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EVALUATION OF BETA-DECAY
II. Finite mass and size effects *

D.H. WILKINSON

University of Sussex, Falmer, Brighton, England

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The effects on the rate of allowed beta-decay of finite masses for the neutrino and for the nucleus are evaluated. The effect of the finite size of the nucleus in determining the magnitude of the electron wavefunction within the nucleus is evaluated for the case that the nuclear charge distribution is represented by a uniformly charged sphere; this evaluation is parameterized analytically with an accuracy of 1 part in 10^4 for values of the nuclear radius corresponding to the region between the neutron and proton drip lines for $Z \le 60$. The effects of the convolution of the nucleon and lepton wavefunctions are evaluated in alternative approximations for the case that the nucleon wavefunctions are constant within the nuclear volume. The results of the convolution are presented graphically and are also parameterized analytically.

1. Nomenclature

As in the first paper in this series [1] plus: $R = 2.5896 \times 10^{-3} \times R$ [fm].

2. Introduction

In ref. [1] the question was addressed of the evaluation of the traditional phase-space factors based on the Fermi function

$$F(Z, W) = 2(\gamma+1)\Gamma(2\gamma+1)^{-2}(2pR)^{2(\gamma-1)} \times e^{\pi\alpha ZW/p}\,|\Gamma(\gamma + i\alpha ZW/p)|^2, \qquad (1)$$

so that the traditional integrated phase space factor of I is

$$f = \int_0^{p_0} p^2(W_0 - W)^2 F(Z, W)\,dp.$$

This traditional f is based on a point nuclear charge which results in solutions to the Dirac equation for the electron or positron that diverge at the origin and that are therefore evaluated at a distance R from the origin that is taken as a conventional representation of the nuclear radius although it is of no physical significance. A realistic phase-space factor must evaluate the electron or positron wavefunctions as solutions to the Dirac equation in a finite charge distribution that adequately represents that of the real nucleus; it must also convolute those wavefunctions appropriately with the neutrino wavefunction and with the single-nucleon wavefunctions of the orbitals between which the beta-decay transition is taking place. These are referred to as the finite size effects; they have been the object of considerable study by many authors; we here particularly follow the work of Behrens and Bühring [2].

The traditional f also assumes that the mass of the neutrino is zero and that that of the nucleus is infinite; relaxation of these assumptions gives the finite mass effects.

3. The effect of finite neutrino mass

If the neutrino has finite mass m_ν then the primitive ($Z = 0$) phase space factor is reduced by $m_\nu^2 p_0^3/6$ (where p_0 is here not the electron end point momentum appropriate to finite m_ν but is rather what it would have been for $m_\nu = 0$). Since for high p_0 f tends to $p_0^5/30$, the fractional change in f due to finite neutrino mass tends to $5(m_\nu/p_0)^2$ or, with a neutrino mass of less than 30 eV, to less than $2 \times 10^{-8}/p_0^2$ which may be safely neglected.

Note, however, that this ignores: (i) the possibility of neutrino mixing which, if it obtains, could have profound effects; (ii) the so-called relativistic spinor term in the phase-space factor which for finite m_ν vanishes for exact V–A which is itself established only approximately.

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4. The effects of finite nuclear mass

The effect of the recoil of a nucleus of mass M $(=1837A)$ is [3] to multiply the phase space by the factor

$$R(W, W_0, M) = 1 + r_0 + r_1/W + r_2W + r_3W^2,$$

where, for vector decay:

$$r_0^V = W_0^2/(2M^2) - 11/(6M^2),$$
$$r_1^V = W_0/(3M^2),$$
$$r_2^V = 2/M - 4W_0/(3M^2),$$
$$r_3^V = 16/(3M^2),$$

and, for axial decay:

$$r_0^A = -2W_0/(3M) - W_0^2/(6M^2) - 77/(18M^2),$$
$$r_1^A = -2/(3M) + 7W_0/(9M^2),$$
$$r_2^A = 10/(3M) - 28W_0/(9M^2),$$
$$r_3^A = 88/(9M^2).$$

Fig. 1 shows the effect of this correction upon the integrated phase-space factor (for vector decay); it is seen to be small although not negligible for certain work of the highest precision. The figure also shows that it is always perfectly adequate to evaluate the correction in the $Z = 0$ limit where the integrations are elementary.

