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ELSEVIER

Evaluation of beta-decay Part IV. The complex gamma function; practicalities ☆

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Abstract

Two easily-evaluated approximations for the square of the modulus of the complex gamma-function as it appears in $F(Z, W)$, the Fermi function for beta-decay, are presented, one for high and the other for low electron energies. When grafted together at the appropriate electron energy as a function of Z the greatest error increases from less than $10^{-5}\%$ for low Z and reaches $10^{-2}\%$ only at $Z \simeq 90$.

1. Introduction [1]

The immediately-preceding paper in this series [1] presented two elementary methods by which the square of the modulus of the complex gamma-function, as it appears in the Fermi function $F(Z, W)$ for beta-decay, might be evaluated with high accuracy throughout the periodic table and for all energies of electrons.

The first of these methods was based upon:

$$\begin{aligned}\ln|\Gamma(\gamma+iy)|^2 &\simeq \sum_{n=0}^{N-1}\ln\left[\frac{n^2+y_1^2}{(n+\gamma)^2+y^2}\right]+\ln\left[\frac{\pi}{y_1\sinh\pi y_1}\right]\\&+\ln\left(N^2+y_1^2\right)+(1-\gamma)\\&\times\left\{2-\ln\left[(N+\gamma)^2+y^2\right]+\frac{2y}{N+\gamma}\arctan\frac{y}{N+\gamma}\right.\\&\left.+\frac{1}{6a}\left[\frac{1}{(N+\gamma)^2+y^2}\right]\right\}\\&-(2N+1)\ln a,\end{aligned}\tag{1}$$

where $y = aZW/p, y_1 = ay, a = (N+1)/(N+\gamma)$, and where N is any integer greater than or equal to unity.

This expression, with $N = 1$, was shown to better an accuracy of $10^{-2}\%$ everywhere except for $Z > 70$ with electron energies greater than about 1 MeV while with $N = 3$ it bettered $10^{-3}\%$ except for approximately the same excluded region of Z and energy. With $N = 2$, $10^{-2}\%$ was bettered everywhere for $Z \leq 100$ while with $N = 4$ the same was true at the $10^{-3}\%$ level.

Eq. (1) therefore yields, by elementary means, accuracy adequate for any practical purpose. It is, however, rather opaque and is completely unsuitable for further algebraic manipulation when combined with the other algebraic functions of Z and of electron energy that enter into the full expressions for beta-decay; this is particularly true if, as is often the case, integration over electron energy is involved. It is therefore desirable to have available a much simpler algebraic expression, of adequate accuracy, more suited to such continued manipulation and that, as is the case for Eq. (1), improves in accuracy as the electron energy tends to zero. To provide such an expression is the first object of this paper.

The second of the methods presented in Part III [1] was based upon the expansion:

$$\Gamma(1+z) - -\gamma_E z + \sum_{n=2}^{\infty}(-)^n\zeta(n)z^n/n,\tag{2}$$

where γ_E is the Euler gamma, and $\zeta(n)$ the Riemann zeta function and remembering that:

$$\mathscr{R}\ln\Gamma(z) = \ln|\Gamma(z)|.\tag{3}$$

On application of Eq. (3) to Eq. (2) we have:

$$|\Gamma(\gamma+i\alpha ZW/p)|^2 \simeq \sum_{b\leq a} Z_a K_b(\alpha Z)^a(W/p)^b.\tag{4}$$

This expression is well-suited to algebraic manipulation and, for electron energies in excess of 1 MeV or so, has an

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[1] The nomenclature is as in the previous papers in this series.

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Table 1
The coefficients $A(j, k)$ for the product expansion as given in Eq. (12), $a[b] = a \times 10^b$

