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EVALUATION OF BETA-DECAY
I. The traditional phase space factors

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The evaluation of beta-decay involves consideration of a large number of influences. These are enumerated. For purposes of highest accuracy it is desirable that these influences be exposed as far as possible in analytical form so that their interplay and their dependence upon their own internal parameterization may be transparent. A programme to this effect is defined and is embarked upon by presenting the analytical forms for the traditional differential and integral phase space factors for (allowed) beta-decay as expansions in powers of αZ. The accuracy of these expansions is indicated in detail.

1. Nomenclature

E = electron kinetic energy in MeV, $W = E/0.51100 + 1$, $p = (W^2 - 1)^{1/2}$, E_0, W_0, p_0 = respective end points, $\alpha = e^2/\hbar c = 137.036$, $\gamma = [1 - (\alpha Z)^2]^{1/2}$. (The above values of the natural constants are exact to the decimal places quoted [1].) Z everywhere positive for (negative) electron emission and negative for positron emission.

2. Introduction

The evaluation of beta-decay involves consideration of a large number of influences. These increase in number, and the care with which they must be considered also increases, as we aim at higher precision in evaluation of the energy spectrum – the differential phase space factor – and of the total transition rate – the integrated phase space factor. These influences include:

(1) the primitive phase space factor $p^2(W_0 - W)^2\,\mathrm{d}p$ that, for allowed beta-decay, describes the partition of energy between the electron and the (massless) neutrino;

(2) the effect of a finite mass for the neutrino;

(3) the effect of the finite mass of the recoiling daughter nucleus which makes its own (coupling-dependent) contribution to the phase space;

(4) other p,W-dependent factors that enter the phase space for other than allowed beta-decay in lowest order;

(5) the effect of possible S, T as well as V, A interactions, interferences between which introduce $1/W$ (Fierz) terms into the primitive phase space factor proportional to $g_S g_V$ and $g_T g_A$;

(6) the effect of possible right-hand currents (viz. left–right symmetric nonstandard electroweak models);

(7) the "traditional" Fermi function $F(Z, W)$ that multiplies the primitive phase space factor and that describes the effect of the Coulomb field of the (point) daughter nucleus on the departing electron;

(8) the modification of the wave function of the departing electron by magnetic interaction with the daughter nucleus;

(9) the screening of the Coulomb field of the daughter nucleus by the atomic electrons;

(10) rearrangement (atomic mismatch, excitation and ionization) effects of the electron shells due to the transition;

(11) the possibility of (negative electron) beta-decay directly into a bound orbital state of the daughter atom;

(12) the effect, in the case of positron emission, of the competing capture of orbital electrons;

(13) the effect of the finite size, radial distribution and shape of the nucleus in modifying the solutions of the Dirac equation for the electron inside the volume of the nucleus and in modifying the convolution between the electron and neutrino wavefunctions;

(14) radiative corrections involving both real and virtual photons;

(15) effects, other than the building-up of the overall charge distribution of the nucleus, to do with the details of the nucleonic orbitals, e.g., their convolution with the lepton wave functions and their own relativistic structure;

(16) many-body nuclear structure effects properly speaking.

Some of these influences (e.g. (2) and (3)) are sufficiently small to ignore for all practical purposes except, perhaps, in extreme and special cases; some are very large and may or may not be susceptible of precise evaluation in which case good estimation of their relia-

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bility becomes critical (e.g. (9)); some are of imperfectly known magnitude, only not-very-sharp limits being presently available for them, and await more detailed consideration in due course (e.g. (5)).

It is the purpose of this series of papers to examine these various influences in critical detail strictly from the point of view of their practical evaluation and with particular regard to the reliability of that evaluation.

Of course, the eventual result of any beta-decay is the folding-together of all these influences which therefore may not properly be treated as separate the one from the other; the treatment given to them individually must therefore be appropriate to handle this folding while keeping track of the separate effects and displaying their mutual interactions. A consequence of this desideratum, the making transparent of the interrelationships, is that as far as possible the effects should be treated algebraically rather than in terms of the transcendental functions through which they chiefly find their most succinct expression. Algebraic representation is also desirable to facilitate exploration of the effects of various changes of important parameters such as the nuclear radius and the natural constants. As far as possible in these papers the algebraic approach will be pursued from a fundamental basis although there must be occasional recourse to parameterization.

This first paper of the programme concerns itself with the basic matter of the "traditional" differential and integrated phase space factors.

3. Necessary accuracy

In considering each influence at work, as listed (1)–(16) above, we must have regard for the magnitude of the others and the confidence that we can repose in their evaluation. We must have particular regard for influence (16), namely our knowledge of, or ignorance about, the many-body nuclear wave functions between which the beta-decay has play; this limits the purposes for which we seek to provide the information as to the differential and integral phase space factors and defines reasonable bounds to the precision to which we should aspire.

As examples of the first of these points consider effects (9) and (14), namely screening and radiative corrections. The screening effects on the total emission probability of electrons by a nucleus of $Z = 60$ are roughly 3% for $E_0 = 0.1$ MeV, 2% for $E_0 = 1$ MeV and 0.4% for $E_0 = 10$ MeV; for positron emission the corresponding corrections are about 60%, 4% and 0.5% respectively. Since, as will be discussed in a subsequent paper, considerable uncertainties attach to the estimates of screening corrections in the heavier elements it is not reasonable to strive for accuracy in our evaluation of the basic phase space superior to, at most, a few percent of the above screening corrections themselves by the time we consider $Z = 60$, viz. at best the order of a few times 10^{-2}% for the more energetic transitions and considerably poorer for those of lower energy, particularly for positrons. For the lighter nuclei and the more energetic transitions ($E_0 > 1$ MeV or so) the screening correction derived via the standard WKB approximation and using screening potentials from atomic Hartree–Fock prescriptions amounts to some $0.04Z^{4/3}/W_0$%. So from this point of view we might reasonably aim at a precision in our evaluation of the integrated phase space factor of, say, $10^{-3}Z^{4/3}/W_0$% viz. about 10^{-3}% for $Z = 10$, $E_0 = 10$ MeV and correspondingly for other Z, E.