A further effect of the finite mass of the nucleus is that the source of the Coulomb field that distorts the outgoing electron wave is not stationary but recoils from the combined lepton momenta. This may be allowed for exactly in the nonrelativistic limit; plausibly [3] in general we may allow for this effect by multiplying the phase space by a factor whose leading term is

$$1 - \pi\alpha Z\{1 + \Omega[W_0 - W]/(3W)\}/(Mp),$$

where $\Omega = 1$ for vector decay and $= -1/3$ for axial decay. As W_0 becomes large the fractional correction tends to $5(1 + \Omega)\pi\alpha Z/(2W_0M)$. Since $\pi\alpha Z/M \approx 10^{-5}$ this correction is everywhere negligible.

5. The effect of finite nuclear size

When we move from a point nucleus to one of finite size the electron wavefunctions become finite at the

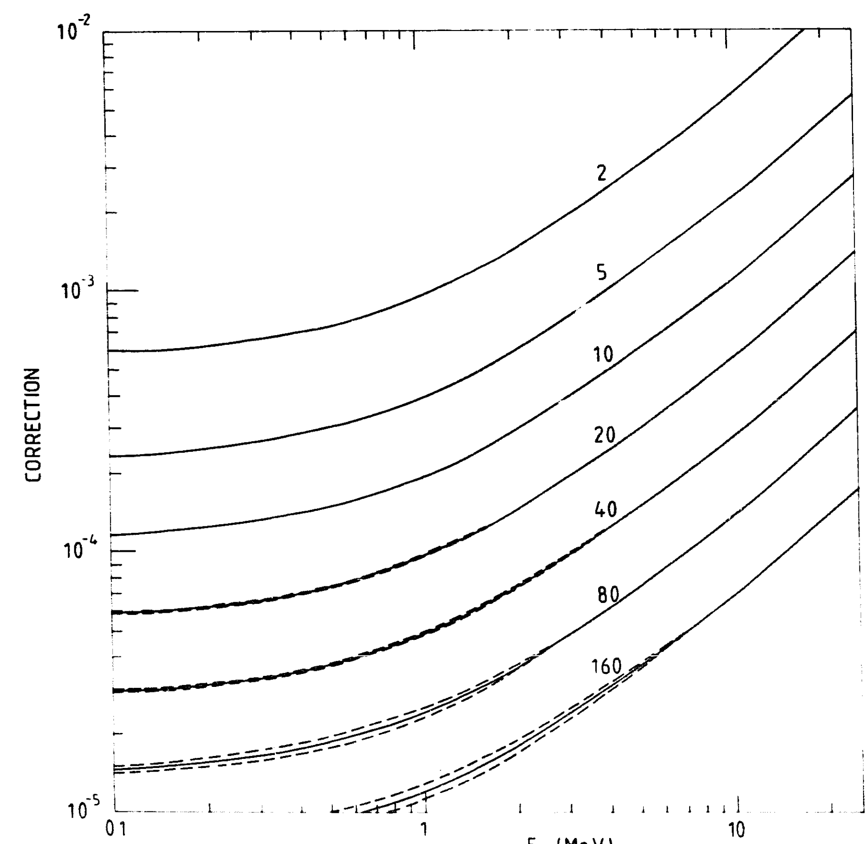

Fig. 1. The effect of finite nuclear mass on the rate of allowed vector beta-decay. The A-values are shown on the curves which include both M^{-1} and M^{-2} correction terms. For $A = 80$ and 160 the dashed curves are for $Z = \pm 35$ and $Z = \pm 66$, respectively (positrons being the upper curves and electrons the lower). The solid lines are for $Z = 0$. For $A < 80$ the thickness of the lines covers the appropriate $\pm Z$.

