Spatial Description of Rigid Bodies and Coordinate Frames

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Spatial Description of a Rigid Body

❑ Position and orientation of a rigid body is completely described in space by its position and orientation with respect to a reference frame.

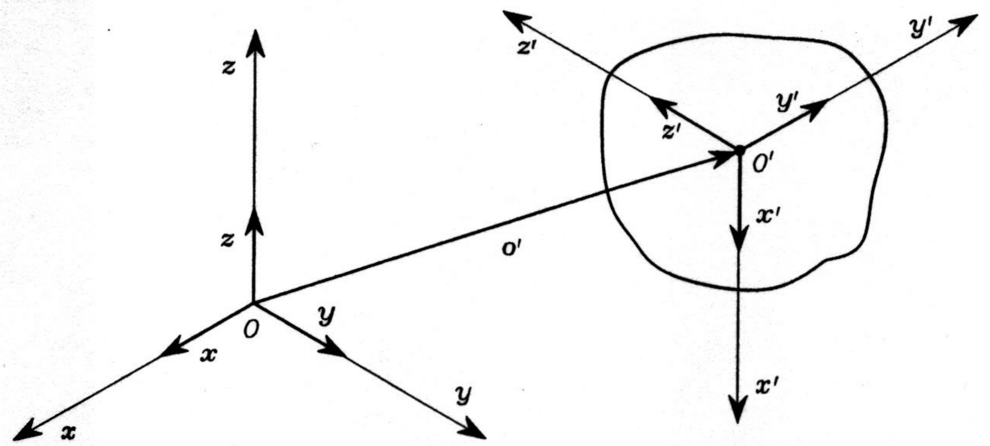


Figure 2.1 Position and orientation of a rigid body.

Reference frame: O-xyz *is orthonormal.*

Position of a point O' on the body:

$$\mathbf{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z},$$

$$\mathbf{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$

Spatial Description of a Rigid Body

Orientation of the body: Attach an orthonormal frame O'-$x'y'z'$ to the body at O'. Find the orientation of the body frame O-$x'y'z'$ wrt. the reference frame O-xyz.

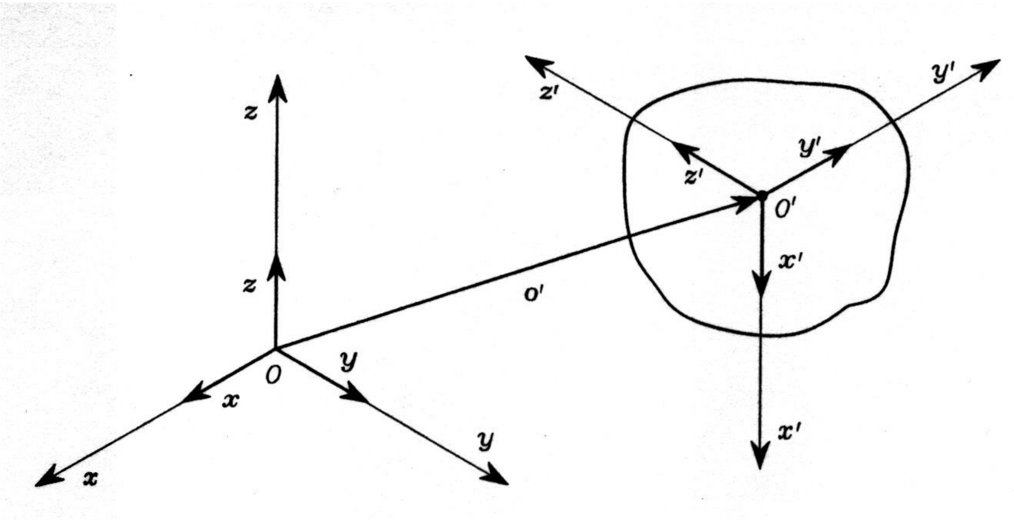


Figure 2.1 Position and orientation of a rigid body.

$$x' = x'_x x + x'_y y + x'_z z$$
$$y' = y'_x x + y'_y y + y'_z z$$
$$z' = z'_x x + z'_y y + z'_z z.$$

The parameters z'_x, z'_y, z'_z are the projection of $\underline{z}'$ on the $\underline{x}$, $\underline{y}$ and $\underline{z}$ directions, that is

$$z'_x = \langle \underline{z}', \underline{x} \rangle \,, \quad z'_y = \langle \underline{z}', \underline{y} \rangle \,, \quad z'_z = \langle \underline{z}', \underline{z} \rangle$$

and so forth for the other parameters.

Spatial Description of a Rigid Body

Rotation Matrix (R):

$$R = \begin{bmatrix} x' & y' & z' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} x'^T x & y'^T x & z'^T x \\ x'^T y & y'^T y & z'^T y \\ x'^T z & y'^T z & z'^T z \end{bmatrix}$$

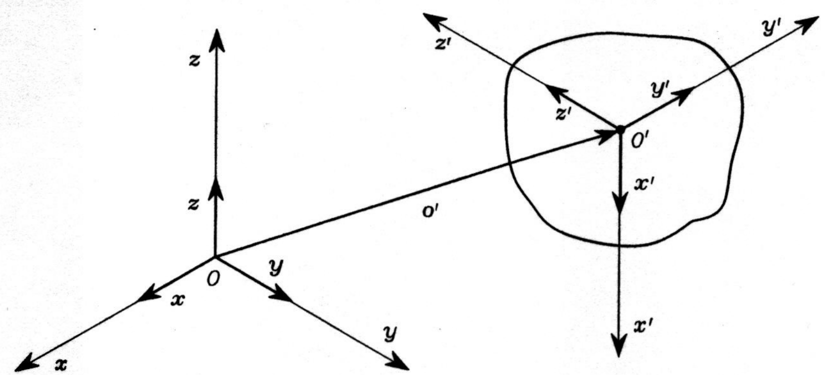


Figure 2.1 Position and orientation of a rigid body.