j, k	$A(j, k)$
2, 0	−0.57721 56649 [0]
2, 2	0.16449 34066 [1]
4, 0	−0.38894 84701 [0]
4, 2	0.25257 51920 [0]
4, 4	0.81174 24252 [0]
6, 0	−0.89327 87159 [−1]
6, 2	−0.22138 40123 [0]
6, 4	0.47182 61514 [0]
6, 6	0.19075 18240 [0]
8, 0	0.19885 33603 [−2]
8, 2	−0.18189 64119 [0]
8, 4	0.16262 24132 [0]
8, 6	0.16832 71329 [0]
8, 8	0.26147 84779 [−1]
10, 0	0.13601 54565 [−1]
10, 2	−0.10264 03354 [0]
10, 4	0.51044 96304 [−1]
10, 6	0.10090 33523 [0]
10, 8	0.29143 03361 [−1]
10, 10	0.23460 81030 [−2]
12, 0	0.10871 41989 [−1]
12, 2	−0.59211 34356 [−1]
12, 4	0.20438 60507 [−1]
12, 6	0.58475 30733 [−1]
12, 8	0.21895 77027 [−1]
12, 10	0.30574 81398 [−2]
12, 12	0.14842 87924 [−3]
14, 0	0.45882 67607 [−2]
14, 2	−0.37410 26726 [−1]
14, 4	0.11762 07123 [−1]
14, 6	0.36918 37772 [−1]
14, 8	0.15160 58663 [−1]
14, 10	0.26381 68399 [−2]
14, 12	0.21702 05763 [−3]
14, 14	0.69758 73604 [−5]
16, 0	0.12688 01642 [−1]
16, 2	−0.30413 29897 [−1]
16, 4	0.88225 79939 [−2]
16, 6	0.25641 63038 [−1]
16, 8	0.10726 91836 [−1]
16, 10	0.20305 00528 [−2]
16, 12	0.20662 38473 [−3]
16, 14	0.11156 62659 [−4]
16, 16	0.25312 17368 [−6]
18, 0	−0.11898 83670 [−1]
18, 2	−0.97170 36422 [−2]
18, 4	0.52080 70127 [−2]
18, 6	0.19211 69028 [−1]
18, 8	0.79489 32754 [−2]
18, 10	0.15434 03358 [−2]
18, 12	0.17148 32722 [−3]
18, 14	0.11456 61890 [−4]
18, 16	0.43537 38308 [−6]
18, 18	0.73047 11653 [−8]
20, 0	0.15614 26935 [−1]
20, 2	−0.33818 35675 [−1]
20, 4	0.10288 49148 [−1]
20, 6	0.14666 19037 [−1]
20, 8	0.61629 23560 [−2]
20, 10	0.11974 12912 [−2]
20, 12	0.13745 84861 [−3]
20, 14	0.10080 10973 [−4]
20, 16	0.47519 16185 [−6]
20, 18	0.13351 23213 [−7]
20, 20	0.17165 38413 [−9]

accuracy bettering 10^{-2}% for modest values of cut-off for the power a of αZ, ranging from $a = 2$ for $Z = 10$ to $a = 10$ for $Z = 100$. Unfortunately Eq. (2) is valid only for $|z| < 1$, i.e. for

$$W > [\gamma(2-\gamma)/(2\gamma-1)]^{1/2}. \quad (5)$$

This limit is as high as about 0.3 MeV for $Z = 100$ and so severely curtails the useful range of applicability of this method as was demonstrated quantitatively in Part III [1]. To provide an alternative to Eq. (4), also in the form of an expansion in powers of αZ and of W/p, but of adequate accuracy to much lower values of electron energy, is the second object of this paper.

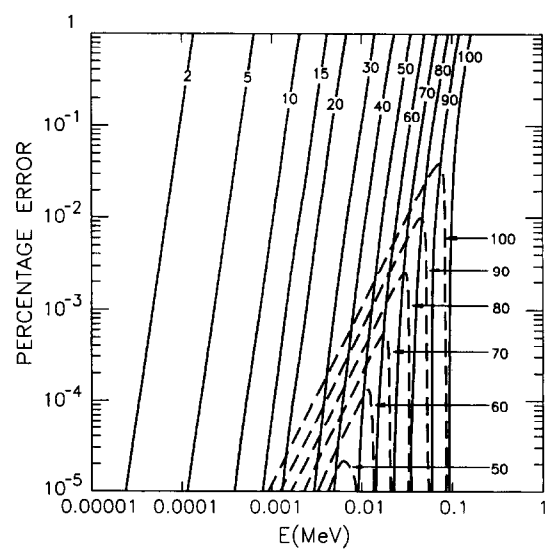

Fig. 1. The accuracy of the low-energy approximation of Eq. (7) defined as 100×(approx./exact-1). The solid lines are for negative errors and the dashed are for positive.

