Notes on generation interval

Define survival function as follows:

$$s_i(a) = \begin{cases} \exp(-\beta a), & a < r_i \\ \exp(-\beta r_i) \end{cases}$$

Define φ:

$$\varphi_i(a) = \begin{cases} \beta, & a < r_i \\ 0 \end{cases}$$

Define

$$l_i(a) = \varphi_i(a) s_i(a) = \begin{cases} \beta \exp(-\beta a), & a < r_i \\ 0 \end{cases}$$

Now assume that we have exponentially distributed infectious period with mean γ to get

$$\begin{aligned} \Big\langle \beta \exp(-\beta a) \Big\rangle_i &= \int_a^\infty \beta \exp(-\beta a) \gamma \exp(-\gamma t) dt \\ &= \beta \exp(-(\beta + \gamma) a) \end{aligned}$$

Normalizing this, we get

$$g(a) = (\beta + \gamma) \exp(-(\beta + \gamma) a)$$

Let's do SEIR. Survival function:

$$s_i(a) = \begin{cases} 1, & a < m_i \\ \exp(-\beta(a - m_i)), & m_i \leq a < m_i + r_i \\ \exp(-\beta r_i) \end{cases}$$

Here, we have

$$\varphi_i(a) = \begin{cases} 0, & a < m_i \\ \beta, & m_i \leq a < m_i + r_i \\ 0 \end{cases}$$

Similarly, define

$$l_i(a) = \varphi_i(a) s_i(a) \begin{cases} 0, & a < m_i \\ \beta \exp(-\beta(a - m_i)), & m_i \leq a < m_i + r_i \\ 0 \end{cases}$$

We have to consider following conditions:

$$\begin{cases} m_i < a \\ r_i > a - m_i \end{cases}$$

If latent and infectious periods have probability density function of f_E and f_I, this is what we're interested in:

$$\int_0^a f_E(s) \left(\beta e^{-\beta(a-s)} \int_{a-s}^\infty f_I(t) dt \right) ds$$

Now, assume that latent and infectious periods are exponentially distributed with mean σ and γ.

$$\begin{aligned}
&\beta \int_0^a f_E(s) \left(e^{-\beta(a-s)} \int_{a-s}^\infty f_I(t) dt \right) ds \\
&= \beta \int_0^a f_E(s) \left(e^{-\beta(a-s)} \int_{a-s}^\infty \gamma e^{-\gamma t} dt \right) ds \\
&= \beta \int_0^a f_E(s) \left(e^{-\beta(a-s)} e^{-\gamma(a-s)} \right) ds \\
&= \beta \int_0^a f_E(s) \left(e^{-(\beta+\gamma)(a-s)} \right) ds \\
&= \beta e^{-(\beta+\gamma)a} \int_0^a \sigma e^{-\sigma s} e^{(\beta+\gamma)s} ds \\
&= \beta \sigma e^{-(\beta+\gamma)a} \int_0^a e^{(\beta+\gamma-\sigma)s} ds \\
&= \beta \sigma (e^{-\sigma a} - e^{-(\beta+\gamma)a}) / (\beta + \gamma - \sigma)
\end{aligned}$$

And we need to normalize to get the following generation distribution:

$$g(a) = \frac{\sigma(\beta+\gamma)}{\beta+\gamma-\sigma} (e^{-\sigma a} - e^{-(\beta+\gamma)a})$$