STAT 764 Homework 2

39.5/40

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Notation of Matrix : uppercase A

lowercase $\underline{a}$

Question 1

* Problems 1, 2, + 4 graded on completion – See solutions

Random vector $\underline{y}$ $p\times 1$

Mean vector $\underline{\mu}$ $p\times 1$

Covariance Matrix Σ $p\times p$

$\underline{y} \sim N_p(\underline{\mu}, \Sigma)$

A New variable : $\underline{x} = A\underline{y} + \underline{b}$

$A: q\times p$ $\quad \mu_x$

$\underline{b}: q\times 1$

$\underline{x}: q\times 1$

Determine $E[(\underline{x}-\mu_x)(\underline{x}-\mu_x)']$

$\underline{x} = A\underline{y} + \underline{b}$

$\underline{\mu_x} = A\underline{\mu} + \underline{b}$

$E(\underline{x}) = E(A\underline{y}+\underline{b}) = AE(\underline{y})+b = A\underline{\mu}+b$

$Cov(\underline{x}) = Cov(A\underline{y}+b) = A\Sigma A'$

$\underline{x} \sim N_q(A\underline{\mu}+b,\ A\Sigma A')$

Definition of Covariance

$Cov(x,y) = \sigma_{xy} = E[(x-\mu_x)(y-\mu_y)]$

$\Sigma = E[(\underline{y}-\underline{\mu})(\underline{y}-\underline{\mu})'] = Cov(\underline{y})$

$\therefore E[(\underline{x}-\mu_x)(\underline{x}-\mu_x)'] = Cov(\underline{x}) = Cov(A\underline{y}+b) = A\Sigma A'$

Another way :)

replace $\underline{x} = A\underline{y}+\underline{b}$, $\mu_x = A\underline{\mu}+\underline{b}$

$E[(A\underline{y}+\cancel{b} - A\underline{\mu}-\cancel{b})(A\underline{y}+\cancel{b}-A\underline{\mu}-\cancel{b})']$

$= E[(A\underline{y}-A\underline{\mu})(A\underline{y}-A\underline{\mu})']$

$= E[A(\underline{y}-\underline{\mu})(\underline{y}-\underline{\mu})'A']$

$= A\,E[(\underline{y}-\underline{\mu})(\underline{y}-\underline{\mu})']\,A'$

$= A\Sigma A'$

Question 2

Random vector $\underline{y}_{p\times 1}$

mean vector $\underline{\mu}_{p\times 1}$

Covariance Matrix $\Sigma_{p\times p}$

$\underline{y} \sim N_p(\underline{\mu}, \Sigma)$

property of multivariate Normal Random Variables:

$\underline{a}'\underline{y} + b \sim N(\underline{a}'\underline{\mu} + b, \underline{a}'\Sigma\underline{a})$ ← A new variable by $\underline{y}$

$A\underline{y} + \underline{b} \sim N_q(A\underline{\mu} + \underline{b}, A\Sigma A')$ ← A new vector variable by $\underline{y}$

a. $(T')^{-1}(\underline{y} - \underline{\mu})$

$= \underset{p\times p}{\boxed{(T')^{-1}}}\underset{p\times 1}{\underline{y}} \underset{p\times 1}{\boxed{-\underset{p\times p}{(T')^{-1}}\underset{p\times 1}{\underline{\mu}}}}$

$A(B-C) = AB - AC$

$\Sigma = T'T$

$T'_{p\times p}$

$(T')^{-1}_{p\times p}$

$A\underline{\mu} + \underline{b}$

$= (T')^{-1}\underline{\mu} - (T')^{-1}\underline{\mu}$

$= \underline{0}_{p\times 1}$

$A\Sigma A'$

$= (T')^{-1}\Sigma((T')^{-1})'$

$= (T')^{-1}\Sigma((T^{-1})')'$

$= (T')^{-1}\Sigma T^{-1}$

$= (T')^{-1} T'T T^{-1}$

$= (T')^{-1}T'(TT^{-1})$

$= (T')^{-1}T' I$

$= [(T')^{-1}(T')] I$

$= I \cdot I$

$= I$

$(A^{-1})' = (A')^{-1}$

$(A')' = A$

$\Sigma = T'T$

$AA^{-1} = A^{-1}A = I$

$ABC = A(BC) = (AB)C$

$\therefore (T')^{-1}(\underline{y} - \underline{\mu}) \sim N\left((T')^{-1}\underline{\mu} - (T')^{-1}\underline{\mu}, (T')^{-1}\Sigma((T')^{-1})'\right)$

