Constructing hyperelliptic pillowcase covers from meanders

Luke Jeffreys

University of Bristol

19/07/2022

Rough objective

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Construct surfaces using a minimal number of squares

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Construct surfaces using a minimal number of squares

with restricted combinatorics

Rough objective

Construct surfaces using a minimal number of squares

with restricted combinatorics

giving rise to prescribed singularity data.

What is a pillowcase cover?

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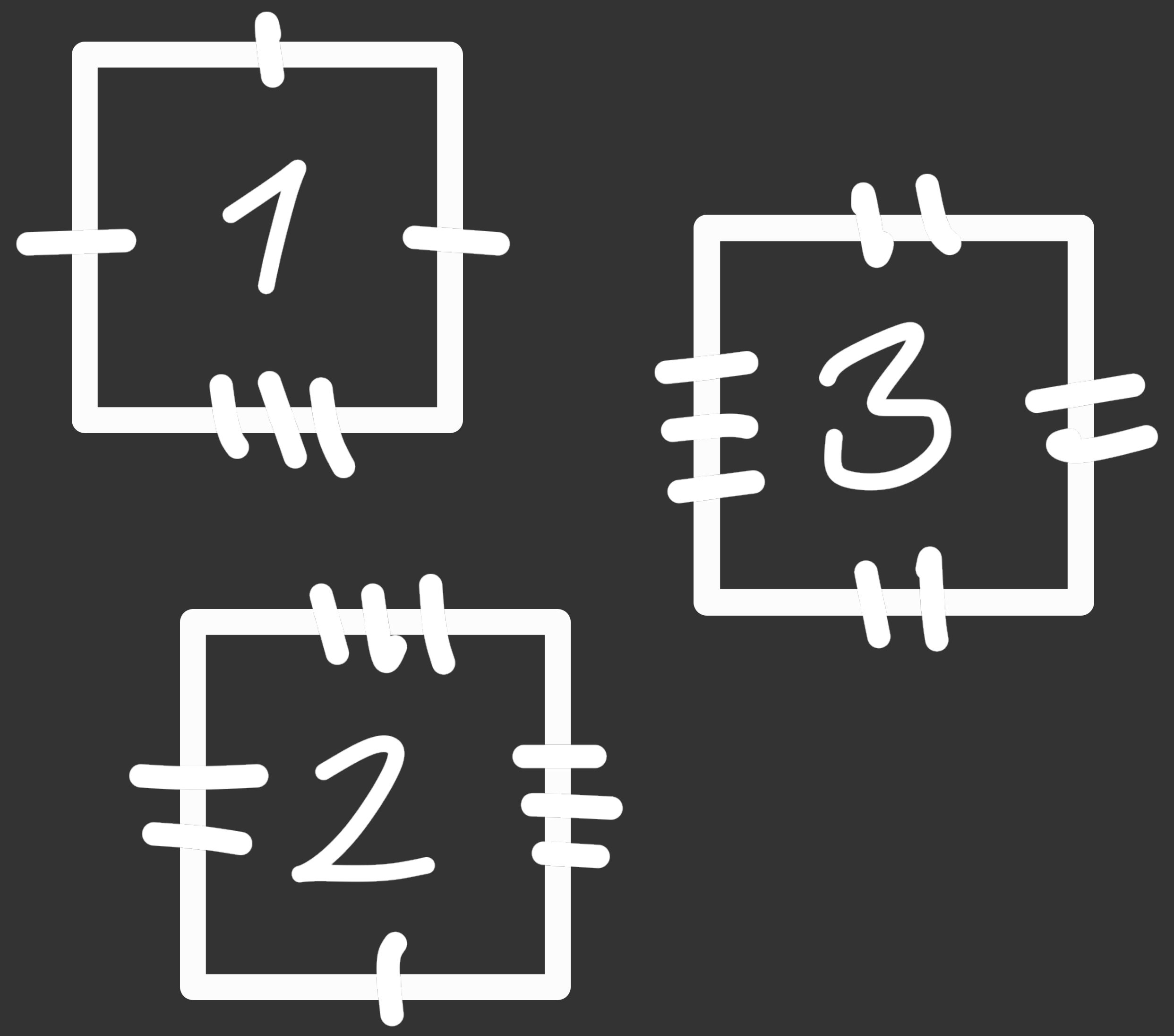

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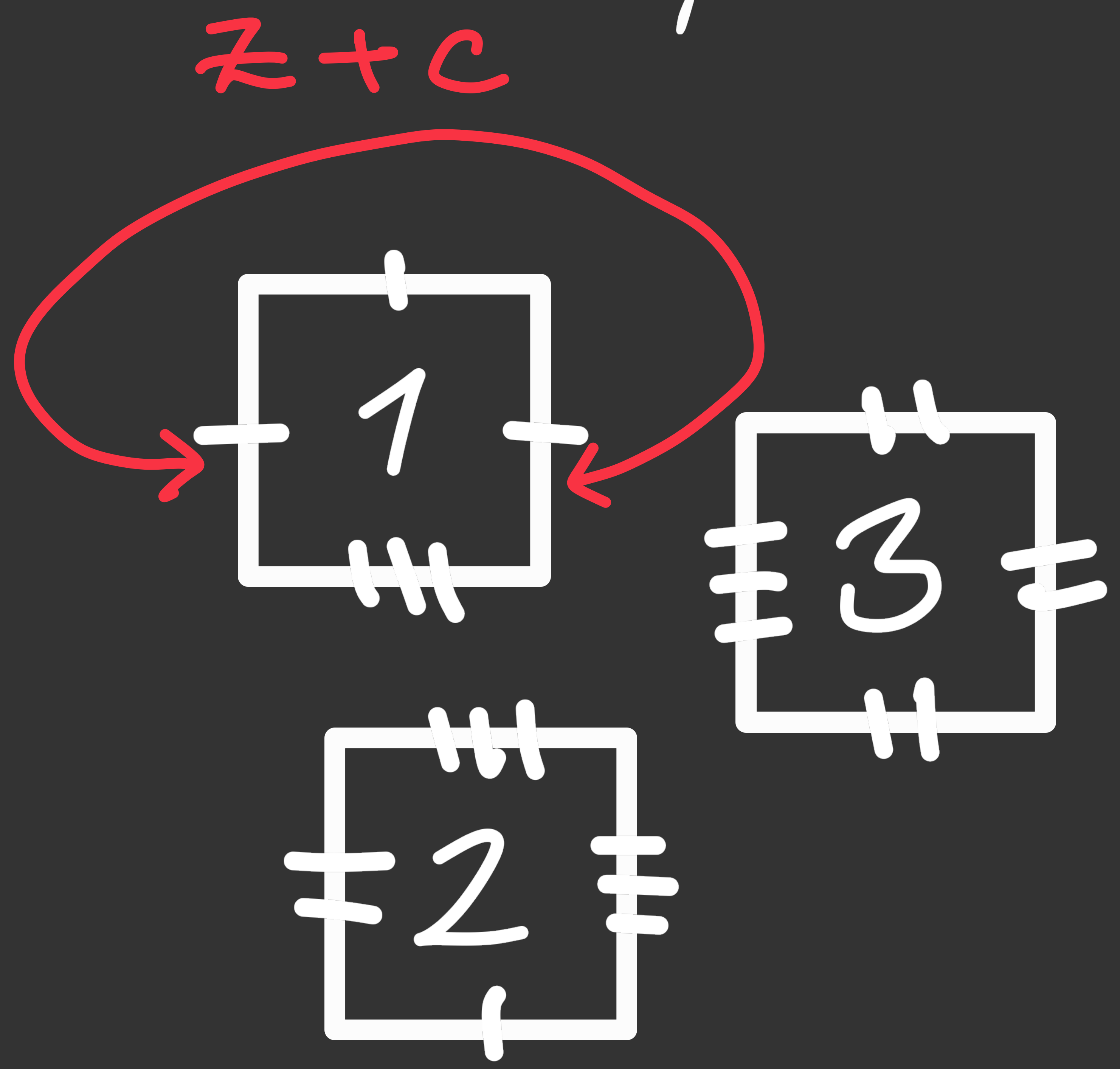

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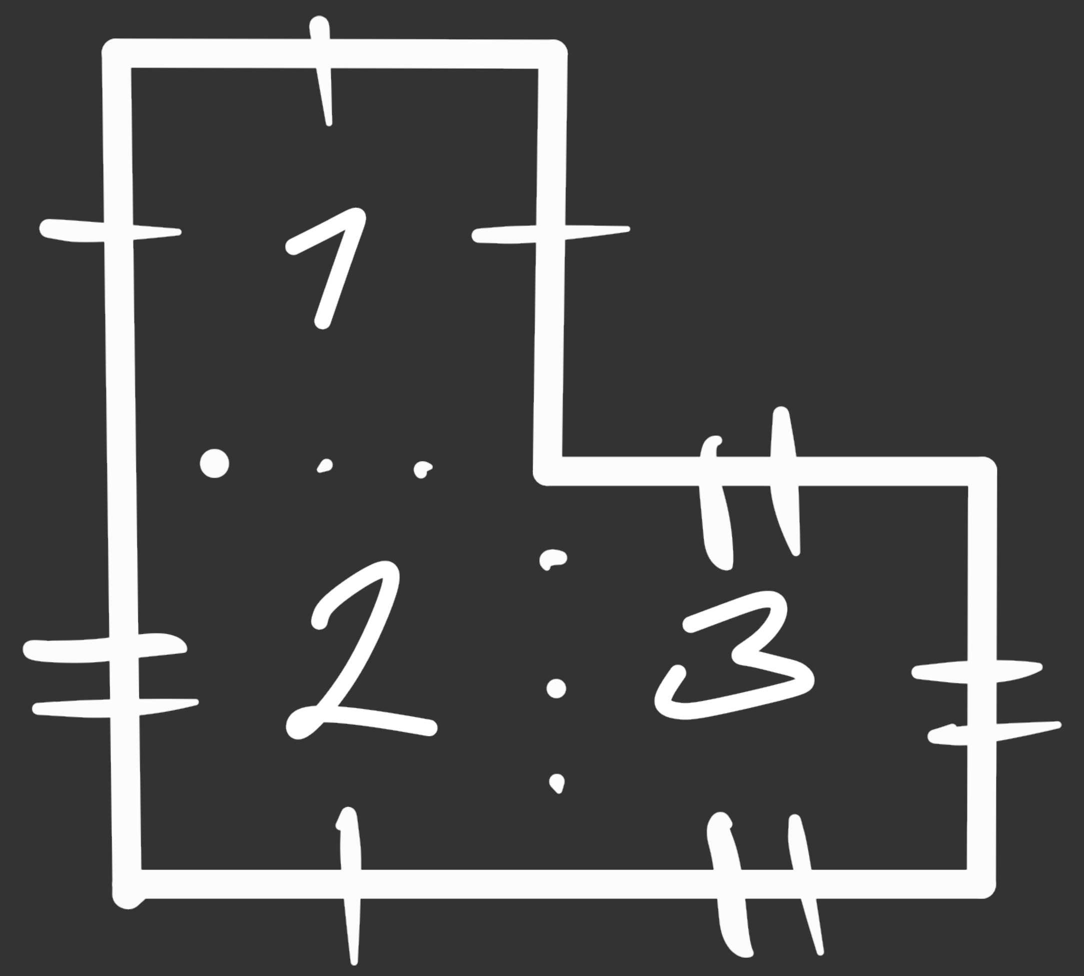

