ELSEVIER

Nuclear Instruments and Methods in Physics Research A 401 (1997) 275–280

NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH Section A

# Evaluation of beta-decay, Part VI: The $Z$-dependent outer radiative corrections for allowed decay[1]

D.H. Wilkinson[a,b,*]

[a] TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada V6T 2A3

[b] University of Sussex, Falmer, Brighton, BN1 9QH, UK

Received 21 July 1997

**Abstract**

A previous paper (D.H. Wilkinson, Nucl. Instr. and Meth. A 365 (1995) 497) presented detailed evaluations of the $Z$-independent radiative corrections (predominantly of order $\alpha$) for allowed beta-decay for $|Z| \leq 100$ and for end-point energies ranging up to 20 MeV. The present paper presents similar evaluations, of varying degrees of certainty, for the $Z$-dependent corrections.

## 1. Introduction[2]

Part V in this series [1] gave a general introduction to the question of the radiative corrections to allowed beta-decay and presented detailed evaluations for the correction of order $\alpha$ with brief comments on $Z$-independent corrections of higher order in $\alpha$. The correction of order $\alpha$, by Sirlin [2], is the same for vector and for axial transitions and is exact *pace* terms of order $\alpha(W/M)\ln(M/W)$ and $\alpha q/M$; its magnitude is about 1%. There are, however, further radiative corrections that are $Z$-dependent and that are additional to those that figure in the Fermi function $F(Z,W)$ and its elaborations. $F(Z,W)$ is itself, in effect, the dominant radiative correction in almost all practical cases; it has been extensively discussed in earlier papers in this series.

Evaluation of the $Z$-dependent radiative corrections properly speaking, that is to say those additional to those contained within the elaborated $F(Z,W)$, is a complicated affair with a long and sometimes tangled history. It is not, indeed, naively obvious that huge terms of order $(Z^2\alpha)^n$ should not occur but it is fortunately the case that they do not [3]. It is also likely that proper radiative corrections of order $(Z\alpha)^n$ are absent although this has not been rigorously demonstrated [3]. What remains clear is that we must eventually consider all radiative correction terms $\delta_{Z^n\alpha^m}$ of order $Z^n\alpha^m$ with

[1] Part I was published in Nucl. Instr. and Meth. A 275 (1989) 378, Part II in Nucl. Instr. and Meth. A 290 (1990) 509, Part III in Nucl. Instr. and Meth. A 335 (1993) 305, Part IV in Nucl. Instr. and Meth. A 365 (1995) 203 and Part V in Nucl. Instr. and Meth. A 365 (1995) 497.

* Correspondence address. Gayles Orchard, Friston, Eastbourne, BN20 0BA, UK. Tel.: +44 1323 423333; fax: +44 1323 42 3329.

[2] The nomenclature is as in the previous members of this series with additions as noted.

0168-9002/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved

PII S0168-9002(97)00971-6

$m \geq n+1$ giving the total radiative correction $\delta_\Sigma$ that speeds the transition by the factor $1 + \delta_\Sigma$:

$$\delta_\Sigma = \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} \delta_{Z^n\alpha^m}. \tag{1}$$

In defining the $\delta_{Z^n\alpha^m}$, it is obviously necessary to avoid double-counting: for example, $\delta_{Z^2\alpha^3}$ must be defined to exclude those terms of order $Z^2\alpha^3$ that arise from the product of $\delta_{Z\alpha^2}$ with $F(Z,W)$. As we shall see, for larger $Z$-values, $\delta_\Sigma$ may amount to several times $\delta_\alpha$ which itself dominates only for low-$Z$.

Of what importance, in practice, are the radiative corrections? The only circumstance in which, at present, we need to know their contribution to the total beta-decay rate to significantly better than 1% is that of the $J^\pi = 0^+ \to 0^+$ super-allowed Fermi transitions within isospin multiplets whose CVC-determined decay is used to determine the $V_{ud}$-element of the Cabibbo–Kobayashi–Maskawa matrix. This special case suffers a small, but very important, correction due to the $Z$-dependence, albeit within an isospin multiplet, of the nuclear structure itself: the nuclear mismatch; this correction amounts to only a few tenths of a percent and may be estimated directly theoretically or semi-empirically; this leaves us with the necessity of knowing the radiative corrections proper[3] to, at worst, a few hundredths of a percent. In such super-allowed Fermi cases as are at present accessible to precise experimentation the radiative correction of order $Z\alpha^2$ ranges up to more than 0.5% (for the decay of $^{54}$Co), and so is of significant importance, while that of order $Z^2\alpha^3$ approaches 0.05% over the same range and so it also demands serious consideration.

