Abstract

Viral spread refers to a rapid rise in interest in something. Prior work has focused on which characteristics something must have to spread more virally than its competitors. Comparatively little work has been done on which networks are more receptive to a given meme. (By meme, we mean any idea, behavior or social phenomenon.) To better understand which types of social networks support the viral spread of a meme, we analyze generic social networks, derive an expression that predicts the most amenaable networks for a meme, and validate those predictions with computational simulations. Here, we show that (i) interaction between cliques in a social network retards viral activity if it exceeds a critical range, (ii) interaction within cliques promotes viral activity, and (iii) viral activity is more likely to happen in networks with a modest number of cliques. Our model provides a quantitative way to determine which phenomena are most likely to go viral, if any, in a social network. We hope that our work motivates further research into the relationship between the topology and dynamics of social networks.
Keywords: social media, computational social science, modeling, going viral

1 Introduction

This paper aims to model the relationship between the structure of a social network the patterns of activity it can sustain. We focus how the structure of a network can promote the rapid spread of interest in an idea throughout the network. As the early appearance of the word *viral* suggests, we are most interested in the spread of information through online social networking sites, for example the rapid rise in popularity of the music video Gangam Style by Korean musician PSY. Our main contribution is our derivation of an analytic relationship between network structure and the chance that a pattern of network activity will go viral.

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Previous work focused on which types of activity are most likely to spread, for example which videos are most likely to go viral on YouTube. Comparatively little work has been done on how the same pattern of activity would fare in networks with different structures. Investigating this could provide insight into why an idea finds traction only in certain groups. In this paper we use techniques from linear algebra to relate the structure of social networks to the dynamics they can support. Similar approaches have provided insight into the dynamics of large neural networks (Rajan & Abbott 2006) and signal transduction networks (Alon 2007).

In the following section we present a general mathematical model of a group of people who interact with each other and external influences.

2 Methods

In this section, we describe our mathematical model of the dynamics of interest in an idea in a network. (We shorten the phrase *interest in an idea in a network* to *interest* for the sake of brevity.) We do not specify the details of the idea because we assume that members of the network differ only in their degree of interest in the same idea. The model considers a defined group with static membership. By *static membership* we mean that no new members join the group and no members leave the group. Members interact with each other and outside influences. Each person in the network weights outside influences differently. We only specify pairwise interactions in the network. We assume that the strength of influence of one person on another does not change for the time scale of the phenomena we are studying. As mentioned above, we assume that all people in the network perceive the idea in the same fashion such that people differ from another only by the level of interest in the idea.

In the **Methods** section we present the equations and detail the environment we used for simulation. In the **Results** section we analyze these equations and present the results of simulating them.

External virality vs internal virality. For external virality the question is what the maximal number of people who can be sustainably interested in an idea? For internal virality, the question is what eigenvalue has the largest real component.

Presentation of equations Equation (1) expresses the dynamics of interest for all members of a network at some time t, $\mathbf{a}(t)$. Table 1 describes the variables in further detail.

$$\frac{d}{dt}\mathbf{a}(t) = -\mathbf{a}(t) + \mathbf{M}\mathbf{a}(t) + \mathbf{h} \tag{1}$$

Variable	Description
$\mathbf{a}(t)$	Vector of level of interest in THE IDEA of all members in the network at time t
$\mathbf{M}$	Matrix describing the strength of influence between pairs of members of the network
$\mathbf{h}$	Vector describing the feed forward input to the network, for example the input needed to make the network go viral
t	Time, we assume all members continuously communicate with their first neighbors
$\mathbf{I}$	Identity matrix

Table 1: Description of variabltes in Equation (1)

Equation (2) rewrites Equation (1) using the nondimensionalization that the **Appendix** describes. We now discuss each of the variables in Equation (2).

$$\frac{d}{d\hat{t}}\mathbf{a}(\hat{t}) = -\mathbf{a}(\hat{t}) + \mathbf{M}\mathbf{a}(\hat{t}) + \phi\mathbf{h}(\hat{t}) \tag{2}$$

Equation (2) is a system of linear ordinary differential equations. We choose this formalism to represent interactions between members and between members and the environment without assuming anything about the network or composition of its members. This reflects the goal of this paper, which is to better understand the general relationship between network structure and dynamics before analyzing specific cases.

