Nuclear Instruments and Methods in Physics Research A 335 (1993) 305–309
North-Holland

NUCLEAR
INSTRUMENTS
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IN PHYSICS
RESEARCH
Section A

Evaluation of beta-decay Part III. The complex gamma function *

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Received 23 April 1993

Two real, analytical, approximations for the square of the modulus of the complex gamma function as it appears in $F(Z, W)$, the Fermi function for beta-decay, are evaluated; an accuracy bettering $10^{-4}\%$ can easily be achieved for all electron energies throughout the periodic table.

1. Introduction [#1]

A critical factor in the Fermi function $F(Z, W)$ that represents the effect of the Coulomb field on the charged lepton emitted in beta-decay is the square of the modulus of the complex gamma function: $|\Gamma(\gamma + i\alpha ZW/p)|^2$. Evaluation of this quantity poses no great problem if the appropriate machinery for handling complex numbers is available but there are two circumstances in which it is desirable that explicit real expressions for it be to hand: i) if one does not conveniently dispose of the above machinery; ii) if the quantity must be made the object of transparent analytical manipulation. Such explicit real analytical expressions must cover the whole range of beta-particle energy and nuclear Z-value that might realistically be encountered, say $0 < E < 20$ MeV and $0 < |Z| \leq 100$, with adequate accuracy, which we may take to mean handsomely to better 1%.

It is the object of the present paper to investigate quantitatively, over the whole above range of E and $|Z|$, the accuracy that may achieved by two such explicit real expressions and to show that that accuracy may easily be raised to $10^{-4}\%$, over this whole range, if so desired.

* Part I was published in Nucl. Instr. and Meth. A275 (1989) 378, and Part II in Nucl. Instr. and Meth. A290 (1990) 509.

[#1] The nomenclature is as in Part I [1] and Part II [2] in this series.

2. The first expression

It was pointed out some time ago [3] that:

$$\ln|\Gamma(\gamma + iy)|^2 \simeq \sum_{n=0}^{N-1} \ln\left[\frac{n^2 + y_1^2}{(n+\gamma)^2 + y^2)}\right] + \ln\left[\frac{\pi}{y_1 \sinh \pi y_1}\right] + \ln(N^2 + y_1^2) + (1-\gamma) \times \left\{2 - \ln\left[(N+\gamma)^2 + y^2\right] + \frac{2y}{N+\gamma}\arctan\frac{y}{N+\gamma} + \frac{1}{6a}\left[\frac{1}{(N+\gamma)^2 + y^2}\right]\right\} - (2N+1)\ln a, \quad (1)$$

where $y = \alpha ZW/p$, $y_1 = ay$, $a = (N+1)/(N+\gamma)$, and where N is any integer greater than or equal to unity. The accuracy of eq. (1) increases with y, i.e. as p gets smaller for a given Z, and with the number of terms taken in the summation, i.e. as N is increased. Fig. 1 shows the error in the value of $|\Gamma(\gamma + i\alpha ZW/p)|^2$ following eq. (1) for $N = 1$ the accuracy of which is seen to be adequate for essentially all practical purposes. Extremely high accuracy may easily be achieved by increasing N as is shown in fig. 2 for the worst-case $p = 10$ which has $W/p \simeq 1$. Fig. 3 shows the region of the p–Z plane for which $N = 1$ betters an accuracy of $10^{-2}\%$ ($N = 2$ betters that accuracy everywhere) figs. 4a, 4b and 4c similarly show the regions within which an accuracy of $10^{-3}\%$ is bettered for $N = 1$, 2 and 3 respectively ($N = 4$ betters that accuracy everywhere) while figs. 5a, 5b and 5c show the bettering of $10^{-4}\%$

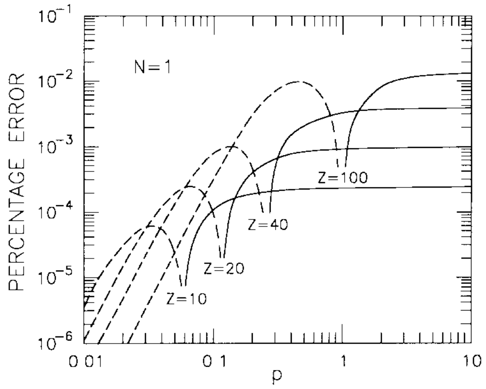

Fig 1. Accuracy of the approximation afforded by expression (1) of the text for $|\Gamma(\gamma + i\alpha ZW/p|^2$ with $N = 1$. The percentage error is defined as 100 (approximation/exact − 1). The error is positive for the solid lines and negative for the dashed.

