Math 471 Spring 2025
Topology Lucia

Homework 2

1. Let $f : \mathbb{R} \to \mathbb{Z}$ be the "floor" function which rounds a real number x down to the nearest integer:

$$f(x) = n \text{ provided that } n \in \mathbb{Z} \text{ and } n \leq x < n+1.$$

   (a) Determine whether or not f is continuous.

   f isn't continuous, for example, $U = \{n\}$, $U \subseteq \mathbb{Z}$ is open, so $f^{-1}(U)$ should also be open to make the function continuous, but $f^{-1}(U) \subseteq \mathbb{R}$, $[n, n+1)$ isn't open.

   (b) Next, determine if f is continuous for $f : \mathbb{R}_{\ell\ell} \to \mathbb{Z}$, where $\mathbb{R}_{\ell\ell}$ is the lower limit topology which we define as the topology on $\mathbb{R}$ with basis given by

$$\mathcal{B} = \{[a, b) | a < b\}.$$

   f is continuous. Because we have discrete topology on $\mathbb{Z}$, $\forall U \subseteq \mathbb{Z}$ is open. For every element $i \in U$, we can get $[i, i+1) \subset \mathbb{R}_{\ell\ell}$, which means we can write $f^{-1}(U) = \cup_{j=1}^{n} [i_j, i_j + 1)$. Since any $[i, i+1)$ are open in $\mathbb{R}_{\ell\ell}$, the union $f^{-1}(U)$ is also open.

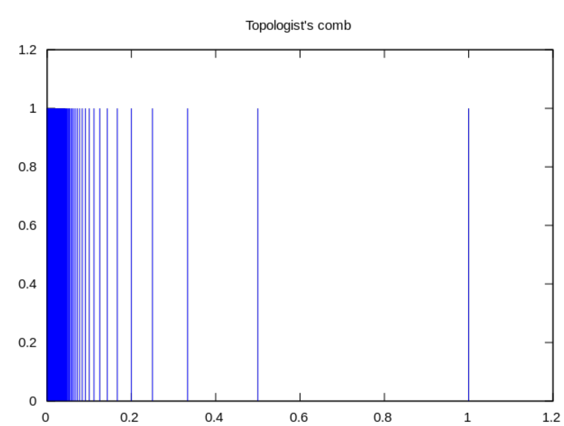
2. Show that the collection $\big\{\{a\} \times (b, c) \subseteq \mathbb{R}^2 | a, b, c \in \mathbb{R}\big\}$ of vertical intervals in the plane is a basis for a topology on $\mathbb{R}^2$. We call this topology the *vertical interval topology*.

   (a) For every $(a, b) \in \mathbb{R}^2$, $\exists \{a\} \times (b-1, b+1) \in \textit{collection}$ with $(a, b) \in \{a\} \times (b-1, b+1)$

   (b) Let $B_1 = \{a_1\} \times (b_1, c_1) \subseteq \mathbb{R}^2$ and $B_2 = \{a_2\} \times (b_2, c_2) \subseteq \mathbb{R}^2$. If $a_1 \neq a_2$, $B_1 \cap B_2 = \emptyset$. If $a_1 = a_2$, and let the intersection be $\{a_1\} \times (b_2, c_1)$. We can find any point (a_1, d) that $b_2 < d < c_1$. Then take $\epsilon < min(|b_2 - d|, |c_1 - d|)$, so that $(a_1, d) \in \{a_1\} \times (d - \epsilon, d + \epsilon) \subset \{a_1\} \times (b_2, c_1)$

3. The topologist's comb C is the subset of $\mathbb{R}^2$ defined by

$$C = \big\{(x, 0) \,\big|\, 0 \leq x \leq 1\big\} \cup \big\{(\frac{1}{2^n}, y) \,\big|\, n = 0, 1, 2, \ldots \text{ and } 0 \leq y \leq 1\big\}.$$



(a) Decide (with justification, of course) if C is closed in the standard topology on $\mathbb{R}^2$.

C isn't closed in the standard topology. If C is closed, $R\backslash C$ should be open, which means for any points $p \in R\backslash C$ we can find an $\epsilon-$ball inside the boundary. It's true for any point except points on y-axis $\{0\} \times (0,1)$. If we pick any point on the y-axis, $\forall \epsilon$, we can find n so that $\frac{1}{2^n} < \epsilon$, which means we can never find a $\epsilon-$ ball for points on y-axis. Therefore $R\backslash C$ isn't open, and C isn't closed.

(b) Decide (again with justification) if C is closed in the vertical interval topology on $\mathbb{R}^2$.

C is closed in the vertical intercal topology.In this topological space, we no longer need to find $\epsilon-$ ball, but a vertical line. So the points on y-axis won't be the problems for $R\backslash C$ now, and for any points we can find a ϵ vertical line that is inside the boundary. Therefore $R\backslash C$ is open, then C is closed.

4. Recall we say that a sequence of points $(x_1, x_2, x_3, \ldots)$ in a topological space X *converges* to a point $x \in X$ if for every neighborhood U of x there exists a positive integer N such that $x_n \in U$ for all $n \geq N$.

