**Math 220 Homework 4 Solutions**

# Chapter 4

**Problem 4:** Suppose that $x, y \in \mathbb{Z}$. If $x$ and $y$ are odd, then $xy$ is odd.

*Proof.* Let $x$ and $y$ be odd integers. By the definition of odd integers this means that $x = 2m + 1$ and $y = 2n + 1$ for some integers $m, n \in \mathbb{Z}$. It follows that

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1. \end{aligned}$$

Since $2mn + m + n \in \mathbb{Z}$ by the definition of an odd integer this implies that $xy$ is an odd integer. Thus, whenever $x$ and $y$ are odd we have that $xy$ is odd. □

**Problem 6:** Suppose $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$, then $a|(b + c)$.

*Proof.* Let $a$, $b$, and $c$ be integers and suppose that $a|b$ and $a|c$. Since $a|b$ there is some integer $m$ such that $b = am$. Because $a|c$ there exists some integer $n$ such that $c = an$. We then have

$$b + c = am + an = a(m + n).$$

By definition of divisibility this implies that $a|(b + c)$. □

**Problem 14:** If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.

*Proof.* Let $n \in \mathbb{Z}$. There are two cases for $n$ which are $n$ is even or $n$ is odd.

First let us consider the case where $n$ is even. By the definition of an even integer we then have $n = 2k$ for some $k \in \mathbb{Z}$. So

$$5n^2 + 3n + 7 = 5(2k)^2 + 3(2k) + 7 = 2(10k^2 + 3k + 3) + 1.$$

Because $10k^2 + 3k + 3 \in \mathbb{Z}$ by the definition of an odd integer this implies that $5n^2 + 3n + 7$ is odd.

Our second case considers when $n$ is odd. By the definition of an odd integer we then have $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$5n^2 + 3n + 7 = 5(2k + 1)^2 + 3(2k + 1) + 7 = 2(10k^2 + 13k + 7) + 1.$$

Because $10k^2 + 13k + 7 \in \mathbb{Z}$ by the definition of an odd integer this implies that $5n^2 + 3n + 7$ is odd.

Thus, in any case $5n^2 + 3n + 7$ is odd. □

**Problem 20:** If $a$ is an integer and $a^2|a$, then $a \in \{-1, 0, 1\}$.

*Proof.* Suppose that $a$ is an integer and $a^2|a$. By the definition of divides there must be some $k \in \mathbb{Z}$ where $a = ka^2$. So

$$\begin{aligned} a &= ka^2 \\ a - ka^2 &= 0 \\ a(1 - ka) &= 0. \end{aligned}$$

This gives us two cases, either $a = 0$ or $ka = 1$. If $a = 0$ then $a \in \{-1, 0, 1\}$ and we are done.

We now consider the case where $ka = 1$. This equation implies by definition that $a$ divides 1. The only divisors of 1 are 1 and $-1$, so $a$ is either 1 or $-1$. In this case $a \in \{-1, 0, 1\}$. Hence, if $a^2|a$, then $a \in \{-1, 0, 1\}$. □

# Chapter 5

**Problem 6:** Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$ then $x > -1$.

*Proof.* We will prove this statement by proving the contrapositive that if $x \leq -1$ then $x^3 - x \leq 0$. Assume that $x$ is a real number such that $x \leq -1$. Because $x \leq -1$ we know $x \leq 0$, $x + 1 \leq 0$, and $x - 1 \leq 0$. The multiplication of three nonpositive numbers is nonpositive so

$$x(x+1)(x-1) = x^3 - x$$

is nonpositive. Thus, $x^3 - x \leq 0$. □

**Problem 10:** Suppose $x, y, z \in \mathbb{Z}$ and $x \neq 0$. If $x \nmid yz$, then $x \nmid y$ and $x \nmid z$.

*Proof.* Let $x, y, z \in \mathbb{Z}$. To prove this statement we will prove the contrapositive that if $x|y$ or $x|z$, then $x|yz$. We have two cases $x|y$ or $x|z$.

First let us assume that $x|y$ which by definition of divides implies that there exists a $k \in \mathbb{Z}$ such that $y = kx$. It follows that

$$yz = kxz = x(kz)$$

so since $kz \in \mathbb{Z}$ by the definition of divides $x|yz$.