$\sim N(\underline{0}, I)$

b. $(\Sigma^{\frac{1}{2}})^{-1}(\underline{y}-\underline{\mu})$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad A\underline{\mu}+b \qquad\qquad\qquad\qquad A\Sigma A'$$

$$= \boxed{(\Sigma^{\frac{1}{2}})^{-1}}\,\underline{y}\ \boxed{-(\Sigma^{\frac{1}{2}})^{-1}\underline{\mu}} \sim N\left((\Sigma^{\frac{1}{2}})^{-1}\underline{\mu}-(\Sigma^{\frac{1}{2}})^{-1}\underline{\mu},\ (\Sigma^{\frac{1}{2}})^{-1}\Sigma\left(\left(\Sigma^{\frac{1}{2}}\right)^{-1}\right)'\right)$$

$A\underline{\mu}+b$

$= (\Sigma^{\frac{1}{2}})^{-1}\underline{\mu} - (\Sigma^{\frac{1}{2}})^{-1}\underline{\mu}$

$= \underline{0}$

$A\Sigma A'$

$= (\Sigma^{\frac{1}{2}})^{-1}\Sigma\left((\Sigma^{\frac{1}{2}})^{-1}\right)'$

$= (\Sigma^{\frac{1}{2}})^{-1}\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}\left((\Sigma^{\frac{1}{2}})^{-1}\right)'$

$= I\,\Sigma^{\frac{1}{2}}\left((\Sigma^{\frac{1}{2}})'\right)^{-1}$

$= I\,\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^{-1}$

$= I \cdot I$

$= I$

$\Sigma = \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}$
$A^{-1}A = AA^{-1} = I$
If A is symmetric $A = A'$
Σ is symmetric, $(\Sigma^{\frac{1}{2}})' = \Sigma^{\frac{1}{2}}$

$$\therefore (\Sigma^{\frac{1}{2}})^{-1}(\underline{y}-\underline{\mu}) \sim N\left(\underline{(\Sigma^{\frac{1}{2}})^{-1}\underline{\mu}-(\Sigma^{\frac{1}{2}})^{-1}\underline{\mu}},\ (\Sigma^{\frac{1}{2}})^{-1}\Sigma\left((\Sigma^{\frac{1}{2}})^{-1}\right)'\right)$$

$$\sim N(\underline{0},\ I)$$

Question 3

Data Matrix Y

		Variable 1 y_1	Variable 2 y_2	$\cdots$	Variable p y_p	
units	1					
	2					← random vector $\underline{y}$
	$\vdots$					
	n					
		μ_1	μ_2		μ_p	← mean vector $\underline{\mu}$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad \underline{\mu} = \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix}$$

$$\Sigma = Cov(\underline{y})$$

$$= \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & & \\ \sigma_{31} & \sigma_{32} & & \\ \vdots & \vdots & & \\ \sigma_{p1} & \sigma_{p2} & & \sigma_{pp} \end{pmatrix}$$

$\sigma_{11} = \sigma_1^2 = 25$

$\sigma_{22} = \sigma_2^2 = 16$

$\sigma_{33} = \sigma_3^2 = 64$

$\sigma_{44} = \sigma_4^2 = 100$

$$= \begin{pmatrix} 25 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & 16 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & 64 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & 100 \end{pmatrix} = \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix}$$

$$P_\rho(\rho_{jk}) = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.8 & 0.5 & -0.4 \\ 0.8 & 1 & 0.75 & -0.3 \\ 0.5 & 0.75 & 1 & -0.35 \\ -0.4 & -0.30 & -0.35 & 1 \end{pmatrix}$$

$$\rho_{jk} = \frac{\sigma_{jk}}{\sigma_j \sigma_k}$$

$\rho_{21} = \rho_{12} = 0.8 = \dfrac{\sigma_{21}}{\sigma_2 \sigma_1} = \dfrac{\sigma_{21}}{4 \times 5}$ $\quad \sigma_{21} = 4 \times 5 \times 0.8 = 16$

