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ANALYSIS OF NEUTRON β-DECAY

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Abstract: The factors determining neutron β-decay are analyzed in detail. The 19 "small" terms entering the lifetime of the neutron within the V, A framework are quantitatively evaluated. The Coulomb effect, the finite mass and the size of the nucleon are all taken into account analytically. To a neutron/hydrogen-atom mass difference of 782.332 ± 0.017 keV corresponds the phase space factor $f^{R} = 1.71465 \pm 0.00015$ where the supercript R implies inclusion of the "outer" radiative corrections through order α^3. Further effects of order α, not included in the present f^{R}, are indicated. Corrections to angular correlations in neutron β-decay due to weak magnetism, recoil and the Coulomb effect are evaluated analytically. The following values are recommended:

$$g_V = (1.41271 \pm 0.00032) \times 10^{-49}\ \mathrm{erg} \cdot \mathrm{cm}^3\,,$$

$$g_A = (1.781 \pm 0.011) \times 10^{-49}\ \mathrm{erg} \cdot \mathrm{cm}^3\,,$$

$$\lambda = 1.2605 \pm 0.0075\,,$$

$$t = 623.6 \pm 6.2\ \mathrm{sec}\,.$$

Here $\lambda = |g_A/g_V|$, and t is the neutron half-life (these g_A and t are not independent).

1. Nomenclature

Natural units ($\hbar = m_e = c = 1$) unless otherwise stated

$\ln = \log_e$,

W = electron total energy (end point = W_0),

$p = (W^2 - 1)^{1/2}$, $\quad p_\nu = W_0 - W$,

$\alpha = e^2/\hbar c = 1/137.036$, $\quad \gamma = \{1 - (\alpha Z)^2\}^{1/2}$,

$y = \alpha Z W/p$, $\quad R$ = nuclear/nucleon radius,

$\bar{W} = W + 3\alpha Z/(2R)$, $\quad \bar{p} = (\bar{W}^2 - 1)^{1/2}$,

$H_2 = -\frac{1}{6}(\bar{p}R)^2$, $\quad N_1 = \frac{1}{3}p_\nu R$,

$N_2 = -\frac{1}{6}(p_\nu R)^2$, $\quad D_1 = \frac{1}{3}\bar{W}R$,

$d_1 = \frac{1}{3}R$, $\quad M$ = nucleon mass ($\simeq 1837$),

$F(Z, W) = 2(1+\gamma)(2pR)^{-2(1-\gamma)}|\Gamma(\gamma + iy)|^2\{\Gamma(2\gamma+1)\}^{-2}\, e^{\pi y}$.

(We here give expressions that include Z for familiarity; as appropriate in the text we particularize to our case of interest: $Z = 1$.)

2. Introduction

Within the assumption that the primary weak currents are exclusively vector and axial in nature and to a good first approximation, the adequacy of which we explore in sects. 8–10, the β-decay of the neutron is determined by the vector and the axial vector weak-coupling constants g_V and g_A†. The vector weak hadronic current appears to be well conserved and so the vector coupling constant g_V may, within certain limits to do with uncertainties as to nuclear structure, be accurately determined from studies of complex nuclei: the $J^\pi = 0^+ \rightarrow J^\pi = 0^+$ "super-allowed Fermi transitions" (appendix 1). However, the axial weak hadronic current is not conserved and so the axial coupling constant g_A could not be accurately extracted from the β-decay of complex nuclei even were we able to repose full confidence in the (nucleonic) nuclear wave functions involved; only for the β-decay of the free neutron does the axial coupling constant suffer no further effective renormalization beyond that associated with the intra-nucleonic strong, electromagnetic and weak structures themselves and only for the free neutron do we have no uncertainty due to complex nuclear wave functions. Free neutron β-decay is, therefore, our sole accurate source of information as to the (electromagnetically renormalized) axial coupling constant for NNeν and so we must investigate and understand it as well as we possibly can.

Neutron β-decay may be used in two chief ways to approach the axial coupling constant:

(i) The neutron half-life is determined by a combination of (the squares of) the vector and axial coupling constants so that if the former constant is taken from "super-allowed Fermi transitions" the latter may be deduced;

(ii) Angular correlations among the products of polarized or unpolarized neutron β-decay reveal the ratio of the axial to vector coupling constants.

We note (appendix 1) that the square of the vector coupling constant is now known to about 0.05%; it appears rather unlikely that this accuracy will be substantially bettered in the near future since the chief uncertainty no longer resides in the experimental data but rather in our theoretical understanding of those nuclear structure dependent effects that, owing to the slight *de facto* lack of charge-independence of the nuclear forces, effectively subvert the perfect conservation of the vector current in the context of complex nuclei by slightly upsetting the precise identity of nuclear wave functions along an isobaric multiplet [1]). We may, however, aim to match this 0.05% accuracy in our corresponding knowledge of the square of the axial coupling constant.

In approach (i) listed above, the extraction of the square of the axial coupling constant from the neutron lifetime is limited by the accuracy of our knowledge of the energy released in neutron beta-decay and by the accuracy of our measurement of the neutron lifetime. As we shall see, the neutron-proton mass difference is now known so accurately that the uncertainty that it reflects into the analysis of the

† General background references for this paper can be found through ref. [1]).

neutron lifetime is only 0.01%. This then implies that we need to understand all those factors that link the energy release and the coupling constants to the lifetime also to an accuracy bettering 0.01%; chief among these factors are the phase space factor and the contributions from (induced) coupling constants other than the vector and axial. At the moment the neutron lifetime is itself known by direct measurement to rather poorer than 1% accuracy (appendix 2) so that care for other contributions to 0.01% is not immediately demanded; however, we may confidently anticipate that the accuracy of the lifetime measurement will considerably improve particularly with the development of confinement techniques and so it is appropriate to sharpen our potential understanding of the lifetime's relationship to the coupling constants to the 0.01% level.

It is the chief object of the present paper to examine the factors entering into the neutron lifetime with a view to moving towards such confidence at the 0.01% level. In doing this we shall examine all terms that might, *prima facie*, contribute at the 0.001% level or which might entrain uncertainties of this order or greater; we shall systematically ignore all couplings and processes contributing below 0.001%. We shall also, as a secondary objective, examine, in appendix 3, the correlation methods of approach (ii) above with a view to supplying corrections to their analysis that have not so far been applied. As a third objective we shall, in appendix 4, recommend values for the various quantities of concern, particularly the axial coupling constant and the neutron lifetime.

As far as possible we shall carry out the analyses analytically so that the contributions of the various terms are made transparent and so as to make easier numerical evaluation for alternative sets of input data.

3. The energy release

The best measurement of the neutron/hydrogen-atom mass difference is, in ordinary units [2]),

$$M_{\mathrm{n}}-M_{\mathrm{H}}=782.332\pm0.017\ \mathrm{keV}\,.$$

This we reduce by 13.6 eV on account of the hydrogen atom electronic binding energy and combine with an electron rest mass of 511.0034 ± 0.0014 keV [ref. [3])], to yield the total energy release in the decay of a neutron into a proton plus a free electron and an electron anti-neutrino (but see sect. 4) of, in natural units:

$$\Delta=2.530946\pm0.000033\,.$$

4. Neutrino mass

All the analyses of this paper will be done under the "standard" assumption of zero mass for the electron neutrino and no mixing of that neutrino with others with or without mass. To allow for a finite mass for the electron neutrino itself is straightforward (appendix 5) but complications due to neutrino mixing could be formidable and we mention these matters no further here, important as they are.

5. Electron end-point

Within the assumptions as to neutrinos in sect. 4 we may write the electron end-point:

$$W_0 = \frac{M_n^2 + 1 - M_p^2}{2M_n}$$

$$= \Delta - \frac{\Delta^2 - 1}{2M_n},$$

which, using sect. 3, with $M_n = 939.5731$ MeV [ref. [3]], gives

$$W_0 = 2.529476 \pm 0.000033 .$$

6. Terms in neutron β-decay

To discuss the total rate for neutron β-decay we accept that only vector and axial currents operate and that the current-current representation is totally valid noting that both these assumptions are susceptible of challenge. We also accept that time-reversal invariance holds to an appropriate degree for the processes that we consider [1]†.

We therefore write the weak interaction energy density (in the absence of electromagnetic intervention) as:

$$H^{\mathrm{weak}} = (g_\beta/\sqrt{2}) J_\mu L_\mu ,$$

where, in standard notation,

$$L_\mu = \bar{\psi}_e \gamma_\mu (1 + \gamma_5) \psi_\nu$$

and, to first order in momentum transfer,

$$J_\mu = \bar{\psi}_N \{ \gamma_\mu + i g_{\mathrm{WM}} \sigma_{\mu\nu} k_\nu + g_{\mathrm{IS}} k_\mu + \gamma_5 (\lambda \gamma_\mu + i g_{\mathrm{IT}} \sigma_{\mu\nu} k_\nu + g_{\mathrm{IP}} k_\mu) \} \tau_\pm \psi_N . \qquad (1)$$

In (1) the first three terms in the curly brackets are the vector current and the second three terms are the axial current so that, in particular, λ is the ratio of the numerical values of the axial to the vector weak-coupling constants.

