Appendix V

A mathematical lemma used in the rigorous derivation of the Lorentz–Lorenz formula (§2.4.2)

In this appendix we shall establish a lemma used in §2.4, according to which

$$\operatorname{curl}\operatorname{curl}\int_\sigma^\Sigma \mathbf{Q}(\mathbf{r}')G(R)\mathrm{d}V' \to \int_\sigma^\Sigma \operatorname{curl}\operatorname{curl}\mathbf{Q}(\mathbf{r}')G(R)\mathrm{d}V' + \frac{8\pi}{3}\mathbf{Q}(\mathbf{r}), \tag{1}$$

as $a \to 0$. Here $\mathbf{Q}(\mathbf{r})$ is an arbitrary vector function of position and $G(R) = \mathrm{e}^{\mathrm{i}kR}/R$. The integrals are taken throughout the volume bounded externally by a surface Σ and internally by a sphere σ of radius a, which is centred at a point P specified by the position vector $\mathbf{r}(x, y, z)$. R denotes the distance $|\mathbf{r}-\mathbf{r}'|$, where the vector $\mathbf{r}'(x', y', z')$ specifies the position of a typical volume element $\mathrm{d}V'$.

Let $\mathbf{A}$ be an arbitrary vector function of position. The components of curl curl $\mathbf{A}$ are

$$(\operatorname{curl}\operatorname{curl}\mathbf{A})_x = \frac{\partial^2 A_y}{\partial y\partial x} + \frac{\partial^2 A_z}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2},$$

etc., so that

$$\left(\operatorname{curl}\operatorname{curl}\int_\sigma^\Sigma \mathbf{Q}G\,\mathrm{d}V'\right)_x = \frac{\partial^2}{\partial y\partial x}\int_\sigma^\Sigma Q_y G\,\mathrm{d}V' + \frac{\partial^2}{\partial z\partial x}\int_\sigma^\Sigma Q_z G\,\mathrm{d}V' - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\int_\sigma^\Sigma Q_x G\,\mathrm{d}V'. \tag{2}$$

Now we have for any differentiable scalar function $F(\mathbf{r}, \mathbf{r}')$,

$$\frac{\partial}{\partial x}\int_\sigma^\Sigma F\,\mathrm{d}V' = \int_\sigma^\Sigma \frac{\partial F}{\partial x}\,\mathrm{d}V' + \operatorname*{Lim}_{\delta x\to 0}\frac{1}{\delta x}\left(\int_{\sigma'}^\Sigma F\,\mathrm{d}V' - \int_\sigma^\Sigma F\,\mathrm{d}V'\right), \tag{3}$$

where σ' denotes a small sphere of radius a, centred on the point $T(x+\delta x, y, z)$. To evaluate the limit in (3), we note that the difference of the two integrals represents contributions from the two regions shown shaded in Fig. 9. The volume element may be expressed in the form $\delta V' = -\delta S' \times \delta x \times \rho_x$, where $\delta S'$ is the surface element and ρ_x is the x-component of the unit radial vector $\boldsymbol{\rho}$ pointing away from the point P. Hence

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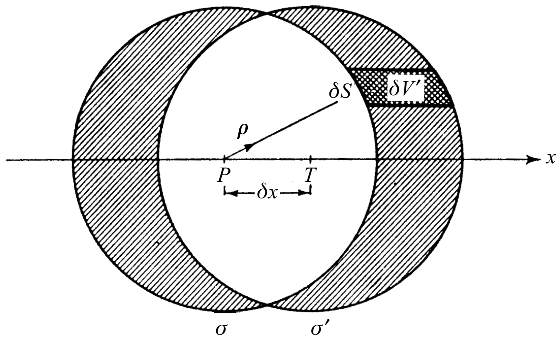

Fig. 9 Evaluation of $\displaystyle\operatorname*{Lim}_{\delta x\to 0}\frac{1}{\delta x}\left[\int_{\sigma'}^{\Sigma}F\,\mathrm{d}V'-\int_{\sigma}^{\Sigma}F\,\mathrm{d}V'\right]$. The centre of σ is at $P(x, y, z)$, the centre of σ' is at $T(x+\delta x, y, z)$.

$$\operatorname*{Lim}_{\delta x\to 0}\frac{1}{\delta x}\left(\int_{\sigma'}^{\Sigma}F\,\mathrm{d}V'-\int_{\sigma}^{\Sigma}F\,\mathrm{d}V'\right)=-\int_{\sigma}F\rho_x\,\mathrm{d}S'. \tag{4}$$

