf{A}^{(d)} \right], \tag{12}$$

where

$$\left. \begin{array}{l} \mathbf{E}^{(d)} = (n^2 - 1)k_0^2 \mathbf{A}^{(d)} \, e^{-i\omega t}, \\[2mm] \mathbf{A}^{(d)} = \displaystyle\int_\sigma^\Sigma \text{curl curl } \mathbf{Q}(\mathbf{r}') G(R) \, dV', \end{array} \right\} \tag{13}$$

and

$$G = \frac{e^{ik_0 R}}{R}. \tag{14}$$

It is shown in Appendix V that, when the radius of the sphere σ is sufficiently small,

$$\int_\sigma^\Sigma \text{curl curl } \mathbf{Q}(\mathbf{r}') G(R) \, dV' = \text{curl curl} \int_\sigma^\Sigma \mathbf{Q}(\mathbf{r}') G(R) \, dV' - \frac{8\pi}{3} \mathbf{Q}(\mathbf{r}). \tag{15}$$

Now the function G represents a spherical wave in the vacuum and therefore satisfies the Helmholtz equation

$$\nabla^2 G + k_0^2 G = 0, \tag{16}$$

except when $R = 0$. Hence it follows from (10) and (16) that the term under the integral sign on the right of (15) may be expressed in the form

$$\mathbf{Q}G = \frac{1}{(n^2 - 1)k_0^2} (\mathbf{Q}\nabla^2 G - G\nabla^2 \mathbf{Q}), \tag{17}$$

giving on integration with the help of Green's theorem,

$$\int_\sigma^\Sigma \mathbf{Q}(\mathbf{r}') G(R) \, dV' =$$
$$\frac{1}{(n^2 - 1)k_0^2} \left[\int_\Sigma \left(\mathbf{Q} \frac{\partial G}{\partial \nu'} - G \frac{\partial \mathbf{Q}}{\partial \nu'} \right) dS' - \int_\sigma \left(\mathbf{Q} \frac{dG}{dR} - G \frac{\partial \mathbf{Q}}{\partial R} \right) dS' \right], \tag{18}$$

where $\partial/\partial\nu'$ denotes differentiation along the outward normal to the boundary Σ. A straightforward calculation gives (see §8.3.1) for the limit of the surface integral over σ as $a \to 0$ the value $-4\pi\mathbf{Q}(\mathbf{r})$, so that

$$\int_\sigma^\Sigma \mathbf{Q}(\mathbf{r}') G(R) \, dV' \to \frac{1}{(n^2 - 1)k_0^2} \left[\int_\Sigma \left(\mathbf{Q} \frac{\partial G}{\partial \nu'} - G \frac{\partial \mathbf{Q}}{\partial \nu'} \right) dS' + 4\pi\mathbf{Q} \right]. \tag{19}$$

From (13), (15) and (19) we obtain, in the limit,

$$\mathbf{A}^{(d)} = \int_{\sigma}^{\Sigma} \text{curl curl } \mathbf{Q}(\mathbf{r}')G(R)\,dV'$$

$$= \frac{1}{(n^2 - 1)k_0^2} \left[4\pi \text{ curl curl } \mathbf{Q} + \text{curl curl} \int_{\Sigma} \left(\mathbf{Q}\frac{\partial G}{\partial v'} - G\frac{\partial \mathbf{Q}}{\partial v'} \right) dS' \right] - \frac{8\pi}{3} \mathbf{Q}(\mathbf{r}).$$

$$(20)$$

Next we use the identity

$$\text{curl curl } \mathbf{Q} \equiv \text{grad div } \mathbf{Q} - \nabla^2 \mathbf{Q}.$$

