Algorithms
Chapter 4 Recurrences

Instructor: Ching-Chi Lin
林清池 助理教授
chingchi.lin@gmail.com

Department of Computer Science and Engineering
National Taiwan Ocean University

Outline

- **The substitution method**
- The recursion-tree method
- The master method

The purpose of this chapter

- When an algorithm contains a recursive call to itself, its running time can often be described by a recurrence.
- A **recurrence** is an equation or inequality that describes a function in terms of its value on small inputs.
  - For example: the worst-case running time of Merge-Sort

$$T(n) = \begin{cases} \theta(1) & \text{if } n = 1, \\ 2T(n/2) + \theta(n) & \text{if } n > 1. \end{cases}$$

  - The solution is $T(n) = \Theta(n \lg n)$.
- Three methods for solving recurrences
  - the substitution method
  - the recursion-tree method
  - the master method

Technicalities$_{1/2}$

- The running time $T(n)$ is only defined when n is an integer, since the size of the input is always an integer for most algorithms.
  - For example: the running time of Merge-Sort is really

$$T(n) = \begin{cases} \theta(1) & \text{if } n = 1, \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \theta(n) & \text{if } n > 1. \end{cases}$$

- Typically, we ignore the boundary conditions.
- Since the running time of an algorithm on a constant-sized input is a constant, we have $T(n) = \Theta(1)$ for sufficiently small n.
- Thus, we can rewrite the recurrence as

$$T(n) = 2T(n/2) + \Theta(n).$$

Technicalities$_{2/2}$

- When we state and solve recurrences, we often omit floors, ceilings, and boundary conditions.
- We forge ahead without these details and later determine whether or not they matter.
- They usually don't, but it is important to know when they do.

The substitution method$_{1/3}$

- The substitution method entails two steps:
  - Guess the form of the solution
  - Use mathematical induction to find the constants and show that the solution works
- This method is powerful, but it obviously can be applied only in cases when it is easy to guess the form of the answer

The substitution method$_{2/3}$

- For example: determine an upper bound on the recurrence

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \text{ where } T(1)=1$$

  - guess that the solution is $T(n)=O(n\lg n)$
  - prove that there exist positive constants $c>0$ and n_0 such that $T(n)\leq cn\lg n$ for all $n\geq n_0$
  - **Basis step**:
    - when $n=1$, $T(1)\leq c_1 1\lg 1=0$, which is odds with $T(1)=1$
    - since the recurrence does not depend directly on $T(1)$, we can replace $T(1)$ by $T(2)=4$ and $T(3)=5$ as the base cases
    - $T(2)\leq c_1 2\lg 2$ and $T(3)\leq c_1 3\lg 3$ for any choice of $c_1\geq 2$
    - thus, we can choose $c_1=2$ and $n_0=2$

The substitution method$_{3/3}$

- **Induction step**:
  - assume $T(\lfloor n/2 \rfloor) \leq c_2 \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$ for $\lfloor n/2 \rfloor$
  - then, $T(n) \leq 2(c_2 \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n$

$$\begin{aligned} &\leq c_2 n \lg(n/2) + n \\ &= c_2 n \lg n - c_2 n \lg 2 + n \\ &= c_2 n \lg n - c_2 n + n \\ &\leq c_2 n \lg n \end{aligned}$$

  - the last step holds as long as $c_2 \geq 1$

- There exist positive constants $c = \max\{2, 1\}$ and $n_0 = 2$ such that $T(n) \leq cn \lg n$ for all $n \geq n_0$

Making a good guess

- **Experience**: $T(n) = 2T(\lfloor n/2 \rfloor + 17) + n$
  - when n is large, the difference between $T(\lfloor n/2 \rfloor)$ and $T(\lfloor n/2 \rfloor + 17)$ is not that large: both cut n nearly evenly in half.
  - we make the guess that $T(n) = O(n \lg n)$
- **Loose upper and lower bounds**: $T(n) = 2T(\lfloor n/2 \rfloor) + n$
  - prove a lower bound of $T(n) = \Omega(n)$ and an upper bound of $T(n) = O(n^2)$
  - gradually lower the upper bound and raise the lower bound until we converge on the tight solution of $T(n) = \Theta(n \lg n)$
- **Recursion trees**:
  - will be introduced later