Radiative corrections are of the form, and broadly of the order, $Z^m\alpha^{m+1}$; it is therefore useful to compare this correction with the $(\alpha Z)^n$ term in the (αZ)-expansion of the integrated phase space factor to determine the order n at which they become comparable. The point of this comparison is that although the radiative correction of order α is known exactly, and is the same for both Fermi and Gamow–Teller transitions, that of order $Z\alpha^2$ is complicated in evaluation and is known only for Fermi transitions while those of higher order are known only very incompletely. There is therefore no point in proceeding far beyond the value of n, in the (αZ)-expansion, at which $(\alpha Z)^n \simeq Z\alpha^2$, viz. $n = 3$ for $Z = 10$; $n = 7$ for $Z = 60$. Since $(\alpha Z)^{n+1} \simeq 3 \times 10^{-5}$ for $Z = 10$ and $\simeq 10^{-3}$ for $Z = 60$ this consideration of justifiable and useful accuracy is not very different from that deriving from consideration of the screening correction.

When we turn to the uses to which the calculations of the phase space factor might be put our objectives become even more modest. The only case where the beta-decay transition rate is independent of the details of the nuclear wave functions involved is that of, within the conserved vector current (CVC) hypothesis, the superallowed Fermi transitions and that only to the degree that we ignore the isospin-breaking mismatch between the initial and final nuclear wave functions. This latter mismatch affects the transition rate by several tenths of a percent; its estimation depends on nuclear dead-reckoning and is unsure by a significant fraction of its own value so that evaluation of the integrated phase space factor to better than 1 part in 10^4 is difficult to justify and to better than 1 part in 10^5 cannot be justified. Cases of Gamow–Teller or mixed Fermi/Gamow–Teller transitions, with the exception of that of the neutron itself, are limited in interpretability by nuclear structure considerations, including the effects of mesonic exchange, isobar excitation and so on, to, at best, 1% or so: evaluation of the phase space effects to better than 0.1% in such cases can certainly never be justified.

The discussion so far has been chiefly in terms of the

integrated phase space factor, viz. the total transition rate. Many of the influences such as we have noted as limiting the interpretability of the transition rate do not affect, or affect only weakly in terms of their dependence on energy, the differential phase space factor, viz. the shape of the beta-spectrum; evaluation of that differential factor to a significantly greater degree of precision than for the integrated phase space factor may then be justified; we might then aim at a precision of 1 part in 10^4 or a little better, over the whole range of Z, E of concern.

We have so far, without discussion, addressed the range of Z up to 60. The heaviest isotope for which usefully precise values for the parameters of superallowed Fermi beta-decay are available, or likely to be available in the envisionable future, is ^{54}Co, viz. $Z = 26$ for the daughter. The necessary $J^\pi = 0^+$ bodies having their beta-decay within isospin multiplets are becoming available for significantly greater Z-values, and might so continue even towards $Z = 60$, but it is scarcely possible that their vital parameters of energy release and partial lifetime will be measurable to a mutual precision that will permit the extraction of the *ft*-values with the accuracy of 0.1% or so necessary to make a meaningful contribution to our testing of "weak" CVC and extraction of the vector coupling constant particularly in view of the increasing severity of the radiative corrections and of the nuclear mismatch factor.

Again, in our interpretation of Gamow–Teller transitions, which necessarily entrain explicit discussion of nuclear many-body wave functions, we are limited by the range of Z (at present ≤ 20) within which we believe that those wave functions are "complete" in the (still limited) sense of comprising all the configurations of the relevant major shells of the underlying independent particle model. Outside that very restricted range we become increasingly dependent upon explicit or implicit truncation of the independent particle model's basis states or upon models that even lack the elementarity of the independent particle model's starting point. To aim, as we shall, at a precision of 1 part in 10^3 up to $Z = 60$ is certainly luxurious for these cases.

Indeed, for $Z > 60$ there seems to be no present case for enlarging upon the technologies of evaluation already available; we therefore restrict ourselves to the range $Z < 60$ in the present series of papers.

4. The phase space factors

The differential phase space factor for allowed beta-decay is $p^2(W_0 - W)^2 F(Z, W)\,\mathrm{d}p$. Forbidden beta-decay introduces other p, W-dependent terms that may be handled using the techniques to be described here; they will not be discussed further in this paper.

The "traditional" Fermi function is:

$$F(Z, W) = 2(\gamma + 1)\Gamma(2\gamma + 1)^{-2}(2pR)^{2(\gamma-1)} \times e^{\pi\alpha ZW/p}\,|\Gamma(\gamma + i\alpha ZW/p)|^2.$$

$F(Z, W)$ effectively evaluates the electron wave function at the distance R (the "nuclear radius") from the fictitious point charge that, in this simplest approximation, is taken to represent the daughter nucleus; R is introduced to avoid the divergence at the origin and has no physical significance; the important effect of introducing a spatially finite charge distribution for the nucleus, following which the electron wave function remains finite everywhere, will be discussed in a subsequent paper.

Following the discussion in section 3 above on necessary accuracy we shall expand the energy-dependent part of $F(Z, W)$ in powers of αZ. We do this in stages beginning with $|\Gamma(\gamma + i\alpha ZW/p)|^2$. An expansion valid for $W > [\gamma(2 - \gamma)/(2\gamma - 1)]^{1/2}$ (which ranges from $E = 1.4$ keV at $Z = 10$ to 58 keV at $Z = 60$) is:

$$|\Gamma(\gamma + i\alpha ZW/p)|^2 = \sum_{a,b} Z_a K_b (\alpha Z)^a (W/p)^b. \quad (1)$$

There is no interest in the lengthy and opaque sums and products of Euler's constant and the Riemann zeta functions that the exactly expressed $Z_a K_b$ comprise (some of them containing more than 30 such terms) so we here simply give their numerical values, as we shall do also for the analogous coefficients of the subsequent expansions following their exact derivation, to appropriate accuracy. Table 1 gives the $Z_a K_b$ of eq. (1) up to $a = 10$. The accuracy of expression (1), taken through its various orders a of αZ, is displayed in figs. 1 and 2; the expression is seen to be of high utility.