Property: The column vectors of rotation matrix R are mutually orthogonal and have unit length:

$$x'^T y' = 0 \qquad y'^T z' = 0 \qquad z'^T x' = 0.$$

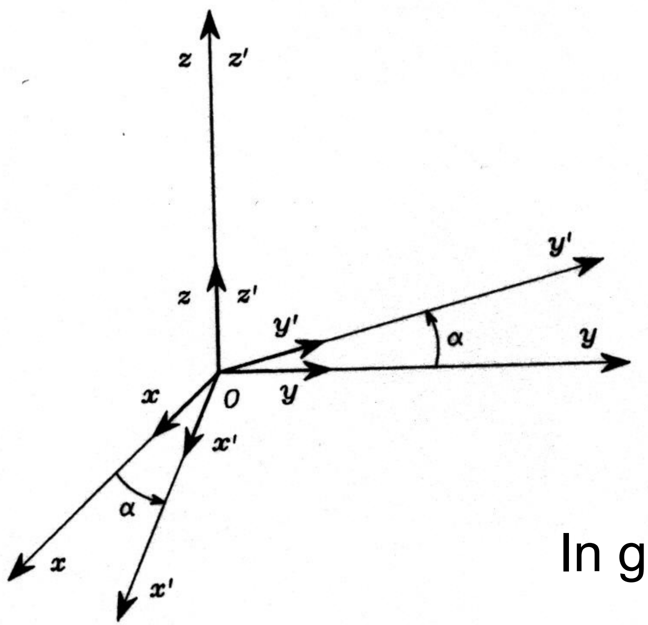
$$x'^T x' = 1 \qquad y'^T y' = 1 \qquad z'^T z' = 1.$$

Therefore, $R^T R = I_{3x3}$ or $R^{-1} = R^T$ ⇒ R is always an orthogonal matrix.

Spatial Description of a Rigid Body

Elementary Rotations: Consider the body frame O'-$x'y'z'$ is obtained by rotating the reference frame O-xyz by angle α wrt. the $\underline{z}$ axis (Figure). Then:

$$\boldsymbol{x}' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{z}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Similarly, for elementary rotations by angles γ and β about axes $\underline{x}$ and $\underline{y}$:

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

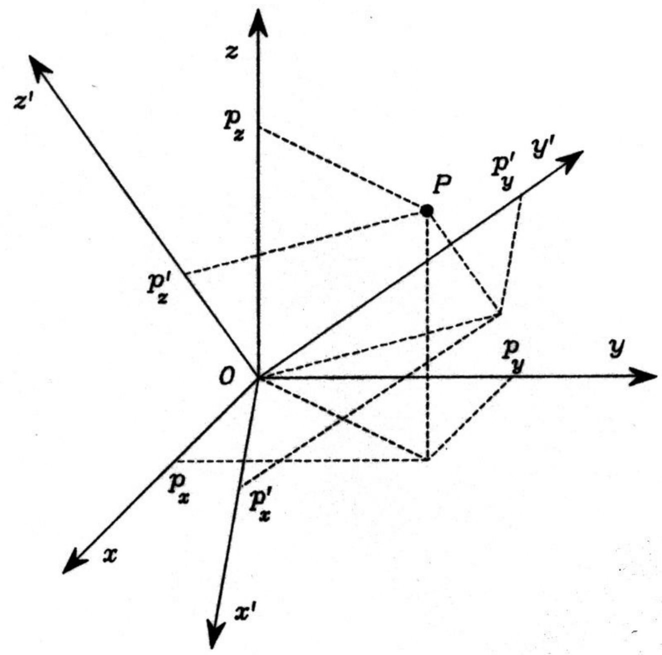
$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

In general, $\boldsymbol{R}_k(-v) = \boldsymbol{R}_k^{-1}(v) = \boldsymbol{R}_k^{T}(v) \quad , \quad k = x,y,z$

Interpretation I: Rotation matrix $\boldsymbol{R}$ describes the rotation about an axis in space needed to align the reference frame O-xyz with the body frame O-$x'y'z'$.

Spatial Description of a Rigid Body

Representation of a Vector: Consider the point *"P"* in space and also the two frames O-xyz and O-$x'y'z'$ with coinciding centers:



$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

"P" represented in O-xyz

$$p' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

"P" represented in O'-$x'y'z'$

Since $\underline{p}$ and $\underline{p}'$ represent the same point *"P"*:

$$p = p'_x x' + p'_y y' + p'_z z' = \begin{bmatrix} x' & y' & z' \end{bmatrix} p' = Rp'$$

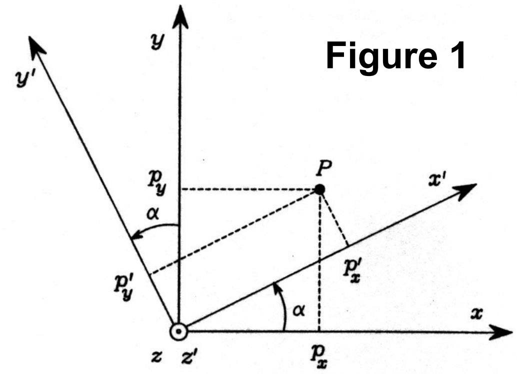
$$\boxed{p = Rp'}$$

Interpretation II: The rotation matrix R represents the transformation of a vector coordinates in the body frame O-$x'y'z'$ into the coordinates in the reference frame O-xyz.

Spatial Description of a Rigid Body

Example 1: $O\text{-}x'y'z'$ is $O\text{-}xyz$ rotated by angle α about z axis (**Figure 1**):

Figure 1



$$\boxed{\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'}$$

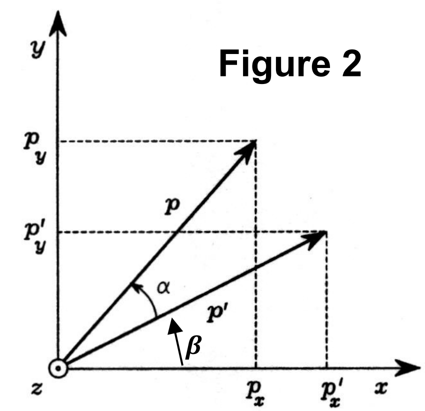
$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

$$\begin{aligned} p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\ p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\ p_z &= p'_z. \end{aligned}$$