Subtleties

- Consider the recurrence:

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1$$

  - **guess** that the solution is $O(n)$, i.e., $T(n) \leq cn$
  - then, $T(n) \leq c\lfloor n/2 \rfloor + (c\lceil n/2 \rceil + 1$
    $= cn + 1$
  - c does not exist
  - **new guess** $T(n) \leq cn - b$
  - then, $T(n) \leq (c\lfloor n/2 \rfloor - b) + (c\lceil n/2 \rceil - b) + 1$
    $= cn - 2b + 1$
    $\leq cn - b$ for $b \geq 1$
  - also, the constant c must be chosen large enough to handle the boundary conditions

Avoiding pitfalls

- It is easy to err in the use of asymptotic notation.
- For example: $T(n) = 2T(\lfloor n/2 \rfloor) + n$
  - guess that the solution is $O(n)$, i.e., $T(n) \leq cn$
  - then, $T(n) \leq 2(c\lfloor n/2 \rfloor) + n$

    $\leq cn + n$

    $= O(n)$ ← **wrong**
  - the error is that we haven't proved the **exact form** of the inductive hypothesis, that is, that $T(n) \leq cn$

Changing variables

- Sometimes, a little algebraic manipulation can make an unknown recurrence similar to one you have seen before.
- For example, consider the recurrence
  - $T(n) = 2T(\lfloor n^{1/2} \rfloor) + \lg n$
- Renaming $m = \lg n$ yields
  - $T(2^m) = 2T(2^{m/2}) + m$
- Renaming $S(m) = T(2^m)$ produces
  - $S(m) = 2S(m/2) + m$
    $S(m) = O(m \lg m)$
- Changing back $S(m)$ to $T(m)$, we obtain
  - $T(n) = T(2^m) = S(m) = O(m \lg m) = O(\lg n \lg \lg n)$