7. The first-order Coulomb effect

But for the final-state interaction due to the Coulomb force between the proton and the departing electron the interference terms between the vector and axial

† Ref. [3]) gives $180.11° \pm 0.17°$ for the g_A/g_V phase angle.

currents, integrated across the electron spectrum, would be identically zero so that there would be no contribution to the total decay rate from $g_V g_A$, $g_A g_{WM}$ etc. [4]). However, the Coulomb force entrains such interference terms in the total rate so that we must consider them in addition to such terms as $g_V g_{WM}$ and $g_A g_{IP}$ that involve only the one or the other weak hadronic current and that therefore contribute to the total decay rate in the absence of final-state interactions.

To evaluate these Coulomb-entrained interference terms it is adequate to use only the first W-dependent term in the α-expansion of $F(Z, W)^{\dagger}$ viz. $\pi\alpha ZW/p$ (see subsect. 15.2).

8. The small terms

Our first task, in discussing the β-decay of the free neutron, is to evaluate the contributions of the induced terms in g_{WM} (weak magnetism), g_{IS} (induced scalar), g_{IT} (induced tensor) and g_{IP} (induced pseudoscalar), relative to the bulk of the decay due to the vector coupling constant g_V which is associated with the first, γ_μ, term in eq. (1) for J_μ and therefore equal to unity (multiplied by g_β in absolute terms) and due to the axial coupling constant g_A which is associated with the $\gamma_5\gamma_\mu$ term in (1) and therefore numerically equal to λg_β where $\lambda \simeq 1.26$ (appendix 4).

In addition to the induced terms we must also consider the cross term $g_V g_A$ that, as remarked in sect. 7, is entrained by the final-state Coulomb interaction. This term plus the induced terms we refer to collectively as the "small terms".

9. The induced coupling constants

Before we evaluate the small terms we must adopt values for the relevant induced coupling constants that appear in J_μ of eq. (1).

9.1. THE CONSERVED VECTOR CURRENT

We assume full "strong" conservation of the vector current which implies $g_{IS} \equiv 0$ and disposes of that term; it also implies

$$g_{WM} = (\mu_p^a - \mu_n^a)/2M = \mu^a/2M \simeq 3.71/2M \simeq 1.01 \times 10^{-3}\,,$$

where μ_p^a and μ_n^a are the proton and neutron anomalous magnetic moments respectively.

9.2. THE SECOND-CLASS COUPLING CONSTANT g_{IT}

The second-class coupling constant g_{IT} poses a special problem: it is quite likely that g_{IT} is zero because second-class currents cannot be satisfactorily accommodated

† We are here concerned only with $Z = 1$ but shall frequently retain the Z in formulae where that seems appropriate to make them look more familiar.

within the successful modern gauge schemes that unify the weak and electromagnetic interactions [1]; however, all that can be said experimentally with 90% confidence is that $g_{IT} < 4 \times 10^{-4}$ [ref. [5]]. We use this limit below but note that the associated contributions to the decay rate are probably much smaller than those resulting from use of this limit.

9.3. THE INDUCED PSEUDO-SCALAR CONSTANT g_{IP}

We use the canonical value:

$$g_{IP} = 2g_A M/M_\pi^2 \simeq 0.063 .$$

9.4. $g_V g_A$ INTERFERENCE

We have noted in sect. 7 that $g_V g_A$ interference makes a finite contribution to the total decay rate through the final-state interaction of the Coulomb force. As we see in table 1 and sect. 10 this term has the same form as the largest term associated with weak magnetism, namely that in $g_A g_{WM}$, and is of numerical value unity in relation to the $\mu^a = \mu_p^a - \mu_n^a$ of the weak magnetism term; the sum of the $g_V g_A$ and $g_A g_{WM}$ terms can therefore be written as proportional to $\mu_p - \mu_n$ where μ_p and μ_n are the *total* proton and neutron magnetic moments. This lumping together may cause confusion unless one remains aware of the different physical origins of the component parts.

10. The spectra and numerical values for the small terms

Table 1 displays the 19 small terms as the spectrum factor $X(W, W_0)$ that multiplies the allowed phase space factor to give the differential energy spectrum of the term in question. Only terms of lowest relevant order are listed; the convenient compilation of Bender *et al.* [6]) has been used in deriving them. In the case of the cross terms between the vector and axial hadronic currents we do not display the first-order term in the differential energy spectrum but rather that of lowest order relevant for the total decay rate namely that brought in by the first-order Coulomb term in $F(Z, W)$, as discussed in sect. 7; this because the lowest order α-independent term in the energy differential spectrum vanishes identically on integration across the spectrum without final-state interaction [4]).

The column "Contribution, C_{ab}" of table 1 is the relative contribution of the term in question to the total decay rate viz.

$$C_{ab} = \frac{g_a g_b \int_1^{W_0} pW(W_0 - W)^2 X_{ab}(W, W_0)\,dW}{(1+3\lambda^2)\int_1^{W_0} pW(W_0 - W)^2 F(Z, W)\,dW} .$$

TABLE 1

The small terms to lowest order, in neutron β-decay as discussed in sect. 8

Term, $g_a g_b$	Spectrum factor, $X_{ab}(W, W_0)$	Contribution, C_{ab}
g_{WM}^2	$2W_0^2-\frac{14}{3}+\frac{8}{3}\frac{W_0}{W}-\frac{20}{3}W_0W+\frac{20}{3}W^2$	5.6×10^{-7}
g_{IT}^2	$2W_0^2-\frac{1}{3}+\frac{4}{3}\frac{W_0}{W}-\frac{4}{3}W_0W+\frac{4}{3}W^2$	$<3.5\times10^{-7}$
g_{IS}^2	1	0
g_{IP}^2	$\frac{1}{M^2}\left(\frac{1}{4}W_0^2-\frac{5}{12}+\frac{1}{6}\frac{W_0}{W}-\frac{2}{3}W_0W+\frac{2}{3}W^2\right)$	1.1×10^{-10}
$g_V g_{WM}$	$\frac{1}{M}\left(2W_0^2-\frac{13}{3}+\frac{7}{3}\frac{W_0}{W}-\frac{16}{3}W_0W+\frac{16}{3}W^2\right)$	4.6×10^{-7}
$g_A g_{IT}$	$4W_0+\frac{2}{W}$	$<9.7\times10^{-4}$
$g_V g_{IS}$	$\frac{2}{W}$	0
$g_A g_{IP}$	$\frac{1}{M}\left(-1+\frac{W_0}{W}\right)$	4.7×10^{-6}
$g_{WM} g_{IS}$	$\frac{1}{M}\left(W_0+\frac{1}{W}-2W\right)$	0
$g_{IT} g_{IP}$	$\frac{1}{M}\left(W_0+\frac{1}{W}-2W\right)$	$<4.8\times10^{-13}$
$g_V g_A$	$\frac{2\pi\alpha}{Mp}(1+W_0W-2W^2)$	1.5×10^{-6}
$g_{WM} g_{IT}$	$\frac{4\pi\alpha W_0}{p}(1+W_0W-2W^2)$	$<4.4\times10^{-9}$
$g_{IS} g_{IP}$	0	0
$g_V g_{IT}$	$\frac{2\pi\alpha W_0}{Mp}(1+W_0W-2W^2)$	$<1.2\times10^{-9}$
$g_A g_{WM}$	$\frac{4\pi\alpha}{p}(1+W_0W-2W^2)$	5.5×10^{-6}
$g_V g_{IP}$	0	0
$g_A g_{IS}$	0	0
$g_{WM} g_{IP}$	0	0
$g_{IT} g_{IS}$	0	0

The respective partial decay rates due to the various small terms are $g_a g_b \int_1^{W_0} pW(W_0-W)^2 X_{ab}(W, W_0)\,dW$; the column "Contribution, C_{ab}" gives these values divided by the total decay rate which is taken as $(1+3\lambda^2)\int_1^{W_0} pW(W_0-W)^2F(Z, W)\,dW$ as in sect. 10. Where a zero is entered under "Spectrum factor" it means that this cross term vanishes identically for all W. The values of the coupling constants are taken from sect. 9, with $g_V=1$, $g_A=1.26$. The integrals across the spectra are taken from appendix 6. For the cross terms between the vector and axial currents the spectrum factor shown is the lowest that does not vanish on integration across the spectrum.

The various necessary integrals involved in evaluating this expression are listed in appendix 6 (table 1 has no regard for the relative *sign* of the various terms).

As remarked in sect. 2 our present objective is to discuss the decay rate to the level of 0.001% and we shall therefore systematically ignore contributions below this level. As seen from table 1 all the small terms may be so ignored with the possible exception of that in $g_A g_{IT}$. Since, as remarked in sect. 9, it is likely that g_{IT}, if finite at all, is substantially less than the present experimental limit entering into table 1 it seems best at this stage to set $g_A g_{IT}$ on one side and await developments.

11. Neutron decay to a bound hydrogen atom

We have now examined, and eliminated from further consideration at the 0.001% level, all factors other than those depending on g_V^2 and g_A^2. There is, however, one more process that must be examined before proceeding: the β-decay of the neutron not into the proton-electron continuum but rather to bound states of the hydrogen atom. To an adequate approximation this branching ratio (including relative branches of about 20% to excited bound states) is [7])

$$2\pi\alpha^3\zeta(3)(W_0-1)^2/f \simeq 4.1\times10^{-6}$$

(ζ is the Riemann zeta function). This effect is, therefore, also negligible at our level of present enquiry although it would, of course, be interesting to study experimentally in its own right.