"P" represented in $O\text{-}xyz$

Example 2 (Rotation of Vectors): Consider vector $\underline{\boldsymbol{p}}$ in the xy plane that is obtained by rotating a vector $\underline{\boldsymbol{p}}'$ about the z *axis* of the reference frame $O\text{-}xyz$ by angle α (Fig. 2). In this case, from Figure 2

Figure 2



$p_x' = ||\underline{p}'|| \cos(\beta)$

$p_x = ||\underline{p}|| \cos(\alpha+\beta)$

$p_y' = ||\underline{p}'|| \sin(\beta)$

$p_y = ||\underline{p}|| \sin(\alpha+\beta)$

$||\underline{p}|| = ||\underline{p}'||$

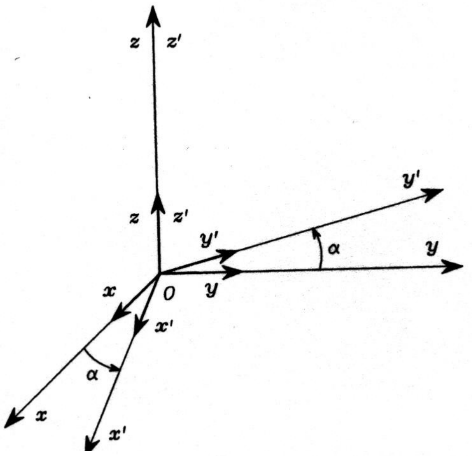
$$\begin{aligned} p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\ p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\ p_z &= p'_z. \end{aligned}$$

$$\boxed{\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'}$$

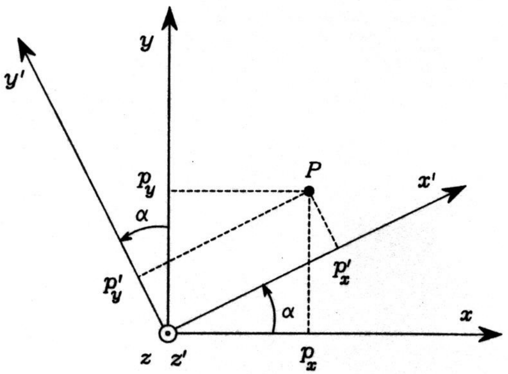
Spatial Description of a Rigid Body

Interpretations of the Rotation Matrix (R):

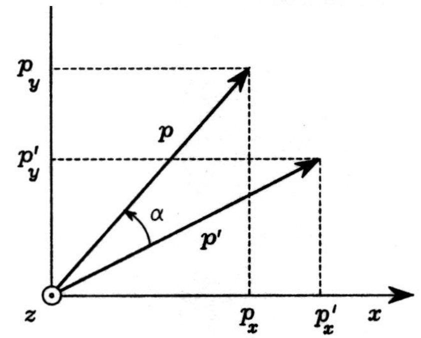
1. ***R*** describes the orientation between two coordinate frames.



2. ***R*** represents the coordinate transformation between the coordinates of a point *"P"* in two different frames.



3. ***R*** acts as an operator that can rotate a vector in the same coordinate frame.



Spatial Description of a Rigid Body

Combination of Rotation Matrices:

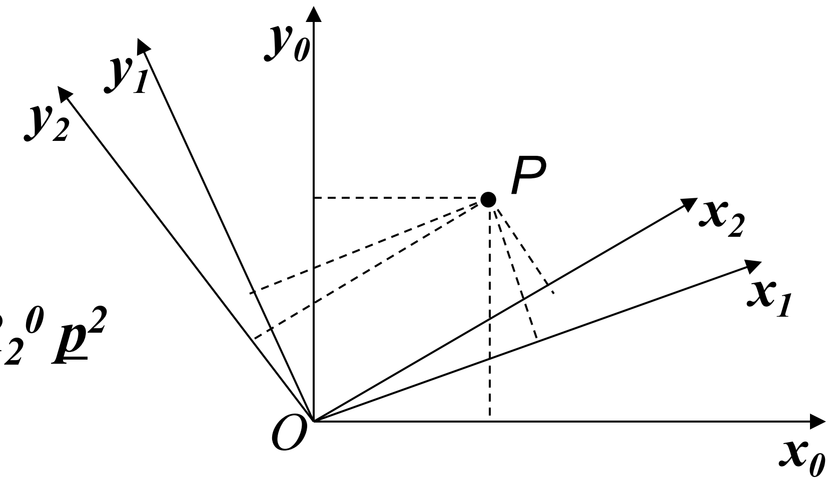
- **Coordinate transformation:** If $\underline{p}^0$, $\underline{p}^1$ and $\underline{p}^2$ denote the coordinates of point *"P"* in frames $O\text{-}x_0y_0z_0$, $O\text{-}x_1y_1z_1$ and $O\text{-}x_2y_2z_2$ (Figure), then

$$\underline{p}^0 = R_1^0\,\underline{p}^1 \quad , \quad \underline{p}^1 = R_2^1\,\underline{p}^2$$

$$\underline{p}^0 = R_2^0\,\underline{p}^2$$

$$\underline{p}^0 = R_1^0\,\underline{p}^1 = R_1^0\,(R_2^1\,\underline{p}^2) = (R_1^0\,R_2^1)\,\underline{p}^2 = R_2^0\,\underline{p}^2$$

