STAT 764 Homework 1

34/35

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Notation of Matrix: uppercase Letter: A

Lowercase letter: $\underline{a}$

Question 1.

$$A=\begin{pmatrix}6&5&7\\5&6&7\\7&7&18\end{pmatrix}$$

$$\text{Diag}(A)=\begin{pmatrix}6&0&0\\0&6&0\\0&0&18\end{pmatrix}$$

Transpose $A'=\begin{pmatrix}6&5&7\\5&6&7\\7&7&18\end{pmatrix}$

$$|A| = a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{12}a_{23}a_{31}$$
$$-a_{31}a_{22}a_{13}-a_{23}a_{32}a_{11}-a_{33}a_{12}a_{21}$$

Trace: $tr(A)=6+6+18=30$

$$AA^{-1}=A^{-1}A=I$$

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$J=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

Eigenvalues: $\lambda_1=25$ $\quad\lambda_2=1$ $\quad\lambda_3=?$

$\underline{x}_1=\begin{pmatrix}1/\sqrt{6}\\1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix}$ $\quad x_2=?$ $\quad\underline{x}_3=?$

a) Find λ_3

$$(A-\lambda I)\underline{x}=\underline{0}$$
$$|A-\lambda I|=0$$

$$|A-\lambda I|=\begin{vmatrix}6-\lambda&5&7\\5&6-\lambda&7\\7&7&18-\lambda\end{vmatrix}=(6-\lambda)(6-\lambda)(18-\lambda)+5\times7\times7+5\times7\times7$$
$$-7\times7\times(6-\lambda)-7\times7\times(6-\lambda)-5\times5\times(18-\lambda)$$

$$=\lambda^3-30\lambda^2+129\lambda-100=0$$

$$f(\lambda)=(\lambda-a)(\lambda-b)(\lambda-c)$$
$$=\lambda^3+(-a-b-c)\lambda^2+(ab+bc+ac)\lambda-abc$$

$$\begin{cases}-a-b-c=-30\\ab+bc+ac=129\\abc=100\end{cases}\Rightarrow\quad\begin{matrix}a=25\\b=1\\c=4\end{matrix}$$

✓

Thus, the third eigenvalue $\lambda_3=4$

b) Find Normalized eigenvectors

$(A-\lambda I)\underline{x} = \underline{0}$

$$\begin{pmatrix} 6-\lambda & 5 & 7 \\ 5 & 6-\lambda & 7 \\ 7 & 7 & 18-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} (6-\lambda)x_1 + 5x_2 + 7x_3 = 0 \\ 5x_1 + (6-\lambda)x_2 + 7x_3 = 0 \\ 7x_1 + 7x_2 + (18-\lambda)x_3 = 0 \end{cases}$$

By $\lambda_1 = 25$, $\begin{cases} (6-25)x_1 + 5x_2 + 7x_3 = 0 \\ 5x_1 + (6-25)x_2 + 7x_3 = 0 \\ 7x_1 + 7x_2 + (18-25)x_3 = 0 \end{cases}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \xrightarrow{\text{Normalized}} \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

By $\lambda_2 = 1$, $\begin{cases} (6-1)x_1 + 5x_2 + 7x_3 = 0 \\ 5x_1 + (6-1)x_2 + 7x_3 = 0 \\ 7x_1 + 7x_2 + (18-1)x_3 = 0 \end{cases}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \xrightarrow{\text{Normalized}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$ ✓

By $\lambda_3 = 4$, $\begin{cases} (6-4)x_1 + 5x_2 + 7x_3 = 0 \\ 5x_1 + (6-4)x_2 + 7x_3 = 0 \\ 7x_1 + 7x_2 + (18-4)x_3 = 0 \end{cases}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \xrightarrow{\text{Normalized}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$ ✓

c) Spectral Decomposition

$$C = (\underline{x_1}\ \underline{x_2}\ \underline{x_3}) = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix} \qquad C' = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{aligned} A &= ACC' \quad \text{------} \ ① \\ &= A(\underline{x_1}\ \underline{x_2}\ \underline{x_3})C' \\ &= (A\underline{x_1}, A\underline{x_2}, A\underline{x_3})C' \\ &= (\lambda_1\underline{x_1}, \lambda_2\underline{x_2}\ \lambda_3\underline{x_3})C' \\ &= CDC' \quad \text{------} \ ② \end{aligned}$$

$A = CDC'$ shown as ② for a symmetric matrix A in terms of its eigenvectors and eigenvalues is known as spectral decomposition of A.

