STUDYSmarter
Learning Language and Research Skills

Logic and Proof Solutions

Question 1

Which of the following are true, and which are false? For the ones which are true, give a proof. For the ones which are false, give a counterexample.

(a) x is an odd number $\Rightarrow x^2$ is an odd number. (**True**)

Proof: Let x be an odd number. This means that $x = 2n + 1$ where n is an integer.

If we square x we get:

$$x^2 = (2n+1)^2 = (2n+1)(2n+1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

which is of the form 2(integer) + 1, and so is also an odd number. $\square$

(b) y is an even number $\Rightarrow y^3$ is an even number. (**True**)

Proof: Let y be an even number. This means that $y = 2n$ where n is an integer.

If we cube y we get:

$$y^3 = (2n)^3 = (2n)(2n)(2n) = 8n^3 = 2(4n^3)$$

which is of the form 2(integer), and so is also an even number. $\square$

(c) $x + y$ is even $\Rightarrow x$ and y are both odd. (**False**)

Counterexample: $4 + 6$ is even, but 4 and 6 are both not odd!

(d) xy is even $\Rightarrow x$ and y are both even. (**False**)

Counterexample: 2×3 is even, but 2 and 3 are not both even, only one of them is!

(e) $x^2 \geq 0 \Rightarrow x > 0$ (**False**)

Counterexample: $0^2 \geq 0$ but zero is not greater than zero!

(f) $x > 2 \Rightarrow x^2 > 2$ (**True**)

Proof: Assume that $x > 2$. We want to show that $x^2 > 2$. Keeping in mind that x is a positive number (as it is greater than 2), multiplying both sides of the inequality ($x > 2$) by x will keep it true.

Doing this we get $x^2 > 2x$. Using the fact that $x > 2$, we know that $2x > 4$. And we also know that $4 > 2$. So we have a chain of inequalities as follows

$$x^2 > 2x > 4 > 2$$

and so we can conclude that $x^2 > 2$. $\square$

(g) $x^2 \in \mathbb{Z} \Rightarrow x \in \mathbb{Z}$ (**False**)

Counterexample: $(\sqrt{5})^2 = 5 \in \mathbb{Z}$, but $\sqrt{5} \notin \mathbb{Z}$.

(h) $x \neq 0 \wedge y \neq 0 \Rightarrow xy \neq 0$ (**True**)

Proof: This is true by definition. You should *google* the *zero-product property*, which is true by definition. The statement we are dealing with is the contrapositive of the zero-product property, which must also be true. $\square$

(i) $x \neq 0 \wedge y \neq 0 \Rightarrow x + y \neq 0$ (**False**)

Counterexample: Let $x = 3$ and $y = -3$. These are two numbers which are both not equal to zero, however $x + y = 3 + (-3) = 0$.

(j) $x^3 = 8 \Rightarrow x = 2$ (**True**)

Proof: The function $y = x^3$ is a *one-to-one* function and so it has a well defined inverse, namely, the cube root. Taking the cube root of both sides of $x^3 = 8$ we get $x = 2$. $\square$

Question 2

For each of the statements in question 1, write the converse and state whether it's true or not. If true, give a proof. If false, give a counterexample.

(a) x^2 is an odd number $\Rightarrow x$ is an odd number (**False**)

Counterexample: $(\sqrt{7})^2$ is an odd number, but $\sqrt{7}$ is not an odd number, as it's not even an integer!

(b) y^3 is an even number $\Rightarrow$ y is an even number. (**False**)

Counterexample: $(\sqrt[3]{8})^3$ is an even number, but $\sqrt[3]{8}$ is not an even number, as it's not even an integer!

(c) x and y are both odd $\Rightarrow x + y$ is even (**True**)

Proof: Let x and y both be odd numbers. Then $x = 2n + 1$ for some integer n, and $y = 2m + 1$ for some integer m. If we add x and y together we get

$$x + y = 2n + 1 + 2m + 1 = 2n + 2m + 2 = 2(n + m + 1)$$

which is of the form 2(integer), and so is an even number. $\square$

(d) x and y are both even $\Rightarrow xy$ is even (**True**)

Proof: Let x and y both be even numbers. Then $x = 2n$ for some integer n, and $y = 2m$ for some integer m. Multiplying x and y together we get

$$xy = (2n)(2m) = 4mn = 2(2mn)$$

which is of the form 2(integer), and so is an even number. $\square$

(e) $x > 0 \Rightarrow x^2 \geq 0$ (**True**)

Proof: Assume that $x > 0$. As x is positive, we can multiply both sides of the inequality $x > 0$ by x, and it will still be true.