The next stage is represented by:

$$e^{\pi\alpha ZW/p}\,|\Gamma(\gamma + i\alpha ZW/p)|^2 = \sum_{a,b} Z_a K_b (\alpha Z)^a (W/p)^b. \quad (2)$$

Numerical values of the exactly expressed coefficients $Z_a K_b$ are given in table 2.

Finally we have:

$$p^{2(\gamma-1)}\, e^{\pi\alpha ZW/p}\,|\Gamma(\gamma + i\alpha ZW/p)|^2 = \sum_{a,b,c} Z_a K_b L_c (\alpha Z)^a (W/p)^b (\ln p)^c. \quad (3)$$

The numerical values of the coefficients $Z_a k_b L_c$ (whose exact expression sometimes contains more than 50 terms) are given in table 3.

The integrated phase space function is:

$$f = \int_0^{p_0} p^2 (W_0 - W)^2 F(Z, W)\,\mathrm{d}p. \quad (4)$$

We cannot perform the full integration of eq. (4) using eq. (3) because, as remarked, eq. (3) is not valid to $p = 0$

Table 1
Coefficients Z_aK_b of the (αZ)-expansion of $|\Gamma(\gamma + i\alpha ZW/p)|^2 = \Sigma_{a,b} Z_aK_b(\alpha Z)^a(W/p)^b$

Z_0K_0	1
Z_2K_0	0.577216
Z_2K_2	−1.644934
Z_4K_0	0.722126
Z_4K_2	−2.151539
Z_4K_4	1.894066
Z_6K_0	0.730658
Z_6K_2	−2.993953
Z_6K_4	4.107516
Z_6K_6	−1.971102
Z_8K_0	0.752192
Z_8K_2	−3.786525
Z_8K_4	7.190587
Z_8K_6	−6.128553
Z_8K_8	1.992466
$Z_{10}K_0$	0.768082
$Z_{10}K_2$	−4.593408
$Z_{10}K_4$	11.047845
$Z_{10}K_6$	−13.373173
$Z_{10}K_8$	8.144145
$Z_{10}K_{10}$	−1.998079

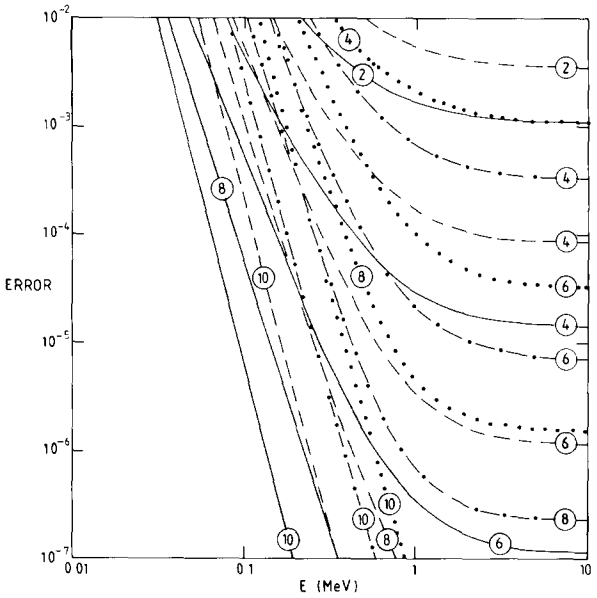

Fig. 2. As fig. 1 but with the full, dashed, dash-dotted and dotted curves for $Z = 30$, 40, 50 and 60 respectively.

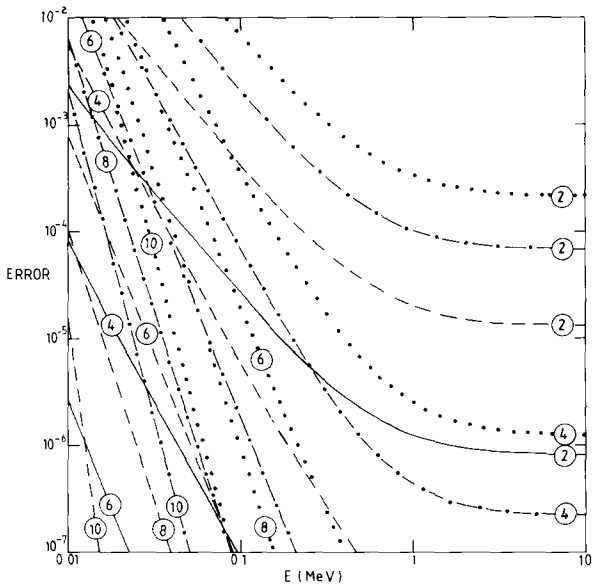

Fig. 1. Accuracy of eq. (1) of the text for the square of the modulus of the complex gamma function. The error is defined as the ratio of the approximate to the exact values less unity (without regard for sign). The full, dashed, dash-dotted and dotted curves are for $Z = 5$, 10, 15 and 20 respectively. The numbers within the circles are the values of a through which eq. (1) is taken.