Table 1
Coefficients for the parameterization of $L_0(Z, W)$ for electrons

	b_1	b_2	b_3	b_4	b_5	b_6
a_{-1}	0.115	−1.8123	8.2498	−11.223	−14.854	32.086
a_0	−0.00062	0.007165	0.01841	−0.53736	1.2691	−1.5467
a_1	0.02482	−0.5975	4.84199	−15.3374	23.9774	−12.6534
a_2	−0.14038	3.64953	−38.8143	172.1368	−346.708	288.7873
a_3	0.008152	−1.15664	49.9663	−273.711	657.6292	−603.7033
a_4	1.2145	−23.9931	149.9718	−471.2985	662.1909	−305.6804
a_5	−1.5632	33.4192	−255.1333	938.5297	−1641.2845	1095.358

Table 2
Coefficients for the parameterization of $L_0(Z, W)$ for positrons

	b_1	b_2	b_3	b_4	b_5	b_6
a_{-1}	−0.0701	−2.572	−27.5971	−128.658	−272.264	−214.925
a_0	0.002308	0.066483	0.6407	2.83606	5.6317	4.0011
a_1	−0.07936	−2.09284	−18.45462	−80.9375	−160.8384	−124.8927
a_2	0.93832	22.02513	197.00221	807.1878	1566.6077	1156.3287
a_3	−4.276181	−96.82411	−835.26505	−3355.8441	−6411.3255	−4681.573
a_4	8.2135	179.0862	1492.1295	5872.5362	11038.7299	7963.4701
a_5	−5.4583	−115.8922	−940.8305	−3633.9181	−6727.6296	−4795.0481

centre of the nucleus and so it becomes convenient and natural to refer their values there although the centre of the nucleus has no special physical significance. For work of the highest accuracy we should solve the Dirac equation in the field of a charge distribution that approximates as closely as possible to the actual charge distribution of the daughter nucleus in question. However, for practical purposes it is adequate to represent the extended nucleus by a uniformly charged sphere of radius R adjusted to give the experimental $\langle r^2\rangle^{1/2}$, viz. $R=(5/3)^{1/2}\langle r^2\rangle^{1/2}$.

In order to effect the transfer to finite nuclear size, multiply the $F(Z, W)$ of I, eq. (1) above, by $[2/(1+\gamma)]L_0(Z, W)$. Here $L_0(Z, W)$ represents the results of (numerical) integration of the Dirac equation; it has been extensively tabulated [4] for $R=r_0A^{1/3}$ with $r_0=$ 1.2 fm and for A-values appropriate to the floor of the stability valley. (And for a value, viz. 1/137.0388, for α slightly different from the modern value, viz. 1/137.036, used here.) However, for our present purposes it is desirable to be able to set R appropriate to the specific circumstance since not only does r_0 vary through the periodic table on the floor of the stability valley but we must also be free to vary A, and hence R, as between the neutron and proton drip lines for a given Z. The tabulation of [4] has therefore been considerably extended * to cover the whole area between the drip lines for $|Z|\le 60$, namely the range of Z to which restriction was made in ref. [1]. Consonant with the philosophy of this series of papers, the results are here presented in analytical form.

It is convenient to start from a slightly generalized version of the form of $L_0(Z, W)$ appropriate to low Z [2] and then to afforce this by a number of empirical terms appropriate to secure the desired precision. For electrons:

$$\begin{aligned} L_0(Z, W) = {} & 1 + 13(\alpha Z)^2/60 - WR\alpha Z(41-26\gamma) \\ & /[15(2\gamma-1)] \\ & - \alpha ZR\gamma(17-2\gamma)/[30W(2\gamma-1)] \\ & + a_{-1}R/W + \sum_{n=0}^{5} a_n(WR)^n \\ & + 0.41(R-0.0164)(\alpha Z)^{4.5}. \end{aligned} \tag{2}$$

The a-values are given by the parameterization

$$a=\sum_{x=1}^{6} b_x(\alpha Z)^x,$$

where the b_x are given in table 1.

For positrons, in the last term in eq. (2) 0.41 is replaced by 0.22 and Z is replaced by $|Z|$: the b_x are given in table 2. These parameterizations yield $L_0(Z, W)$ accurately to 1 part in 10^4 for $p\le 45$ and for $|Z|\le 60$ everywhere between the neutron and proton drip lines.

