# *Appendices*

# Appendix I

# The calculus of variations

It is a general feature of equations of classical physics that they can be derived from variational principles. Two early examples are Fermat's principle in optics (1657) and Maupertuis' principle in mechanics (1744). The equations of elasticity, hydrodynamics and electrodynamics can also be represented in this way.

However, when one deals with field equations, involving as a rule four or more independent variables $x, y, z, t, \ldots,$ one makes little use, owing to the great complexity of partial differential equations, of the property that the solution expresses stationary values of certain integrals. The only essential advantage of the variational approach in such cases is connected with the derivation of conservation laws — e.g. for energy. The situation is quite different in problems involving one independent variable (time in mechanics, or length of a ray in geometrical optics). Then one deals with a set of ordinary differential equations and it turns out that a study of the behaviour of the solution is greatly facilitated by a variational approach. This approach is in fact a straightforward generalization of ordinary geometrical optics in every detail. Its modern representation owes much to David Hilbert, on whose unpublished lectures, given at Göttingen in about 1903, we base the considerations of the following sections. The theory is presented here for a three-dimensional space $(x, y, z)$ only, but can easily be extended to more dimensions.

### 1 Euler's equations as necessary conditions for an extremum

Let $F(u, v, x, y, z)$ be a given function with continuous partial derivatives up to the second order in all the five variables. Further, let $C$ be any curve $x = x(z)$, $y = y(z)$ in the $x,y,z$ space. The derivatives of $x$ and $y$ will also be assumed to be continuous up to second order. If we set

$$u = x', \qquad v = y'$$

(the prime denoting differentiation with respect to $z$), the integral

$$I = \int_{z_1}^{z_2} F(x', y', x, y, z)\mathrm{d}z \tag{1}$$



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is a function of the curve $C$, i.e. of the two functions $x(z)$, $y(z)$, in other words a *functional*. The fundamental problem of the calculus of variation is:

*To determine a curve C between two given points* $P_1[x_1 = x(z_1),\ y_1 = y(z_1),\ z_1]$ and $P_2[x_2 = x(z_2),\ y_2 = y(z_2),\ z_2]$ *for which the integral is an extremum* (*minimum or maximum*).

The necessary conditions which such a curve $C$, called an *extremal*, must satisfy may be determined by the simple process of linear variation. For this purpose we choose a function $\xi(z)$ with a continuous first-order derivative, which vanishes at the end points,

$$\xi(z_1) = \xi(z_2) = 0, \tag{2}$$

and form the 'varied' curve $C'$ by replacing the $x$ coordinate of the extremal by $x + \varepsilon\xi$, where $\varepsilon$ is a small parameter. Eq. (1) then becomes a function of $\varepsilon$,

$$I(\varepsilon) = \int_{z_1}^{z_2} F(x' + \varepsilon\xi',\ y',\ x + \varepsilon\xi,\ y,\ z)\mathrm{d}z. \tag{3}$$

The quantity

$$(\delta I)_x = \left(\frac{\partial I}{\partial \varepsilon}\right)_{\varepsilon=0} = \int_{z_1}^{z_2} \left(\frac{\partial F}{\partial x'}\xi' + \frac{\partial F}{\partial x}\xi\right)\mathrm{d}z \tag{4}$$

is called *the first variation* with respect to $x$. The vanishing of the first variation

$$(\delta I)_x = 0 \tag{5}$$

is clearly a necessary condition for an extremum.

Now if we integrate the first term in (4) by parts and use the boundary conditions (2), we obtain

$$(\delta I)_x = \int_{z_1}^{z_2} \left(F_x - \frac{\mathrm{d}}{\mathrm{d}z}F_{x'}\right)\xi\,\mathrm{d}z, \tag{6a}$$

where $F_x$ stands for $\partial F/\partial x$, etc. Similarly, replacing $y$ by $y + \varepsilon\eta$, we find that

$$(\delta I)_y = \int_{z_1}^{z_2} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}z}F_{y'}\right)\eta\,\mathrm{d}z. \tag{6b}$$

Since $\xi$ and $\eta$ may be chosen arbitrarily in the interval $z_1 \leqslant z \leqslant z_2$, it follows from (6a) and (6b) that the conditions $(\delta I)_x = 0$ and $(\delta I)_y = 0$ may be expressed in the form of two differential equations, known as *Euler's equations*,

$$F_x - \frac{\mathrm{d}}{\mathrm{d}z}F_{x'} = 0, \tag{7a}$$

$$F_y - \frac{\mathrm{d}}{\mathrm{d}z}F_{y'} = 0. \tag{7b}$$

Eqs. (7a) and (7b) are two differential equations of the second order for $x(z)$ and $y(z)$. The leading terms, i.e. those with derivatives of the highest order, will be written out fully:

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$$F_{x'x'}x'' + F_{x'y'}y'' + \cdots = 0, \tag{8a}$$

$$F_{y'x'}x'' + F_{y'y'}y'' + \cdots = 0. \tag{8b}$$

These equations can be solved for $x''$ and $y''$ provided the associated determinant does not vanish, i.e. provided that

$$F_{uu}F_{vv} - F_{uv}^2 \neq 0. \tag{9}$$

We shall assume that this condition holds throughout the five-dimensional region with which we are concerned.

The solutions of two differential equations of the second order contain four arbitrary constants of integration so that the extremals form a four-parameter family of curves ($\infty^4$ extremals).

## 2 Hilbert's independence integral and the Hamilton–Jacobi equation

In order to discuss the properties of these extremals it is convenient to consider another related problem. The variables $u$ and $v$ will be regarded as functions in the $x,y,z$ space

$$u = u(x, y, z), \qquad v = v(x, y, z). \tag{10}$$

Then $F[u(x, y, z), v(x, y, z), x, y, z]$ as well as its partial derivatives $F_u$, $F_v$ are functions of $x$, $y$, $z$. Let us now choose a curve $C[x = x(z), y = y(z)]$, and form the integral

$$S = \int_{z_1}^{z_2} [F + (x' - u)F_u + (y' - v)F_v]\mathrm{d}z. \tag{11}$$

The problem is this: *to find such functions $u$, $v$ which secure that $S$ is independent of the choice of the curve $C$*: $S$ is then a function of the end points $P_1$ and $P_2$ only, where $P_1$ has the coordinates $x_1 = x(z_1)$, $y_1 = y(z_1)$, $z_1$, and $P_2$ the coordinates $x_2 = x(z_2)$, $y = y(z_2)$, $z_2$. $S$ will be called *Hilbert's independence integral*.

