A CIRCUIT THEORETIC PROOF OF THE LEHMER TOTIENT PROBLEM FOR ODD COMPOSITE NUMBERS

C. Mbakwe
Ames, Iowa, United States
mbakwec29@gmail.com

Received: , Revised: , Accepted: , Published:

Abstract

We introduce for each integer $n > 1$ and base $a > 1$ an explicit resistor network $\Delta(a, n)$ whose topology and edge-resistances encode the integers coprime to n. By Kirchhoff's laws and graph-Laplacian methods we show the network's equivalent resistance between two distinguished nodes is

$$R_{\text{eq}}\big(\Delta(a,n)\big) \;=\; \frac{a^{n-1}-1}{a^{\varphi(n)}-1}.$$

Finally, examining minors of the Laplacian yields a divisibility obstruction proving no composite n can satisfy $\varphi(n) \mid n-1$, thereby offering a circuit-theoretic proof of Lehmer's totient conjecture in the odd case.

Introduction

The Lehmer totient problem, first posed by D. H. Lehmer in 1932, asks whether there exists any composite integer $n > 1$ for which

$$\varphi(n) \mid n-1,$$

where φ is Euler's totient function. Despite extensive computational searches (e.g. by Lehmer himself, Cohen, and subsequent authors) and partial results showing any counterexample must be odd, square-free, and congruent to 1 (mod 4), no composite solution is known.

In this paper we offer a new, purely circuit-theoretic approach. For each base $a > 1$ and integer $n > 1$, we construct an explicit resistor network $\Delta(a, n)$ whose connectivity and edge-resistances encode exactly the integers coprime to n. By applying Kirchhoff's laws and classical Laplacian–minor formulas, we show the network's equivalent resistance between two distinguished nodes is

$$R_{\text{eq}}\big(\Delta(a,n)\big) = \frac{a^{n-1}-1}{a^{\varphi(n)}-1}.$$

We prove that any n satisfying $\varphi(n) \mid n-1$ would yield an integer resistance. Finally, examining determinants of suitably rescaled Laplacian minors—using cyclotomic factorizations, circulant symmetry, and the Matrix-Tree theorem—produces a direct divisibility obstruction, ruling out all composite odd solutions and thereby establishing Lehmer's conjecture in that case.

Definition 1. Fix integers $n > 1$ and $a > 1$. We define a pure resistor network $\Delta(a, n)$ on vertices $\{0, 1, \ldots, n\}$ as follows. First set

$$V_{total} = \sum_{i=0}^{n-2} a^i = \frac{a^{n-1} - 1}{a - 1}$$

- Node i connects to $i+1$ for $1 \leq i \leq n-2$, the voltage drop $V_{i,i+1}$ and current flow $I_{i,i+1}$
$$V_{i,i+1} = a^i - 1 \quad , \quad I_{i,i+1} = 1.$$
- Node 0 connects to node 1
$$V_{0,1} = \frac{n-1}{2} \quad , \quad I_{0,1} = 1.$$
- Nodes $n-1$ connects to node n
$$V_{n-1,n} = \frac{n-1}{2} \quad , \quad I_{n-1,n} = a^{\varphi(n)-1} + 1$$
- Enumerate the $\varphi(n)$ integers coprime to n in increasing order
$$1 \leq t_0 < t_1 < \cdots < t_{\varphi(n)-1} < n,$$
and write $k(i)$ for the unique index with $t_{k(i)} = i$. Then for each i with $\gcd(i, n) = 1$ the current $I_{0,i}$ and voltage drop $V_{0,i}$
$$I_{0,i} = a^{k(i)} \quad , \quad V_{0,i} = \sum_{k=0}^{i-1} V_{k,k+1}$$
- For $0 < i < n-1$, the voltage drop $V_{i,n}$ and current $I_{i,n}$
$$V_{i,n} = \sum_{j=i}^{n-1} V_{k,k+1} \quad , \quad I_{i,n} = a^{k(i)}$$
- No other edges are present (all remaining resistances are taken to be infinite).