$\rho_{31} = \rho_{13} = 0.5 = \dfrac{\sigma_{31}}{\sigma_3 \sigma_1} = \dfrac{\sigma_{31}}{8 \times 5}$ $\quad \sigma_{31} = 0.5 \times 8 \times 5 = 20$

$\rho_{41} = \rho_{14} = -0.4 = \dfrac{\sigma_{41}}{\sigma_4 \sigma_1} = \dfrac{\sigma_{41}}{10 \times 5}$ $\quad \sigma_{41} = -0.4 \times 10 \times 5 = -20$

$\rho_{32} = \rho_{23} = 0.75 = \dfrac{\sigma_{32}}{\sigma_3 \sigma_2} = \dfrac{\sigma_{32}}{8 \times 4}$ $\quad \sigma_{32} = 0.75 \times 8 \times 4 = 24$

$\rho_{42} = \rho_{24} = -0.30 = \dfrac{\sigma_{42}}{\sigma_4 \sigma_2} = \dfrac{\sigma_{42}}{10 \times 4}$ $\quad \sigma_{42} = -0.30 \times 10 \times 4 = -12$

$\rho_{43} = \rho_{34} = -0.35 = \dfrac{\sigma_{43}}{\sigma_4 \sigma_3} = \dfrac{\sigma_{43}}{10 \times 8}$ $\quad \sigma_{43} = -0.35 \times 10 \times 8 = -28$

a. Σ of $\underline{y}$

$$\Sigma = \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix}$$

b. difference in the sum of the first 2 variables (y_1 and y_2) and the sum of the last 2 variables (y_3 and y_4) ← a new variable by $\underline{y}$

Let $D = (y_1 + y_2) - (y_3 + y_4)$

$= y_1 + y_2 - y_3 - y_4 + 0$

$$= \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \underline{a}'\underline{y} \sim N(\underline{a}'\underline{\mu} + b,\ \underline{a}'\Sigma\underline{a})$$

$\underline{a}' = (1 \ 1 \ -1 \ -1)$

$b = 0$

$$\underline{\mu} = \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix}$$

$$\underline{a}'\underline{\mu} = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix} = 43 + 22 - 65 - 58 = -58$$

$$\underline{a}'\Sigma\underline{a} = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 20 & 8 & -104 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$= 157$

$\therefore D \sim N(-58, 157)$

c. A new vector $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ ~~A~~ a new vector

$$\begin{pmatrix} 4y_1 - 3y_2 + 2y_3 - y_4 \\ y_1 + 2y_2 + 2y_3 + 3y_4 \\ 5y_1 + 0y_2 - 4y_3 + 0y_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \\ 2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 4 & -3 & 2 & -1 \\ 1 & 2 & 2 & 3 \\ 5 & 0 & -4 & 0 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ -10 \\ 2 \end{pmatrix}}_{\underline{b}} \sim N_3(\underbrace{A\underline{\mu} + \underline{b}}_{3\times 1},\ \underbrace{A\Sigma A'}_{3\times 3})$$

$A\underline{\mu} + \underline{b}$

$$= \begin{pmatrix} 4 & -3 & 2 & -1 \\ 1 & 2 & 2 & 3 \\ 5 & 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 178 \\ 391 \\ -45 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 178 \\ 381 \\ -43 \end{pmatrix}$$ ✓