To determine $u$ and $v$ we first re-write (11) in the form

$$S = \int_{P_1}^{P_2} (U\,\mathrm{d}x + V\,\mathrm{d}y + W\,\mathrm{d}z), \tag{12}$$

where

$$U = F_u, \qquad V = F_v, \qquad W = F - uF_u - vF_v. \tag{13}$$

Now it is well known that the necessary and sufficient conditions for (12) to be independent of the curve are the vanishing of the components of the curl of the vector **A** whose components are $U$, $V$, $W$, i.e.

$$\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z} = 0, \qquad \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} = 0, \qquad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0. \tag{14}$$

This is a set of three partial differential equations for $u(x, y, z)$ and $v(x, y, z)$; the equations are, however, not quite independent as the identity

$$\frac{\partial}{\partial x}\left(\frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial z} - \frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) = 0 \tag{15}$$

holds for any $U$, $V$ and $W$ ($\operatorname{div}\operatorname{curl}\mathbf{A} = 0$ for any vector $\mathbf{A}$). If (14) are satisfied, then $U\,\mathrm{d}x + V\,\mathrm{d}y + W\,\mathrm{d}z$ is a total differential,

$$\mathrm{d}S = U\,\mathrm{d}x + V\,\mathrm{d}y + W\,\mathrm{d}z, \tag{16}$$

and $S$ is then a function of $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ alone. Writing for short $x$, $y$, $z$ in place of $x_2$, $y_2$, $z_2$ we have

$$U = \frac{\partial S}{\partial x}, \qquad V = \frac{\partial S}{\partial y}, \qquad W = \frac{\partial S}{\partial z}. \tag{17}$$

We now take an arbitrary surface $T(x, y, z) = 0$ and at each point $P_1$ of the surface we choose the vector $(U, V, W)$ normal to the surface. Then by (16) $\mathrm{d}S = 0$, and therefore

$$S(x, y, z) = S_1 \tag{18}$$

is constant on the surface. From two of the (compatible) equations (13) we can solve for $u$ and $v$ as functions on the surface

$$u = u(U, V, x, y, z), \qquad v = v(U, V, x, y, z). \tag{19}$$

If we next solve the differential equations (14) with these boundary values we obtain a special solution of our problem, namely a solution which has a constant value $S_1$ of $S$ on the selected surface $T(x, y, z) = 0$. This solution can be determined from a single partial differential equation for the function $S$. For on substituting from (19) into the remaining equation of (13) we obtain

$$W = W(U, V, x, y, z), \tag{20}$$

which, on using (17), becomes

$$\frac{\partial S}{\partial z} = W\left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, x, y, z\right); \tag{21}$$

this is known as *the Hamilton–Jacobi equation* of the problem.

A function $S(x, y, z)$ which has a given constant value on the surface $T(x, y, z) = 0$ and satisfies (21) represents a solution of the problem. The two functions $u$, $v$ which make the integral independent of the path can be found by solving any two of the (compatible) equations

$$F_u = \frac{\partial S}{\partial x}, \qquad F_v = \frac{\partial S}{\partial y}, \qquad F - uF_u - vF_v = \frac{\partial S}{\partial z}, \tag{22}$$

obtained by combining (13) and (17).

## 3 The field of extremals

We now establish *the connection between the two problems* treated in §1 and §2. It is this:

*If* $u(x, y, z)$ *and* $v(x, y, z)$ *are two functions which make the Hilbert integral* $S$ *defined by* (11) *independent of the path, then the differential equations*

$$x' = u(x, y, z), \qquad y' = v(x, y, z), \tag{23}$$

*have as solutions a two-parameter set* ($\infty^2$) *of extremals, namely those which satisfy*

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*the condition that they are 'transversal' to the surfaces* $S(x, y, z) = S_1$. By 'transversality' we mean here that the relation

$$U\,\mathrm{d}x + V\,\mathrm{d}y + W\,\mathrm{d}z = 0, \tag{24}$$

is satisfied: it states that the vector $(U, V, W)$, defined by (13) in terms of $u$ and $v$, is orthogonal to any element $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ of the surface.

Consider a region in the $x,y,z$ space and associate with each point of this region a vector $(u,v)$, which is continuous and has continuous first-order partial derivatives. The set of these vectors defined over the given region is called a *field*. In the present context one speaks of a *field of extremals* and $u(x, y, z)$, $v(x, y, z)$ are then called *slope functions* of the field.

The following converse of the theorem just enunciated also holds: *If a field of* $\infty^2$ *extremals is constructed in such a way that it is transversal to a given surface* $T(x, y, z) = 0$, *and if u and v are its slope functions, defined by* (23)*, then the Hilbert integral S, given by* (11)*, is independent of the path.*

Before proving these theorems we note the following corollary:

Let the curve $C$ in (11) be an extremal of the field; the Hilbert integral (11) then reduces to the variational integral $I = \int_{z_1}^{z_2} F\,\mathrm{d}z$. Hence the value of this integral taken between each pair of 'corresponding' points on the surfaces $S(x, y, z) = S_1$ and $S(x, y, z) = S_2$ (i.e. between points on the same extremal transversal to $S_1$ and $S_2$) is the same for all such pairs of points (see Fig. 1). The surfaces $S(x, y, z) =$ constant and the $\infty^2$ transversals may be regarded as generalizations of wave-fronts and rays of geometrical optics.

To prove the first theorem we consider a fixed curve $C$ which satisfies (23) and is transversal to the surface $S(x, y, z) = S_1$, and apply a linear variation to it, i.e. we replace $x$ by $x + a\xi$ and $y$ by $y + b\eta$ where $a$ and $b$ are small parameters and $\xi$ and $\eta$ are arbitrary but fixed differentiable functions of $z$, which vanish for $z_1$ and $z_2$. Now since, by hypothesis, $S$ is independent of the path,

$$\left(\frac{\partial S}{\partial a}\right)_0 = 0, \qquad \left(\frac{\partial S}{\partial b}\right)_0 = 0, \tag{25}$$

where the suffix 0 indicates that we substitute $a = b = 0$ after differentiation.

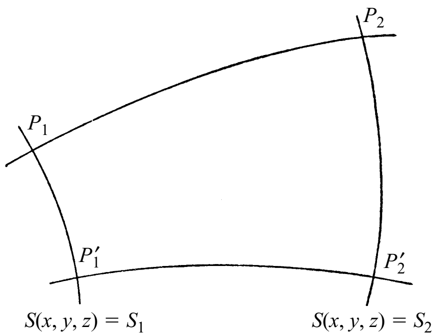


Fig. 1 Illustrating a generalization of the concept of wave-fronts and rays of geometrical optics. The variational integral (1) has a constant value for all extremals such as $P_1P_2$, $P_1'P_2'$, ... which are transversal to the surfaces $S_1$ and $S_2$.

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Differentiation of $F[u(x, y, z), v(x, y, z), x, y, z]$ gives

$$\frac{\partial F}{\partial a} = (F_u u_x + F_v v_x + F_x)\xi. \tag{26}$$

From (11) we obtain, with the help of (26),

$$\begin{aligned}\left(\frac{\partial S}{\partial a}\right)_0 = \int_{z_1}^{z_2} \bigg[ & (F_u u_x + F_v v_x + F_x)\xi + (\xi' - u_x \xi)F_u - v_x \xi F_v \\ & + (x' - u)\frac{\partial F_u}{\partial a} + (y' - v)\frac{\partial F_v}{\partial a}\bigg]_0 \mathrm{d}z.\end{aligned} \tag{27}$$

The terms in the second line vanish, since the curve is assumed to satisfy (23). In the first line several terms cancel and we obtain

$$\left(\frac{\partial S}{\partial a}\right)_0 = \int_{z_1}^{z_2} (F_x \xi + F_u \xi')\mathrm{d}z, \tag{28}$$

or, on integrating by parts,

$$\left(\frac{\partial S}{\partial a}\right)_0 = \int_{z_1}^{z_2} \left(F_x - \frac{\mathrm{d}}{\mathrm{d}z}F_u\right)\xi\, \mathrm{d}z, \tag{28a}$$

and similarly

$$\left(\frac{\partial S}{\partial b}\right)_0 = \int_{z_1}^{z_2} \left(F_y - \frac{\mathrm{d}}{\mathrm{d}z}F_v\right)\eta\, \mathrm{d}z. \tag{28b}$$

The right-hand sides of (28a) and (28b) are the first variations of $I$ [cf. (6a) and (6b)], and (25) shows that they vanish. Hence the curve $C$ satisfies Euler's equations, i.e. it is an extremal, and the theorem is proved.