The first term on the right vanishes on account of (11), and the second is by (10) equal to $n^2 k_0^2 \mathbf{Q}$. Hence, on substituting into (20), we finally obtain the following expression for the dipole field:

$$\mathbf{A}^{(d)} = \frac{4\pi}{3} \left(\frac{n^2 + 2}{n^2 - 1} \right) \mathbf{Q} + \frac{1}{(n^2 - 1)k_0^2} \text{ curl curl} \int_{\Sigma} \left(\mathbf{Q}\frac{\partial G}{\partial v'} - G\frac{\partial \mathbf{Q}}{\partial v'} \right) dS'. \qquad (21)$$

According to (10) the first term on the right represents a wave propagated with the velocity c/n, whereas the second is seen to represent a wave which, like G, is propagated with the vacuum velocity c. Hence, as anticipated, the basic equation (12) splits into two groups of terms, each of which represents a wave propagated with a different velocity of propagation. This is only possible if each group vanishes separately, so that

$$\frac{4\pi}{3} N\alpha = \frac{n^2 - 1}{n^2 + 2}, \qquad (22)$$

and

$$\mathbf{A}^{(i)} + \text{curl curl} \int_{\Sigma} \left(\mathbf{Q}\frac{\partial G}{\partial v'} - G\frac{\partial \mathbf{Q}}{\partial v'} \right) dS' = 0. \qquad (23)$$

The relation (23) expresses the *extinction* of the incident wave

$$\mathbf{E}^{(i)} = \mathbf{A}^{(i)} e^{-i\omega t}$$

at any point inside the medium, by interference with part of the dipole field. The incident wave is replaced by another wave, namely by

$$\mathbf{E}' = \frac{1}{N\alpha}\mathbf{P} = \frac{1}{N\alpha}(n^2 - 1)k_0^2 \mathbf{Q} e^{-i\omega t} = \frac{4\pi}{3}(n^2 + 2)k_0^2 \mathbf{Q} e^{-i\omega t}, \qquad (24)$$

which is propagated with velocity c/n inside the medium. n is given by (22) in terms of the number density N and the polarizability α; this relation will be recognized as the *Lorentz–Lorenz formula*, tentatively introduced in §2.3.3.

We must now connect the effective field $\mathbf{E}'$ with the ordinary field $\mathbf{E}$ of Maxwell's theory. To do this, we write down the two definitions of the electric displacement $\mathbf{D}$:

$$\mathbf{D} = \varepsilon\mathbf{E} = n^2\mathbf{E} \qquad (25a)$$

and

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \qquad (25b)$$

Elimination of **D** between these two equations gives

$$\mathbf{E} = \frac{4\pi}{n^2 - 1}\mathbf{P}. \tag{26}$$

Now from (3) and (22),

$$\mathbf{E}' = \frac{1}{N\alpha}\mathbf{P} = \frac{4\pi}{3}\frac{n^2 + 2}{n^2 - 1}\mathbf{P},$$

so that, using (26), we obtain

$$\mathbf{E}' = \mathbf{E} + \frac{4\pi}{3}\mathbf{P}, \tag{27}$$

in agreement with §2.3 (12). Finally from (26) and (9) it is seen that, in terms of **Q**, the field **E** *inside the medium* ($n \neq 1$) is given by

$$\mathbf{E} = 4\pi k_0^2 \mathbf{Q}\, e^{-i\omega t}. \tag{28}$$

To determine the electric field *outside the medium* we must return to (4). The integrals occurring in these equations now extend over the whole medium, so that the operator curl curl may be taken outside the respective integrals. Further, since outside the medium $\mathbf{D} = \mathbf{E}$, we have $\mathbf{P} = 0$ there, and consequently, by (27), we have, formally, $\mathbf{E}' = \mathbf{E}$. In place of (6) we now obtain

$$\mathbf{E} = \mathbf{E}^{(i)} + \mathbf{E}^{(r)},$$

where

$$\mathbf{E}^{(r)} = \text{curl curl} \int \frac{\mathbf{P}(\mathbf{r}', t - R/c)}{R}\, dV'. \tag{29}$$

For the time harmonic case here considered, we have, on substituting from (9), using (17) and Green's theorem,

$$\mathbf{E}^{(r)} = \mathbf{A}^{(r)}\, e^{-i\omega t},$$

where

$$\mathbf{A}^{(r)} = \text{curl curl} \int_\Sigma \left(\mathbf{Q}\frac{\partial G}{\partial \nu'} - G\frac{\partial \mathbf{Q}}{\partial \nu'} \right) dS', \tag{30}$$

and **Q** is the same function on the surface Σ as before.