In all cases other than the special one just mentioned the beta-decay rate involves a nuclear matrix element whose magnitude we cannot usually estimate with high precision because it involves detailed consideration of the structure of the initial and final nuclear states. However, such detailed consideration is already approaching possibility in some simple cases such as "closed-shell-plus-or-minus-one" nuclei where corrections to the naive nucleonic wave functions that lie behind that sobriquet are becoming increasingly refined. Of course all cases other than that just mentioned of super-allowed Fermi transitions within isospin multiplets of $J = 0$ are additionally complicated by the incompletely understood phenomenon of the apparent quenching of $g_A$ in the nuclear medium. Be all this as it may, however, it is evidently desirable to know the magnitude of the radiative corrections throughout the periodic table, particularly, since $Z\alpha$ approaches unity in the heaviest cases so that convergence with $n$ in corrections of order $Z^n\alpha^m$ may not be very rapid.

It is the object of this paper to consider all values of $n,m$ in $Z^n\alpha^m$.

## 2. Generalities

For a beta-transition of electron energy $W$ and end-point energy $W_0$ we write the differential radiative correction of order $Z^n\alpha^m$ as $Z^n\alpha^m G_{Z^n\alpha^m}(W,W_0)$ so that the total correction of that order, integrated across the beta-spectrum, becomes[4]:

$$\delta_{Z^n\alpha^m} = Z^n\alpha^m \int \mathrm{PS}\, G_{Z^n\alpha^m}(W,W_0)\,\mathrm{d}W \Big/ \int \mathrm{PS}\,\mathrm{d}W, \tag{2}$$

where PS stands for the phase-space of the decay.

It is now important to consider the appropriate choice for the phase-space factor PS to use for the various $n$, $m$-values. Most primitively, we have

$$\mathrm{PS}(1) = pW(W_0 - W)^2, \tag{3}$$

which must everywhere be replaced, for our purposes, by, at least:

$$\mathrm{PS}(2) = \mathrm{PS}(1)\, F(Z,W), \tag{4}$$

[3] "Proper" in this context because one may consider the nuclear mismatch itself to be a kind of radiative correction; it is, however, conventional to separate it from the radiative corrections as defined here.

[4] In the expressions for the $\delta_{Z^n\alpha^m}$ $Z$ is always positive; in those involving the Fermi function $F(Z,W)$ $Z$ is, as usual, positive for negative electron emission and negative for positron emission.

where $F(Z,W)$ is the familiar Fermi function as discussed at length in earlier papers in this series and written out in, e.g., Eq. (1) of Part V [1]. However, as has already been seen in Part V, it may be necessary to refine the phase-space factor further and to use

$$\mathrm{PS}(3) = \mathrm{PS}(2)\, L_0(Z,W)\, C(Z,W), \tag{5}$$

where, as also discussed in Part V, $L_0(Z,W)$ effects the transformation of the appropriate solution to the Dirac equation at a distance $R$ from a point charge, as represented by $F(Z,W)$, to the corresponding solution at the centre of a uniformly charged sphere of radius $R$ while $C(Z,W)$, which has also been written out in Part V, accounts for the appropriate convolution of the nucleonic and leptonic wave functions throughout the nuclear volume. ($C(Z,W)$ differs for the cases of vector and axial decay; the effect of those differences is negligible for our present consideration: we, in fact, here use the form appropriate for axial decay.)

## 3. The correction of order $Z\alpha^2$

This correction was first considered in the pioneering work of Jaus and Rasche [4] followed by further numerical [5] and analytical [6] treatment the latter of which we largely follow here. This correction evidently involves the nucleus as a whole and so must reflect its dimensions and form which is done most directly through the parameter:

$$\Lambda \equiv \sqrt{6}/\langle r^2\rangle^{1/2}, \tag{6}$$

where $\langle r^2\rangle^{1/2}$ is in natural units so that in terms of the $R$-values of Table 2 of Ref. [7]

$$\Lambda \equiv \sqrt{10}/R. \tag{7}$$

($\Lambda$ ranges from about 400 at $A = 10$ to about 160 at $A = 250$.)
Write [6]:

$$G_{Z\alpha^2} = \sum_{i=1}^{4} \Delta_i, \tag{8}$$

where $\Delta_1$ is further split:

$$\Delta_1 = \Delta_1^0 + \Delta_1^F. \tag{9}$$

$\Delta_1^0$ refers to a point nucleus and is energy-sensitive; $\Delta_1^F$ is, neglecting terms of order $W/\Lambda$, energy-insensitive but nuclear-structure-sensitive. $\Delta_1^0$ is combined with $\Delta_4$ and evaluated analytically [6] in the extreme-relativistic approximation

$$\Delta_1^0 + \Delta_4 = \ln M - \tfrac{5}{3}\ln(2W) + \tfrac{43}{18}. \tag{10}$$

