On the Lagarias Inequality and Superabundant Numbers

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Abstract

We study the Lagarias inequality, an elementary criterion equivalent to the Riemann Hypothesis. Using a continuous extension of the harmonic numbers, we show that the sequence

$$B_n := \frac{H_n + \exp(H_n)\log(H_n)}{n}$$

is strictly increasing for $n \geq 1$. As a consequence, if the Lagarias inequality has counterexamples, then the least counterexample must be a superabundant number; equivalently, it suffices to verify the inequality on the superabundant numbers.

1 Introduction

Let $H_n = \sum_{j=1}^{n} \frac{1}{j}$ denote the nth harmonic number, and let $\sigma(n) = \sum_{d|n} d$ denote the sum-of-divisors function. Lagarias [2] proved that the Riemann Hypothesis is true if and only if, for every $n \geq 1$,

$$\sigma(n) \leq H_n + \exp(H_n)\log(H_n). \tag{1}$$

A number $n \in \mathbb{N}$ is called *superabundant* if for all $m < n$,

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

2 The Lagarias Inequality

Let $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denote the digamma function and γ the Euler–Mascheroni constant. For $x > 0$, define the continuous extension of the harmonic numbers by

$$H(x) := \psi(x+1) + \gamma,$$

which satisfies $H(n) = H_n$ for all $n \in \mathbb{N}$.

Lemma 2.1. *Define*

$$L(x) := \frac{H(x) + e^{H(x)} \log(H(x))}{x}.$$

Then

$$L'(x) = \frac{N(x)}{x^2},$$

where the numerator $N(x)$ is

$$N(x) = xH'(x) - H(x) + e^{H(x)} \left((xH'(x) - 1) \log(H(x)) + x \frac{H'(x)}{H(x)} \right).$$

Proof. By the quotient rule,

$$L'(x) = \frac{\left(\frac{d}{dx} \left[H(x) + e^{H(x)} \log(H(x)) \right] \right) x - \left(H(x) + e^{H(x)} \log(H(x)) \right)}{x^2}.$$

Also

$$\frac{d}{dx} \left(e^{H(x)} \log(H(x)) \right) = e^{H(x)} H'(x) \log(H(x)) + e^{H(x)} \frac{H'(x)}{H(x)}.$$

Substituting gives $L'(x) = \frac{N(x)}{x^2}$ with $N(x)$ as stated. $\square$

Lemma 2.2. *For every $x \geq 1$, the following hold:*

$$H(x) \geq \log(x+1), \tag{2}$$

$$H'(x) \geq \frac{1}{x+1}. \tag{3}$$

Proof. Define $f(x) := H(x) - \log(x+1) = \psi(x+1) + \gamma - \log(x+1)$. Then

$$f'(x) = \psi'(x+1) - \frac{1}{x+1}.$$

For $y > 0$, one has the series representation

$$\psi'(y) = \sum_{k=0}^{\infty} \frac{1}{(y+k)^2}.$$

Apply this with $y = x + 1 > 0$:

$$\psi'(x+1) = \sum_{k=0}^{\infty} \frac{1}{(x+1+k)^2}.$$

Consider $g(t) = \frac{1}{(x+1+t)^2}$ on $[0, \infty)$. Since g is positive and strictly decreasing, for each integer $k \geq 0$ and all $t \in [k, k+1]$ we have $g(t) \leq g(k)$, so

$$\int_k^{k+1} g(t)\, dt \leq g(k) = \frac{1}{(x+1+k)^2}.$$

Summing over $k \geq 0$ gives

$$\int_0^\infty g(t)\,dt = \sum_{k=0}^\infty \int_k^{k+1} g(t)\,dt \leq \sum_{k=0}^\infty \frac{1}{(x+1+k)^2} = \psi'(x+1).$$

Compute the integral:

$$\int_0^\infty \frac{1}{(x+1+t)^2}\,dt = \frac{1}{x+1}.$$

Hence $\psi'(x+1) \geq \frac{1}{x+1}$ and therefore $f'(x) \geq 0$ for all $x \geq 0$. Since $f(0) = 0$, we have $f(x) \geq 0$ for all $x \geq 0$, i.e. $H(x) \geq \log(x+1)$ for $x \geq 0$, in particular for $x \geq 1$.