Thus, if we can prove $A = ACC'$ shown as ① is true, then we can obtain ②.

$$ACC' = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

ok

$$= \begin{pmatrix} \frac{25}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{4}{\sqrt{3}} \\ \frac{25}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{4}{\sqrt{3}} \\ \frac{50}{\sqrt{6}} & 0 & \frac{-4}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

All you need to show is $\underline{C}\underline{C}' = \underline{I}$

$$= \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix} = A$$

Thus, $A = ACC' \Rightarrow A = CDC'$.

d) write A using spectral decomposition.

$$A = CDC'$$

$$= \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

✓

$$= \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix}$$

e) The symmetric matrix A is said to be positive definite if $\underline{x}' A \underline{x} > 0$ for all possible vectors $\underline{x}$ (except $\underline{x} = \underline{0}$)

A positive definite matrix A can be factored into $A = T'T$,

where T is a nonsingular upper triangular matrix,

one way to obtain T is cholesky decomposition.

~~If~~ Explain why A can be factored using cholesky decomposition,

we only need to prove A is positive definite. ✓

or calculate T by cholesky decomposition, prove $A = T'T$.

$t_{11} = \sqrt{a_{11}}$

$t_{1j} = \frac{a_{1j}}{t_{11}}$ for $2 \le j \le n$

$t_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} t_{ki}^2}$ for $2 \le i \le n$

$t_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} t_{ki} t_{kj}}{t_{ii}}$ for $2 \le i < j \le n$

$t_{ij} = 0$ for $1 \le j < i \le n$

$$A = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix}$$

$t_{11} = \sqrt{a_{11}} = \sqrt{6}$

$t_{12} = \frac{a_{12}}{t_{11}} = \frac{5}{\sqrt{6}}$

$t_{13} = \frac{a_{13}}{t_{11}} = \frac{7}{\sqrt{6}}$

$t_{22} = \sqrt{a_{22} - t_{12}^2} = \sqrt{6 - \frac{25}{6}} = \sqrt{\frac{11}{6}}$

$t_{23} = \frac{a_{23} - t_{12} t_{13}}{t_{22}} = \frac{7 - \frac{5}{\sqrt{6}} \cdot \frac{7}{\sqrt{6}}}{\sqrt{\frac{11}{6}}} = \frac{7}{\sqrt{6} \cdot \sqrt{11}}$

$t_{33} = \sqrt{a_{33} - (t_{13}^2 + t_{23}^2)} = \sqrt{18 - (\frac{49}{6} + \frac{49}{66})} = \frac{10}{\sqrt{11}}$

$$T = \begin{pmatrix} \sqrt{6} & \frac{5}{\sqrt{6}} & \frac{7}{\sqrt{6}} \\ 0 & \frac{\sqrt{11}}{\sqrt{6}} & \frac{7}{\sqrt{6} \cdot \sqrt{11}} \\ 0 & 0 & \frac{10}{\sqrt{11}} \end{pmatrix} \quad T' = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ \frac{5}{\sqrt{6}} & \frac{\sqrt{11}}{\sqrt{6}} & 0 \\ \frac{7}{\sqrt{6}} & \frac{7}{\sqrt{6} \cdot \sqrt{11}} & \frac{10}{\sqrt{11}} \end{pmatrix}$$

ok

$$A = T'T = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 5/\sqrt{6} & \sqrt{11}/\sqrt{6} & 0 \\ 7/\sqrt{6} & 7/\sqrt{6} \cdot \sqrt{11} & \frac{10}{\sqrt{11}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{6} & 5/\sqrt{6} & 7/\sqrt{6} \\ 0 & \sqrt{11}/\sqrt{6} & 7/\sqrt{6} \cdot \sqrt{11} \\ 0 & 0 & 10/\sqrt{11} \end{pmatrix} = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix}$$

Question 2.

$$A = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix}$$

a) $|A| = 6\times6\times18 + 5\times7\times7 + 5\times7\times7 - 7\times6\times7 - 7\times7\times6 - 5\times5\times18$

$= 648 + 245 + 245 - 294 - 294 - 450$

$= 100$ ✓

b) symmetric ?