We can interpret the terms of this equation in a sociological context. If there is any relationship between the interests of people in the same idea, we may approximate that relationship with a power series, the first term of which is $\mathbf{Ma}$ (see **Appendix**). This formulation allows us to explore the effect of network structure on network dynamics in a general way.

Internal interest The variable $a_i(t)$ denotes the increase in interest above baseline for the ith member of the group. By *interest* we mean the feeling or emotion that causes attention and engagement with an idea. We choose to model the spread of the predisposition to engagement for the sake of parsimony. The means of engagement will differ among networks. Including them in such an early model would obscure more than clarify.

Internal influence The matrix $\mathbf{M}$ denotes the influence that one member of the network has on another member such that M_{ij} denotes the degree to which the interest of the jth member of the network in THE IDEA influences the interest of the ith member. We assume that (i) all important interactions within in the network can be expressed as combinations of pairwise (dyadic) interactions and (ii) the strength of influence that one member has over another is approximately constant over the span of time we study.

External interest The variable $h_i(t)$ denotes the increase in interest from ah external source. We distinguish between internal and external interest because we conceptualize member of the network as individuals but impose no such restriction on external sources. Sources of influence external to the network could be individuals or groups of individuals.

Time An span of time $\Delta\hat{t}$ is long enough for all members of the network to communicate with their first neighbors and incorporate influence from the outside. In reality, people communicate at various rates. We ignore those faster dynamics, because we assume that the spread of interest through a network happens much more slowly than an individual changing his or her level of interest. This assumption simplifies formalism at the expense of temporal resolution.

Assumptions We assume that (i) all important interactions between the network and its surroundings can expressed in terms of pairwise interactions and (ii) the strength of influence that the surroundings have on the network is approximately constant over the time we study. It possible, mathematically, to combine the terms for internal and external influence into one matrix. We choose not to because doing so would complicate the analysis and obscure the correspondance with sociological constructs with no obvious analytic or computational benefit.

Expressing activity in terms of structure

A productive initial step in relating the dynamics of a network to its structure is to mathematically express the former in terms of the latter. To do this we use techniques from linear algebra. For further background, the reader is referred to Strang (2003).

To express the level of interest of the network, $\mathbf{a}$, in terms of the connection matrix, $\mathbf{M}$, we represent the former as a weighted sum of the generalized eigenvectors,$\{\mathbf{e}_\mu\}$, of the latter (Equation (3)). The weights can vary over time. We choose generalized eigenvectors because we do not assume that agents interact reciprocally. Making that assumption would simplify the math at the expense of realism.

$$\mathbf{a}\,(t) = \sum_{\mu} c_\mu\,(t)\,\mathbf{e}_\mu \tag{3}$$

Simulations All simulations were performed on 1.86 GHz Macintosh with 4 GB RAM, running OS X version 10.6.8. Random asymmetric matrices of size $100,000 \times 100,000$ were generated using numeric routines in SciPy (Jones *et al.* 2001) according to Sommers (1988). Viral propensity was calculated as the fraction of eigenvalues with no imaginary component and real component greater than 1. All code is available at `https://github.com/mac389/jass`.

3 Results

This section analyzes the equations that **Methods** introduced, and presents simulations of those equations with various network structures to validate out theory.

Analysis of model

Interest spreads virally when some eigenvectors of the connection matrix have eigenvalues with positive real parts.

Importance of sign of eigenvalues Equation (4) shows the general solution of a linear ordinary differential equation with a chain of length r of generalized eigenvectors with an eigenvalue λ.

$$\mathbf{a}\,(t) = e^{\lambda t}\left(\sum_{i-1}^{j} \frac{t^{j-i}}{(j-i)!} v_i\right) \tag{4}$$

In Equation (4), v_i refers to the ith generalized eigenvector in the chain. Equation (4) demonstrates that the amplitude of the dynamics of interest depend on the exponential of the eigenvalue.

To demonstrate the importance of the real and imaginary components of the eigenvalues, Equation (5) rewrites the first term on the right-hand side of Equation (4) using Euler's identity.

$$\begin{aligned} e^{\lambda t} &= e^{(\alpha+\beta i)t} \\ &= e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right) \end{aligned} \tag{5}$$

The first line of Equation (5) expresses the complex number λ as the sum of a real component, α, and an imaginary component, β. The second line uses Euler's identity to express the exponential of an imaginary number in terms of trigonometric functions. The second line describes an oscillating wave with amplitude $e^{\alpha t}$. If α is negative, then as time progresses $e^{\alpha t}$ becomes progressively smaller and the interest of the network approaches 0. If α is 0, then the interest oscillates in a fixed range. If α is greater than 0, then the interest exponentially increases as time progresses. Figure 1 demonstrates this.