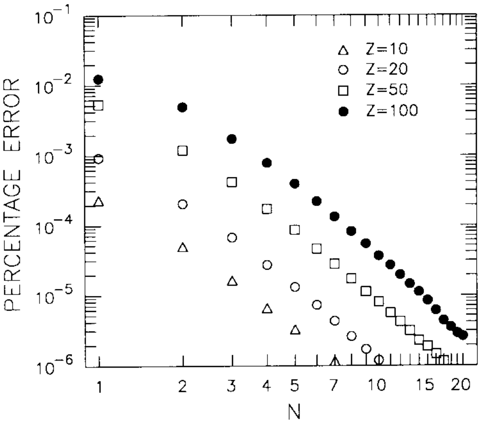

Fig. 2. Accuracy of expression (1) for $p = 10$ as a function of N and for $Z = 10$, 20, 50 and 100.

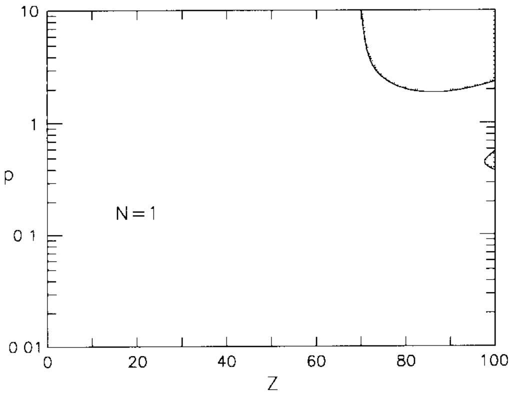

Fig. 3. Accuracy of expression (1) with $N = 1$: $10^{-2}\%$ is bettered everywhere except within the stippled areas. ($N = 2$ betters $10^{-2}\%$ everywhere)

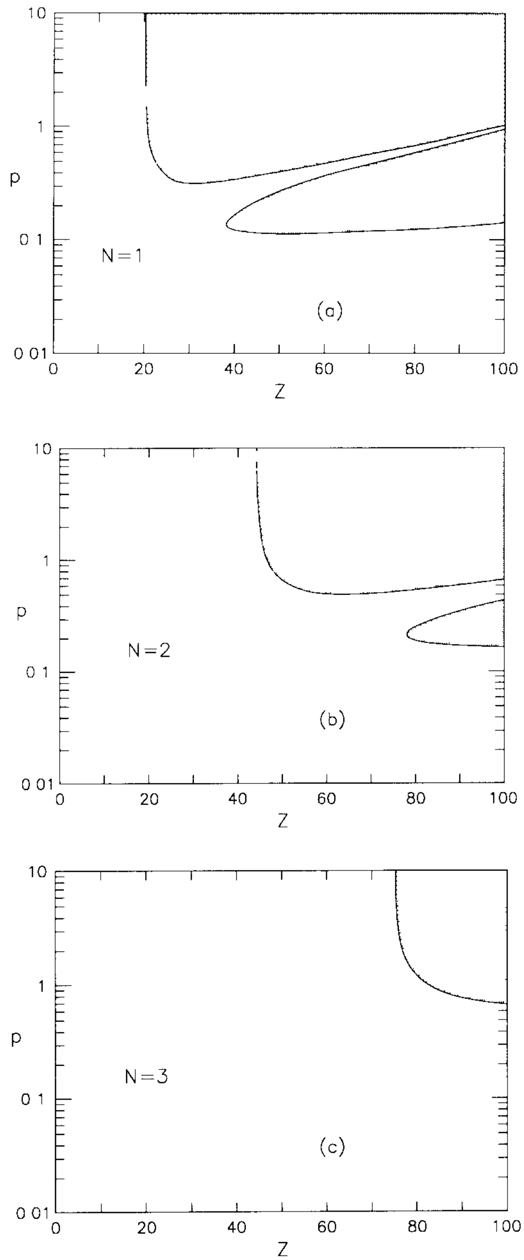

Fig. 4. As fig. 3 but with reference to an accuracy of $10^{-3}\%$ with (a) $N = 1$; (b) $N = 2$; (c) $N = 3$. ($N + 4$ betters $10^{-3}\%$ everywhere).

by $N = 3$, 5 and 7 respectively ($N = 8$ bettering that accuracy everywhere).