$A\Sigma A'$

$$= \underbrace{\begin{pmatrix} 4 & -3 & 2 & -1 \\ 1 & 2 & 2 & 3 \\ 5 & 0 & -4 & 0 \end{pmatrix}}_{3\times 4} \underbrace{\begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix}}_{4\times 4} \underbrace{\begin{pmatrix} 4 & 1 & 5 \\ -3 & 2 & 0 \\ 2 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}}_{4\times 3}$$

$$= \underbrace{\begin{pmatrix} 112 & 76 & 164 & -200 \\ 37 & 60 & 112 & 200 \\ 45 & -16 & -156 & 12 \end{pmatrix}}_{3\times 4} \underbrace{\begin{pmatrix} 4 & 1 & 5 \\ -3 & 2 & 0 \\ 2 & 2 & -4 \\ -1 & 3 & 0 \end{pmatrix}}_{4\times 3}$$

$$= \begin{pmatrix} 748 & -8 & -96 \\ -8 & 981 & -263 \\ -96 & -263 & 849 \end{pmatrix}$$

d. if variable x and y are independent, then $\sigma_{xy} = 0$

$$\sigma_{xy} = E(xy) - \mu_x\mu_y = E(x)E(y) - \mu_x\mu_y = \mu_x\mu_y - \mu_x\mu_y = 0$$

independence implies $\sigma_{xy} = 0$,
but $\sigma_{xy} = 0$ doesn't imply independence.

But if (x, y) follow a bivariate normal distribution then $\sigma_{xy} = 0 \Leftrightarrow$ independence.

$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ covariance matrix $\begin{pmatrix} 748 & -8 & -96 \\ -8 & 981 & -263 \\ -96 & -263 & 849 \end{pmatrix}$

No zeros
thus, There aren't any pairs of ✓ variables among w_1 w_2 w_3 are independent.

e. Distribution of $\underline{y_2}$ ← a variable

$$y_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \underline{a}'\underline{y} + b \sim N(\underline{a}'\underline{\mu} + b,\ a'\Sigma a)$$

$\underline{a}' = (0,\ 1,\ 0\ \ 0)$

$b = 0$

$$\underline{a}'\underline{\mu} + b = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix} + 0 = 22$$

$$\underline{a}'\Sigma\underline{a} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 16 & 24 & -12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= 16$$

$y_2 \sim N(22,\ 16)$

f. Distribution of $\underline{x}_1 = \begin{pmatrix} y_1 \\ y_4 \end{pmatrix}$ ← a new vector

$$\begin{pmatrix} y_1 \\ y_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\underline{b}}$$

$$= A\,\underline{y} + \underline{b} \sim N_2(A\underline{\mu} + b,\ A\Sigma A')$$

$$A\underline{\mu} + \underline{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 43 \\ 22 \\ 65 \\ 58 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 43 \\ 58 \end{pmatrix}$$

$$A\Sigma A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & 16 & 20 & -20 \\ -20 & -12 & -28 & 100 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix}$$

✓

$$x_1 = \begin{pmatrix} y_1 \\ y_4 \end{pmatrix} \sim N_2\left(\begin{pmatrix} 43 \\ 58 \end{pmatrix}, \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} \right)$$

9. $\underline{z} = A(\underline{x}_1 - \underline{b}) \sim N_2(\underline{0}, I)$

$A\underline{x}_1 - A\underline{b}$

Find A and $\underline{b}$

property 2 of standardized variables

$* \quad \underline{z} = (T')^{-1}(\underline{y} - \underline{\mu}) \sim N_p(\underline{0}, I)$

$(T')^{-1}\underline{y} - (T')^{-1}\underline{\mu}$ given $\underline{y} \sim N(\underline{\mu}, \Sigma)$

$x_1 = \begin{pmatrix} y_1 \\ y_4 \end{pmatrix} \quad \mu_{x_1} = \begin{pmatrix} 43 \\ 58 \end{pmatrix} \quad \Sigma_{x_1} = \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} \quad \Sigma_{x_1} = T'T$

$\therefore \quad (T')^{-1}(\underline{x}_1 - \underline{\mu}_{x_1}) \sim N_2(0, I)$ $\quad \underline{x}_1 \sim N(\underline{\mu}_{x_1}, \Sigma_{x_1})$

