Algorithms
Chapter 8
Sorting in Linear Time

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Outline

- **Lower bounds for sorting**
- Counting sort
- Radix sort
- Bucket sort

Overview

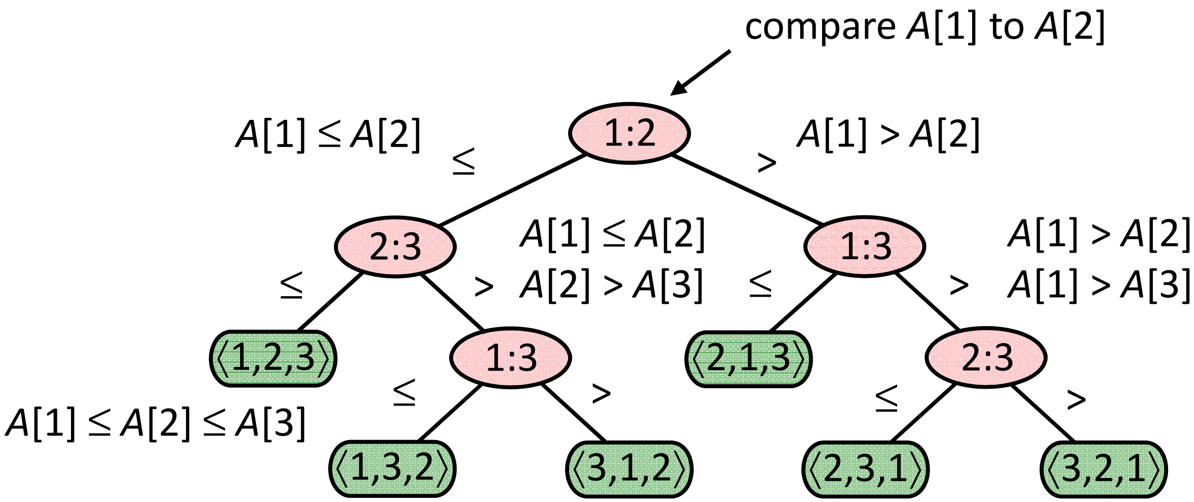
- Sort n numbers in $O(n \lg n)$ time
  - Merge sort and heapsort achieve this upper bound in the worst case.
  - Quicksort achieves it on average.
  - For each of these algorithms, we can produce a sequence of n input numbers that causes the algorithm to run in $\Theta(n \lg n)$ time.
- **Comparison sorting**
  - The only operation that may be used to gain order information about a sequence is comparison of pairs of elements.
  - All sorts seen so far are comparison sorts: insertion sort, selection sort, merge sort, quicksort, heapsort.

Lower bounds for sorting

- **Lower bounds**
  - $\Omega(n)$ to examine all the input.
  - All sorts seen so far are $\Omega(n \lg n)$.
  - We'll show that $\Omega(n \lg n)$ is a lower bound for comparison sorts.
- **Decision tree**
  - Abstraction of any comparison sort.
  - A full binary tree.
  - Represents comparisons made by
    - a specific sorting algorithm
    - on inputs of a given size.
  - Control, data movement, and all other aspects of the algorithm are ignored.

Decision tree

- For insertion sort on 3 elements:



- How many leaves on the decision tree?
  - There are $\geq n!$ leaves, because every permutation appears at least once.

Properties of decision trees$_{1/3}$

- **Lemma 1** Any binary tree of height h has $\leq 2^h$ leaves.
  - *Proof*: By induction on h.
  - **Basis:**
    - $h = 0$. Tree is just one node, which is a leaf. $2^h = 1$.
  - **Inductive step:**
    - Assume true for height = $h - 1$.
    - Extend tree of height $h - 1$ by making as many new leaves as possible.
    - Each leaf becomes parent to two new leaves.
    - # of leaves for height h = 2 · (# of leaves for height $h - 1$)

      = $2 \cdot 2^{h-1}$ (ind. hypothesis)

      = 2^h .

