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NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH
Section A

ELSEVIER

Evaluation of beta-decay Part V. The Z-independent outer radiative corrections for allowed decay ☆

D.H. Wilkinson

TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada V6T 2A3
University of Sussex, Falmer, Brighton, BN1 9QH, UK

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Abstract

Before interpreting nuclear beta-decay in terms of nucleonic wave functions and weak coupling constants allowance must be made for the influence of virtual photon exchange and real photon emission together with Z-boson exchange—the radiative corrections. This paper discusses in detail the lowest order 'outer' radiative correction of Sirlin, energy-dependent but weak-interaction-mechanism-independent, and presents graphical and other material to enable its accurate evaluation for allowed decay for all beta-decay transition energies up to 20 MeV and for $0 \leq |Z| \leq 100$. The need to use electron spectra due to a nuclear charge distribution of finite spatial extension with appropriate convolution of the leptonic and nucleonic wave functions throughout the nuclear volume is emphasized; the variation of these effects between the neutron and proton drip lines is explored. Exponentiation and higher-order Z-independent effects are also considered.

1. Introduction [1]

Central to the analysis of nuclear beta-decay is the Fermi function:

$$F(Z,W) = 2(1+\gamma)[\Gamma(2\gamma+1)]^{-2}(2pR)^{2(\gamma-1)} \times e^{\pi\alpha ZW/p}|\Gamma(\gamma + i\alpha ZW/p)|^2 \tag{1}$$

in which, as everywhere in these papers, we take Z positive for negative electron emission and negative for positron emission. The simplest definition of the phase space factor f for an allowed transition now becomes:

$$f = \int_1^{W_0} pW(W_0 - W)^2 F(Z,W)\,\mathrm{d}W\,. \tag{2}$$

However, Eq. (1) is valid only for a point nuclear charge, the weak divergence of the electron wave function at the origin being cut off by its evaluation at the radial distance R, sometimes spoken of as the 'radius of the nucleus', but, in fact, having no physical significance beyond one of formal definition. Numerical solution of the Dirac equation in the field of a finite nuclear charge distribution removes the divergence at the origin which now becomes the natural place to which to refer the magnitude of the lepton wave functions. If, specifically, we choose as the nuclear charge distribution a uniform sphere of radius R, we multiply Eq. (1) by $[2/(1+\gamma)]L_0(Z,W)$ where $L_0(Z,W)$, a function of R, is extensively tabulated [1] for $R = r_0 A^{1/3}$ with $r_0 = 1.2$ fm and A chosen as a function of Z to correspond to the floor of the stability valley. In order to permit accurate discussion away from the floor of the stability valley a detailed parameterization of $L_0(Z,W)$ for $1.0 \leq r_0 \leq 1.5$ fm and for $Z \leq 60$ was presented in Part II of this series of papers [2].[2] In the following presentation we use for $L_0(Z,W)$ the parameterization just referred to for $Z \leq 60$ together with experimental values of r_0 as a function of Z appropriate to the floor of the stability valley; for $Z > 60$ we use the tabulation of Ref. [1] parameterized for each Z-value as a polynomial in p through p^8. Other values for r_0 in the region $Z \leq 60$ are considered in Section 4.

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[1] The nomenclature is as in the previous members of this series with additions as noted.

[2] Note that two misprints in Table 2 of Ref. [2] were reported in Ref. [3].

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Of course, the use of a uniformly-charged sphere for generating $L_0(Z,W)$ is itself an artificiality. There have been extensive discussions of the effect of moving to a more realistic radial profiling of a spherically-symmetrical charge distribution [3,4] and to an ellipsoidally-deformed charge distribution of uniform density [5]. These latter effects become of interest only when the highest accuracy in the phase space factor itself is sought and are of no sensible significance for the issue addressed in this present paper; however, the effect of moving from a point charge to a finite charge distribution, through the introduction of the $L_0(Z,W)$ appropriate to a uniformly-charged sphere, can be important under certain circumstances for the present problem as we shall see.

With the introduction of a nuclear charge distribution of finite spatial extension we must also recognize that the beta-decay spectrum must reflect the appropriate convolution of nucleonic and leptonic wave functions through that extended nuclear volume. This has the effect of introducing into Eq. (1) the further multiplicative shape factors ${}^{V}C(Z,W)$ or ${}^{A}C(Z,W)$ for allowed vector or axial transitions respectively that, to appropriate order in the small quantities in terms of our present needs, read [4]:

$${}^{V}C(Z,W) \simeq 1 + {}^{V}C_0 + {}^{V}C_1 W + {}^{V}C_2 W^2,$$

where

$$\begin{aligned} {}^{V}C_0 &= -233(\alpha Z)^2/630 - (W_0R)^2/5 - 6W_0R\alpha Z/35, \\ {}^{V}C_1 &= -13R\alpha Z/35 + 4W_0R^2/15, \\ {}^{V}C_2 &= -4R^2/15; \end{aligned} \tag{3}$$

and:

$${}^{A}C(Z,W) = 1 + {}^{A}C_0 + {}^{A}C_1 W + {}^{A}C_2 W^2,$$

where

$$\begin{aligned} {}^{A}C_0 &= -233(\alpha Z)^2/630 - (W_0R)^2/5 + 2W_0R\alpha Z/35, \\ {}^{A}C_1 &= -21R\alpha Z/35 + 4W_0R^2/9, \\ {}^{A}C_2 &= -4R^2/9. \end{aligned} \tag{4}$$

With the introduction of these factors, relating to finite nuclear size, into Eq. (1) we designate the various resultant quantities of interest by the subscripts L or LC following the introduction of just the factor $L_0(Z,W)$ or both factors $L_0(Z,W)$ and $C(Z,W)$, respectively.

Following these considerations, refined as necessary for factors mentioned in passing, and with allowance, which may be made with high accuracy, for finite nuclear mass effects, screening by the orbital electrons, in the case of positron emission for orbital electron capture, and for the inevitable excitation of the electronic entourage that accompanies the beta-transition [3,6], the Q-value for the transition may be reduced to the phase space factor with a precision of order 0.01% given accurate knowledge of the nuclear charge distribution and given knowledge of the relevant single-nucleonic wave functions (taken as simple rectangles of radial extension R in writing down Eqs. (3) and (4), an approximation amply adequate for the purposes of this paper). This intrinsic precision of, say, 0.01% is the gauge against which we must set any other effects that may influence the beta-decay transition rate and that we must hope to be able to discuss with matching confidence. Of these other effects this paper addresses itself to those so-called outer radiative corrections that do not depend upon the nuclear charge Z.

2. The radiative corrections primarily of order α

When a nucleon beta-transforms itself into one of opposite t_3 the primary process is the transformation of a u-quark into a d-quark, or vice versa, with the emission of a W-boson which then decays into an electron and a neutrino. However, this is not all: photons may be additionally exchanged between appropriate participant particles or be emitted into the free state—inner bremsstrahlung—and Z-bosons may also be exchanged. These additional, γ, Z-related, processes are the radiative corrections that, in sum, can amount to several per cent in their influence upon the transition rate and so are of major importance for accurate discussion.