So far we have considered the electric field only. To determine the magnetic field we have only to substitute for $\mathbf{E}'$ into (5) and evaluate the integral. This can be done by a similar calculation as above, but the evaluation is somewhat simpler now, since the operator curl in (5) may be taken outside the integral sign, as is evident from discussion of Appendix V. On comparing the resulting expressions for **H** with those for **E** we find that the solutions are consistent with Maxwell's equations. Since these calculations are straightforward and do not bring out any important new features, they will not be given here.

It is of very great interest that an approach based on the physical remark that the field in matter is more naturally characterized by the polarization than by the displacement vector should lead so elegantly, via the integro-differential equation (4), to a strict derivation of the Lorentz–Lorenz formula and the extinction theorem. This

powerful method has so far been little used in treating more specific problems,[*] but an example of its application will be given in Chapter XIV.

2.4.3 Refraction and reflection of a plane wave, treated with the help of the Ewald–Oseen extinction theorem

The Ewald–Oseen extinction theorem, expressed by the formula (23), will now be applied to the case of a plane monochromatic wave entering a homogeneous medium which fills the half-space $z < 0$. It will be shown that it leads to the laws of refraction and reflection and to the Fresnel formulae.

The incident wave will be written in the form

$$\mathbf{E}^{(i)} = \mathbf{A}^{(i)}(\mathbf{r}) \, e^{-i\omega t} = \mathbf{A}_0^{(i)} \, e^{i[k_0(\mathbf{r} \cdot \mathbf{s}^{(i)}) - \omega t]}, \tag{31}$$

where $k_0 = \omega/c$, $\mathbf{A}_0^{(i)}$ is a constant vector and $\mathbf{s}^{(i)}$ the unit vector in the direction of propagation.

We choose the z-axis so that it passes through the point of observation P, assumed first to be inside the medium at a distance r from the boundary Σ (see Fig. 2.4). The x-axis will be chosen so that $\mathbf{s}^{(i)}$ lies in the x,z-plane. Hence, if θ_i denotes the angle of incidence, we have

$$s_x^{(i)} = -\sin\theta_i, \qquad s_y^{(i)} = 0, \qquad s_z^{(i)} = -\cos\theta_i. \tag{32}$$

In accordance with the analysis of the preceding section, the transmitted wave will be assumed to have the same frequency as the incident wave, but a different velocity
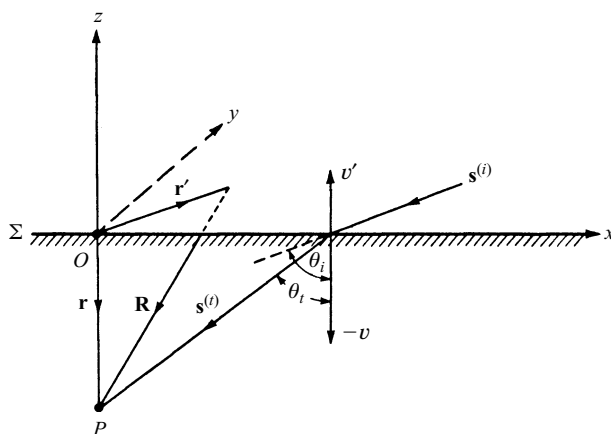

Fig. 2.4 Penetration of a wave into a homogeneous medium which is considered as a system of dipoles.