$$\Rightarrow \boxed{R_2^0 = R_1^0\,R_2^1}$$

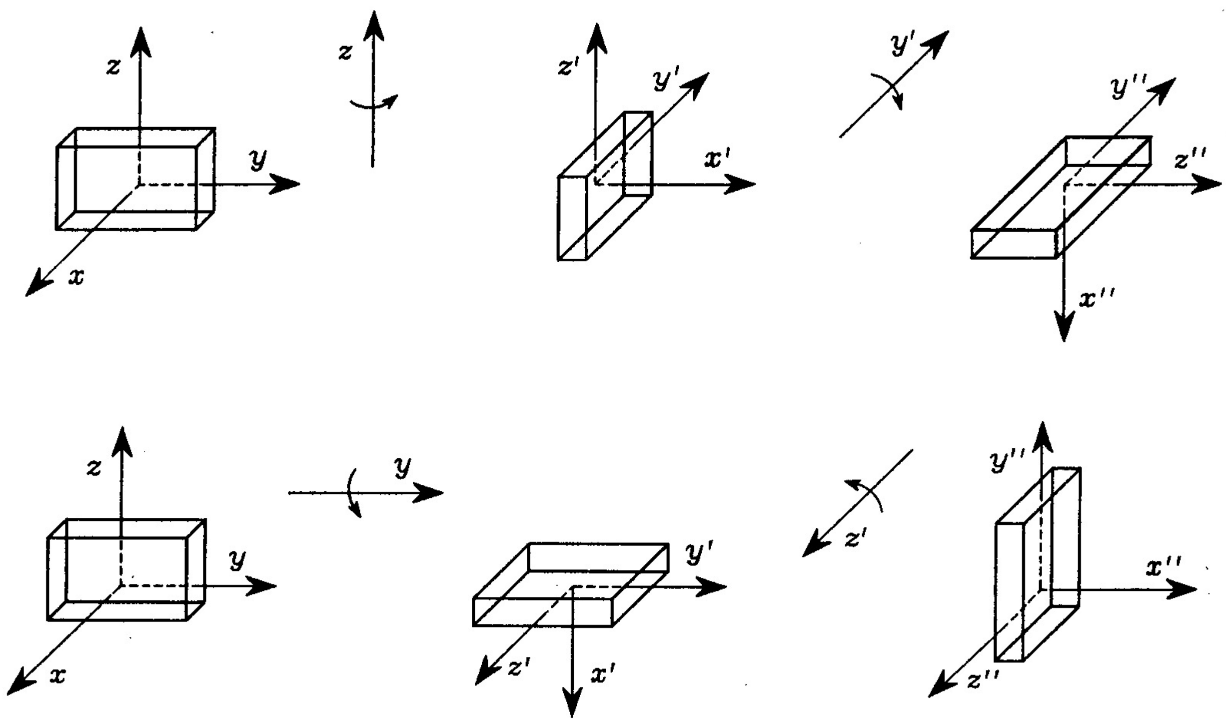


- **Frame Alignment (Rotation):** Consider a frame originally aligned with $O\text{-}x_0y_0z_0$
  - Rotate the given frame to align it with $O\text{-}x_1y_1z_1$ using R_1^0
  - Rotate the given frame (now aligned with $O\text{-}x_1y_1z_1$) to align it with $O\text{-}x_2y_2z_2$ using R_2^1

Spatial Description of a Rigid Body

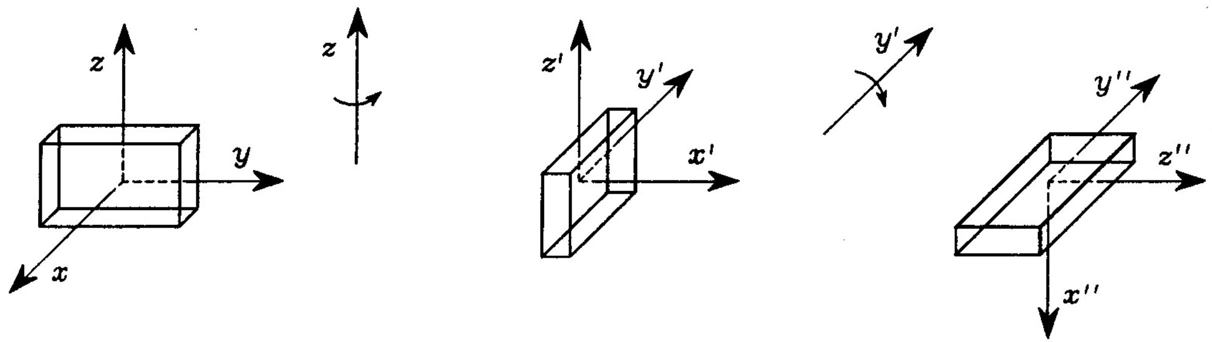
Combination of Rotation Matrices:

- Order of frame rotation does not commute.



Spatial Description of a Rigid Body

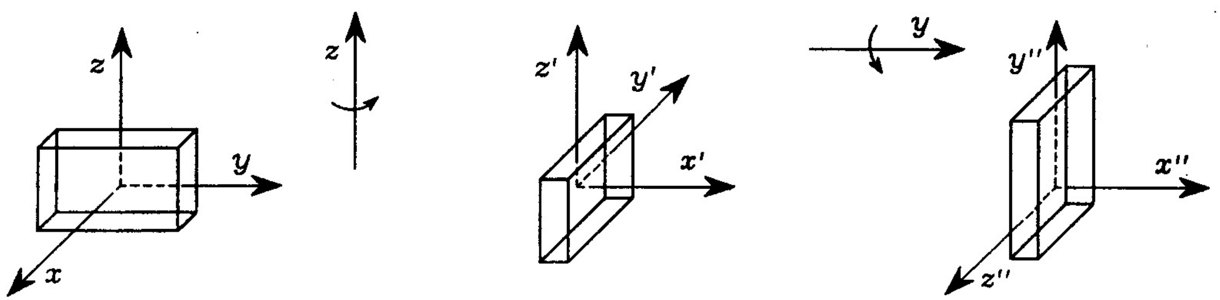
Frame rotation about <u>current</u> axis:



$$R_2^0 = R_{1z}^0\, R_{2y'}^1$$

Post-multiplication

Frame rotation about <u>fixed</u> axis:

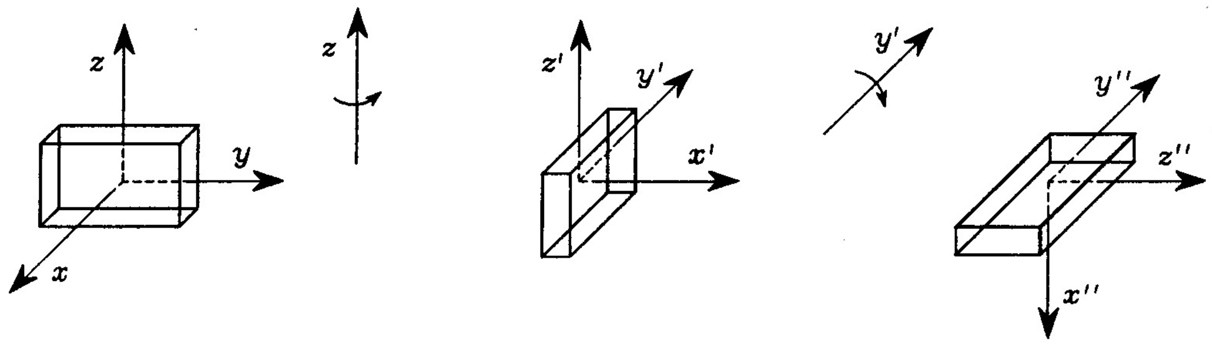


$$R_2^0 = R_{2y}^1\, R_{1z}^0$$

Pre-multiplication

Spatial Description of a Rigid Body

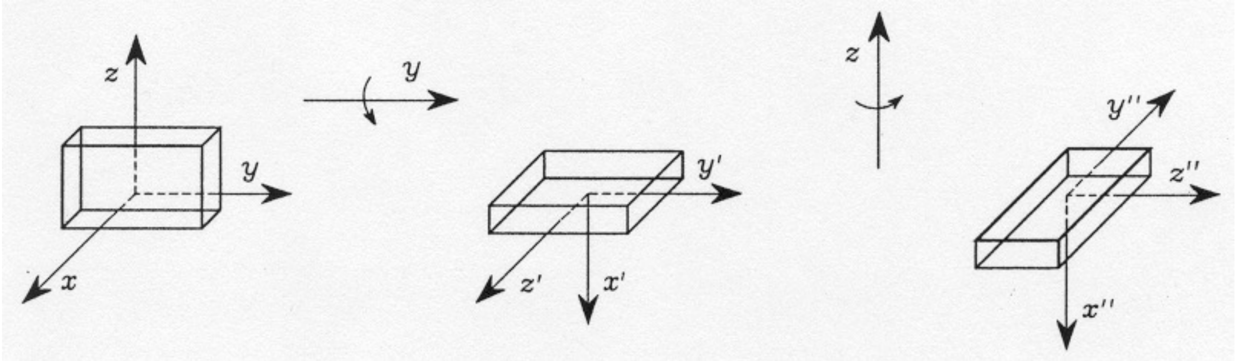
Frame rotation about current axis:



$R_2^0 = R_z R_{y'}$

Post-multiplication

Frame rotation about fixed axis: The same result as of the rotation about current axes case is obtained if the order of rotation is changed.



$R_2^0 = R_z R_y$

Pre-multiplication

Spatial Description of a Rigid Body

Euler Angles: Rotation matrix is characterized by nine elements:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

Minimal representation: Due to the six constraints ($\boldsymbol{R}^T\boldsymbol{R}=\boldsymbol{I}$)

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

$$\boldsymbol{x}'^T\boldsymbol{x}' = 1 \qquad \boldsymbol{y}'^T\boldsymbol{y}' = 1 \qquad \boldsymbol{z}'^T\boldsymbol{z}' = 1.$$

The rotation matrix can be represented by three independent parameters.

Spatial Description of a Rigid Body

Euler Angles: Minimal representation of orientation can be obtained using a set of three "*Euler angles*"

$$\phi = [\varphi \quad \vartheta \quad \psi]^T$$

- by composing a sequence of three elementary rotations, while
- the axes of two consecutive rotations are not in parallel.

The following combinations of 12 distinct Euler angles are possible:

- XYZ , XZY, YXZ , YZX , ZXY, ZYX
- XYX , XZX , YXY, YZY, ZXZ , ZYZ

Spatial Description of a Rigid Body

ZYX Euler Angles (Roll-Pitch-Yaw) $\Phi = \begin{bmatrix} \varphi & \vartheta & \psi \end{bmatrix}^T$:

$R_x(\psi)$: Rotation about $\underline{x}$ axis (**Roll**) by the angle ψ

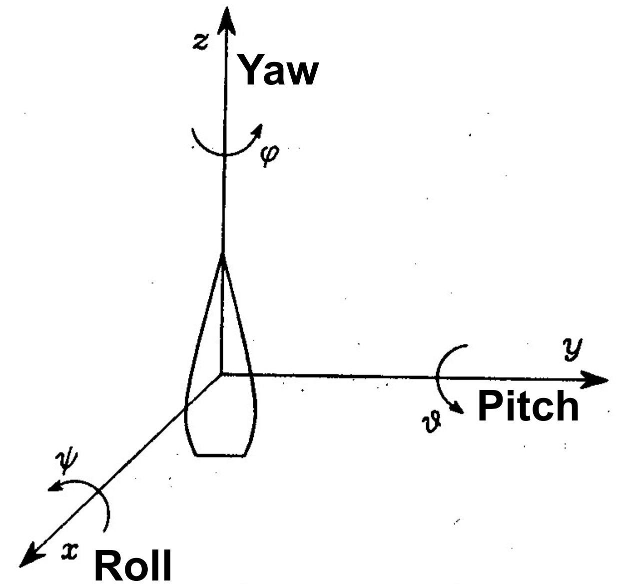
$R_y(\vartheta)$: Rotation about the fixed axis $\underline{y}$ (**Pitch**) by the angle ϑ

$R_z(\varphi)$: Rotation about the fixed axis $\underline{z}$ (**Yaw**) by the angle φ

$$\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$

Equivalent rotation matrix

$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$



Spatial Description of a Rigid Body

ZYX Euler Angles (Roll-Pitch-Yaw) $\Phi = \begin{bmatrix} \varphi & \vartheta & \psi \end{bmatrix}^T$**:**

Reverse procedure: Given the rotation matrix

$$R(\Phi) = \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

what are the possible sets of ZYX Euler (roll-pitch-yaw) angles?

$$\varphi = \mathrm{Atan2}(r_{21}, r_{11})$$
$$\vartheta = \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$\psi = \mathrm{Atan2}(r_{32}, r_{33}),$$

$$-90 < \vartheta < 90$$

$$\varphi = \mathrm{Atan2}(-r_{21}, -r_{11})$$
$$\vartheta = \mathrm{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right)$$
$$\psi = \mathrm{Atan2}(-r_{32}, -r_{33}).$$

$$-180 < \vartheta < -90 \;\&\; 90 < \vartheta < 180$$

Special case: when $\vartheta = -90 \; or \; 90 \Rightarrow \cos(\vartheta) = 0$ and $r_{11} = r_{21} = r_{32} = r_{33} = 0.$
In this case, we have two consecutive rotations about two parallel axes and only $\varphi + \psi$ or $\varphi - \psi$ can be found.