To a good approximation these radiative corrections divide themselves into two classes: (i) the 'inner' corrections that are concerned with the detailed W-boson-dependent anatomy of the primary beta-decay process; in the case of the vector $N \to Ne\nu$ they are finite and are critical for the inter-comparison of nuclear and muon decay [7] leading to assessment of the V_{ud} element of the Cabibbo–Kobayashi–Maskawa matrix; primarily of first order in α they are extensible to all orders in α in leading-log approximation [8]; in the case of the axial $N \to Ne\nu$ there is no similar unambiguous framework for discussion and the effect of the inner correction, whatever it might be, is simply lumped into the operational definition of the axial coupling constant; these inner corrections are independent of the transition energy; (ii) the 'outer' corrections that divide into those dependent upon the nuclear charge Z and those primarily independent of it; they do not depend in lowest order upon the quark-based anatomy of the beta-decay process, although, as we shall see, there are higher-order effects that are quark-dependent, but they do depend upon the transition energy.

This paper concerns itself primarily with the outer radiative correction of order α, particularly with practical details of its numerical evaluation in respect of the influence of the nuclear charge Z. The nuclear charge Z enters the evaluation of the radiative correction that is nominally just of order α because that correction, for a given total energy release, W_0, in the transition, depends upon the energy, W, carried away just by the electron itself and therefore, in sum, upon the electron spectrum over which it must be integrated to assess its total impact upon the transition rate. This influence, via the electron spectrum, of the nuclear charge Z upon the outer radiative correction primarily of order α effectively entrains terms of order $Z\alpha^2$, $Z^2\alpha^3$.... Additional terms of such higher order, not discussed in the present paper, also exist in their own right and must be added to those of higher order that arise from that of order α, via $F(Z, W)$, as just described; double counting must obviously be avoided.

2.1. The outer radiative correction of order $\alpha : Z = 0$

The outer radiative correction of order α is given by multiplying the phase space by Sirlin's factor $g(W, W_0)$ [9]:

$$g(W, W_0) = 3\ln M - \frac{3}{4} + 4\left[\frac{\operatorname{arctanh}\beta}{\beta} - 1\right] \times\left[\frac{W_0 - W}{3W} - \frac{3}{2} + \ln\{2(W_0 - W)\}\right] + \frac{4}{\beta}L\frac{2\beta}{1+\beta} + \frac{1}{\beta}[\operatorname{arctanh}\beta] \times\left[2(1+\beta^2) + \frac{(W_0 - W)^2}{6W^2} - 4\operatorname{arctanh}\beta\right]. \quad (5)$$

Here $\beta = p/W$ and $L(x)$ is the Spence function:

$$L(x) = \int_0^x \frac{\ln(1-t)}{t}\,dt. \quad (6)$$

The effect of $g(W, W_0)$ is to multiply the transition rate by the factor $1 + \delta(Z, W_0)$ where:

$$\delta(Z, W_0) = \frac{\alpha}{2\pi} f^{-1} \int_1^{W_0} pW(W_0 - W)^2 \times F(Z, W) g(W, W_0)\,dW \quad (7)$$

and where $F(Z, W)$ and f are given by Eqs. (1) and (2) respectively or with the additional introduction of L_0 and/or $C(Z, W)$ as discussed above. We may note immediately that $\delta(Z, W_0)$ is of order 10^{-2}.

Eq. (5) is valid for both vector and axial transitions. It is very largely nucleon-structure-independent although there are small strong-interaction effects that are not accounted for. Terms omitted from Eq. (5) are of order $\alpha(W/M)\ln(M/W)$ and $\alpha q/M$ in their impact upon Eq. (7) where q is the leptons' transfer of momentum to the nucleon. For our practical purposes we shall consider $E_0 \leq 20$ MeV so that the larger of these omitted terms could be of order, integrating across the decay spectrum in the $Z = 0$ approximation, ranging from about 6×10^{-5} at $E_0 \simeq 2$ MeV to 3×10^{-4} at $E_0 \simeq 20$ MeV to be compared with the corresponding values of about 1.3×10^{-2} and 5×10^{-3} deriving from Eqs. (5) and (7) for $\delta(0, W_0)$.[3]

It is useful for purposes of orientation, and also as a point of reference for our discussion of $\delta(Z, W_0)$ for the lower range of E_0, as we shall see, to consider $\delta(0, W_0)$ in some detail.

Eq. (5) is somewhat complicated; it is not possible to give an analytical approximation that is useful over the whole range of electron energy. We rather begin as earlier [10] by noticing that:

$$\delta(0, 1) = \frac{\alpha}{2\pi}\left\{3\ln M - \frac{27}{4}\right\} = 1.8348 \times 10^{-2} \quad (8)$$

and that:

$$\delta(0, W_0 \to \infty) = \frac{\alpha}{2\pi}\left\{3\ln\frac{M}{2W_0} + \frac{81}{10} - \frac{4}{3}\pi^2\right\} \quad (9)$$

so that a parameterization, useful overall, might be one adapted from Eq. (9) so as to give the correct value for Eq. (8):

$$\delta(0, W_0) \simeq \frac{\alpha}{2\pi}\{15.409 - 3\ln(W_0 - \varepsilon) + [0.3889 + 3\ln(1-\varepsilon)]\,e^{-k(W_0 - 1)}\} \quad (10)$$

where an appropriate least-squares fitting over the range of electron energy up to 10 MeV yields the constants: $\varepsilon = 0.48$; $k = 1.072$. The accuracy of this semi-empirical formula is shown in Fig. 1 where it is seen everywhere to better an absolute error of $5 \times$

[3] Eq. (5), derived for a free nucleon, is also applicable to the bound nucleons of nuclear beta-decay with which we are concerned here. Off-hand one might have expected, on dimensional grounds, that nuclear binding would introduce additional uncertainty of order $\alpha v/c$ where v is a nucleon velocity within the nucleus; such terms would be large, of order $\alpha/3$, but, fortunately, they do not occur. (I am grateful to Professor Sirlin for this information.) A further consideration relating to bound nucleons is the value, effective in that context, of the nucleon mass M (for which we use 1837.3). Now, as we shall see, the values of $\delta(Z, W_0)$ that we might encounter in practice range from about 0.5 to 1.8%, i.e. the value of $g(W, W_0)$, averaged over phase space, ranges from about 4 to 15. Now the contribution to Eq. (5) from the term in M is 22.55 if we use $M = 1837.3$; if, most crudely, the effective value of M were to be reduced by a typical relevant nucleonic binding energy, say 10 MeV, the contribution of M to Eq. (5) would decrease by 0.032 so that the greatest effect on $\delta(Z, W_0)$ would be only about 1% of its own value.