If the transpose of a matrix is the same as the original matrix, the matrix is said to be symmetric. That is, A is symmetric if $A = A'$.

$A' = \begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix} \Rightarrow A = A' \Rightarrow$ A is symmetric. ok

c) Non-Singular ? e) Inverse ?

If matrix A is square and of full rank, then A is said to be non-singular, and A has a unique inverse A^{-1}, with property: $A^{-1}A = AA^{-1} = I$

Size of A : 3x3, so A is square.

Rank(A) = number of linearly independent rows of A,
= number of linearly independent columns of A,

If A is $n \times p$, the maximum possible rank of A is the smaller of n and p, in which case A is said to be of full rank.

A has rank 3, because either rows or columns are linearly independent.

Thus, A is full rank. → How do you know this? -1

Thus, A is non-singular. (c)

and A has a unique inverse A^{-1}. (e)

Calculate A^{-1}.

$$\begin{pmatrix} 6 & 5 & 7 \\ 5 & 6 & 7 \\ 7 & 7 & 18 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} 6a_{11} + 5a_{21} + 7a_{31} &= 1 \\ 5a_{11} + 6a_{21} + 7a_{31} &= 0 \\ 7a_{11} + 7a_{21} + 18a_{31} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_{11} &= 0.59 \\ a_{21} &= -0.41 \\ a_{31} &= -0.07 \end{aligned}$$

$$\begin{aligned} 6a_{12} + 5a_{22} + 7a_{32} &= 0 \\ 5a_{12} + 6a_{22} + 7a_{32} &= 1 \\ 7a_{12} + 7a_{22} + 18a_{32} &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_{12} &= -0.41 \\ a_{22} &= 0.59 \\ a_{32} &= -0.07 \end{aligned}$$

$$\begin{aligned} 6a_{13} + 5a_{23} + 7a_{33} &= 0 \\ 5a_{13} + 6a_{23} + 7a_{33} &= 0 \\ 7a_{13} + 7a_{23} + 18a_{33} &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_{13} &= -0.07 \\ a_{23} &= -0.07 \\ a_{33} &= 0.11 \end{aligned}$$

Thus, $A^{-1} = \begin{pmatrix} 0.59 & -0.41 & -0.07 \\ -0.41 & 0.59 & -0.07 \\ -0.07 & -0.07 & 0.11 \end{pmatrix}$ ✓

d) positive Definite ?

a symmetric matrix A is positive definite (P.d.) if $\underline{x}'A\underline{x} > 0 \ \forall \underline{x} \neq 0$

$A = CDC'$ by spectral decomposition

$C'AC = D = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\underline{x}'A\underline{x} = \underline{x}'CDC'\underline{x}$$
$$= (x'C)D(C'x) \quad \text{let } \underline{y} = C'\underline{x} \quad \text{then } \underline{y} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}_{3\times3} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times1}$$
$$= \underline{y}'D\underline{y}$$

$$= \begin{pmatrix} 1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3 \\ 1/\sqrt{2}x_1 - 1/\sqrt{2}x_2 \\ 1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3 \end{pmatrix}_{3\times1}$$

$$\underline{y}' = \begin{pmatrix} 1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3 & 1/\sqrt{2}x_1 - 1/\sqrt{2}x_2 & 1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3 \end{pmatrix}$$

