Problem Sets1

IE509/AI533 Advanced Quality Control

Lee hyo jeong
AIGS 20215539

September 16, 2022

Problem 1

Prove that the sample variance is an unbiased estimator of the population variance. Let $X_1, X_2, ..., X_n$ are independent observations from a population with mean μ and variance σ^2

$$E(s^2) = E(\frac{\sum(X_i - \bar{X})^2}{n-1}) = \sigma^2$$

Expand the equation $E(\sum(X_i - \bar{X})^2)$

$$E(\sum(X_i - \bar{X})^2) = E(\sum X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2)$$

$$= E(\sum X_i^2 - 2\bar{X} \cdot n\bar{X} + n\bar{X}^2)$$

$$= E(\sum X_i^2 - n\bar{X}^2)$$

$$= \sum E(X_i^2) - E(n\bar{X}^2)$$

Given $E(X^2) = \sigma^2 + \mu^2$ and $E(\bar{X}^2) = \frac{\mu^2}{n} + \mu^2$

$$\sum E(X_i^2) - E(n\bar{X}^2) = n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2$$

$$= (n-1)\sigma^2$$

Therefore,

$$E(s^2) = E(\frac{\sum(X_i - \bar{X})^2}{n-1}) = \frac{1}{n-1}E(\sum(X_i - \bar{X})^2)$$

$$= \frac{(n-1)\sigma^2}{n-1} = \sigma^2$$

Problem 2

Suppose we wish to test the hypothesis

$$H_0 : \mu_0 = 45$$

$$H_1 : \mu_1 \neq 45$$

Where we know that $\sigma_0^2 = 16$. Assume that $\alpha = 0.05$. Since $\sigma_0^2 = 16$, $\sigma_0 = 4$

(a) Mean shift to $\mu_1 = 50$

Hypothesized mean: $\mu_0 = 45$ True mean $\mu_1 = 50$

$$Z_{\alpha/2} = Z_{0.025} = 1.96$$

$$Z_\beta = Z_0.1 = 1.28$$

$$n = \frac{(Z_{\frac{\alpha}{2}} + Z_\beta)^2\sigma_0^2}{(\mu_0 - \mu_1)^2}$$

$$\frac{(1.96 + 1.28)^2 \times 16}{(45 - 50)^2}$$

$$n = 6.71 \approx 7$$

(b) For $H_1 : \mu_0 > 45$
Given Z_α : 1.65

$$n = \frac{(Z_\alpha + Z_\beta)^2\sigma_0^2}{(\mu_0 - \mu_1)^2}$$

$$\frac{(1.65 + 1.28)^2 \times 16}{(45 - 50)^2}$$

$$n = 5.49 \approx 6$$

Problem 3

Given that each individual sample variance s_i^2 is an unbiased estimator of σ^2

$$E(s_i^2) = \sigma^2$$

Formula of UCL and LCL, σ is standard deviation.

$$UCL = E(s^2) + 3\sigma(s^2)$$

$$LCL = E(s^2) - 3\sigma(s^2)$$

From (1)

$$V[\frac{(m-1)s_i^2}{\sigma^2}] = 2(m-1)$$

$$= \frac{(m-1)^2}{\sigma^4}V(s_i^2) = 2(m-1)$$

$$= V(s_i^2) = \frac{2(m-1)}{(m-1)^2}\sigma^4$$

$$= V(s_i^2) = \frac{2\sigma^4}{m-1}$$

$$= \sigma(s_i^2) = \sqrt{\frac{2\sigma^4}{m-1}} = \sigma^2\sqrt{\frac{2}{m-1}}$$

$$UCL = \sigma^2 + 3\sigma^2\sqrt{\frac{2}{m-1}}$$

$$LCL = \sigma^2 - 3\sigma^2\sqrt{\frac{2}{m-1}}$$

Problem 4

$$H_0 : \mu = 150$$

$$H_1 : \mu > 150$$

$$n = 20, \bar{x} = 153.7, s = 11.5, \alpha = 0.05$$

(a) Test Hypothesis

$$Teststatistic : t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Degrees of freedom = n-1 = 19

$$t : t \geq t_{0.05,19} = 1.729$$

$$t^* = \frac{153.7 - 150}{11.5/\sqrt{20}} = 1.439$$

Do not reject H_0 since value of the test statistic, $t^* = 1.44$, is less than the critical value 1.729
(b) Compute the type II error if the weld strength is actually 160

$$\beta = P_{H_1}(failtorejectH_0)$$
$$= P(\frac{\bar{x} - \mu}{s/\sqrt{n}} \leq \frac{154.4 - 160}{11.5/\sqrt{20}})$$
$$P(T \leq -2.178) = 0.0214$$

Problem 5

(a) For 90% confidence Given $\sigma = 4$, $E = 2$

$$Z_{\frac{\alpha}{2}} = 1.64$$

$$n = (\frac{Z_{\frac{\alpha}{2}}\sigma}{E})$$
$$= (\frac{1.64 * 4}{2})^2$$
$$= 10.76$$
$$\approx 11$$

(b) For 90% confidence

$$Z_{\frac{\alpha}{2}} = 1.64$$

$$CI = 21.1 \pm 1.645(\frac{4}{\sqrt{49}})$$
$$= 21.1 \pm 0.73$$
$$= (20.37, 21.83)$$