ECE 351 - Section #53

Lab #8 Report

Fourier Series Approximation of a Square Wave

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Contents

1 Introduction

The goal of this weeks lab was to learn to approximate periodic time-domain signals with Fourier Series in Python, practicing using a basic Square Wave function. This lab was extremely helpful for my understanding of the related class topics. This lab was completed using *Python* through the *Spyder-IDE*. The packages used in the completion of this lab were `numpy` for definitions of mathematical functions and `matplotlib.pyplot` to plot outputs of functions.

All code for this lab, including this report, can be found on my Github.

2 Equations

The equations used within this lab are shown in this section. The equations will be referenced by number throughout the rest of the report.

General Fourier Series Equations/Formulas:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \tag{1}$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt \tag{2}$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt \tag{3}$$

$$w_0 = \frac{2\pi}{T} \tag{4}$$

Fourier Series found from preliminary:

$$x(t) = \sum_{k=1}^{N} \frac{2}{k\pi}(1 - \cos(k\pi))\sin(k\omega_0 t) \tag{5}$$

3 Methodology

3.1 Preliminary

A crucial part of this lab was the preliminary work. Before the lab started, we found the Fourier Series approximation of the function seen in Figure 1. Using Equations (1), (2), (3), and (4), the Fourier Series approximation found is seen in Equation (5). My preliminary work can be found in the Attachments section of this report.

Unfortunately, my first attempt at finding this approximation was a failure, but we covered the correct method during lab. Both methods used can be found in the attached work.

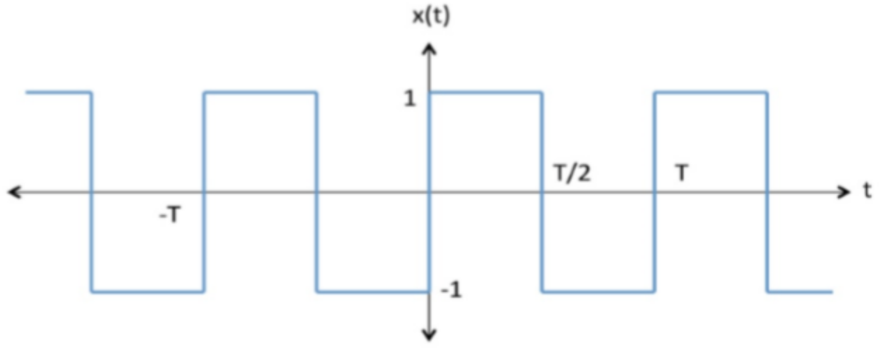


Figure 1: Square Wave to Approximate

3.2 Lab: Part 1

In Part 1 of this lab, the only part, we explored different approximations of the square wave function given in the lab handout (Figure 1) using the Fourier Series approximation found in the preliminary (Equation (5)).

We started by finding the first four values of a_k & b_k after giving them function definitions. After this, we found the Fourier Series Approximation for $N = \{1, 3, 15, 50, 150, 1500\}$.

Below can be seen the code implementation of the tasks carried out in Part 1 of this lab.

```
#GLOBAL
step_size = 1e-3

# PART 1

#1
def b(k) :
    b_k = (2/(k*np.pi))*(1-np.cos(k*np.pi))
    return b_k

# a_k = 0; Thus is not defined.

print("a_k & b_k for 4 values.")
for i in range(1, 5, 1) :
```

```
    print("a_", i, "=", 0, " ; ", "b_", i, "=", b(i))

#2
T = 8;
w0 = 2*np.pi/T
def x(t, N) :
    x = 0
    for k in range(1, N+1, 1) :
        x += b(k) * np.sin(k*w0*t)
    return x

fList = [1, 3, 15, 50, 150, 1500]

for i in fList :
    t = np.arange(0, 20 + step_size, step_size)
    y1 = x(t, i)

    plt.figure(figsize = (10, 11))
    plt.subplot(1, 1, 1)
    plt.plot(t, y1, "b-")
    plt.grid()
    plt.ylabel('x(t) [N = ' + str(i) + ']')
    plt.xlabel('t')
    plt.title(Fourier Series Output for N = ' + str(i))
```

4 Results

The results of this lab are very straightforward. The implementation of all functions worked as expected and the results are as expected.