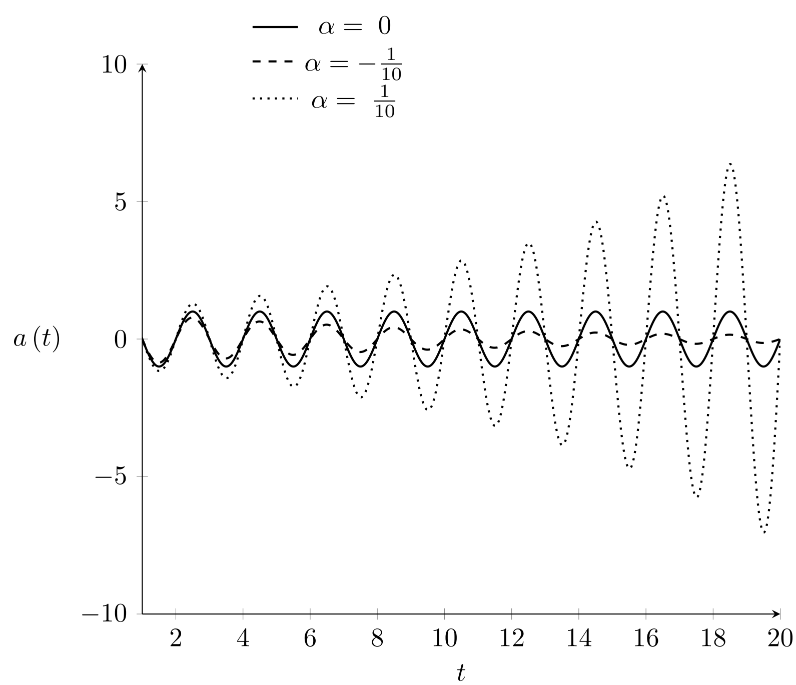

Figure 1: **Effect of real part of eignvalue on amplitude of solution.** Plot with three traces of the bottom line of Equation (5) for different values of α, as indicated by legend. For all traces, $\beta = 1$.

Figure 1 demonstrates that as time progresses, eigenvectors whose corresponding real parts are positive will come to dominate the sum on the right hand side of Equation (3). A network has the capability to support viral activity only if its connection matrix has at least one eigenvector whose corresponding eigenvalues has a positive real part.

Warping of viral activity Having identified the structural properties a network must have in our model to support the virality we now identify which patterns can go viral within structures that possess those properties. As in the previous section, we begin by translating our question from sociological to mathematical terms. In our model a pattern goes viral if it has a nonzero projection onto those eigenvectors of the connection matrix whose eigenvalues have positive real parts. Interestingly, what goes viral is not the network activity, but the projection of the network activity onto those eigenvectors. This suggests that as interest spreads in the idea, the interest in unavoidably filtered by the network, except in the unusual case where network activity is completely parallel to one eigenvector with a positive real eigenvalue.

Examples with dyads Before discussing larger networks with simulations, we analyze the interactions between a network of two agents.

No interaction If the members of the network do not interact, then the off-diagonal elements of the connection matrix are 0 (Equation (6)). The diagonal elements are zero because this iteration of the model assumes that members have no internal dialogue that affects their level of interest.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

Figure shows the evolution of the level of interest of one member of the dyad.

Interacting members In this case, the off-diagonal elements are nonzero. A reciprocal interaction corresponds to the special case where the off-diagonal elements are nonzero and equal to each other. We consider a connection matrix of the general form in Equation (7)

$$\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} \tag{7}$$

Reciprocal interactions correspond to when $a = b$. Table 2 lists the eigenvalues and eigenvectors for Equation (7)

Connections within groups The diagonal terms of the connection matrix correspond to the degree to which an agent influences that agent's own behavior. For a system of n agents, Equation (8) quantifies the average strength of connections *within* a clique.

$$i_{\text{within}} = \frac{1}{n} \frac{\text{tr }(\mathbf{M})}{\det(\mathbf{M})} \tag{8}$$