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[*] The method has been applied to the propagation of electromagnetic waves through a stratified medium by D. R. Hartree, *Proc. Cambr. Phil. Soc.*, **25** (1929), 97.

c/n, n being given in terms of the polarizability and the density by the Lorentz–Lorenz formula. As our trial solution for the transmitted wave we take a plane wave propagated in the direction of a unit vector $\mathbf{s}^{(t)}$ which is assumed to lie in the x,z-plane:

$$s_x^{(t)} = -\sin\theta_t, \qquad s_y^{(t)} = 0, \qquad s_z^{(t)} = -\cos\theta_t. \tag{33}$$

(28) then becomes

$$\mathbf{E} = 4\pi k_0^2 \mathbf{Q}\, e^{-i\omega t} = 4\pi k_0^2 \mathbf{Q}_0\, e^{i[k_0 n(\mathbf{r}\cdot\mathbf{s}^{(t)})-\omega t]}, \tag{34}$$

$\mathbf{Q}_0$ being a constant vector which, on account of (11), is orthogonal to $\mathbf{s}^{(t)}$.

The solution of the integro-differential equation can easily be obtained if we only consider points of observation P whose distance from the boundary $z = 0$ is large compared to the wavelength, i.e. if

$$\frac{2\pi r}{\lambda} = k_0 r \gg 1. \tag{35}$$

This condition is merely imposed to simplify the calculations and does not correspond to a limitation on the physical validity of the integro-differential equation.[*]

The derivatives occurring in the integral over Σ are now

$$\frac{\partial \mathbf{Q}(\mathbf{r}')}{\partial v'} = \frac{\partial}{\partial v'} \mathbf{Q}_0\, e^{ink_0(\mathbf{r}'\cdot\mathbf{s}^{(t)})} = ink_0\left(\frac{\partial \mathbf{r}'}{\partial v'}\cdot\mathbf{s}^{(t)}\right)\mathbf{Q}_0\, e^{ink_0(\mathbf{r}'\cdot\mathbf{s}^{(t)})}, \tag{36a}$$

$$\frac{\partial G(R)}{\partial v'} = \frac{\partial}{\partial v'}\frac{e^{ik_0 R}}{R} = ik_0\frac{\partial R}{\partial v'}G\left(1 + \frac{i}{k_0 R}\right). \tag{36b}$$

The last term in (36b) may be neglected because of (35), and the integral occurring in (23) becomes

$$\mathbf{J} = \int_\Sigma\left(\mathbf{Q}\frac{\partial G}{\partial v'} - G\frac{\partial \mathbf{Q}}{\partial v'}\right)dS' = ik_0\mathbf{Q}_0\int_\Sigma\left[\frac{\partial R}{\partial v'} - n\left(\frac{\partial \mathbf{r}'}{\partial v'}\cdot\mathbf{s}^{(t)}\right)\right]\frac{e^{ik_0[R+n(\mathbf{r}'\cdot\mathbf{s}^{(t)})]}}{R}\,dS'. \tag{37}$$

The components of the vectors $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{R}$ are (see Fig. 2.4),

$$
\begin{array}{rccc}
\mathbf{r}: & 0, & 0, & -r, \\
\mathbf{r}': & x', & y', & 0, \\
\mathbf{R}: & -x', & -y', & -r,
\end{array}
$$

so that

$$\left.\begin{array}{ll}
R = \sqrt{x'^2 + y'^2 + r^2}, & \mathbf{r}'\cdot\mathbf{s}^{(t)} = -x'\sin\theta_t, \\[2mm]
\dfrac{\partial R}{\partial v'} = \dfrac{r}{R}, & \dfrac{\partial \mathbf{r}'}{\partial v'}\cdot\mathbf{s}^{(t)} = s_z^{(t)} = -\cos\theta_t.
\end{array}\right\} \tag{38}$$

Eq. (37) becomes on substitution

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[*] The exact solution of the integro-differential equation for the problem of refraction and reflection was given by É. Lalor and E. Wolf, *J. Opt. Soc. Amer.*, **62** (1972), 1165.

$$\mathbf{J} = ik_0\mathbf{Q}_0 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{R}\left(\frac{r}{R} + n\cos\theta_t\right) e^{ik_0(R-nx'\sin\theta_t)}\, dx'\, dy'. \tag{39}$$