Doing this we get $x^2 > 0$. But if $x^2 > 0$, then clearly $x^2 \geq 0$. $\square$

(f) $x^2 > 2 \Rightarrow x > 2$ (**False**)

Counterexample: $(-4)^2 = 16 > 2$, but -4 is not greater than 2.

(g) $x \in \mathbb{Z} \Rightarrow x^2 \in \mathbb{Z}$ (**True**)

Proof: The fact that squaring an integer results in an integer is a property too basic to prove! It arises from the definition of the set of integers.

(h) $xy \neq 0 \Rightarrow x \neq 0 \wedge y \neq 0$ (**False**)

Counterexample: If $x = 0$ and $y = 3$, then $xy = 0$, but not both of x and y are zero!

(i) $x + y \neq 0 \Rightarrow x \neq 0 \land y \neq 0$ (**False**)

Counterexample: Let $x = 0$ and $y = 1$. Then $x+y \neq 0$ but not both of our numbers are not equal to zero!

(j) $x = 2 \Rightarrow x^3 = 8$ (**True**)

Proof: This is clearly true, as $2^3 = 2 \times 2 \times 2 = 8$.

Question 3

For each of the statements in question 1, write the contrapositive and state whether it's true or not. If true, give a proof. If false, give a counterexample.

Note: We are taking the contrapositives of each of the statements from Question 1. This will preserve the truth of each of the statements. There is no need to prove or disprove each statement, as the truth or falsity of the contrapositive of each statement has been shown in the solutions to Question 1. However, feel free to come up with an original proof or counterexample!

(a) x^2 is not an odd number $\Rightarrow$ x is not an odd number (**True**)

(b) y^3 is not an even number $\Rightarrow$ y is not an even number. (**True**)

(c) At least one of x and y is not odd $\Rightarrow$ $x + y$ is not even (**False**)

(d) At least one of x and y is not even $\Rightarrow$ xy is not even (**False**)

(e) $x \leq 0 \Rightarrow x^2 < 0$ (**False**)

(f) $x^2 \leq 2 \Rightarrow x \leq 2$ (**True**)

(g) $x \notin \mathbb{Z} \Rightarrow x^2 \notin \mathbb{Z}$ (**False**)

(h) $xy = 0 \Rightarrow x = 0 \lor y = 0$ (**True**)

(i) $x + y = 0 \Rightarrow x = 0 \lor y = 0$ (**False**)

(j) $x \neq 2 \Rightarrow x^3 \neq 8$ (**True**)

Question 4

Use an algebraic proof to prove each of the following true statements.

(a) The product of two odd numbers is an odd number.

Proof: Take two odd numbers $2n + 1$ and $2m + 1$ where n and m are integers. Multiplying these together and arranging we get

$$(2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1$$

which is of the form $2(\text{ integer }) + 1$, and so is odd. $\square$

(b) The product of two square numbers is a square number.

Proof: Take two square numbers, n^2 and m^2, where n and m are integers. Multiplying these together we get

$$n^2m^2 = (nm)^2$$

which is also a square number. $\square$

(c) The sum of three consecutive numbers is divisible by three.

Proof: Take three consecutive numbers. Here we write these three numbers as

$$n, n + 1, n + 2$$

where n is an integer. Adding these together we get

$$n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$$

which is of the form $3(\text{ integer })$, and so is a multiple of 3, and is therefore divisible by 3. $\square$

(d) The sum of three consecutive even numbers is divisible by six.

Proof: Take three consecutive even numbers. Here we write these three numbers as

$$2n, 2n + 2, 2n + 4$$

where n is an integer. Adding these together we get

$$2n + (2n + 2) + (2n + 4) = 6n + 6 = 6(n + 1)$$

which is of the form $6(\text{ integer })$, and so is a multiple of 6, and is therefore divisible by 6. $\square$

(e) The sum of an odd number and an even number is odd.

Proof: Take an odd number $2n+1$ and an even number $2m$, where n and m are both integers. Adding these together we get

$$(2n+1)+2m = 2n+2m+1 = 2(n+m)+1$$

which is of the form $2(\text{ integer })+1$, which is an odd number. $\square$

Question 5

Use proof by contradiction to prove each of the following true statements.

