

# CS 3341 – Probability and Statistics

Cheat Sheet

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## Chapter 3 – Discrete Random Variables And Their Distributions

### Bernoulli distribution

- $p$  – probability of success
- $\mathbf{E}(X) = p$
- $\text{Var}(X) = pq$

### Binomial distribution

- $n$  – number of trials
- $p$  – probability of success
- $q$  – probability of failure
- $P(x) = \binom{n}{x} p^x q^{n-x}$
- $\mathbf{E}(X) = np$
- $\text{Var}(X) = npq$

### Geometric distribution

- $p$  – probability of success
- $P(x) = (1 - p)^{x-1} p, x = 1, 2, \dots$
- $\mathbf{E}(X) = \frac{1}{p}$
- $\text{Var}(X) = \frac{1-p}{p^2}$

### Negative Binomial distribution

- $k$  – number of successes
- $p$  – probability of success

- $P(x) = \binom{x-1}{k-1} (1-p)^{x-k} p^k, x = k, k+1, \dots$
- $\mathbf{E}(X) = \frac{k}{p}$
- $\text{Var}(X) = \frac{k(1-p)}{p^2}$

### Poisson distribution

- $\lambda$  – frequency, average number of events
- $P(x) = e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, 2, \dots$
- $\mathbf{E}(X) = \lambda$
- $\text{Var}(X) = \lambda$

## Chapter 4 – Continuous Distributions

### Uniform distribution

- $\mathbf{E}(X) = \frac{a+b}{2}$
- $\text{Var}(X) = \frac{(b-a)^2}{12}$

### Exponential distribution

- $\lambda$  – frequency parameter, the number of events per time unit
- $\mathbf{E}(x) = \frac{1}{\lambda}$
- $\text{Var}(X) = \frac{1}{\lambda^2}$

### Gamma distribution

- $\alpha$  – shape parameter
- $\lambda$  – frequency parameter
- $\mathbf{E}(X) = \frac{\alpha}{\lambda}$
- $\text{Var}(X) = \frac{\alpha}{\lambda^2}$

### Gamma-Poisson formula

- $\mathbf{P}\{T > t\} = \mathbf{P}\{X < \alpha\}$
- $\mathbf{P}\{T \leq t\} = \mathbf{P}\{X \geq \alpha\}$

## Normal distribution

- $\mu$  – expectation, location parameter
- $\sigma$  – standard deviation, scale parameter
- $\mathbf{E}(X) = \mu$
- $\text{Var}(X) = \sigma^2$
- $Z = \frac{x-\mu}{\sigma}$
- $Z_n = \frac{\bar{S}_n - n\mu}{\sigma/\sqrt{n}}$  – Central limit theorem

Normal can be approximated to Binomial with  $\text{Binomial}(n, p) \approx \text{Normal}(\mu = np, \sigma = \sqrt{np(1-p)})$

## Chapter 6 – Stochastic processes

### Binomial counting process

- $\lambda = \frac{p}{\Delta}$
- $n = \frac{t}{\Delta}$
- $X(n) = \text{Binomial}(n, p) = np$
- $Y = \text{Geometric}(p) = \frac{1}{p}$
- $T = Y\Delta$
- $\mathbf{E}(T) = \mathbf{E}(Y)\Delta = \frac{\Delta}{p} = \frac{1}{\lambda}$
- $\text{Var}(T) = \text{Var}(Y)\Delta^2 = (1-p)(\frac{\Delta}{p})^2 = \frac{1-p}{\lambda^2}$

### Poisson process

- $\mathbf{E}X(t) = \lambda t$
- $X(t) = \text{Poisson}(\lambda t)$
- $T = \text{Exponential}(\lambda)$
- $T_k = \text{Gamma}(k, \lambda)$
- $\mathbf{P}\{T_k \leq t\} = \mathbf{P}\{X(t) \geq k\}$
- $\mathbf{P}\{T_k > t\} = \mathbf{P}\{X(t) < k\}$

## Chapter 7 – Queuing Systems

### Parameters of a queuing system

- $\lambda_A$  = arrival rate, i.e. the expected arrivals per 1 unit of time.
- $\lambda_S$  = service rate, the average number of jobs processed by a continuously working server during one unit of time.
- $\mu_A = 1/\lambda_A$  mean interarrival time