$\Delta_1^0 + \Delta_4$ has also been evaluated numerically [5] without invoking the extreme-relativistic approximation; agreement between the two approaches is good; we use Eq. (10) for our present evaluations. We now have [6]

$$\Delta_1^F = 1 - \gamma_E - 4\pi \int_0^\infty \rho(r) r^2 \ln(Mr)\,\mathrm{d}r - (8/M) \times \int_0^\infty \rho(r) r[1 + \gamma_E + \ln(Mr)]\,\mathrm{d}r, \tag{11}$$

$$\Delta_2 = (4/M) \int_0^\infty \rho(r) r(1 - \pi/(4Mr))\,\mathrm{d}r, \tag{12}$$

$$\Delta_3 = \frac{8 g_A(1 + \mu_V)}{M} \times \int_0^\infty \rho(r) r \left\{\gamma_E + \ln(Mr) - \frac{1}{2} + \frac{\pi}{8Mr}\right\} \mathrm{d}r, \tag{13}$$

where the nuclear charge density $\rho(r)$ is normalized:

$$4\pi \int_0^\infty \rho(r) r^2\,\mathrm{d}r = 1. \tag{14}$$

We take $g_A = 1.265$; $\mu_V = 3.706$. It is here appropriate to remark that these equations are derived for the specific case referred to above of vector transitions within isospin multiplets of $J = 0$ and that we have no quantitative guidance as to their reliability in other cases of vector or axial decay; it is, however, unlikely that they might there be grossly in error.

### 3.1. Numerical evaluation of $\delta_{Z\alpha^2}$

We now numerically evaluate Eqs. (10–13), for both negative electron and positron emission, for $1 \le E_0 \le 20$ MeV and for $|Z| \le 100$. Evaluation has been carried out for $\Lambda$-values appropriate, as a function of $Z$, to the floor of the stability valley and for $r_0$-values, for use in the expression

$R = r_0 A^{1/3}$, appropriate to those $Z$, $A$-values. In this evaluation we have used a Fermi form for $\rho(r)$

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-c)/a}} \tag{15}$$

with $c$, $a$-values appropriate to $Z$, $A$.

For purposes of presentation it is useful to note that a useful first approximation to $\delta_{Z\alpha^2}$ is the leading-log term;

$$\delta_{Z\alpha^2} \approx Z\alpha^2 \overline{\ln \frac{A}{W}} \tag{16}$$

and that its crudest evaluation, using the zero-order phase-space factor $W^2(W_0 - W)^2$, appropriate for high-$W_0$, and integrating over all $W$ reads

$$\delta_{Z\alpha^2} \approx Z\alpha^2 \left[\frac{47}{60} - \ln \frac{W_0}{A}\right]. \tag{17}$$

It is now useful to define a reduced form of $\delta_{Z\alpha^2}$

$$\delta_{2r} = \delta_{Z\alpha^2}/Z - \alpha^2 \left[\frac{47}{60} - \ln \frac{W_0}{A}\right] \tag{18}$$

following which we find that a good first approximation to $\delta_{2r}$ is for electrons:

$$\delta_{2ra} = 5.43 \times 10^{-5} - 6.85 \times 10^{-6} E_0 + 1.56 \times 10^{-7} E_0^2 \tag{19}$$

for positrons:

$$\delta_{2ra} = 4.50 \times 10^{-5} - 5.35 \times 10^{-6} E_0 + 9.12 \times 10^{-8} E_0^2, \tag{20}$$

where $E_0$ is the kinetic-energy end-point of the beta-spectrum in MeV.