We must now consider the convolution of the lepton and nucleon wavefunctions. For work of the highest accuracy we should take the single-nucleon wavefunc-

* I am very much indebted to Dr. H. Behrens for various communications and for making available to me his computer programs.

tions as generated by the most realistic single-nucleon potentials just as we should use a realistic nuclear charge distribution in evaluating $L_0(Z, W)$. However, for practical purposes it is adequate to take single-nucleon wavefunctions as rectangles of radius R just as it was adequate to use a uniform spherical charge distribution of radius R to generate the $L_0(Z, W)$ as presented above.

The effect of convolution is to introduce an additional term $C(Z, W)$ into the phase-space factor. The optimum analytical form for $C(Z, W)$ has been the object of deep consideration [2]. The choice is by no means unambiguous and various forms for $C(Z, W)$ are available depending on alternative ways of defining the convolution prior to reducing it to tractable analytical form. We recommend, and primarily follow here, the latest recommendation [2] although also examining an alternative to gain some impression of the dependence of the integrated phase-space factor on reasonable choices of $C(Z, W)$.

The simplest situation is that for the superallowed Fermi vector transitions of $J = 0 \to 0$; with adequate accuracy we may write

$$^{V}C(Z, W) = 1 + {^{V}C_0} + {^{V}C_1}W + {^{V}C_{-1}}/W + {^{V}C_2}W^2,$$

where

$$^{V}C_0 = -233(\alpha Z)^2/630 - (W_0R)^2/5 - 6W_0R\alpha Z/35,$$

$$^{V}C_1 = -13R\alpha Z/35 + 4W_0R^2/15,$$

$$^{V}C_{-1} = 2\gamma W_0R^2/15 + \gamma R\alpha Z/70,$$

$$^{V}C_2 = -4R^2/15.$$

($^{V}C_{-1}$ also contains a Z, W-dependent factor μ that is tabulated in ref. [4] but that may be set equal to unity without sensible error.)

This $^{V}C(Z, W)$ accounts for all terms in vector $J = 0 \to 0$ transitions with the exception of the relativistic term familiar from old notation as $\int(\alpha \cdot r)$; this term is very small and model-dependent and is not appropriate for incorporation into a standard definition, unlike the terms listed above which simply represent the effects of the finite de Broglie wavelengths in changing the magnitudes of the lepton wavefunctions through the nuclear volume.

Vector transitions between states of $J \neq 0$ entrain other terms that must be explicitly added to those subsumed through $C(Z, W)$ into our definition of the phase-space factor. Our definition, through its regard for the variation of the lepton wavefunctions through the nuclear volume, automatically includes certain terms that could alternatively be separately evaluated and labelled second-forbidden and so on; but this would be an artificial procedure since the r-dependences in question do not reflect higher j,l-values in their lepton waves but simply the r-dependence of those of $j = \frac{1}{2}$ to which we restrict ourselves in our definition of the phase-space factor.

The term $^{V}C_{-1}/W$ in $^{V}C(Z, W)$ is small compared with the other correction terms. It is worth retaining it for vector transitions for the sake of completeness, $^{V}C(Z, W)$ so defined being a full account of the important $J = 0 \to 0$ transitions, apart from the relativistic term, and because the Conserved Vector Current (CVC) encourages the making of evaluations of the highest precision. But it is not worth keeping it for our evaluation of the axial phase-space factor because its effect is negligible in relation to our ability to analyze axial transitions where no such principle as CVC exists to determine the nuclear matrix elements. We therefore drop the $1/W$ term from our definition of $^{A}C(Z, W)$ writing

$$^{A}C(Z, W) = 1 + {^{A}C_0} + {^{A}C_1}W + {^{A}C_2}W^2,$$

where

$$^{A}C_0 = -233(\alpha Z)^2/630 - (W_0R)^2/5 + 2W_0R\alpha Z/35,$$

$$^{A}C_1 = -21R\alpha Z/35 + 4W_0R^2/9,$$

$$^{A}C_2 = -4R^2/9.$$

It is of some interest to compare these correction terms with those arising from an earlier treatment of the problem [2]. In this earlier treatment the electron wavefunctions were, like the neutrino wavefunctions, simply expanded in powers of the radial coordinate. The newer, and superior, treatment followed here, expands the electron wavefunctions in powers of mass, energy and charge, the coefficients of the expansion being functions of the radial coordinate dependent on the charge distribution. The correction terms resulting from the earlier method are slightly different from those just presented because, in particular, the newer treatment automatically entrains certain higher powers of the radial coordinate.