$A(\underline{x}_1 - \underline{b})$

$A = (T')^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{\sqrt{84}}{105} & \frac{\sqrt{84}}{84} \end{pmatrix}$

$\underline{b} = \underline{\mu}_{x_1} = \begin{pmatrix} 43 \\ 58 \end{pmatrix}$ ✓

$\underline{\mu}_{x_1} = \begin{pmatrix} 43 \\ 58 \end{pmatrix}$

$\Sigma_{x_1} = \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} = T'T = \Sigma_{x_1}^{\frac{1}{2}} \Sigma_{x_1}^{\frac{1}{2}}$

$A = (T')^{-1}$

$A = (\Sigma_{x_1}^{\frac{1}{2}})^{-1}$

$\star \quad \underline{z} = (\Sigma_{x_1}^{\frac{1}{2}})^{-1}(\underline{x}_1 - \underline{\mu}_{x_1}) \sim N_2(0, I)$

$A(\underline{x}_1 - b)$

$A = (\Sigma_{x_1}^{\frac{1}{2}})^{-1}$

$\underline{b} = \underline{\mu}_{x_1}$

$t_{11} = \sqrt{\Sigma_{x_1 11}} = \sqrt{25} = 5$

$t_{12} = \frac{\Sigma_{x_1 12}}{t_{11}} = \frac{-20}{5} = -4$

$t_{22} = \sqrt{\Sigma_{x_1 22} - t_{12}^2} = \sqrt{100 - 16} = \sqrt{84}$

$T = \begin{pmatrix} 5 & -4 \\ 0 & \sqrt{84} \end{pmatrix}$

$T' = \begin{pmatrix} 5 & 0 \\ -4 & \sqrt{84} \end{pmatrix}$

$(T') \cdot (T')^{-1} = I$

$\begin{pmatrix} 5 & 0 \\ -4 & \sqrt{84} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$5x_1 = 1 \qquad x_1 = \frac{1}{5}$

$-4x_1 + \sqrt{84}\, y_1 = 0 \qquad y_1 = \frac{\sqrt{84}}{105}$

$5x_2 = 0 \qquad x_2 = 0$

$-4x_2 + \sqrt{84}\, y_2 = 1 \qquad y_2 = \frac{\sqrt{84}}{84}$

$(T')^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{\sqrt{84}}{105} & \frac{\sqrt{84}}{84} \end{pmatrix}$ ✓

h.

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad \Sigma = \text{Cov}(\underline{y}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} = \begin{pmatrix} 25 & 16 & 20 & -20 \\ 16 & 16 & 24 & -12 \\ 20 & 24 & 64 & -28 \\ -20 & -12 & -28 & 100 \end{pmatrix}$$

$\underline{x_1} | \underline{x_2}$

$$\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_4 \\ \hline y_2 \\ y_3 \end{pmatrix} \qquad \Sigma = \text{Cov}\left(\begin{pmatrix} \underline{x_1} \\ \underline{x_2} \end{pmatrix}\right) = \begin{array}{c|cc|cc} & 1 & 4 & 2 & 3 \\ \hline 1 & 25 & -20 & 16 & 20 \\ 4 & -20 & 100 & -12 & -28 \\ \hline 2 & 16 & -12 & 16 & 24 \\ 3 & 20 & -28 & 24 & 64 \end{array}$$

$\Sigma_{x_1x_1}$, $\Sigma_{x_1x_2}$, $\Sigma_{x_2x_1} = \Sigma'_{x_1x_2}$, $\Sigma_{x_2x_2}$

$$E(\underline{x_1}|\underline{x_2}) = \mu_{x_1} + \Sigma_{x_1x_2}\Sigma_{x_2x_2}^{-1}(\underline{x_2} - \mu_{x_2})$$

$$= \begin{pmatrix} 43 \\ 58 \end{pmatrix} + \underbrace{\begin{pmatrix} 16 & 20 \\ -12 & -28 \end{pmatrix}\begin{pmatrix} 16 & 24 \\ 24 & 64 \end{pmatrix}^{-1}}\begin{pmatrix} y_2 - 22 \\ y_3 - 65 \end{pmatrix}$$