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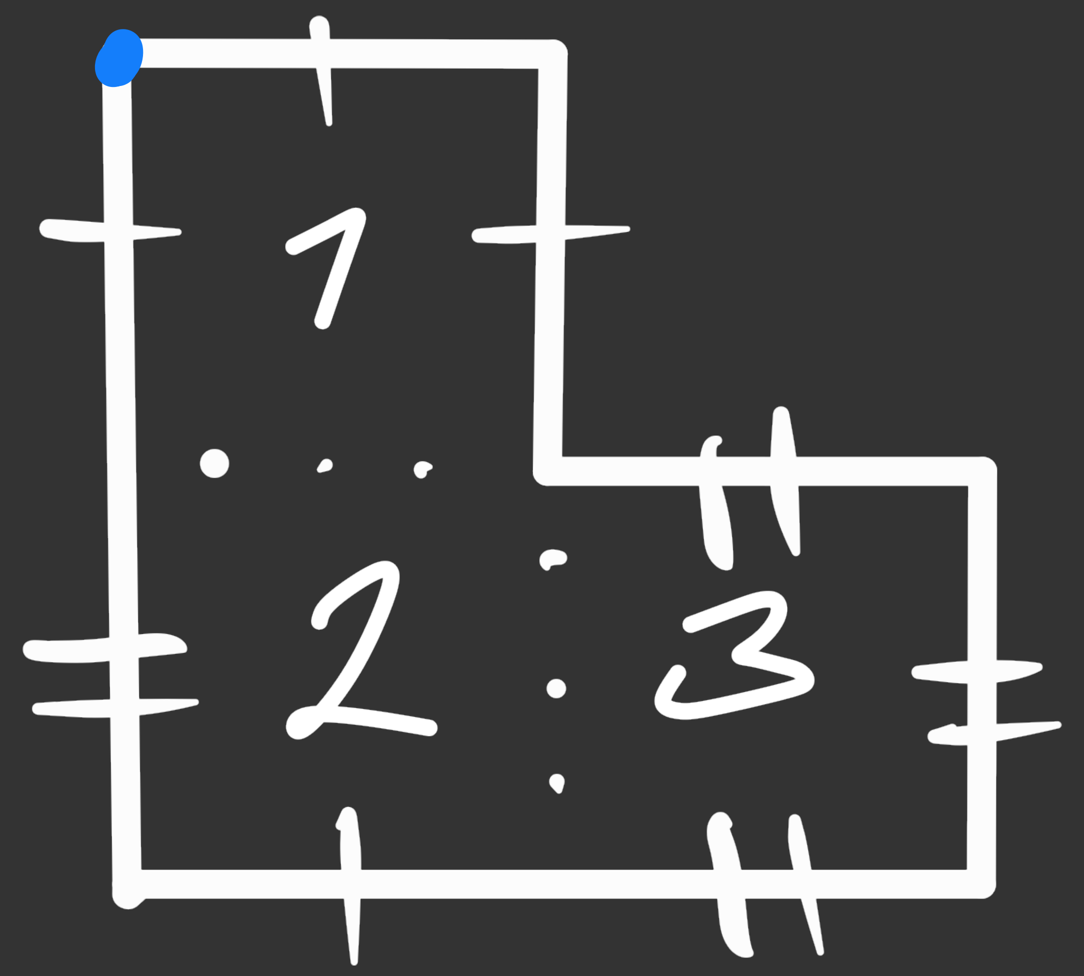

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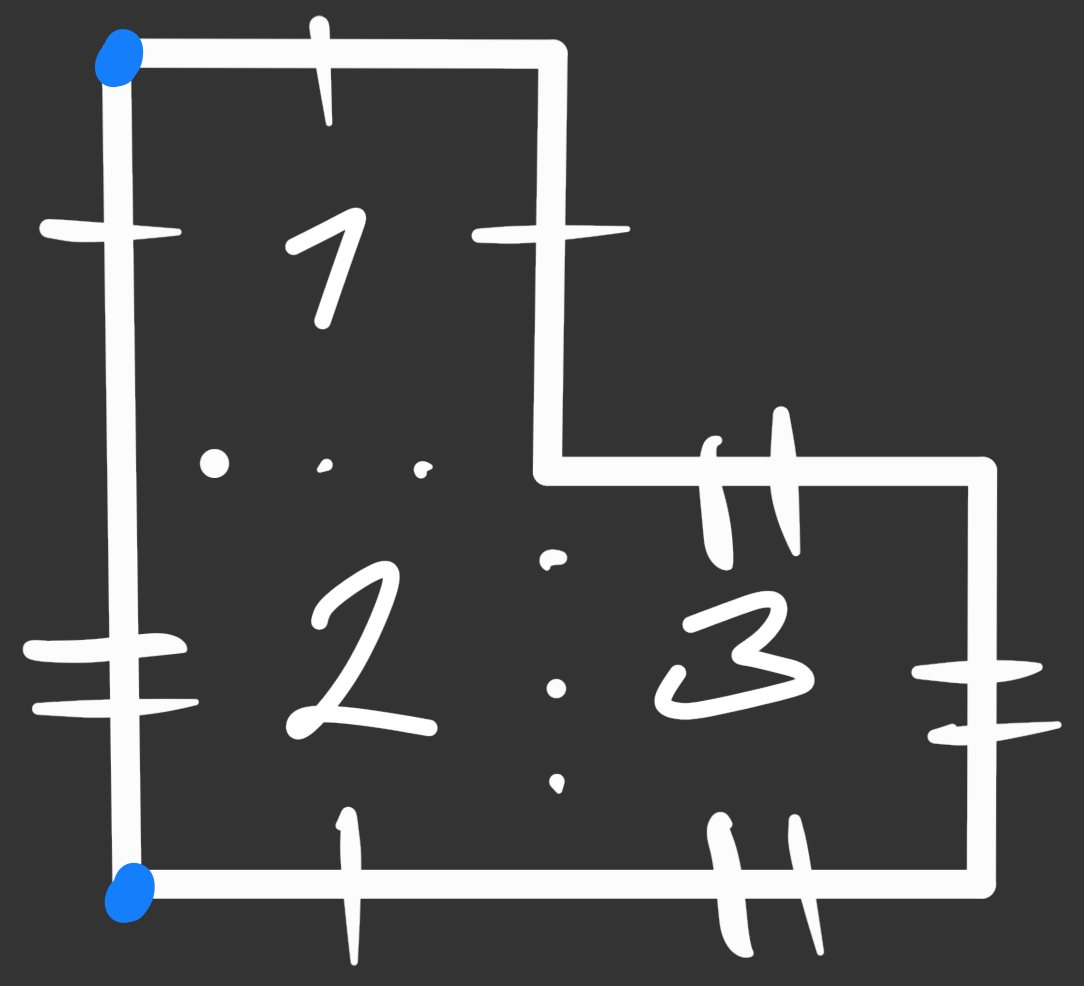

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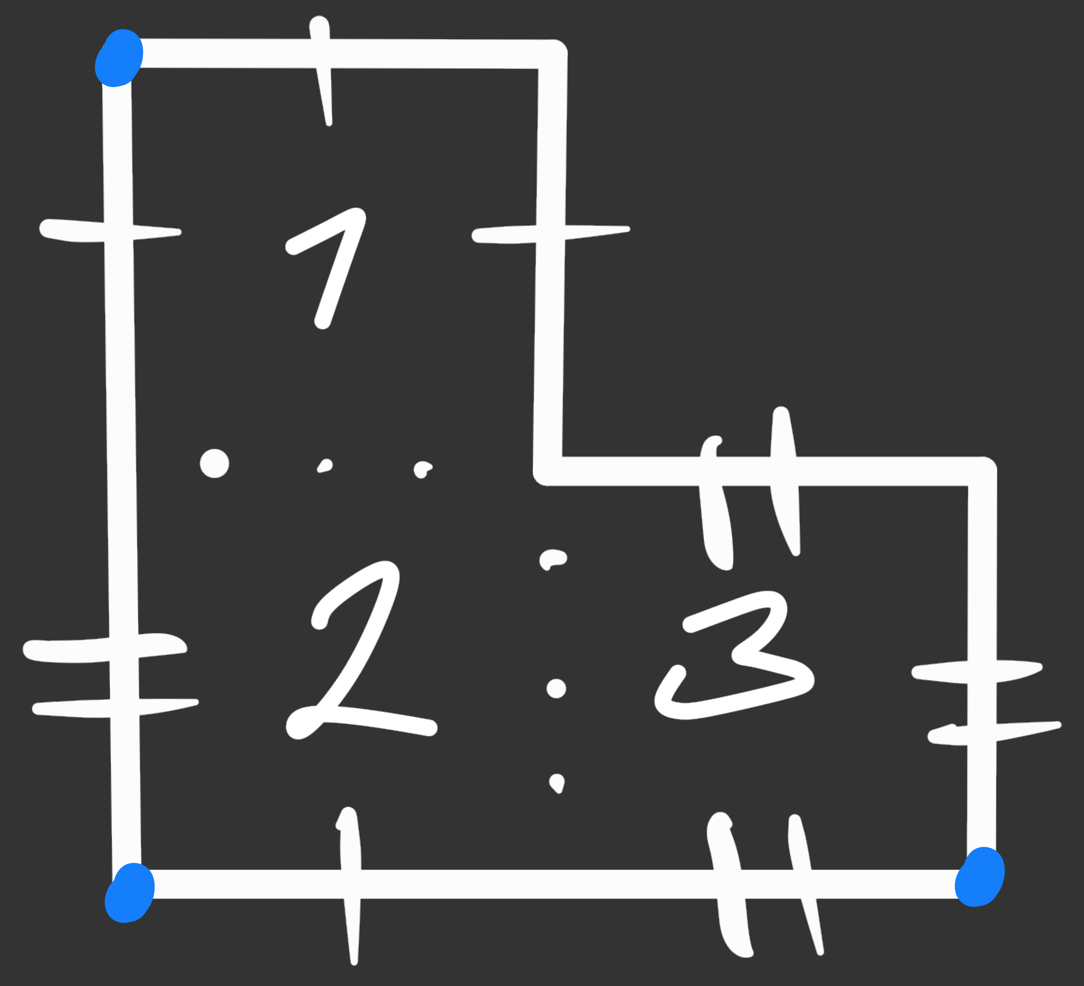

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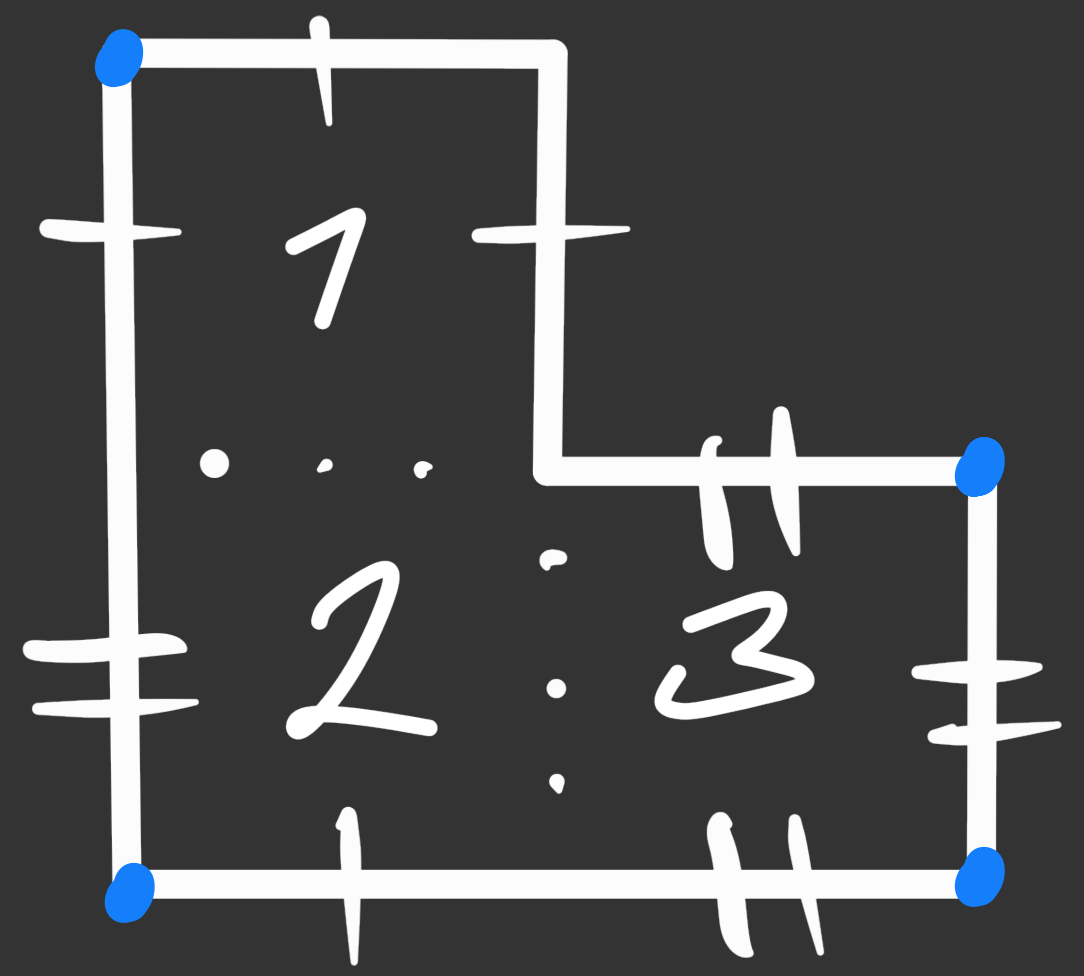