Finally, (3) follows since $H'(x) = \psi'(x+1) \geq \frac{1}{x+1}$. □

Lemma 2.3. *For every $x \geq 1$,*

$$H(x) \leq 1 + \log x. \tag{4}$$

Proof. Define $f(x) := 1 + \log x - H(x)$. Then $f'(x) = \frac{1}{x} - \psi'(x+1)$. For $x \geq 1$,

$$\psi'(x+1) = \sum_{k=0}^\infty \frac{1}{(x+1+k)^2} = \sum_{k=1}^\infty \frac{1}{(x+k)^2}.$$

Consider $g(t) = \frac{1}{(x+t)^2}$ on $[0, \infty)$. Since g is strictly decreasing, for each integer $k \geq 1$ and all $t \in [k-1, k]$ we have $g(t) \geq g(k)$, so

$$\int_{k-1}^k g(t)\,dt \geq g(k) = \frac{1}{(x+k)^2}.$$

Summing over $k \geq 1$ gives

$$\int_0^\infty g(t)\,dt = \sum_{k=1}^\infty \int_{k-1}^k g(t)\,dt \geq \sum_{k=1}^\infty \frac{1}{(x+k)^2} = \psi'(x+1).$$

Compute the integral:

$$\int_0^\infty \frac{1}{(x+t)^2}\,dt = \frac{1}{x}.$$

Hence $\psi'(x+1) \leq \frac{1}{x}$, so $f'(x) \geq 0$ for all $x \geq 1$. Since $f(1) = 1 + \log 1 - H(1) = 0$, we have $f(x) \geq 0$ for $x \geq 1$, i.e. $H(x) \leq 1 + \log x$. □

Lemma 2.4. *For every $x \geq 1$,*

$$N(x) \geq \frac{x}{x+1} - H(x) + \frac{e^{H(x)}}{x+1}\left(\frac{x}{H(x)} - \log(H(x))\right). \tag{5}$$

Proof. From (3), $H'(x) \geq \frac{1}{x+1}$, so $xH'(x) \geq \frac{x}{x+1}$ and hence

$$xH'(x) - 1 \geq -\frac{1}{x+1}.$$

Also

$$x\frac{H'(x)}{H(x)} \geq \frac{x}{(x+1)H(x)}.$$

Since $x \geq 1$ implies $H(x) \geq \log(x+1) \geq \log 2 > 0$, we have $\log(H(x)) \geq 0$. Therefore

$$(xH'(x) - 1)\log(H(x)) \geq -\frac{1}{x+1}\log(H(x)).$$

Substitute these bounds into the definition of $N(x)$ in Lemma 2.1:

$$N(x) \geq xH'(x) - H(x) + e^{H(x)}\left(-\frac{1}{x+1}\log(H(x)) + \frac{x}{(x+1)H(x)}\right).$$

Factor:

$$N(x) \geq xH'(x) - H(x) + \frac{e^{H(x)}}{x+1}\left(\frac{x}{H(x)} - \log(H(x))\right).$$

Finally use $xH'(x) \geq \frac{x}{x+1}$ to obtain (5). $\square$

Lemma 2.5. *For every $x \geq 1$,*

$$\frac{x}{1+\log x} - \log(1+\log x) \geq 0. \tag{6}$$

Proof. Let $t = \log x \geq 0$. Then $x = e^t$ and (6) becomes

$$\frac{e^t}{1+t} - \log(1+t) \geq 0 \quad \Longleftrightarrow \quad p(t) := e^t - (1+t)\log(1+t) \geq 0.$$

Compute derivatives:

$$p'(t) = e^t - \log(1+t) - 1, \qquad p''(t) = e^t - \frac{1}{1+t}.$$

For $t > 0$, we have $\frac{1}{1+t} < 1 < e^t$, so $p''(t) > 0$ for all $t > 0$. Thus p' is strictly increasing on $(0, \infty)$. Since $p'(0) = 1 - 0 - 1 = 0$, it follows that $p'(t) > 0$ for all $t > 0$. Therefore p is strictly increasing on $(0, \infty)$. Since $p(0) = 1-(1)\cdot 0 = 1 > 0$, we have $p(t) \geq 1 > 0$ for all $t \geq 0$. This proves (6). $\square$

Lemma 2.6. *For every $x \geq 1$, $N(x) \geq G(x)$, where*

$$G(x) := \frac{x}{x+1} - (1+\log x) + \frac{x}{1+\log x} - \log(1+\log x). \tag{7}$$

Proof. Start from (5):

$$N(x) \geq \frac{x}{x+1} - H(x) + \frac{e^{H(x)}}{x+1}\left(\frac{x}{H(x)} - \log(H(x))\right).$$

From (4), $H(x) \leq 1 + \log x$, so

$$-H(x) \geq -(1 + \log x).$$

Also since $H(x) \leq 1 + \log x$ and both are positive, we have

$$\frac{1}{H(x)} \geq \frac{1}{1+\log x}, \qquad \log(H(x)) \leq \log(1 + \log x).$$

Thus

$$\frac{x}{H(x)} - \log(H(x)) \geq \frac{x}{1+\log x} - \log(1 + \log x).$$

Using (2), $e^{H(x)} \geq x + 1$, so $\frac{e^{H(x)}}{x+1} \geq 1$. By Lemma 2.5, the quantity

$$\frac{x}{1+\log x} - \log(1 + \log x)$$

is nonnegative for $x \geq 1$. Therefore multiplying it by a factor ≥ 1 preserves a lower bound:

$$\frac{e^{H(x)}}{x+1}\left(\frac{x}{1+\log x} - \log(1+\log x)\right) \geq \left(\frac{x}{1+\log x} - \log(1+\log x)\right).$$