Table 2
Coefficients Z_aK_b of the (αZ)-expansion of $e^{\pi\alpha ZW/p}|\Gamma(\gamma + i\alpha ZW/p)|^2 = \Sigma_{a,b} Z_aK_b(\alpha Z)^a(W/p)^b$

Z_0K_0	1	Z_8K_0	0.752192
		Z_8K_2	−0.180874
Z_1K_1	3.141593	Z_8K_4	−4.653076
		Z_8K_6	6.179487
Z_2K_0	0.577216	Z_8K_8	−2.008155
Z_2K_2	3.289868		
		Z_9K_1	2.363082
Z_3K_1	1.813376	Z_9K_3	−8.119890
Z_3K_3	0	Z_9K_5	8.959546
		Z_9K_7	−3.167823
Z_4K_0	0.722126	Z_9K_9	0
Z_4K_2	0.696907		
Z_4K_4	−2.164646	$Z_{10}K_0$	0.768082
		$Z_{10}K_2$	−0.881488
Z_5K_1	2.268627	$Z_{10}K_4$	−4.672377
Z_5K_3	−3.776373	$Z_{10}K_6$	10.923585
Z_5K_5	0	$Z_{10}K_8$	−8.164857
		$Z_{10}K_{10}$	2.001989
Z_6K_0	0.730658		
Z_6K_2	0.569598	$Z_{11}K_1$	−6.277150
Z_6K_4	−4.167149	$Z_{11}K_3$	−10.543503
Z_6K_6	2.034686	$Z_{11}K_5$	17.003454
		$Z_{11}K_7$	−12.056502
Z_7K_1	2.295429	$Z_{11}K_9$	3.147902
Z_7K_3	−5.674039	$Z_{11}K_{11}$	0
Z_7K_5	3.257605		
Z_7K_7	0		

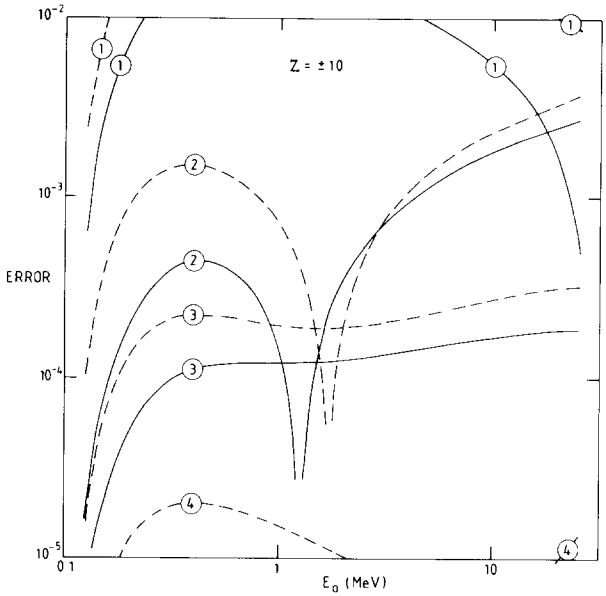

Fig. 3. Accuracy of the algebraic evaluation of the integrated phase space factor as described in the text. The solid curve is for electron emission ($Z = 10$); the dashed curve is for positron emission ($Z = -10$). The error is defined as the ratio of the approximate to the exact values less unity (without regard for sign).

and because most of the separate integrals implied in eq. (4) diverge at $p = 0$. We may still, however, in a manner satisfactory for our handling of the vast majority of cases of practical interest where high accuracy is required, split eq. (4) into two parts: $f = f_< + f_>$. Here

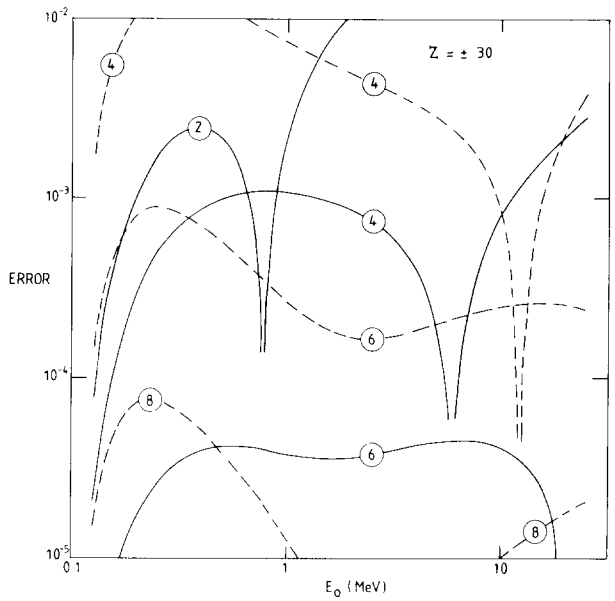

Fig. 5. As fig. 3 but for $Z = \pm 30$.

$f_<$ expresses the integral taken from $p = 0$ to some intermediate value $p = p_1$, while $f_>$ carries the integration from p_1 to p_0.

$f_<$ we evaluate precisely numerically which we can do once-and-for-all writing:

$$f_< = W_0^2 f_<(1) - 2W_0 f_<(2) + f_<(3).$$

The $f_<(1, 2, 3)$ associated with the energy-dependent parts of $F(Z, W)$ we designate $f_<^{\mathrm{E}}(1, 2, 3)$ and list in table 4 and 5.

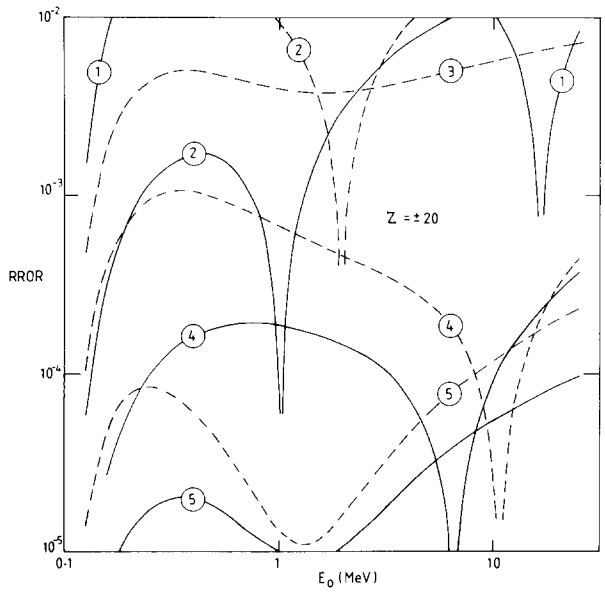

Fig. 4. As fig. 3 but for $Z = \pm 20$.