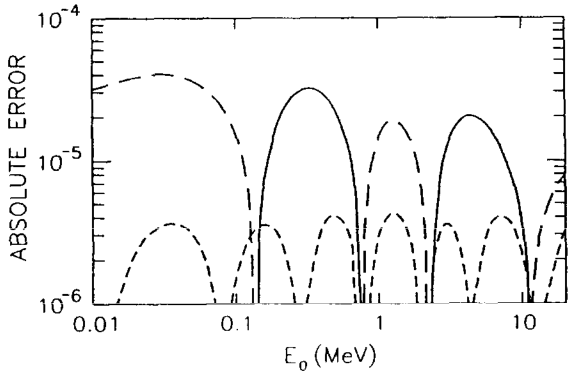


Fig. 1. Error in approximate expressions for $\delta(0, W_0)$: the upper curves show Eq. (10) minus the exact $\delta(0, W_0)$ (solid lines are for negative values; dashed are for positive); the lower curves show the effect of afforcing Eq. (10) by the trimming term of Eq. (11) (the error alternates in sign with increasing E_0 beginning with negative).

10^{-5}. Should higher accuracy be required it is provided by adding to Eq. (10) a polynomial trimming term:

$$\delta_t = (8.48281x - 491.818x^2 + 2563.87x^3 - 5347.53x^4 + 5649.82x^5 - 3209.77x^6 + 936.278x^7 - 110.207x^8) \times 10^{-5}, \qquad (11)$$

where $x = (\ln W_0)^{1/2}$. As seen from Fig. 1, Eq. (10) plus the trimming δ_t of Eq. (11) now everywhere betters an accuracy of 5×10^{-6}.

We may easily write down reductions of Eq. (5) in the regime $W_0 \to 1$ where we have:

$$g(W, W_0 \to 1) = 3 \ln M - \frac{27}{4} - \frac{26}{9} p^2 + \frac{4}{3} p^2 \ln(p_0^2 - p^2), \qquad (12)$$

and

$$\delta(0, W_0 \to 1) = \frac{\alpha}{2\pi} \left\{ 3 \ln M - \frac{27}{4} + \frac{8}{9} p_0^2 \left[\ln 2p_0 - \frac{668}{315} \right] \right\}; \qquad (13)$$

Eq. (13) betters an accuracy of 2×10^{-4} for $E_0 <$ 0.5 MeV. However, these reductions are only of exceptional use: Eq. (12) does not lend itself readily to discussion of the correction for finite Z over a useful range of E_0 and Z.

2.2. *The outer radiative correction of order α: low Z and high E_0*

Much of the interest in precise analysis of allowed beta-decay resides in the regime of fairly low Z ($Z \leq$ 20) and fairly high E_0 ($E_0 \geq 3$ MeV); here we may derive useful analytical approximations to $g(W, W_0)$ and to $\delta(Z, W_0)$.

As $\beta \to 1$ we have [11]:

$$g(W, W_0) \simeq 3 \ln M - \frac{3}{4} - \frac{2}{3}\pi^2 + 4(\ln x - 1)\left[\frac{1-x}{3x} - \frac{3}{2} + \ln\frac{1-x}{x}\right] + \frac{(1-x)^2}{6x^2} \ln x + (\ln 2W_0) \times \left[\frac{4(1-x)}{3x} - 6 + \frac{(1-x)^2}{6x^2} + 4 \ln\frac{1-x}{x}\right], \qquad (14)$$

where $x = W/W_0$. The behaviour of $g(W, W_0)$ and the accuracy of Eq. (14) afforded in the regime of higher E_0 are shown in Figs. 2(a) and (b). It is seen that $g(W, W_0)$ falls rapidly with increasing electron energy so that factors such as the introduction of L_0 and $C(Z, W)$ that we discussed above, which modify the electron spectrum, will affect $\delta(Z, W_0)$, perhaps substantially, as we shall illustrate below. It is also seen that, represented as a function of E/E_0 rather than as a function of $x = W/W_0$, the $g(W, W_0)$ curves intersect almost at a point. This is presumably just an algebraic curiousity but is not unexpected from the approximation of Eq. (14) which affords exact such a crossing as a function of x, on the vanishing of the coefficient of the term in $\ln 2W_0$ which occurs at $x = 0.3363$. Using the exact $g(W, W_0)$ of Eq. (5), as in Fig. 2, the crossing of the $g(W, W_0)$ curves as E_0 increases varies by only 0.0027 either side of the value $E/E_0 = 0.3590$ as E_0 varies from 5 to 20 MeV and has fallen only to $E/E_0 =$ 0.322 at $E_0 = 2$ MeV. We do not make use of this directly but note here that it implies that the sense of the effect of L_0 and $C(Z, W)$ upon $\delta(Z, W_0)$ will be general although its magnitude will, of course, depend greatly upon E_0 and Z. We take this as an indication that for accurate work we should include L_0 and $C(Z, W)$ in our presentation of $\delta(Z, W_0)$, as will be done below, then designating it as $\delta(Z, W_0)_{LC}$.

We now apply the $g(W, W_0)$ of Eq. (14) to the evaluation of $\delta(Z, W_0)$ at lowish Z. As just pointed out, it will be important to use realistic L_0 and $C(Z, W)$ corrections for best results. However, the analytical incorporation of L_0 and $C(Z, W_0)$ into the electron spectrum over which the $g(W, W_0)$ of Eq. (14) is then integrated, although possible, leads to excessive complication: we rather integrate $g(W, W_0)$ over the simple spectrum of Eq. (2), in its lowest-order expansion in power of αZ, i.e. in effect approximating $F(Z, W)$ by $1 + \pi\alpha ZW/p$, and separately presenting the effect of introducing L_0 and $C(Z, W)$. We carry out the integration in the $\beta \to 1$ approximation of Eq. (14)