$$= \begin{pmatrix} 43 \\ 58 \end{pmatrix} + \begin{pmatrix} 1.208 & -0.144 \\ -0.204 & -0.36 \end{pmatrix}_{2\times2}\begin{pmatrix} y_2 - 22 \\ y_3 - 65 \end{pmatrix}_{2\times1}$$

$$= \begin{pmatrix} 43 \\ 58 \end{pmatrix} + \begin{pmatrix} 1.208(y_2-22) - 0.144(y_3-65) \\ -0.204(y_2-22) - 0.36(y_3-65) \end{pmatrix}$$

$$= \begin{pmatrix} 1.208y_2 - 0.144y_3 + 43 - 22\times1.208 + 65\times0.144 \\ -0.204y_2 - 0.36y_3 + 58 + 22\times0.204 + 65\times0.36 \end{pmatrix} = \begin{pmatrix} 1.208y_2 - 0.144y_3 + 25.784 \\ -0.204y_2 - 0.36y_3 + 85.888 \end{pmatrix}$$

$$\Sigma_{x_1x_2}\Sigma_{x_2x_2}^{-1} = \begin{pmatrix} 16 & 20 \\ -12 & -28 \end{pmatrix}\begin{pmatrix} 16 & 24 \\ 24 & 64 \end{pmatrix}^{-1} = \begin{pmatrix} 16 & 20 \\ -12 & -28 \end{pmatrix}\begin{pmatrix} 0.143 & -0.054 \\ -0.054 & 0.036 \end{pmatrix} = \begin{pmatrix} 1.208 & -0.144 \\ -0.204 & -0.36 \end{pmatrix}$$

* Keep calculations in exact form -0.5

$$\text{Cov}(\underline{x_1}|\underline{x_2}) = \Sigma_{x_1x_1} - \Sigma_{x_1x_2}\Sigma_{x_2x_2}^{-1}\Sigma_{x_2x_1}$$

$$= \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} - \underbrace{\begin{pmatrix} 16 & 20 \\ -12 & -28 \end{pmatrix}\begin{pmatrix} 16 & 24 \\ 24 & 64 \end{pmatrix}^{-1}}\begin{pmatrix} 16 & -12 \\ 20 & -28 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} - \begin{pmatrix} 1.208 & -0.144 \\ -0.204 & -0.36 \end{pmatrix}\begin{pmatrix} 16 & -12 \\ 20 & -28 \end{pmatrix}$$

$$= \begin{pmatrix} 25 & -20 \\ -20 & 100 \end{pmatrix} - \begin{pmatrix} 16.448 & -10.464 \\ -10.464 & 12.528 \end{pmatrix}$$

$$= \begin{pmatrix} 8.552 & -9.536 \\ -9.536 & 87.472 \end{pmatrix}$$

$$x_1|x_2 \sim N_2\left(\begin{pmatrix} 1.208y_2 - 0.144y_3 + 25.784 \\ -0.204y_2 - 0.36y_3 + 85.888 \end{pmatrix}, \begin{pmatrix} 8.552 & -9.536 \\ -9.536 & 87.472 \end{pmatrix}\right)$$

✓

i. $x_3 | y_2 = \begin{pmatrix} y_1 \\ y_3 \\ y_4 \\ \hline y_2 \end{pmatrix}$

$$\Sigma = \mathrm{cov}\left(\begin{pmatrix} x_3 \\ y_2 \end{pmatrix}\right) = \begin{array}{c|ccc|c} & \multicolumn{3}{c|}{\Sigma x_3 x_3} & \Sigma x_3 y_2 \\ & 1 & 3 & 4 & 2 \\ \hline 1 & 25 & 20 & -20 & 16 \\ 3 & 20 & 64 & -28 & 24 \\ 4 & -20 & -28 & 100 & -12 \\ \hline 2 & 16 & 24 & -12 & 16 \\ & \multicolumn{3}{c|}{\Sigma y_2 x_3} & \Sigma y_2 y_2 \end{array}$$