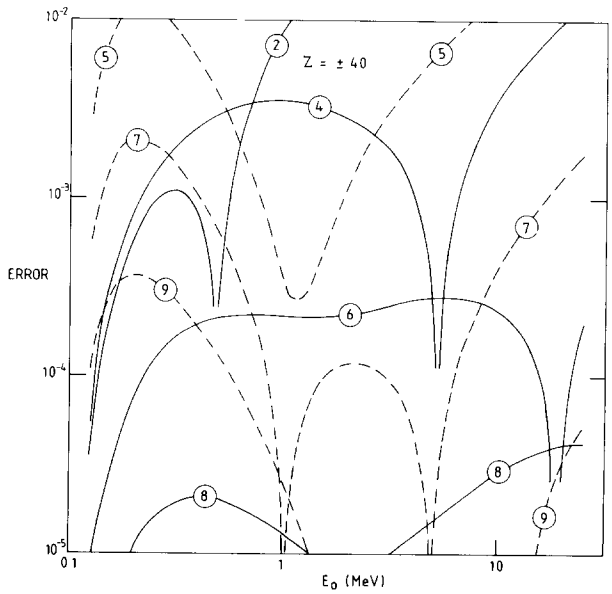

Fig. 6. As fig. 3 but for $Z = \pm 40$.

Table 3
Coefficients $Z_a K_b L_c$ of the (αZ)-expansion of $p^{2(\gamma-1)}$ $e^{\pi\alpha ZW/p}|\Gamma(\gamma+i\alpha ZW/p)| = \Sigma_{a,b,c} Z_a K_b L_c (\alpha Z)^a (W/p)^b$ $(\ln p)^c$

$Z_0K_0L_0$	1	$Z_9K_1L_0$	2.363082
		$Z_9K_1L_1$	−3.334694
$Z_1K_1L_0$	3.141593	$Z_9K_1L_2$	2.078531
		$Z_9K_1L_3$	−0.694928
$Z_2K_0L_0$	0.577216	$Z_9K_1L_4$	0.130900
$Z_2K_0L_1$	−1	$Z_9K_3L_0$	−8.119890
$Z_2K_2L_0$	3.289868	$Z_9K_3L_1$	6.618132
		$Z_9K_3L_2$	−1.888187
$Z_3K_1L_0$	1.813376	$Z_9K_5L_0$	8.959546
$Z_3K_1L_1$	−3.141593	$Z_9K_5L_1$	−3.257605
		$Z_9K_7L_0$	−3.167823
$Z_4K_0L_0$	0.722126		
$Z_4K_0L_1$	−0.827216	$Z_{10}K_0L_0$	0.768082
$Z_4K_0L_2$	0.5	$Z_{10}K_0L_1$	−1.124905
$Z_4K_2L_0$	0.696907	$Z_{10}K_0L_2$	0.745425
$Z_4K_2L_1$	−3.289868	$Z_{10}K_0L_3$	−0.286256
$Z_4K_4L_0$	−2.164646	$Z_{10}K_0L_4$	0.065717
		$Z_{10}K_0L_5$	−0.008333
$Z_5K_1L_0$	2.268627	$Z_{10}K_2L_0$	−0.881488
$Z_5K_1L_1$	−2.598775	$Z_{10}K_2L_1$	−0.305660
$Z_5K_1L_2$	1.570796	$Z_{10}K_2L_2$	0.973067
$Z_5K_3L_0$	−3.776373	$Z_{10}K_2L_3$	−0.527385
		$Z_{10}K_2L_4$	0.137078
$Z_6K_0L_0$	0.730658	$Z_{10}K_4L_0$	−4.672377
$Z_6K_0L_1$	−0.991430	$Z_{10}K_4L_1$	5.965444
$Z_6K_0L_2$	0.538608	$Z_{10}K_4L_2$	−2.624736
$Z_6K_0L_3$	−0.166667	$Z_{10}K_4L_3$	0.360774
$Z_6K_2L_0$	0.569598	$Z_{10}K_6L_0$	10.923585
$Z_6K_2L_1$	−1.519374	$Z_{10}K_6L_1$	−6.688159
$Z_6K_2L_2$	1.644934	$Z_{10}K_6L_2$	1.017343
$Z_6K_4L_0$	−4.167149	$Z_{10}K_8L_0$	−8.164857
$Z_6K_4L_1$	2.164646	$Z_{10}K_8L_1$	2.008155
$Z_6K_6L_0$	2.034686	$Z_{10}K_{10}L_0$	2.001989
$Z_7K_1L_0$	2.295429	$Z_{11}K_1L_0$	2.413001
$Z_7K_1L_1$	−3.114670	$Z_{11}K_1L_1$	−3.533993
$Z_7K_1L_2$	1.692086	$Z_{11}K_1L_2$	2.341823
$Z_7K_1L_3$	−0.523599	$Z_{11}K_1L_3$	−0.899301
$Z_7K_3L_0$	−5.674039	$Z_{11}K_1L_4$	0.206457
$Z_7K_3L_1$	3.776373	$Z_{11}K_1L_5$	−0.026180
$Z_7K_5L_0$	3.257605	$Z_{11}K_3L_0$	−10.543503
		$Z_{11}K_3L_1$	10.010446
$Z_8K_0L_0$	0.752192	$Z_{11}K_3L_2$	−3.781113
$Z_8K_0L_1$	−1.061466	$Z_{11}K_3L_3$	0.629396
$Z_8K_0L_2$	0.661617	$Z_{11}K_5L_0$	17.003454
$Z_8K_0L_3$	−0.221203	$Z_{11}K_5L_1$	−9.773947
$Z_8K_0L_4$	0.041667	$Z_{11}K_5L_2$	1.628802
$Z_8K_2L_0$	−0.180873	$Z_{11}K_7L_0$	−12.056502
$Z_8K_2L_1$	−1.155058	$Z_{11}K_7L_1$	3.167823
$Z_8K_2L_2$	1.170920	$Z_{11}K_9L_0$	3.147902
$Z_8K_2L_3$	−0.548311		
$Z_8K_4L_0$	−4.653076		
$Z_8K_4L_1$	4.708310		
$Z_8K_4L_2$	−1.082323		
$Z_8K_6L_0$	6.179487		
$Z_8K_6L_1$	−2.034686		
$Z_8K_8L_0$	−2.008155		