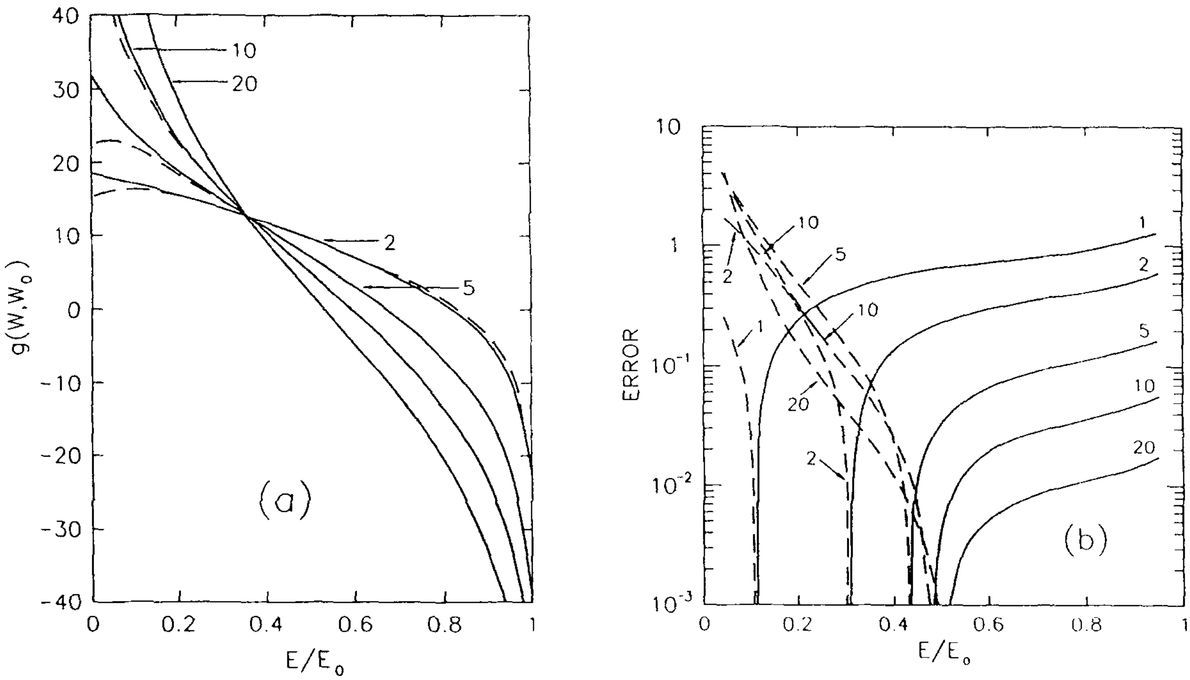


Fig. 2. (a) The solid lines show the exact $g(W, W_0)$ of Eq. (5); the dashed lines show the approximation of Eq. (14); where no dashed line shows it is within the thickness of the solid. (b) The accuracy afforded by the approximation of Eq. (14); the error is defined as $g_{\text{approx}} - g_{\text{exact}}$; full lines mean positive error, dashed negative. In both figures the numbers labelling the curves are the E_0 values in MeV.

namely by replacing p by W except in the Coulomb term $\alpha ZW/p$ where we use $W/p \simeq 1 + 1/2W^2$. All the integrals involved in the application of Eq. (14) to this phase space factor to first order in αZ can be performed exactly with the exception of one whose result is given in terms of the Spence function. Carrying everything to fifth order in $1/W_0$ and expanding the Spence function and $\ln(1 - 1/W_0)$ to this same order we find:

$$
\begin{aligned}
\frac{2\pi}{\alpha}\delta(Z, W_0) = {} & 3\ln M + \frac{81}{10} - \frac{4\pi^2}{3} - 3w - 3y \\
& + z(5 - 5y) + z^2(25 - 10y) \\
& + z^3\left(-\frac{347}{18} - \frac{20\pi^2}{3} + \frac{70w}{3} + \frac{140y}{3} - 40wy\right) + z^4(6 + 10\pi^2 - 30w - 75y \\
& + 60wy) + z^5\left(\frac{11271}{50} - 4\pi^2 + \frac{54w}{5} - \frac{204y}{5} - 24wy\right) + \pi\alpha Z\left[z\left(\frac{5}{2} + \frac{5y}{2}\right)\right. \\
& + z^2\left(\frac{877}{36} - \frac{100w}{3} + 5w^2 - \frac{40y}{3} + 10wy\right) \\
& + z^3\left(\frac{11}{4} - 10\pi^2 + 75w + 55y - 60wy\right) \\
& + z^4\left(-\frac{341}{4} + 10\pi^2 - 45w - 10y + 60wy\right) + z^5\left(\frac{2597}{4} + 30\pi^2 - \frac{1315w}{3} \right. \\
& \left.\left. + 50w^2 - 495y + 280wy\right)\right], \qquad (15)
\end{aligned}
$$

where $w = \ln W_0$; $y = \ln 2$; $z = 1/W_0$.

Figs. 3(a) and 3(b) show the accuracy of Eq. (15) compared with Eq. (7) in which we use the full $F(Z, W)$ of Eq. (1) and the exact $g(W, W_0)$ of Eq. (5) with f similarly given exactly by Eq. (2).

As remarked above, we must now additionally consider the effect of using f_{LC} as the phase space factor and similarly introducing L_0 and $C(Z, W)$ into the integral of Eq. (7) to generate $\delta(Z, W_0)_{LC}$; this is shown in Fig. 4.

2.3. The outer radiative correction of order α: general

We have seen that a useful analytical approximation to $\delta(Z, W_0)$ can be found for the interesting regime of lowish Z and highish E_0. We must, however, be able to present $\delta(Z, W)_{LC}$ with adequately high accuracy, i.e. with an absolute error of 10^{-4} or better, for all Z and

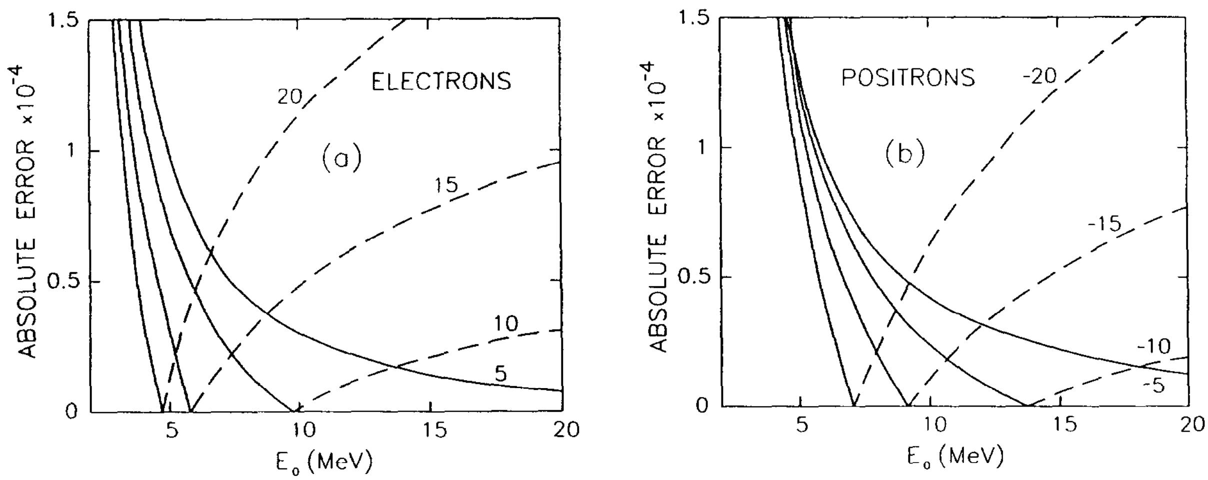


Fig. 3. (a) The approximate $\delta(Z, W_0)$ of Eq. (15) for electrons minus the exact; solid lines denote positive error, dashed mean negative. (b) As (a) but for positrons. The numbers labelling the curves are the Z-values.

all E_0. This can be done only in tabular and graphical form; we here make a graphical presentation.