Outline

- The substitution method
- **The recursion-tree method**
- The master method

The recursion-tree method

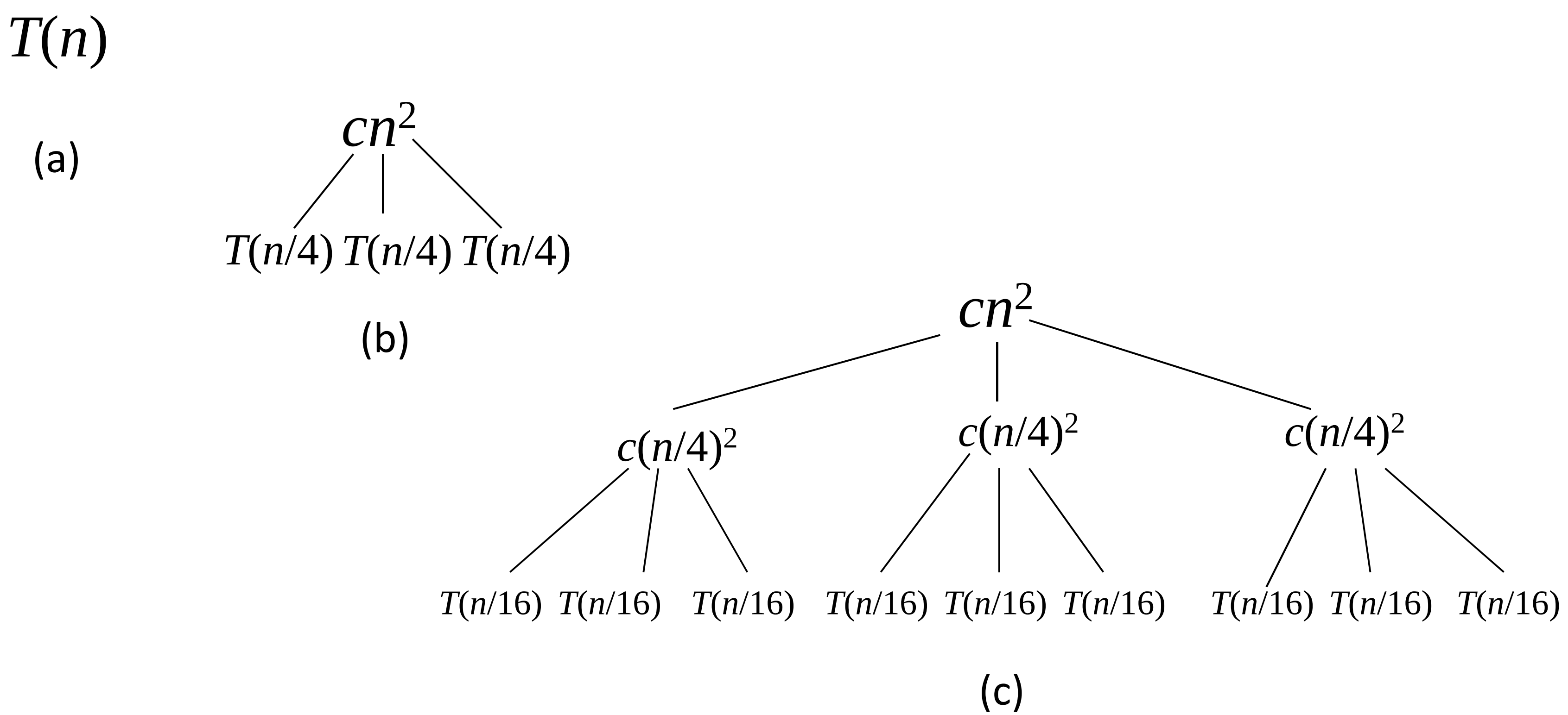
- A recursion tree is best used to generate a good guess, which is then verified by the substitution method.
- Tolerating a small amount of "sloppiness", we could use recursion-tree to generate a good guess.
- One can also use a recursion tree as a direct proof of a solution to a recurrence.
- Ideas:
  - in a **recursion tree**, each node represents the cost of a single subproblem
  - sum the costs within each level to obtain a set of per-level costs
  - sum all the per-level costs to determine the total cost

An example

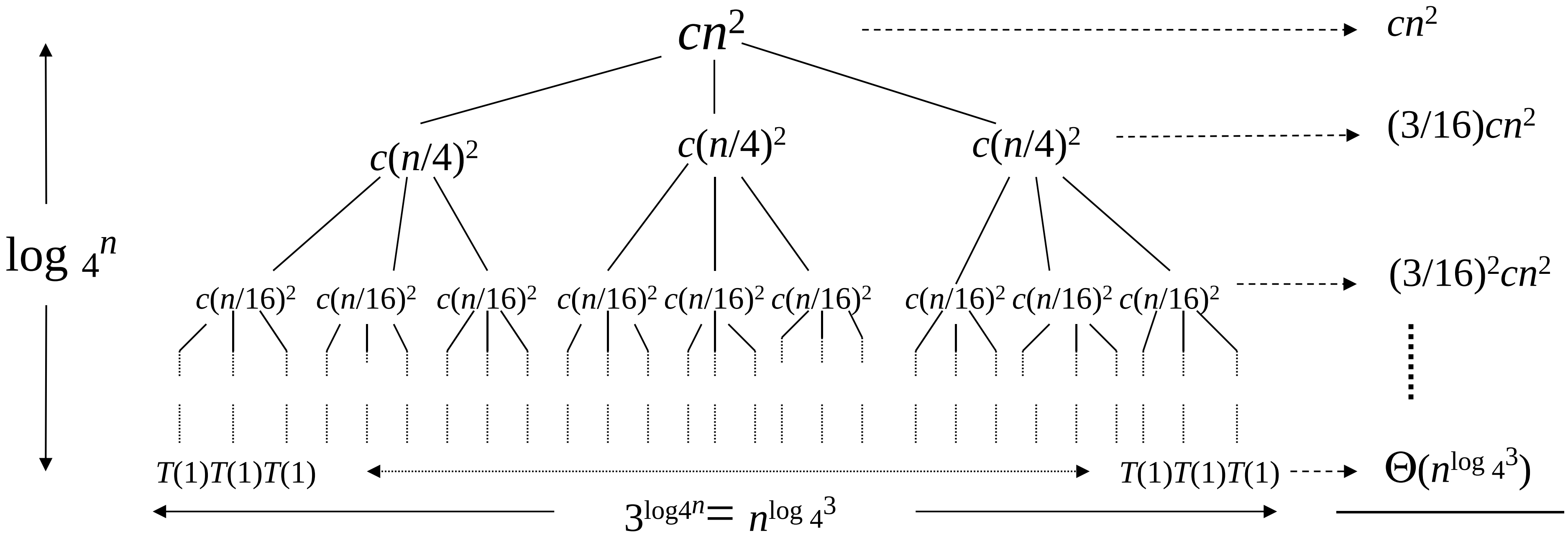
- For example: $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$
- Tolerating the sloppiness:
  - ignore the floor in the recurrence
  - assume n is an exact power of 4
- Rewrite the recurrence as $T(n) = 3T(n/4) + cn^2$