Thus, $\underline{x}'A\underline{x} = \underline{y}'D\underline{y}$

$$= \begin{pmatrix} 1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3 & 1/\sqrt{2}x_1 - 1/\sqrt{2}x_2 & 1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3 \end{pmatrix}_{1\times3} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{3\times3} \begin{pmatrix} 1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3 \\ 1/\sqrt{2}x_1 - 1/\sqrt{2}x_2 \\ 1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3 \end{pmatrix}$$

$$= \begin{pmatrix} (1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3)\lambda_1 & (1/\sqrt{2}x_1 - 1/\sqrt{2}x_2)\lambda_2 & (1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3)\lambda_3 \end{pmatrix}_{1\times3} \begin{pmatrix} 1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3 \\ 1/\sqrt{2}x_1 - 1/\sqrt{2}x_2 \\ 1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3 \end{pmatrix}_{3\times1}$$

ok

$$= (1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3)^2\lambda_1 + (1/\sqrt{2}x_1 - 1/\sqrt{2}x_2)^2\lambda_2 + (1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3)^2\lambda_3$$
$$= (1/\sqrt{6}x_1 + 1/\sqrt{6}x_2 + 2/\sqrt{6}x_3)^2 \cdot 25 + (1/\sqrt{2}x_1 - 1/\sqrt{2}x_2)^2 \cdot 1 + (1/\sqrt{3}x_1 + 1/\sqrt{3}x_2 - 1/\sqrt{3}x_3)^2 \cdot 4$$
$$> 0$$

Proof of positive definite using principal axes theorem.

A is symmetric, by the principal axes theorem,

Let $P^T A P = D = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$ where $P^{-1} = P^T$ and λ_i are eigenvalues of A.

Given a column $\underline{x}$ in $\mathbb{R}^n$, write $\underline{y} = P^T \underline{x} = [y_1\ y_2 \cdots y_n]^T$.

$$\underline{x}^T A \underline{x} = \underline{x}^T (PDP^T)\underline{x}$$

$$= \underline{y}^T D \underline{y}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

some $y_i \neq 0$ and every $\lambda_i > 0$,

Thus, $x^T A x > 0$

Thus, A is positive definite.

f) As proved in d), A is positive definite, So $A^{\frac{1}{2}} = C D^{\frac{1}{2}} C'$

$$A^{\frac{1}{2}} = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{25}\ 5 & 0 & 0 \\ 0 & \sqrt{1}\ 1 & 0 \\ 0 & 0 & \sqrt{4}\ 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{3}} \\ \frac{5}{\sqrt{6}} & -1/\sqrt{2} & \frac{2}{\sqrt{3}} \\ \frac{10}{\sqrt{6}} & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

✓

$\frac{5}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{5}{6} + \frac{1}{2} + \frac{2}{3} = \frac{5+3+4}{6} = \frac{12}{6} = 2$

$\frac{5}{\sqrt{6}}\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{5}{6} - \frac{1}{2} + \frac{2}{3} = \frac{6}{6} = 1$

$\frac{10}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{10}{6} - \frac{2}{3} = \frac{10-4}{6} = 1$

$\frac{5}{\sqrt{6}} \cdot \boxed{\frac{1}{\sqrt{6}}} + \frac{1}{\sqrt{2}} \cdot \boxed{\frac{-1}{\sqrt{2}}} + \frac{2}{\sqrt{3}} \cdot \boxed{\frac{1}{\sqrt{3}}} = \frac{5}{6} - \frac{1}{2} + \frac{2}{3} = \frac{5-3+4}{6} = \boxed{1}$

$\frac{5}{\sqrt{6}}\ \frac{1}{\sqrt{6}}\quad \frac{-1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} + \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{5}{6} + \frac{1}{2} + \frac{2}{3} = \frac{5+3+4}{6} = \boxed{2}$

$\frac{10}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}\quad \frac{-2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{10}{6} - \frac{2}{3} = \frac{10-4}{6} = \boxed{1}$

$\frac{5}{\sqrt{6}} \boxed{\frac{2}{\sqrt{6}}} + \frac{2}{\sqrt{3}} \boxed{\frac{-1}{\sqrt{3}}} = \frac{10}{6} - \frac{2}{3} = \frac{10-4}{6} = 1$

$\frac{5}{\sqrt{6}} \cdot \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{3}} \cdot \frac{-1}{\sqrt{3}} = 1$