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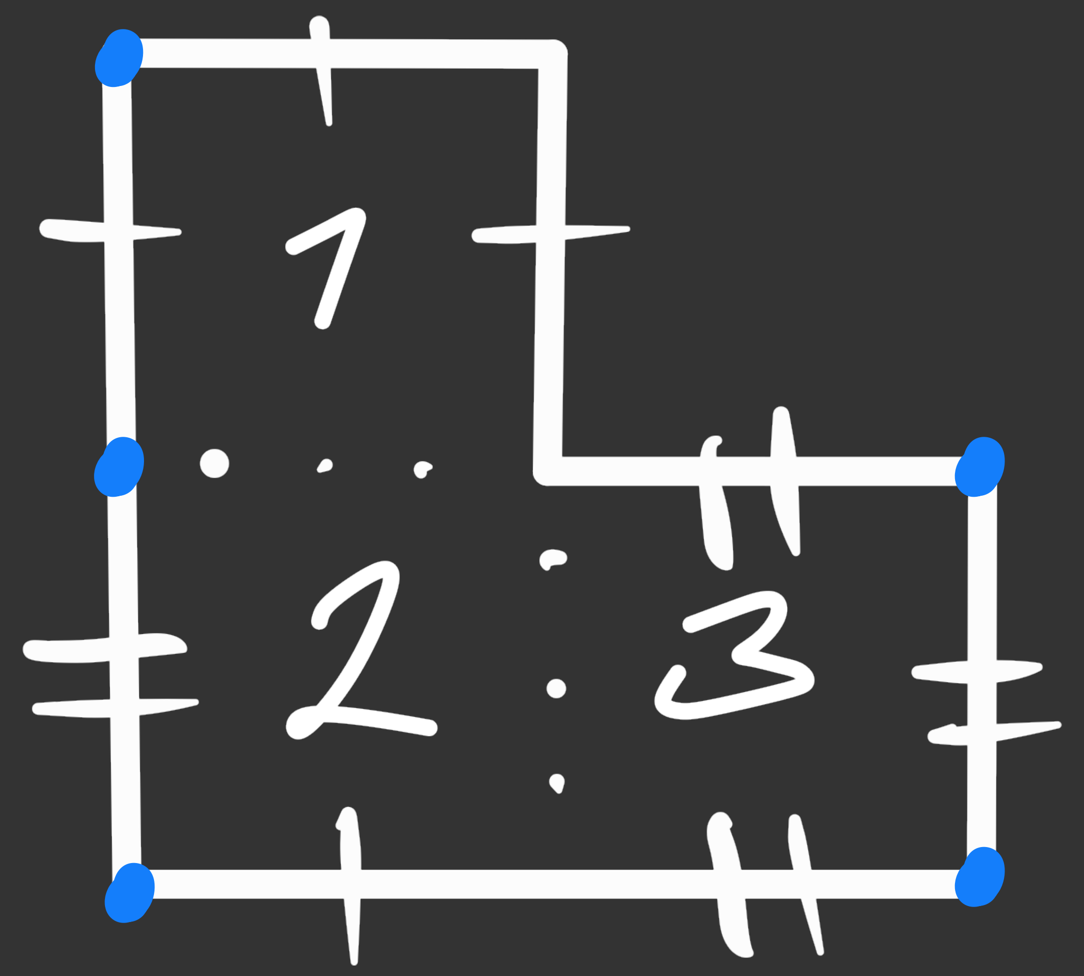

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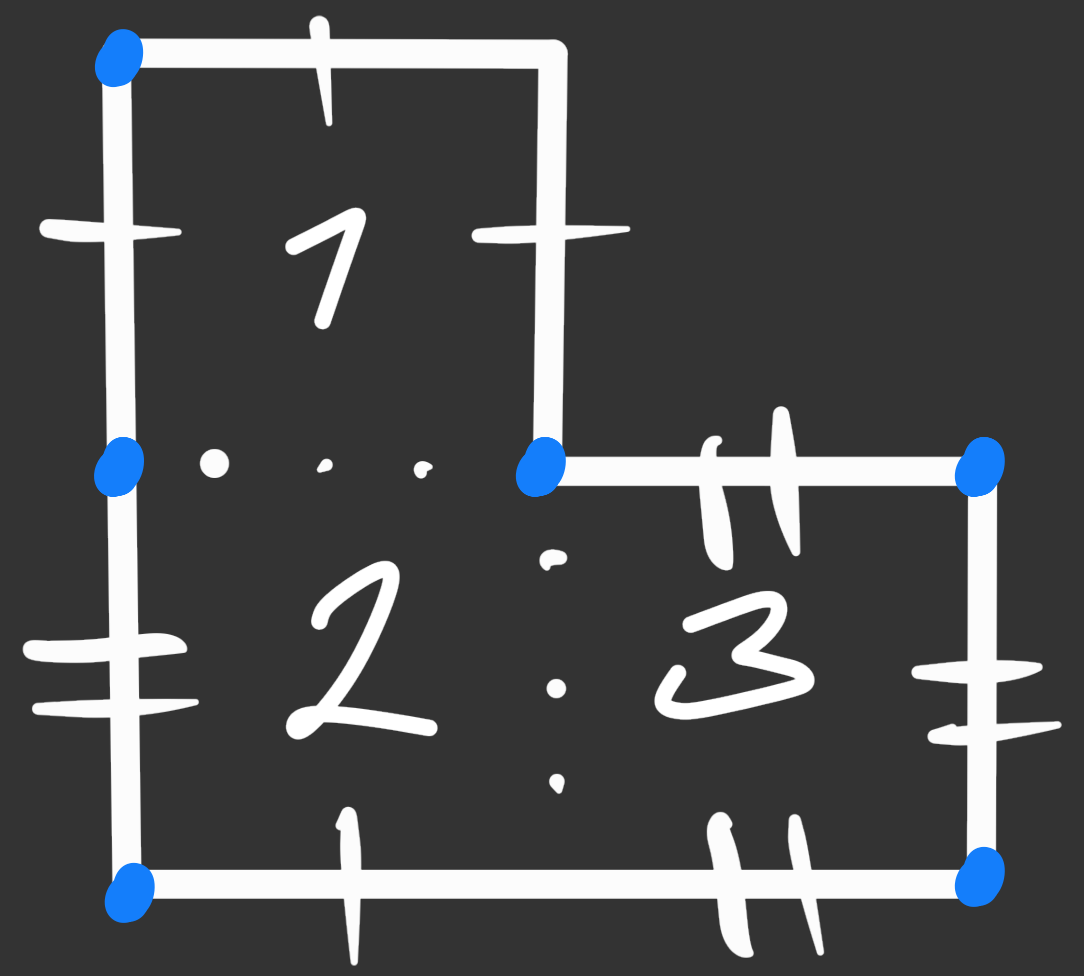

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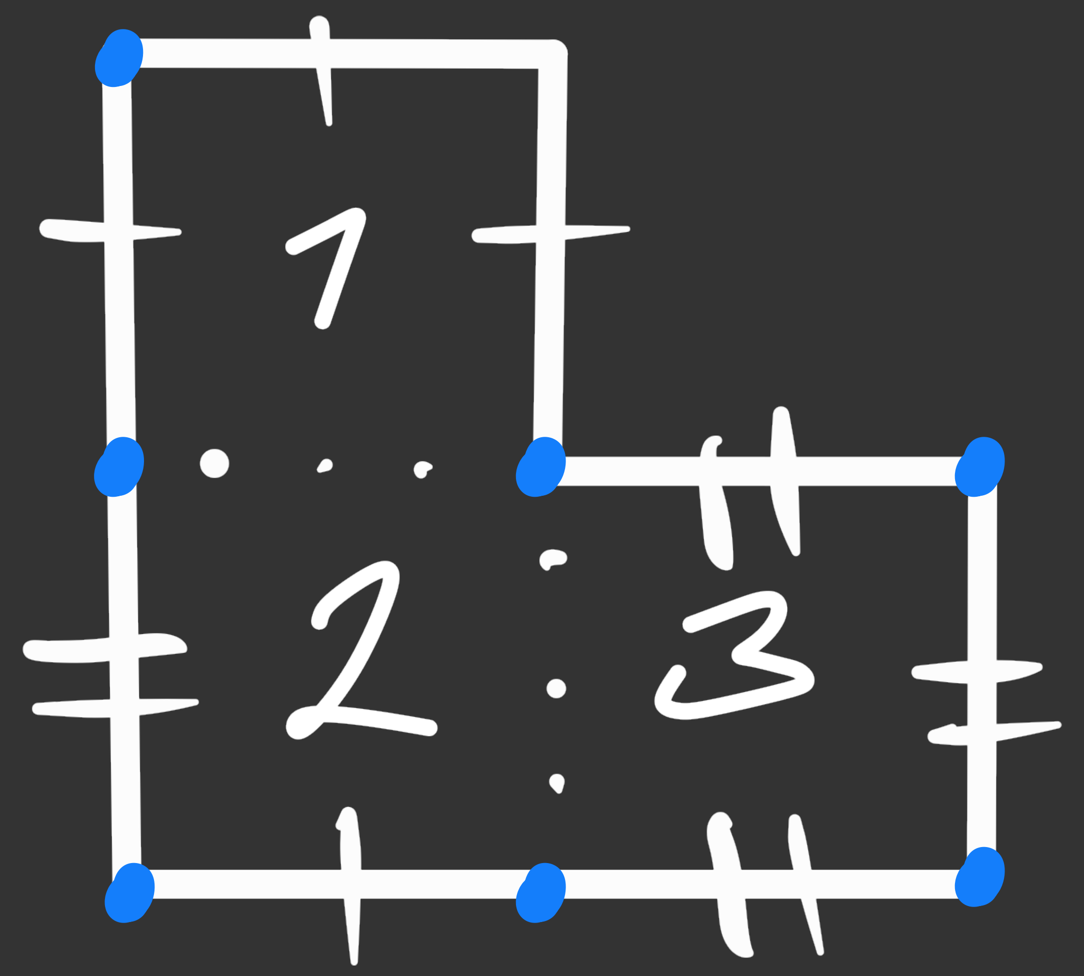

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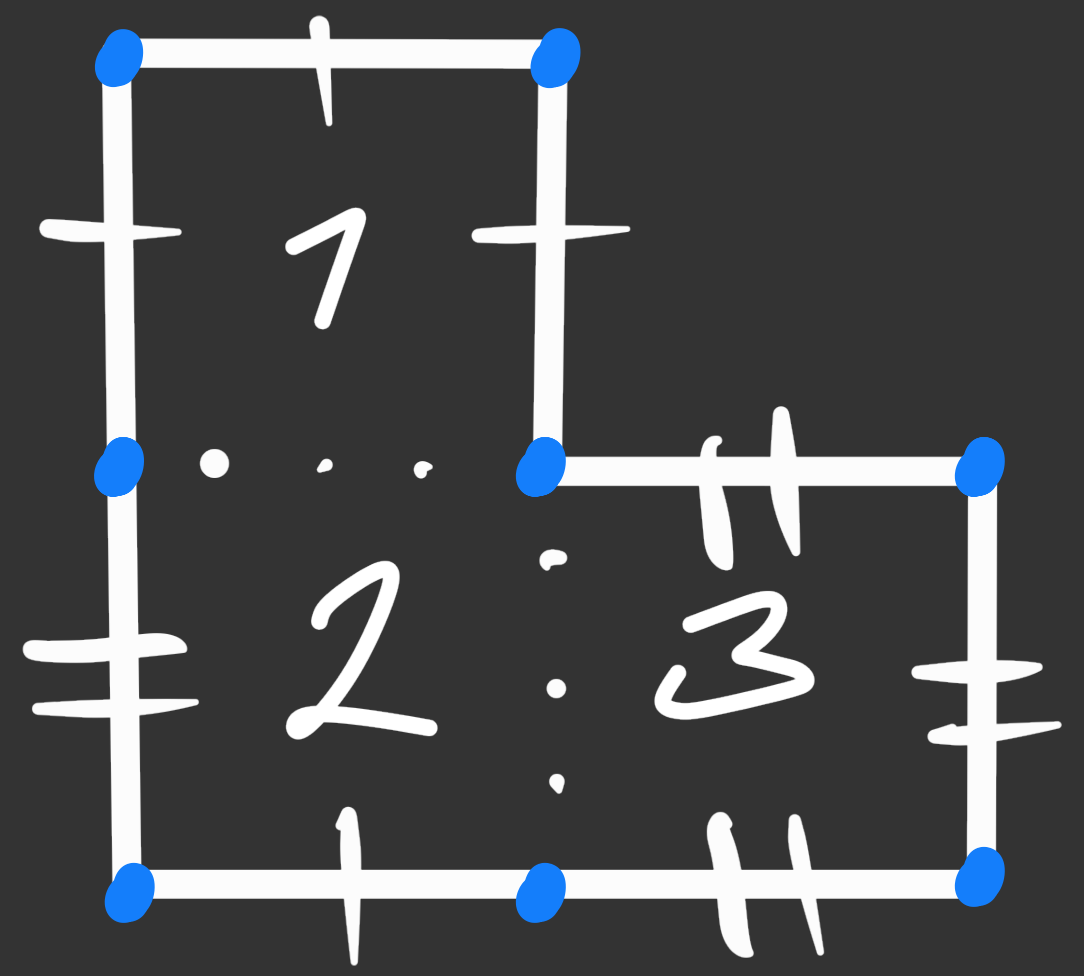