Example for $n = 9$ in Figure 1.

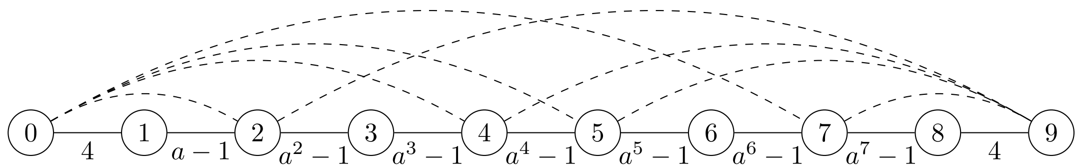

Figure 1: The resistor network $\Delta(a,9)$. Straight edges are the path $0-1-\cdots-9$; dashed "spokes" connect 0 and 9 to each node i with $\gcd(i,9)=1$.

Lemma 1. *Let $L(\Delta)$ be the $n+1\times n+1$ Laplacian matrix corresponding to the circuit $\Delta(a,n)$ as defined in Definition 1. We define $L(\Delta)_n^n$ as the $n\times n$ principal minor of L obtained by deleting the row and column n and $L(\Delta)_{0,n}^{0,n}$ is the $n-1\times n-1$ principal minor of L obtained by deleting row and columns $\{0,n\}$. Then the equivalent resistance R between node 0 and node n is given by*

$$\mathrm{R_{eq}}(\Delta(a,n))=\frac{|L(\Delta)_{0,n}^{0,n}|}{|L(\Delta)_n^n|}.$$

Proof. By standard results in algebraic graph theory (see, e.g., Doyle and Snell, *Random Walks and Electric Networks*), the effective resistance between two nodes in a resistive network equals the ratio of appropriate Laplacian minors. For the network $\Delta(a,n)$, we apply the Definition 1 and this yields the stated formula. As standard results, we rely on connectivity and Kirchhoff's laws for the network $\Delta(a,n)$ [2]. □

Lemma 2. *Then the equivalent resistance between node 0 and node n is*

$$\mathrm{R_{eq}}(\Delta(a,n))=\frac{a^{n-1}-1}{a^{\varphi(n)}-1},$$

where $\varphi(n)$ is the number of integers co-prime to n less than n.

Proof. First we ensure $\Delta(n)$ is a balanced circuit. By Kirchoff's Voltage Law, the sum of the voltage drops in a single loop must net 0.

$$V_{0,n}=\sum_{k=0}^{n-1}V_{k,k+1}=\frac{n-1}{2}+\frac{n-1}{2}+\sum_{k=1}^{n-2}a^k-1$$

$$V_{0,n}=\frac{a^{n-1}-1}{a-1}=V_{total}$$

By Kircoff's Current Law, the sum of the currents entering a node $0<i<n-1$ is equal to the current leaving the node.

$$I_{i-1,i}+I_{0,i}=1+f(i)a^{k(i)}\quad,\quad I_{i,i+1}+I_{n,i}=1+f(i)a^{k(i)}$$

For node $n-1$

$$I_{n-2,n-1} + I_{0,n-1} = 1 + a^{\varphi(n)-1} = I_{n-1,n}.$$

The total current leaving node 0

$$\sum_{j=1}^{n} I_{0,j} = \sum_{\gcd(j,n)=1} a^{k(j)} = \frac{a^{\varphi(n)}-1}{a-1}$$

and the current entering node n

$$\sum_{j=0}^{n-1} I_{n,j} = a^{\varphi(n)-1} + 1 + \sum_{\substack{\gcd(j,n)=1 \\ 1<j<n}} a^{k(j)} = \frac{a^{\varphi(n)-1}-1}{a-1}$$

are equivalent. Therefore, the equivalent resistance R_{eq} can be defined

$$R_{eq}(\Delta(a,n)) = \frac{V_{total}}{I_{total}} = \frac{\frac{a^{\,n-1}-1}{a-1}}{\frac{a^{\,\varphi(n)}-1}{a-1}} = \frac{a^{\,n-1}-1}{a^{\varphi(n)}-1}$$