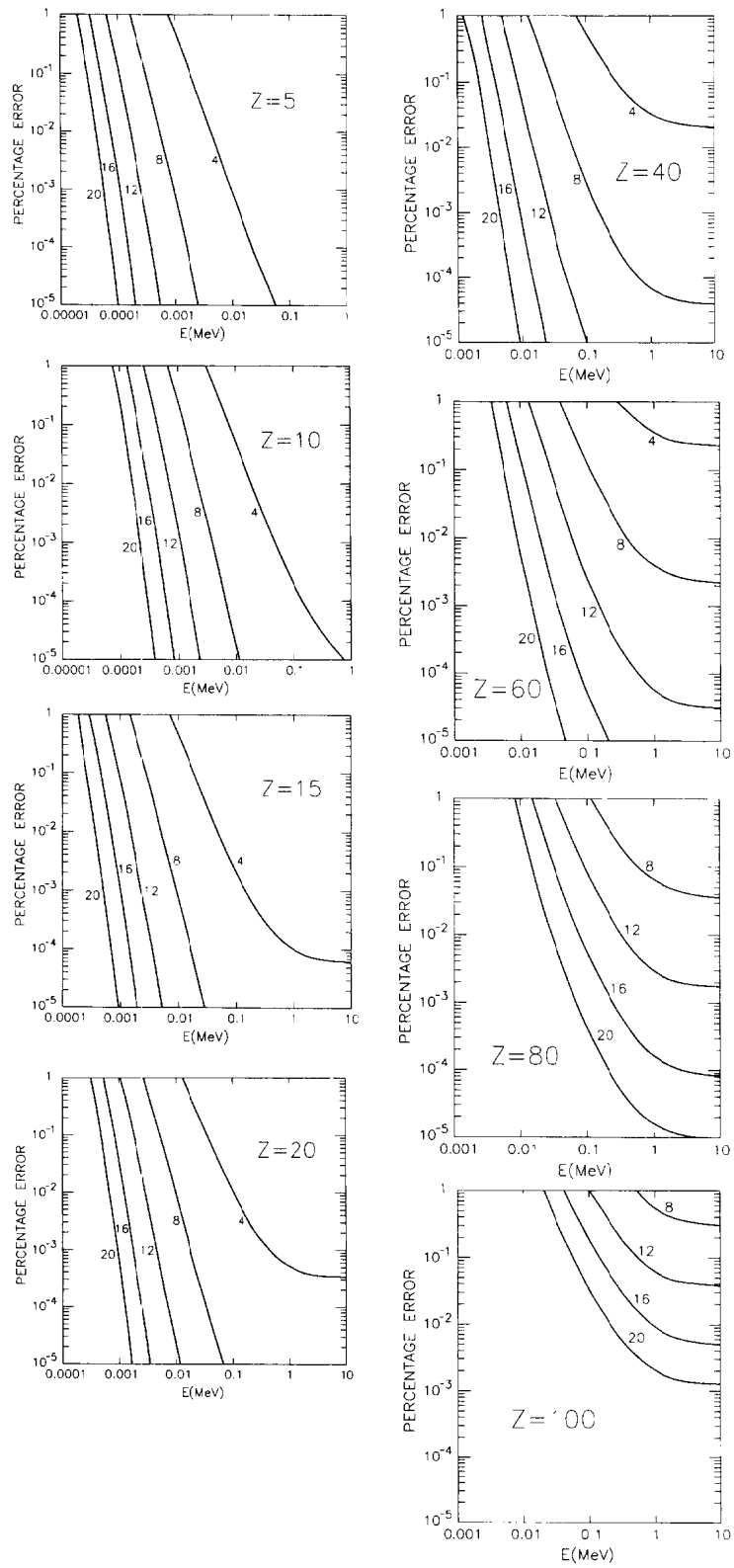

Fig. 2. The accuracy of the high-energy approximation of Eq. (12) and Table 1 defined as for Fig. 1; all errors are positive. The numbers labelling the curves give the values of j through which Eq. (12) is evaluated.

2. The first expression

A simple expression of increasing accuracy as the electron energy tends to zero exploits the fact that:

$$|\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh \pi y}. \tag{6}$$

For finite Z we may make a Taylor expansion based on Eq. (6) using standard properties of the polygamma function.

Carrying the expansion to appropriate order for our present purposes we find, writing $K = 1/y^2$; $G = \gamma - 1$:

$$|\Gamma(\gamma+iy)|^2 \simeq \frac{\pi y}{K^G \sinh \pi y} \exp\{GK(A + BG + CG^2)\}, \tag{7}$$

where

$$A = \frac{1}{6}\left(1 + \frac{K}{10} + \frac{K^2}{21} + \frac{K^3}{20} + \frac{K^4}{11} + \frac{691K^5}{2730} + K^6\right), \tag{8}$$

$$B = \frac{1}{2}, \tag{9}$$

$$C = \frac{1}{3}\left(1 - \frac{K}{2} - \frac{K^2}{6} - \frac{K^3}{6} - \frac{3K^4}{10} - \frac{5K^5}{6}\right). \tag{10}$$

The accuracy of Eq. (7) is displayed in Fig. 1.

3. The second expression

To avoid the restriction in electron energy, as given by Eq. (5), involved in the use of Eq. (2) we may, of course, use the standard series expansion for $1/\Gamma(z)$ which is valid for all z (see e.g. 6.1.34 of Ref. [2]).