12. The decay rate

We may now approach our evaluation of the decay rate knowing that it depends effectively only on g_V^2 and g_A^2. These coupling constants determine the half-life, t, through the phase space factor f. Begging an important question we write, using ordinary units for this equation,

$$f^R t = \frac{2\pi^3\hbar^7 \ln 2}{m^5c^4\{g_V^{R2}+3g_A^{R2}\}} = \frac{1.23062\times10^{-94}}{g_V^{R2}+3g_A^{R2}}\,\mathrm{erg}^2\cdot\mathrm{cm}^6\cdot\mathrm{sec}\,. \tag{2}$$

We have written f and the coupling constants with superscripts R to emphasize the role of the radiative corrections which for present purposes we take as: (i) renormalizing the coupling constants in an energy-independent way (the "inner" radiative corrections) that is determined by nucleon (quark/QCD) structure and the intermediate vector boson anatomy of the β-decay process and (ii) contributing to the phase space in an energy-dependent way (the "outer" radiative corrections). g_V^R and g_A^R are, of course, the electromagnetically renormalized weak-coupling constants as normally encountered since the inner radiative corrections are always present; we should be interested in the underlying, unrenormalized, constants only if we wished to enquire about the renormalization itself [1]). We are here interested only in the effective, overall, coupling constants and sò, having inserted the supercripts R on the coupling constants in the above equation to remind ourselves of this inner renormalization we will drop them henceforth and write just g_A and g_V as usual. However, we cannot similarly drop the R on f: the "outer" radiative corrections must be explicitly evaluated as we do in subsects. 15.7 and 16.6.

The important question that we have begged in writing (2) is whether the phase space factor is the same for the vector and the axial components of the decay; this question we explicitly address at appropriate points in what follows; the conclusions are summarized in sect. 17.

13. The philosophy

When we ask questions at the 0.001% level it is not adequate to treat the nucleon as structureless. Indeed we know that the nucleon has a finite size and has an underlying quark structure. We do not yet know the details of the relationship between the quarks and the overall electromagnetic and weak structures, in particular the spatial distribution of the quarks in relation to the whole – where "whole" means different things depending upon the question asked.

In the face of these uncertainties we carry out our present evaluation in terms of a very simple model of the nucleon namely a uniform sphere of radius R, the uniformity being the same in respect of the proton's charge distribution and in respect of the distribution of the seat of β-decay. In other words our philosophy is to treat the nucleon as a very simple sort of "little nucleus" in order to determine the possible sensitivity of its β-decay to its dimensions. We should note that there is no fundamental basis for this assumption; in particular it could well be that the charge and weak "radii" for the nucleon are significantly different; it would be easy to effect our evaluations for two differing charge and weak radii but this complication, with no clear guidance as to the form it might best take, would certainly not be justified at this stage.

The charge radius of the proton is $\langle r_E^2\rangle^{1/2} = 0.90 \pm 0.05$ fm while its magnetic radius is $\langle r_M^2\rangle^{1/2} = 0.83 \pm 0.07$ fm [ref. [8])]. We may use $\langle r^2\rangle^{1/2} = 0.87$ fm for our present purposes, corresponding to $R = 1.12$ fm. The "matter radius" is probably smaller than this by 10% or so and we shall therefore use $R = 1.0$ fm for our illustrations. We shall have a brief word later (subsect. 16.4) about the assumption of uniformity in the electromagnetic context.

It will be recognized that by adopting such a model for the nucleon we shall automatically incorporate in the calculations the momentum-transfer dependence of the weak coupling constants so that what emerges will be the constants appropriate to zero momentum transfer.

We must also recognize that if we were to take our "little nucleus" model of the nucleon seriously we should introduce into our consideration certain terms, such as the relativistic $\int i(\alpha \cdot r)/R$, now in respect of the quarks, that evidently have no concern for an "elementary particle" nucleon. Since we do not know how to address these problems we ignore them.

14. The phase space factor

We write the phase space factor as

$$f^{\mathrm{R}} = \int_1^{W_0} S(W, W_0, M) F_0(Z, W) L_0(Z, W) Q(Z, W, M) C(Z, W) \times G(W, W_0) J(Z) K(\alpha)\, \mathrm{d}W. \tag{3}$$

The terms are as follows:

$\underline{S(W, W_0, M)}$. This is the phase space for a point nucleon in the absence of the Coulomb force; for an infinitely massive nucleon it would take the familiar form $pW(W_0 - W)^2$ and so we write

$$S(W, W_0, M) = pW(W_0 - W)^2\{1 + R(W, W_0, M)\},$$

where $R(W, W_0, M)$ is the recoil correction due to the finite nucleon mass carried to the order appropriate. (It is at this point that we meet our first distinction between the vector and axial terms since this recoil correction obviously depends on the electron/anti-neutrino correlation which is different in the vector and axial cases.)

$\underline{F_0(Z, W)}$. The familiar classical Fermi function $F(Z, W)$ – see sect. 1 – represents the distortion of the outgoing electron wave by the Coulomb field of a residual point nucleus. Owing to the weak divergence at the origin it used to be conventional to evaluate the electron wave function at the "nuclear radius" R which is what $F(Z, W)$ does. However, we nowadays determine the electron wave function by solving the Dirac equation for an electron moving under the influence of a spatially finite nuclear charge distribution which, in our present model, is a uniformly charged sphere of radius R. $F_0(Z, W) = 2F(Z, W)/(1+\gamma)$ now becomes the more appropriate form of the Fermi function.

$\underline{L_0(Z, W)}$. Having moved to the solution of the Dirac equation for an extended charge distribution there is no longer a divergence at the origin which becomes the more logical place for the evaluation of the electron wave function rather than the nucleon radius R which has no special relevance for the physics of the matter; $L_0(Z, W)$ effects this transfer from R to the origin.

$\underline{Q(Z, W, M)}$. The Fermi function $F(Z, W)$ is defined for an infinitely massive nucleus so that the $1/r$ Coulomb potential within which the retreating electron moves is given by the charge Z fixed at the origin. However, the nucleus recoils from the emission of the leptons so that the Coulomb field that distorts the electron wave comes from a moving source; $Q(Z, W, M)$ takes account of this.

$\underline{C(Z, W)}$. The electron wave function, evaluated at the centre of the nucleon through F_0L_0, is not constant through the nucleon volume, nor is the anti-neutrino wave function. The decay rate must therefore be found by an appropriate convolution, through the nucleon volume, of the electron, anti-neutrino and nucleon wave functions, the latter in our present case being generalized to the seat of the β-decay process that we are taking to be uniform throughout the nucleon. $C(Z, W)$ represents this convolution and, as with $S(W, W_0, M)$, is obviously different for the vector and axial cases. $C(Z, W)$ also takes into account all possible "allowed" processes defined as those involving only leptons of $l=\frac{1}{2}$. (Note that these "allowed" processes involve r^2 terms in the lepton wave functions and so contain contributions sometimes represented as "second-forbidden".)

$\underline{G(W, W_0)}$. This term represents the "outer" radiative correction of order α. It is exact for a point Dirac nucleon in the absence of strong forces.

$\underline{J(Z)}$. This term represents "outer" radiative corrections of higher order in α.

$\underline{K(\alpha)}$. This term represents various further electromagnetic corrections of order α.

Before we discuss these terms separately we should pause to consider the orders in their expansion, and the combinations between them, to which we should go. We are aiming at "uncertainty" at the level of 0.01% and so must consider possible contributions down to the 0.001% level. We have: $M \simeq 1800$; $R \simeq 3\times10^{-3}$; $W \simeq 1$; $\alpha \simeq 10^{-2}$. We will therefore carry our enquiry systematically to such orders e.g. $1/M^2$, α/M, $\alpha(WR)^2$, α^3 as appear unlikely to contribute above the level of 0.001% but which might do so if large numerical coefficients were involved.

15. Details of the terms in the phase space factor

In eq. (3) in sect. 14 we have set out the terms that we must consider in the phase space factor f^{R} and in subsequent subsects we have described them qualitatively; we now examine them in detail.

15.1. THE NEUTRAL PHASE SPACE: $S(W, W_0, M)$

In Shekhter's compact notation [9], consider the expression:

$$SS(W, W_0, M) = M_n p(W_0 - W)T/(4R_e^3),$$

where:

$$R_e = M_n^2 + 1 - 2M_n W,$$

$$T = (\lambda+1)^2\{(M_n W - 1)(M_n - W)\Gamma_1 - \tfrac{1}{3}M_n p^2 \Gamma_2\} + (\lambda-1)^2 W R_e(\Gamma_1+\Gamma_2)$$
$$-2(\lambda^2-1)M_p(M_n W - 1)R_e(R_e - M_p^2),$$

$$\Gamma_1 = R_e^2 - M_p^4; \qquad \Gamma_2 = 4M_n^2(W_0 - W)^2.$$

$S(W, W_0, M)$ for vector neutron β-decay is now that part of the expansion of

$SS(W, W_0, M)$ that is independent of λ while $S(W, W_0, M)$, axial, is one-third of the coefficient of λ^2 in the expansion. (The coefficient of λ relates to the $g_V g_A$ interference term in the spectrum but this we do not need to concern ourselves with because we have already (sect. 10) established that its contribution is negligible in the total decay rate.)