The deliverables for Part 1 of this lab are seen in all figures given below. As can be seen in all graphs, as N increases the approximation of the function gets better. This explains why in the definition of the Fourier Series as $N \to \infty$, the desired function should be perfectly approximated.

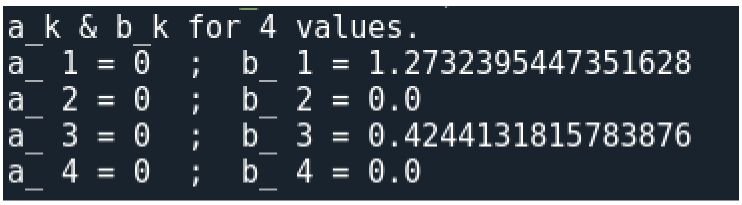

```
a_k & b_k for 4 values.
a_ 1 = 0  ;  b_ 1 = 1.2732395447351628
a_ 2 = 0  ;  b_ 2 = 0.0
a_ 3 = 0  ;  b_ 3 = 0.4244131815783876
a_ 4 = 0  ;  b_ 4 = 0.0
```

Figure 2: First Four Terms of Fourier coefficients (a_k & b_k)

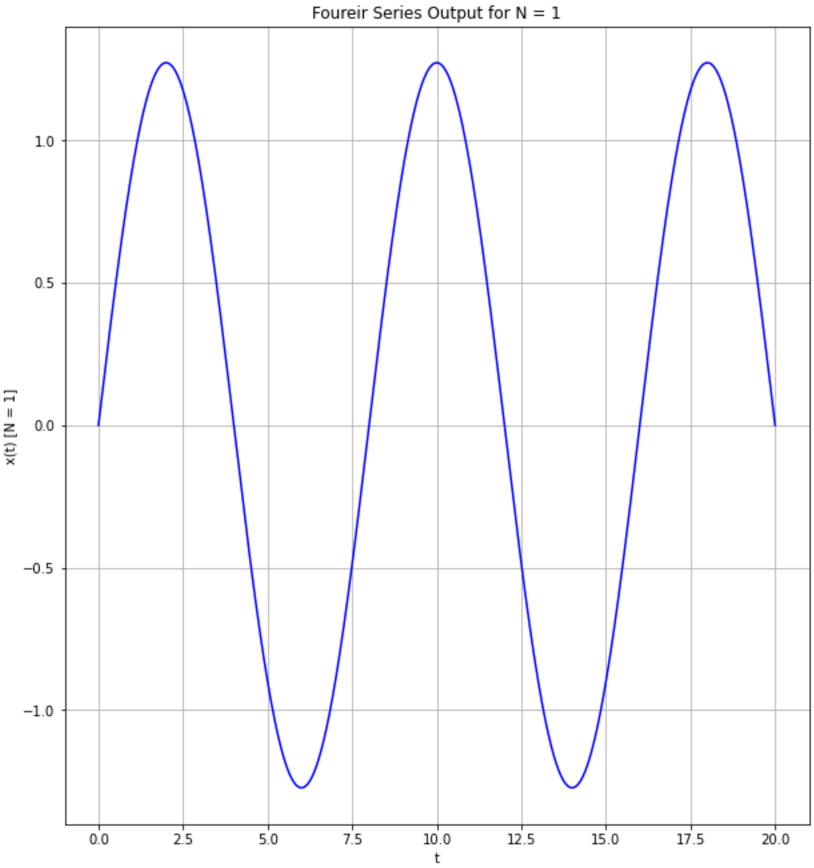


Figure 3: Fourier Approximation for $N = 1$

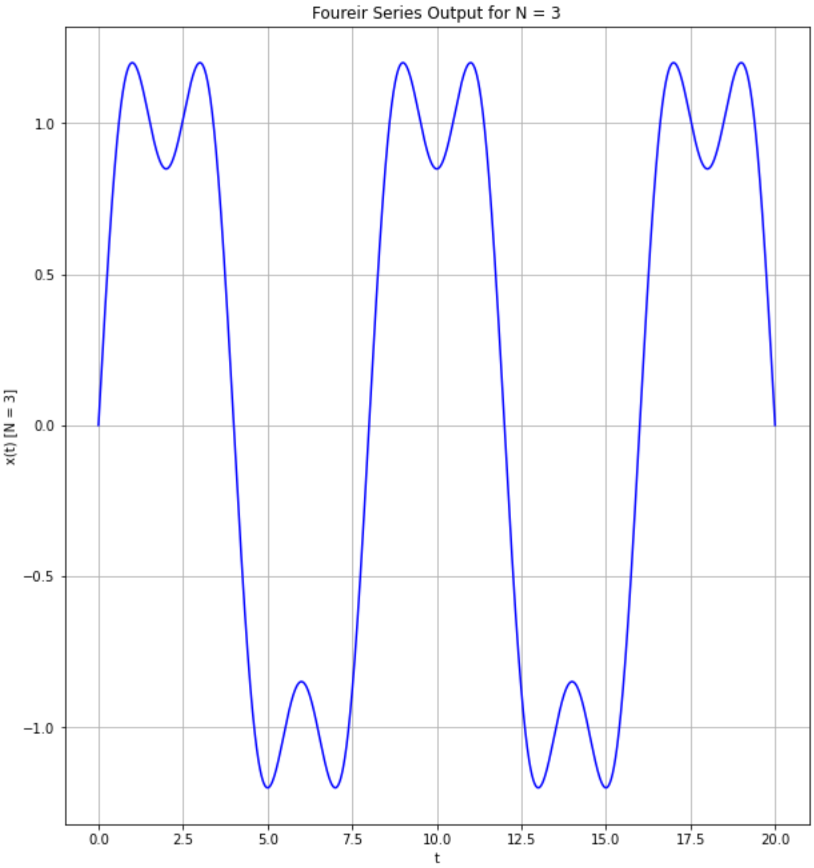


