

# Notes on generation interval

Define survival function as follows:

$$s_i(a) = \begin{cases} \exp(-\beta a), & a < r_i \\ \exp(-\beta r_i) & \end{cases}$$

Define the probability that an infected person is still in the state,  $\varphi$ :

$$\varphi_i(a) = \begin{cases} 1, & a < r_i \\ 0 & \end{cases}$$

Define

$$l_i(a) = \varphi_i(a)s_i(a) = \begin{cases} \exp(-\beta a), & a < r_i \\ 0 & \end{cases}$$

Now assume that we have exponentially distributed infectious period with mean  $\gamma$  to get

$$\begin{aligned} \left\langle \exp(-\beta a) \right\rangle_i &= \int_a^\infty \exp(-\beta a) \gamma \exp(-\gamma t) dt \\ &= \exp(-(\beta + \gamma)a) \end{aligned}$$

Normalizing this, we get

$$g(a) = (\beta + \gamma) \exp(-(\beta + \gamma)a)$$

Let's do SEIR. Survival function:

$$s_i(a) = \begin{cases} 1, & a < m_i \\ \exp(-\beta(a - m_i)), & m_i \leq a < m_i + r_i \\ \exp(-\beta r_i) & \end{cases}$$

Similarly, define

$$l_i(a) = \begin{cases} 0, & a < m_i \\ \exp(-\beta(a - m_i)), & m_i \leq a < m_i + r_i \\ 0 & \end{cases}$$

We have to consider following conditions:

$$\begin{cases} m_i < a \\ r_i > a - m_i \end{cases}$$

If latent and infectious periods have probability density function of  $f_E$  and  $f_I$ , this is what we're interested in:

$$\int_0^a f_E(s) \left( e^{-\beta(a-s)} \int_{a-s}^{\infty} f_I(t) dt \right) ds$$

Now, assume that latent and infectious periods are exponentially distributed with mean  $\sigma$  and  $\gamma$ .

$$\begin{aligned} & \int_0^a f_E(s) \left( e^{-\beta(a-s)} \int_{a-s}^{\infty} f_I(t) dt \right) ds \\ &= \int_0^a f_E(s) \left( e^{-\beta(a-s)} \int_{a-s}^{\infty} \gamma e^{-\gamma t} dt \right) ds \\ &= \int_0^a f_E(s) \left( e^{-\beta(a-s)} e^{-\gamma(a-s)} \right) ds \\ &= \int_0^a f_E(s) \left( e^{-(\beta+\gamma)(a-s)} \right) ds \\ &= e^{-(\beta+\gamma)a} \int_0^a \sigma e^{-\sigma s} e^{(\beta+\gamma)s} ds \\ &= \sigma e^{-(\beta+\gamma)a} \int_0^a e^{(\beta+\gamma-\sigma)s} ds \\ &= \sigma (e^{-\sigma a} - e^{-(\beta+\gamma)a}) / (\beta + \gamma - \sigma) \end{aligned}$$

And we need to normalize to get the following generation distribution:

$$g(a) = \frac{\sigma(\beta + \gamma)}{\beta + \gamma - \sigma} (e^{-\sigma a} - e^{-(\beta+\gamma)a})$$