We now need to obtain $S(W, W_0, M)$ to order $1/M^2$; recalling

$$S(W, W_0, M) = pW(W_0 - W)^2\{1 + R(W, W_0, M)\},$$

and writing

$$R(W, W_0, M) = 1 + r_0 + r_1/W + r_2 W + r_3 W^2,$$

we have, expanding $SS(W, W_0, M)$,

$$r_0^V = \frac{1}{2}\frac{W_0^2}{M^2} - \frac{11}{6}\frac{1}{M^2},$$

$$r_1^V = \frac{1}{3}\frac{W_0}{M^2},$$

$$r_2^V = 2\frac{1}{M} - \frac{4}{3}\frac{W_0}{M^2},$$

$$r_3^V = \frac{16}{3}\frac{1}{M^2},$$

and for the axial part,

$$r_0^A = -\frac{2}{3}\frac{W_0}{M} - \frac{1}{6}\frac{W_0^2}{M^2} - \frac{77}{18}\frac{1}{M^2},$$

$$r_1^A = -\frac{2}{3}\frac{1}{M} + \frac{7}{9}\frac{W_0}{M^2},$$

$$r_2^A = \frac{10}{3}\frac{1}{M} - \frac{28}{9}\frac{W_0}{M^2},$$

$$r_3^A = \frac{88}{9}\frac{1}{M^2}.$$

15.2. THE FERMI FUNCTION: $F(Z, W)$

Although it is $F_0(Z, W)$ that figures in eq. (3) above for f^R it will be interesting, if only for historical reasons, to replace $F_0(Z, W)L_0(Z, W)$ by $F(Z, W)L_0^*(Z, W)$ where the factor $\frac{1}{2}(1+\gamma)$ that relates $F_0(Z, W)$ to $F(Z, W)$ is absorbed into the replacement of $L_0(Z, W)$ by $L_0^*(Z, W)$. This will enable us the more clearly to see the effect of replacing the "old" computation of f using a point proton with evaluation at a fictitious radius R by the present computation that uses a uniformly charged proton of finite size, radius R.

We have:

$$F(Z, W) = \sum_{n=0}^{\infty} a_n(\alpha Z)^n ,$$

where, to appropriate order for our present concern [10]),

$$\begin{aligned} a_0 &= 1 , \qquad a_1 = \pi W/p , \\ a_2 &= \tfrac{11}{4} - \gamma_E - \ln(2pR) + \tfrac{1}{3}\pi^2(W/p)^2 , \\ a_3 &= \pi(W/p)\{\tfrac{11}{4} - \gamma_E - \ln(2pR)\} , \end{aligned} \tag{4}$$

where γ_E is Euler's constant 0.577215 ....

Although we may be rather confident that expansion of $F(Z, W)$ through α^3 is adequate for our present purpose it may be advisable to check this by exact computation of $F(Z, W)$ (see appendix 7); the results are shown in table 2 for $R = 1.0$ fm. Expansion (4), while excellent over the bulk of the range of W of interest, patently breaks down in the non-relativistic limit as $p \to 0$ where the exact $F(Z, W)$ goes as $1/p$ while the expansion contains terms in $1/p^2$. This requires a numerical check on the overall acceptability of the resultant integrated phase space factor; this we provide in subsect. 16.1.

TABLE 2

Comparison of the expansion F_{exp} [expression (4) of the text] with F_{exact}, namely $F(Z, W)$ as exactly computed (see appendix 8)

W	2.5	2.0	1.5	1.2	1.1	1.05
error (10^{-5}%)	1.3	1.4	1.8	2.5	3.1	2.2

W	1.01	1.001	1.0001	1.00001
error (%)	-9.7×10^{-4}	-9.1×10^{-2}	-3.5	-31

% Error is defined as $100(F_{exact}/F_{exp} - 1)$. $Z = 1$, $R = 1.0$ fm.

15.3. THE FINITE CHARGE RADIUS: $L_0(Z, W)$

For small Z-values an analytical expression is available for $L_0(Z, W)$ [ref [11])]:

$$L_0(Z, W) \simeq 1 + \tfrac{13}{60}(Z\alpha)^2 - WR\alpha Z - \frac{1}{2}\frac{Z\alpha R}{W} .$$

The adequacy of this analytical expression for our present problem is illustrated by comparing it with an accurate parameterization, for $|Z| < 10$, of numerically computed L_0 values [1]): agreement for $Z = 1$ is to one unit in the seventh decimal place.

As noted in subsect. 15.2, we have chosen to work with $F(Z, W)$ rather than with $F_0(Z, W)$ and so absorb the difference, to appropriate order, through replacing $L_0(Z, W)$ by

$$L_0^*(Z, W) = 1 + l_0 + l_1/W + l_2 W,$$

where $(Z = 1)$: $l_0 = \frac{7}{15}\alpha^2$, $l_1 = -\frac{1}{2}R\alpha$, $l_2 = -R\alpha$.

15.4. THE EFFECT OF RECOIL ON THE FERMI FUNCTION: $Q(Z, W, M)$

We have noted in subsect. 14.4 that account must be taken of the correction of $F(Z, W)$ by the recoil of the source of the Coulomb field in which the electron moves. We may estimate this effect by noting that the solution is known for the non-relativistic case where we have:

$$F(Z, W)_{\mathrm{NR}} = 2\pi\nu/(1 - e^{-2\pi\nu}),$$

where $\nu = Z\alpha/v$ and v is the relative electron-nucleus velocity. We now note that the relativistic $F(Z, W)$ reduces as $Z \to 0$ to precisely the above non-relativistic form so long as we interpret v as p/W. It therefore seems likely that we shall gain a good estimate, for neutron decay, of the correction to $F(Z, W)$ due to nuclear recoil, in lowest order of α, by replacing $\pi\alpha W/p = \pi\alpha/v_e$, where v_e is the electron velocity, by $\pi\alpha/\bar{v}_r$ where $\bar{v}_r$ is the mean value of the relativistically compounded electron-proton relative velocity.

We may easily show that $\bar{v}_r$ is the relativistic sum of v_e and $\bar{v}_M$ where $\bar{v}_M$ is the mean proton recoil velocity projected along the line of electron emission and that

$$\bar{v}_M = \frac{p}{M}\left(1 + B\frac{W_0 - W}{3W}\right),$$

where the electron/anti-neutrino correlation is $1 + (Bp/W)\cos\theta$, θ being the angle between the electron and anti-neutrino directions; $B = (1 - \lambda^2)/(1 + 3\lambda^2) \simeq -0.099$ (see appendix 3).

The relativistic composition now gives:

$$\bar{v}_r = \frac{p}{W} + \frac{p}{M}\left(1 + B\frac{W_0 - W}{3W}\right) \Big/ W^2,$$

so that

$$Q(Z, W, M) = 1 - \frac{\pi\alpha}{M}\frac{1}{p}\left(1 + B\frac{W_0 - W}{3W}\right).$$

(Note that this term is, by its nature, one that depends on the V/A mix and so, unlike the other correction terms, is inappropriate for separate evaluation for the vector and axial parts.)

15.5. THE LEPTON-NUCLEON CONVOLUTION: $C(Z, W)$

We have [12]:

$$C(Z, W) = 1 + \tfrac{6}{5}(H_2 + \xi N_1 D_1 + N_2) + \eta \frac{N_1 d_1}{5W},$$

where factors of γ and the parameter μ have been set equal to unity in the very small last term and where

$$\text{for vector decay:} \quad \xi^{\mathrm{V}} = -1\,; \quad \eta^{\mathrm{V}} = 6\,,$$
$$\text{for axial decay:} \quad \xi^{\mathrm{A}} = \tfrac{1}{3}\,; \quad \eta^{\mathrm{A}} = -2\,.$$

Writing

$$C(Z, W) = 1 + c_0 + c_1/W + c_2 W + c_3 W^2,$$

we have

$$c_0 = -\tfrac{9}{20}\alpha^2 + \tfrac{1}{5}R^2 + \tfrac{1}{5}\xi R\alpha W_0 - \tfrac{1}{5}W_0^2 R^2 - \tfrac{1}{45}\eta R^2,$$
$$c_1 = \tfrac{1}{45}\eta R^2 W_0,$$
$$c_2 = R(-\tfrac{3}{5}\alpha + \tfrac{2}{15}\xi R W_0 - \tfrac{1}{5}\xi\alpha + \tfrac{2}{5}W_0 R),$$
$$c_3 = -\tfrac{2}{5}R^2(1 + \tfrac{1}{3}\xi).$$

15.6. THE "OUTER" RADIATIVE CORRECTION OF ORDER α: $G(W, W_0)$

We have

$$G(W, W_0) = 1 + \frac{\alpha}{2\pi} g(W, W_0),$$

where $g(W, W_0)$ is an explicit but rather lengthy function [13] that we handle numerically; it is not useful to write it out here.

15.7. THE HIGHER-ORDER "OUTER" RADIATIVE CORRECTIONS: $J(Z)$

We expect these to be very small. In leading order they are identical for vector and axial transitions and, in that order, are energy independent:

$$J(Z) = 1 + (\delta^{Z\alpha^2} + \delta^{Z^2\alpha^3} + \delta^{Z\alpha^3})/(1 + \delta^{\alpha}).$$

δ^{α} is the result of integrating $G(W, W_0)$ (subsect. 15.6) across the energy spectrum (subsect. 16.6) and [14]:

$$\delta^{Z\alpha^2} \simeq \alpha^2 \ln M = 0.000400$$
$$\delta^{Z^2\alpha^3} \simeq \alpha^3 \pi^{-1}(3 \ln 2 - \tfrac{3}{2} + \tfrac{1}{3}\pi^2) \ln M = 0.000004\,,$$
$$\delta^{Z\alpha^3} \simeq \alpha^3 (2\pi)^{-1} (\ln M)^2 = 0.000003\,,$$

where we have set $Z = 1$ after identifying the origin of the terms.