Figure 4: Fourier Approximation for $N = 3$

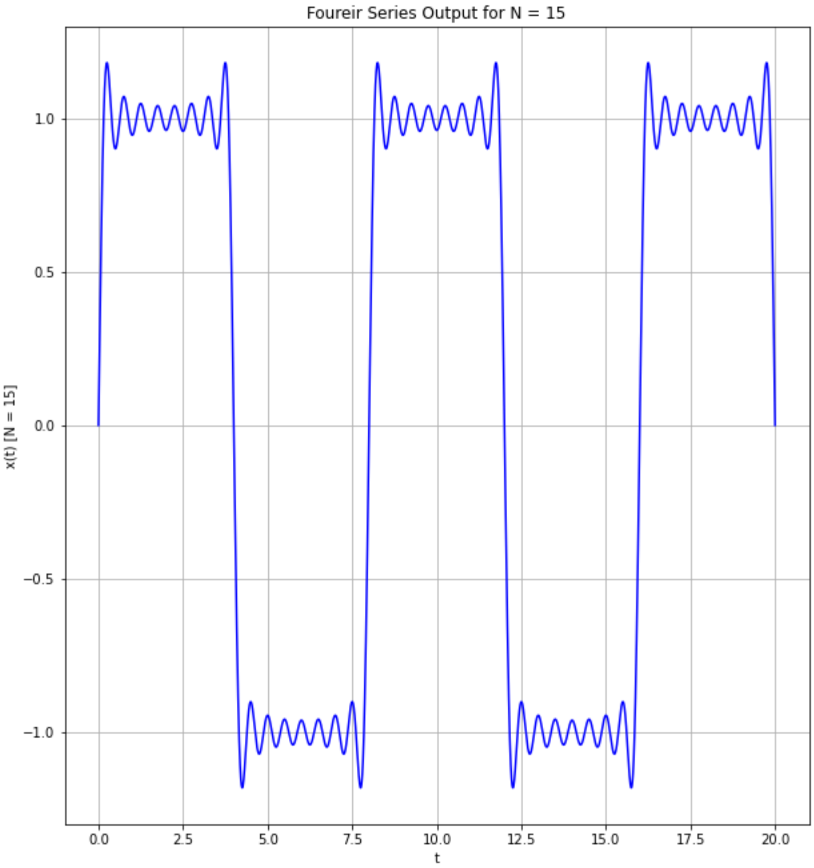


Figure 5: Fourier Approximation for $N = 15$

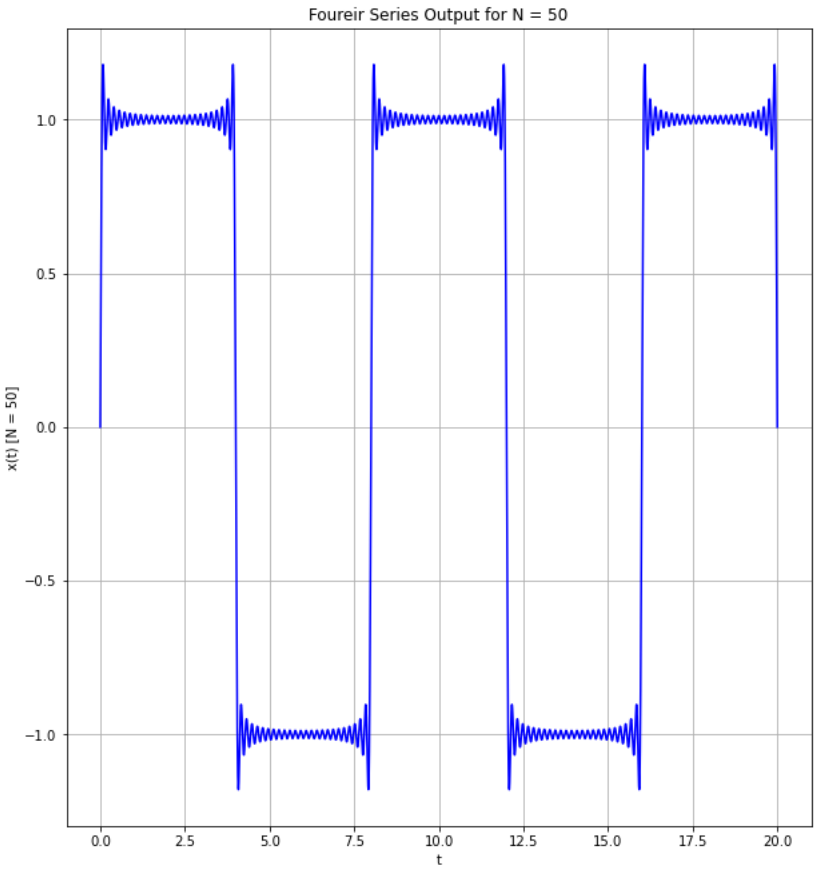


Figure 6: Fourier Approximation for $N = 50$

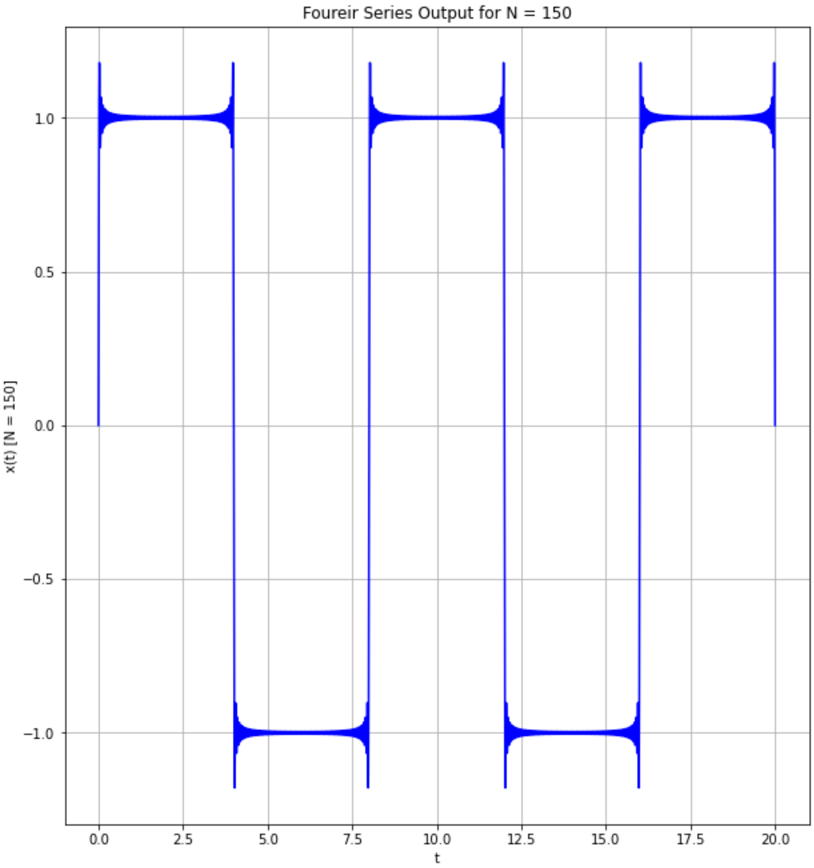


Figure 7: Fourier Approximation for $N = 150$

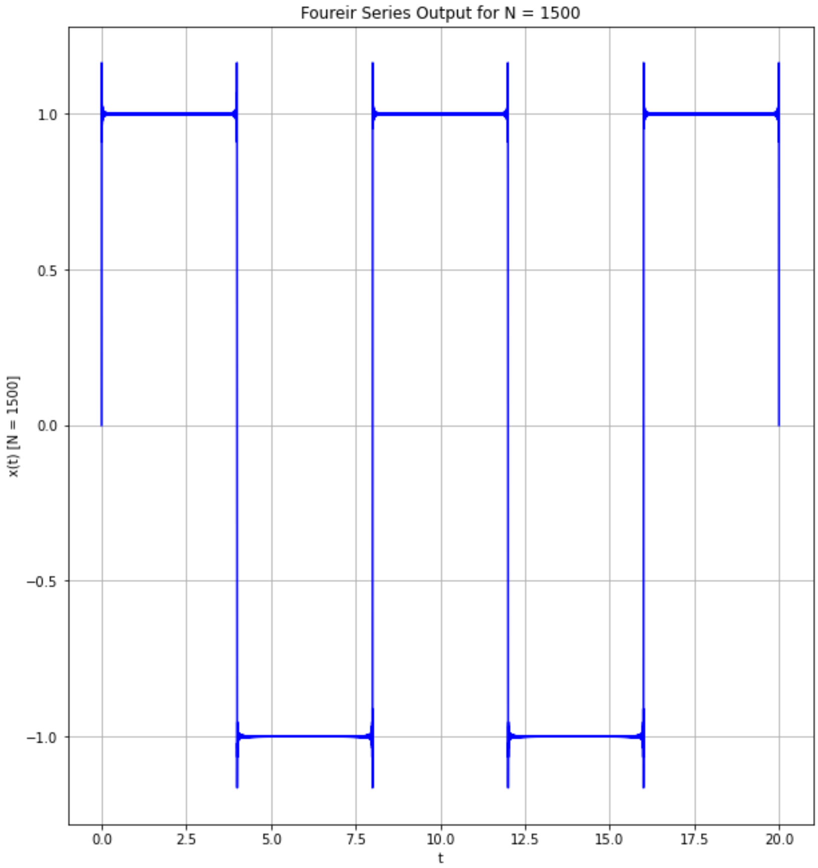


Figure 8: Fourier Approximation for $N = 1500$

5 Error Analysis

No sources of error were seen throughout this lab, and I did not run into any real problems while implementing anything lab lab asked. The implementation of Task 2 of this lab took some time to implement properly, but as seen from Lab: Part 1 the code is not that complicated to understand.