Fig. 5 provides an orientation into two matters. The first is that the radiative correction initially falls rapidly with E_0 from its value of 1.835% at $E_0 = 0$ as given by Eq. (8). (This value is, of course, independent of Z and of our detailed treatment of the electron spectrum.) This indicates that in the regime of lower E_0, say for $E_0 < 3$ MeV, it will be best to present results for $\delta(Z, W_0)_{LC}$ relative to those for $\delta(0, W_0)$ while for $E_0 > 3$ MeV they may be presented directly. The second matter is that the radiative correction can indeed differ significantly as between $\delta(Z, W)$, $\delta(Z, W)_L$, and $\delta(Z, W)_{LC}$: in the extreme case of high energy and heavy nuclei the differences approach a factor of two; the qualitative effects are as expected, for example for electrons, as displayed in Fig. 6, L_0 favours lower electron energies and hence, as seen from Fig. 2, its introduction will raise the radiative correction while for positrons the opposite is the case.

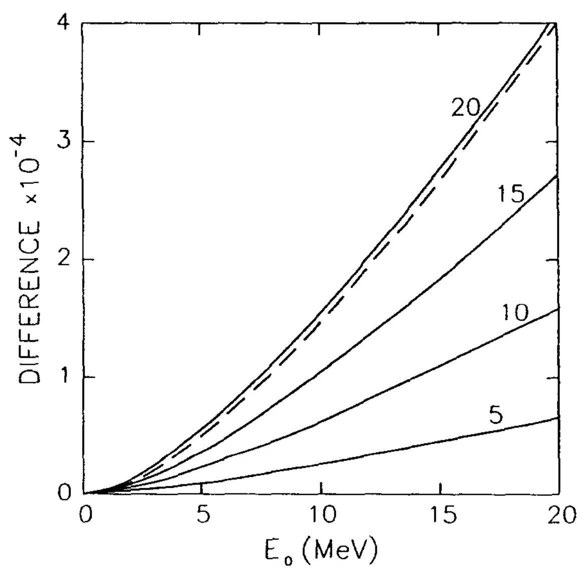


Fig. 4. The difference $\delta(Z, W_0) - \delta(Z, W_0)_{LC}$ with $\delta(Z, W_0)$ exact for the Z-values labelling the curves. The solid curves are for electrons. The dashed curve is for positrons of $Z = -20$; elsewhere the curves for positrons are very close to those for electrons of the same $|Z|$. The sign of the difference is negative for electrons, positive for positrons.

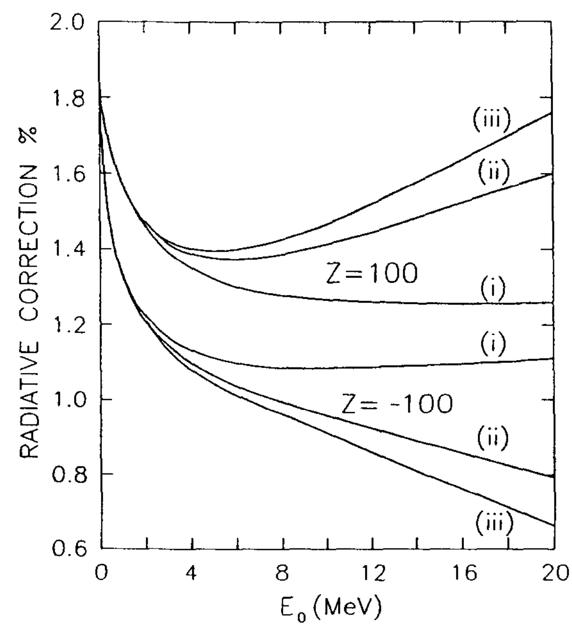


Fig. 5. $\delta(Z, W_0)$ (curves(i)); $\delta(Z, W_0)_L$ (curves(ii)) and $\delta(Z, W_0)_{LC}$ (curves(iii)) for $Z = 100$ and $Z = -100$.

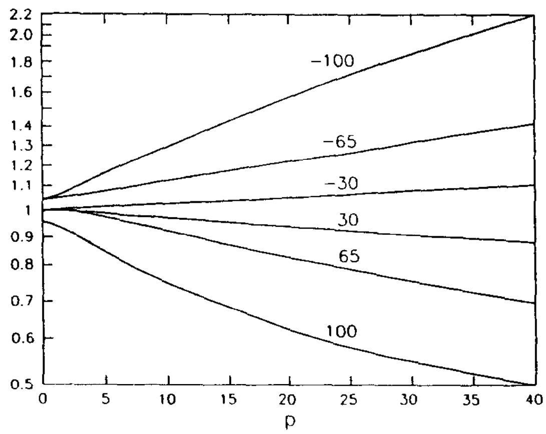


Fig. 6. L_0 versus p for the Z-values shown.

We have already seen from Fig. 4 that $\delta(Z,W)_{LC}$ differs from $\delta(Z,W)$ by significant amounts even for lightish nuclei if the highest accuracy justified by today's experimental precision is to be aimed at.

The regime $E_0 < 3$ MeV is presented in Fig. 7. It is seen that electrons and positrons behave remarkably differently within this regime, a difference further illustrated in Fig. 8. These differences are illuminated by Fig. 9 which shows the electron and positron spectra, suitably normalized, for low E_0 and high $|Z|$. We see, in particular, that the positron spectra are much more sensitive to E_0 than are those of electrons and that they are also much more sensitive to $|Z|$.

This insensitivity of the electron spectra to Z and to E_0 is easily understood in terms of the behaviour of $F(Z,W)$ in the Coulomb-dominated regime where we have:

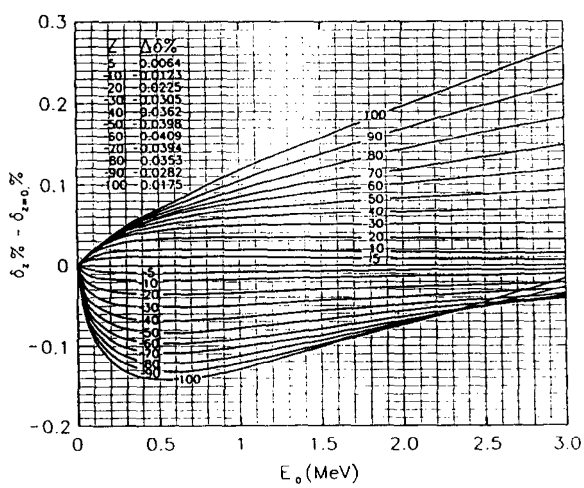


Fig. 7. The difference $\delta(Z,W_0)_{LC} - \delta(0,W_0)$ for $E_0 \leq 3$ MeV. The inset table gives the difference values at $E_0 = 3$ MeV for positrons.