Spatial Description of a Rigid Body

Equivalent Angle (ϑ) and Axis ($\underline{r}$) : A rotation

about a unit vector $\underline{r} = [r_x \; r_y \; r_z]^T$ (expressed in the reference frame O-xyz), and by the angle ϑ

can be composed of the following five elementary rotations

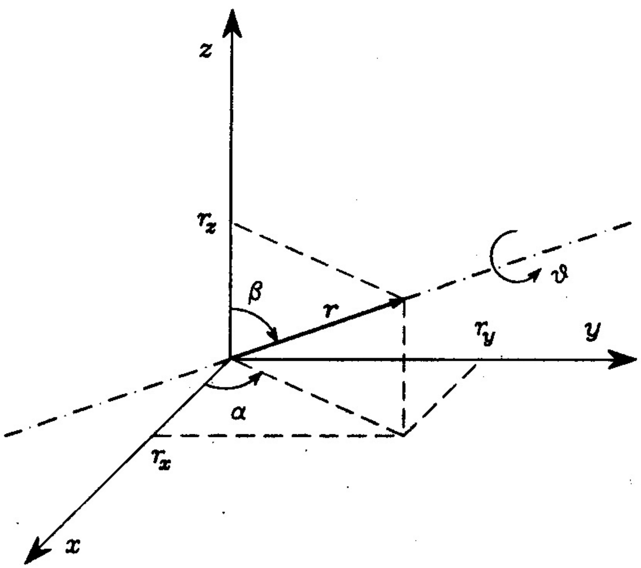
align $\underline{r}$ with $\underline{z}$ by

- ❖ $R_z(-\alpha)$: rotation about the fixed axis $\underline{z}$ by the angle $-\alpha$
- ❖ $R_y(-\beta)$: rotation about the fixed axis $\underline{y}$ by the angle $-\beta$

$R_z(\vartheta)$: rotation about the fixed axis $\underline{z}$ by the angle ϑ

re-align with the initial direction of $\underline{r}$ by

- ❖ $R_y(\beta)$: rotation about the fixed axis $\underline{y}$ by the angle β
- ❖ $R_z(\alpha)$: rotation about the fixed axis $\underline{z}$ by the angle α

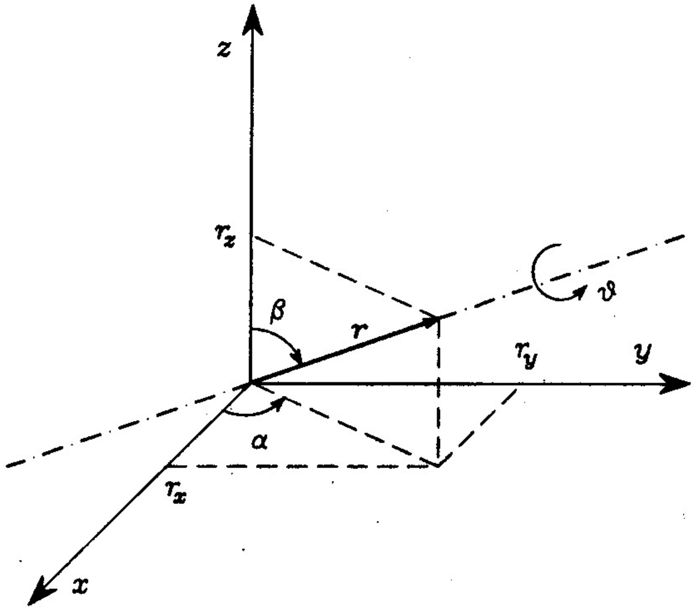


$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha).$$

Spatial Description of a Rigid Body

Equivalent Angle (ϑ) and Axis ($\underline{r}$) :

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha).$$



$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$||\boldsymbol{r}||^2 = r_x{}^2 + r_y{}^2 + r_z{}^2 = 1$ **unit vector**

$$\sin\beta = \sqrt{r_x^2 + r_y^2}$$

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1 - c_\vartheta) + c_\vartheta & r_x r_y(1 - c_\vartheta) - r_z s_\vartheta & r_x r_z(1 - c_\vartheta) + r_y s_\vartheta \\ r_x r_y(1 - c_\vartheta) + r_z s_\vartheta & r_y^2(1 - c_\vartheta) + c_\vartheta & r_y r_z(1 - c_\vartheta) - r_x s_\vartheta \\ r_x r_z(1 - c_\vartheta) - r_y s_\vartheta & r_y r_z(1 - c_\vartheta) + r_x s_\vartheta & r_z^2(1 - c_\vartheta) + c_\vartheta \end{bmatrix}$$

Spatial Description of a Rigid Body

Equivalent Angle (ϑ) and Axis ($\underline{r}$) :

Reverse Procedure: Given the rotation matrix

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_y r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

what are the possible equivalent axes and angles of rotation?

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \qquad \vartheta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

Property: The rotation matrix representation is not unique. $\boldsymbol{R}(-\vartheta, -\boldsymbol{r}) = \boldsymbol{R}(\vartheta, \boldsymbol{r})$,

Special case : when $\vartheta = 0$, the unit vector $\boldsymbol{r}$ is arbitrary. How about $\vartheta = 180\,\text{deg}$?

Spatial Description of a Rigid Body

Unit Quaternion (Euler parameters) $Q=\{\eta,\epsilon\}$**:** A general rotation about an axis $\underline{r}$ by an angle ϑ can be represented by the Euler parameters

Scalar parameter $\eta=\cos(\vartheta/2)$

Vector parameter $\epsilon=\begin{bmatrix}\epsilon_x & \epsilon_y & \epsilon_z\end{bmatrix}^T=\sin(\vartheta/2)r=\sin(\vartheta/2)\begin{bmatrix}r_x & r_y & r_z\end{bmatrix}^T$

Properties:

$\eta^2+\|\epsilon\|^2=1$ (**unit** Quaternion)

ϵ is along the direction of $\underline{r}$.