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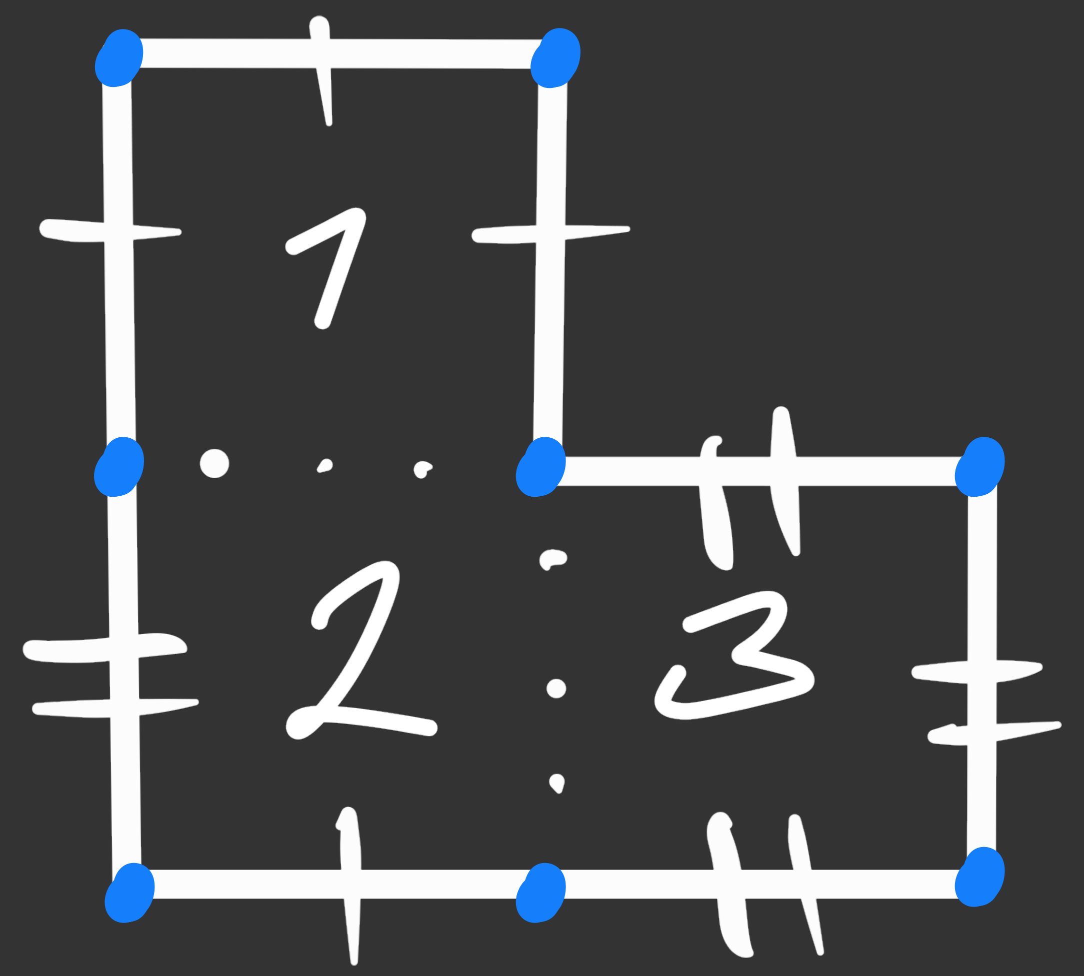

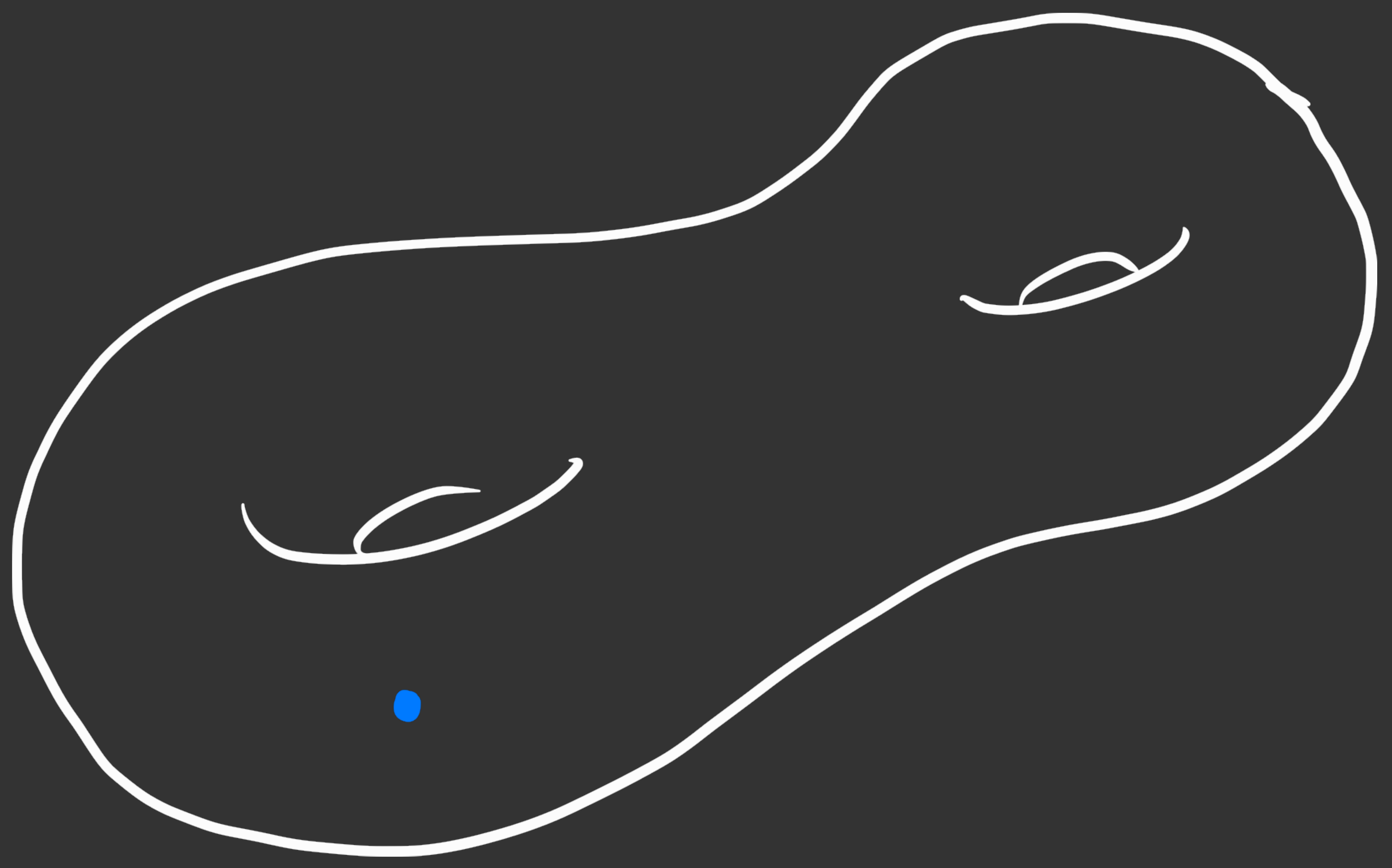

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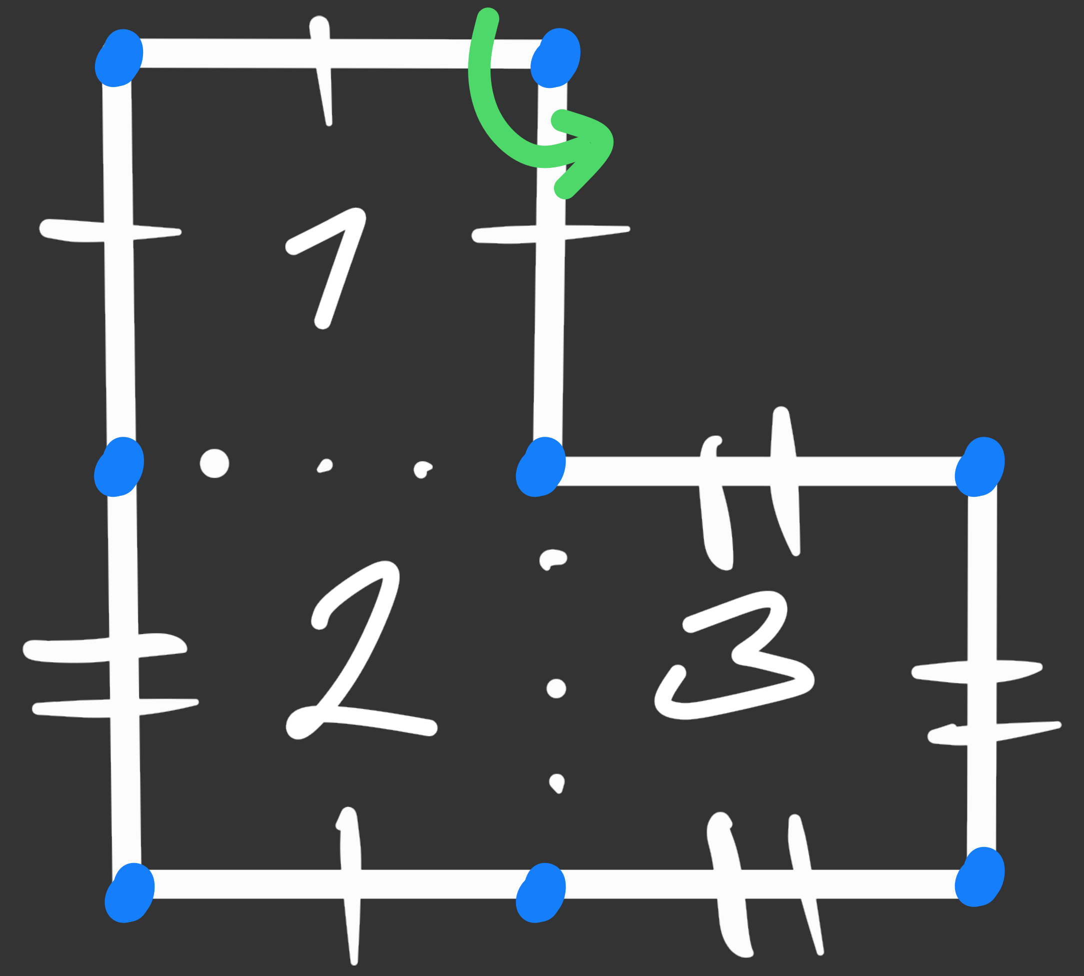