With identical restrictions upon the lepton quantum numbers the older treatment has

$$^{V}C_0 = -9(\alpha Z)^2/20 - (W_0R)^2/5 - W_0R\alpha Z/5 + R^2(1 - 2\gamma/3)/5,$$

$$^{V}C_1 = -2R\alpha Z/5 + 4W_0R^2/15,$$

$$^{V}C_{-1} = 2\gamma W_0R^2/15,$$

$$^{V}C_2 = -4R^2/15;$$

$$^{A}C_0 = -9(\alpha Z)^2/20 - (W_0R)^2/5 + W_0R\alpha Z/15 + R^2/5,$$

$$^{A}C_1 = -2R\alpha Z/3 + 4W_0R^2/9,$$

$$^{A}C_2 = -4R^2/9.$$

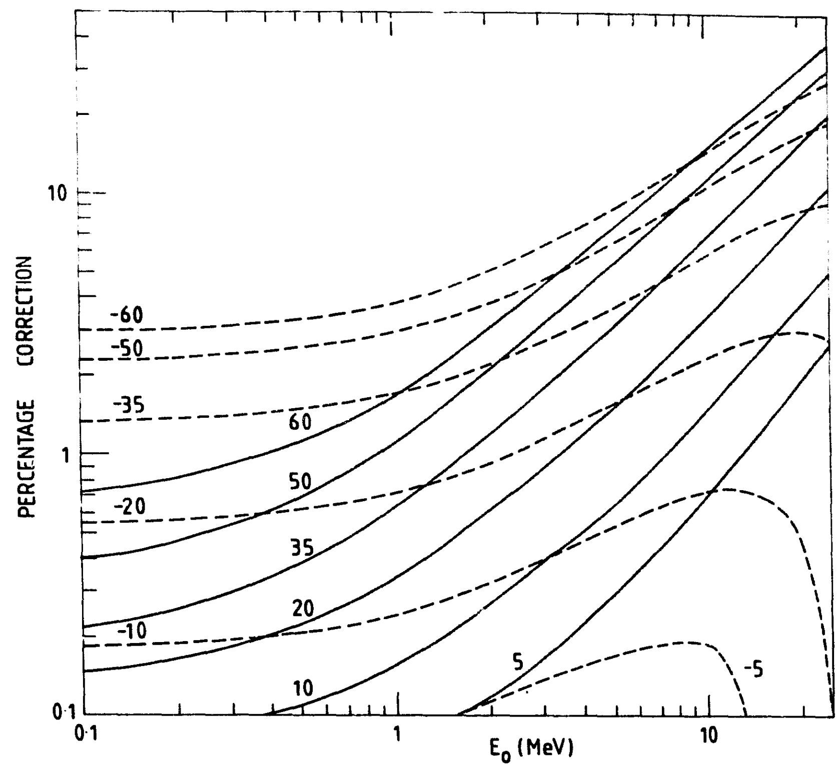

Fig. 2. Correction to the rate of allowed vector beta-decay resulting from the finite nuclear size effects (variation of the lepton wavefunctions through the nuclear volume plus their convolution with the nucleonic wavefunctions). The percentage correction is defined as $100(f_F/f-1)$ where f_F is the integrated phase-space factor allowing for the finite nuclear size effects and f is the traditional factor for a point nuclear charge. The numbers on the curves, which are solid for electrons and dashed for positrons, are the Z-values. The correction is negative for electrons and positive for positrons; it is here displayed for the floor of the stability valley as defined by $r_0 = 1.2$ fm.

