No. 2

a)

We start with the two dimensional wave equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right)$$

we know want a discretization of $u(x,y,t)$

$$u(x,y,t) \to u^n_{i,j} \equiv u(i\Delta x, j\Delta y, n\Delta t)$$

the second derivitive is given as

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} \to \frac{u^{n-1}_{i,j} - 2u^n_{i,j} + u^{n+1}_{i,j}}{\Delta t^2}$$

this leaves us with

$$\frac{u^{n-1}_{i,j} - 2u^n_{i,j} + u^{n+1}_{i,j}}{\Delta t^2} = c^2 \left(\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{\Delta x^2} + \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j}}{\Delta y^2} \right)$$

$\Leftrightarrow$

$$u_{i,j}^{n-1} - 2u_{i,j}^{n} + u_{i,j}^{n+1} = c^2 \Delta t^2 \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j}^{n}}{\Delta y^2} \right)$$

b)

Von Neumann Stability

the complex i is red.

$$\delta u_{i,j}^{n} = \tilde{c}_{\omega,k_x,k_y}\, e^{i(k_x i\Delta x + k_y j\Delta y - \omega n \Delta t)}$$

let's plug it into the wave equation

$$\tilde{c}^{\,n+1}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y j\Delta y - \omega(n+1)\Delta t)}$$

$$-2\tilde{c}^{\,n}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y j\Delta y - \omega n\Delta t)}$$

$$+\tilde{c}^{\,n-1}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y j\Delta y - \omega(n-1)\Delta t)}$$

$$= c^2 \Delta t^2 \left(\frac{1}{\Delta x^2} \left(\tilde{c}^{\,n}_{\omega,k_x,k_y} e^{i(k_x(i+1)\Delta x + k_y j\Delta y - \omega n\Delta t)} \right.\right.$$

$$- 2\tilde{c}^{n}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y j\Delta y - \omega n\Delta t)}$$

$$+ \tilde{c}^{n}_{\omega,k_x,k_y} e^{i(k_x(i-1)\Delta x + k_y j\Delta y - \omega n\Delta t)} \Big)$$

$$+ \frac{1}{y^2}\Big(\tilde{c}^{n}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y(j+1)\Delta y - \omega n\Delta t)}$$

$$- 2\tilde{c}^{n}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y j\Delta y - \omega n\Delta t)}$$

$$+ \tilde{c}^{n}_{\omega,k_x,k_y} e^{i(k_x i\Delta x + k_y(j-1)\Delta y - \omega n\Delta t)} \Big)\Big)$$

$$\Leftrightarrow \tilde{c}_{\omega,k_x,k_y} e^{i\omega\Delta t} - 2 + \tilde{c}^{-1}_{\omega,k_x,k_y} e^{-i\omega\Delta t}$$

$$= c^2\Delta t^2 \Big(\frac{1}{\Delta x^2}\Big(e^{i\Delta x k_x} - 2 + e^{-ik_x\Delta x}\Big)$$

$$+ \frac{1}{\Delta y^2}\Big(e^{i\Delta y k_y} - 2 + e^{-ik_y\Delta y}\Big)\Big)$$

$\Rightarrow$

$$\tilde{c}_{\omega,k_x,k_y} = c^2 \Delta t^2 \left(\frac{2\cos(k_x \Delta x) - 2}{\Delta x^2} - \frac{2\cos(k_y \Delta y) - 2}{\Delta y^2} \right) e^{\tilde{\omega} \Delta t}$$

$$- 2 e^{\omega \Delta t} - \frac{e^{2\omega \Delta t}}{\tilde{c}_{\omega,k_x,k_y}}$$

c) $$\hat{u}_{i,j} = c^2 \Delta t^2 \left(\frac{u^0_{i+1,j} - 2u^0_{i,j} + u^0_{i-1,j}}{\Delta x^2} + \frac{u^0_{i,j+1} - 2u^0_{i,j} + u^0_{i,j-1}}{\Delta y^2} \right)$$

$$+ 2 u^0_{i,j} - u^{-1}_{i,j}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$u^{-1}_{i,j} = -2 \Delta t \left. \frac{\partial u(x,y,t)}{\partial t} \right|_{t=0} + u^0_{i,j}$$ (3.2) Kierfeld

$$\Rightarrow \quad u^{1}_{i,j} = c^2 \Delta t^2 \left(\frac{u^{0}_{i+1,j} - 2u^{0}_{i,j} + u^{0}_{i-1,j}}{\Delta x^2} + \frac{u^{0}_{i,j+1} - 2u^{0}_{i,j} + u^{0}_{i,j-1}}{\Delta y^2} \right)$$

$$+ 2u^{2}_{i,j} - 4\Delta t \, u^{0}_{i,j} \left. \frac{\partial u(x,y,t)}{\partial t} \right|_{t=0}$$

d)

$$\tilde{x} = \frac{x}{\ell} \qquad \tilde{y} = \frac{y}{\ell} \qquad \tilde{t} = \frac{t}{\tau} \qquad \tilde{c} = \frac{c}{v} \qquad v = \frac{\ell}{\tau}$$

$$\tilde{c} \overset{!}{=} 1 \rightarrow v = c$$

$$\frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\cancel{\tau^2} \, \partial \tilde{t}^2} =$$

$$\underbrace{\cancel{v^2}}_{= \frac{\ell^2}{\tau^2}} \tilde{c}^2 \cdot \left(\frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\cancel{\ell^2} \, \partial \tilde{x}^2} + \frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\cancel{\ell^2} \, \partial \tilde{y}^2} \right)$$

$$\Leftrightarrow \quad \frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\partial \tilde{t}^2} = \tilde{c}^2 \left(\frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\partial \tilde{x}^2} + \frac{\partial^2 u(\tilde{x}\ell, \tilde{y}\ell, \tilde{t}\tau)}{\partial \tilde{y}^2} \right)$$

$$\Leftrightarrow \tilde{c}_{\omega,k_x,k_y} e^{i\omega\Delta t} - 2 + \tilde{c}^{-1}_{\omega,k_x,k_y} e^{-i\omega\Delta t}$$

$$= c^2 \Delta t^2 \left(\frac{1}{\Delta x^2} \left(e^{i\Delta x k_x} - 2 + e^{-i k_x \Delta x} \right) \right.$$

$$\left. + \frac{1}{\Delta y^2} \left(e^{i\Delta y k_y} - 2 + e^{-i k_y \Delta y} \right) \right)$$

$$-2 + ab + \frac{1}{ab} = c(d+e)$$

$$-2ab + (ab)^2 = abc(d+e)$$

$$0 = ab(-ab + 2 + c(d+e)) \qquad ab = 2 + c(d+e)$$

$$\tilde{c}_{\omega,k_x,k_y} e^{i\omega\Delta t} = 2 + c^2 \Delta t^2$$

$$\cdot \left(\frac{2\cos(k_x \Delta x) - 2}{\Delta x^2} - \frac{2\cos(k_y \Delta y) - 2}{\Delta y^2} \right)$$

$$\tilde{c}_{\omega,k_x,k_y} = e^{-i\omega\Delta t} \left(2 + c^2 \Delta t^2 2 \left(\frac{\cos(k_x \Delta x) - 1}{\Delta x^2} - \frac{\cos(k_y \Delta y) - 1}{\Delta y^2} \right) \right)$$

This is probably wrong because it would mean that it is stable for every Δt as long as ω is real. And also that higher values of $\Delta x, \Delta y$ make it more stable.