(a) $\dfrac{a}{b}+\dfrac{b}{a} \geq 2$ for all $a,b \in \mathbb{R}$

Proof: We assume that

$$\frac{a}{b}+\frac{b}{a} < 2$$

for all $a,b \in \mathbb{R}$. We then manipulate the statement to try and arrive at a contradiction. The algebra follows:

$$\begin{aligned}
\frac{a}{b}+\frac{b}{a} &< 2 \\
\frac{a^2+b^2}{ab} &< 2 \\
a^2+b^2 &< 2ab \text{ (If } ab>0) \\
a^2+b^2-2ab &< 0 \\
(a-b)^2 &< 0
\end{aligned}$$

However, the square of a number can never be negative so this is a contradiction. Therefore the original statement is true, but we have only proved it in the case $ab>0$.

We have ignored the case where $ab<0$, as this implies that exactly one of a or b will be negative and so both $\frac{a}{b}$ and $\frac{b}{a}$ will be negative and so $\frac{a}{b}+\frac{b}{a}$ will be negative and clearly less than 2.

We also ignore the case where $ab=0$ as this implies that at least one of a or b is zero, and one or more of the fractions will become undefined. $\square$

(b) $x^2+y^2 \geq 2xy$ for all $x,y \in \mathbb{R}$

Proof: We assume that

$$x^2+y^2 < 2xy$$

for all $x, y \in \mathbb{R}$. We then manipulate the statement to try and arrive at a contradiction. The algebra follows:

$$\begin{array}{rcl} x^2 + y^2 & < & 2xy \\ x^2 + y^2 - 2xy & < & 0 \\ (x - y)^2 & < & 0 \end{array}$$

However, the square of a number can never be negative so this is a contradiction. Therefore the original statement is true. □

(c) Zero is an even number.

Proof: To prove this, we assume that zero is not an even number. As zero is an integer, and every integer is either even or odd, we are assuming that zero is odd.

Now we observe what happens when we multiply zero with four:

$$0 \times 4 = 0$$

Under our assumption that 0 is odd, the above is of the form

$$\text{odd} \times \text{even} = \text{odd}$$

However, the product of an odd number and an even number should always be even! Thus, we have reached a contradiction with an earlier true result. We are forced to reject our assumption that zero is not even, and instead accept that it is. □

(d) There exist no integers a and b for which $18a + 6b = 1$.

Proof: Assume to the contrary, that there does exist integers a and b for which $18a + 6b = 1$. If this is true then:

$$2(9a) + 2(3b) = 1$$

The above equation is of the form

$$2(\text{integer}) + 2(\text{integer}) = 1$$

which in turn can be viewed as

$$\text{even} + \text{even} = 1$$

This is a contradiction to the fact that the sum of two even numbers is even. Therefore the original statement is true. □

(e) The square of an odd number is odd.

Proof: Let $2n + 1$ be an odd number, where n is an integer.

If we square this odd number we get:

$$(2n + 1)^2 = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

which is of the form $2(\text{ integer }) + 1$, and so is also an odd number. □

Question 6

Look up the proof that $\sqrt{2}$ is irrational. Using this as a guide, construct a proof that $\sqrt{3}$ is irrational.

Proof: Assume to the contrary. We assume that $\sqrt{3}$ is a rational number, that is, that $\sqrt{3} = \frac{a}{b}$ where a and b are integers having no common factor (except 1).

Squaring both sides of $\sqrt{3} = \frac{a}{b}$ we get

$$3 = \frac{a^2}{b^2}.$$

We rearrange this to get

$$3b^2 = a^2.$$

As a^2 is the product of 3 and another integer, this means that a^2 is a multiple of 3. If a^2 is a multiple of 3 then we must conclude that a is also a multiple of 3 (try and prove this separately!).

This means that we can write $a = 3n$ where n is an integer. Substituting this into our equation above we get

$$3b^2 = (3n)^2.$$

Carrying out the square we get

$$3b^2 = 9n^2.$$

Dividing through by 3 we see that:

$$b^2 = 3n^2$$

As b^2 is the product of 3 and another integer, this means that b^2 is a multiple of 3. If b^2 is a multiple of 3 then we must conclude that b is also a multiple of 3.

So both a and b are multiples of 3, that is, they share a common factor of 3. However, we assumed that a and b had no common factors. Thus we have reached a contradiction! Therefore, $\sqrt{3}$ is irrational. $\square$