6 Questions

1. In Figure 1, the square wave function seen is odd as $f(-x) = -f(x)$.
2. As seen from the preliminary in Attachments, the values of $a_2, a_3, ..., a_n$ do not really matter as the term associated with that coefficient will go to zero.
3. As seen in the Results section of this report, as N increases the better the approximation of the Square wave becomes. With only 1500 terms, the approximation looks pretty good but is far from perfect. As seen in majority of Fourier Series, there is a ringing at the peak edges of the square wave which is the biggest visual difference in the approximation.
4. Mathematically, as the value of N increases more terms of monotonically increasing harmonics are added to the series. This further fills the inaccuracies of the current approximation and leaves you with a better looking approximation as $N \to \infty$.
5. Purpose, deliverables, and expectations were very clear for this lab.

7 Conclusion

In conclusion, I feel this lab was very successful. The implementation of the code in this lab was quite simple and I really enjoyed seeing how you could find the total transfer function with some symbolic math and Python. All in all, I am very satisfied with what this lab has taught me and feel it was an excellent use of time.

8 Attachments

1. Pre-Lab

Muc
Edmondson

Fourier Series Approximation of a Square Wave

Prelab 8

Spring 2021

1 Purpose

Find a general expression for the coefficients of a Fourier series.

2 Deliverables

Typed and properly formatted derivation of the k-th Fourier series coefficients a_k and b_k.

3 Tasks

Consider the square wave in figure 1. Assume this is a real-valued function and can be explained by the Fourier series:

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos(k\omega_0 t) + b_k sin(k\omega_0 t) \tag{1}$$

Where,

$$a_k = \frac{2}{T}\int_0^T x(t)cos(k\omega_0 t)dt \tag{2}$$

$$b_k = \frac{2}{T}\int_0^T x(t)sin(k\omega_0 t)dt \tag{3}$$

$$\omega_0 = \frac{2\pi}{T} \tag{4}$$

1. By hand, find the expressions for a_k, b_k, and $x(t)$.

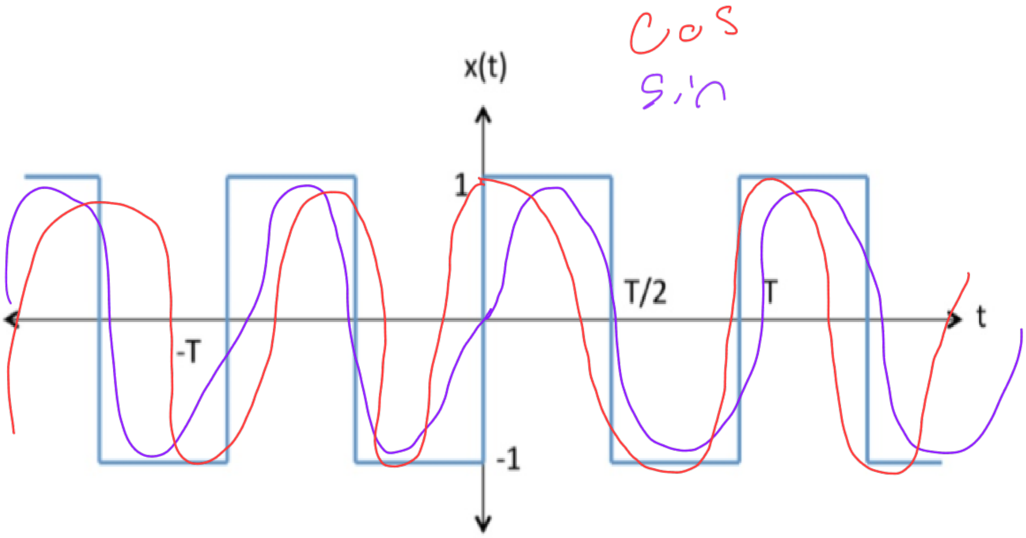
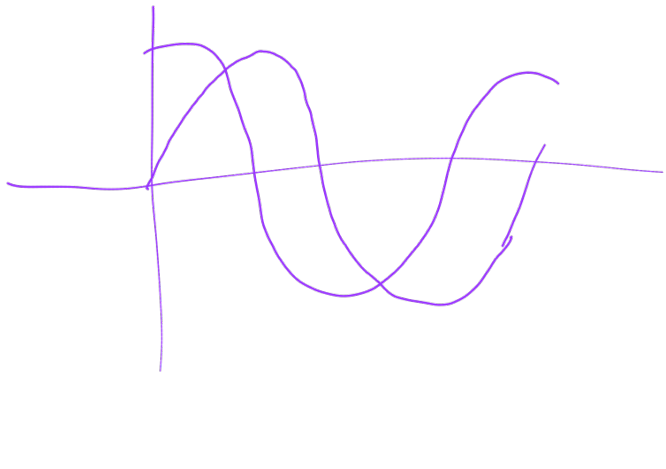