$\frac{10}{\sqrt{6}} \cdot \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{3}}\ \frac{-1}{\sqrt{3}} = \frac{20}{6} + \frac{2}{3} = \frac{20+4}{6} = 4$

Question 3

$$B=\begin{pmatrix}6&0\\-2&-6\\1&-1\\0&-2\end{pmatrix}\qquad BB'=\begin{pmatrix}36&-12&6&0\\-12&40&4&12\\6&4&2&2\\0&12&2&4\end{pmatrix}\qquad B'B=\begin{pmatrix}41&11\\11&41\end{pmatrix}$$

$$B'=\begin{pmatrix}6&-2&1&0\\0&-6&-1&-2\end{pmatrix}$$

$$\begin{vmatrix}41-\lambda^2&11\\11&41-\lambda^2\end{vmatrix}=0$$

$$(41-\lambda^2)^2-11^2=0$$

$$41-\lambda^2=\pm 11$$

$$\lambda_1^2=30$$

$$\lambda_2^2=52$$

$$\underline{x}_1=\begin{pmatrix}\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}\end{pmatrix}\qquad \underline{x}_2=\begin{pmatrix}\frac{-1}{\sqrt2}\\\frac{-1}{\sqrt2}\end{pmatrix}$$

$$\lambda_1^2=30\qquad \lambda_2^2=52$$

$$\underline{x}_1=\begin{pmatrix}\frac{3}{\sqrt{15}}\\\frac{2}{\sqrt{15}}\\\frac{1}{\sqrt{15}}\\\frac{1}{\sqrt{15}}\end{pmatrix}\qquad \underline{x}_2=\begin{pmatrix}\frac{-3}{\sqrt{26}}\\\frac{4}{\sqrt{26}}\\0\\\frac{1}{\sqrt{26}}\end{pmatrix}$$

$$B=UDV'\qquad D=\text{diag}(\lambda_1,\lambda_2)=\begin{pmatrix}\sqrt{30}&0\\0&\sqrt{52}\end{pmatrix}$$

$$U=\begin{pmatrix}\frac{3}{\sqrt{15}}&\frac{-3}{\sqrt{26}}\\\frac{2}{\sqrt{15}}&\frac{4}{\sqrt{26}}\\\frac{1}{\sqrt{15}}&0\\\frac{1}{\sqrt{15}}&\frac{1}{\sqrt{26}}\end{pmatrix}$$

$$V=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}\qquad V'=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{-1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}$$

$$B=UDV'$$

$$=\begin{pmatrix}\frac{3}{\sqrt{15}}&\frac{-3}{\sqrt{26}}\\\frac{2}{\sqrt{15}}&\frac{4}{\sqrt{26}}\\\frac{1}{\sqrt{15}}&0\\\frac{1}{\sqrt{15}}&\frac{1}{\sqrt{26}}\end{pmatrix}_{4\times2}\cdot\begin{pmatrix}\sqrt{30}&0\\0&\sqrt{52}\end{pmatrix}_{2\times2}\cdot\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{-1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}_{2\times2}$$

$$=\begin{pmatrix}\frac{3}{\sqrt{15}}&\frac{-3}{\sqrt{26}}\\\frac{2}{\sqrt{15}}&\frac{4}{\sqrt{26}}\\\frac{1}{\sqrt{15}}&0\\\frac{1}{\sqrt{15}}&\frac{1}{\sqrt{26}}\end{pmatrix}_{4\times2}\begin{pmatrix}\frac{\sqrt{30}}{\sqrt2}&\frac{-\sqrt{30}}{\sqrt2}\\\frac{-\sqrt{52}}{\sqrt2}&\frac{-\sqrt{52}}{\sqrt2}\end{pmatrix}_{2\times2}=\begin{pmatrix}6&0\\-2&-6\\1&-1\\0&-2\end{pmatrix}$$