We finally display, in Fig. 1, the residual $\delta_{2rar}$, where

$$\delta_{Z\alpha^2} = Z \left\{ \delta_{2rar} + \delta_{2ra} + \alpha^2 \left[\frac{47}{60} - \ln \frac{W_0}{A}\right] \right\}. \tag{21}$$

Now, Fig. 1 has been constructed for $r_0$-values appropriate to the floor of the stability valley: specifically using the parameterization, good for $A \geq 6$:

$$r_0(\mathrm{fm}) = 1.614 - 0.1067 \ln A + 0.005456 \ln^2 A + 6.112/(A - 1.76)^2, \tag{22}$$

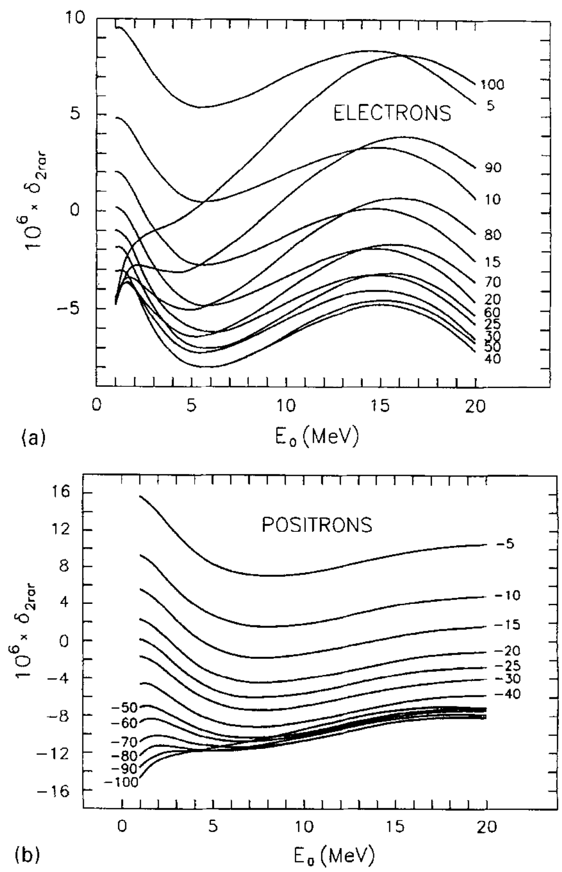


Fig. 1. (a) The reduced correction $\delta_{2rar}$ of order $Z\alpha^2$ as defined in Eq. (21) for negative electron emission. $E_0$ is the kinetic energy end-point of the transition. The numbers labelling the curves are the relevant $Z$-values of the daughter nuclei. (b) As for in (a) but for positron emission.

so the question arises as to the effect upon $\delta_{Z\alpha^2}$ of using some alternative $r_0$-value, $r_0^a$, instead of $r_0$, such as would, in effect, be the case for an $A$-value far out of the stability valley at a particular $Z$-value. Fig. 2 reports the exploration of this question for $Z = 60$; $A = 142$ using $r_0 = 1.2$ fm (Eq. (22) gives $r_0 = 1.22$ fm) for $1.0 \leq r_0^a \leq 1.5$ fm. It is seen that errors of up to 20% of its own value might arise in $\delta_{Z\alpha^2}$ in an extreme case through the use of Fig. 1. This is not likely to be of practical importance but we may note, as also shown in Fig. 2, that if, following Eq. (17), we add to the $\delta_{Z\alpha^2}$ derived from Fig. 1 the quantity $Z\alpha^2 \ln(r_0/r_0^a)$ the error is significantly reduced.

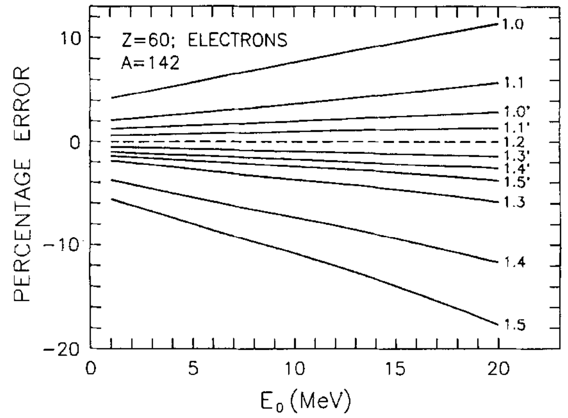


Fig. 2. The effect on $\delta_{Z\alpha^2}$, for electrons, at $Z = 60$; $A = 142$ of changes to the radius constant $r_0$ from a canonical value of 1.2 fm. The percentage error is defined as the error, as a percentage of its own value, entrained in $\delta_{Z\alpha^2}$ by the use of $r_0 = 1.2$ fm rather than a more-correct value as given by the unprimed numbers labelling the curves. (A positive error means that the correct $\delta_{Z\alpha^2}$ is larger than that for $r_0 = 1.2$ fm.) The primed numbers labelling the curves show the effect of adding $Z\alpha^2 \ln(1.2/r_0)$ to the $\delta_{Z\alpha^2}$ for $r_0 = 1.2$ fm.