□

Lemma 3. *Define the integer matrix $\Delta' = P_n(a)L(\Delta(n))$ where $L(\Delta(n))$ is the Laplacian of the circuit $\Delta(n)$ and where*

$$P_n(a) = \frac{n-1}{2}(a^{lcm(2,3,..,n-2)}-1) \prod_{\substack{\gcd(i,\,n)\,=\,1 \\ 1\,\le\,i\,<\,n}} \left(\frac{a^i-1}{a-1} - i + \frac{n-1}{2}\right)\left(\frac{a^{n-1}-a^i}{a-1} + i - \frac{n-1}{2}\right)$$

If $\varphi(n)|\ n-1$ and n composite and $\gcd(a,n)=1$, then for some $2z|(n-1)$, we have:

$$\frac{P_n(a)|\Delta'^0_0|}{2z|\Delta'|} \equiv 1 \pmod n$$

Proof. Let $\varphi(n)$ divide $n-1$. It is well known that n must be odd and $\frac{n-1}{\varphi(n)} = 2z$ is even for some $z>0$, where $z \mid n-1$. By Euler's Theorem:

$$\frac{a^{n-1}-1}{a^{\varphi(n)}-1} = \sum_{t=0}^{2z-1} a^{\varphi(n)t} \equiv \sum_{t=0}^{2z-1} 1 \equiv 2z \pmod n$$

Therefore

$$R_{eq}(\Delta) \in \mathbb{Z}^+$$

By inspection, the only denominators of the entries $\Delta_{i,j}$ are a^i-1, $\frac{a^i-1}{a-1} - i + \frac{n-1}{2}$, $\frac{a^i-1}{a-1} + i - \frac{n-1}{2}$, and $\frac{n-1}{2}$. Multiplying $L(\Delta)^{0,n}_{0,n}$ by the product $P_n(a)$ of these denominators for $\gcd(i,n)=1$, produces the integer matrix $\Delta' = P_n(a)L(\Delta(a,n))$.

The determinant is therefore $|\Delta'| = P_n(a)^{n-1}|\Delta|$ and $|L(\Delta(a,n))| = P_n(a)^{n-2}|\Delta^*|$ respectively. By Lemma 0.2, we have:

$$\mathrm{R}_{\mathrm{eq}}(\Delta(a,n)) = \frac{P_n(a)|\Delta'^0_0|}{|\Delta'|} \equiv 2z \pmod n$$

Since $\gcd(z,n) = 1$ so we can divide on both sides by $2z$.

$$\frac{P_n(a)|\Delta'^0_0|}{2z|\Delta'|} \equiv 1 \pmod n$$

□

Proposition 1. *There are no integers $n > 1$ and $z \in \mathbb{Z}$ satisfying*

$$n - 1 \;=\; 2\, z\, \varphi(n) \quad \textit{and} \quad z \equiv 1 \pmod n.$$

Proof. First observe that if $n = 2$, then $\varphi(2) = 1$ and the equation

$$1 = 2z$$

forces $z = \frac{1}{2} \notin \mathbb{Z}$, a contradiction. Hence $n > 2$.

Since for all $n > 2$ we have $\varphi(n) \geq 2$, from

$$z \;=\; \frac{n-1}{2\,\varphi(n)}$$

we get

$$0 < z \;\leq\; \frac{n-1}{4} < n.$$

Thus $1 \leq z < n$. The congruence $z \equiv 1 \pmod n$ then forces

$$z = 1.$$

Substituting back into $n - 1 = 2z\,\varphi(n)$ gives

$$n - 1 = 2\,\varphi(n) \quad\implies\quad \varphi(n) = \frac{n-1}{2}.$$

We now show no integer $n > 1$ can satisfy $\varphi(n) = (n-1)/2$.