We must note that for the (pure vector) cases in light complex nuclei, where the matter has been considered, the terms additional to the logarithm in $\delta^{Z\alpha^2}$, involving R and W_0, are about an order of magnitude smaller than the logarithmic term; we must therefore expect a possible error in our final f^R of a few units in the fifth decimal place on account of our present neglect of such terms.

15.8. THE ELECTROMAGNETIC CORRECTION TERMS: $K(\alpha)$

The electromagnetic influences separate into the distortion of the outgoing electron wave, represented in eq. (3) through F_0L_0Q, and the radiative corrections, represented in eq. (3) through GJ. However, this separation is not completely clean nor are its individual components fully described by the formulae that have been presented so far. We now review some as-yet-unquantified corrections of order α that we have represented collectively by $K(\alpha)$ in eq. (3).

Distortion. Within the uncertainties already discussed the distorting effects due to the Coulomb field are fully described by F_0L_0Q [after further account of the finite de Broglie wavelength effects that are included in C of eq. (3)]. However, in addition to this Coulomb interaction there are further interactions associated with the magnetic moments of the electron and proton that remain to be evaluated. Consider, illustratively, the spin-spin interaction. Owing to parity non-conservation the electron emerges with average longitudinal polarization (neglected signs) $P_e = p/W$ while the proton recoils with average polarization, measured along the axis of electron emission, of

$$P_p = \frac{p}{W}\frac{2\lambda(\lambda-1)}{1+3\lambda^2} \simeq 0.11(p/W)\,.$$

Classically, such average polarizations would result in a static moment-moment potential of

$$V_\mu(r) = \frac{2\mu_e\mu_p}{r^3}P_eP_p\,,$$

which has, just outside the proton (taken as $r = \hbar/m_\pi c$) a ratio to the electrostatic potential of about

$$0.15(p/W)^2\frac{m_\pi^2}{m_eM} \simeq 6(p/W)^2\,.$$

Of course, the moment-moment interaction falls off much more rapidly than the Coulomb interaction but its effect on the distortion of the outgoing electron wave may not necessarily be totally negligible and remains to be considered as do other magnetic effects.

Radiative corrections. Comment has already been made (subsects. 15.7 and 17) on neglect in J of eq. (3) of nucleon/nuclear structure effects. But uncertainties still attach to G of eq. (3). The canonical G is exact for an electron linking "Dirac" point nucleons in a V–A interaction. It does not allow for: (i) parity non-conservation such as results in a residual average proton polarization. (This we have included for our present purposes under the previous paragraph but it could, alternatively, be treated as a radiative correction.); (ii) the anomalous nucleon magnetic moments; (iii) nucleon structure. (ii) may be largely electron-energy independent [13]) and therefore better absorbed into the electromagnetic renormalization of the coupling constants themselves rather than into the phase space factor but this remains to be quantitatively investigated. (The same might possibly be true of the effect under (i), above, that we are presently classifying as contributing to the phase space.) (iii) reflects the QCD structure of the nucleon and can be evaluated in terms of explicit nucleon models. Strong interaction effects are probably by no means negligible when we ask questions at the level of $10^{-2}\%$; an explicit estimate within a current algebra approach [the third of ref. [13]] has indeed given about $4\times10^{-2}\%$ for their contribution in the Weinberg–Salam scheme but this can probably be largely absorbed into the renormalization of the coupling constants.

At this point were merely note that our $K(\alpha)$ of eq. (3) is used to repose any W-dependent corrections of order α not accounted for by the other terms.

16. Evaluation of f^{R}

We are concerned with corrections to the zero-order phase space f_0, the $I_{0,0}$ of appendix 6. As mentioned in sect. 2 we shall carry the evaluation to such orders of correction as appear unlikely to contribute at the 0.001% level and shall therefore quote all numerical values to six decimal places.

16.1 THE COULOMB CORRECTION

We begin with the Coulomb correction as represented through $F(Z, W)$ i.e. setting $M=\infty$ in $S(W, W_0, M)$ but preserving W_0 as the limit of integration and setting all other factors in eq. (3) to unity. Using eq. (4) and appendix 6 we have, in obvious notation,

$$f_0 = I_{0,0}$$

$$= 1.62972 \pm 0.00014\,,$$

$$\delta f_\alpha = \pi\alpha I_{1,1}$$

$$= 0.054646 \pm 0.000004\,,$$

$$\delta f_{\alpha^2} = \alpha^2\Big[(\tfrac{11}{4} - \gamma_E - \ln 2R)I_{0,0} + \tfrac{1}{3}\pi^2\{(\tfrac{1}{30}W_0^4 + \tfrac{11}{60}W_0^2 + \tfrac{8}{15})p_0 - \tfrac{3}{4}W_0 \ln(W_0 + p_0)\}$$

$$- (\tfrac{1}{30}W_0^4 - \tfrac{3}{20}W_0^2 - \tfrac{2}{15})p_0 \ln p_0 - (-\tfrac{47}{1800}W_0^4 + \tfrac{57}{400}W_0^2 + \tfrac{16}{225})p_0$$

$$- \tfrac{1}{4}W_0(\ln p_0 - \tfrac{3}{4})\ln(W_0 + p_0) + \tfrac{1}{4}W_0 \sum_{r=1}^{\infty} c_r\{\ln(W_0 + p_0)\}^{1+2r}\Big]$$

$$= 0.001361\,,$$

$$\delta f_{\alpha^3} = \pi\alpha^3[(\tfrac{11}{4} - \gamma_E - \ln 2R)I_{1,1} - \{(\tfrac{1}{30}W_0^5 - \tfrac{1}{3}W_0^2 + \tfrac{1}{2}W_0 - \tfrac{1}{5})\ln p_0$$

$$+ (\tfrac{1}{3}W_0^2 + \tfrac{1}{5})\ln(W_0 + 1) - \tfrac{47}{1800}W_0^5 - \tfrac{3}{20}W_0^3 + W_0^2(\tfrac{4}{9} - \tfrac{1}{3}\ln 2)$$

$$- \tfrac{23}{40}W_0 + \tfrac{23}{75} - \tfrac{1}{5}\ln 2\}]$$

$$= 0.000022\,.$$

(The numerical values above and for the rest of sect. 16 are, apart from the dominant f_0, given to six decimal places to establish significance, or otherwise, at the 0.001% level below which we are systematically considering all contributions to be negligible; as everywhere in this paper values are given for the illustrative $R = 1.0$ fm (sect. 13) with $W_0 = 2.529476 \pm 0.000033$ (sect. 5). Where no error is entered the result – in respect to the error in W_0 – is exact to the order stated.)

The only awkward term entering the above Coulomb evaluation is in δf_{α^2} where the c_r are related to the Bernoulli numbers B_{2r} by

$$c_r = \frac{2^{2r}B_{2r}}{(1+2r)!}\,,$$

so that

$$c_1 = \tfrac{1}{9}\,, \qquad c_2 = -\tfrac{1}{225}\,, \qquad c_3 = \tfrac{2}{6615}\,, \qquad c_4 = -\tfrac{1}{42525}\,, \qquad c_5 = \tfrac{2}{1029105}\,, \text{ etc.}$$

The contribution to δf_{α^2} from $r = 2$ is only 1.5×10^{-6} while that from $r = 3$ is 2.5×10^{-7}; $r = 3$ and higher terms may therefore be safely neglected.

It may be noted that the numerical coefficients of the terms in successive powers α^n become quite large; normalized to f_0 they are approximately: 4.6 for $n = 1$, 16 for $n = 2$ and 34 for $n = 3$. Such phenomena are well known in QED-type calculations and we might have some slight anxiety that, despite table 2, the full $F(Z, W)$ might, in view of our remark in subsect. 15.2 about the failure of our expansions of $F(Z, W)$ in the non-relativistic limit, as also shown in table 2, give on integration an answer differing significantly at our level of interest from our above result through order α^3; this is not so: the above $f_0 + \delta f_\alpha + \delta f_{\alpha^2} + \delta f_{\alpha^3}$ agrees with numerical integration, using the exact $F(Z, W)$, to the sixth decimal place.