Properties of decision trees$_{2/3}$

- **Theorem 1** Any decision tree that sorts n elements has height $\Omega(n \lg n)$.

  Proof:

  - $\ell \geq n!$, where ℓ = # of leaves.
  - By lemma 1, $n! \leq \ell \leq 2^h$ or $2^h \geq n!$.
  - Take logs: $h \geq \lg(n!)$.
  - Use Stirling's approximation: $n! > (n/e)^n$

  $$\begin{aligned} h &> \lg(n/e)^n \\ &= n \lg(n/e) \\ &= n \lg n - n \lg e \\ &= \Omega(n \lg n). \end{aligned}$$

Properties of decision trees$_{3/3}$

- **Corollary 1** Heapsort and merge sort are asymptotically optimal comparison sorts.

  Proof:

  - The $O(n \lg n)$ upper bounds on the running times for heapsort and merge sort match the $\Omega(n \lg n)$ worst-case lower bound from Theorem 1.

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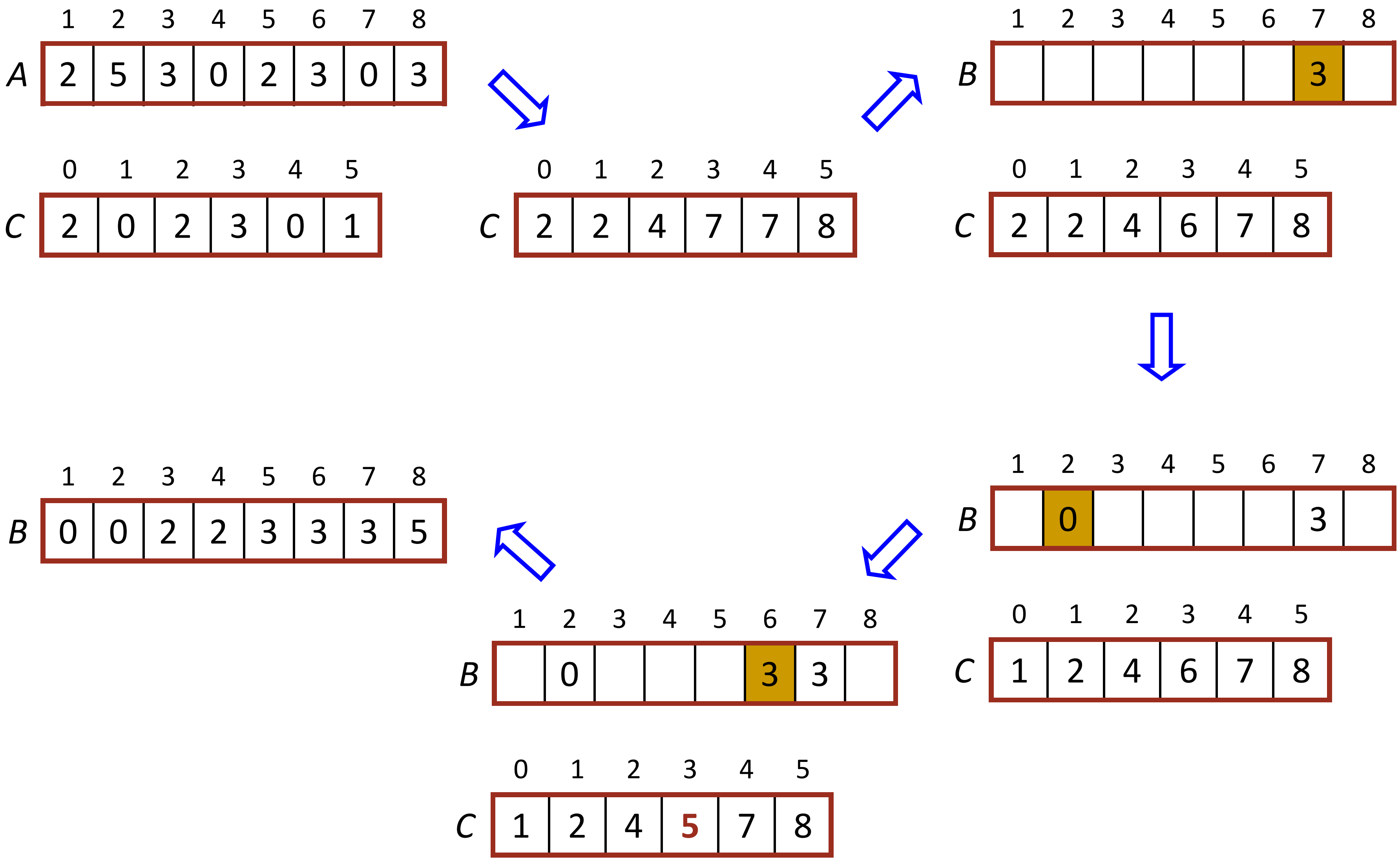
Counting sort