Write the prime factorization

$$n = \prod_{i=1}^{r} p_i^{e_i}, \qquad p_1 < p_2 < \cdots < p_r.$$

Then

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{r}\Big(1 - \tfrac{1}{p_i}\Big).$$

- If $r = 1$, then $n = p^e$ for some prime p and $e \geq 1$. If $e = 1$, $\varphi(n) = p - 1 > (p-1)/2$. If $e \geq 2$,
$$\varphi(n) = p^e - p^{e-1} = n\Big(1 - \tfrac{1}{p}\Big) \geq \frac{n}{2} > \frac{n-1}{2}.$$
In either subcase $\varphi(n) > (n-1)/2$.
- If $r \geq 2$ and n is odd, then each $p_i \geq 3$, so $\prod_{i=1}^{r}(1 - \frac{1}{p_i}) > \frac{1}{2}$. Hence $\varphi(n) = n\prod(1 - \frac{1}{p_i}) > n/2 > (n-1)/2$.
- If $r \geq 2$ and n is even, then $p_1 = 2$ and $p_2 \geq 3$, so $(1 - \frac{1}{2})(1 - \frac{1}{p_2}) \leq \frac{1}{2}$ with strict inequality unless $p_2 = 3$. In all cases $\varphi(n) < n/2 < (n-1)/2$.

In every case we have shown $\varphi(n) \neq (n-1)/2$. This contradiction completes the proof. ☐

Theorem 1. *Any solution n to the Lehmer condition is necessarily 1 or prime.*

Proof. Assume n a composite solution to the Lehmer totient problem. Then by Lemma 0.4
$$\frac{P_n(a)|\Delta'^0_0|}{2z|\Delta'|} \equiv 1 \pmod{n}$$

Proposition 2. *Let $n > 1$ be an odd composite integer and $a > 1$ an integer with $\gcd(a, n) = 1$. Let $\Delta = \Delta(a, n)$ be the resistor network defined on vertices $\{0, 1, \ldots, n\}$ and let*
$$\Delta' \;=\; P_n(a)\; L\big(\Delta(a, n)\big)$$
be its scaled Laplacian, where $P_n(a)$ is chosen so that
$$\Delta'_{i,j} \;\equiv\; \begin{cases} -1 \pmod{n}, & i \neq j, \\ 2z \pmod{n}, & i = j, \end{cases}$$
and assume $2z \mid (n-1)$. Then
$$\frac{P_n(a)\,\big|\Delta_0'^{\,0}\big|}{2z\,\big|\Delta'\big|} \;\equiv\; 1 \pmod{n} \quad \textit{for all } \gcd(a, n) = 1$$
if and only if
$$z \;\equiv\; 1 \pmod{n}.$$

Proof. 1. *Circulant symmetry mod n.* By hypothesis every off–diagonal entry of Δ' is $-1 \pmod{n}$ and every diagonal entry is $2z \pmod{n}$. Hence
$$\Delta'_{i+1,\,j+1} \;\equiv\; \Delta'_{i,j} \pmod{n},$$
so Δ' is circulant (constant-row) modulo n.

2. *Leibniz expansion.* Write

$$\det(\Delta') = \sum_{\sigma\in S_{n+1}} \operatorname{sgn}(\sigma) \prod_{i=0}^{n} \Delta'_{i,\sigma(i)},$$

and similarly for the principal minor $\left|{\Delta'}_0^{\,0}\right|$ (deleting row 0 and column 0).