(a) Let $x_n = \frac{(-1)^n}{n} \in \mathbb{R}$ and discuss the convergence of this sequence in $\mathbb{R}$ under the standard topology and

$x_n = \frac{(-1)^n}{n} \in \mathbb{R}$ will converges to 0 as n goes in infinity. For any ϵ, we can let $N < \frac{1}{\epsilon}$ so that $x_n \in U$ for all $n \geq N$.

(b) in $\mathbb{R}_{\ell\ell}$ under the lower-limit topology.

$x_n = \frac{(-1)^n}{n} \in \mathbb{R}$ won't converge to 0 or any number. Since $[0, \epsilon)$ is a neighborhood, it's impossible to find a negative number in this neighborhood so x_n doesn't converges to 0.

(c) Consider sequences in $\mathbb{R}$ with the finite complement topology. Which sequences converge? To what value(s) do they converge?

In the finite complement topology on $\mathbb{R}$, a subset $U \subseteq \mathbb{R}$ is open if either U is the empty set or U has a finite complement in $\mathbb{R}$ ($\mathbb{R}\backslash U$ is finite). For (x_n) converges to x, we want to make sure that for any open set U containing x, $\exists N$ such that $x_n \in U$ for all $n \geq N$. If the sequence has infinite points and converges in $\mathbb{R}_{std}$, it would , because for any open set $U \subseteq R$, since there are only finite numbers exclude from $\mathbb{R}$, we can always find the N, so that $x_n \in U,\ \forall n \geq N$.

However, if the sequence converges to a number but there are finite points in this sequence, there must exist some $U \subseteq \mathbb{R}$ that excludes all the points except x_n. This means for $U \subseteq \mathbb{R}$, we cannot find any N, so that $x_n \in U,\ \forall n \geq N$.

5. (a useful property) Let X be a topological space and let A and B be closed subsets of X such that $X = A \cup B$. Let Y be another topological space and let $f : A \to Y$ and $g : B \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Prove that the function $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A, \\ g(x), & \text{if } x \in B, \end{cases}$$

is a continuous function.

Since $f : A \to Y$ and $g : B \to Y$ are continuous functions, for any open set $U \subseteq Y$, the preimage $f^{-1}(U)$ is open in A and also $g^{-1}(U)$ is open in B. $h(x) = f(x) \cup g(x)$, so $h : A \cup B \to Y$. We want to show that for any open set $V \subseteq Y$, the preimage $h^{-1}(V)$ is open in X.

Let $V = V_1 \cup V_2$ and $h^{-1}(V) = h^{-1}(V_1) \cup h^{-1}(V_2)$ that both V_1 and V_2 are open and also $h^{-1}(V_1) = f^{-1}(V_1) \subseteq A$, $h^{-1}(V_2) = g^{-1}(V_2) \subseteq B(V_1,\ V_2$ can be empty). Since we know that V_1 and V_2 are open, $f^{-1}(V_1)$ and $h^{-1}(V_2)$ are also open. Therefore $h^{-1}(V) = h^{-1}(V_1) \cup h^{-1}(V_2) = f^{-1}(V_1) \cup h^{-1}(V_2)$ the union of open set is also open.

We show that the preimage $h^{-1}(V)$ is open in X.

6. Let X be a topological space. Let $f : X \to \mathbb{R}$ be a function such that for all $a \in \mathbb{R}$, the sets

$$\{x \mid f(x) > a\} \text{ and } \{x \mid f(x) < a\}$$

are open (that is these form the topology on X). Prove that f is continuous.

For any open set $(i, j) \subseteq \mathbb{R}$, $f^{-1}((i, j)) = \{x | i < f(x) < j\} = \{x | f(x) < j\} \cap \{x | i < f(x)\}$. Since both $\{x | f(x) < j\}$ and $\{x | i < f(x)\}$ are open in the topological space, the finite intersection should also be open. Therefore, the preimage of any openset (i, j) is open.

7. Let X be a set with a topology $\mathcal{T}$ on it, such that $\mathcal{B}$ is a basis for the topological space $(X, \mathcal{T})$. For $Y \subset X$, define $\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$. Show that $\mathcal{B}_Y$ is a basis for Y under the subspace topology.

$\mathcal{T}_Y = \{Q \cap Y \mid Q \in \mathcal{T}\}$

Using Munkres

(a) Since $Y \subset X$, for every $y \in Y$, $\exists B \in \mathcal{B}$ that $y \in B$. Because $y \in Y$ and $y \in B$, $y \in Y \cap B = B_i$. By the definition of $\mathcal{B}_Y$, $B_i \in \mathcal{B}_Y$. Therefore for every $y \in Y$, $\exists B_i \in \mathcal{B}_Y$ with $y \in B_i$.

(b) For $B_1 \cap Y \in \mathcal{B}_Y, B_2 \cap Y \in \mathcal{B}_Y$ where $B_1, B_2 \in \mathcal{B}$, and $y \in (B_1 \cap Y) \cap (B_2 \cap Y)$, since y is an elemt of X, it's true that there exist $B_3 \in \mathcal{B}$ such that $y \in B_3 \subseteq B_1 \cap B_2$. Because $y \in Y$, insert Y to the previous one and we get

$$y \in B_3 \cap Y \subseteq B_1 \cap B_2 \cap Y$$
$$y \in B_3 \cap Y \subseteq (B_1 \cap Y) \cap (B_2 \cap Y)$$

Because $B_3 \cap Y \in \mathcal{B}_Y$, we prove B2 for Munkres.