The construction of a recursion tree$_{1/2}$

- $T(n) = 3T(n/4) + cn^2$



The construction of a recursion tree$_{2/2}$



(d)

Total: $O(n^2)$

Determine the cost of the tree$_{1/2}$

- The subproblem size for a node at depth i is $n/4^i$.
- Thus, the tree has $\log_4 n + 1$ levels (0, 1, 2,..., $\log_4 n$).
- Each node at depth i, has a cost of $c(n/4^i)^2$ for $0 \le i \le \log_4 n - 1$.
- So, the total cost over all nodes at depth i is $3^i * c(n/4^i)^2 = (3/16)^i cn^2$.
- The last level, at $\log_4 n$, has $3^{\log_4 n} = n^{\log_4 3}$ nodes.
- The cost of the entire tree:

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + (\frac{3}{16})^2 cn^2 + \ldots + (\frac{3}{16})^{\log_4 n - 1} cn^2 + \theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n - 1} (\frac{3}{16})^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})
\end{aligned}
$$

Determine the cost of the tree$_{2/2}$

- Take advantage of small amounts of sloppiness, we have

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1-(3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2)
\end{aligned}
$$

- Thus, we have derived a guess of $T(n)=O(n^2)$.

Verify the correctness of our guess

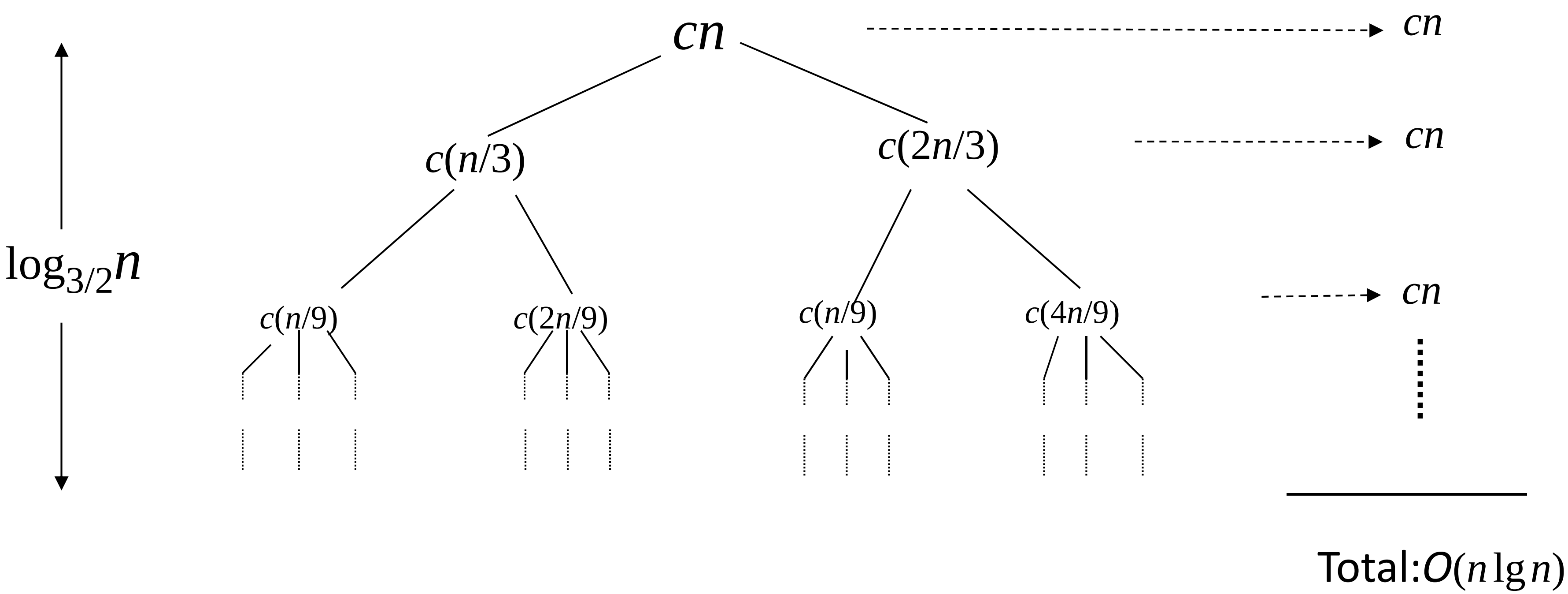
- Now we can use the substitution method to verify that our guess is correct.
- We want to show that T(n) $\leq dn^2$ for some constant $d > 0$.
- Using the same constant $c > 0$ as before, we have

$$
\begin{aligned}
T(n) &\leq 3T(\lfloor n/4 \rfloor) + cn^2 \\
&\leq 3d\lfloor n/4 \rfloor^2 + cn^2 \\
&\leq 3d(n/4)^2 + cn^2 \\
&= 3/16dn^2 + cn^2 \\
&\leq dn^2,
\end{aligned}
$$

where the last step holds for $d \geq (16/13)c$.

Another example

- Another example: $T(n) = T(n/3)+T(2n/3)+O(n)$.
- The recursion-tree:



Determine the cost of the tree

- The height of tree is $\log_{3/2} n$.
- The recursion tree has fewer than $2^{\log_{3/2} n} = n^{\log_{3/2} 2}$ leaves.
- The total cost of all leaves would then be $\theta(n^{\log_{3/2} 2})$, which is $\omega(n \lg n)$.
- Also, not all levels contribute a cost of exactly cn.
- Thus, we derived a guess of $T(n) = O(n \lg n)$.

Verify the correctness of our guess

- We can verify the guess by the substitution method.
- We have $$\begin{aligned} T(n) &\le T(n/3) + T(2n/3) + cn \\ &\le d(n/3)\lg(n/3) + d(2n/3)\lg(2n/3) + cn \\ &= (d(n/3)\lg n - d(n/3)\lg 3) \\ &\quad + (d(2n/3)\lg n - d(2n/3)\lg(3/2)) + cn \\ &= dn\lg n - d((n/3)\lg 3 + (2n/3)\lg(3/2)) + cn \\ &= dn\lg n - d((n/3)\lg 3 + (2n/3)\lg 3 - (2n/3)\lg 2) + cn \\ &= dn\lg n - dn(\lg 3 - 2/3) + cn \\ &\le dn\lg n \end{aligned}$$

for $d \ge c/(\lg 3 - (2/3))$.

Outline

- The substitution method
- The recursion-tree method
- **The master method**

The master method$_{1/2}$

- The master method provides a "cookbook" method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

  - $a \geq 1$ and $b > 1$ are constants
  - $f(n)$ is an asymptotically positive function

- It requires memorization of three cases, but then the solution of many recurrences can be determined quite easily.

