Mathematical Framework for Optimal Order Allocation

Problem Formulation

This document aims to determine the optimal allocation vector $x = [x_1, x_2, \ldots, x_n]$ that minimizes the total temporary impact cost, considering a total order size S that must be executed over N trading periods. The problem is formulated as follows:

Minimize: Σ (from i=1 to N) $g_i(x_i)$

Subject to:

- Σ (from i=1 to N) $x_i = S$
- $x_i \geqslant 0$ for all $i \in \{1,2,\ldots,N\}$

Where $g_i(x_i)$ is the temporary impact function at time i, representing the slippage cost of executing an order of size x_i at that time.

Temporary Impact Function

Based on our analysis, we model the temporary impact function as a power-law function:

$$g_i(x_i) = \alpha_i \cdot x_i^\beta \cdot \sigma_i$$

Where:

- α_i : A time-varying coefficient reflecting market conditions at time i.
- β : The power-law exponent (typically between 0.5 and 1).
- σ_i : Market volatility at time i.

Solution Approach

To solve this optimization problem, we propose a Dynamic Programming approach combined with Convex Optimization techniques. The algorithm includes the following steps:

Step 1: Parameter Estimation

For each time period i, parameters α_i and σ_i are estimated based on historical data and current market conditions. This can involve analyzing historical data, adjusting parameters based on market conditions (e.g., volatility, spread, order book imbalance), and forecasting parameters for future periods using time series models or market microstructure models.

Step 2: Convex Optimization

With the estimated parameters, the optimization problem becomes convex when $\beta \geqslant 1$. For the case where $\beta < 1$ (which is often true in practice), numerical optimization techniques such as Sequential Quadratic Programming (SQP) or interior-point methods can be used.

The problem can be solved using Karush-Kuhn-Tucker (KKT) conditions to obtain the optimal solution. For the case where all $x_i > 0$, an analytical solution for x_i can be derived in terms of the Lagrange multiplier λ, and then λ is solved numerically using the constraint $\Sigma x_i = S$.

Step 3: Adaptive Execution

As execution progresses, estimates of α_i and σ_i are updated based on observed market conditions, and remaining allocations are adjusted accordingly. This involves updating parameters after executing x_i at time i, re-optimizing the problem for the remaining periods with updated parameters and remaining order size, and adjusting the execution plan based on re-optimization results. This adaptive approach allows for responsiveness to changing market conditions and improved execution quality.

Implementation Considerations

Important implementation considerations include:

- **Time Discretization:** Choosing appropriate time intervals (e.g., 5-minute intervals for liquid securities).
- **Parameter Estimation:** Using a rolling window of historical data and incorporating real-time market data to update estimates.
- **Risk Constraints:** Additional constraints such as maximum order size per period or risk limits can be integrated into the optimization framework.
- **Computational Efficiency:** Utilizing efficient numerical optimization libraries and warm-start techniques for real-time applications.

Conclusion

The proposed mathematical framework provides a principled approach to optimal order allocation that minimizes temporary impact costs. By modeling the impact function as a power-law and using convex optimization techniques, the optimal allocation vector can be determined, balancing immediate execution and market impact. The adaptive nature of the approach allows it to respond to changing market conditions and improve execution quality over time.