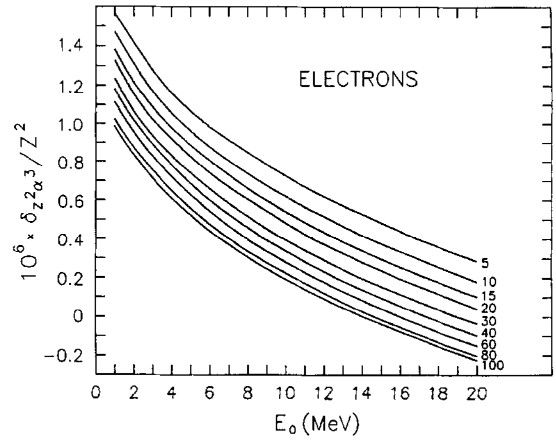


Fig. 3. $\delta_{Z^2\alpha^3}$ for electron emission. The numbers labelling the curves are the $Z$-values of the daughter nuclei. The results for positron emission are closely similar.

## 4. The correction of order $Z^2\alpha^3$

While $\delta_{Z\alpha^2}$ is, apart from small terms as noted, exact within its terms of reference the higher corrections are progressively less certain. That of order $Z^2\alpha^3$, namely $\delta_{Z^2\alpha^3}$, is due to Sirlin [8], who refers to its treatment as heuristic:

$$G_{Z^2\alpha^3} = \left[ a \ln \frac{A}{W} + bf(W) + dg(W) + h \ln(2W_0) \right], \tag{23}$$

where

$$a = \left( \frac{\pi^2}{3} - \frac{3}{2} \right) \Big/ \pi \quad \text{(see Ref. [10] of Ref. [6])}, \tag{24}$$

$$b = \frac{4}{3\pi} \left( \frac{11}{4} - \gamma_E - \frac{\pi^2}{6} \right), \tag{25}$$

$$d = 4/(3\pi), \qquad (26)^5$$

$$h = -0.649, \tag{27}$$

$$f(W) = \ln(2W) - \tfrac{5}{6}, \tag{28}$$

$$g(W) = \tfrac{1}{2}[\ln^2 R - \ln^2(2W)] + \tfrac{5}{3}\ln(2RW). \tag{29}$$

### 4.1. Numerical evaluation of $\delta_{Z^2\alpha^3}$

We evaluate $\delta_{Z^2\alpha^3}$, following Eqs. (23–29), for both electron and positron emission, for $|Z| \leq 100$ and for $1 \leq E_0 \leq 20$ MeV. As may be anticipated from the logarithmic appearance of $W$, the results for electron and for positron emission are very closely similar: they are presented for electrons in Fig. 3 those for positrons being within the thickness of the lines.

Fig. 3 was prepared using the PS(2) of Eq. (4) rather than the full PS(3) of Eq. (5) that was used for $\delta_{Z\alpha^2}$, as may again be anticipated, a broad exploration showed that the differences between the use of PS(2) and of PS(3) are everywhere of negligible importance.

## 5. Higher-order terms

We have little guidance as to the values of the $\delta_{Z^n\alpha^m}$ beyond those already exposed. In very general terms, by naive loop-counting, we may anticipate

[5] This coefficient is misprinted in Ref. [9]: the evaluation of $\delta_{Z^2\alpha^3}$ was, however, there carried out using its correct value.

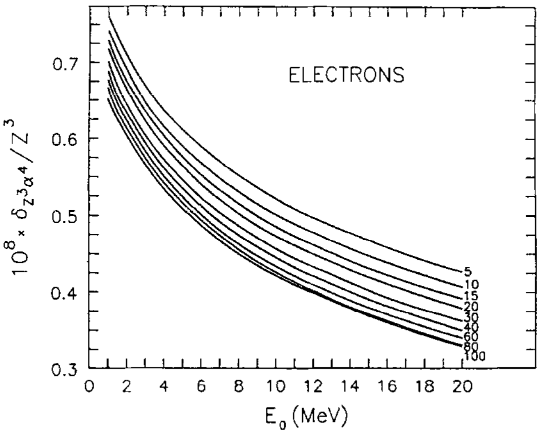


Fig. 4. $\delta_{Z^3\alpha^4}$ for electron emission. The numbers labelling the curves are the $Z$-values of the daughter nuclei. The results for positron emission are closely similar.