To establish the converse theorem we construct a field $f_1$ of $\infty^2$ extremals transversal to a given surface $T(x, y, z) = 0$, and compare it with another field $f_2$ of $\infty^2$ extremals. The latter field is constructed in the following way. We solve the Hamilton–Jacobi equation (21) with the boundary condition that $S(x, y, z)$ shall be constant on $T(x, y, z) = 0$. If $u$ and $v$ are determined from (22), the solution can be represented by the integral (11). Then, according to the theorem just established, (23) define a field $f_2$ of extremals which are transversal to $T = 0$. These two fields $f_1$ and $f_2$ must, however, be identical, since they satisfy the same differential equations and the same boundary conditions on $T = 0$. Hence, for the given field $f_1$, the integral $S$ is independent of the path.

## 4 Determination of all extremals from the solution of the Hamilton–Jacobi equation

So far we have only considered an $\infty^1$ set of solutions $S(x, y, z) =$ constant of the Hamilton–Jacobi equation, which correspond to an $\infty^2$ set of transversal extremals. To obtain all $\infty^4$ extremals we must consider a larger set of solutions $S$, namely $\infty^4$; these can be obtained by rotating the surface $T = 0$ round a point, and taking, in each case, $S(x, y, z) = S_1$. Assume that we have found the 'complete' solution $S(x, y, z, \alpha, \beta)$ of (21), involving two parameters $\alpha$ and $\beta$. The function $S$, corresponding to any pair of values of $\alpha$ and $\beta$, can be represented by an integral of the form (11) which is independent of the path by making an appropriate choice of the two functions $u$ and $v$: $u = u(x, y, z, \alpha, \beta)$, $v = v(x, y, z, \alpha, \beta)$. Hence not only $S$ but also $\partial S/\partial\alpha$ and $\partial S/\partial\beta$ are independent of the path. Now since $F = F[u(x, y, z, \alpha, \beta), v(x, y, z, \alpha, \beta), x, y, z]$

$$\frac{\partial F}{\partial \alpha} = F_u u_\alpha + F_v v_\alpha, \tag{29}$$

and we obtain from (11)

$$\left.\begin{aligned} S_\alpha &= \int_{z_1}^{z_2} \left[(x' - u)\frac{\partial F_u}{\partial \alpha} + (y' - v)\frac{\partial F_v}{\partial \alpha}\right] \mathrm{d}z, \\ S_\beta &= \int_{z_1}^{z_2} \left[(x' - u)\frac{\partial F_u}{\partial \beta} + (y' - v)\frac{\partial F_v}{\partial \beta}\right] \mathrm{d}z, \end{aligned}\right\} \tag{30}$$

where

$$\left.\begin{aligned} \frac{\partial F_u}{\partial \alpha} &= F_{uu} u_\alpha + F_{uv} v_\alpha, \\ \frac{\partial F_v}{\partial \alpha} &= F_{vu} u_\alpha + F_{vv} v_\alpha, \end{aligned}\right\} \tag{31}$$

with similar expressions for the other two derivatives.

Since the integrals (30) are independent of the path, it follows that the expressions

$$\left.\begin{aligned} \mathrm{d}S_\alpha &= \left[(x' - u)\frac{\partial F_u}{\partial \alpha} + (y' - v)\frac{\partial F_v}{\partial \alpha}\right] \mathrm{d}z, \\ \mathrm{d}S_\beta &= \left[(x' - u)\frac{\partial F_u}{\partial \beta} + (y' - v)\frac{\partial F_v}{\partial \beta}\right] \mathrm{d}z, \end{aligned}\right\} \tag{32}$$

are total differentials; $S_\alpha$ and $S_\beta$ are thus functions of the end points $P_1$ and $P_2$ only, so that surfaces defined by constant values $S_\alpha$ and $S_\beta$ exist, say

$$\frac{\partial S(x, y, z, \alpha, \beta)}{\partial \alpha} = A, \qquad \frac{\partial S(x, y, z, \alpha, \beta)}{\partial \beta} = B, \tag{33}$$

$A$, $B$ being constants. These equations must therefore represent the solutions of the differential equations

$$x' = u(x, y, z, \alpha, \beta), \qquad y' = v(x, y, z, \alpha, \beta), \tag{34}$$

for on the surfaces defined by the equations (33), $\mathrm{d}S_\alpha = 0$, $\mathrm{d}S_\beta = 0$, and by (32) this implies (34), provided that the associated determinant does not vanish. Now, according to (31) the determinant may be written in the form

$$\begin{vmatrix} \dfrac{\partial F_u}{\partial \alpha} & \dfrac{\partial F_v}{\partial \alpha} \\ \dfrac{\partial F_u}{\partial \beta} & \dfrac{\partial F_v}{\partial \beta} \end{vmatrix} = \begin{vmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{vmatrix} \begin{vmatrix} u_\alpha & u_\beta \\ v_\alpha & v_\beta \end{vmatrix}. \tag{35}$$

The first factor on the right-hand side is expression (9), which was assumed to differ from zero. The second term can only vanish at points where (34) have no solutions for $\alpha$ and $\beta$, in other words at points where the field does not cover the space uniquely. If

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we exclude such cases, (34) represents $\infty^2$ sets of differential equations, each with $\infty^2$ solutions; all these together must therefore be identical with the totality of $\infty^4$ extremals. We have proved that the solutions

$$x = x(z, \alpha, \beta, A, B), \qquad y = y(z, \alpha, \beta, A, B), \tag{36}$$

of (33) are just this total set of $\infty^4$ extremals; hence the whole set of $\infty^4$ extremals can be obtained from a complete solution $S(x, y, z, \alpha, \beta)$ of the Hamilton–Jacobi equation by differentiation and elimination only, according to (33) and (36).