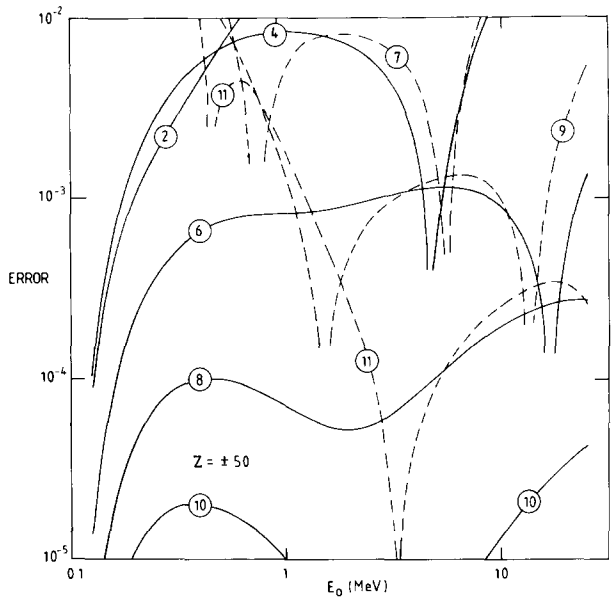

Fig. 7. As fig. 3 but for $Z = \pm 50$.

$f_>$ contains the full flexibility of the present treatment for all transitions of $p_0 > p_1$. We arbitrarily choose p_1 to correspond to $E_0 = 0.1$ MeV, viz. $p_1 = 0.655504$, which is safely within the range of validity of eq. (1) as noted above.

Writing:

$$F(Z) = 2(\gamma+1)[\Gamma(2\gamma+1)]^{-2}(2R)^{2(\gamma-1)},$$

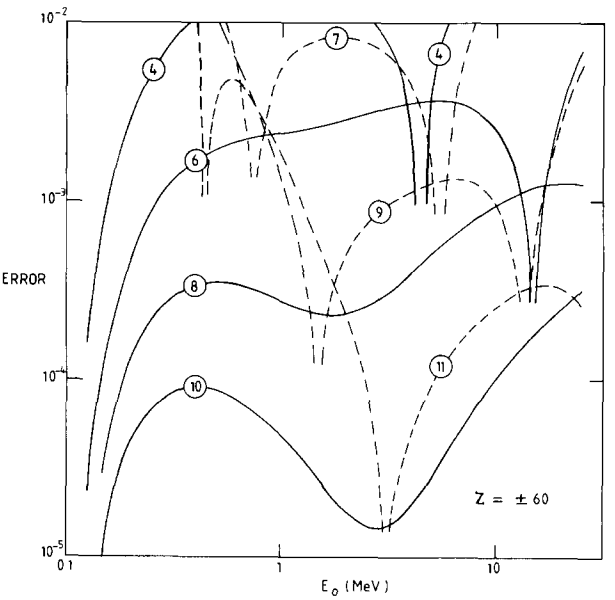

Fig. 8. As fig. 3 but for $Z = \pm 60$.

Table 4
The integrals $f_<^E(1)$, $f_<^E(2)$ and $f_<^E(3)$ for electron emission

Z	$f_<^E(1)$	$f_<^E(2)$	$f_<^E(3)$	Z	$f_<^E(1)$	$f_<^E(2)$	$f_<^E(3)$
1	0.0994455	0.1113197	0.1248722	31	0.3690406	0.4076125	0.4513841
2	0.1052738	0.1177416	0.1319669	32	0.3807167	0 4204578	0.4655528
3	0.1113660	0.1244487	0.1393708	33	0.3925227	0.4334472	0.4798819
4	0.1177165	0.1314354	0.1470784	34	0.4044546	0.4465763	0.4943664
5	0.1243197	0.1386962	0.1550844	35	0.4165084	0.4598408	0.5090015
6	0.1311702	0.1462260	0.1633833	36	0.4286801	0.4732362	0.5237824
7	0.1382629	0.1540193	0.1719701	37	0.4409659	0.4867583	0.5387042
8	0.1455925	0.1620711	0.1808396	38	0.4533618	0.5004025	0.5537619
9	0.1531542	0.1703764	0.1899866	39	0.4658637	0.5141645	0.5689507
10	0.1609430	0.1789300	0.1994060	40	0.4784677	0.5280398	0.5842656
11	0.1689542	0.1877271	0.2090927	41	0.4911698	0.5420239	0.5997015
12	0.1771829	0.1967629	0.2190419	42	0.5039658	0.5561124	0.6152535
13	0.1856246	0.2060325	0.2292484	43	0.5168519	0.5703007	0.6309166
14	0.1942747	0.2155311	0.2397072	44	0.5298237	0.5845842	0.6466855
15	0.2031286	0.2252541	0.2504135	45	0.5428772	0.5989583	0.6625552
16	0.2121820	0.2351967	0.2613623	46	0.5560083	0.6134184	0.6785204
17	0.2214305	0.2453543	0.2725486	47	0.5692127	0.6279597	0.6945760
18	0.2308697	0.2557223	0.2839677	48	0.5824862	0.6425777	0.7107167
19	0.2404954	0.2662962	0.2956145	49	0.5958244	0.6572673	0.7269370
20	0.2503033	0.2770714	0.3074843	50	0.6092230	0.6720239	0.7432316
21	0.2602892	0.2880435	0.3195721	51	0.6226777	0.6868425	0.7595950
22	0.2704489	0.2992078	0.3318732	52	0.6361840	0.7017182	0.7760218
23	0.2807785	0.3105600	0.3443827	53	0.6497373	0.7166459	0.7925062
24	0.2912736	0.3220957	0.3570958	54	0.6633332	0.7316206	0.8090427
25	0.3019304	0.3338104	0.3700077	55	0.6769669	0.7466371	0.8256254
26	0.3127448	0.3456997	0.3831136	56	0.6906339	0.7616902	0.8422486
27	0.3237127	0.3577592	0.3964085	57	0.7043293	0.7767747	0.8589064
28	0.3348303	0.3699846	0.4098879	58	0.7180483	0.7918851	0.8755928
29	0.3460934	0.3823715	0.4235468	59	0.7317862	0.8070161	0.8923018
30	0.3574981	0.3949156	0.4373805	60	0.7455378	0.8221621	0.9090271