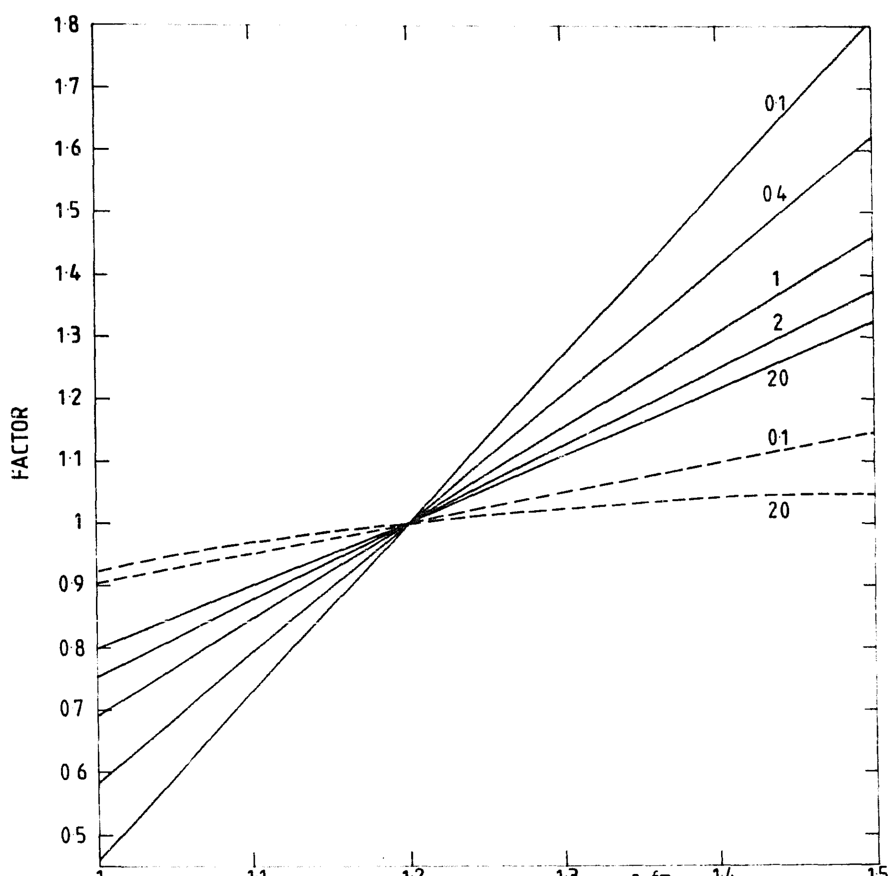

Fig. 3. The variation of the finite nuclear size correction – for a given Z, A – with the nuclear size. "Factor" is the correction (as defined in fig. 2) for a given r_0 divided by that for $r_0 = 1.2$ fm. The numbers on the lines are E_0; the solid lines are for electrons and the dashed for positrons. (Vector beta-decay for $Z = 30$; $A = 67$.)