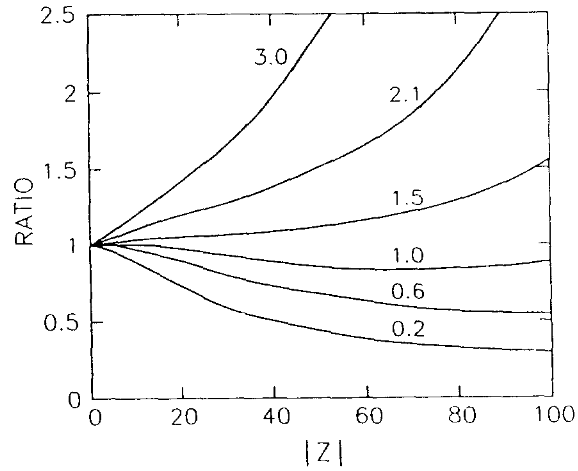


Fig. 8. The modulus of the difference, as defined for Fig. 7, for electrons divided by that for positrons. The numbers labelling the curves give E_0 in MeV.

$$|\Gamma(\gamma + iy)|^2 \to \frac{\pi y}{\sinh \pi y} y^{2(\gamma - 1)}, \tag{16}$$

where $y = \alpha ZW/p$ so that with the further approximation for negative electrons, increasingly valid within the Coulomb-dominated regime:

$$\sinh \pi y \sim e^{\pi y} \tag{17}$$

we find that the $F(Z,W)$ of Eq. (1) becomes simply proportional to W/p as $p \to 0$.

This reduction of $F(Z,W)$ means that, in the Coulomb-dominated regime, the electron spectrum:

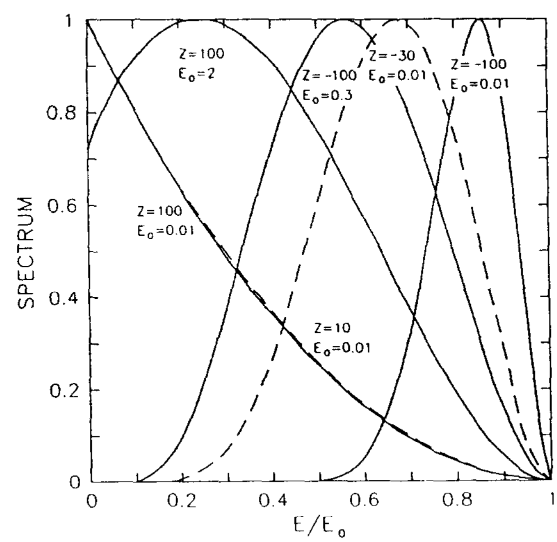


Fig. 9. Some electron and positron spectra normalized at the peak. The differential sensitivity of electrons and positrons in the Coulomb-dominated regime is seen by comparing the full and dashed lines for the same E_0.

$$pW(W_0 - W)^2F(Z, W) \sim W^2(W_0 - W)^2 \tag{18}$$

that is $x^2(1-x)^2$ independent of both Z and E_0 on the scale of $x = W/W_0$.[4]

This independence of the form of the negative electron spectrum from Z for a given E_0 at low E_0 means that $\delta(Z, W_0)$ becomes similarly independent of Z. This tendency for the $\delta(Z, W)_{LC}$ values to clump together at high Z for electrons with no such striking tendency for positrons (for which we have no similar cancellation of energy-dependent and Z-dependent terms) is illustrated in Fig. 10(a) where we see that the $\delta(Z, W)_{LC}$ values are approximately uniformly spread in Z as between electrons and positrons at $E_0 \simeq 1$ MeV but that electron clumping takes place increasingly for higher Z and lower E_0. This clumping for electrons and their distinction from positrons is seen more clearly in Fig. 10(b) for lower E_0 and is dramatized in Fig. 10(c) for yet lower E_0. Figs. 10(b) and 10(c) also show some (weaker) tendency for the positrons' $\delta(Z, W)_{LC}$ values themselves to clump together at high Z at the lowest E_0 values. This is due to the (weaker) tendency for the positron spectra to become similar under those conditions owing to the Coulomb repulsion which drives the maximum of the beta-spectrum towards E_0 as seen in Fig. 9. Figs. 10(b) and 10(c) show the effect of evaluating $\delta(Z, W)_{LC}$ directly using $g(W, W_0)$ appropriate to $E = 0.9E_0$ (cf. the spectrum in Fig. 9 for $Z = -100$ and $E_0 = 0.01$ MeV) rather than integrating $g(W, W_0)$ across the beta-spectrum; it is seen that the actual $\delta(Z, W)_{LC}$ values for positrons are roughly asymptotic to this line at the highest $|Z|$ and lowest E_0.

As mentioned above, for the higher E_0 values it is more convenient to present $\delta(Z, W)_{LC}$ directly which is done for electrons in Fig. 11. A similar presentation is made for positrons in Fig. 12; however, we see that in this case there is confusion in that the $\delta(Z, W)_{LC}$ values criss-cross with Z at a given E_0; it is here more convenient to present $\delta(Z, W_0)_{LC}$ versus Z for various E_0 values; this is done in Fig. 13.

3. Vector and axial transitions

The vast majority of allowed beta-transitions are axial in nature. We have therefore presented all our data so far using the ${}^AC(Z, W)$ of Eq. (4). We should, however, be aware of the difference in $\delta(Z, W)_{LC}$ as between axial and vector transitions. This is displayed in Fig. 14. We see that for the only region in which the question is likely seriously to arise in practice, namely for lightish nuclei, the differences are negligible.

[4] It is often said that within the Coulomb-dominated regime we may write Eq. (16) without its last factor of $y^{2(\gamma-1)}$; this factor is, however, present and cancels the $p^{2(\gamma-1)}$ of Eq. (1) as $p \to 0$ giving the simplicity of Eq. (18).

4. Dependence upon *R*

Everything so far has been presented in terms of Z–A relationships appropriate to the floor of the stability valley and for r_0-values, in $R = r_0A^{1/3}$, of $r_0 = 1.2$ fm for $|Z| > 60$ and for experimental values for $|Z| \leq 60$ (ranging from about $r_0 = 1.21$ fm at $|Z| = 60$ up to 1.35 fm at $|Z| = 10$ and 1.43 fm at $|Z| = 5$). We should, however, enquire into the change in $\delta(Z, W)_{LC}$ as we move away from the floor of the stability valley in either direction towards the neutron and proton drip lines. The largest such movement in R, in relative terms, takes place in the middle of the periodic table where it may be parameterized for a given Z by retaining the A-value of the stability valley and permitting r_0 to range between $r_0 = 1.0$ and 1.5 fm. The effect for electrons at $Z = 30$ and 60 is shown in Fig. 15. The magnitude of the effect for positrons is closely similar but of opposite sign.