## 5 Hamilton's canonical equations

Each Euler equation (7) is a differential equation of the second order. It is often convenient to replace these two second-order equations by four differential equations of the first order. This can be done in many ways. The most symmetrical way gives the so-called Hamilton canonical equations, obtained as follows:

Eqs. (13) are regarded as a Legendre transformation (see p. 144), which replaces the variables $u$, $v$ by $U$, $V$ (retaining $x$, $y$, $z$), and the function $F(u, v, x, y, z)$ by $W(U, V, x, y, z)$. We can write the last equation in (13) as

$$W = F - uU - vV, \tag{37}$$

so that

$$\mathrm{d}W = \mathrm{d}F - u\,\mathrm{d}U - v\,\mathrm{d}V - U\,\mathrm{d}u - V\,\mathrm{d}v.$$

Now

$$\begin{aligned} \mathrm{d}F &= F_u\,\mathrm{d}u + F_v\,\mathrm{d}v + F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z \\ &= U\,\mathrm{d}u + V\,\mathrm{d}v + F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z, \end{aligned}$$

and therefore

$$\mathrm{d}W = -u\,\mathrm{d}U - v\,\mathrm{d}V + F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z.$$

Since $W$ is to be regarded as a function of $U$, $V$, $x$, $y$, $z$, it follows that

$$W_U = -u, \qquad W_V = -v, \qquad W_x = F_x, \qquad W_y = F_y, \qquad W_z = F_z. \tag{38}$$

If we now consider a curve $x = x(z)$, $y = y(z)$ which satisfies the equations

$$x' = u(x, y, z), \qquad y' = v(x, y, z), \tag{39}$$

these two equations, together with Euler's equations (7), may be written in the form

$$\left.\begin{aligned} x' &= -W_U, \qquad & y' &= -W_V, \\ U' &= \phantom{-}W_x, \qquad & V' &= \phantom{-}W_y. \end{aligned}\right\} \tag{40}$$

Eqs. (40) are four differential equations of the first order for $x$, $y$, $U$, $V$ as functions of $z$, and are called *Hamilton's canonical equations*. They may be regarded as the Euler equations of the variational integral expressed in terms of the function $W(U, V, x, y, z)$. If we substitute from (37) and (39) into (1), the integral goes over into

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$$I = \int_{z_1}^{z_2} [W(U, V, x, y, z) + x'U + y'V]\mathrm{d}z. \tag{41}$$

If $U$, $V$, $x$, $y$ are regarded here as four unknown functions of $z$ and the Euler equation for each of them is formed, then (40) is immediately obtained.

## 6 The special case when the independent variable does not appear explicitly in the integrand

The case when $F$ does not explicitly depend on $z$ deserves special consideration.

We have in general for $F(x', y', x, y, z)$ that

$$\frac{\mathrm{d}F}{\mathrm{d}z} = F_{x'}x'' + F_{y'}y'' + F_x x' + F_y y' + F_z.$$

Assume now that $F_z = 0$, and substitute for $F_x$, $F_y$ from the Euler equations (7). This gives

$$\frac{\mathrm{d}F}{\mathrm{d}z} = F_{x'}x'' + F_{y'}y'' + x'\frac{\mathrm{d}}{\mathrm{d}z}F_{x'} + y'\frac{\mathrm{d}}{\mathrm{d}z}F_{y'} = \frac{\mathrm{d}}{\mathrm{d}z}(x'F_{x'} + y'F_{y'}).$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}z}(F - x'F_{x'} - y'F_{y'}) = 0,$$

so that

$$F - x'F_{x'} - y'F_{y'} = \text{constant}. \tag{42}$$

This expression, which is just the quantity $W$ on the extremal, is independent of $z$; $W$ is a constant of integration. The same result can be seen directly from the canonical equations (40), for if $F$ does not explicitly depend on $z$, $W$ is also independent of $z$, $W_z = 0$, and

$$\frac{\mathrm{d}W}{\mathrm{d}z} = W_U U' + W_V V' + W_x x' + W_y y';$$

this expression vanishes by virtue of (40).

In this case one may reduce the variational problem from three to two dimensions. Consider $y$ as a function of $x$; then $y' = (\mathrm{d}y/\mathrm{d}x)x'$, and $F(x', y', x, y) = F(x', \mathrm{d}y/\mathrm{d}x, x, y)$. In the same way $F_{x'}$, $F_{y'}$, and also $F - x'F_{x'} - y'F_{y'}$ can be regarded as functions of $x'$, $\mathrm{d}y/\mathrm{d}x$, $x$, $y$. Now the equation

$$F - x'F_{x'} - y'F_{y'} = W \tag{43}$$

can be solved with respect to $x'$:

$$\left.\begin{aligned} x' &= \Phi\left(\frac{\mathrm{d}y}{\mathrm{d}x}, x, y, W\right), \\ y' &= \frac{\mathrm{d}y}{\mathrm{d}x}\Phi\left(\frac{\mathrm{d}y}{\mathrm{d}x}, x, y, W\right). \end{aligned}\right\} \tag{44}$$

Let us consider all curves for which $W$ has some given value; then the integral (1) can be replaced by the difference of two integrals, or

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$$J=\int_{z_1}^{z_2}(F-W)\mathrm{d}z=\int_{z_1}^{z_2}(x'F_{x'}+y'F_{y'})\mathrm{d}z=\int_{x_1}^{x_2}\left(F_{x'}+\frac{\mathrm{d}y}{\mathrm{d}x}F_{y'}\right)\mathrm{d}x.$$

If now the abbreviation

$$F_{x'}+\frac{\mathrm{d}y}{\mathrm{d}x}F_{y'}=f \tag{45}$$

is used, and $x'$, $y'$ are eliminated with the help of (44), $f$ becomes a function of $\mathrm{d}y/\mathrm{d}x$, $x$, $y$, $W$, and

$$J=\int_{x_1}^{x_2}f\left(\frac{\mathrm{d}y}{\mathrm{d}x},\,x,\,y,\,W\right)\mathrm{d}x. \tag{46}$$

Thus we have, for each value of $W$, a variational integral of one dimension less. (This reduction corresponds in mechanics to the transition from Hamilton's principle to Maupertuis' principle: see (88) below.) If $y(x)$ is found from the Euler equation corresponding to (46), the complete extremal is obtained on integrating (44).

## 7 Discontinuities

It may happen that the function $F(u, v, x, y, z)$ is not everywhere continuous. The most important case (which frequently occurs in optics) is that in which $F$ has a finite discontinuity along a surface $\sigma(x, y, z) = 0$ for all values of $u$, $v$.

It is obvious that outside the surface the extremals are again the solutions of the Euler equations (7); but if an extremal crosses the surface, it will have a discontinuity of direction (refraction). Let us distinguish the space to the left and right of the surface by the indices 1 and 2 (Fig. 2).

In order to find the 'law of refraction' we have to establish the condition which ensures that the Hilbert integral $S$, defined by (11), extended from a point $P_1$ on the left of the surface to a point $P_2$ on the right of it, is independent of the path. Consider two paths $P_1AP_2$ and $P_1BP_2$, $A$ and $B$ being points on the surface. Then (with obvious notation) we demand that $S(P_1AP_2) = S(P_1BP_2)$ or

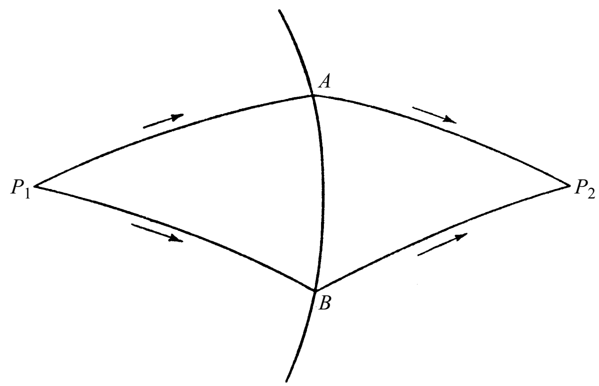


Fig. 2 Illustrating the variational analogue of the optical law of refraction.