$$E(x_3|y_2) = \mu_{x_3} + \Sigma_{x_3 y_2} \Sigma_{y_2 y_2}^{-1} (y_2 - \mu_{y_2})$$

$$= \begin{pmatrix} 43 \\ 65 \\ 58 \end{pmatrix} + \begin{pmatrix} 16 \\ 24 \\ -12 \end{pmatrix} (16)^{-1} \cdot (y_2 - 22)$$

$$= \begin{pmatrix} 43 \\ 65 \\ 58 \end{pmatrix} + \begin{pmatrix} 1 \\ 1.5 \\ -0.75 \end{pmatrix} \cdot (y_2 - 22)$$

$$= \begin{pmatrix} y_2 + 43 - 22 \\ 1.5y_2 + 65 - 22\times 1.5 \\ -0.75y_2 + 58 + 0.75\times 22 \end{pmatrix} = \begin{pmatrix} y_2 + 21 \\ 1.5y_2 + 32 \\ -0.75y_2 + 74.5 \end{pmatrix}$$ ✓

$$\Sigma_{x_3 y_2} \cdot \Sigma_{y_2 y_2}^{-1} = \begin{pmatrix} 16 \\ 24 \\ -12 \end{pmatrix} (16)^{-1}$$

$$= \begin{pmatrix} 16 \\ 24 \\ -12 \end{pmatrix} 0.0625$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ -0.75 \end{pmatrix}$$

$\mathrm{Cov}(x_3|y_2)$

$$= \Sigma_{x_3 x_3} - \Sigma_{x_3 y_2} \Sigma_{y_2 y_2}^{-1} \Sigma_{y_2 x_3}$$

$$= \begin{pmatrix} 25 & 20 & -20 \\ 20 & 64 & -28 \\ -20 & -28 & 100 \end{pmatrix} - \underset{3\times 1}{\begin{pmatrix} 1 \\ 1.5 \\ -0.75 \end{pmatrix}} \cdot \underset{1\times 3}{(16 \quad 24 \quad -12)}$$

$$= \begin{pmatrix} 25 & 20 & -20 \\ 20 & 64 & -28 \\ -20 & -28 & 100 \end{pmatrix} - \begin{pmatrix} 16 & 24 & -12 \\ 24 & 36 & -18 \\ -12 & -18 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -4 & -8 \\ -4 & 28 & -10 \\ -8 & -10 & 91 \end{pmatrix}$$ ✓

$$x_3 | y_2 \sim N_3\left(\begin{pmatrix} y_2 + 21 \\ 1.5y_2 + 32 \\ -0.75y_2 + 74.5 \end{pmatrix}, \begin{pmatrix} 9 & -4 & -8 \\ -4 & 28 & -10 \\ -8 & -10 & 91 \end{pmatrix} \right)$$ ✓

Question 4

a.

$$\underline{y} \sim N_p(\underline{\mu}, \Sigma)$$

$$\bar{\underline{y}} = \frac{1}{n} \Sigma_i \underline{y_i}$$

$$\bar{\underline{y}} \sim N_p(\underline{\mu}, \frac{\Sigma}{n}) \quad n=20$$

$$\bar{\underline{y}} \sim N_p(\underline{\mu}, \frac{\Sigma}{20})$$

b.

Sample variance S

$$W = (n-1)S = 19S \sim W_p(19, \Sigma)$$

wishart distribution with 19 degree of freedom.

Question 5.

we begin to assess multivariate normality by first looking at distribution of each variable seperately.

By Q-Q plot for y_1 y_2 y_3, All the points are close to a straight line, there is no indication of departure from normality. Thus, all variables of y_1 y_2 y_3 follow Normality seperately.

By Scatterplots of y_1 vs y_2 y_1 vs. y_3 y_2 vs. y_3,

There's No indication of a curved trend, outliers, or other nonnormal appearance.

Thus, we can infer each pair of variables has a bivariate normal distribution.

There's No indication of a clear deviation from MVN distribution. It is appropriate to assume that these observation vectors come from a multivariate normal distribution.