16.2. THE FINITE NUCLEON MASS CORRECTION

We now move to the finite nucleon mass factor $S(W, W_0, M)$. It will be of interest to evaluate δf_M i.e. the correction to first order in $1/M$, δf_{M^2} i.e. the correction to order $1/M^2$ and also $\delta f_{M\alpha}$ that results from the cross term between the $1/M$ factor in $S(W, W_0, M)$ and the term in α in the expansion of $F(Z, W)$. Referring to subsect. 15.1 we have:

$$\delta f_M^{\mathrm{V}} = \frac{2}{M} I_{1,0} = 0.002824 \, ,$$

$$\delta f_M^{\mathrm{A}} = \frac{1}{M}(-\tfrac{2}{3} W_0 I_{0,0} - \tfrac{2}{3} I_{-1,0} + \tfrac{10}{3} I_{1,0}) = 0.002824 \, ,$$

$$\delta f_{M^2}^{\mathrm{V}} = \frac{1}{M^2}\{(\tfrac{1}{2} W_0^2 - \tfrac{11}{6}) I_{0,0} + \tfrac{1}{3} W_0 I_{-1,0} - \tfrac{4}{3} W_0 I_{1,0} + \tfrac{16}{3} I_{2,0}\} = 0.000005 \, ,$$

$$\delta f_{M^2}^{\mathrm{A}} = \frac{1}{M^2}\{(-\tfrac{1}{6} W_0^2 - \tfrac{77}{18}) I_{0,0} + \tfrac{7}{9} W_0 I_{-1,0} - \tfrac{28}{9} W_0 I_{1,0} + \tfrac{88}{9} I_{2,0}\} = 0.000004 \, ,$$

$$\delta f_{M\alpha}^{\mathrm{V}} = \frac{2}{M} \pi\alpha I_{2,1} = 0.000090 \, ,$$

$$\delta f_{M\alpha}^{\mathrm{A}} = \frac{\pi\alpha}{M}(-\tfrac{2}{3} W_0 I_{1,1} - \tfrac{2}{3} I_{0,1} + \tfrac{10}{3} I_{2,1}) = 0.000086 \, .$$

16.3 THE FINITE NUCLEON RADIUS CORRECTION

We now turn to the effects in eq. (3) associated with the finite nucleon radius [apart from that contained in $F(Z, W)$]. These are in $L_0^*(Z, W)$, that is concerned with the effect of finite proton extension on the solution of the Dirac equation, and in $C(Z, W)$ that is concerned with convolution through the nucleon volume. Although these effects are of distinct physical origin and although, as we have noted, it is by no means clear that the same "radius" for the nucleon should be used in both of them, we will lump them together as effects associated with finite nucleon radius R and evaluate, referring to subsects. 15.3 and 15.5,

$$\delta f_R = (l_0 + c_0) I_{0,0} + (l_1 + c_1) I_{-1,0} + (l_2 + c_2) I_{1,0} + c_3 I_{2,0} \, ,$$

so that $\delta f_R^{\mathrm{V}} = -0.000100$, $\delta f_R^{\mathrm{A}} = -0.000091$. We must also consider the cross term $\delta f_{R\alpha}$ with the lowest $-\alpha$ term in $F(Z, W)$ for which we have

$$\delta f_{R\alpha} = \pi\alpha\{(l_0 + c_0) I_{1,1} + (l_1 + c_1) I_{0,1} + (l_2 + c_2) I_{2,1} + c_3 I_{3,1}\} \, ,$$

so that $\delta f_{R\alpha}^{\mathrm{V}} = -0.000003$ and $\delta f_{R\alpha}^{\mathrm{A}} = -0.000003$.

16.4. SENSITIVITY TO THE NUCLEON RADIUS AND FORM

It is appropriate at this point to examine the sensitivity of the phase space factor to R and also to enquire as to the possible importance of our obviously unphysical assumption that the nucleon is a uniform sphere. Reference to eq. (3) and sects. 14 and 15 show that the nucleon radius R enters through $F_0(Z, W)$, $L_0(Z, W)$ and $C(Z, W)$ – also through $J(Z)$ and $K(\alpha)$ but that we reserve for separate remark (subsects. 15.7 and 15.8 respectively). To bring out the dependence of the phase space factor on R we show, in table 3 the percentage increase, $\delta f(\%)$, in the phase space factor on moving away from our illustrative $R = 1.0$ fm.

TABLE 3

The effect on the phase space factor, through $F_0(Z, W)$, $L_0(Z, W)$ and $C(Z, W)$, of changing the nucleon radius R

R (fm)	$\delta f(\%)^{\mathrm{V}}$	$\delta f(\%)^{\mathrm{A}}$
0.4	1.5×10^{-2}	1.4×10^{-2}
0.6	8.9×10^{-3}	8.5×10^{-3}
0.8	4.2×10^{-3}	4.0×10^{-3}
1.0	0	0
1.2	-3.9×10^{-3}	-3.7×10^{-3}
1.4	-7.6×10^{-3}	-7.1×10^{-3}
1.6	-1.1×10^{-2}	-1.0×10^{-2}

$\delta f(\%)$ is the percentage increase in the phase space factor on moving away from $R = 1.0$ fm; it is shown separately for the vector and axial cases.

We should note that although the conserved vector current enables us to associate the vector weak and electromagnetic properties of the nucleon rather closely we have no such assurance for the (dominant) axial part of the decay and so it is appropriate to consider a rather wider range of R than might seem indicated by the accuracy, noted in sect. 13, to which the effective electromagnetic radius of the proton has been established.

It is not profitable, at this stage, to speculate as to the best value to use for R; as we have noted, here and in Sect. 13 the "weak" and "electromagnetic" radii will certainly differ, if only for axial decay, but we have no real guidance at present as to what to do. However, we may at least use table 3 to indicate that use of our illustrative $R = 1.0$ fm is most unlikely to entrain an error of more than a few times 0.001%.

We also have the open question as to the sensitivity of our results to the *form* of the nucleon. It is a common experience from nuclear structure physics that the determining factor is $\langle r^2\rangle$ and that the form is secondary, within reasonable limits [15]). We may illustrate this, in our present case, with reference to the possibly more physically prepossessing form for the proton's charge distribution,

$$\text{charge density} \sim \{1+(r/a)^2\}\, e^{-(r/a)^2}\,.$$

Such a distribution results, for the same value of $\langle r^2\rangle$, in a change of phase space factor for neutron decay of substantially less than 0.001%.

16.5. THE COULOMB RECOIL CORRECTION

Integration of the Coulomb recoil correction $Q(Z, W, M)$ of subsect. 15.4 gives:

$$\delta f_r = -\frac{\pi\alpha}{M}\{I_{0,1}+\tfrac{1}{3}B(W_0 I_{-1,1}-I_{0,1})\}$$

$$= -0.000020\,.$$

16.6. THE "OUTER" RADIATIVE CORRECTION OF ORDER α

The "outer" radiative correction of order α, $G(W, W_0)$ of subsect. 15.6, is complicated and it is not profitable to attempt its analytical integration across the spectrum particularly since it is not significantly sensitive to other parameters of the calculation and so may be handled on a "once-and-for-all" basis. This has already been done [16]) and we here merely quote the result for neutron decay writing the effect of integrating $G(W, W_0)$ across the spectrum using the full $F(Z, W)$ as

$$1+\delta^{\alpha} = 1+0.015056\,.$$

[Note that since $G(W, W_0)$ is of order α, this integration entrains terms of order $\alpha^2 \ldots$ through $F(Z, W)$; these terms are explicitly excluded from the radiative corrections of higher order in α that we represent by $J(Z)$ in eq. (3) and subsect. 14.7.]

17. Summary

We now draw together in table 4 the various contributions to the phase space factor f^{R} that we have exposed separately above. We give the various terms to six decimal places in order to establish their relevance or otherwise in the fifth decimal place to which our present knowledge of the energy release permits us to proceed.

For consistency in comparing the various terms we list as δf^{α} etc. the absolute contributions to f^{R} made by the various radiative corrections e.g. in table 4 $\delta f^{\alpha} = f \times \delta^{\alpha}$ where f refers to the phase space factor derived without reference to the radiative corrections and δ^{α} is as defined and evaluated in subsect. 16.6 and similarly for the higher order radiative corrections as discussed in subsect. 15.7 i.e. $\delta f^{\alpha^2} = f \times \delta^{Z\alpha^2}$ etc.

TABLE 4

Contributions to the phase factor f^R for neutron β-decay as defined in the text

Term	Definition (see sect.)	Value
f_0	16.1	1.629715
δf_α	16.1	0.054646
δf_{α^2}	16.1	0.001361
δf_{α^3}	16.1	0.000022
δf_M^V	16.2	0.002824
δf_M^A	16.2	0.002824
$\delta f_{M^2}^V$	16.2	0.000005
$\delta f_{M^2}^A$	16.2	0.000004
$\delta f_{M\alpha}^V$	16.2	0.000090
$\delta f_{M\alpha}^A$	16.2	0.000086
δf_R^V	16.3	−0.000100
δf_R^A	16.3	−0.000091
$\delta f_{R\alpha}^V$	16.3	−0.000003
$\delta f_{R\alpha}^A$	16.3	−0.000003
δf_r	16.5	−0.000020
δf^α	16.6; 17	0.025423
δf^{α^2}	15.7; 17	0.000675
$\delta f^{(Z^2)\alpha^3}$	15.7; 17	0.000007
$\delta f^{(Z)\alpha^3}$	15.7; 17	0.000005
	total f^R (vector)	1.714650
	toal f^R (axial)	1.714654

(Note that the outer radiative corrections δf^α etc. refer to the contributions to the absolute value of f – see sect. 17.)

From table 4 we draw our total phase space factors:

$$f^R(\text{vector}) = 1.714650\,,$$

$$f^R(\text{axial}) = 1.714654\,.$$

We see that to within the accuracy permitted by our knowledge of the energy release we may adopt a single value of f^R appropriate for both vector and axial components of neutron β-decay, namely

$$f^R = 1.71465 \pm 0.00015\,.$$

Through the analytical breakdown of the components of the phase space factor as presented in this paper we may easily adjust this value of f^R for any future changes in the best value for the energy release and other parameters.

We must, however, recall particularly the caveat of subsect. 15.8 concerning $K(\alpha)$, the residual terms of order α and be sensitive to the fact that the above-stated error in f^R is associated solely with the lack of precision in the energy release and has no regard for these further uncertainties; we must also recall that the radiative corrections of order α^2 and α^3 are, in any case, correct as given in subsect. 15.7, only to their leading terms and the uncertainties as to effective "weak" nucleon size that reside in our treatment of the finite size factors.