5. Exponentiation

It will be observed that Eq. (5) for $g(W, W_0)$ displays an unphysical singularity at $W = W_0$. This singularity is only logarithmic and disappears harmlessly on integration over the spectrum to gain $\delta(Z, W)$. We must wonder, however, whether this finite value may not conceal a significant error in $\delta(Z, W)$ due to the end-point singularity in $g(W, W_0)$ itself. This singularity is associated with the emission of soft real photons and may be eliminated by including such contributions to all orders in perturbation theory; this converts the logarithmic singularity into a power behaviour which is finite at W_0 [12]:

$$X \ln(W_0 - W) \to (W_0 - W)^X - 1 \tag{19}$$

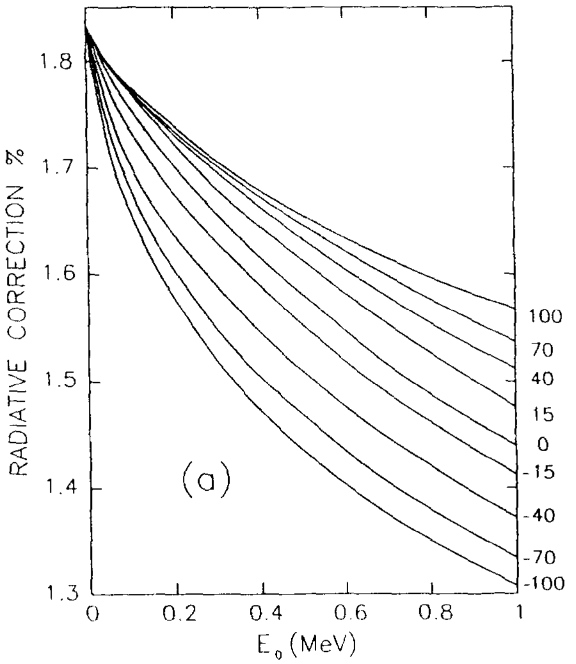
where:

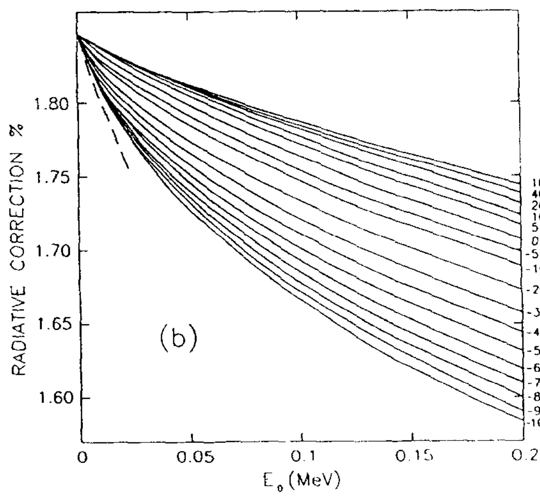
$$X = \frac{2\alpha}{\pi}\left[\frac{\operatorname{arctanh}\beta}{\beta} - 1\right]. \tag{20}$$

This exponentiation of the $g(W, W_0)$ of Eq. (5) results in an increase of $\delta(Z, W_0)$ that is shown in Fig. 16.

6. Higher powers of α

Eq. (5) gives the outer radiative correction of order α. We have seen how its integration across the beta-spectrum, following Eq. (7), leads to contributions to





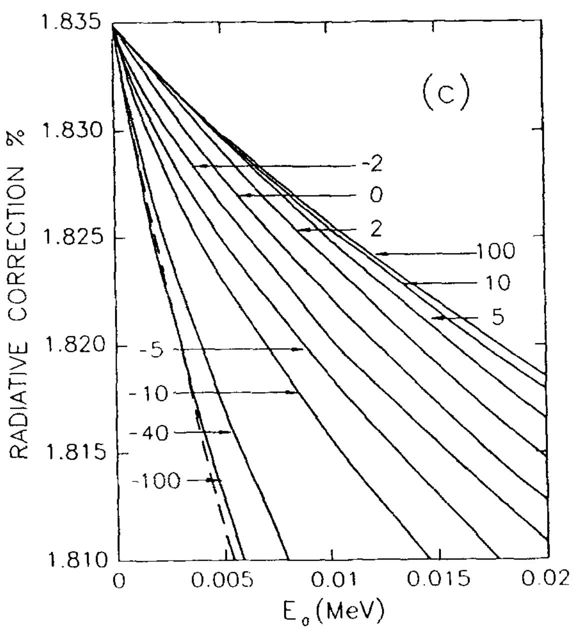


Fig. 10. (a) $\delta(Z, W_0)_{LC}$ for small E_0-values showing the tendency for independence of $\delta(Z, W_0)_{LC}$ from Z for electrons as E_0 falls and the weaker tendency for positrons. In (b) and (c) the dashed lines show $(\alpha/2\pi)\ g(W, W_0)\%$ where W is evaluated at $E = 0.9E_0$. The numbers labelling the curves are the Z-values.

$\delta(Z, W)$ of order $Z\alpha^2$, $Z^2\alpha^3$ ... that result from further virtual photon exchange with the nucleus as a whole additional to that lying behind Eq. (5) and that have, in effect, been the chief object of study in this paper. There are, however, authentically higher-order, Z-independent, contributions to the outer radiative correction that result from multiple real and virtual photon effects associated with the primary single-nucleonic process itself. These contributions of higher order in α are too complicated to evaluate in toto but may be evaluated approximately in leading-log order using renormalization group methods [8]. This results

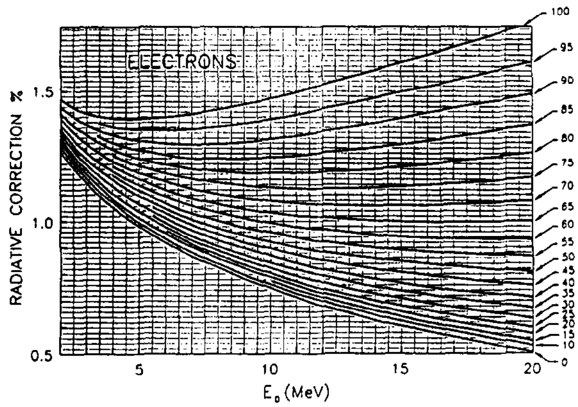


Fig. 11. $\delta(Z,W)_{LC}$ for electrons. The numbers labelling the curves are the Z-values.

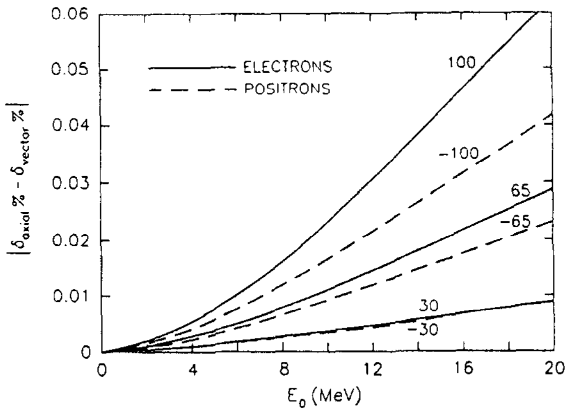


Fig. 14. The difference between $\delta(Z,W)_{LC}$ for axial and for vector transitions. The numbers labelling the curves are the Z-values. The differences are positive for electrons, negative for positrons.