It will be convenient to introduce an angle ϕ and an associated unit vector $\mathbf{s}$, defined by

$$n\sin\theta_t = \sin\phi, \tag{40}$$

$$s_x = -\sin\phi, \qquad s_y = 0, \qquad s_z = -\cos\phi. \tag{41}$$

Since $k_0 r$ was assumed to be large compared with unity, the exponent in (39) will also be large compared with unity and the exponential term will oscillate rapidly and change sign many times as the point (x', y') explores the domain of integration. Under these conditions, a good approximation to the value of $\mathbf{J}$ is obtained by the application of the following formula, which is a consequence of the principle of stationary phase [see Appendix III, (20)]:

$$\iint_D g(x', y')\, e^{ik_0 f(x',y')}\, dx'\, dy' \sim \frac{2\pi i}{k_0}\sum_j \frac{\sigma_j}{\sqrt{|\alpha_j\beta_j - \gamma_j^2|}}\, g(x_j', y_j')\, e^{ik_0 f(x_j',y_j')}. \tag{42}$$

Here (x_j', y_j') are the points in the domain D of integration at which f is stationary and

$$\alpha_j = \left(\frac{\partial^2 f}{\partial x'^2}\right)_{x_j',y_j'}, \qquad \beta_j = \left(\frac{\partial^2 f}{\partial y'^2}\right)_{x_j',y_j'}, \qquad \gamma_j = \left(\frac{\partial^2 f}{\partial x_j'\partial y_j'}\right)_{x_j',y_j'} \tag{43a}$$

and

$$\sigma_j = \begin{cases} +1 & \text{when} \quad \alpha_j\beta_j > \gamma_j^2, \quad \alpha_j > 0, \\ -1 & \text{when} \quad \alpha_j\beta_j > \gamma_j^2, \quad \alpha_j < 0, \\ -i & \text{when} \quad \alpha_j\beta_j < \gamma_j^2. \end{cases} \tag{43b}$$

In the present case, we have, if we also use (40) and (41),

$$g = \frac{1}{R}\left(\frac{r}{R} + \frac{\sin\phi}{\sin\theta_t}\cos\theta_t\right), \qquad f = R - x'\sin\phi, \tag{44}$$

and consequently

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2 + r^2}} - \sin\phi, \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2 + r^2}}.$$

For f to be stationary,

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial y'} = 0.$$

Hence (noting that the positive square root is to be taken in evaluating the partial derivatives), f is seen to be stationary only when

$$x' = x_1', \qquad y' = y_1',$$

where

$$x_1' = r \tan \phi, \qquad y_1' = 0. \tag{45}$$

The expressions in (43) and (44) take the following values at this point:

$$\alpha_1 = \left(\frac{\partial^2 f}{\partial x'^2} \right)_{x_1', y_1'} = \frac{\cos^3 \phi}{r}, \qquad \beta_1 = \left(\frac{\partial^2 f}{\partial y'^2} \right)_{x_1', y_1'} = \frac{\cos \phi}{r}, \qquad \gamma_1 = 0, \tag{46a}$$

$$\sigma_1 = 1, \tag{46b}$$

$$R_1 = (R)_{x_1', y_1'} = \frac{r}{\cos \phi}, \tag{46c}$$

$$f(x_1', y_1') = r \cos \phi = \mathbf{r} \cdot \mathbf{s}, \tag{46d}$$

$$g(x_1', y_1') = \frac{\cos \phi}{r} \frac{\sin(\phi + \theta_t)}{\sin \theta_t}. \tag{46e}$$