18. Parameterization

Although the formulae presented in this paper permit of straightforward computation for various values of the energy release in the β-decay it may be convenient to note that a linear dependence of f^R on the energy release, namely an increase of 8.6394×10^{-6} in f^R for each 1 eV increase in the energy release, is valid to the 6th place in f^R for a range of 0.25 keV change of energy release in either direction from the value used illustratively in this paper, viz. for a range exceeding ten times the error presently attaching to the energy release.

Appendix 1

VECTOR β-DECAY: g_V

The well known "super-allowed Fermi transitions" within isospin multiplets of $J^\pi=0^+$ allow, as noted in sect. 2, the determination of g_V (renormalized by the "inner" radiative corrections as noted in sect. 12). If charge independence were perfect and the vector current perfectly conserved all these transitions would have exactly the same f^Rt value namely $(f^Rt)_{0\to0}$ from which g_V could be extracted. In practice Coulomb and the other charge-dependent forces introduce a slight mismatch between successive wave functions along an isospin multiplet so that f^Rt increases with Z; $(f^Rt)_{0\to0}$ is then the common value of f^Rt after correction for this nuclear mismatch [1]).

The problem of correction for the nuclear mismatch has been widely discussed. We follow here the method of extrapolation of the experimental f^Rt values to $Z=0$ guided by the Behrends–Sirlin–Ademollo–Gatto (BSAG) theorem [1]). Since the last such analysis [17]) the only significant new datum is a re-measurement of the ^{14}O mass the considerable importance of which has been emphasized [17]). The new measurement, which implies a kinetic energy release of 1808.25 ± 0.10 keV for the transition in question [18]), is significantly more accurate than that adopted in the most recent analysis [17]) namely 1808.62 ± 0.35 keV, and is in agreement with it. However, the new value is reported as preliminary only so it has seemed wise, while admitting it, provisionally to double its stated error gaining a combined value of 1808.34 ± 0.17 keV. Repeat of the earlier analysis now gives:

$$(f^Rt)_{0\to0}=3083.1\pm1.4\ \mathrm{sec}\,,$$

$$g_V=(1.41271\pm0.00032)\times10^{-49}\ \mathrm{erg\cdot cm^3}\,.$$

We should note the pivotal role played by the ^{14}O datum in the BSAG method of analysis providing, as it does, an accurate effective anchor point now closest to $Z=0$ for the extrapolation of the $f^R t$ values; we emphasize the high desirability of gaining a comparably accurate value for "super-allowed Fermi" β-decay of ^{10}C should that be possible. However, spuriously high accuracy may be inferred in $(f^R t)_{0\to 0}$ if the BSAG method is used when the accuracy of a point or points at low-Z is significantly greater than for the other points and so caution must be exercised.

Appendix 2

DIRECT NEUTRON LIFETIME MEASUREMENTS

Direct measurements of the neutron half-life of significant accuracy are:

Ref.	t (sec)
[19])	702 ± 18
[20])	636.6 ± 9.6
[21])	607.8 ± 5.4
[22])	649 ± 12

Unfortunately these direct determinations are not mutually consistent ($\chi^2/\nu = 11$). However, we note that refs. [20,22]) are mutually consistent and, if combined, yield 641.6 ± 7.5 sec. The combination of this value with ref. [19]) would have $\chi^2/\nu = 5$ or with ref. [21]) would have $\chi^2/\nu = 7$ so it may be held that these combinations could reasonably be rejected and that for the directly measured half-life we should adopt the above value:

$$t_{\mathrm{dir}}^{\mathrm{I}} = 641.6 \pm 7.5 \text{ sec}.$$

An alternative, more conservative, approach could be to consider all the "modern" measurements viz. refs. [20–22]) [it might, in any case, appear that ref. [22]) replaces ref. [19])] and, recognizing the extreme difficulty of these measurements, inflate all stated errors proportionately until $\chi^2/\nu = 1$. This yields

$$t_{\mathrm{dir}}^{\mathrm{II}} = 620 \pm 12 \text{ sec}.$$

At this stage it may seem wiser to adopt the more conservative $t_{\mathrm{dir}}^{\mathrm{II}}$.

Appendix 3

CORRELATION MEASUREMENTS OF g_A/g_V

There are two practical approaches to g_A/g_V by correlation experiments on the decay of free neutrons:

(i) Measure the angular correlation between the direction of the electron and the spin direction of the neutron.

(ii) Measure the angular correlation between the electron and the anti-neutrino (inferred from the momentum distribution of the recoil proton).

Each sort of experiment is very tricky; the former involves observation of the decay of polarized neutrons, the latter depends upon a close study of the shape of the proton recoil spectrum whose upper limit is only 751 eV.

For experiment type (i), if we ignore (a) the recoil of the proton due to the departing electron and anti-neutrino and (b) the effects of weak magnetism (g_{WM}), and $g_V g_A$ interference, we find the angular correlation

$$1 + A_0(p/W)\cos\theta .$$

θ is the angle between the direction of the electron of velocity $v = p/W$ and the neutron spin direction, λ is the ratio of axial to vector coupling constant strengths as defined by eq. (1) of the main text and

$$A_0 = 2\lambda \frac{1-\lambda}{1+3\lambda^2} .$$

For experiment type (ii) we similarly, to first approximation, have a correlation now in the angle θ between electron and anti-neutrino directions, of

$$1 + B_0(p/W)\cos\theta ,$$

where

$$B_0 = \frac{1-\lambda^2}{1+3\lambda^2} .$$

In both cases we must integrate these correlations, as appropriate to the experimental circumstances, across the electron spectrum, having due regard for the Coulomb force between the proton and the departing electron.

Three accurate measurements of the asymmetry parameter $\bar{A}$, the bar indicating that the value is inferred from measurements integrated across the greater part of the electron spectrum, are available:

$$\bar{A} = \begin{cases} -0.113 \pm 0.006 & [\text{ref}\ ^{23})] \\ -0.118 \pm 0.010 & [\text{ref.}\ ^{24})] \\ -0.115 \pm 0.006 & [\text{ref.}\ ^{25})] \end{cases} .$$

These values are nicely mutually consistent and we combine them to gain

$$\bar{A} = -(0.1147 \pm 0.0039) .$$

To transform this adopted asymmetry parameter into a value of λ we must correct the above simple theoretical expression for A_0 on account of weak magnetism,

$g_V g_A$ interference, and nucleon recoil. To appropriate order:

$$A \simeq A_0\{1 + A_{\mu M}(A_1 W_0 + A_2 W + A_3/W)\},$$

where:

$$A_{\mu M} = \frac{\lambda + \mu}{\lambda(1-\lambda)(1+3\lambda^2)} \frac{1}{M},$$

$$A_1 = \lambda^2 + \tfrac{2}{3}\lambda - \tfrac{1}{3},$$

$$A_2 = -\lambda^3 - 3\lambda^2 - \tfrac{5}{3}\lambda + \tfrac{1}{3},$$

$$A_3 = 2\lambda^2(1-\lambda).$$

Here $\mu = \mu_p - \mu_n$ is the difference between the *total* magnetic moments of a proton and a neutron, viz. 4.71. This term in μ is compounded of μ^a from weak magnetism (see subsect. 9.1) plus unity from $g_V g_A$ interference (see subsect. 9.4 for further comment). This expression must now be integrated appropriately across the electron spectrum with due allowance for the Coulomb effects. For integration across the full energy spectrum we expect the correlation $1 + \bar{A} \cos\theta$ where, to appropriate order,

$$\bar{A} \simeq A_0 \frac{I_{-1,-1}}{I_{0,0}} \left[1 + \pi\alpha\left\{\frac{I_{0,0}}{I_{-1,-1}} - \frac{I_{1,1}}{I_{0,0}}\right\} + \frac{A_{\mu M}}{I_{-1,-1}}\{A_1 W_0 I_{-1,-1} + A_2 I_{0,-1} + A_3 I_{-2,-1}\}\right].$$

The $I_{n,m}$ are the integrals across the spectrum as defined and evaluated in appendix 6.

Use of such integration allowing for the Coulomb, recoil and weak magnetism effects permits us to analyze the experimental data, expressed through the above-adopted value of $\bar{A}$ which was derived without account of them. We find

$$\lambda = 1.259 \pm 0.010.$$

It is interesting to note that analysis using the simple A_0 gives $\lambda = 1.263$ so that the small effects reduce the inferred λ by 0.004 which although within the experimental error is not insignificant in relation to it. [Separately: the Coulomb effect decreases the asymmetry by 0.22%; the recoil effect (including $g_V g_A$ interference) increases the asymmetry by 0.61%; weak magnetism increases the asymmetry by 0.99%.]

For experiment type (ii) only one accurate experiment exists and yields [26])

$$\lambda = 1.259 \pm 0.017.$$

We combine these concordant results of the two types of experiment and adopt

$$\lambda = 1.2590 \pm 0.0086.$$

[We note that although, as seen particularly in subsect. 16.6, the effect of radiative corrections on the absolute decay rate is major, their effect on the correlations is negligible [27]).]