3. The second expression

Although the accuracy of our first expression, eq. (1), can be made, as we have seen, adequate for any imaginable practical purpose, that expression is rather opaque and is not readily susceptible of analytical

Table 1
Coefficients Z_a, K_b of expression (4) of the text

Z_0K_0	1
Z_2K_0	0.577215664902
Z_2K_2	−1.64493406685
Z_4K_0	0.722126394842
Z_4K_2	−2.15153861428
Z_4K_4	1.89406565899
Z_6K_0	0.730657619934
Z_6K_2	−2.99395303313
Z_6K_4	4.10751647417
Z_6K_6	−1.97110218259
Z_8K_0	0.752192203532
Z_8K_2	−3.78652479361
Z_8K_4	7.19058681675
Z_8K_6	−6.12855282403
Z_8K_8	1.99246600370
$Z_{10}K_0$	0.768082119803
$Z_{10}K_2$	−4.59340777340
$Z_{10}K_4$	11.0478454141
$Z_{10}K_6$	−13.3731733037
$Z_{10}K_8$	8.14414529689
$Z_{10}K_{10}$	−1.99807901520
$Z_{12}K_0$	0.781406949930
$Z_{12}K_2$	−5.40680581378
$Z_{12}K_4$	15.7034683612
$Z_{12}K_6$	−24.4717335218
$Z_{12}K_8$	21.5441015459
$Z_{12}K_{10}$	−10.1508822505
$Z_{12}K_{12}$	1.99951537029
$Z_{14}K_0$	0.792686875154
$Z_{14}K_2$	−6.22686033957
$Z_{14}K_4$	21.1650107539
$Z_{14}K_6$	−40.2230580903
$Z_{14}K_8$	46.0442662231
$Z_{14}K_{10}$	−31.7064383935
$Z_{14}K_{12}$	12.1532984038
$Z_{14}K_{14}$	−1.99987834069
$Z_{16}K_0$	0.802397663526
$Z_{16}K_2$	−7.05299920545
$Z_{16}K_4$	27.4405487370
$Z_{16}K_6$	−61.4329566043
$Z_{16}K_8$	86.2965025178
$Z_{16}K_{10}$	−77.7643278220
$Z_{16}K_{12}$	43.8641567405
$Z_{16}K_{14}$	−14.1540867280
$Z_{16}K_{16}$	1.99996952843
$Z_{18}K_0$	0.810869227626
$Z_{18}K_2$	−7.88472625713
$Z_{18}K_4$	34.5371973952
$Z_{18}K_6$	−88.9155423732
$Z_{18}K_8$	147.758784082
$Z_{18}K_{10}$	−164.076086037
$Z_{18}K_{12}$	121.634342722
$Z_{18}K_{14}$	−58.0198291738
$Z_{18}K_{16}$	16.1543298642
$Z_{18}K_{18}$	−1.99999237574

Table 1
(Continued)

$Z_{20}K_0$	0.818342895931
$Z_{20}K_2$	−8.72158676254
$Z_{20}K_4$	42.4613497530
$Z_{20}K_6$	−123.492101870
$Z_{20}K_8$	236.703398048
$Z_{20}K_{10}$	−311.851331190
$Z_{20}K_{12}$	285.717579841
$Z_{20}K_{14}$	−179.656500709
$Z_{20}K_{16}$	74.1746984971
$Z_{20}K_{18}$	−18.1544021403
$Z_{20}K_{20}$	1.99999809322