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$$S_1(P_1A) + S_2(AP_2) = S_1(P_1B) + S_2(BP_2). \tag{47}$$

Now on the closed path $P_1ABP_1$ on the left of the surface,

$$S_1(P_1A) + S_1(AB) + S_1(BP_1) = 0, \tag{48}$$

and on the closed path on the right

$$S_2(P_2B) + S_2(BA) + S_2(AP_2) = 0. \tag{49}$$

If we add (48) and (49), and use (47) and the relation $S(XY) = -S(YX)$, it follows that

$$S_1(AB) = S_2(AB). \tag{50}$$

The integral (11) taken along any path on the surface $\sigma = 0$ has, therefore, the same value, whether one takes as $u$, $v$ the values $u_1$, $v_1$ on the left, or the values $u_2$, $v_2$ on the right. Thus the integrands must be equal in the two cases and the *law of refraction* is equivalent to the assertion that the expression

$$F + (x' - u)F_u + (y' - v)F_v \tag{51}$$

is continuous on the surface $\sigma = 0$. According to (13) this condition may be expressed in the form

$$(Ux' + Vy' + W)_1 = (Ux' + Vy' + W)_2, \tag{52}$$

where $x'$, $y'$ are the derivatives of $x(z)$, $y(z)$ for any curve on the surface. This can also be expressed by saying that the vector $U_2 - U_1$, $V_2 - V_1$, $W_2 - W_1$ is normal to the discontinuity surface:

$$(U_2 - U_1)\mathrm{d}x + (V_2 - V_1)\mathrm{d}y + (W_2 - W_1)\mathrm{d}z = 0. \tag{53}$$

Very similar to the problem of refracted extremals is that of reflected extremals. One has to connect two points $P_1$ and $P_2$, situated in a region where $F$ is a continuous function of $x$, $y$, $z$, by a curve $P_1AP_2$ which has a discontinuity in direction at a point $A$ on a given surface $\sigma(x, y, z) = 0$, the points $P_1$ and $P_2$ being on the same side of $\sigma$ (Fig. 3).

It is obvious that the parts $P_1A$ and $AP_2$ must be extremals, and it follows, by a

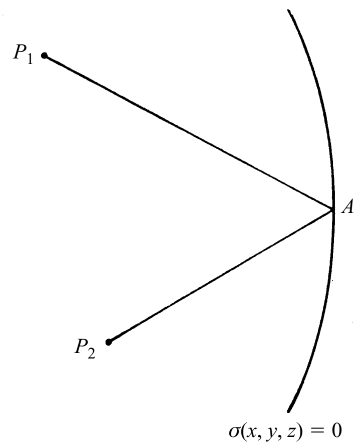


Fig. 3 Illustrating the variational analogue of the optical law of reflection.

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consideration like that leading to the law of refraction [(53)], that the condition for the validity of the *independence theorem* for an incident field (suffix 1) and a reflected field (suffix 2) is the *law of reflection*,

$$(U_1 + U_2)\mathrm{d}x + (V_1 + V_2)\mathrm{d}y + (W_1 + W_2)\mathrm{d}z = 0. \tag{54}$$

The independence theorem also holds for any fields with a finite number of refracting or reflecting discontinuities. It will be shown in the next section that in all these cases, whether the extremal which connects $P_1$ and $P_2$ is continuous or has (directional) discontinuities, it will give rise to a minimum of the integral (1), provided that the function $F$ satisfies certain simple conditions along this curve.

## 8 Weierstrass' and Legendre's conditions (sufficiency conditions for an extremum)

So far no distinction has been made between maxima and minima; the extremals considered (smooth or with a 'kink') may even correspond to stationary cases which are not true extrema. We shall now derive conditions necessary for a real minimum.

Let $\bar{x}(z)$, $\bar{y}(z)$ be a fixed extremal $\overline{C}$ embedded in a field $u(x, y, z)$, $v(x, y, z)$, and let $x(z)$, $y(z)$ be any neighbouring curve $C$ also completely embedded in the field and with the same end points $P_1$ and $P_2$ as $\overline{C}$ (Fig. 4). The extremum will be a real minimum if

$$\int_C F(x', y', x, y, z)\mathrm{d}z - \int_{\overline{C}} F(\bar{x}', \bar{y}', \bar{x}, \bar{y}, z)\mathrm{d}z > 0. \tag{55}$$

According to §2 and §3 we may replace the second integral by one extended not over $\overline{C}$ but over $C$, namely by

$$\int_C [F(u, v, x, y, z) + (x' - u)F_u + (y' - v)F_v]\mathrm{d}z;$$

this integral is independent of the path and reduces to $\int_{\overline{C}} F\,\mathrm{d}z$, if the path is taken to coincide with $\overline{C}$. Hence (55) becomes

$$\int_C F\,\mathrm{d}z - \int_{\overline{C}} F\,\mathrm{d}z = \int_C \mathcal{E}(x', y', u, v, x, y, z)\mathrm{d}z > 0, \tag{56}$$

where

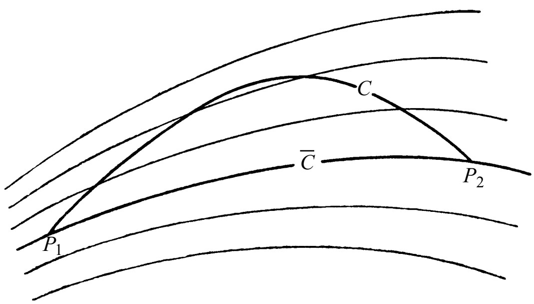


Fig. 4 Illustrating the definition of the Weierstrass' $\mathcal{E}$-function.

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$$\mathcal{E}(x', y', u, v, x, y, z) = F(x', y', x, y, z) - F(u, v, x, y, z) \\ - (x' - u)F_u - (y' - v)F_v, \tag{57}$$

and the arguments in $F_u$, $F_v$ are the same as in $F(u, v, x, y, z)$. The function defined by (57) is called *the* $\mathcal{E}$*-function* (*or the excess function*) *of Weierstrass*; the arguments $x, y, z, x', y'$ refer to a point on the curve $C$ and to its direction, while $u, v$ refer to the direction of the extremal of the field, which passes through the point $x, y, z$.

It is seen that $\mathcal{E}$ vanishes on any portion of $C$ which coincides with a field extremal. Now we can choose the field as the set of all $\infty^2$ extremals going through $P_1$. Then we construct a special curve $C$ which is such that between $P_1$ and a point $A$ the curve coincides with a field extremal, from $A$ to a point $B$ on the given extremal it is a straight line, and from $B$ to $P_2$ it coincides with the given extremal (Fig. 5). Then $\mathcal{E}$ vanishes on the parts $P_1A$ and $BP_2$, and there remains

$$\int_A^B \mathcal{E}\,\mathrm{d}z > 0.$$

By letting $A$ approach $B$, it is seen that this inequality is only possible if

$$\mathcal{E}(x', y', \bar{x}', \bar{y}', \bar{x}, \bar{y}, \bar{z}) > 0; \tag{58}$$

here $\bar{x}, \bar{y}, \bar{z}$ refer to a typical point ($B$) on the given extremal $\bar{C}$ and $x', y'$ refer to the direction $AB$ which is quite arbitrary. Formula (58) is *Weierstrass' condition for a strong minimum*; it is certainly a necessary condition. But on the assumption that the function $F$ is continuous in all its five arguments (hence $\mathcal{E}$ is continuous in its seven arguments), it follows that if (58) is satisfied for all points on the given extremal and for arbitrary directions $x', y'$, then the inequality (56) must hold for any neighbouring curve $C$ of arbitrary directions in a certain region surrounding $\bar{C}$. Hence condition (58) is also sufficient for a strong minimum. This minimum is, of course, only relative, for there may be several extremals having the property of giving a minimum value to the integral (1) in comparison with all neighbouring curves; which of them gives the absolute minimum cannot be decided in this way.