Appendix 4

THE NEUTRON LIFETIME, λ AND g_A

The neutron lifetime and λ are linked through

$$f^{R}t = \frac{2(ft)_{0\to 0}}{1+3\lambda^2},$$

where $(ft)_{0\to 0}$ has been discussed in appendix 1 and the numerical value 3083.1 ± 1.4 sec adopted for it. We may use this equation for $f^{R}t$ to test the mutual consistency of the four experimental or experimentally derived quantities that it contains. Taking the directly determined $\lambda\,(=\lambda_{\mathrm{dir}}) = 1.2590 \pm 0.0086$ from appendix 3 we have

$$\mathrm{RHS} = 1071 \pm 12\ \mathrm{sec}.$$

Taking $f^{R} = 1.71465 \pm 0.00015$ from sect. 17 and the conservatively preferred $t^{\mathrm{II}}_{\mathrm{dir}}$ from appendix 2 we have

$$\mathrm{LHS} = 1063 \pm 21\ \mathrm{sec}.$$

There is thus consistency within the above equation for $f^{R}t$. We should, however, note that the situation is basically unsatisfactory and that use of $t^{\mathrm{I}}_{\mathrm{dir}}$ of appendix 2 would give an LHS of 1100 ± 13 sec that would sit uncomfortably with the RHS. It seems unlikely that this situation will be resolved without further direct measurements of the neutron lifetime. If a definite discrepancy were to be established within the above equation for $f^{R}t$ then that may signal the effect of neutrino mixing which, as noted in appendix 5, could possibly affect the neutron lifetime by several percent. It is clear that difficult as neutron lifetime measurements may be, their importance is such that more are urgently demanded.

If our interest is in one or the other of t and λ, *but not both*, then the above equation for $f^{R}t$ may be used to derive an inferred value of t or λ to combine with the directly measured quantity.

Using λ_{dir} as above we infer, from the above equation for $f^{R}t$, $t_{\mathrm{inf}} = 624.9 \pm 7.2$ sec; this we combine with the preferred $t^{\mathrm{II}}_{\mathrm{dir}}$ to give a present recommended

$$t = 623.6 \pm 6.2\ \mathrm{sec}.$$

Alternatively, using $t^{\mathrm{II}}_{\mathrm{dir}}$ we infer, from the above equation for $f^{R}t$, $\lambda_{\mathrm{inf}} = 1.265 \pm 0.015$; this we combine with λ_{dir} to give a present recommended

$$\lambda = 1.2605 \pm 0.0075$$

to which corresponds, using g_V from appendix 1,

$$g_A = (1.781 \pm 0.011) \times 10^{-49}\ \text{erg} \cdot \text{cm}^3 .$$

Appendix 5

FINITE NEUTRINO MASS

Except in the simplest schemes there is no fundamental reason why the mass of the electron neutrino should be zero and, indeed, a finite value has been experimentally reported [29]); it is therefore important to know what effect on the decay rate a finite value of m_ν might have. With no complications beyond m_ν, the phase space $p_e W_e p_\nu W_\nu$ becomes, neglecting recoil and all refinements,

$$S(W, W_0^m) = pW(W_0^m - W + m_\nu)\{(W_0^m - W + m_\nu)^2 - m_\nu^2\}^{1/2} ,$$

where, in the no-recoil approximation, $W_0^m = W_0 - m_\nu$. Integration of $S(W, W_0^m)$ gives $I_{0,0}^m - \delta I$, δI measuring the loss of decay rate due to a finite m_ν. It is elementary to show that the lowest term in δI is of order m_ν^2; to that order,

$$\delta I = \tfrac{1}{6} p_0^3 m_\nu^2 .$$

Since, with good confidence [28]), $m_\nu < 2 \times 10^{-4}$ this effect is negligible to well below our level of concern. [The above expressions for the phase space and for δI omit the so-called relativistic spinor [29]) term which is linear in m_ν but which vanishes for the "standard" vector/axial combination with full parity non-conservation that we are adopting here.]

Although finite m_ν as such is of negligible effect for us, the possibility of finite masses for the electron, muon and tauon neutrinos also entrains the possibility of their mixing, which in turn could affect the neutron decay rate by several per cent for certain masses and certain mixing schemes. This situation is so fluid that it does not seem profitable even to give reference to the rapidly burgeoning literature but we must recognize the possibly major importance of such effects.

Appendix 6

Integrals of the following form have to be considered:

$$I_{m,n} = \int_1^{W_0} pW(W_0 - W)^2 W^m / p^n \, dW .$$

The integrals that we need and their numerical values for $W_0 = 2.529476 \pm 0.000033$ are

$$I_{-1,0} = \tfrac{1}{12} W_0^3 p_0 + \tfrac{13}{24} W_0 p_0 - \tfrac{1}{2} W_0^2 \ln(W_0 + p_0) - \tfrac{1}{8} \ln(W_0 + p_0)$$
$$= 1.066 ,$$

$$I_{0,0}=\tfrac{1}{30}W_0^4p_0-\tfrac{3}{20}W_0^2p_0-\tfrac{2}{15}p_0+\tfrac{1}{4}W_0\ln(W_0+p_0)$$
$$=1.62972\pm0.00014\,,$$
$$I_{1,0}=\tfrac{1}{60}W_0^5p_0-\tfrac{1}{30}W_0^3p_0+\tfrac{49}{240}W_0p_0-\tfrac{1}{8}W_0^2\ln(W_0+p_0)-\tfrac{1}{16}\ln(W_0+p_0)$$
$$=2.594\,,$$
$$I_{2,0}=\tfrac{1}{105}W_0^6p_0-\tfrac{1}{84}W_0^4p_0-\tfrac{13}{280}W_0^2p_0+\tfrac{1}{8}W_0\ln(W_0+p_0)-\tfrac{8}{105}p_0$$
$$=4.296\,,$$
$$I_{-2,1}=W_0^2\ln W_0-\tfrac{3}{2}W_0^2+2W_0-\tfrac{1}{2}$$
$$=0.899\,,$$
$$I_{-1,1}=\tfrac{1}{3}W_0^3-W_0^2+W_0-\tfrac{1}{3}$$
$$=1.193\,,$$
$$I_{0,1}=\tfrac{1}{12}W_0^4-\tfrac{1}{2}W_0^2+\tfrac{2}{3}W_0-\tfrac{1}{4}$$
$$=1.649\,,$$
$$I_{1,1}=\tfrac{1}{30}W_0^5-\tfrac{1}{3}W_0^2+\tfrac{1}{2}W_0-\tfrac{1}{5}$$
$$=2.384\,,$$
$$I_{2,1}=\tfrac{1}{60}W_0^6-\tfrac{1}{4}W_0^2+\tfrac{2}{5}W_0-\tfrac{1}{6}$$
$$=3.611\,,$$
$$I_{3,1}=\tfrac{1}{105}W_0^7-\tfrac{1}{5}W_0^2+\tfrac{1}{3}W_0-\tfrac{1}{7}$$
$$=5.731\pm0.001\,,$$
$$I_{-2,-1}=I_{0,1}-I_{-2,1}$$
$$=0.749\,,$$
$$I_{-1,-1}=I_{1,1}-I_{-1,1}$$
$$=1.191\,,$$
$$I_{0,-1}=I_{2,1}-I_{0,1}$$
$$=1.962\,.$$

With the exception of $I_{0,0}$ all these integrals are needed only in correction terms; their numerical values are exact as given unless an error is associated with them.

Appendix 7

Should it, for some reason, be desirable to gain $F(Z, W)$ to an accuracy greater than afforded by expansion (4) of the main text it may be noted that, for $Z=$

$1, \Gamma(2\gamma+1) = 1.99990172$ while $|\Gamma(\gamma + i\alpha W/p)|$ may be conveniently found by one or other of the following methods:

(i) For $\alpha W/p < 1$,

$$\ln|\Gamma(\gamma+i\alpha W/p)|$$
$$= -\gamma_{\mathrm{E}} + \sum_{n=2}^{\infty} (-)^n \frac{\zeta(n)}{n}$$
$$\times\left\{\varepsilon^n - \frac{n!}{(n-2)!2!}\varepsilon^{n-2}y^2 + \frac{n!}{(n-4)!4!}\varepsilon^{n-4}y^4 + \cdots\right\},$$

where $\varepsilon = \gamma - 1$; $y = \alpha W/p$; γ_{E} is Euler's constant ($=0.577215\ldots$); the finite series in curly brackets stops when the power of ε is unity or zero; $\zeta(n)$ is the Riemann zeta-function. This expression converges extremely rapidly over the great bulk of the range of W of concern.

(ii) For any value of W[ref. [30]],

$$\ln\{|\Gamma(\gamma+i\alpha W/p|^2\}$$
$$\approx \ln\{y_1^2/(\gamma^2+y^2)\} + \ln\{\pi/(y_1 \sinh \pi y_1)\}$$
$$+\ln(1+y_1^2) + (1-\gamma)[2 - \ln\{(1+\gamma)^2 + y^2\}$$
$$+\{2y/(1+\gamma)\}\arctan\{y/(1+\gamma)\} + \{1/(\{1+\gamma\}^2+y^2)\}/6a]$$
$$-3\ln a,$$

where

$$y_1 = \{2/(1+\gamma)\}y = ay = a\alpha W/p.$$

This expression is complementary to expression (1) in that it converges all the more rapidly the lower the value of p. [It is directly related to the non-relativistic expression for $F(Z, W)$.] Over the entire range of W of present concern (down to $W = 1$) it is never in error by more than 3 parts in 10^8.

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