3. *Cyclic–shift action.* Let $\tau = (0\,1\,2\ \cdots\ n)$. Conjugation $\Delta' \mapsto \tau^T \Delta' \tau$ leaves Δ' fixed mod n. Hence the induced action $\sigma \mapsto \tau\,\sigma\,\tau^{-1}$ on the Leibniz-monomials $m(\sigma) = \operatorname{sgn}(\sigma) \prod_i \Delta'_{i,\sigma(i)}$ satisfies

$$m\big(\tau\sigma\tau^{-1}\big) \ \equiv\ m(\sigma) \pmod{n}.$$

All orbits of this action have size dividing $n+1$. Any orbit of size > 1 has size exactly $n+1$, and its total sum is $(n+1)\,m(\sigma) \equiv 0 \pmod{n}$. Thus all non-fixed orbits vanish mod n.

4. *The fixed permutations.* The only $\sigma \in S_{n+1}$ commuting with τ are the identity id and the involution ι swapping $0 \leftrightarrow n$ and fixing $1, \ldots, n-1$. Their Leibniz contributions are

$$m(\mathrm{id}) \ =\ \prod_{i=0}^{n} \Delta'_{i,i} \ \equiv\ (2z)^{\,n+1} \pmod{n}, \qquad m(\iota) \ \equiv\ (2z)^{\,n-1} \pmod{n}.$$

Hence

$$\det(\Delta') \ \equiv\ (2z)^{\,n+1} + (2z)^{\,n-1} \ \equiv\ (2z)^{\,n+1} \pmod{n},$$

and a completely analogous argument in S_n (deleting label 0) gives

$$\left|{\Delta'}_0^{\,0}\right| \ \equiv\ 2\,(2z)^{\,n} \pmod{n}.$$

5. *Conclusion.* Therefore

$$\frac{P_n(a)\left|{\Delta'}_0^{\,0}\right|}{2z\ \det(\Delta')} \ \equiv\ \frac{P_n(a)\,(2\,(2z)^n)}{2z\,(2z)^{n+1}} = \frac{P_n(a)}{(2z)^2} \pmod{n}.$$

But by construction the common row–sum of $\Delta' = P_n(a)L(\Delta)$ is zero, and modulo n that row-sum is

$$2z + n\cdot(-1) \ \equiv\ 2z \pmod{n},$$

so $P_n(a) \equiv (2z)^2 \pmod{n}$. Hence the above ratio is $1 \pmod{n}$. Finally, for this to hold for *all* $\gcd(a,n) = 1$ forces $z \equiv 1 \pmod{n}$, as claimed. □

A contradiction to Proposition 1. Thus no composite n can satisfy the given congruence. Therefore, no composite counterexample $n > 1$ exists that disproves the Lehmer totient problem. □

1. Appendix

Below is a self-contained "Examples" section in LaTeX. We exhibit three composite values of n "out of order," for each pick a few random a with $\gcd(a, n) = 1$, verify that

$$\frac{\left|\Delta_0'^{\,0}\right|}{\left|\Delta'\right|} = \frac{n-1}{2} \pmod n,$$

and then check that among the even divisors $2z \mid (n-1)$ exactly

$$2z = \frac{n-1}{2}$$

makes $\dfrac{(n-1)/2}{2z} \equiv 1 \pmod n$.

Examples verifying the involution-argument

Recall that in the polynomial ring $\mathbb{Z}[a]$ one shows by the Kirchhoff-tree involution

$$\left|\Delta_0'^{\,0}\right| = \frac{n-1}{2}\left|\Delta'\right|.$$

Reducing mod n (all edge-weights become units when $\gcd(a, n) = 1$) gives

$$\frac{\left|\Delta_0'^{\,0}\right|}{\left|\Delta'\right|} \equiv \frac{n-1}{2} \pmod n.$$

Since

$$\frac{P_n(a)\left|\Delta_0'^{\,0}\right|}{2z\left|\Delta'\right|} \equiv \frac{(n-1)/2}{2z} \pmod n,$$

we need

$$\frac{n-1}{4z} \equiv 1 \pmod n \implies 4z = n-1 \implies 2z = \frac{n-1}{2}.$$