From (3) and (4) we have, if we set $F=Q_j(\mathbf{r})G(R)$, where Q_j ($j=x$, y or z) is any of the Cartesian components of $\mathbf{Q}$,

$$\frac{\partial}{\partial x}\int_{\sigma}^{\Sigma}(Q_jG)\mathrm{d}V'=\int_{\sigma}^{\Sigma}\frac{\partial}{\partial x}(Q_jG)\mathrm{d}V'-\int_{\sigma}(Q_jG)\rho_x\,\mathrm{d}S'. \tag{5}$$

Next we consider the partial derivatives of the second order. We differentiate (5) with respect to x and use (5) again. This gives

$$\frac{\partial^2}{\partial x^2}\int_{\sigma}^{\Sigma}(Q_jG)\mathrm{d}V'=\int_{\sigma}^{\Sigma}\frac{\partial^2}{\partial x^2}(Q_jG)\mathrm{d}V'-\int_{\sigma}\frac{\partial}{\partial x}(Q_jG)\rho_x\,\mathrm{d}S'-\frac{\partial}{\partial x}\int_{\sigma}(Q_jG)\rho_x\,\mathrm{d}S'. \tag{6}$$

Since

$$\left.\begin{aligned}\frac{\partial G}{\partial x}&=\frac{\mathrm{d}G}{\mathrm{d}R}\frac{\partial R}{\partial x}=-\rho_x\frac{\mathrm{d}}{\mathrm{d}R}\frac{\mathrm{e}^{\mathrm{i}kR}}{R}=\rho_x\left(\frac{1}{R}-\mathrm{i}k\right)\frac{\mathrm{e}^{\mathrm{i}kR}}{R},\\ \mathrm{d}S'&=a^2\,\mathrm{d}\Omega,\end{aligned}\right\} \tag{7}$$

and

where $\mathrm{d}\Omega$ is an element of the solid angle, it follows that

$$\begin{aligned}\int_{\sigma}\frac{\partial}{\partial x}(Q_jG)\rho_x\,\mathrm{d}S'&=\int_{\sigma}\rho_xQ_j\frac{\partial G}{\partial x}\,\mathrm{d}S'\\ &=\int_{\Omega}\rho_x^2Q_j\,\mathrm{e}^{\mathrm{i}ka}(1-\mathrm{i}ak)\mathrm{d}\Omega\to\frac{4\pi}{3}Q_j(\mathbf{r})\quad\text{as}\quad a\to 0,\end{aligned} \tag{8}$$

Ω denoting the surface of the unit sphere. The last integral in (6) tends to zero with a, so that we have, as $a\to 0$,

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$$\frac{\partial^2}{\partial x^2}\int_\sigma^\Sigma (Q_j G)\mathrm{d}V' \rightarrow \int_\sigma^\Sigma \frac{\partial^2}{\partial x^2}(Q_j G)\mathrm{d}V' - \frac{4\pi}{3}Q_j(\mathbf{r}). \tag{9}$$

The mixed second-order partial derivatives may be evaluated in the same way, and we have, for example,

$$\frac{\partial^2}{\partial y \partial x}\int_\sigma^\Sigma (Q_j G)\mathrm{d}V' \rightarrow \int_\sigma^\Sigma \frac{\partial^2}{\partial y \partial x}(Q_j G)\mathrm{d}V'. \tag{10}$$

The term $-(4\pi/3)Q_j(\mathbf{r})$ is now absent, since the integral corresponding to (8) is

$$\int_\sigma \frac{\partial}{\partial x}(Q_j G)\rho_y\,\mathrm{d}S' = \int_\Omega \rho_x \rho_y Q_j(1 - \mathrm{i}ak)\mathrm{d}\Omega$$

and this tends to zero with a.

On substituting from equations of the form (9) and (10) into (2) one finds that, as $a \rightarrow \infty$,

$$\left[\operatorname{curl}\operatorname{curl}\int_\sigma^\Sigma (\mathbf{Q}G)\mathrm{d}V'\right]_x \rightarrow \left[\int_\sigma^\Sigma \operatorname{curl}\operatorname{curl}(\mathbf{Q}G)\mathrm{d}V'\right]_x + \frac{8\pi}{3}Q_x. \tag{11}$$

Similar expressions are obtained for the y- and z-components. Combining these three expressions into vector form, (1) follows.

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