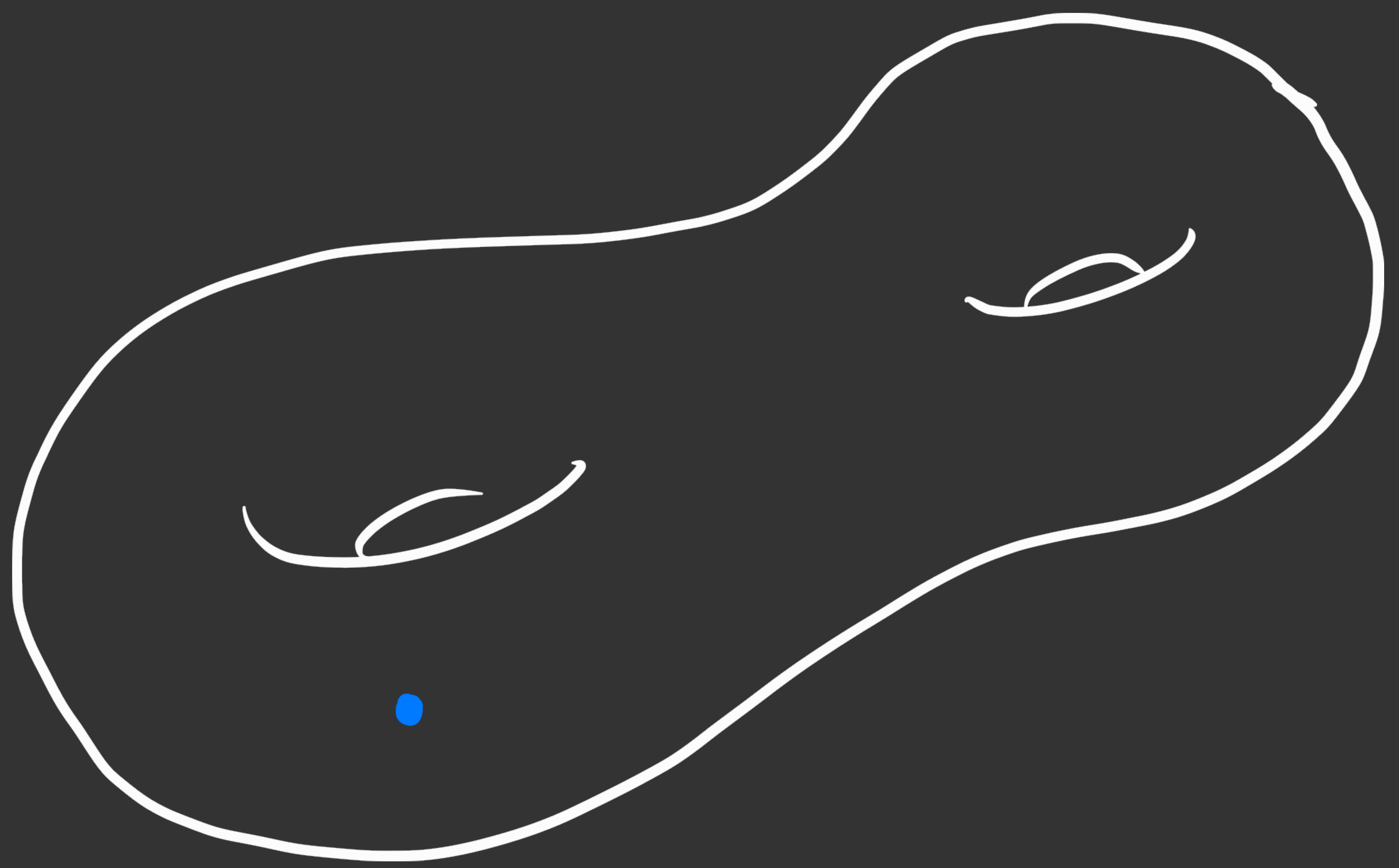

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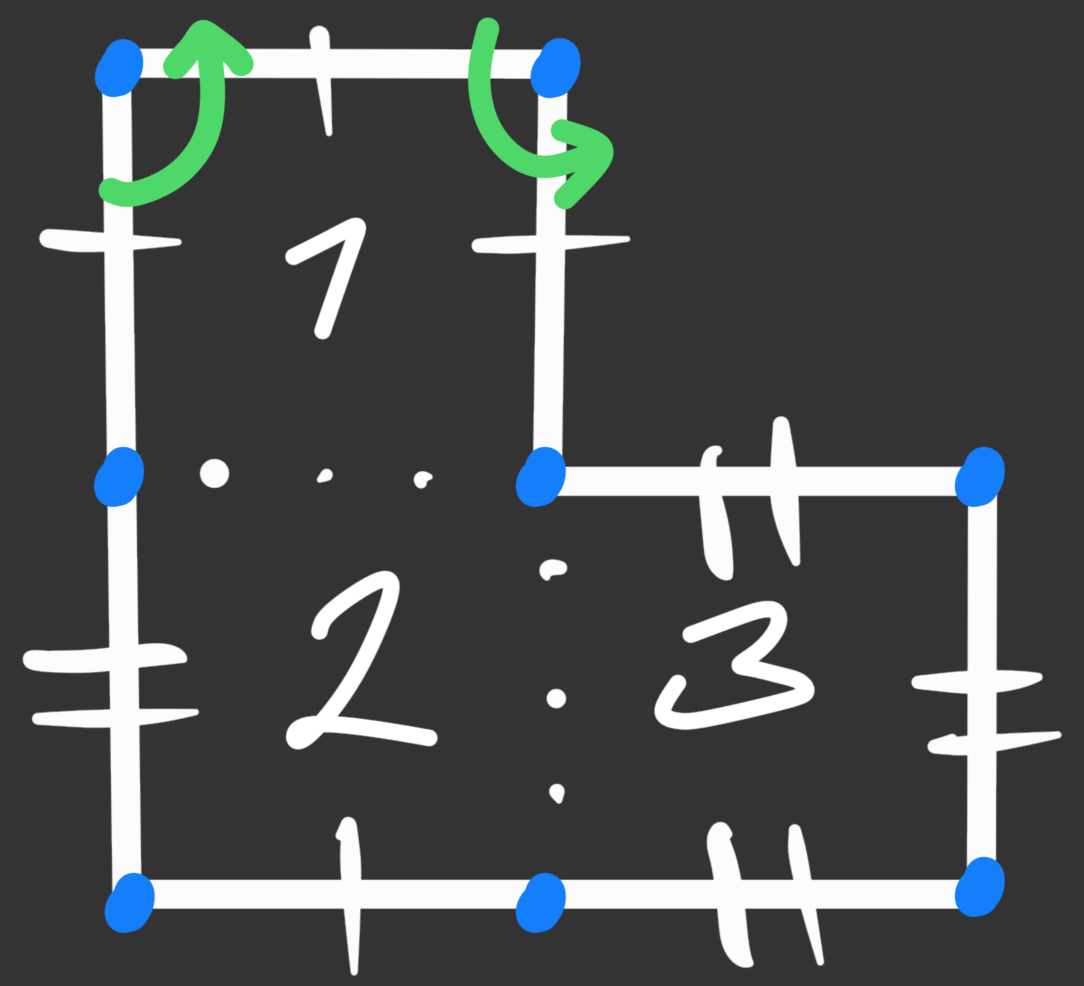

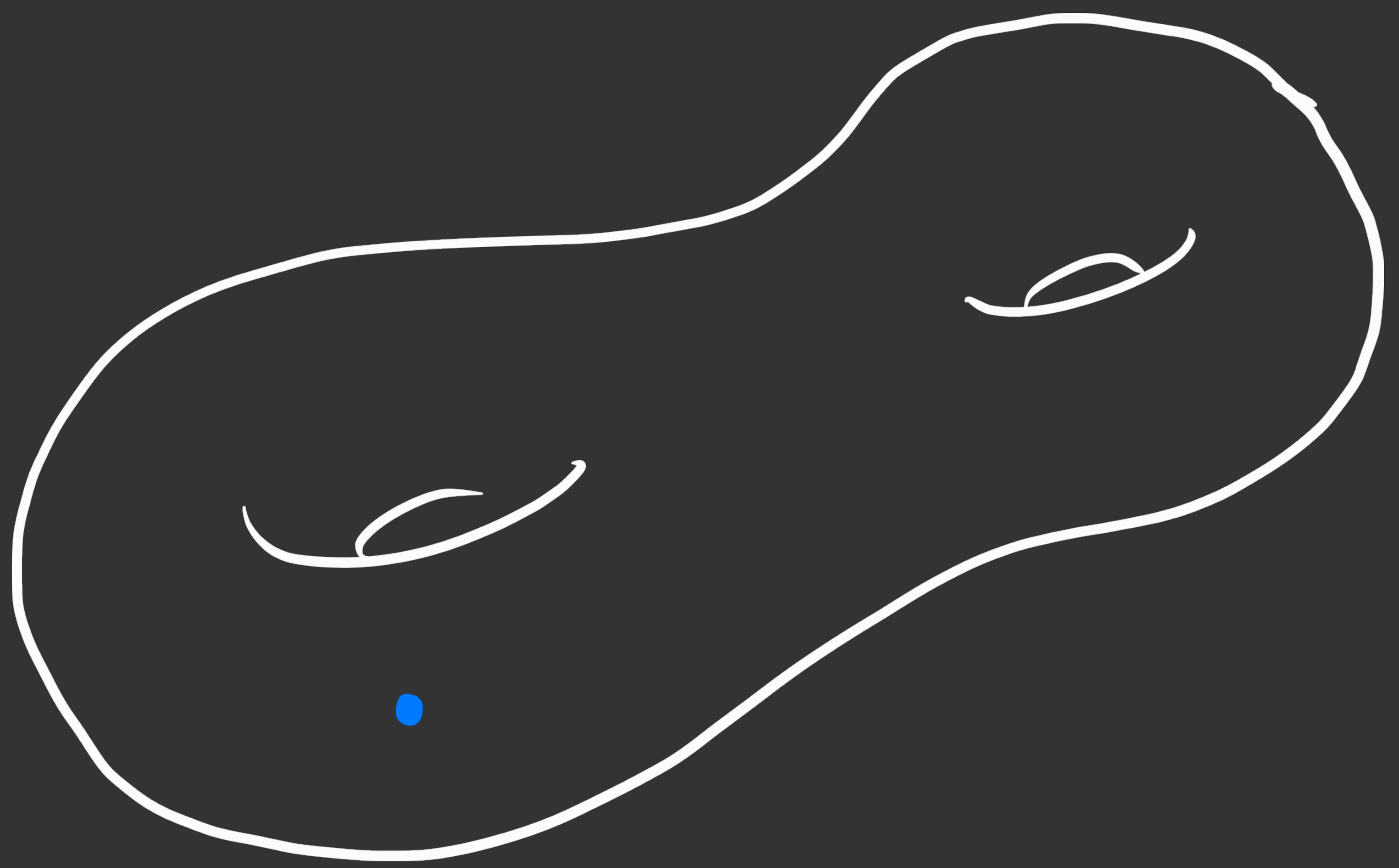

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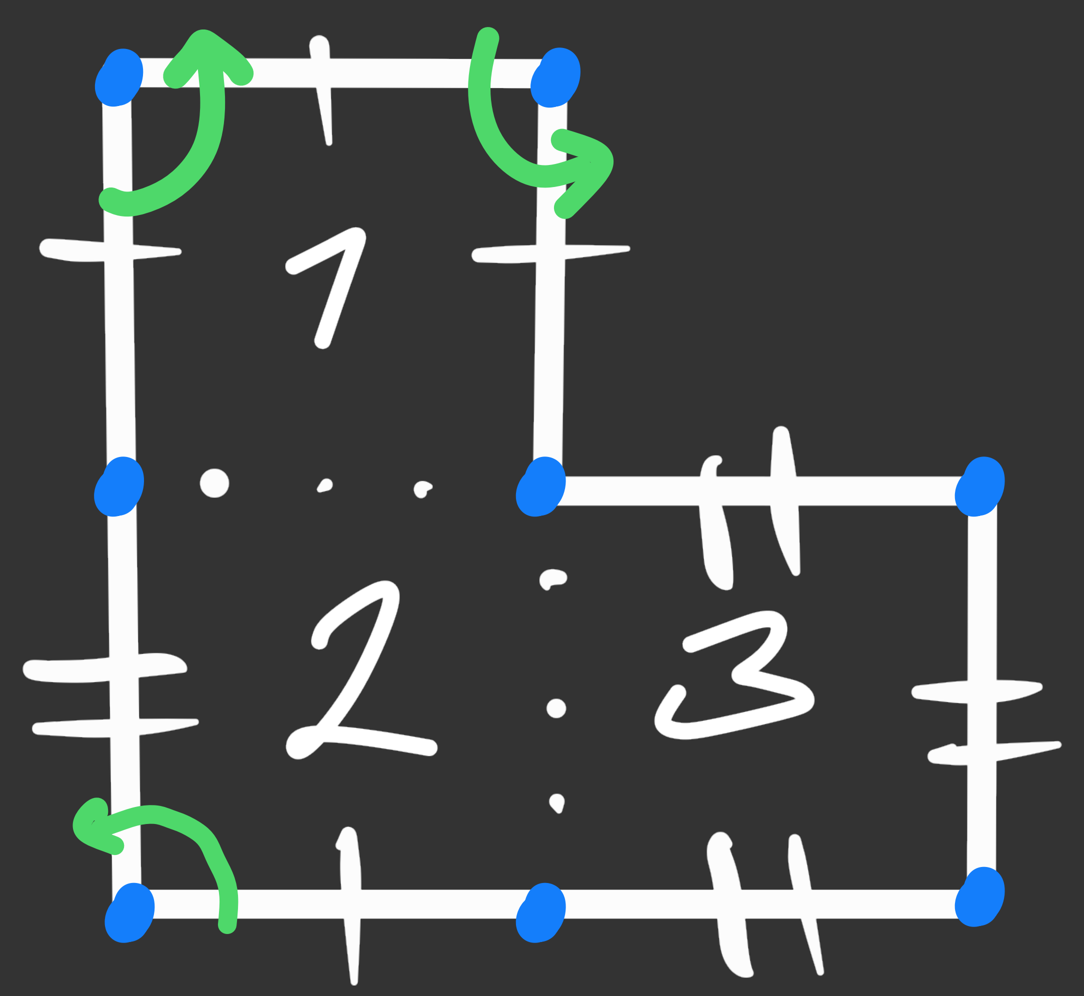

