Derivatives with no expiries

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The biggest problem that we face with expiries is that it forces LPs and traders alike to fragment liquidity and attention span. Traditionally there is some payoff function $f(x)$ that gives the payoff of a derivative corresponding to some maturity time. That contract is valued at some current time and traded. Turning this problem on its head, rather than proposing a derivative whose terminal value is $f(x)$ we want a derivative with a current value of $f(x)$ and ask how much we would have to pay over time to obtain that exposure.

Jensen's inequality gives a hint of how to accomplish this. To state it plainly, Jensen's inequality states that when f is convex, X is a random variable and E is the expected value

$$f(E[X]) \leq E[f(X)]$$

If we take this one step further and consider f to be what we want the current value function of our derivative and consider X to be the discounted price of the underlying. To simplify expressions for the time being we will assume that dividends and the risk free rate are zero, $r = q = 0$. Because the discounted price of the underlying is a martingale under the risk neutral measure, $f(E_0^\star[S_t]) = S_0$

$$\begin{aligned}
\mathcal{J} &\equiv \frac{f\left(E_0^\star\left[S_t\right]\right)}{E_0^\star\left[f(S_t)\right])} \\
&= \frac{f\left(S_0\right)}{E_0^\star\left[f(S_t)\right])} \quad \text{(martingale)}
\end{aligned}$$

Note we drop the $E_0^\star$ and just go with E from now for readability and without loss of generality. If there is a need to specify a filtration other than $\mathcal{F}_0$ it will be specified with a subscript and if there is a need to specify a measure other than the risk neutral measure it will be specified with a superscript and most likely noted.

If the payoff function for a new derivative is $g(x) = f(x)\mathcal{J}(x)$ then the price is

$$\begin{aligned}
E[g(S_t)] &= E[f(S_t)\mathcal{J}(S_t)] \\
&= E[f(S_t)f(S_0)/E[f(S_t)]] \\
&= f(S_0)E[f(S_t)/E[f(S_t)]] \\
&= f(S_0)E[f(S_t)]/E[f(S_t)] \\
&= f(S_0)
\end{aligned}$$