Figure 1: Square Wave for ECE 351 Prelab 8



As seen on graph,

$a_0 = \emptyset$. (No DC component)

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$= \frac{2}{T}\left(\int_0^{T/2}\cos(k\omega_0 t)\,dt - \int_{T/2}^{T}\cos(k\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(\frac{\sin(kT\omega_0/2)}{k\omega_0} - \frac{\sin(kT\omega_0)-\sin(kT\omega_0/2)}{k\omega_0}\right)$$

$$= \frac{\cancel{2}}{\cancel{T}}\left(\frac{\cancel{T}\sin(k\pi)}{k\cancel{2}\pi} - \frac{\cancel{T}(\sin(2\pi k)-\sin(k\pi))}{k\cancel{2}\pi}\right)$$

$$= \frac{\sin(k\pi) - \sin(2\pi k) + \sin(k\pi)}{k\pi}$$

$$= \frac{2\sin(k\pi) - \sin(k2\pi)}{k\pi}$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt = \frac{2}{T}\left(\int_0^{T/2}\sin(k\omega_0 t)\,dt - \int_{T/2}^{T}\sin(k\omega_0 t)\,dt\right)$$

$$= \frac{2}{T}\left(\left[-\frac{\cos(k\omega_0 t)}{k\omega_0}\right]_0^{T/2} + \left[\frac{\cos(k\omega_0 t)}{k\omega_0}\right]_{T/2}^{T}\right)$$

$$= \frac{2}{T}\left[\left(\cancel{\frac{-\cos(k\omega_0\frac{T}{2})}{k\omega_0}} + \frac{1}{k\omega_0}\right) + \left(\frac{\cos(k\omega_0 T)}{k\omega_0} - \cancel{\frac{\cos(k\omega_0\frac{T}{2})}{k\omega_0}}\right)\right]$$

$$= \frac{2}{T}\left(\frac{\cos(k\omega_0 T)}{k\omega_0} + \frac{1}{k\omega_0}\right) = \frac{\cancel{2}}{\cancel{T}}\left(\frac{\cancel{T}\cos(k2\pi)}{k2\pi} + \frac{\cancel{T}}{k2\pi}\right)$$

$$= \frac{\cos(k2\pi)+1}{k\pi}$$

$$\Rightarrow X(t) = \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

In-Lab: (gone-over with Kate)

$$\cancel{a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt}$$

we know this goes to $\emptyset$.

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$b_k = 2 \cdot \frac{2}{T}\int_0^{T/2} (1)\sin(k\omega_0 t)\,dt$$

$$= \frac{4}{T} - \frac{1}{k\omega_0}\left[\cos\left(\frac{k\omega_0 t}{2}\right) - 0\right]$$

$$= \frac{2}{k\pi}\left(1 - \cos(k\pi)\right)$$