Notes on network and generation

Suppose we have one infected individual and one susceptible individual. Infected individual makes effective contact at a constant rate β and has a constant recovery rate of λ. Then, we can write the probability that the susceptible remains uninfected at time t as follows:

$$s(\tau) = \exp(-\beta \int_{t=0}^{\tau} \exp(-\lambda t) dt)$$

Solving the integral, we have

$$s(\tau) = \exp\left(-\frac{\beta}{\lambda}(1 - \exp(-\lambda\tau))\right)$$

Define $K(\tau) = F(\tau)\beta$ as the rate of secondary case, where $F(\tau) = \exp(-\lambda\tau)$ is the probability that an infectious individual has not recovered t time units after infection.

We define network generation interval as follows:

$$g(\tau) = \frac{K(\tau)s(\tau)}{\int_0^\infty K(\tau)s(\tau)d\tau}$$

which can be simplified into

$$g(\tau) = \frac{F(\tau)s(\tau)}{\int_0^\infty F(\tau)s(\tau)d\tau}$$

Let's try to solve the integral... First, let's do substitution:

$$\begin{aligned} dF &= -\lambda \exp(-\lambda\tau)d\tau \\ &= -\lambda F d\tau \end{aligned}$$

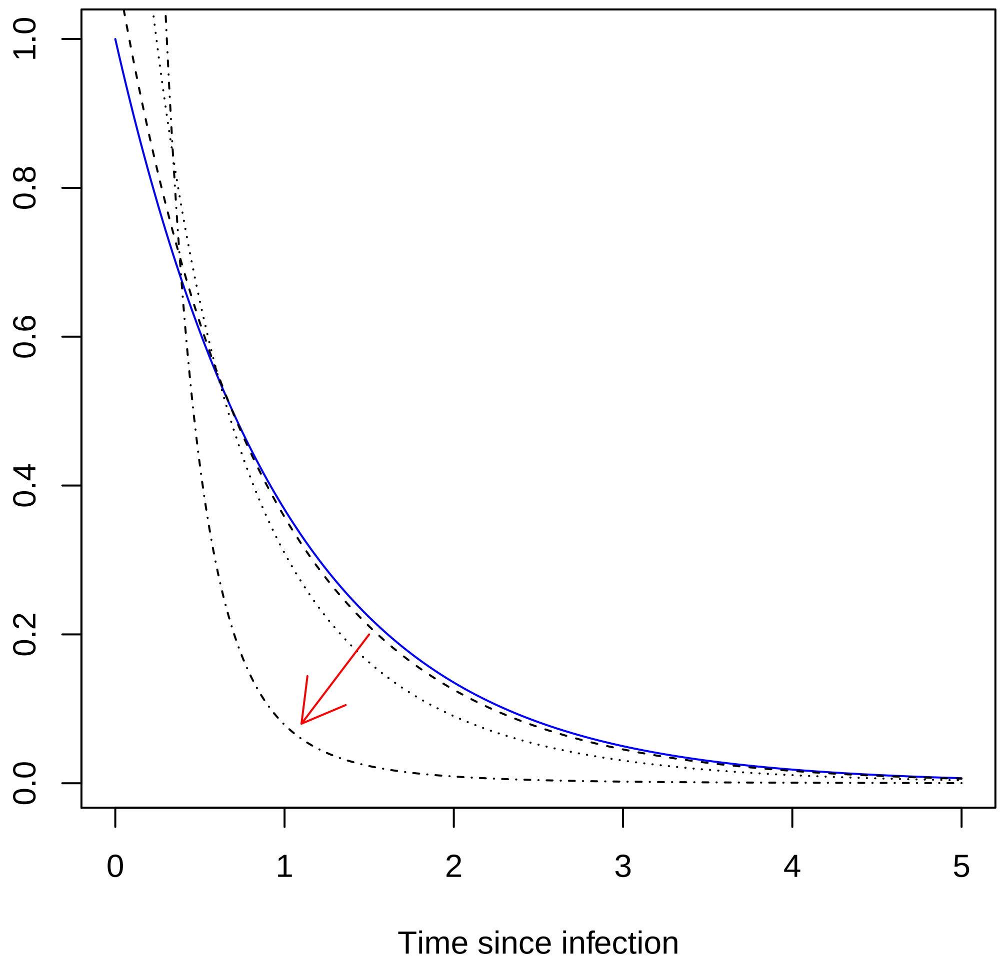
Substitute it:

$$\begin{aligned} &\int_0^\infty F(\tau)s(\tau)d\tau \\ &= -\frac{1}{\lambda}\int_1^0 \exp(-\frac{\beta}{\lambda}(1-F))dF \\ &= \frac{1}{\beta}\left[\exp(-\frac{\beta}{\lambda}(1-F))\right]_0^1 \\ &= \frac{1}{\beta}\left(1 - \exp(-\frac{\beta}{\lambda})\right) \end{aligned}$$

Therefore, the generation interval is

$$g(\tau) = \beta\left(1 - \exp(-\frac{\beta}{\lambda})\right)^{-1} \exp(-\lambda\tau)\exp\left(-\frac{\beta}{\lambda}(1 - \exp(-\lambda\tau))\right)$$

Comparison of intrinsic generation with network generation distribution for SIR model:



As we increase effective transmission rate, network generation follows the red arrow. Let's try to write a general equation.

In a homogeneous population, generation interval is defined s follows (Svensson 2015):

$$g(\tau) = \frac{P(X + Y > \tau) - P(X > \tau)}{E(Y)}$$

where X is the duration of the latent period and Y is the duration of the infection period. To check that it works, let's try $SEIR$ model where both latent and infectious periods are exponentially distributed with rate γ and λ:

$$
\begin{aligned}
P(X+Y>\tau) &= 1 - \int_0^\tau \int_0^s f_Y(s-t) f_X(t) dt ds \\
&= 1 - \int_0^\tau \int_0^s \lambda \exp(-\lambda(s-t)) \gamma \exp(-\gamma t) dt ds \\
&= 1 - \int_0^\tau \gamma\lambda \exp(-\lambda s)[\exp((\lambda-\gamma)t)/(\lambda-\gamma)]_0^s ds \\
&= 1 - \int_0^\tau \frac{\gamma\lambda}{\lambda-\gamma} \exp(-\lambda s)(\exp((\lambda-\gamma)s)-1) ds \\
&= 1 - \frac{\gamma\lambda}{\lambda-\gamma} \int_0^\tau (\exp(-\gamma s) - \exp(-\lambda s)) ds \\
&= 1 - \frac{\gamma\lambda}{\lambda-\gamma} \left[\exp(-\lambda s)/\lambda - \exp(-\gamma s)/\gamma \right]_0^\tau \\
&= 1 - \frac{1}{\lambda-\gamma} \left[\gamma \exp(-\lambda s) - \lambda \exp(-\gamma s) \right]_0^\tau \\
&= 1 - \frac{1}{\lambda-\gamma} \left[\gamma \exp(-\lambda \tau) - \lambda \exp(-\gamma \tau) - \gamma + \lambda \right] \\
&= \frac{\gamma \exp(-\lambda\tau) - \lambda \exp(-\gamma\tau)}{\gamma - \lambda}
\end{aligned}
$$

And we also have

$$
\begin{aligned}
P(X>\tau) &= 1 - \int_0^\tau \gamma \exp(-\gamma t) dt \\
&= 1 + (\exp(-\gamma t))_0^\tau \\
&= \exp(-\gamma\tau)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{P(X+Y>\tau) - P(X>\tau)}{E(Y)} &= \lambda \left(\frac{\gamma \exp(-\lambda\tau) - \lambda \exp(-\gamma\tau)}{\gamma-\lambda} - \exp(-\gamma\tau) \right) \\
&= \lambda\gamma \left(\frac{\exp(-\lambda\tau) - \exp(-\gamma\tau)}{\gamma-\lambda} \right)
\end{aligned}
$$

Got the same result as Svensson, which is a good sign. Now, how do we apply this to a network.. Going back to Trapman et al, we have the following survival function:

$$
\begin{aligned}
s(\tau) &= \exp\left(-\beta \left(\int_0^\tau F(t) dt \right) \right) \\
&= \exp\left(-\beta \left(\int_0^\tau P(X+Y>t) - P(X>t) dt \right) \right)
\end{aligned}
$$