Appendix VIII

Proof of the inequality $|\mu_{12}(\nu)| \leqslant 1$ for the spectral degree of coherence (§10.5)

In this appendix we provide a proof of §10.5 (10) according to which the upper bound of the spectral degree of coherence is unity.

Let $v_T(P, \nu)$ be the Fourier transform of the truncated field variable $V_T^{(r)}(P, t)$ [§10.3 (25) and §10.3 (14)] and let a_1 and a_2 be arbitrary complex numbers. Evidently

$$|a_1 v_T(P_1, \nu) + a_2 v_T(P_2, \nu)|^2 \geqslant 0, \tag{1}$$

or, more explicitly,

$$\begin{aligned} &a_1^\star a_1 v_T^\star(P_1, \nu) v_T(P_1, \nu) + a_2^\star a_2 v_T^\star(P_2, \nu) v_T(P_2, \nu) \\ &\quad + a_1^\star a_2 v_T^\star(P_1, \nu) v_T(P_2, \nu) + a_1 a_2^\star v_T(P_1, \nu) v_T^\star(P_2, \nu) \geqslant 0. \end{aligned} \tag{2}$$

Let us divide this inequality by $2T$, take the ensemble average and proceed to the limit as $T \to \infty$. Recalling the definitions of the spectral density $S(P, \nu)$ [§10.3 (32)] and the cross-spectral density $G(P_1, P_1, \nu)$ [§10.3 (28)] one obtains at once the inequality

$$a_1^\star a_1 S(P_1, \nu) + a_2^\star a_2 S(P_2, \nu) + a_1^\star a_2 G(P_2, P_1, \nu) + a_1 a_2^\star G(P_1, P_2, \nu) \geqslant 0. \tag{3}$$

Since this inequality must hold for all values of a_1 and a_2 it follows from a well-known property of nonnegative definite quadratic forms that the determinant*

$$\begin{vmatrix} S(P_1, \nu) & G(P_2, P_1, \nu) \\ G(P_1, P_2, \nu) & S(P_2, \nu) \end{vmatrix} \geqslant 0. \tag{4}$$

If we next use the fact that $G(P_2, P_1, \nu) = G^\star(P_1, P_2, \nu)$ which follows from the definition of the cross-spectral density, it follows that

$$|G(P_1, P_2, \nu)|^2 \leqslant S(P_1, \nu) S(P_2, \nu). \tag{5}$$

Hence

$$|\mu_{12}(\nu)| \equiv \frac{|G(P_1, P_2, \nu)|}{\sqrt{S(P_1, \nu)}\sqrt{S(P_2, \nu)}} \leqslant 1, \tag{6}$$

which is the inequality §10.5 (10).

* F. R. Gantmacher, *The Theory of Matrices*, Vol. I (New York, Chelsea Publishing Company, 1959), Theorem 20, p. 337.

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