Quaternion is a unique set of parameters in the sense that both rotation parameter sets (ϑ,r) and $(-\vartheta,-r)$ map to the same set of Quaternion "Q".

$$\boldsymbol{R}(\eta,\boldsymbol{\epsilon})=\begin{bmatrix}2(\eta^2+\epsilon_x^2)-1 & 2(\epsilon_x\epsilon_y-\eta\epsilon_z) & 2(\epsilon_x\epsilon_z+\eta\epsilon_y)\\ 2(\epsilon_x\epsilon_y+\eta\epsilon_z) & 2(\eta^2+\epsilon_y^2)-1 & 2(\epsilon_y\epsilon_z-\eta\epsilon_x)\\ 2(\epsilon_x\epsilon_z-\eta\epsilon_y) & 2(\epsilon_y\epsilon_z+\eta\epsilon_x) & 2(\eta^2+\epsilon_z^2)-1\end{bmatrix}$$

Spatial Description of a Rigid Body

Unit Quaternion (Euler parameters) $Q = \{\eta, \epsilon\}$**:**

Reverse procedure: Given the rotation matrix

$$\boldsymbol{R}(\eta,\epsilon)=\begin{bmatrix} 2(\eta^2+\epsilon_x^2)-1 & 2(\epsilon_x\epsilon_y-\eta\epsilon_z) & 2(\epsilon_x\epsilon_z+\eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y+\eta\epsilon_z) & 2(\eta^2+\epsilon_y^2)-1 & 2(\epsilon_y\epsilon_z-\eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z-\eta\epsilon_y) & 2(\epsilon_y\epsilon_z+\eta\epsilon_x) & 2(\eta^2+\epsilon_z^2)-1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

what is the set of unit Quaternion (Euler parameters)?

$$\eta = \frac{1}{2}\sqrt{r_{11}+r_{22}+r_{33}+1}$$

$$\epsilon = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32}-r_{23})\sqrt{r_{11}-r_{22}-r_{33}+1} \\ \operatorname{sgn}(r_{13}-r_{31})\sqrt{r_{22}-r_{33}-r_{11}+1} \\ \operatorname{sgn}(r_{21}-r_{12})\sqrt{r_{33}-r_{11}-r_{22}+1} \end{bmatrix}$$

$\eta > 0 \Rightarrow -180 < \vartheta < 180$

Inverse rotation: $\mathcal{Q}^{-1} = \{\eta, -\epsilon\}.$

Successive rotations: $\mathcal{Q}_1 * \mathcal{Q}_2 = \{\eta_1\eta_2 - \epsilon_1^T\epsilon_2, \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1 \times \epsilon_2\}$

Spatial Description of a Rigid Body

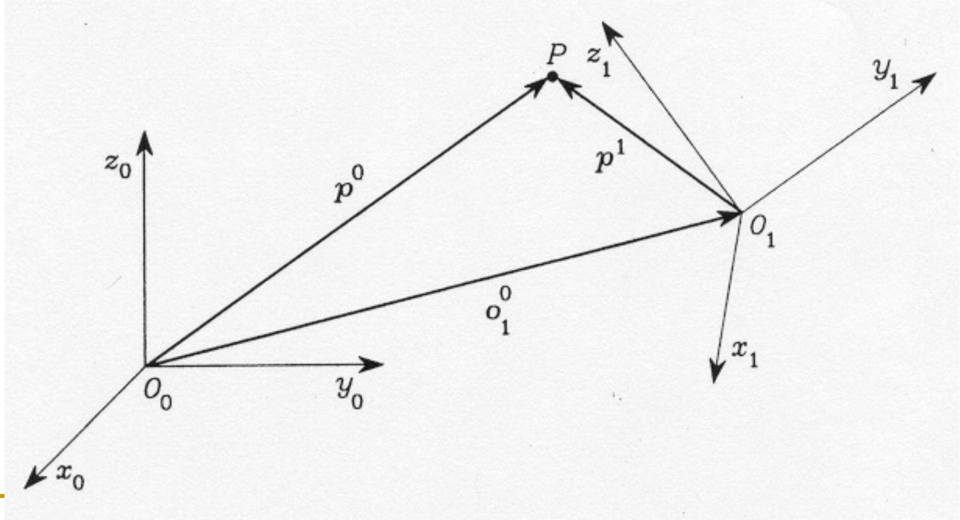
Homogeneous Transformation:

- $\boldsymbol{p^0}$: vector of coordinates of point "P" wrt. frame $O_0\text{-}x_0y_0z_0$ expressed in $O_0\text{-}x_0y_0z_0$
- $\boldsymbol{p^1}$: vector of coordinates of point "P" wrt. frame $O_1\text{-}x_1y_1z_1$ expressed in $O_1\text{-}x_1y_1z_1$
- $\boldsymbol{o^0_1}$: vector describing origin of $O_1\text{-}x_1y_1z_1$ wrt. $O_0\text{-}x_0y_0z_0$ expressed in $O_0\text{-}x_0y_0z_0$
- $\boldsymbol{R^0_1}$: rotation matrix of $O_1\text{-}x_1y_1z_1$ wrt. $O_0\text{-}x_0y_0z_0$

Coordinate transformation:

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0\boldsymbol{p}^1.$$

1. Rotation
2. Translation



Spatial Description of a Rigid Body

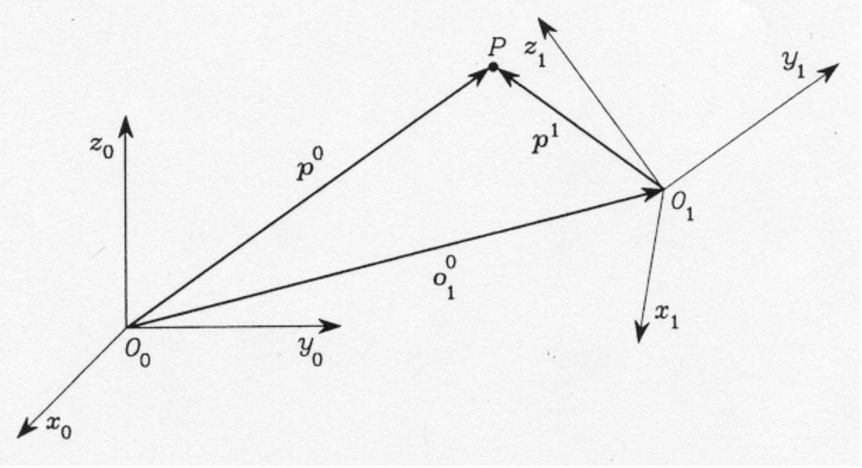
Homogeneous Transformation:

- $p^0 = o^0_1 + R^0_1 p^1$: Coordinates of *"P"* wrt. frame O_0-$x_0y_0z_0$ expressed in O_0-$x_0y_0z_0$
- $R^0_1\, p^1$: Coordinates of *"P"* wrt. frame O_1-$x_1y_1z_1$ expressed in O_0-$x_0y_0z_0$

$$p^0 = o^0_1 + R^0_1 p^1 \Rightarrow (R^0_1)^{-1} p^0 = (R^0_1)^{-1} o^0_1 + p^1$$

$$\Rightarrow p^1 = (R^0_1)^T p^0 - (R^0_1)^T o^0_1 \text{ or } p^1 = R^1_0 p^0 - R^1_0 o^0_1$$

- $p^1 = R^1_0 p^0 - R^1_0 o^0_1$: coordinates of *"P"* wrt. frame O_1-$x_1y_1z_1$ expressed in O_1-$x_1y_1z_1$
- $R^1_0 p^0$: coordinates of *"P"* wrt. frame O_0-$x_0y_0z_0$ expressed in O_1-$x_1y_1z_1$



Spatial Description of a Rigid Body

Homogeneous Transformation:

Homogeneous Representation: to achieve a compact representation of the relationship between the coordinates of the same point in two different frames

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}$$

Homogeneous representation of $\boldsymbol{p}$

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

Homogeneous transformation matrix

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1. \Rightarrow \begin{bmatrix} \boldsymbol{p}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1}$$

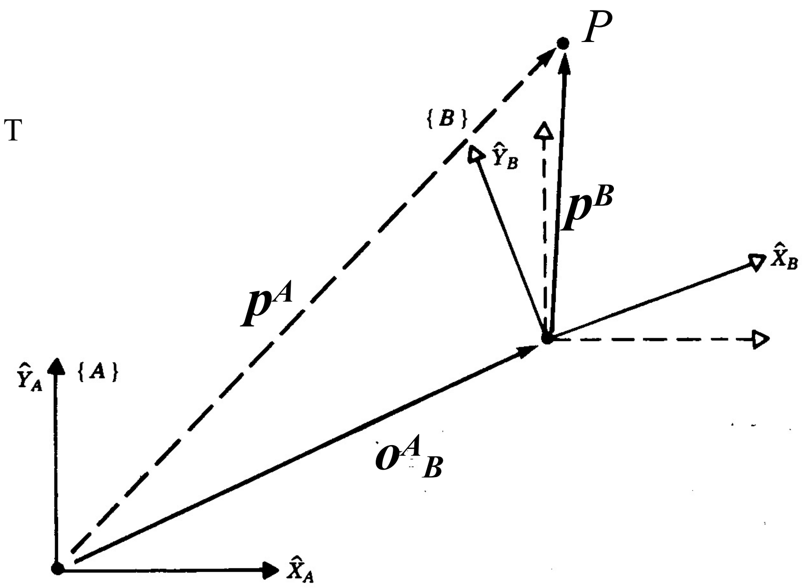
Compact form

Spatial Description of a Rigid Body

Example: Figure below shows a frame *"B"* which is rotated relative to the frame *"A"* about z axis by α=30 degrees, and translated 10 units in X_A and 5 units in Y_A directions. Find p_A given p_B = [3.0 , 7.0 , 0.0]^T.

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$o^A{}_B = [10 \;\; 5 \;\; 0]^T$$



$$A^A{}_B = \left[\begin{array}{ccc|c} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$p^B = [3.0 , 7.0 , 0.0 , 1]^T$$

$$p^A = A^A{}_B \, p^B \;\Rightarrow\; p^A = [9.1 , 12.6 , 0.0 , 1]^T$$

$$p^A = [9.1 , 12.6 , 0.0]^T$$

Spatial Description of a Rigid Body

Homogeneous Transformation:

$$p^1 = R^1{}_0\, p^0 - R^1{}_0\, o^0{}_1$$

$$p^1 = (R^0{}_1)^T p^0 - (R^0{}_1)^T o^0{}_1$$

$$\tilde{p}^1 = A_0^1 \tilde{p}^0 = \left(A_1^0\right)^{-1} \tilde{p}^0.$$

$$(A^0{}_1)^{-1} = A_0^1 = \begin{bmatrix} R_1^{0T} & -R_1^{0T} o_1^0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_0^1 & -R_0^1 o_1^0 \\ 0^T & 1 \end{bmatrix}$$

Inverse transformation

$$A_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0^T & 1 \end{bmatrix}$$

$$A^{-1} \neq A^T$$

Composite transformations: If A_i^{i-1} relates the coordinates of a point in two frames "*i-1*" and "*i*", then a sequence of "*n*" coordinate transformations is described by

$$\tilde{p}^0 = A_1^0 A_2^1 \ldots A_n^{n-1} \tilde{p}^n$$

$$A_n{}^0 = A_1{}^0 A_2{}^1 \cdots A_n{}^{n-1}$$

Special Case: 2D Spatial Representation

- In this course, the vehicle motion and LiDAR scan data are constrained to the two-dimensional *x-y* plane
- In such cases, it would be convenient to ignore movement along z-axis and constrain rotations to those just one around *z*-axis:

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \qquad R = R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

- All previous interpretations and rules still apply, with reduced matrix and vector sizes