($\Gamma(2\gamma+1)$ is tabulated in ref. [2]) we now have:

$$f_< = F(Z)\left[W_0^2 f_<^E(1) - 2W_0 f_<^E(2) + f_<^E(3)\right],$$

$$f_> = F(Z)\int_{p_1}^{p_0} p^2 (W_0 - W)^2 \times \sum_{a,b,c} Z_a K_b L_c (\alpha Z)^a (W/p)^b (\ln p)^c \, dp. \quad (5)$$

The great majority of the integrals within eq. (5) are elementary; a few are "awkward" and are discussed, and instruction as to their appropriate evaluation presented, in the appendix.

The accuracy of this algebraic evaluation of the integrated phase space factor is displayed in figs. 3–8 (which are themselves based upon precisely numerically computed values of all the "awkward" integrals plus the analytical forms for the elementary ones); it is seen that it satisfactorily exceeds the requirements established in section 3 above. We do not, in presenting these figures, pursue the accuracy beyond 1 part in 10^5 since this has no practical utility.

It is interesting to compare the errors of these (αZ)-expansions, as defined and expressed in the figures, with the order of αZ through which the expansion is taken. We might a priori expect that the error on taking the expansion through order a would be comparable with $(\alpha Z)^{a+1}$; therefore call the error divided by $(\alpha Z)^{a+1}$ the relative error. The behaviour of the individual relative errors as a function of a is obviously somewhat capricious, as is seen from the figures. We may, however, for a particular Z and a particular E_0, average over the associated relative errors to get an impression of their general behavior. Taking, illustratively, $E_0 = 2$ MeV we find that for electron emission this average of the (absolute values of) the relative errors falls smoothly from about 2 for low-Z to about unity for high-Z over the range of Z discussed here. As is to be expected, the relative errors are larger for positron emission: the corresponding numbers increase smoothly from about 4 at low-Z to about 16 at high-Z, the fact of the increase being due to the increasing

Table 5
The integrals $f_<^E(1)$, $f_<^E(2)$ and $f_<^E(3)$ for positron emission

$-Z$	$f_<^E(1)$	$f_<^E(2)$	$f_<^E(3)$	$-Z$	$f_<^E(1)$	$f_<^E(2)$	$f_<^E(3)$
1	0.0886014	0.0993518	0.1116295	31	0.0131776	0.0150579	0.0172264
2	0.0835806	0.0938001	0.1054755	32	0.0123051	0.0140674	0.0161002
3	0.0788136	0.0885227	0.0996186	33	0.0114872	0.0131382	0.0150431
4	0.0742897	0.0835084	0.0940476	34	0.0107207	0.0122668	0.0140512
5	0.0699986	0.0787467	0.0887511	35	0.0100026	0.0114500	0.0131209
6	0.0659303	0.0742270	0.0837185	36	0.0093300	0.0106845	0.0122487
7	0.0620751	0.0699393	0.0789391	37	0.0087003	0.0099675	0.0114312
8	0.0584234	0.0658736	0.0744025	38	0.0081109	0.0092960	0.0106653
9	0.0549662	0.0620203	0.0700986	39	0.0075595	0.0086675	0.0099479
10	0.0516947	0.0583702	0.0660176	40	0.0070438	0.0080792	0.0092762
11	0.0486002	0.0549142	0.0621498	41	0.0065615	0.0075289	0.0086475
12	0.0456745	0.0516434	0.0584860	42	0.0061107	0.0070143	0.0080593
13	0.0429097	0.0485496	0.0550171	43	0.0056895	0.0065331	0.0075091
14	0.0402980	0.0456243	0.0517343	44	0.0052959	0.0060834	0.0069946
15	0.0378321	0.0428598	0.0486292	45	0.0049284	0.0056633	0.0065137
16	0.0355049	0.0402485	0.0456936	46	0.0045853	0.0052708	0.0060644
17	0.0333094	0.0377828	0.0429195	47	0.0042650	0.0049043	0.0056445
18	0.0312392	0.0354558	0.0402992	48	0.0039662	0.0045622	0.0052525
19	0.0292879	0.0332606	0.0378254	49	0.0036874	0.0042429	0.0048864
20	0.0274494	0.0311907	0.0354909	50	0.0034274	0.0039450	0.0045448
21	0.0257179	0.0292397	0.0332889	51	0.0031850	0.0036672	0.0042260
22	0.0240879	0.0274016	0.0312127	52	0.0029591	0.0034081	0.0039286
23	0.0225540	0.0256706	0.0292561	53	0.0027486	0.0031666	0.0036513
24	0.0211111	0.0240411	0.0274129	54	0.0025525	0.0029416	0.0033928
25	0.0197544	0.0225078	0.0256774	55	0.0023698	0.0027319	0.0031518
26	0.0184793	0.0210656	0.0240438	56	0.0021997	0.0025366	0.0029273
27	0.0172811	0.0197096	0.0225069	57	0.0020414	0.0023547	0.0027181
28	0.0161558	0.0184352	0.0210615	58	0.0018941	0.0021853	0.0025233
29	0.0150993	0.0172379	0.0197027	59	0.0017570	0.0020277	0.0023420
30	0.0141078	0.0161135	0.0184258	60	0.0016295	0.0018811	0.0021732

severity with increase of Z of the cancellation between successive terms of $e^{\pi\alpha ZW/p}$.