Eigenvalue	Eigenvector
$\sqrt{ab}$	$\begin{pmatrix} \sqrt{\frac{a}{b}} \\ 1 \end{pmatrix}$
$-\sqrt{ab}$	$\begin{pmatrix} -\sqrt{\frac{a}{b}} \\ 1 \end{pmatrix}$

Table 2: My caption

Connections between groups All terms but the diagonal terms in the connection matrix correspond to the degree to which one agent influences or is influenced by other agents. Accordingly, Equation (9) quantifies the average influence *between* cliques.

$$\begin{aligned} i_{\text{between}} &= \frac{1}{n}\left(1 - \frac{\text{tr }(\mathbf{M})}{\det\ (\mathbf{M})}\right) \\ &= 1 - i_{\text{within}}` \end{aligned} \tag{9}$$

3.1 Going viral

We consider that a pattern of activity has gone viral when the activity of the network starts to resemble that pattern of activity. Mathematically, our model selectively amplifies those dimensions of the input that are parallel to the eigenvectors of the connection matrix that have the largest eigenvalues. This observation is made clearer by expressing the steady-state activity of the network, $\mathbf{a}_\infty$ in terms of the eigenvectors and eigenvalues of the connection matrix, as Equation (10).

$$\mathbf{a}_\infty = \sum \frac{\mathbf{e_i} \cdot \mathbf{h}}{1 - \lambda_i} \mathbf{e_i}. \tag{10}$$

The sum in Equation (10) is over all eigenvectors and eigenvalues of the connection matrix. The larger the ith eigenvalue, the larger the weight of the ith eigenvector in determining the steady-state activity of the network.

The eigenvalues of the connection matrix can be expressed as a function of the trace and determinant of the matrix for a network of two agents, as Equation (11) does. Equation (11) relates the propensity of a dimension to dominate network activity to the strength of interactions between and within agents or groups of agents.

$$\lambda_\pm = \frac{1}{2}\left(\text{tr }(\mathbf{M}) \pm \sqrt{(\text{tr }(\mathbf{M}))^2 - 4\det\mathbf{M}}\right) \tag{11}$$

Equation (12) expresses Equation (11) in terms of Equations (8) and (9). In Equation (12), the term i_w refers to i_{within} and i_b refers to i_{between}.

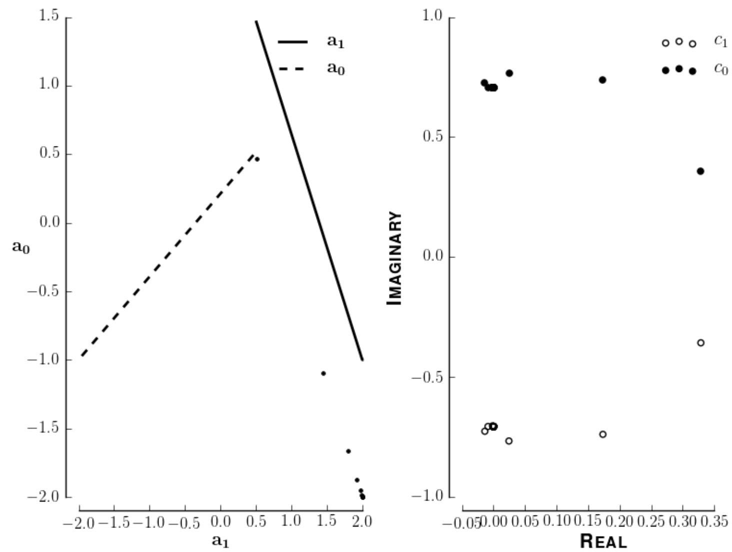

Figure 2: **Dynamics of dyad with one positive and one negative influence. Left:** Phase-plane trajectory of dyad. Lines represent nullclines.**Right:** Argand diagram of strength of projections of activity onto the 1^{st} and 2^{nd} eigenvectors.

$$\begin{aligned}
\frac{\partial \lambda_\pm}{\partial i_w} &= \frac{1}{2}\left(n \det \mathbf{M} \pm \frac{2 i_w}{\sqrt{\left(n \det \mathbf{M}\; i_w\right)^2 - 4 \det \mathbf{M}}} \right) \\
\frac{\partial \lambda_\pm}{\partial i_b} &= \frac{1}{2}\left(-n \det \mathbf{M} \pm \frac{2 - 2 i_b}{\sqrt{\left(n \det \mathbf{M}\;\; (1 - i_b)\right)^2 - 4 \det \mathbf{M}}} \right)
\end{aligned} \tag{12}$$

Equation (12) demonstrates that interactions between and within groups have opposite effects for a network of two homogeneous groups that interact. Interaction between groups dampens one eigenvalue faster than interaction within groups increases it. Equation (12) assumes that the connection matrix is invertible; otherwise, the rightmost terms in each line are undefined. In a sociological context, requiring an invertible matrix means the model considers the total strengths of interactions between groups and within groups to be different.