The inequality (58) has a simple geometrical interpretation. For a fixed point $\bar{x}, \bar{y}, \bar{z}$ in space, $F$ is a function of $\bar{x}', \bar{y}'$ alone; this function $F(\bar{x}', \bar{y}')$ can be represented by a surface in the three-dimensional $\bar{x}', \bar{y}', F$-space. (In Fig. 6 the two-dimensional cross-section $\bar{x}'F$ is drawn.) Then

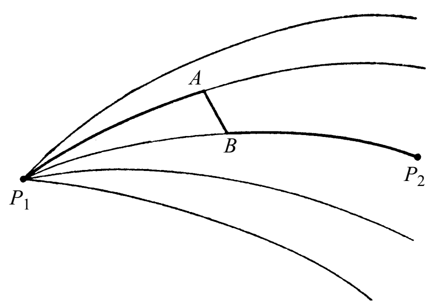


Fig. 5 Derivation of Weierstrass' condition for a strong minimum.

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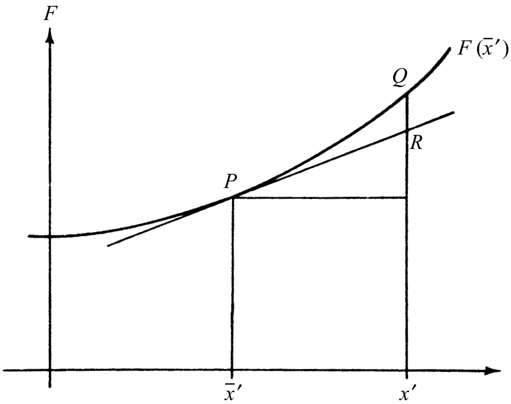


Fig. 6 A geometrical interpretation of Weierstrass' condition for a strong minimum.

$$\mathcal{E}(x', y', \bar{x}', \bar{y}') = F(x', y') - [F(\bar{x}', \bar{y}') + (x' - \bar{x}')F_{\bar{x}'} + (y' - \bar{y}')F_{\bar{y}'}] \tag{59}$$

is obviously the distance $QR$ along the ordinate in $x'$, $y'$, between the point $Q$ on the surface $F = F(\bar{x}', \bar{y}')$ and the point $R$ (see Fig. 6) where the tangential plane at $P(\bar{x}', \bar{y}')$ intersects this ordinate. Hence $\mathcal{E}(x', y', \bar{x}', \bar{y}') > 0$ if the surface $F$ is above the tangential plane at $P$. If this holds for all $x'$, $y'$ there is a *strong minimum*.

If, however, (58) holds only for small intervals of $\xi = x' - \bar{x}'$, $\eta = y' - \bar{y}'$, there is a weak minimum; in this case we may expand $\mathcal{E}$ in powers of $\xi$, $\eta$ and obtain (if again we omit the arguments $x$, $y$, $z$)

$$\begin{aligned}\mathcal{E}(x', y', \bar{x}', \bar{y}') = F(x', y') - F(\bar{x}', \bar{y}') - \xi F_{\bar{x}'} - \eta F_{\bar{y}'} \\ + \tfrac{1}{2}[F_{\bar{x}'\bar{x}'}\xi^2 + 2F_{\bar{x}'\bar{y}'}\xi\eta + F_{\bar{y}'\bar{y}'}\eta^2] + \cdots .\end{aligned}$$

For small values $\xi$ and $\eta$ the quadratic terms are decisive and must evidently be positive for a minimum. Thus we obtain *Legendre's condition* (necessary and sufficient) *for a weak minimum*:*

$$F_{\bar{x}'\bar{x}'} > 0, \qquad F_{\bar{x}'\bar{x}'}F_{\bar{y}'\bar{y}'} - F^2_{\bar{x}'\bar{y}'} > 0. \tag{60}$$

## 9 Minimum of the variational integral when one end point is constrained to a surface

The $\mathcal{E}$-function provides a simple solution of the problem of finding the minimum of the variational integral (1) with respect to all curves that have one end point $P_1$ in common and the other constrained to a given surface $\sigma(x, y, z) = 0$.

The curve must obviously be one of the $\infty^2$ extremals through $P_1$, the problem is 'which of them?' Now amongst these $\infty^2$ extremals there is just one† that is

* This condition has an immediate generalization when one deals not with two but with a larger number of variables ($n$ say). In order to have a minimum, the quadratic form in the $n$ variables must be positive definite; this implies that the associated determinant and its principal minors must all be positive.

† It is assumed that $P_1$ is near enough to the surface, so that the case of several such extremals is excluded.

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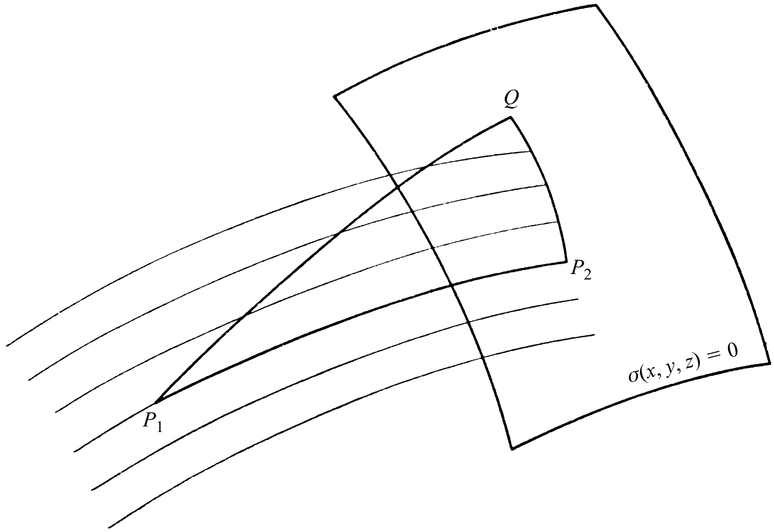


Fig. 7 Determination of the minimum of the variational integral with respect to all curves which have one end point fixed and the other constrained to a surface.

transversal to the surface $\sigma = 0$ and it is easily seen that this represents the solution of the problem. To show this let $P_2$ be the point in which this extremal intersects the surface $\sigma = 0$, and surround it by the field of all extremals that are transversal to the surface. Let $P_1Q$ be any extremal through $P_1$ and $Q$ its point of intersection with the surface (Fig. 7). Then the Hilbert integral $S(P_2, Q)$ vanishes. Hence the integral $S$ taken over the path $P_1QP_2$ is equal to the variational integral $I(P_1, Q)$. The difference $I(P_1, Q) - I(P_1, P_2)$ can now be expressed in terms of the $\mathcal{E}$-function in just the same way as before, and if (58) is satisfied, this difference is positive as long as $Q$ differs from $P_1$.