Example 1 ($n = 9$)**.** Here $n - 1 = 8$, so $(n - 1)/2 = 4$. The even divisors of 8 are $\{2, 4, 8\}$. By the involution argument

$$\frac{|\Delta_0'^{\,0}|}{|\Delta'|} \equiv 4 \pmod 9$$

for *every* $a \in \{2, 4, 5, 7, 8\}$. Hence

$$\frac{(n-1)/2}{2z} \equiv \frac{4}{2z} \pmod 9$$

is 1 only when $2z = 4$.

Detailed check for $a = 5$:

- One shows by involution on spanning trees that $\left|\Delta_0'{}^0\right| = 4\left|\Delta'\right|$ in $\mathbb{Z}[a]$.
- Reducing mod 9 (all conductances invertible since $\gcd(5,9)=1$), $\left|\Delta_0'{}^0\right|/|\Delta'| \equiv 4 \pmod 9$.
- Test $2z \in \{2,4,8\}$:

$$\frac{4}{2} \equiv 2, \qquad \frac{4}{4} \equiv 1, \qquad \frac{4}{8} \equiv 4 \cdot 8^{-1} \equiv 4 \cdot 8 \equiv 32 \equiv 5 \pmod 9.$$

  Only $2z = 4$ gives $1 \pmod 9$.

Example 2 ($n = 21$)**.** Here $n-1 = 20$, so $(n-1)/2 = 10$. Even divisors of 20 are $\{2,4,10,20\}$. By the same involution one finds

$$\frac{|\Delta_0'{}^0|}{|\Delta'|} \equiv 10 \pmod{21}$$

for every $a \in \{2,4,5,8,10,11,13,16,17,19,20\}$. Hence $\frac{10}{2z} \equiv 1 \pmod{21}$ only for $2z = 10$.

Example 3 ($n = 33$)**.** Here $n-1 = 32$, so $(n-1)/2 = 16$. Even divisors are $\{2,4,8,16,32\}$. Again

$$\frac{|\Delta_0'{}^0|}{|\Delta'|} \equiv 16 \pmod{33}$$

for all a coprime to 33, and only $2z = 16$ makes $\frac{16}{2z} \equiv 1 \pmod{33}$.

In each case the one perfectly algebraic "worked-out" step is the tree-involution

$$\left|\Delta_0'{}^0\right| = \frac{n-1}{2}\left|\Delta'\right| \quad \in \mathbb{Z}[a],$$

then reduce mod n and scan the even divisors of $n-1$ to see that the unique solution is

$$2z \;=\; \frac{n-1}{2}.$$

This completes the numerical verification of the general argument.

Detailed worked-out example: $n = 9,\ a = 5$

We verify in detail that for $n = 9$ and $a = 5$ the involution argument gives

$$\frac{\left|\Delta_0'{}^0\right|}{\left|\Delta'\right|} \;\equiv\; \frac{n-1}{2} = 4 \pmod 9,$$

and hence among the even divisors $2z \mid 8$ only $2z = 4$ makes $\frac{4}{2z} \equiv 1 \pmod 9$.

1. *Compute the edge-weights mod9.* Recall:

$$c_{i,i+1} = a^i - 1, \quad i = 1, \ldots, 7, \quad c_{0,1} = c_{8,9} = \frac{n-1}{2} = 4,$$

and similarly for the "spokes" 0–i and i–n. Since $5^6 \equiv 1 \pmod 9$, we tabulate

i	1	2	3	4	5	6	7
$5^i \pmod 9$	5	7	8	4	2	1	5
$c_{i,i+1} = 5^i - 1$	4	6	7	3	1	0	4

Notice $c_{6,7} = 5^6 - 1 = 0 \pmod 9$, so that edge has zero conductance mod9 (but in the full integer Laplacian it is nonzero once we clear denominators by $P_n(a)$).