Instead let us assume our other case that $x|z$ which by definition of divides implies that there exists a $m \in \mathbb{Z}$ such that $z = mx$. It follows that

$$yz = ymx = x(ym)$$

so since $ym \in \mathbb{Z}$ by the definition of divides $x|yz$.

Hence, in all cases $x|yz$.

□

**Problem 14:** If $a, b \in \mathbb{Z}$ and $a$ and $b$ have the same parity, then $3a + 7$ and $7b - 4$ do not.

*Proof.* Assume that $a, b \in \mathbb{Z}$ and $a$ and $b$ have the same parity. This means $a$ and $b$ are both even or both odd.

First let us consider the case where $a$ and $b$ are both even that by definition that $a = 2m$ and $b = 2n$ for some $m, n \in \mathbb{Z}$. We have

$$3a + 7 = 3(2m) + 7 = 2(3m + 3) + 1$$

which is odd and

$$7b - 4 = 7(2n) - 4 = 2(7n - 2)$$

which is even. Thus, $3a + 7$ and $7b - 4$ have opposite parity.

Next let us consider the case where $a$ and $b$ are both odd that by definition that $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. We have

$$3a + 7 = 3(2m + 1) + 7 = 2(3m + 5)$$

which is even and

$$7b - 4 = 7(2n + 1) - 4 = 2(7n + 1) + 1$$

which is odd. Thus, $3a + 7$ and $7b - 4$ have opposite parity.

Therefore in all cases $3a + 7$ and $7b - 4$ have opposite parity. $\square$

**Problem 16:** Suppose that $x \in \mathbb{Z}$. If $x + y$ is even, then $x$ and $y$ have the same parity.

*Proof.* Assume that $x \in \mathbb{Z}$ and $x + y$ is even. By definition of an even integer $x + y = 2k$ for some $k \in \mathbb{Z}$. We will consider two cases for $x$ the first being $x$ is even and the second being $x$ is odd.

Assume that $x$ is even which by definition means that $x = 2n$ for some $n \in \mathbb{Z}$. It follows that

$$\begin{aligned} x + y &= 2k \\ y &= 2k - x \\ &= 2k - 2n \\ &= 2(k - n) \end{aligned}$$

which by definition implies that $y$ is also even. Hence, since $x$ and $y$ are both even they have the same parity.

Assume instead that $x$ is odd which by definition means that $x = 2n + 1$ for some $n \in \mathbb{Z}$. It follows that

$$\begin{aligned} x + y &= 2k \\ y &= 2k - x \\ &= 2k - (2n + 1) \\ &= 2(k - n - 1) + 1 \end{aligned}$$

which by definition implies that $y$ is also odd. Hence, since $x$ and $y$ are both odd they have the same parity.

Thus in all cases $x$ and $y$ have the same parity. $\square$

**Problem 18** For any $a, b \in \mathbb{Z}$, it follows that $(a + b)^3 \equiv a^3 + b^3 \pmod 3$.

*Proof.* Let $a, b \in \mathbb{Z}$. It follows that

$$(a + b)^3 - (a^3 + b^3) = a^3 + 3a^2b + 3ab^2 + b^3 - a^3 - b^3 = 3a^2b + 3ab^2 = 3(a^2b + ab^2).$$

Since $a^2b + ab^2 \in \mathbb{Z}$ by definition of divisibility 3 divides $(a + b)^3 - (a^3 + b^3)$. By definition of congruence modulo 3 this implies that $(a + b)^3 \equiv a^3 + b^3 \pmod 3$. $\square$

**Problem 20:** If $a \in \mathbb{Z}$ and $a \equiv 1 \pmod 5$, then $a^2 \equiv 1 \pmod 5$.

*Proof.* Assume that $a \in \mathbb{Z}$ and $a \equiv 1 \pmod 5$. By definition of congruence modulo 5 this implies that 5 divides $a - 1$ which further implies that there exists some $k \in \mathbb{Z}$ such that $a - 1 = 5k$ so $a = 5k + 1$. It follows that

$$\begin{aligned}
a &= 5k + 1 \\
a^3 &= (5k + 1)^3 \\
a^3 &= 125k^3 + 75k^2 + 15k + 1 \\
a^3 - 1 &= 125k^3 + 75k^2 + 15k \\
a^3 - 1 &= 5(25k^3 + 15k^2 + 3k).
\end{aligned}$$

Hence 5 divides $a^3 - 1$ and by the definition of congruence $a^3 \equiv 1 \pmod 5$. $\square$