- Non-comparison sorts.
- Depends on a **key assumption:** numbers to be sorted are integers in $\{0, 1, \ldots, k\}$.
- **Input:** $A[1 \,..\, n]$, where $A[j] \in \{0, 1, \ldots, k\}$ for $j = 1, 2, \ldots, n$.
  Array A and values n and k are given as parameters.
- **Output:** $B[1 \,..\, n]$, sorted. B is assumed to be already allocated and is given as a parameter.
- **Auxiliary storage:** $C[0 \,..\, k]$.
- **Worst-case running time:** $\Theta(n+k)$.

The COUNTING-SORT procedure

```
COUNTING-SORT(A, B, k)
    for i ← 0 to k                              } Θ(k)
        do C[i] ← 0
    for j ← 1 to length[A]                      } Θ(n)
        do C[A[j]] ← C[A[j]] + 1
    /* C[i] now contains the number of elements equal to i. */
    for i ← 1 to k                              } Θ(k)
        do C[i] ← C[i] + C[i – 1]
    /* C[i] now contains the number of elements less than or equal to i. */
    for j ← length[A] downto 1                  } Θ(n)
        do B[C[A[j]]] ← A[j]
           C[A[j]] ← C[A[j]] – 1
```

- **The running time:** $\Theta(n+k)$.


1 2 3 4 5 6 7 8
A 2 5 3 0 2 3 0 3
0 1 2 3 4 5
C 2 0 2 3 0 1
0 1 2 3 4 5
C 2 2 4 7 7 8
1 2 3 4 5 6 7 8
B 3
0 1 2 3 4 5
C 2 2 4 6 7 8
1 2 3 4 5 6 7 8
B 0 3
0 1 2 3 4 5
C 1 2 4 6 7 8
1 2 3 4 5 6 7 8
B 0 3 3
0 1 2 3 4 5
C 1 2 4 5 7 8
1 2 3 4 5 6 7 8
B 0 0 2 2 3 3 3 5

Properties of counting sort

- A sorting algorithm is said to be **stable** if keys with same value appear in same order in output as they did in input.
- **Counting sort is stable** because of how the last loop works.
- Counting sort will be used in radix sort.

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Radix sort

- **Key idea:** Sort **least** significant digits first.

RADIX-SORT(A, d)

```
for i ← 1 to d
    do use a stable sort to sort array A on digit i
```

- An example:

sorted

326	690	704	326
453	751	608	435
608	453	326	453
835	704	835	608
751	835	435	690
435	435	751	704
704	326	453	751
690	608	690	835

Correctness of radix sort

- **Proof:** By induction on number of passes (i in pseudocode).
- **Basis:**
  - $i = 1$. There is only one digit, so sorting on that digit sorts the array.
- **Inductive step:**
  - Assume digits 1, 2,…, $i - 1$ are sorted.
  - Show that a stable sort on digit i leaves digits 1, 2,…, i sorted:
    - If 2 digits in position i are different, ordering by position i is correct, and positions 1,…, $i - 1$ are irrelevant.
    - If 2 digits in position i are equal, numbers are already in the right order (by inductive hypothesis). The stable sort on digit i leaves them in the right order.