Formula (42) now gives

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{R} \left(\frac{r}{R} + n \cos \theta_t \right) e^{ik_0(R - nx' \sin \theta_t)} \, \mathrm{d}x' \, \mathrm{d}y' = \frac{2\pi i}{k_0} \frac{\sin(\phi + \theta_t)}{\cos \phi \sin \theta_t} \, e^{ik_0(\mathbf{r} \cdot \mathbf{s})},$$

so that (39) becomes

$$\mathbf{J} = \int_{\Sigma} \left(\mathbf{Q} \frac{\partial G}{\partial v'} - G \frac{\partial \mathbf{Q}}{\partial v'} \right) \mathrm{d}S' = -2\pi \frac{\sin(\phi + \theta_t)}{\cos \phi \sin \theta_t} \mathbf{Q}_0 \, e^{ik_0(\mathbf{r} \cdot \mathbf{s})}. \tag{47}$$

We now substitute this expression into the integro-differential equation (23), which expresses the extinction of the incident wave. We have

$$\operatorname{curl} \mathbf{Q}_0 \, e^{ik_0(\mathbf{r} \cdot \mathbf{s})} = i k_0 (\mathbf{s} \times \mathbf{Q}_0) e^{ik_0(\mathbf{r} \cdot \mathbf{s})},$$

$$\operatorname{curl} \operatorname{curl} \mathbf{Q}_0 \, e^{ik_0(\mathbf{r} \cdot \mathbf{s})} = -k_0^2 \mathbf{s} \times (\mathbf{s} \times \mathbf{Q}_0) e^{ik_0(\mathbf{r} \cdot \mathbf{s})}.$$

Hence (23) reduces to

$$\mathbf{A}_0^{(i)} \, e^{ik_0(\mathbf{r} \cdot \mathbf{s}^{(i)})} = -2\pi k_0^2 \frac{\sin(\phi + \theta_t)}{\cos \phi \sin \theta_t} \mathbf{s} \times (\mathbf{s} \times \mathbf{Q}_0) e^{ik_0(\mathbf{r} \cdot \mathbf{s})}. \tag{48}$$

Now this must be an identity for all points $\mathbf{r}$ on the boundary. Hence

$$\mathbf{s} = \mathbf{s}^{(i)}, \tag{49}$$

and

$$\mathbf{A}_0^{(i)} = -2\pi k_0^2 \frac{\sin(\phi + \theta_t)}{\cos \phi \sin \theta_t} [\mathbf{s}^{(i)} \times (\mathbf{s}^{(i)} \times \mathbf{Q}_0)]. \tag{50}$$

Eq. (49) expresses the *law of refraction* §1.5 (8). For it implies that

$$\phi = \theta_i, \tag{51}$$

or, by (40),

$$n \sin \theta_t = \sin \theta_i. \tag{52}$$

Moreover, (41) and (32) show that the vector $\mathbf{s}^{(t)}$ lies in the plane specified by $\mathbf{s}^{(i)}$ and the normal to Σ.

Eq. (50) connects the amplitudes of the incident wave (31) and the transmitted wave (34). Denoting the (vector) amplitude of the transmitted wave by $\mathbf{T}_0$,

$$\mathbf{T}_0 = 4\pi k_0^2 \mathbf{Q}_0, \tag{53}$$

and expanding the vector triple product, (50) becomes

$$\mathbf{A}_0^{(i)} = \frac{1}{2} \frac{\sin(\theta_i + \theta_t)}{\cos\theta_i \sin\theta_t} [\mathbf{T}_0 - \mathbf{s}^{(i)}(\mathbf{s}^{(i)} \cdot \mathbf{T}_0)]. \tag{54}$$

Let $A_\perp$, $A_\parallel$ and $T_\perp$, $T_\parallel$ be the components of $\mathbf{A}_0^{(i)}$ and of $\mathbf{T}_0$ in directions perpendicular and parallel to the plane of incidence. Then, remembering that $\mathbf{A}_0^{(i)}$ is orthogonal to $\mathbf{s}^{(i)}$ and $\mathbf{T}_0$ is orthogonal to $\mathbf{s}^{(t)}$, and that $\mathbf{s}^{(i)}$ and $\mathbf{s}^{(t)}$ are inclined to each other at an angle $(\theta_i - \theta_t)$, it follows from (54) that

$$A_\perp = \frac{1}{2} \frac{\sin(\theta_i + \theta_t)}{\cos\theta_i \sin\theta_t} T_\perp, \tag{55a}$$

$$A_\parallel = \frac{1}{2} \frac{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}{\cos\theta_i \sin\theta_t} T_\parallel. \tag{55b}$$