- $\mu_S = 1/\lambda_S$  mean service time
- $r = \lambda_A/\lambda_S = \mu_S/\mu_A$  = utilization, or arrival-to-service ratio, or the expected number of jobs receiving service at any given time

## Random values of a queuing system

- $X_s(t)$  = number of jobs receiving service at time  $t$
- $X_w(t)$  = number of jobs waiting in a queue at time  $t$
- $X(t) = X_s(t) + X_w(t)$  = the total number of jobs in the system at time  $t$ . This is a **queuing process** because it may increase or decrease, whereas a **counting process** may only increase.
- $S_k$  = service time of the  $k$ -th job
- $W_k$  = waiting time of the  $k$ -th job
- $R_k = S_k + W_k$  = response time, the total time a job spends in the system from its arrival until the departure
- $\lambda_A \mathbf{E}(R) = \mathbf{E}(X)$  – The Little’s Law (stationary queuing systems only)

## Notes

- A queuing system is *stationary* if the distributions of  $S_k$ ,  $W_k$ , and  $R_k$  are independent of  $k$ .

## Bernoulli single-server queuing process

- $\Delta$  – frame size
- $p_A = \lambda_A \Delta$  – probability of arrival during a frame
- $p_S = \lambda_S \Delta$  – probability of service during a frame

## Markov property

- $p_{00} = \mathbf{P}\{\text{no arrivals}\} = 1 - p_A$
- $p_{01} = \mathbf{P}\{\text{new arrival}\} = p_A$

For all  $i \geq 1$ ,

- $p_{i,i-1} = \mathbf{P}\{\text{no arrivals} \cap \text{one departure}\} = (1 - p_A)p_S$
- $p_{i,i} = \mathbf{P}\{\text{no arrivals} \cap \text{no departures}\} + \mathbf{P}\{\text{one arrival} \cap \text{one departure}\} = (1 - p_A)(1 - p_S) + p_A p_S$
- $p_{i,i+1} = \mathbf{P}\{\text{one arrival} \cap \text{no departures}\} = p_A(1 - p_S)$

Steady state distribution can be computed if  $\lambda_S > \lambda_A$ .

## M/M/1 system

A/S/n/C

- A – distribution of interarrival times
- S – distribution of service times
- n – number of servers
- C – capacity
- $\pi_x = \mathbf{P}\{X = x\} = r^x(1 - r)$  for  $x = 0, 1, 2, \dots$
- $\mathbf{E}\{X\} = \frac{r}{1-r}$
- $Var(x) = \frac{r}{(1-r)^2}$
- $r = \lambda_A/\lambda_S = \mu_S/\mu_A = 1 - \pi_0 = P(busy) = 1 - P(idle)$

The system is functional if  $r < 1$ . IF  $r \geq 1$  the system will be overloaded.

- $W = S_1 + S_2 + S_3 + \dots + S_X$  – waiting time
- $\mathbf{E}(W) = \mathbf{E}(S_1 + \dots + S_X) = \mathbf{E}(S)\mathbf{E}(X) = \frac{\mu_S r}{1-r} = \frac{r}{\lambda_S(1-r)}$  – Expected waiting time
- $\mathbf{E}(R) = \mathbf{E}(W) + \mathbf{E}(S) = \frac{\mu_S r}{1-r} + \mu_S = \frac{\mu_S}{1-r} = \frac{1}{\lambda_S(1-r)}$  – Expected response time

Queue length (number of waiting jobs)

- $X_w = X - X_s$  where  $X_s$  is number of jobs getting service at any time (0 or 1).
- $\mathbf{E}(X_w) = \mathbf{E}(X) - \mathbf{E}(X_s) = \frac{r}{1-r} - r = \frac{r^2}{1-r}$  # Chapter 8 – Introduction to Statistics
- $\sigma(\hat{\theta})$  – standard error of estimator  $\hat{\theta}$  of parameter  $\theta$
- $s(\hat{\theta})$  – estimated standard error – standard deviation of the estimator

## Chapter 9 – Statistical Inference I

- $\mu_k = \mathbf{E}(X^k)$  – k-th population moment
- $m_k = \frac{1}{n} \sum_{i=1}^n X_i^k$  – k-th sample moment