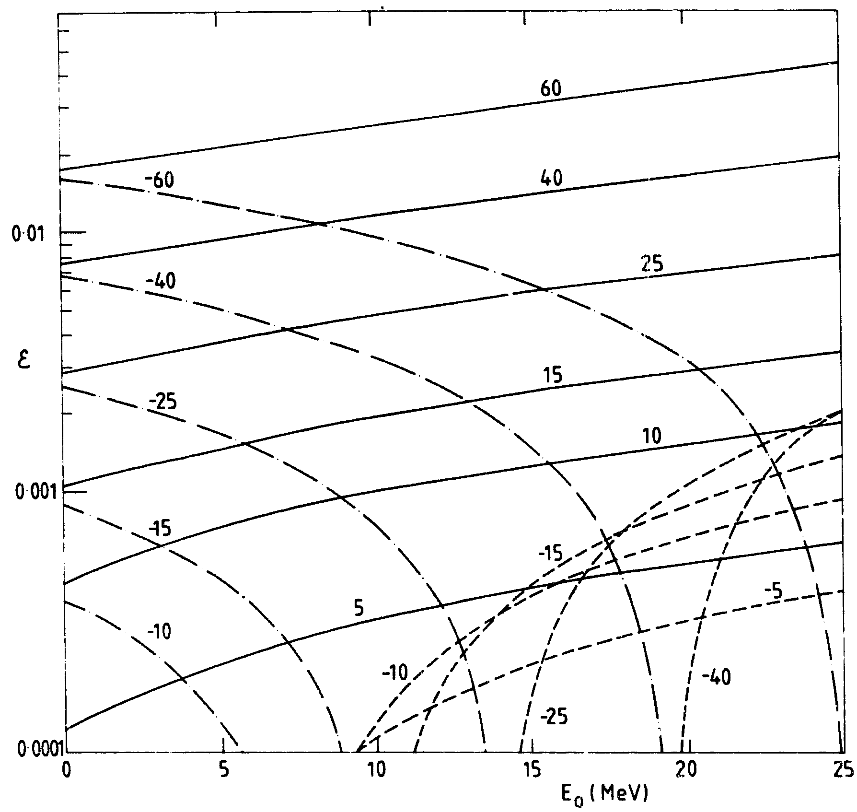

Fig. 4. Comparison of the "new" and "old" finite nuclear size corrections as defined in the text. $1+\epsilon$ is the factor by which the "new" correction (as defined in fig. 2) exceeds the "old". The Z-values are as on the curves. The solid curves are for electrons for which ϵ is positive; for positrons ϵ is positive for the dash-dotted curves and negative for the dashed curves.

We are now in a position analytically to evaluate the integrated phase-space factor f_F where the subscript F stands for finite size, viz.

$$f_F = [2/(1+\gamma)]\int_0^{p_0} p^2(W_0 - W)^2 F(Z, W) \times L_0(Z, W) C(Z, W)\,\mathrm{d}p. \quad (3)$$

This may be done straightforwardly by application of the methods detailed in I. This is, however, exceedingly cumbersome; the results may be presented more simply: Remove $F(Z, W)$ from the above eq. (3) for f_F and call the resultant integral, which may be evaluated in elementary fashion, f'_F. Then

$$f_F/f = (1+\epsilon) f'_F/f(0),$$

where

$$f(0) = \int_0^{p_0} p^2(W_0 - W)^2\,\mathrm{d}p.$$

Empirically, for electron emission:

$${}^{V}\epsilon \simeq A(1 - \exp[E_0/B]) + CE_0,$$

$$A = 1.267 \times 10^{-7} \times Z^{2.21},$$

$$B = 0.252 + 8.40 \times 10^{-3} \times Z,$$

$$C = 2.935 \times 10^{-11} \times Z^{4.00}.$$

For positron emission:

$${}^{V}\epsilon \simeq D(1 - \exp[E_0/F]) + GE_0 + HE_0^2,$$

$$D = -2.13 \times 10^{-8} \times |Z|^{2.82},$$

$$F = 0.396 + 2.67 \times 10^{-3} \times |Z|,$$

$$G = 5.69 \times 10^{-10} \times |Z|^{3.35},$$

$$H = -7.82 \times 10^{-12} \times |Z|^{3.23}.$$

For electrons:

$${}^{A}\epsilon/{}^{V}\epsilon \simeq 1.159 - 8.25 \times 10^{-3} \times \ln E_0.$$

For positrons:

$${}^{A}\epsilon/{}^{V}\epsilon = 1.194 - 7.50 \times 10^{-3} \times \ln E_0.$$

The f_F/f values, derived following these recipes, everywhere better an accuracy of 1 part in 10^4 for $|Z| \le 60$ and for $0.1 \le E_0 \le 25$.

Fig. 2 displays the magnitude of the correction resulting from the combination of the finite nuclear size and convolution effects for vector transitions on the floor of the stability valley defined by $r_0 = 1.2$ fm. The correction depends on the nuclear size for a given Z differently for electrons and positrons; fig. 3 shows this for $Z = 30$.

We finally, in fig. 4, illustrate the sensitivity of the correction to the choice of $C(Z, W)$ as between the newer and older versions referred to and quoted above; we see that although the difference can become quite large for the larger Z, E values with which we have concerned ourselves it is quite small for the region of the periodic table within which high interest resides in respect of the testing of CVC and the determination of the vector coupling constant, viz. positron emitters of $|Z| \leq 26$, $E_0 \leq 7$. Thus, for ^{54}Co decay the correction is of about 2.5% whereas the difference between the corrections is about 0.26% which is small but not negligible.

References

[1] D.H. Wilkinson, Nucl. Instr. and Meth. A275 (1989) 378.
[2] H. Behrens and W. Bühring, Electron Radial Wave Functions and Nuclear Beta-decay (Clarendon, Oxford, 1982).
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