These relations will be recognized as the *Fresnel formulae* §1.5 (20a) *for refraction.*

Finally we consider the case when the point of observation is outside the medium ($z > 0$). The calculations are strictly analogous, only $z' = -r$ must now be replaced by $z' = r$ in the appropriate formulae. This is equivalent to replacing s_z by $-s_z$ in (46d), i.e. to replacing $\phi = \theta_i$ by $\phi = \theta_r$, where

$$\theta_r = \pi - \theta_i. \tag{56}$$

In place of the unit vector $\mathbf{s} = \mathbf{s}^{(i)}$ we now introduce the unit vector $\mathbf{s}^{(r)}$ with components

$$s_x^{(r)} = -\sin\theta_r = -\sin\theta_i, \quad s_y^{(r)} = 0, \quad s_z^{(r)} = -\cos\theta_r = \cos\theta_i, \tag{57}$$

and obtain instead of (47),

$$\mathbf{J}' = \int_\Sigma \left(\mathbf{Q}\frac{\partial G}{\partial \nu'} - G\frac{\partial \mathbf{Q}}{\partial \nu'} \right) \mathrm{d}S' = -2\pi \frac{\sin(\theta_i - \theta_t)}{\cos\theta_i \sin\theta_t} \mathbf{Q}_0\, \mathrm{e}^{\mathrm{i}k_0(\mathbf{r}\cdot\mathbf{s}^{(r)})}. \tag{58}$$

Eq. (30) now gives for the time-independent part of the reflected wave

$$\mathbf{A}_0^{(r)}\, \mathrm{e}^{\mathrm{i}k_0(\mathbf{r}\cdot\mathbf{s}^{(r)})} = 2\pi k_0^2 \frac{\sin(\theta_i - \theta_t)}{\cos\theta_i \sin\theta_t} [\mathbf{s}^{(r)} \times (\mathbf{s}^{(r)} \times \mathbf{Q}_0)]\, \mathrm{e}^{\mathrm{i}k_0(\mathbf{r}\cdot\mathbf{s}^{(r)})}. \tag{59}$$

Eq. (59) represents a plane wave propagated in the direction specified by the unit vector $\mathbf{s}^{(r)}$, this direction being related to the direction $\mathbf{s}^{(i)}$ of the incident wave by (56); relation (56) expresses the *law of reflection*, in agreement with §1.5 (7).

In terms of the amplitude of the transmitted wave, the amplitude $\mathbf{A}_0^{(r)}$ of the reflected wave is

$$\mathbf{A}_0^{(r)} = -\frac{1}{2} \frac{\sin(\theta_i - \theta_t)}{\cos\theta_i \sin\theta_t} [\mathbf{T}_0 - \mathbf{s}^{(r)}(\mathbf{s}^{(r)} \cdot \mathbf{T}_0)]. \tag{60}$$

Denoting by $R_\perp$ and $R_\parallel$ the components of $\mathbf{A}_0^{(r)}$ in directions perpendicular and parallel to the plane of incidence, and using (55), we obtain

$$R_\perp = -\frac{1}{2}\frac{\sin(\theta_i - \theta_t)}{\cos\theta_i \sin\theta_t}\, T_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}\, A_\perp, \tag{61a}$$

and

$$R_\parallel = -\frac{1}{2}\frac{\sin(\theta_i - \theta_t)\cos(\theta_r - \theta_t)}{\cos\theta_i \sin\theta_t}\, T_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}\, A_\parallel. \tag{61b}$$

Relations (61) are the *Fresnel formulae* §1.5 (21a) *for reflection*.