3.2 Reverberant patterns

We use the term *reverberant pattern* to refer to a pattern of network activity that neither vanishes nor goes viral. Mathematically, this corresponds to an

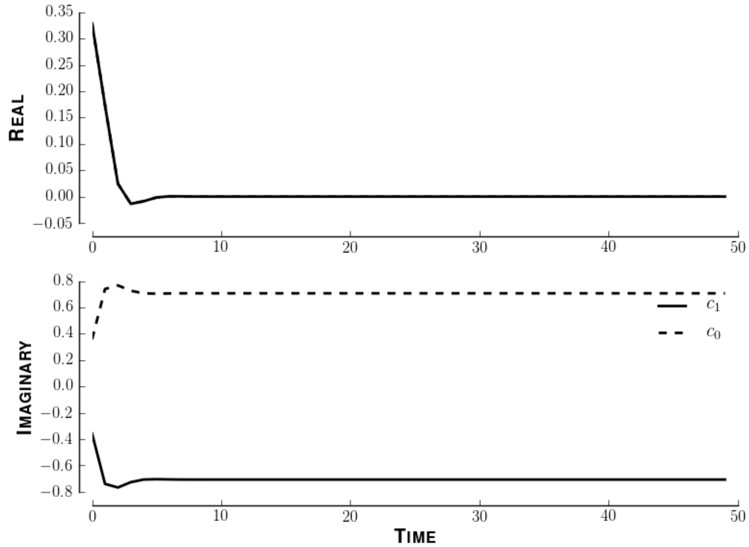

Figure 3: **Time course of activtiy as projected onto eigenvectors. Top:** Real component **Bottom:** Imaginary component. Legend applies to both panels.

eigenvector with apurely imaginary eigenvalue. In this section, we examine the more realistic and general case of an eigenvector with a complex eigenvalue. For a 2×2 connection matrix, either both roots are complex or neither are, according to the complex conjugate root theorem. The eigenvalues of Equation (11) are complex when $4 \det \mathbf{M} > (\mathrm{tr}\ (\mathbf{M}))^2$. In a sociological context, this means that patterns reverberate as long as a minimal amount of interaction between groups, i^*_w, persists. Equation (13) describes that cutoff.

$$\frac{4}{n} > i^*_w \ \mathrm{tr}\ (\mathbf{M}) \tag{13}$$

We analyze Equation (13) for two conditions, where $\mathrm{tr}\ (\mathbf{M}) > 0$ and where $\mathrm{tr}\ (\mathbf{M}) < 0$. These conditions correspond to when interactions within groups promote or discourage further activity within groups, respectively. Equation (14) describes how these conditions set a lower and upper bound on the strength of interactions between groups.

$$\begin{aligned} \frac{4}{n}\left(\frac{1}{\mathrm{tr}\ (\mathbf{M})}\right) - 1 &< i^*_{\text{between}} \qquad \text{if } \mathrm{tr}\ (\mathbf{M}) > 0 \\ \frac{4}{n}\left(\frac{1}{\mathrm{tr}\ (\mathbf{M})}\right) - 1 &> i^*_{\text{between}} \qquad \text{if } \mathrm{tr}\ (\mathbf{M}) < 0 \end{aligned} \tag{14}$$

Computational Results

The preceding section presented theoretical results from the analysis of the interactions of two homogeneous agents or populations of agents. Equations (8) and (9) generalize readily to larger groups. The Abel-Ruffini theorem states that there is no general closed form solution for polynomial equations of degree five or higher, which is to say that Equation (11) cannot generalize to realistically sized populations. To better understand the dynamics of a network with more than two type of agents, this section discusses the results of computational simulations of the dynamics of such networks. We focus on which patterns of input go viral and whether the tendency to go viral depends on the heterogeneity of the social network.