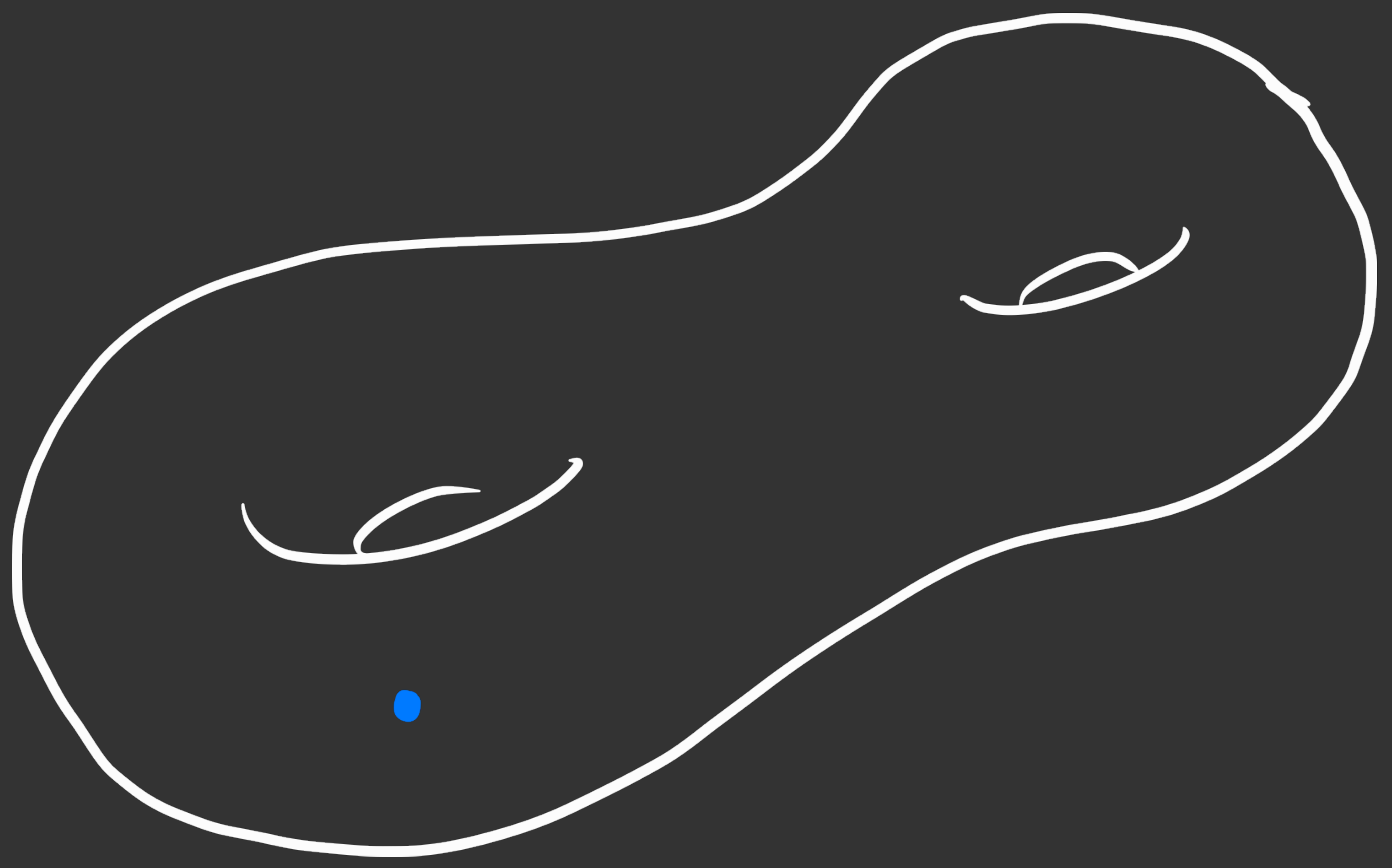

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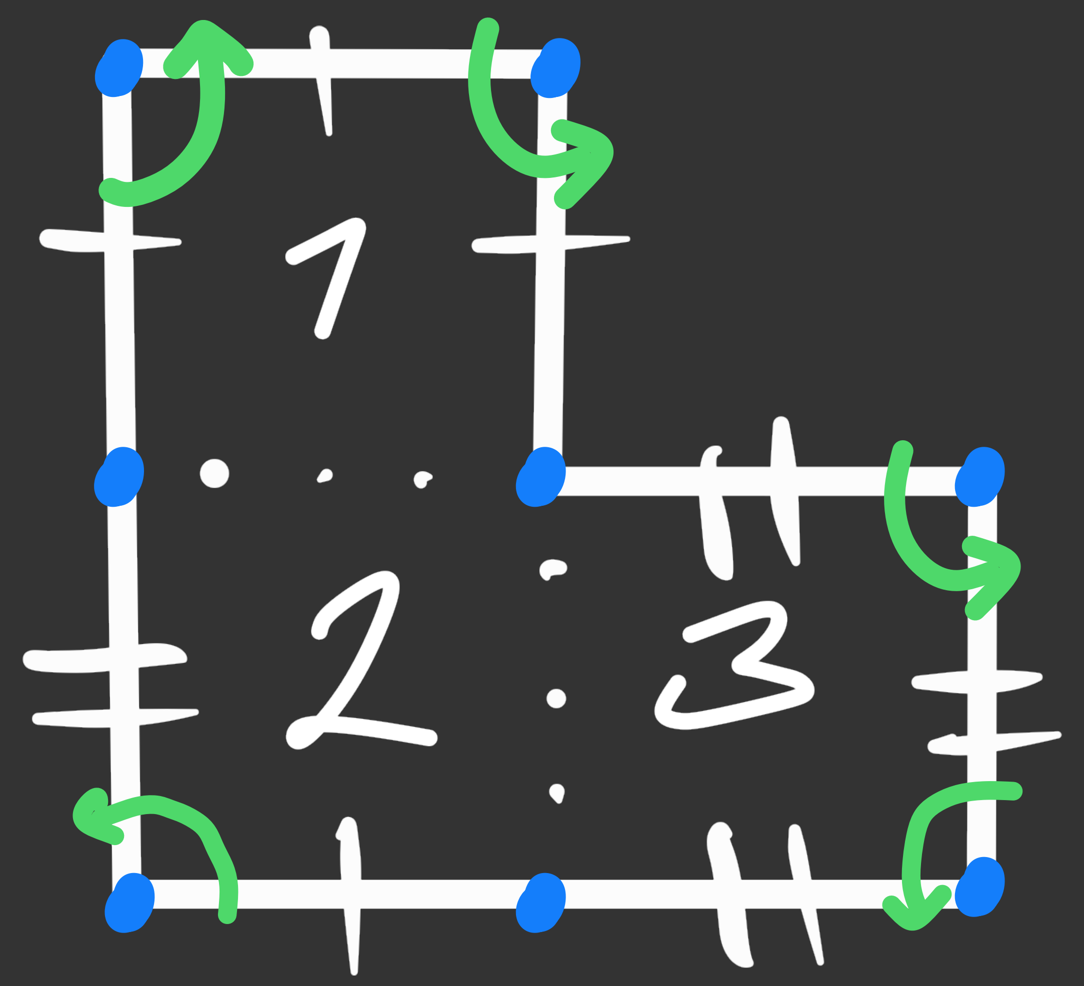