## 10 Jacobi's criterion for a minimum

If an extremal can be embedded in a field and the Legendre condition is satisfied for all points of it between $P_1$ and $P_2$, then the integral $I$, defined by (1), is certainly a (weak) minimum. It remains to find a criterion for the existence of the field.

Let all $\infty^2$ extremals that pass through $P_1$ be given by

$$x = x(z, \alpha, \beta), \qquad y = y(z, \alpha, \beta), \tag{61}$$

and let the given extremal $C$ be characterized by the values $\alpha = 0, \beta = 0$:

$$x = x(z, 0, 0), \qquad y = y(z, 0, 0). \tag{62}$$

The curves (61) form a field as long as there is one such curve through a given point $P(x, y)$ arbitrarily close to $C$, i.e. as long as (61) have unique solutions for $\alpha$, $\beta$ as functions of $x$, $y$; the condition for this is that

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$$\Delta = \begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} \neq 0. \tag{63}$$

This is *Jacobi's criterion for a minimum*.

The determinant $\Delta$ is a function of $z$ along the given extremal (62). The first point $\overline{P}$ where $\Delta = 0$ is called the *conjugate point of* $P_1$; for every interval $P_1P_2$ such that $P_2$ lies between $P_1$ and $\overline{P}$, there is a real minimum.

In $\overline{P}$ the given extremal is intersected by a neighbouring extremal (infinitesimally close to it); it is a point of the envelope of the set (61). Hence the *limit of the field is determined by the envelope of the set of extremals* (61). In optics these envelopes are the caustic surfaces.

## 11 Example I: Optics

The general theory will now be illustrated by a few examples. The first concerns the shortest line in ordinary geometry and the line of shortest optical length in geometrical optics.

Euclidean geometry is based on the theorem of Pythagoras according to which the line element d$s$ is related to its projections d$x$, d$y$, d$z$ on to the axes of a rectangular system by means of the relation

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2. \tag{64}$$

The shortest lines between two points are then given by the minima of the variational integral

$$s = \int_{P_1}^{P_2} \mathrm{d}s = \int_{z_1}^{z_2} \sqrt{x'^2 + y'^2 + 1}\,\mathrm{d}z. \tag{65}$$

Geometrical optics can be based on a generalization of this integral, namely on Fermat's principle of the shortest optical path (see §3.3.2),

$$\int_{P_1}^{P_2} n\,\mathrm{d}s = \int_{z_1}^{z_2} n(x,\ y,\ z)\sqrt{x'^2 + y'^2 + 1}\,\mathrm{d}z, \tag{66}$$

where $n(x,\ y,\ z)$ is the index of refraction. We shall treat only the optical case, as (65) is the special case of (66) for $n = 1$.

We now have

$$F(x',\ y',\ x,\ y,\ z) = n(x,\ y,\ z)\sqrt{x'^2 + y'^2 + 1}. \tag{67}$$

Since $\mathrm{d}s = \sqrt{x'^2 + y'^2 + 1}\,\mathrm{d}z$,

$$\left.\begin{aligned} U &= F_{x'} = \frac{nx'}{\sqrt{x'^2 + y'^2 + 1}} = n\frac{\mathrm{d}x}{\mathrm{d}s} = ns_x, \\ V &= F_{y'} = \frac{ny'}{\sqrt{x'^2 + y'^2 + 1}} = n\frac{\mathrm{d}y}{\mathrm{d}s} = ns_y, \\ W &= F - F_{x'}x' - F_{y'}y' = \frac{n}{\sqrt{x'^2 + y'^2 + 1}} = n\frac{\mathrm{d}z}{\mathrm{d}s} = ns_z, \end{aligned}\right\} \tag{68}$$

where $s_x$, $s_y$, and $s_z$ are the components of the unit vector $\mathbf{s}$ tangential to the curve

$x = x(z)$, $y = y(z)$. The law of refraction on a discontinuity surface of $n(x, y, z)$ may according to (53) be expressed in the form

$$(n_2\mathbf{s}_2 - n_1\mathbf{s}_1)\cdot \mathrm{d}\mathbf{l} = 0, \tag{69a}$$

where $\mathrm{d}\mathbf{l}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is any line element of the surface. This equation implies that $\mathbf{s}_1$, $\mathbf{s}_2$ and the surface normal are coplanar, and that the angles $\theta_1$ and $\theta_2$ which $\mathbf{s}_1$ and $\mathbf{s}_2$ make with the surface normal are related by

$$n_2 \sin\theta_2 = n_1 \sin\theta_1, \tag{69b}$$

in agreement with the law of refraction [§3.2 (19)].

The Euler equations (7) associated with (66) are

$$\frac{\partial n}{\partial x}\sqrt{x'^2 + y'^2 + 1} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{nx'}{\sqrt{x'^2 + y'^2 + 1}} = 0,$$

$$\frac{\partial n}{\partial y}\sqrt{x'^2 + y'^2 + 1} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{ny'}{\sqrt{x'^2 + y'^2 + 1}} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{\partial n}{\partial x}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}y}{\mathrm{d}s}\right) = \frac{\partial n}{\partial y}. \tag{70a}$$

The corresponding equation in $z$, namely

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial n}{\partial z}, \tag{70b}$$

is an identity, as it follows from

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2 = 1. \tag{71}$$

To derive (70b) one may proceed as follows: First one differentiates (71) with respect to $s$. Next one multiplies (71) by $\mathrm{d}n/\mathrm{d}s$, the differentiated equation by $n$ and one adds the two equations. Finally, one uses (70a).* The three scalar differential equations (70) are in agreement with the vector equation §3.2 (2) for the light rays.

Since $U$, $V$ and $W$ now represent the components of the ray-vector [see §4.1 (4)] it is seen from (12) that the $S$ function belonging to the geometrical optics field is just the *Hamilton point characteristic function* [see §4.1 (1)]. Moreover, by following the procedure leading to (21) one finds that the Hamilton–Jacobi equation of the present variational problem is the *eikonal equation*.

To investigate the Legendre condition (60) one has to determine the derivatives $F_{x'x'}$, etc. One has, in the present case,

* Eq. (70b) can be obtained in a more symmetrical way, which, however, introduces a superfluous Euler equation. One regards $x$, $y$ and $z$ as functions of a parameter $\lambda$; one then has three Euler equations connected by an identity and the parameter $\lambda$ is finally identified with $s$.

$$F_{x'x'} = n\frac{1+y'^2}{(1+x'^2+y'^2)^{3/2}},$$

$$F_{y'y'} = n\frac{1+x'^2}{(1+x'^2+y'^2)^{3/2}},$$

$$F_{x'y'} = -n\frac{x'y'}{(1+x'^2+y'^2)^{3/2}},$$

so that

$$F_{x'x'}\cdot F_{y'y'} - F^2_{x'y'} = \frac{n^2}{(1+x'^2+y'^2)^2} > 0. \tag{72}$$

Every extremal therefore gives a weak minimum (p. 868) if the Jacobi condition (63) is satisfied. But as the function $F$, for given $x$, $y$, $z$, i.e. for a given $n$, is convex downwards for all values of $x'$, $y'$, it follows from the geometrical interpretation of Weierstrass' condition that the minimum is strong.