We assume that the connections between agents, that is the entries of $\mathbf{M}$, are drawn from a Gaussian distribution with zero mean, unit autocorrelation, and cross-correlation τ. We further assume that $-1 \leq \tau \leq 1$. If the connection matrix $\mathbf{M}$ meets these assumptions, then Equation (15) gives the density, ρ, of eigenvalues in the imaginary plane (Sommers 1988).

$$\rho\left(\lambda\right) = \frac{2}{\pi\left(1+\tau\right)^2}\left\{\left(1+\tau\right)^2 - \lambda^2\right\} \tag{15}$$

Equation (15) describes a semi-ellipse with axes $1 \pm \tau$ when considered over $[-a, a]$. We use τ as a proxy for network heterogeneity, assuming networks with $|\tau|$ closer to one to be more homogeneous than those with $|\tau|$ closer to zero.

The previous section introduced our mathematical criterion for a pattern of activity to go viral—the eigenvalue of the eigenvector most parallel to the pattern of activity must be greater than one. We define the *viral propensity* ($\aleph$) as the fraction of eigenvalues greater than one (Equation (16)).

$$\begin{aligned} \aleph &= \frac{\int_1^2 \mathrm{d}x\ \rho\left(\lambda\right)}{\int_{-2}^2 \mathrm{d}x\ \rho\left(\lambda\right)} \\ &= \frac{2\sqrt{4+\left(1+\tau\right)^2} - \sqrt{1+\left(1+\tau\right)^2} + \left(1+\tau\right)^2 \ln\frac{2+\sqrt{4+(1+\tau)^2}}{-2+\sqrt{4+(1+\tau)^2}}}{4\sqrt{4+\left(1+\tau\right)^2} + \left(1+\tau\right)^2 \ln\frac{2+\sqrt{4+(1+\tau)^2}}{-2+\sqrt{4+(1+\tau)^2}}} \end{aligned} \tag{16}$$

If we assume that τ is very small, then we may use the Taylor expansion of Equation (16) to visualize the viral propensity of the network with small amounts of heterogeneity (Equation (17)) .

$$\aleph_{\tau\ \text{small}} \approx \frac{5}{2\left(\left(1+\tau\right)^2+4\right)} - \frac{1}{4} \tag{17}$$

Figure 4 show the distribution of eigenvalues on the imaginary plane for $\tau = 0.7$. In contrast to a symmetric matrix, the eigenvalues are not distributed uniformly on the unit circle. The density of eigenvalues is highest near zero, falling off rapidly as one progresses to the circumference of the unit circle. Figure 5 shows the calculation of viral propensity for a network of 100,000 agents as a function of network heterogeneity. The viral propensity is highest in symmetric networks, undergoing a sharp increase when $|\tau| > 0.75$. This corresponds to an increase in the number of eigenvectors with purely real eigenvalues. Connection matrices dominated by eigenvectors with real eigenvalues correspond to social networks with high numbers of intermediaries, agents that aid the propagation of information.

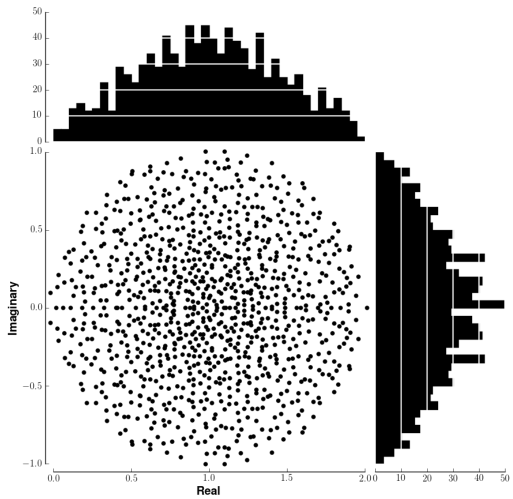

Figure 4: **Distribution of eigenvalues of asymmetric random matrix.** Scatter plot of eigenvalues of a square asymmetric matrix. Values chosen from a multivariate Gaussian distribution as described in Methods. The histogram above the scatter plot shows the distribution of points along the x (real) axis. The histogram below the scatter plot shows the distribution of points along the y (imaginary) axis.