that

$$\delta_{Z^n\alpha^m} \approx Z^n \alpha^m K_{nm} \overline{\ln^{m-n} \frac{\Lambda}{W}}. \qquad (30)$$

We have that the numerical values for the $K_{nm}$ taken from the leading-log terms for $\delta_\alpha$, $\delta_{Z\alpha^2}$, and $\delta_{Z^2\alpha^3}$, already treated, and for $\delta_{Z\alpha^3}$ [10] are, respectively, 0.48, 1, 0.57 and 0.16 which average out at about 0.5, a value that we now use, in an exploratory spirit, in an assessment of the $\delta_{Z^n\alpha^m}$, through Eq. (30), for those $n$, $m$-values not so far explicitly considered.

We begin by noting that for a given value of $n$ the values of $\delta_{Z^n\alpha^m}$ for successively higher values of $m$ decline more rapidly than $\alpha \ln \Lambda$ per unit increment of $m$ so that the effect of ignoring, as we shall, all but $m = n + 1$ is only a very few percent at most. We also note that if we give $K_{nm}$ a fixed value we have that the summed effect of all the higher-order terms is

$$\delta_{\text{higher}} \approx \sum_{n=3}^{n=\infty} \delta_{Z^n\alpha^{n+1}} = \delta_{Z^3\alpha^4}/(1 - Z\alpha). \qquad (31)$$

Fig. 4 presents $\delta_{Z^3\alpha^4}$ for electrons following Eq. (30) with $K_{nm} = 0.5$ and using the PS(2) of Eq. (4); the results for positrons are very closely similar.

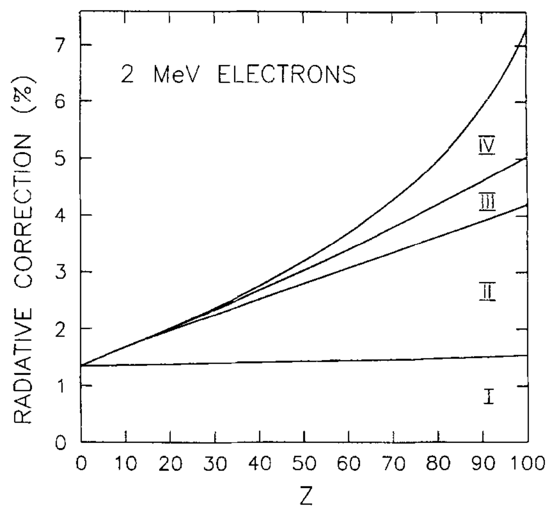


Fig. 5. The build-up of the overall radiative correction $\delta_\Sigma$ for 2 MeV electrons as a function of Z. Region I represents $\delta_{\alpha,\alpha^2 \ldots}$, region II is $\delta_{Z\alpha^2}$, region III is $\delta_{Z^2\alpha^3}$, region IV is $\delta_{\text{higher}}$ as defined in Eq. (31) of the text.

## 6. The summed correction $\delta_\Sigma$

It is evident that the importance of the imperfectly understood corrections of high order, $n \geq 2$ in $\delta_{Z^n\alpha^m}$, increases rapidly with $Z$. This is illustrated in Fig. 5 for electrons of 2 MeV. This, together with the earlier-stated uncertainty as to the reliability of the present evaluations other than for the particular class of Fermi transitions for which they were largely derived, implies that caution must be used in cases were beta-decay probabilities are to be analyzed with high precision, particularly in the heavier nuclei.

## References

[1] D.H. Wilkinson, Nucl. Instr. and Meth. A 365 (1995) 497.
[2] A. Sirlin, Phys. Rev. 164 (1967) 1767.
[3] M.A. Bég, J. Bernstein, A. Sirlin, Phys. Rev. D 6 (1972) 2597.
[4] W. Jaus, G. Rasche, Nucl. Phys. A 143 (1970) 202.
[5] W. Jaus, G. Rasche, Phys. Rev. D 35 (1987) 3420.
[6] A. Sirlin, Phys. Rev. D 35 (1987) 3423.
[7] D.H. Wilkinson, Nucl. Instr. and Meth. A 335 (1993) 182.
[8] A. Sirlin, R. Zucchini, Phys. Rev. Lett. 57 (1986) 1994.
[9] D.H. Wilkinson, Nucl. Instr. and Meth. A 335 (1995) 201.
[10] W. Jaus, Phys. Lett. 40 (1972) 616.