It remains to consider Jacobi's criterion. For $n =$ constant, i.e. in ordinary Euclidean geometry, the extremals are obviously straight lines; since a bundle of straight lines through a point $P_1$ never has an envelope, each straight line gives a strong minimum of the distance between any two points of it. In geometrical optics on the other hand, where $n$ in general depends on $x$, $y$, $z$ (continuously or discontinuously), the bundle of rays from $P_1$ gives rise to envelopes (caustic surfaces). To determine the nature of the extremum these surfaces have to be examined separately in every particular case.

## 12 Example II: Mechanics of material points

As a second example we consider *the mechanics of systems of material points*. Here the independent variable is the time $t$, and the unknown functions are the Lagrangian coordinates $q_\alpha (\alpha = 1, 2, \ldots, n)$ and their derivatives the velocities $u_\alpha = \dot{q}_\alpha$.

The variational problem is given by *Hamilton's principle*

$$\int_{t_1}^{t_2} L(u_1, u_2, \ldots, q_1, q_2, \ldots, t)\mathrm{d}t = \text{extremum}, \tag{73}$$

where $L$ is the Lagrangian. In ordinary nonrelativistic mechanics one has $L = T - \Phi$, where $T$ is the kinetic energy, a quadratic form of the $u_\alpha$, and $\Phi$ the potential energy; but (73) holds also in more general cases, when a magnetic force is acting and when the relativistic variation of mass is taken into account.

The function $F(u, v, x, y, z)$ is here replaced by $L(u, q, t)$; hence [see (13)] $U$, $V$ now correspond to the momenta

$$p_\alpha = \frac{\partial L}{\partial u_\alpha} \tag{74}$$

and $W$ to $-H$, where $H$ is the *Hamiltonian*

$$H = \sum_\alpha u_\alpha \frac{\partial L}{\partial u_\alpha} - L = \sum_\alpha u_\alpha p_\alpha - L. \tag{75}$$

If $L = T - \Phi$, (75) becomes

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$$H = \sum_\alpha u_\alpha \frac{\partial L}{\partial u_\alpha} - T + \Phi.$$

From Euler's theorem on homogeneous functions* it follows that

$$2T = \Sigma u_\alpha \frac{\partial T}{\partial u_\alpha} \left( = \Sigma u_\alpha \frac{\partial L}{\partial u_\alpha} \right),$$

and $H$ reduces to the total energy

$$H = T + \Phi. \tag{76}$$

The Euler equations (7) become Lagrange's equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0; \tag{77}$$

and the canonical equations (40) now are

$$\frac{\mathrm{d}q_\alpha}{\mathrm{d}t} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{\mathrm{d}p_\alpha}{\mathrm{d}t} = -\frac{\partial H}{\partial q_\alpha}, \tag{78}$$

where $H$ is to be regarded as a function of the $p_\alpha$, $q_\alpha$ and $t$. If $H$ is independent of the time $t$, formula (42) expresses the law of conservation of energy

$$H = \sum_\alpha u_\alpha p_\alpha - L = \text{constant} = E. \tag{79}$$

In this case the Hamilton–Jacobi differential equation (21) becomes

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = 0. \tag{80}$$

This gives on integration,

$$S = -Et + S_1(q_\alpha), \tag{81}$$

where, because of (80), $S_1$ satisfies the equation

$$H\left(\frac{\partial S_1}{\partial q_\alpha}, q_\alpha\right) = E. \tag{82}$$

From the solution of the Hamilton–Jacobi equation one obtains the momenta in accordance with (17):

$$p_\alpha = \frac{\partial S}{\partial q_\alpha} = \frac{\partial S_1}{\partial q_\alpha}. \tag{83}$$

It follows that the line integral

$$\int_{P_1}^{P_2} \sum_\alpha p_\alpha \,\mathrm{d}q_\alpha \tag{84}$$

is independent of the path joining $P_1$ and $P_2$ and is, therefore, zero when taken over a closed path in a simply connected region,

* See, for example, R. Courant, *Differential and Integral Calculus*. Vol. II (Glasgow, Blackie and Son, 1936), p. 109.

$$\oint p_\alpha \,\mathrm{d}q_\alpha = 0. \tag{85}$$

If the functions $p_\alpha$ are multivalued it may happen that the integral over a closed path is not zero, but is a multiple of a certain constant period. This result is a generalization of the optical invariant of Lagrange (§3.3.1) and is one of *Poincaré's invariants*. It may also be expressed in the form

$$\frac{\partial p_\alpha}{\partial q_\beta} - \frac{\partial p_\beta}{\partial q_\alpha} = 0. \tag{86}$$

If $H$ is independent of time, $t$ may be eliminated from the minimum principle according to the general procedure expressed by (45) and (46). One then has

$$J = \int_{t_1}^{t_2} (L + E)\mathrm{d}t = \int_{t_1}^{t_2} \sum_\alpha \dot{q}_\alpha p_\alpha \,\mathrm{d}t = \int_{P_1}^{P_2} \sum_\alpha p_\alpha \,\mathrm{d}q_\alpha = \text{extremum}. \tag{87}$$

This is *Maupertuis' principle of least action*, generalized for arbitrary $L$; it has to be understood in the following way: (79) allows the elimination of the time derivatives $u_\alpha = \dot{q}_\alpha$, expressing them in terms of purely geometrical derivatives, say $\mathrm{d}q_\alpha/\mathrm{d}q_1$ ($\alpha = 2, \ldots, n$), using $q_1$ as an independent variable. Eq. (87) represents a purely geometrical principle describing the orbits, not the motions. The latter are then to be found from (78).

If $L = T - \Phi$, then $E = T + \Phi$ and one has Maupertuis' original expression

$$J = 2\int_{t_1}^{t_2} T \,\mathrm{d}t, \tag{88}$$

which is to be understood in the same way. An example for this reduction of the problem of the motion to the problem of the orbit by the transition from Hamilton's to Maupertuis' principle is the treatment of electron optics given in §2 of Appendix II.

The Legendre condition can only be investigated for a given $L$. If $L = T - \Phi$ and $T$ is a quadratic form of the $u_\alpha$, the condition is obviously equivalent to the postulate that $T$ is positive definite. Then Weierstrass' condition is also satisfied and one has a strong minimum as long as Jacobi's condition holds. The latter leads to the investigation of dynamical foci and caustics, but is of little importance in practice.

The Legendre condition for the relativistic electron with the Lagrangian (see Appendix II),

$$L = -mc^2\sqrt{1 - \left(\frac{v}{c}\right)^2} + e\left(\frac{\mathbf{v}}{c}\cdot\mathbf{A} - \phi\right) \tag{89}$$

leads to the quadratic form

$$\frac{m}{\left[1 - \left(\frac{v}{c}\right)^2\right]^{3/2}}\left[\rho^2\left(1 - \frac{v^2}{c^2}\right) + \left(\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{c}\right)^2\right] \tag{90}$$

in the components of the vector $\boldsymbol{\rho}(\xi, \eta, \zeta)$ and is, therefore, always satisfied.

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