4 Conclusions

This paper presented a linear model of the dynamics of a social network to better understand the types of interactions and network topologies that allow viral activity. We demonstrated that (i) interaction between groups retards viral activity if it exceeds a critical range, (ii) interaction within groups promotes viral activity, and (iii) viral activity is more likely to happen in networks with modest heterogeneity. Selecting the most important nodes to maximize the spread of influence through a social network is an NP-hard problem (Kempe *et al.* 2003). Our work suggests a way to restrict the search space of that optimization

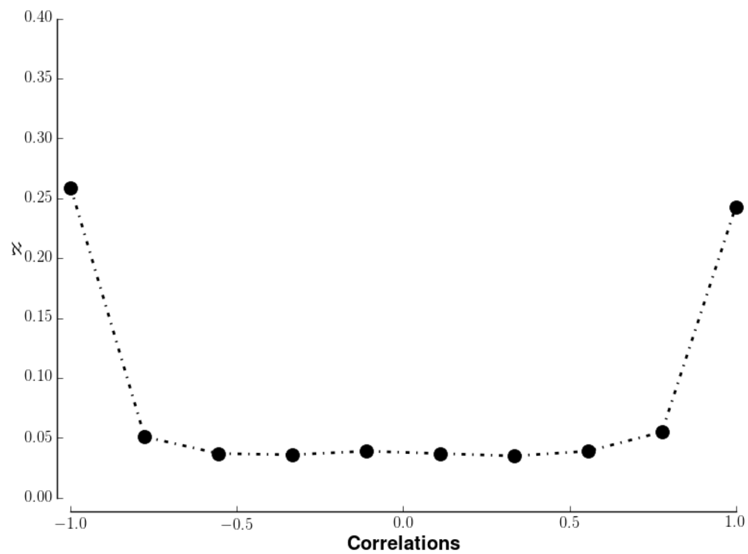

Figure 5: **Viral propensity as a function of network heterogeneity.** Each dot represents the viral propensity, ℵ, as determined by computational simulations. The degree of correlation, τ, is plotted on the x-axis. Higher correlations correspond to networks with lower heterogeneity. See Methods for further detail on the simulations.

problem by considering the nodes (agents) whose patterns of connections lie most parallel to the eigenvectors with the largest real eigenvalues. Stocker *et al.* (2001) found that social networks required a critical level of connectivity to achieve consensus. We found that beyond a critical level of heterogeneity, our model social network was unlikely to allow anything to go viral. Future work could explore how competition between ideas affects these dynamics.

4.1 Limitations

Our work is one of the first to provide a computational model linking viral activity to the topology of a social network. Our model is preliminary and can be enhanced in many ways. It ignores nonlinear interactions between agents. It only considers pairwise interactions. Our model, furthermore, assumes that the influence one agent has on another in the social network is constant over time. Our model, additionally, assumed that every agent interacted with every other agent; real networks may have sparser (Gonçalves *et al.* 2011) or more structured (Leskovec *et al.* 2008) patterns of interactions. These simplifications limit the quantitative accuracy of the model.

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5 Appendix

Generalized eigenvectors Associated with any square matrix, $\mathbf{A} \in \mathbb{C}$ is a characteristic polynomial with p roots (Equation (18)).

$$P(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| \tag{18}$$

Each root occurs k times, which allows us to re-write Equation (18) as Equation (19)

$$P(\lambda) = \prod_{i}^{p} (\lambda - \lambda_i)^{k_i} \tag{19}$$

For standard eigenvalues, k is unity.

For a matrix $\mathbf{A}$, the vector $\mathbf{u}$ is a generalized eigenvector, corresponding to the generalized eigenvalue λ, of rank r if that vector satisfies Equation (20) .

$$\begin{aligned} (\mathbf{A} - \lambda \mathbf{I})^{r} \mathbf{u} &= 0 \\ (\mathbf{A} - \lambda \mathbf{I})^{r-1} &\neq 0 \end{aligned} \tag{20}$$

If we have one generalized eigenvector, $\mathbf{u}_r$ of rank r, then Equation (21) specifies a recurrence relation to generate a chain of generalized eigenvectors of length r.

$$\begin{aligned} \mathbf{u}_{r-1} &= (\mathbf{A} - \lambda \mathbf{I}) \mathbf{u}_r \\ \mathbf{u}_{r-2} &= (\mathbf{A} - \lambda \mathbf{I}) \mathbf{u}_{r-1} \\ &\vdots \\ 0 &= (\mathbf{A} - \lambda \mathbf{I}) \mathbf{u}_1 \end{aligned} \tag{21}$$

The vectors in a chain of generalized eigenvectors are linearly independent, which allows us to use them as we would standard eigenvectors.