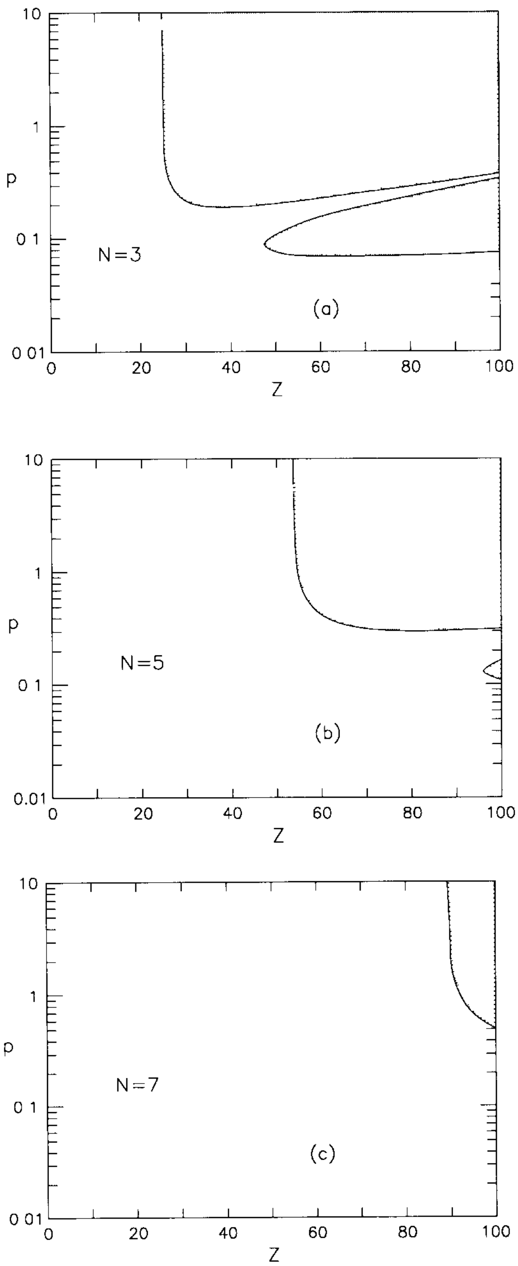

Fig. 5. As fig. 3 but with reference to an accuracy of 10^{-4}% with (a) $N=3$; (b) $N=5$; (c) $N=7$. ($N=8$ betters 10^{-4}% everywhere).

manipulation. Should such manipulation be desired a polynomial representation of $|\Gamma(\gamma + i\alpha ZW/p)|^2$ is by far the most convenient. This is afforded by using:

$$\ln \Gamma(1+z) = -\gamma_E z + \sum_{n=2}^{\infty} (-)^n \zeta(n) z^n/n, \tag{2}$$

where γ_E is the Euler gamma, and $\zeta(n)$ the Riemann zeta function and remembering that:

$$\mathscr{R} \ln \Gamma(z) = \ln |\Gamma(z)|. \tag{3}$$

Our second expression results from cutting off the

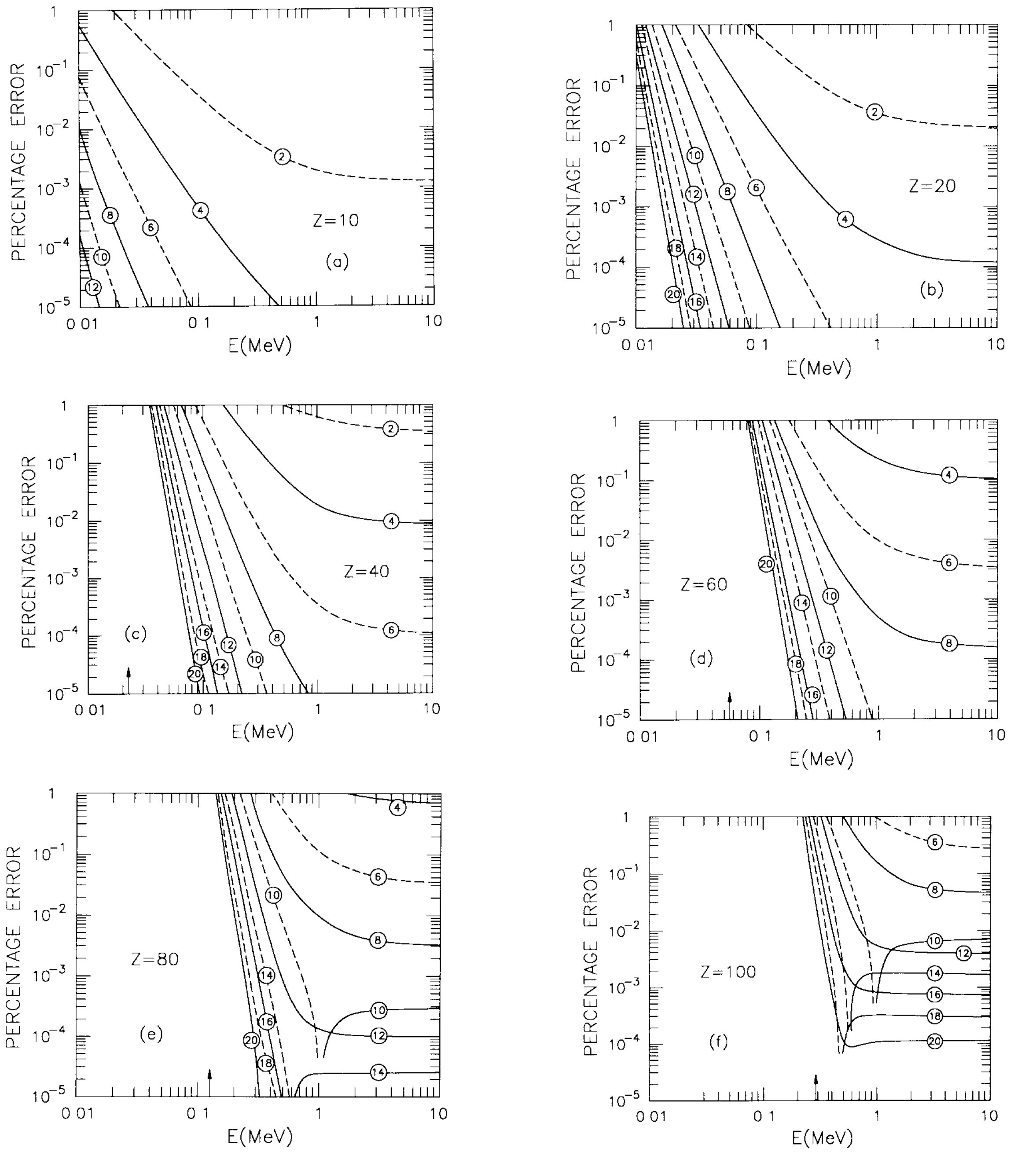

Fig. 6. Accuracy of the approximation afforded by expression (4) of the text, together with the coefficients $Z_a K_b$ listed in table 1, as defined in the caption to fig. 1 for: (a) $Z=10$; (b) $Z=20$; (c) $Z=40$; (d) $Z=60$; (e) $Z=80$; (f) $Z=100$ and for $0.01 \leq E \leq 10$ MeV. The numbers labeling the curves are the break-off values of a in the summation of expression (4) as defined in the text. The arrows on the abscissae indicate the energies corresponding to the condition given by eq. (5) of the text. (Where no arrow shows, the condition lies below the energy scale of the figure).

summation in (2) at some finite value of n and writing, in the notation of part I:

$$|\Gamma(\gamma + i\alpha ZW/p)|^2 \simeq \sum_{a,b} Z_a K_b (\alpha Z)^a (W/p)^b. \quad (4)$$

The present table 1 here extends table 1 of Part I from the a, $b = 10$ of Part I to a, $b = 20$. The summation in eq. (4) extends over all a-values up to and including that at which the summation is broken off (the break-off value of a) and with $b \leq a$ for each a-value so involved.

We must expect that the accuracy of eq. (4) will, contrary to that of eq. (1), fall off as p decreases i.e. as $|z|$ increases: indeed eq. (4) is valid only for $|z| < 1$, i.e. for

$$W > [\gamma(2-\gamma)/(2\gamma-1)]^{1/2}. \quad (5)$$

The accuracy afforded by eq. (4), taken through break-off a-values of $2 \leq a \leq 20$, is shown in figs. 6a–6f for $Z = 10$, 20, 40, 60, 80 and 100 respectively and for $0.01 \leq E \leq 10$ MeV; fig. 7 extends the energy range down to $E = 0.001$ MeV for $Z = 5$, 10 and 15. The figures also show the energies down to which the condition expressed in eq. (5) is met.

It is seen that this second expression (4) for $|\Gamma(\gamma + i\alpha ZW/p)|^2$, like the first, is adequate for all practical purposes in the construction of beta-spectra, despite the condition of eq. (5), although, unlike the first expression, it cannot, of course, be used for *integration* of the spectrum down to $p = 0$, a situation that was discussed in Part I.

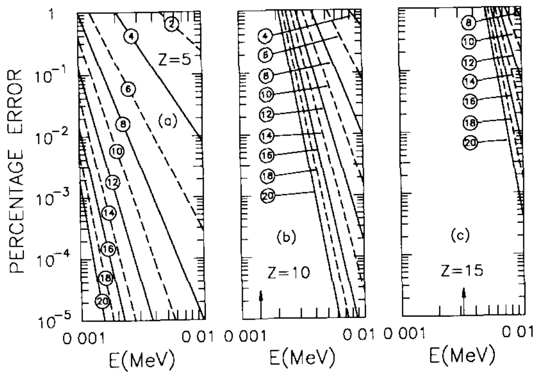

Fig. 7. As for fig. 6 but for $0.001 \leq E \leq 0.01$ (MeV for: (a) $Z = 5$; (b) $Z = 10$; (c) $Z = 15$.

References

[1] D.H. Wilkinson, Nucl. Instr. and Meth. A275 (1989) 378.
[2] D.H. Wilkinson, Nucl. Instr. and Meth. A290 (1990) 509.
[3] D.H. Wilkinson, Nucl. Instr. and Meth. 82 (1970) 122.