✓

$$\frac{3}{\sqrt{15}}\cdot\underbrace{\boxed{\frac{\sqrt{30}}{\sqrt2}}}_{3.873}+\frac{-3}{\sqrt{26}}\cdot\underbrace{\boxed{\frac{-\sqrt{52}}{\sqrt2}}}_{-5.099}\doteq 6.000$$

$$\frac{2}{\sqrt{15}}\qquad\frac{4}{\sqrt{26}}\qquad\doteq -2.000$$

$$\frac{1}{\sqrt{15}}\qquad 0\qquad\doteq 1.000$$

$$\frac{1}{\sqrt{15}}\qquad\frac{1}{\sqrt{26}}\qquad\doteq 0$$

$$\frac{3}{\sqrt{15}}\cdot\underbrace{\boxed{-\frac{\sqrt{30}}{\sqrt2}}}_{-3.873}+\frac{-3}{\sqrt{26}}\cdot\underbrace{\boxed{\frac{-\sqrt{52}}{\sqrt2}}}_{-5.099}\doteq 0$$

$$\frac{2}{\sqrt{15}}\qquad\frac{4}{\sqrt{26}}\qquad\doteq -6$$

$$\frac{1}{\sqrt{15}}\qquad 0\qquad\doteq -1$$

$$\frac{1}{\sqrt{15}}\qquad\frac{1}{\sqrt{26}}\qquad\doteq -2$$

Question 4

Any positive definite matrix C can also be factored as $C = T'T$.
where T is a nonsingular upper triangular matrix.
The cholesky decomposition can be used to obtain T.

$$t_{11} = \sqrt{c_{11}}$$

$$t_{1j} = \frac{c_{1j}}{t_{11}} \quad \text{for } 2 \le j \le n$$

$$t_{ii} = \sqrt{c_{ii} - \sum_{k=1}^{i-1} t_{ki}^2} \qquad \text{for } 2 \le i \le n$$

$$t_{ij} = \frac{c_{ij} - \sum_{k=1}^{i-1} t_{ki} t_{kj}}{t_{ii}} \qquad \text{for } 2 \le i < j \le n$$

$$t_{ij} = 0 \qquad \text{for } 1 \le j < i \le n$$

$$C = \begin{pmatrix} 25 & -15 & 10 & 20 \\ -15 & 18 & -3 & 0 \\ 10 & -3 & 30 & 22 \\ 20 & 0 & 22 & 40 \end{pmatrix}$$

$$t_{11} = \sqrt{c_{11}} = \sqrt{25} = 5$$

$$t_{12} = \frac{c_{12}}{t_{11}} = \frac{-15}{5} = -3$$

$$t_{13} = \frac{c_{13}}{t_{11}} = \frac{10}{5} = 2$$

$$t_{14} = \frac{c_{14}}{t_{11}} = \frac{20}{5} = 4$$

$$t_{22} = \sqrt{c_{22} - t_{12}^2} = \sqrt{18 - (-3)^2} = 3$$

$$t_{23} = \frac{c_{23} - t_{12}t_{13}}{t_{22}} = \frac{-3 - (-3)\times 2}{3} = 1$$

$$t_{24} = \frac{c_{24} - t_{12}t_{14}}{t_{22}} = \frac{0 - (-3)\times 4}{3} = 4$$

$$t_{33} = \sqrt{c_{33} - (t_{13}^2 + t_{23}^2)} = \sqrt{30 - (2^2 + 1^2)} = 5$$

$$t_{34} = \frac{c_{34} - (t_{13}t_{14} + t_{23}t_{24})}{t_{33}} = \frac{22 - (2\times 4 + 1\times 4)}{5} = 2$$

$$t_{44} = \sqrt{c_{44} - (t_{14}^2 + t_{24}^2 + t_{34}^2)} = \sqrt{40 - (4^2 + 4^2 + (-2)^2)} = 2$$

$$T = \begin{pmatrix} 5 & -3 & 2 & 4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

To verify T is correct:

$$T' = \begin{pmatrix} 5 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \\ 4 & 4 & 2 & 2 \end{pmatrix}$$

$$T'T = \begin{pmatrix} 5 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \\ 4 & 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 & 2 & 4 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & -15 & 10 & 20 \\ -15 & 18 & -3 & 0 \\ 10 & -3 & 30 & 22 \\ 20 & 0 & 22 & 40 \end{pmatrix}$$