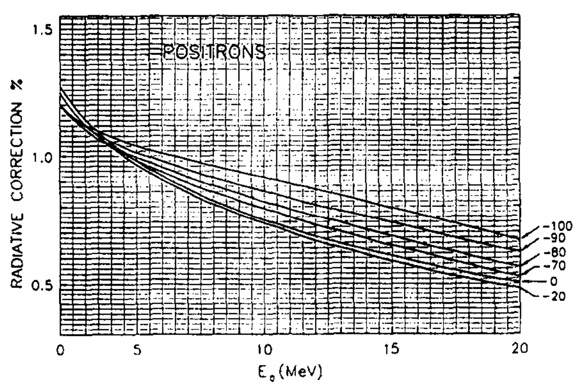


Fig. 12. As Fig. 11 for positrons.

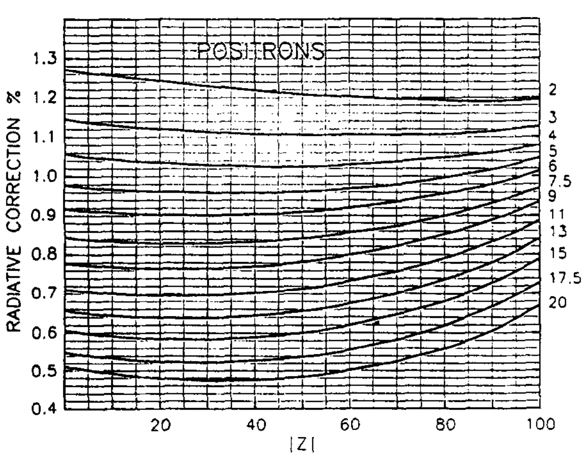


Fig. 13. $\delta(Z,W_0)_{LC}$ for positrons. The numbers labelling the curves give E_0 (MeV).

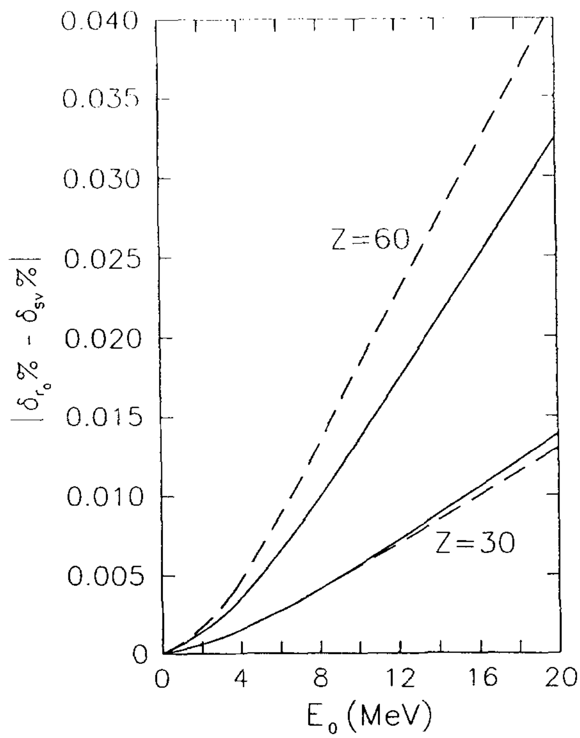


Fig. 15. The change of $\delta(Z,W_0)_{LC}$ with r_0, for which it is designated δ_{r_0}, on movement out of the stability valley, where it is designated δ_{SV}%. Solid lines are for $r_0 = 1.0$ fm (for which the difference is negative); dashed lines are for $r_0 = 1.5$ fm (positive difference). The figure is for electrons; the effect for positrons is closely similar but of opposite sign.

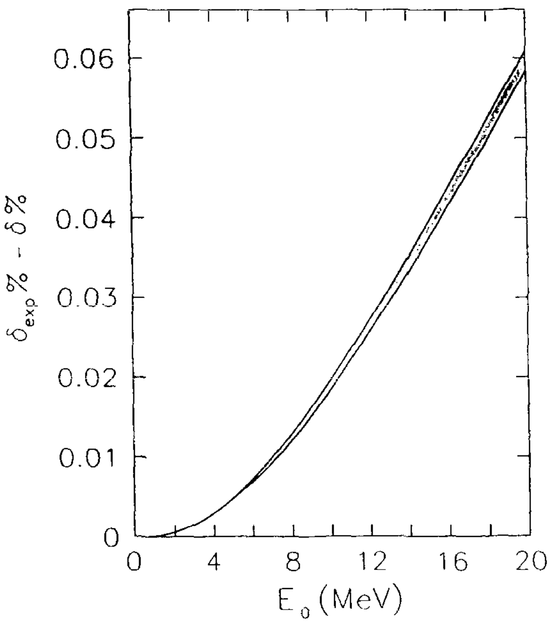


Fig. 16. Effect of exponentiation following the substitution of Eq. (20); δ_{exp} designates the exponentiated version of $\delta(Z, W_0)_{LC}$ and δ that following Eq. (15). The shaded band covers all Z-values $-100 \leq Z \leq 100$.

in the replacement of $\delta(Z, W_0)$ as defined in Eq. (7) by:

$$\delta(Z, W_0)_> = \delta(Z, W_0) \frac{\alpha(M)}{\alpha} S(M, M_Z), \tag{21}$$

where $\alpha(M)$ is the fine structure constant evaluated at M in the $\overline{MS}$ scheme and where the short-range factor $S(M, M_Z)$ is a complicated combination of powers of the ratios between the fine structure constants, evaluated in the $\overline{MS}$ scheme, at the masses of the nucleon, the tauon, and c and b quarks and the W and Z bosons.[5]

The best currently-available values for $\alpha(M)$ and for $S(M, M_Z)$ are due to F. Jegerlehner as reported in Ref. [14] namely:

$$\alpha(M)^{-1} = 133.85 \pm 0.11, \tag{22}$$

$$S(M, M_Z) = 1.022284 \pm 0.000020 \tag{23}$$

which yield:

$$\delta(Z, W_0)_> = (1.0466 \pm 0.0009)\delta(Z, W_0). \tag{24}$$

The best present estimate of the Z-independent outer radiative corrections (pace our comments on their entrainment of Z by integration over the beta-spectrum) is therefore given by enhancing the $\delta(Z, W_0)_{LC}$ of this paper by the 5% or so of Eq. (24). The reliability of this enhancement factor is unknown.

[5] The explicit formula for $S(M, M_Z)$ was given to me by Professor A. Sirlin and is written out in Ref. [13].

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