Time complexity of radix sort

- Assume that we use counting sort as the intermediate sort.
- When each digit is in the range 0 to $k-1$, each pass over n d-digit number takes time $\Theta(n + k)$.
- There are d passes, so the total time for radix sort is $\Theta(d(n + k))$.
- If $k = O(n)$, time = $\Theta(dn)$.

- **Lemma 2:** Given n d-digit numbers in which each digit can take on up to k possible values, RADIXSORT correctly sorts these numbers in $\Theta(d(n + k))$ time.

Break each key into digits$_{1/2}$

- **Lemma 3:** Given n b-bit numbers and any positive integer $r \leq b$, RADIX-SORT correctly sorts these numbers in $\Theta((b/r)(n + 2^r))$ time.
- **Proof**
  - We view each key as having $d = \lceil b/r \rceil$ digits of r bits each.
  - Each digit is an integer in the range 0 to $2^r - 1$, so that we can use counting sort with $k = 2^r - 1$.
  - Each pass of counting sort takes time $\Theta(n+k) = \Theta(n+2^r)$.
  - A total running time of $\Theta(d(n + 2^r)) = \Theta((b/r)(n + 2^r))$.
- **For example:**
  - 32-bit words, 8-bit digits.
  - $b = 32$, $r = 8$, $d = 32/8 = 4$, $k = 2^8 - 1 = 255$.

Break each key into digits$_{2/2}$

- Recall that the running time is $\Theta((b/r)(n + 2^r))$.
- How to choose r?
  - Balance b/r and $n + 2^r$.
- If $b < \lfloor \lg n \rfloor$, then choosing $r = b$ yields a running time of
  $(b/b)(n + 2^r) = \Theta(n)$.
- If $b \geq \lfloor \lg n \rfloor$, then choosing $r \approx \lg n$ gives us $\theta(\frac{b}{\lg n}(n + n)) = \theta(\frac{bn}{\lg n})$.
  - If $r > \lg n$, then 2^r term in the numerator increases faster than the r term in the denominator.
  - If $r < \lg n$, then b/r term increases, and $n + 2^r$ term remains at $\Theta(n)$.

The main reason

- How does radix sort violate the ground rules for a comparison sort?
  - Using counting sort allows us to gain information about keys by means other than directly comparing 2 keys.
  - Used keys as array indices.

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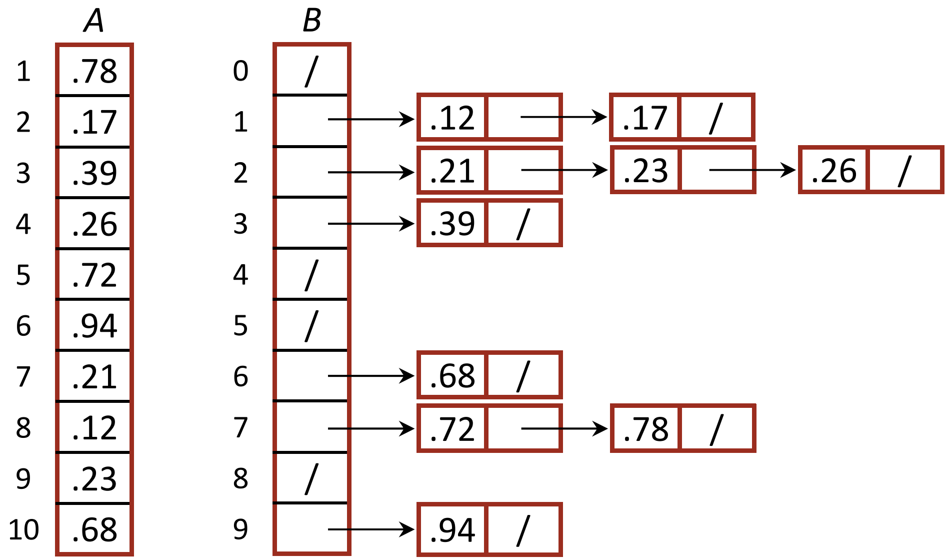
Bucket sort

- Assumes the input is generated by a random process that distributes elements uniformly over [0, 1).
- **Key idea:**
  - Divide [0, 1) into *n* equal-sized **buckets**.
  - Distribute the *n* input values into the buckets.
  - Sort each bucket.
  - Then go through buckets in order, listing elements in each one.