This, however, is rather awkward for our present application; it is more convenient to use the standard expression that gives us directly the desired square of the modulus, viz.

$$|\Gamma(\gamma+iy)|^{-2} = [\Gamma(\gamma)]^{-2} \prod_{n=0}^{\infty}\left[1 + \frac{y^2}{(\gamma+n)^2}\right], \tag{11}$$

which is also valid without restriction.

The result of the expansion is presented in Table 1 in the form:

$$|\Gamma(\gamma+iy)|^{-2} = 1 + \sum_{k\le j} A(j,k)(\alpha Z)^j (W/p)^k. \tag{12}$$

In generating Table 1:
(i) the product in Eq. (11) was directly evaluated algebraically through $n = 100$ and through $(\alpha Z)^{20}$;
(ii) the logarithm of the product from $n = 101$ to infinity was summed by expanding its individual elements through $(\alpha Z)^{20}$, then using the appropriate zeta-functions abated by numerical evaluation of the various inverse-power series through $n = 100$;
(iii) the exponential of the result of (ii) was taken through $(\alpha Z)^{20}$;
(iv) an empirical polynomial expansion was made of $[\Gamma(\gamma)]^{-2}$ through $(\alpha Z)^{20}$;
(v) the product of the output of steps in (i), (iii) and (iv), through $(\alpha Z)^{20}$, gives Eq. (12).

The accuracy of Eq. (12) using Table 1 is displayed in Fig. 2 for various cut-off values of j. By comparing Fig. 2 with Figs. 6 and 7 of Part III [1] we see that for the same values of a, j comparable accuracy is achieved for the higher electron energies but that for the lower electron energies the present product expansion is, as expected, distinctly superior. Specifically, for expansions through $(\alpha Z)^{20}$, the electron energy down to which an accuracy of 10^{-2}% or better is achieved is lower for the present product expansion than for the series expansion of Part III [1] by factors of several for the higher Z-values increasing to factors of 10–20 over the greater part of the periodic table; for an accuracy of 1% the corresponding factors are 10–15 for the highest Z-values increasing to about 30 for the lowest.

As we have just seen, the range of electron energy over which useful accuracy is offered is very much greater for the present product expansion than for the series expansion if Part III [1]; to that degree it is distinctly superior. However, since it gives the inverse squared modulus in

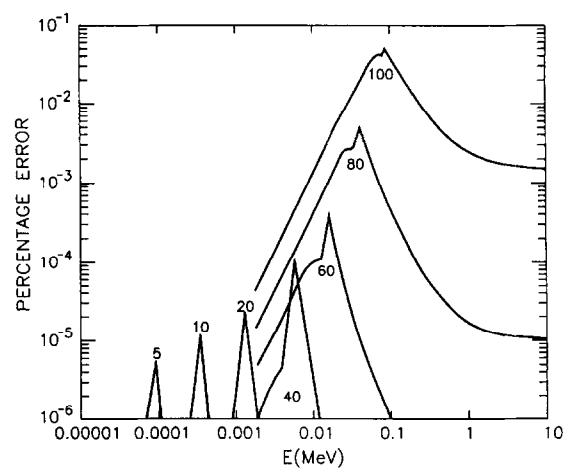

Fig. 3. The (absolute values of) the percentage error, for the Z-values shown, when the low-energy approximation of Eq. (7) is used below the grafting energy E_g and the high-energy approximation of Eq. (12) and Table 1 (through $j = 20$) is used above E_g, E_g being given by Eq. (13). The form of the envelopes for the higher Z-values is due to the change of sign of the error of the low-energy approximation as is seen in Fig. 1; the envelopes are conservative.

terms of power of αZ and of W/p it is not so convenient for further manipulation (although it can, of course, be inverted if need be).

4. The practical combination

As we see by comparing Fig. 1 with Fig. 3, the accuracy of the first expression, Eq. (7), increases to lower electron energy while the opposite is true for the second expression, Eq. (12). The question now arises as to whether we can effect a grafting of the two expressions at some grafting energy of electrons, E_g, at which the accuracies of the two expressions are equal, using Eq. (7) for $E < E_g$ and Eq. (12) for $E > E_g$, such that the resultant overall accuracy remains acceptable throughout the periodic table for all energies of electrons.

Since, as is clear from the figures, the accuracy of the two expressions deteriorates rapidly, above E_g for the first and below E_g for the second, the determination of E_g for practical application is a delicate matter. We find:

$$E_g = \exp\{-15.71904 + 9.094318x - 5.7438059x^2 + 2.243834x^3 - 0.43174199x^4 + 0.033036534x^5\}, \quad (13)$$

where: $x = \ln Z$.

References

[1] D.H. Wilkinson, Nucl. Instr. and Meth. A 335 (1993) 305.
[2] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965).