The master method$_{2/2}$

- The recurrence $T(n) = aT(n/b) + f(n)$ describes the running time of an algorithm that
  - divides a problem of size n into a subproblems, each of size n/b
  - each of subproblems is solved recursively in time $T(n/b)$
  - the cost of dividing and combining the results is $f(n)$
- For example, the recurrence arising from the MERGE-SORT procedure has $a = 2$, $b = 2$, and $f(n) = \Theta(n)$.
- Normally, we omit the floor and ceiling functions when writing divide-and-conquer recurrences of this form.

Master theorem

- **Master theorem:** Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then, $T(n)$ can be bounded asymptotically as follows.

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \theta(n^{\log_b a})$.
2. If $f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c<1$ and all sufficiently large n, then $T(n) = \theta(f(n))$.

Intuition behind the master method

- Intuitively, the solution to the recurrence is determined by comparing the two functions $f(n)$ and $n^{\log_b a}$.
  - Case 1: if $n^{\log_b a}$ is asymptotically larger than $f(n)$ by a factor of n^{ε} for some constant $\varepsilon > 0$, then the solution is $T(n) = \theta(n^{\log_b a})$.
  - Case 2: if $n^{\log_b a}$ is asymptotically equal to $f(n)$, then the solution is $T(n) = \theta(n^{\log_b a} \lg n)$.
  - Case 3: if $n^{\log_b a}$ is asymptotically smaller then $f(n)$ by a factor of n^{ε}, and the function $f(n)$ satisfies the "regularity“ condition that $af(n/b) \le cf(n)$, then the solution is $T(n) = \theta(f(n))$.
- The three cases do not cover all the possibilities for $f(n)$.

Using the master method$_{1/3}$

- Example 1: $T(n) = 9T(n/3) + n$
  - For this recurrence, we have $a = 9$, $b = 3$, $f(n) = n$.
  - Thus, $n^{\log_b a} = n^{\log_3 9} = \theta(n^2)$.
  - Since $f(n) = O(n^{\log_3 9 - \varepsilon})$, where ε = 1, we can apply case 1.
  - The solution is $T(n) = \Theta(n^2)$.

- Example 2: $T(n) = T(2n/3) + 1$
  - For this recurrence, we have $a = 1$, $b = 3/2$, $f(n) = 1$.
  - Thus, $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$.
  - Since $f(n) = O(n^{\log_b a}) = \theta(1)$, we can apply case 2.
  - The solution is $T(n) = \Theta(\lg n)$.

Using the master method$_{2/3}$

- Example 3: $T(n) = 3T(n/4) + n \lg n$
  - For this recurrence, we have $a = 3$, $b = 4$, $f(n) = n \lg n$.
  - Thus, $n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$.
  - For sufficiently large n, $af(n/b) = 3(n/4)\lg(n/4) \le (3/4)n \lg n = cf(n)$ for $c = 3/4$.
  - Since $f(n) = \Omega(n^{\log_4 3+\varepsilon})$ with $\varepsilon \approx 0.2$ and the regularity condition holds for $f(n)$ case 3 applies.
  - The solution is $T(n) = \Theta(n \lg n)$.

Using the master method$_{3/3}$

- Example 4: $T(n) = 2T(n/2) + n\lg n$
  - For this recurrence, we have $a = 2$, $b = 2$, $f(n) = n\lg n$.
  - The function $f(n) = n\lg n$ is asymptotically larger than $n^{\log_b a} = n^{\log_2 2} = n$.
  - But, it is not polynomially larger since the ratio $f(n)/n^{\log_b a} = (n\lg n)/n = \lg n$ is asymptotically less than n^{ε} for any positive constant ε.
  - Consequently, the recurrence falls into the gap between case 2 and case 3.
- If $g(n)$ is asymptotically larger than $f(n)$ by a factor of n^{ε} for some constant $\varepsilon>0$, then we said $g(n)$ is **polynomially larger** than $f(n)$.