Putting these together yields

$$N(x) \geq \frac{x}{x+1} - (1 + \log x) + \frac{x}{1+\log x} - \log(1 + \log x) = G(x).$$

□

Lemma 2.7. *For every $t \geq 4$,*

$$e^t \geq 2t^2 + 3t + 1. \tag{8}$$

Proof. Define $s(t) := e^t - (2t^2 + 3t + 1)$. Then

$$s'(t) = e^t - (4t + 3), \qquad s''(t) = e^t - 4, \qquad s'''(t) = e^t > 0.$$

Since $s'''(t) > 0$, s'' is strictly increasing. At $t = 4$, $s''(4) = e^4 - 4 > 0$, so $s''(t) > 0$ for all $t \geq 4$. Thus s' is strictly increasing on $[4, \infty)$. Since $s'(4) = e^4 - 19 > 0$, we have $s'(t) > 0$ for all $t \geq 4$. Therefore s is strictly increasing on $[4, \infty)$. Since

$$s(4) = e^4 - 45 > 0,$$

we conclude $s(t) > 0$ for all $t \geq 4$, proving (8). □

Proposition 2.1. *For all real $x \geq e^4$, we have $L'(x) > 0$. Consequently, $B_{n+1} > B_n$ for all integers $n \geq 55$, where*

$$B_n = \frac{H_n + e^{H_n}\log(H_n)}{n}.$$

Proof. Fix $x \geq e^4$. By Lemma 2.1, it suffices to show $N(x) > 0$. By Lemma 2.6, it suffices to show $G(x) > 0$.

Set $t = \log x$ and $u = t + 1 = 1 + \log x$. Then $t \geq 4$ and (7) gives

$$G(x) = \frac{x}{x+1} - u + \frac{x}{u} - \log u.$$

By Lemma 2.7,

$$x = e^t \geq 2t^2 + 3t + 1 = (2t+1)(t+1) = (2t+1)u,$$

so $\frac{x}{u} \geq 2t + 1$. Hence

$$G(x) \geq \frac{x}{x+1} - u + (2t+1) - \log u = \frac{x}{x+1} + t - \log(1+t).$$

Since $1 + t < e^t$ for all $t > 0$, we have $\log(1+t) < t$, so

$$G(x) > \frac{x}{x+1} > 0.$$

Therefore $N(x) \geq G(x) > 0$, so $L'(x) = \frac{N(x)}{x^2} > 0$ for all $x \geq e^4$. If $n \geq 55$, then $n \geq e^4$ and $B_{n+1} - B_n = L(n+1) - L(n) > 0$. $\square$

Corollary 2.1. *The sequence*

$$\left\{ \frac{H_n + \exp(H_n)\log(H_n)}{n} \right\}_{n=1}^{\infty} \tag{9}$$

is strictly increasing.

Proof. Proposition 2.1 gives $B_{n+1} > B_n$ for all $n \geq 55$. Direct computation verifies $B_{n+1} - B_n > 0$ for $1 \leq n \leq 54$. $\square$

3 Superabundant Numbers

The proof strategy in this section is inspired by [1].

Theorem 3.1. *If there are counterexamples to the Lagarias inequality, the smallest such counterexample must be a superabundant number.*

Proof. Suppose, for sake of contradiction, that m is the smallest counterexample to (1). Then

$$\frac{\sigma(m)}{m} > \frac{H_m + \exp(H_m)\log(H_m)}{m} =: B_m.$$

Let n be the least integer $1 \leq n < m$ such that

$$\frac{\sigma(n)}{n} = \max_{1 \leq k < m} \frac{\sigma(k)}{k}.$$

Then by minimality of n, for every $k < n$ we have $\frac{\sigma(k)}{k} < \frac{\sigma(n)}{n}$, so n is superabundant. Moreover, by maximality,

$$\frac{\sigma(n)}{n} \geq \frac{\sigma(m)}{m}.$$

By Corollary 2.1, the sequence $\{B_n\}$ is strictly increasing, so since $n < m$ we have $B_m > B_n$. Hence

$$\frac{\sigma(n)}{n} \geq \frac{\sigma(m)}{m} > B_m > B_n = \frac{H_n + \exp(H_n)\log(H_n)}{n}.$$

Thus $n < m$ is also a counterexample to (1), contradicting the minimality of m. $\square$

References

[1] I. Assani, A. Chester, and B. Paschal, *On Robin's Inequality and the Kaneko–Lagarias Inequality*, arXiv:2503.03159 (2025).

[2] J. C. Lagarias, *An Elementary Problem Equivalent to the Riemann Hypothesis*, American Mathematical Monthly **109** (2002), no. 6, 534–543.