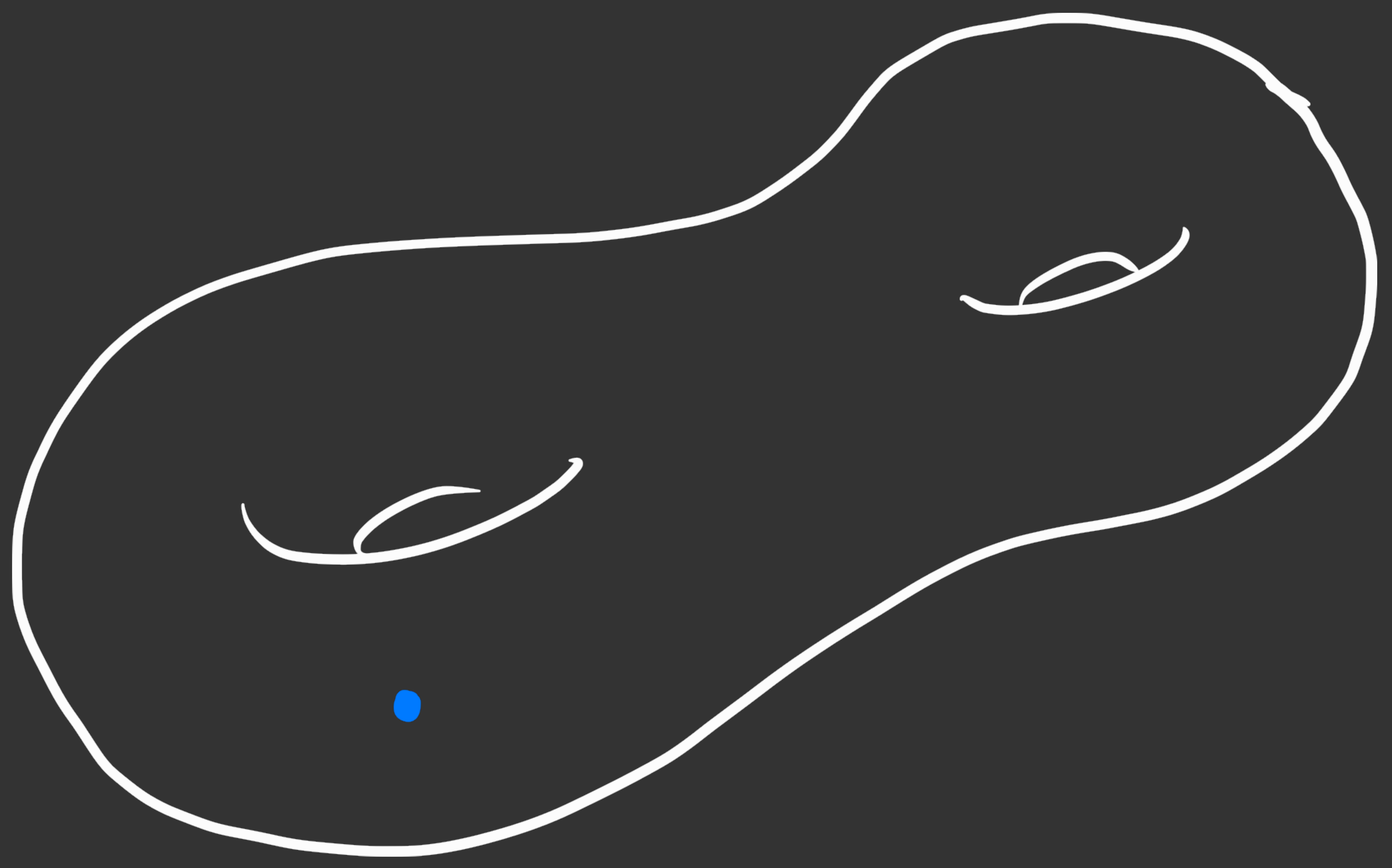

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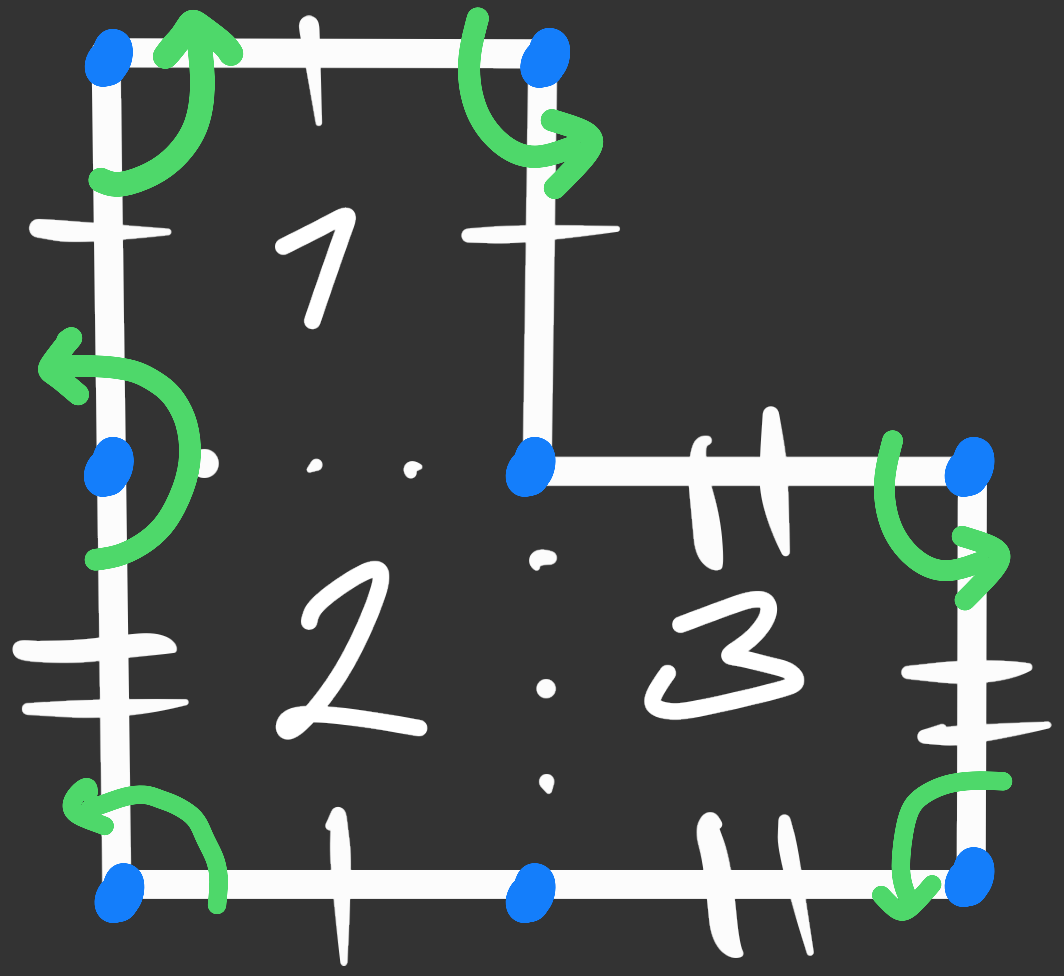

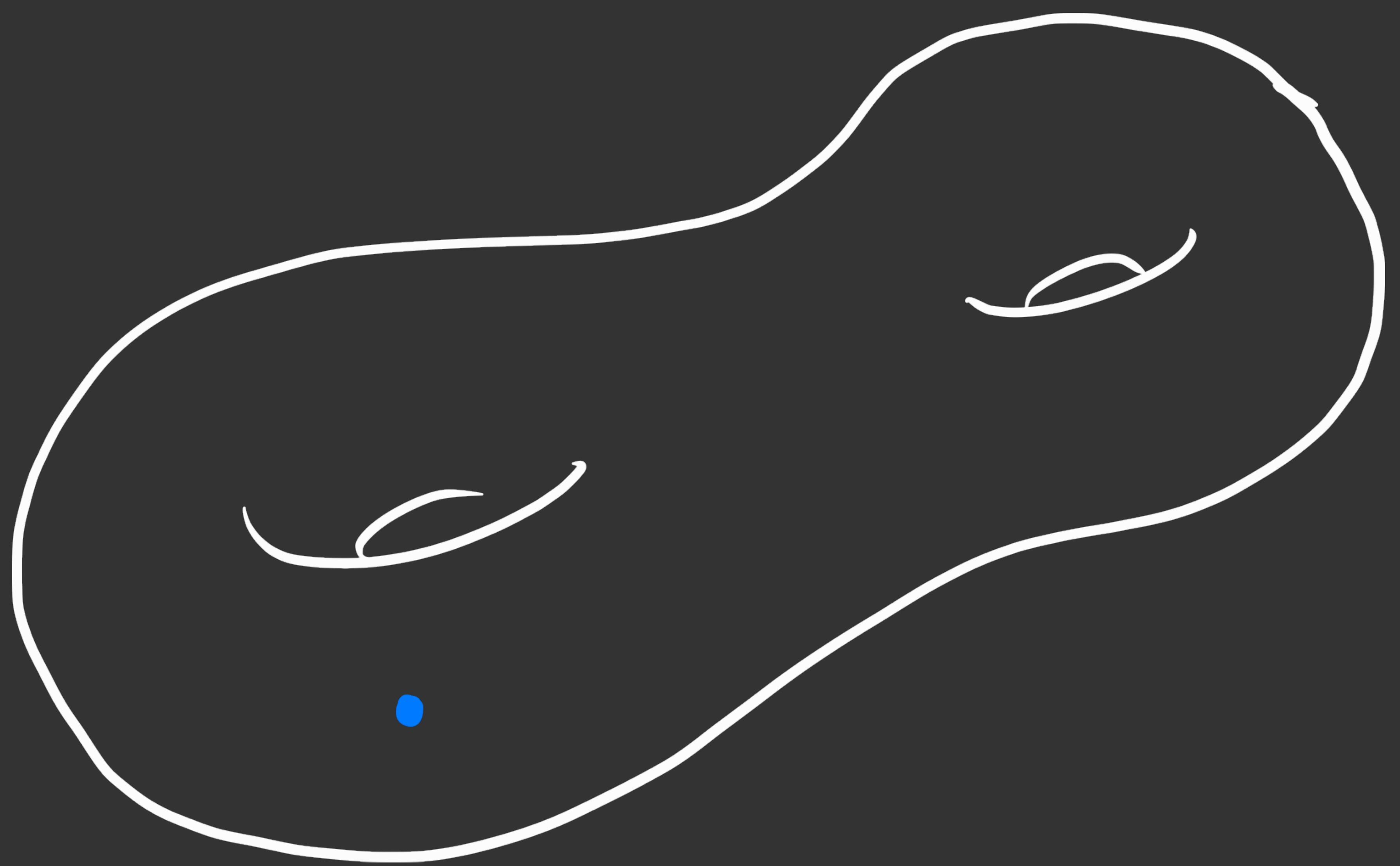

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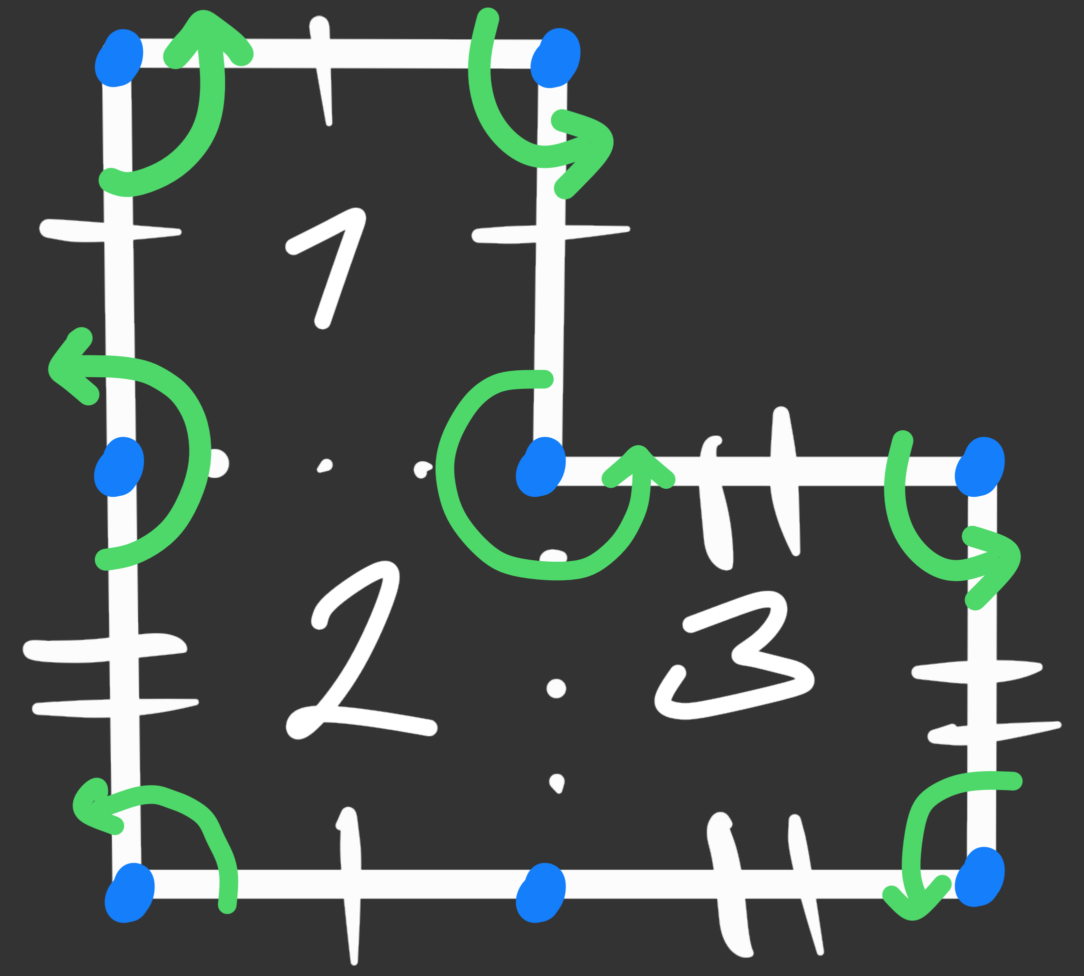