The BUCKET SORT procedure

- **Input:** $A[1..\, n]$, where $0 \le A[i] < 1$ for all i.
- **Auxiliary array:** $B[0..n-1]$ of linked lists, each list initially empty.

BUCKET-SORT(A, n)

for $i \leftarrow 1$ **to** n

 do insert $A[i]$ into list $B[\lfloor n \cdot A[i] \rfloor]$

for $i \leftarrow 0$ **to** $n - 1$

 do sort list $B[i\,]$ with insertion sort

concatenate lists $B[0], B[1], \ldots, B[n-1]$ together in order

return the concatenated lists

Correctness of bucket sort

- Consider $A[i]$, $A[j]$.
  Assume without loss of generality that $A[i] \le A[j]$.
- Then $\lfloor n \cdot A[i] \rfloor \le \lfloor n \cdot A[j] \rfloor$.
- So $A[i]$ is placed into the same bucket as $A[j]$ or into a bucket with a lower index.
- If same bucket, insertion sort fixes up.
- If earlier bucket, concatenation of lists fixes up.

Time complexity of bucket sort

- Relies on no bucket getting too many values.
- All lines of algorithm except insertion sorting take $\Theta(n)$ altogether.
- Intuitively, if each bucket gets a constant number of elements, it takes $O(1)$ time to sort each bucket ➔ $O(n)$ sort time for all buckets.
- We "expect" each bucket to have few elements, since the average is 1 element per bucket.

Time complexity of bucket sort

- Define a random variable: ***n_i*** = the number of elements placed in bucket $B[i]$.
- Because insertion sort runs in quadratic time, bucket sort time is $T(n) = \theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$.
- Take expectations of both sides:

$$E[T(n)] = E\left[\theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right]$$

$$= \theta(n) + \sum_{i=0}^{n-1} E[O(n_i^2)]$$

linearity of expectation

$$= \theta(n) + \sum_{i=0}^{n-1} O(E[n_i^2])$$

$E[aX] = aE[X]$

Claim that $E[n_i^2] = 2 - 1/n$ for $0 \le i \le n-1$.

Therefore, $E[T(n)] = \theta(n) + \sum_{i=0}^{n-1} O(2 - 1/n)$

$$= \theta(n) + O(n)$$

$$= \theta(n).$$

Proof of claim

- **Claim:** $E[n_i^2] = 2 - 1/n$ for $0 \le i \le n - 1$.
- **Proof**
  - Pr{$A[j]$ falls in bucket i} = p = $1/n$.
  - The probability that $n_i = k$ follows the binomial distribution $b(k; n, p)$.
  - So, $E[n_i] = np = 1$ and variance $\mathrm{Var}[n_i] = np(1 - p) = 1 - 1/n$.
  - For any random variable X, we have

$$
\begin{aligned}
E[n_i^2] &= \mathrm{Var}[n_i] + E^2[n_i] \\
&= 1 - \frac{1}{n} + 1^2 \\
&= 2 - \frac{1}{n}.
\end{aligned}
$$

Notes

- Again, not a comparison sort. Used a function of key values to index into an array.
- This is a **probabilistic analysis**. We used probability to analyze an algorithm whose running time depends on the distribution of inputs.
- Different from a **randomized algorithm**, where we use randomization to impose a distribution.
- With bucket sort, if the input isn't drawn from a uniform distribution on [0, 1), all bets are off (performance-wise, but the algorithm is still correct).