Projection of network activity onto eigenvectors The eigenvectors of a $n \times n$ matrix, $\{\mathbf{e}\}$ form a basis for any n-dimensional vector. This allows us to express the dynamics of interest in terms of a weighted sum of the eigenvectors of the connection matrix, where the weights vary over time (Equation (22))

$$\mathbf{a}(t) = \sum_{\mu=1}^{N_v} c_\mu(t)\, \mathbf{e}_\mu \tag{22}$$

In Equation (22), the sum runs over all eigenvalues. Having found the eigenvectors, as described in the previous section, Equation (23) provides an expression for the dynamics of the weights in Equation (22)

$$\frac{dc_v}{dt} = -\left(1-\lambda_v\right)c_v\left(t\right) + \mathbf{e}_v \cdot \left(\mathbf{Wu}\left(t\right)\right) \tag{23}$$

Equation (23) has a closed form solution only if the right-most term is independent in time.

Nondimensionalization Nondimensionalization refers to the removal of units from an equation with units to leave dimensionless quantities. It can reveal the characteristic structure of a system of equations and often simplifies them.

We begin by re-expressing every variable as a multiple of some intrinsic quantity, in the form $x = kx_c$ (Equation (24)).

$$\begin{aligned}
\frac{d}{dt}\mathbf{a}\left(t\right) &= -\mathbf{a}\left(t\right) + \mathbf{Ma}\left(t\right) + \mathbf{Wu}\left(t\right) \\
d\mathbf{a}\left(t\right) &= dt\left(\mathbf{a}\left(t\right) + \mathbf{Ma}\left(t\right) + \mathbf{Wu}\left(t\right)\right) \\
\alpha d\mathbf{a}_c\left(t\right) &= \tau dt_c\left(-\alpha\mathbf{a}\left(t\right) + \alpha\mathbf{Ma}\left(t\right) + \phi\mathbf{Wu}\left(t\right)\right) \\
d\mathbf{a}_c\left(t\right) &= \tau dt\left(-\mathbf{a}_c\left(t\right) + \mathbf{Ma}_c\left(t\right) + \frac{\phi}{\alpha}\mathbf{Wu}_c\left(t\right)\right)
\end{aligned} \tag{24}$$

Equation rewrites the last line from Equation (24) dropping the subscripts and substituting $\hat{t}$ for τt.

$$\frac{d}{d\hat{t}}\mathbf{a}\left(t\right) = -\mathbf{a}\left(t\right) + \mathbf{Ma}\left(t\right) + \frac{\phi}{\alpha}\mathbf{Wu}\left(t\right) \tag{25}$$

We now obtain the form of Equation (24) used in the text by replacing $\hat{t}$ with t and assuming that $\phi = \alpha$.

Form of internal and external influences In this section, we describe the rationale for approximating internal influence with the term $\mathbf{Ma}$. The argument proceeds analogously for $\mathbf{Wu}$.

We assume a function, f, exists that describes the influence of all other people in the network on one person. and is a function of the levels of interest of all those other members in the network (Equation (26)).

$$f_{a_i} = f\left(a_0, a_1, \ldots, a_n\right) = f\left(\mathbf{a}\right) \tag{26}$$

Equation (27) expands Equation (26) with a power series.

$$f_{a_i} = \boldsymbol{\alpha a_0}\mathbf{a} + \boldsymbol{\alpha_1}\mathbf{a}^2 + \boldsymbol{\alpha_2}\mathbf{a}^3 + \ldots \tag{27}$$

If we assume that $\mathbf{a}$ is small—by construction it expresses a deviation from baseline—, then the higher-order terms will be vanishingly small and may be neglected. This assumption simplified Equation (27) to Equation (28)

$$
\begin{aligned}
f_{a_i} &= \boldsymbol{a_0}\mathbf{a} \\
&= \begin{pmatrix} a_{i0} \\ a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_j \end{pmatrix}
\end{aligned} \tag{28}
$$

The term a_{i0} denotes the degree of influence (weighting coefficient) between the 0th and ith members of the network. The entire vector denotes the strength of influence of members of the network on the ith member. It corresponds to the ith column of $\mathbf{M}$.