Appendix

The integrals of eq. (5) that were referred to in the text as "awkward" are those that reduce to terms in:

$$A(n) = \int (1/p)\,\ln(p+W)(\ln p)^n\,\mathrm{d}p,$$

$$B(n) = \int (1/p)\,\ln((W-1)/(W+1))(\ln p)^n\,\mathrm{d}p.$$

It is possible to express these integrals in the form of series, e.g.:

$$A(0) = \sum_{k=0}^{\infty} [c_k/(2k+1)][\ln(p+W)]^{2k+1}, \tag{6}$$

$$A(1) = A(0)\ln p - \sum_{k=0}^{\infty} [c_k/(2k+1)] \times \sum_{l=0}^{\infty} [c_l/(2k+2l+1)] \times [\ln(p+W)]^{2k+2l+1}, \tag{7}$$

$$B(0) = [\ln((W-1)/(W+1))]^2/4 + \sum_{k=1}^{\infty} (1/k^2)[2/(W+1)]^k, \tag{8}$$

etc. where in $A(0)$ and $A(1)$ $c_k = [2^{2k}/(2k)!]B_{2k}$, the B_{2k} being the Bernoulli numbers. Such treatment, however, rapidly becomes excessively cumbersome and is largely unnecessary for our purposes as we shall immediately see. It is, however, useful to consider the low-p ($p<1$) and high-p ($p>1$) expansions of the above integrands for which it is helpful to note:
for low-p:

$$\ln(p+W) = p(1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \ldots),$$

$$\ln(W-1) = \ln(p^2/2) - (p^2/4)(1 + Fp^2 + Gp^4 + Hp^6 + Ip^8 + Jp^{10} + \ldots);$$

for high-p:

$$\ln(p+W) = \ln 2p + (1/4p^2)(1 + F/p^2 + G/p^4 + H/p^6 + I/p^8 + J/p^{10} + \ldots),$$

$$\ln(W-1) = \ln p - (1/p)(1 + A/p^2 + B/p^4 + C/p^6 + D/p^8 + E/p^{10} + \ldots),$$

where:

$A = -1/6$; $B = 3/40$; $C = -5/112$; $D = 35/1152$;

$E = -63/2816$; $F = -3/8$; $G = 5/24$; $H = -35/256$;

$I = 63/640$; $J = -77/1024$.

When it is permissible to use these low-p, high-p expansions all the integrals become elementary.

For low values of n treatment must be careful because although an approximate evaluation of the integral in question may be adequate for the order of αZ in which it first appears a small error there becomes much more important for higher orders of αZ. It is therefore recommended that for $A(0)$ and $A(1)$ eqs. (6) and (7) respectively be used. These series expansions are valid only for $\ln(p + W) < \pi$, viz. for $E_0 < 5.41$ MeV but, in any case, it is most economical to use them only below a certain energy above which they should be supplemented by the appropriate high-p forms. Specifically it is recommended that below $E_0 = 1$ MeV eqs. (6) and (7) be used by themselves; if we are not concerned with errors of $< 10^{-6}$ (in the definition of figs. 3–8) it is everywhere adequate to use $k = 10$ in eq. (6) and k, $l = 5$ in eq. (7); such truncation of the series will nowhere affect the resultant error of the (αZ)-approximation as recorded in figs. 3–8 by more than a small fraction of its own value. Above $E_0 = 1$ MeV it is recommended that precisely computed value of $A(0)$ and $A(1)$ from p_t to p_1, where p_1 corresponds to $E = 1$ MeV, viz. $p_1 = 2.78272$, be supplemented by the relevant high-p expansions of $A(0)$, $A(1)$ from p_1 to p_0. The precisely computed values of $A(0)$, $A(1)$ between p_t and p_1 in question are given in table 6. This recipe also results in estimates nowhere differing significantly from those deriving from the exact evaluation of the $A(0)$, $A(1)$ integrals in question.

Similarly for $B(0)$ it is recommended that the series of eq. (8) be evaluated. For larger values of W the series converges only rather slowly but above a certain value of k it becomes reasonable to replace summation over k to infinity by the corresponding integral over k, viz.:

$$B(0) = [\ln((W-1)/(W+1))]^2/4 + \sum_{k=1}^{k_0} + \ln X[e^{-x_0}/x_0 - E_1(x_0)],$$

Table 6
Values of the integrals $A(n)$ and $B(n)$ of the appendix between p_t (corresponding to $E = 0.1$ MeV) and p_1 (corresponding to $E = 1$ MeV)

Integral	Value
$A(0)$	1.638615
$A(1)$	0.691602
$A(2)$	0.017005
$A(3)$	−0.005285
$A(4)$	0.001763
$B(1)$	0 195359
$B(2)$	−0.056401
$B(3)$	0.018134

where $X = (W+1)/2$, $x_0 = (k_0 + Y) \ln X$. Here the exponential integral E_1 is as defined in ref. [3] in which appropriate instructions for its evaluation are given (approximation 5.1.54 of ref. [3] is wholly adequate for our purposes). We should expect $Y \simeq 0.5$; it is found empirically that, with $k_0 = 10$, $Y = 0.513$ gives the best overall fit. This recipe is everywhere wholly adequate.

It is recommended that for the remaining "awkward" integrals, viz. $A(2, 3, 4)$ and $B(1, 2, 3)$, just the simple forms for low-p and high-p be used as relevant based on the respective forms for $\ln(p + W)$ and $\ln(W - 1)$ given above. Such use nowhere results in overall errors that differ significantly from those registered in figs. 3–8. In applying this prescription the relevant integrals from $p = p_t$ to $p = p_1$ are also needed for the high-p cases; their precise numerical evaluation is given in table 6.

References

[1] E.R. Cohen and P. Giacomo, Symbols, Units, Nomenclature and Fundamental Constants in Physics (SUNAMCO 87-1).

[2] D.H. Wilkinson, Nucl. Instr. and Meth. 82 (1970) 122.

[3] M. Abramowitz and I.E. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965).