Mathematical Framework for Optimal Order Allocation

Problem Formulation

Given a total order size S that must be executed over N trading periods, we need to determine the optimal allocation vector x = [x_1, x_2, ..., x_n] that minimizes the total temporary impact cost. The mathematical formulation of this problem is:

Minimize:
$\sum_{i=1}^{N} g_i(x_i)$

Subject to:
$\sum_{i=1}^{N} x_i = S$
$x_i \geq 0 \quad \forall i \in \{1, 2, ..., N\}$

Where g_i(x_i) is the temporary impact function at time i, representing the slippage cost of executing an order of size x_i at that time.

Temporary Impact Function

Based on our analysis, we model the temporary impact function as a power-law function:

$$g_i(x_i) = \alpha_i \cdot x_i^{\beta} \cdot \sigma_i$$

Where:

- α_i is a time-varying coefficient that captures market conditions at time i
- β is the power-law exponent (typically between 0.5 and 1)
- σ_i is the market volatility at time i

Solution Approach

To solve this optimization problem, we propose a dynamic programming approach combined with convex optimization techniques. The algorithm proceeds as follows:

Step 1: Parameter Estimation

For each time period i, estimate the parameters α_i and σ_i based on historical data and current market conditions:

1. **Historical data analysis**: Use recent market data to estimate the baseline parameters.
2. **Market condition adjustment**: Adjust parameters based on current market conditions such as volatility, spread, and order book imbalance.
3. **Parameter forecasting**: For future periods, forecast parameters using time series models or market microstructure models.

The parameter estimation can be formalized as:

$$\alpha_i = f_\alpha(H_i, C_i)$$
$$\sigma_i = f_\sigma(H_i, C_i)$$

Where:

- H_i represents historical market data up to time i
- C_i represents current market conditions at time i
- f_α and f_σ are estimation functions that can be based on regression models, machine learning algorithms, or market microstructure models

Step 2: Convex Optimization

With the estimated parameters, the optimization problem becomes:

Minimize:
$\sum_{i=1}^{N} \alpha_i \cdot x_i^\beta \cdot \sigma_i$

Subject to:
$\sum_{i=1}^{N} x_i = S$
$x_i \geq 0 \quad \forall i \in \{1, 2, ..., N\}$

This is a convex optimization problem when β ⩾ 1. For β < 1 (which is often the case in practice), we can use numerical optimization techniques such as sequential quadratic programming (SQP) or interior-point methods.

The Lagrangian of this problem is:

$$L(x, \lambda, \mu) = \sum_{i=1}^{N} \alpha_i \cdot x_i^\beta \cdot \sigma_i - \lambda \left(\sum_{i=1}^{N} x_i - S \right) - \sum_{i=1}^{N} \mu_i x_i$$

Where λ and μ_i are Lagrange multipliers.

The Karush-Kuhn-Tucker (KKT) conditions for optimality are:

1. **Stationarity**:
   $\frac{\partial L}{\partial x_i} = \beta \cdot \alpha_i \cdot x_i^{\beta-1} \cdot \sigma_i - \lambda - \mu_i = 0 \quad \forall i \in \{1, 2, ..., N\}$
2. **Primal feasibility**:
   $\sum_{i=1}^{N} x_i = S$
   $x_i \geq 0 \quad \forall i \in \{1, 2, ..., N\}$
3. **Dual feasibility**:
   $\mu_i \geq 0 \quad \forall i \in \{1, 2, ..., N\}$
4. **Complementary slackness**:
   $\mu_i x_i = 0 \quad \forall i \in \{1, 2, ..., N\}$

For the case where all x_i > 0 (i.e., μ_i = 0 for all i), we can solve for x_i in terms of λ:

$$x_i = \left(\frac{\lambda}{\beta \cdot \alpha_i \cdot \sigma_i} \right)^{\frac{1}{\beta-1}} \quad \forall i \in \{1, 2, ..., N\}$$

Then, using the constraint that Σ x_i = S, we can solve for λ:

$$\sum_{i=1}^{N} \left(\frac{\lambda}{\beta \cdot \alpha_i \cdot \sigma_i} \right)^{\frac{1}{\beta-1}} = S$$

This equation can be solved numerically for λ, and then the optimal allocation x_i can be computed.

Step 3: Adaptive Execution

As the execution progresses, we update our estimates of α_i and σ_i based on the observed market conditions and adjust the remaining allocations accordingly:

1. **Parameter update**: After executing x_i at time i, update the parameter estimates for future periods based on the observed market response.
2. **Reoptimization**: Solve the optimization problem again for the remaining periods with the updated parameters and the remaining order size.
3. **Execution adjustment**: Adjust the execution plan based on the reoptimization results.

This adaptive approach allows us to respond to changing market conditions and improve execution quality.

Implementation Considerations

1. Time Discretization

The choice of time periods is crucial. Too fine a discretization may lead to excessive trading and increased costs, while too coarse a discretization may miss opportunities for cost reduction. We recommend using 5-minute intervals for liquid securities and longer intervals for less liquid ones.

2. Parameter Estimation

The accuracy of the parameter estimates directly affects the quality of the allocation. We recommend:

- Using a rolling window of historical data to estimate parameters
- Incorporating real-time market data to update estimates
- Considering market regime changes and adjusting estimates accordingly

3. Risk Constraints

In practice, the optimization problem may include additional constraints such as:

- Maximum order size per period: $x_i \leqslant M_i$

- Maximum deviation from a target trajectory: $|x_i - x_i^{target}| \leqslant D_i$
- Risk limits: $\mathrm{Var}(\text{execution price}) \leqslant R$

These constraints can be incorporated into the optimization framework without changing the fundamental approach.

4. Computational Efficiency

For real-time applications, computational efficiency is important. We recommend:

- Precomputing parameter estimates when possible
- Using efficient numerical optimization libraries
- Implementing warm-start techniques for reoptimization

Conclusion

The proposed mathematical framework provides a principled approach to optimal order allocation that minimizes temporary impact costs. By modeling the impact function as a power-law and using convex optimization techniques, we can determine the optimal allocation vector that balances the trade-off between immediate execution and market impact. The adaptive nature of the approach allows it to respond to changing market conditions and improve execution quality over time.