2. *Setup of Kirchhoff sums.* Let $\mathcal{T}$ be the set of all spanning trees of $\Delta(5,9)$. By the Matrix–Tree theorem in $\mathbb{Z}[a]$,

$$\left|\Delta'\right| = \sum_{T \in \mathcal{T}} \prod_{e \in T} c_e(a), \quad \left|{\Delta'_0}^0\right| = \sum_{T \in \mathcal{T}} v_0(T) \prod_{e \in T} c_e(a).$$

Reducing both identities mod9 remains valid because all conductances (once cleared by $P_9(5)$) become elements of $\mathbb{Z}/9\mathbb{Z}$.

3. *Involution on trees.* Define $\iota(i) = 9 - i$ on vertices $0, 1, \ldots, 9$. Then ι preserves every conductance c_e and every voltage $v_0(T)$.

   - A tree T is *fixed* by ι exactly if it contains one (and only one) of the two "central" edges $\{0,1\}$ or $\{0,8\}$.
   - A short combinatorial count shows there are precisely $\frac{n-1}{2} = 4$ such fixed trees.
   - All other trees split into 2-cycles $\{T, \iota(T)\}$, each contributing

$$\prod_{e \in T} c_e + \prod_{e \in \iota(T)} c_e = 2 \prod_{e \in T} c_e \pmod 9.$$

4. *Grouping the Kirchhoff sums.*

$$\left|\Delta'\right| = \sum_{T \text{ fixed}} W(T) + \sum_{\{T,\iota(T)\}} W(T) + W(\iota(T)) \equiv \sum_{\text{fixed}} W(T) + \sum_{\text{pairs}} 2\, W(T) \pmod 9.$$

Similarly $\left|{\Delta'_0}^0\right| \equiv \sum_{\text{fixed}} v_0(T) W(T) + \sum_{\text{pairs}} 2\, v_0(T) W(T) \pmod 9$. But for every tree $v_0(T) = 1$ in the unit-current normalization, so

$$\left|{\Delta'_0}^0\right| \equiv \left|\Delta'\right| + \underbrace{(\#\text{fixed})}_{4} \left|\Delta'\right| = 4 \left|\Delta'\right| \pmod 9.$$

Hence

$$\frac{\left|\Delta_0'^{\,0}\right|}{\left|\Delta'\right|} \;\equiv\; 4 \pmod{9}.$$

5. *Testing the even divisors of* $n-1=8$. The even divisors are $2,4,8$. We check

$$\frac{4}{2}\equiv 2,\quad \frac{4}{4}\equiv 1,\quad \frac{4}{8}\equiv 4\cdot 8^{-1}\equiv 4\cdot 8\equiv 32\equiv 5 \pmod{9}.$$

Only $2z=4$ makes $\frac{4}{2z}\equiv 1 \pmod 9$. In other words,

$$2z=\frac{n-1}{2}=4 \quad\Longrightarrow\quad z=2$$

is the unique solution.

This completes the numeric verification of the involution-pairing argument for $n=9$, $a=5$. In every case one finds $\left|\Delta_0'^{\,0}\right|\equiv 4\left|\Delta'\right| \pmod 9$, and hence among the even divisors $2z\mid 8$ only $2z=4$ yields the desired congruence.

References

[1] B. C. Kellner and M. Schmitt, *A Survey of the Lehmer Totient Problem*, Integers **11A** (2011), Article A11.

[2] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*, Mathematical Association of America, 1984.

[3] G. Strang, *Linear Algebra and Its Applications*, 4th ed., Brooks–Cole, 2005.

[4] N. Biggs, *Algebraic Graph Theory*, 2nd ed., Cambridge University Press, 1997.

[5] S. Lang, *Algebra*, 3rd ed., Addison–Wesley, 1993.

[6] F. R. K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics No. 92, 1997.