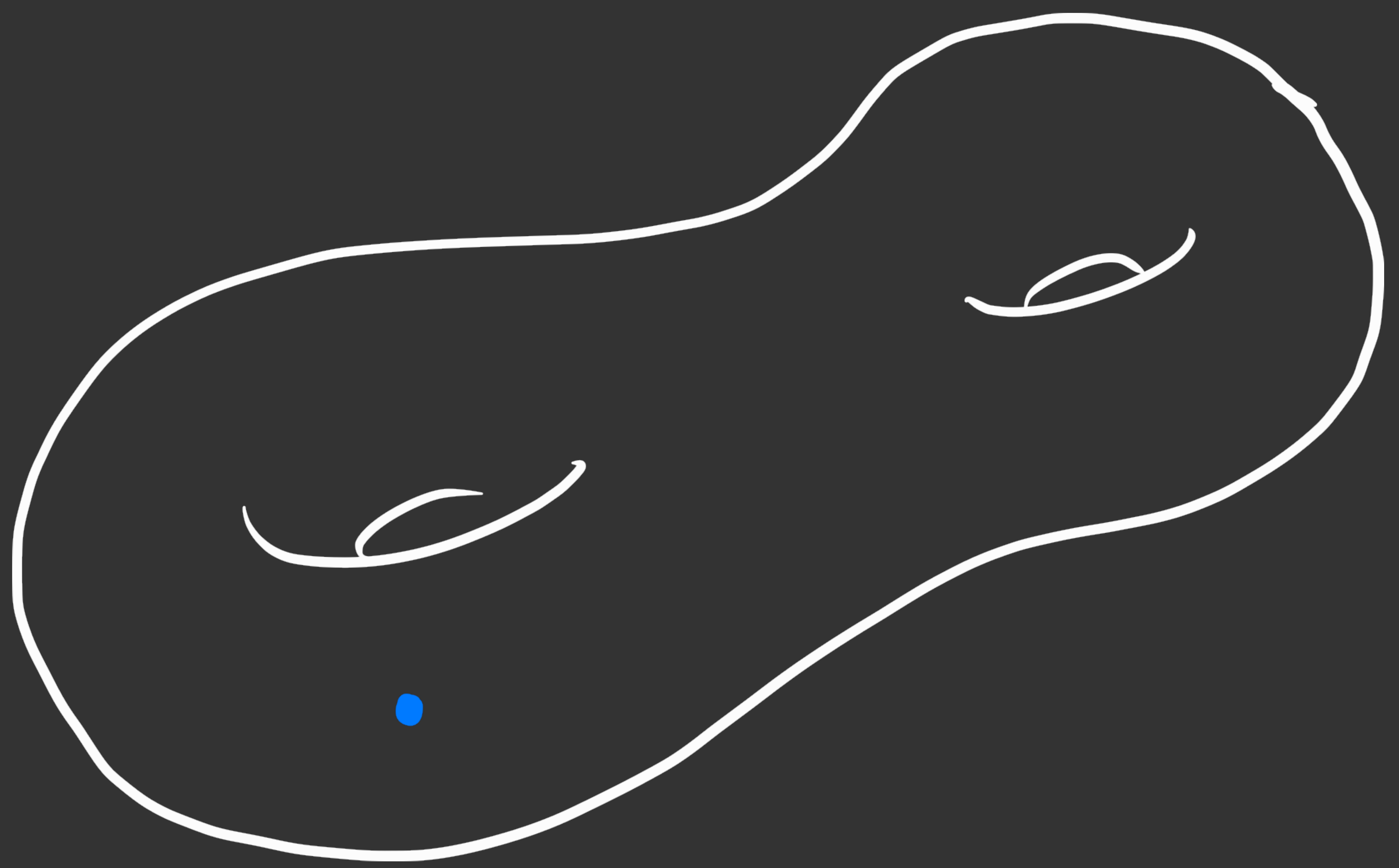

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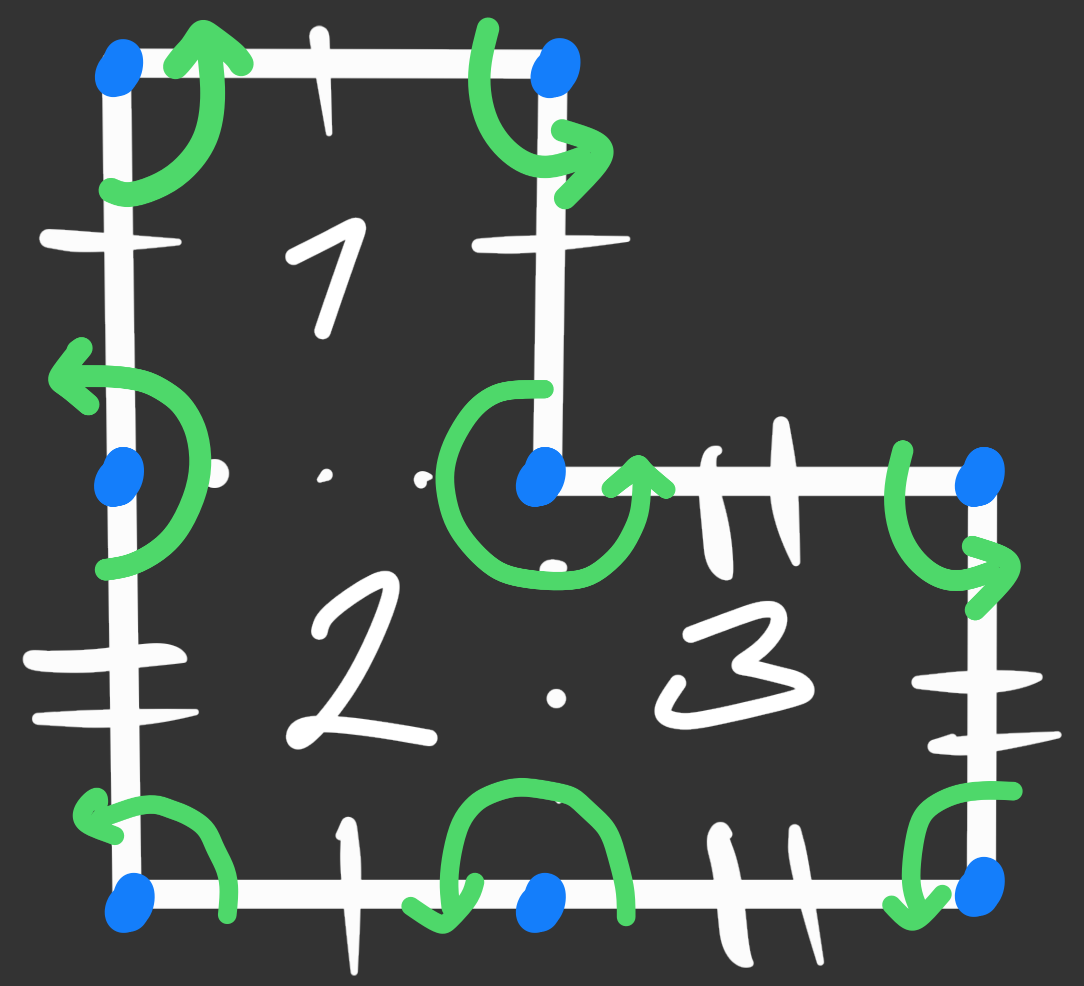

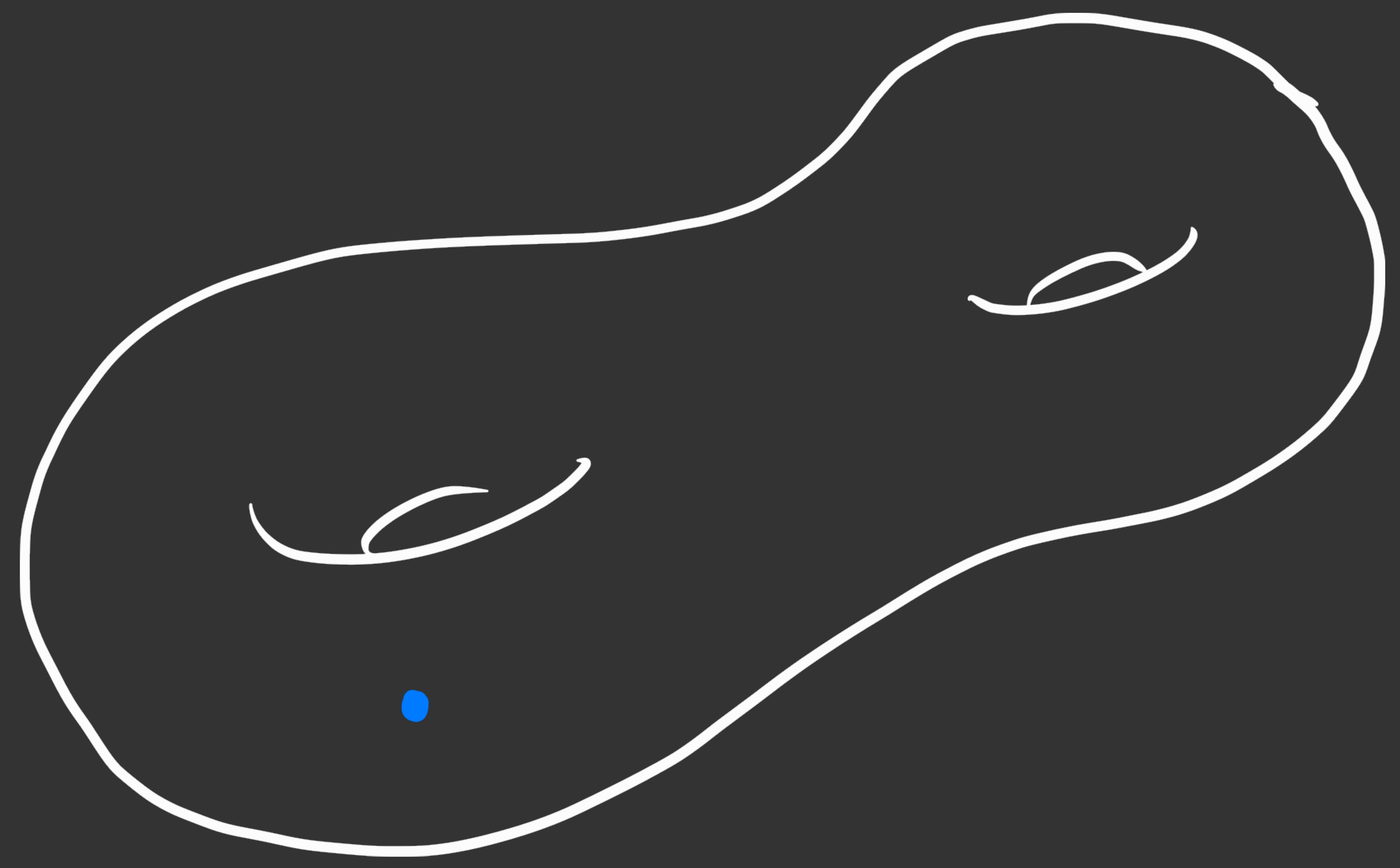

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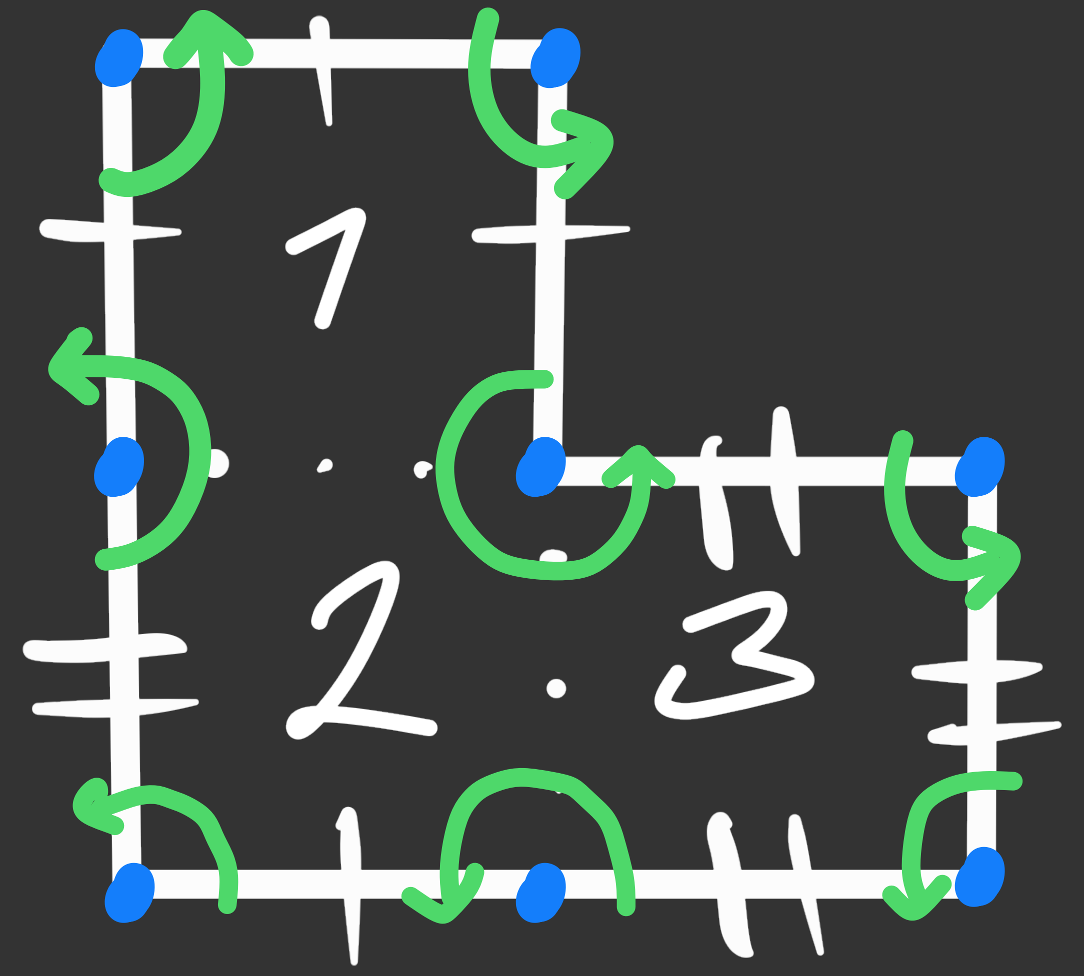

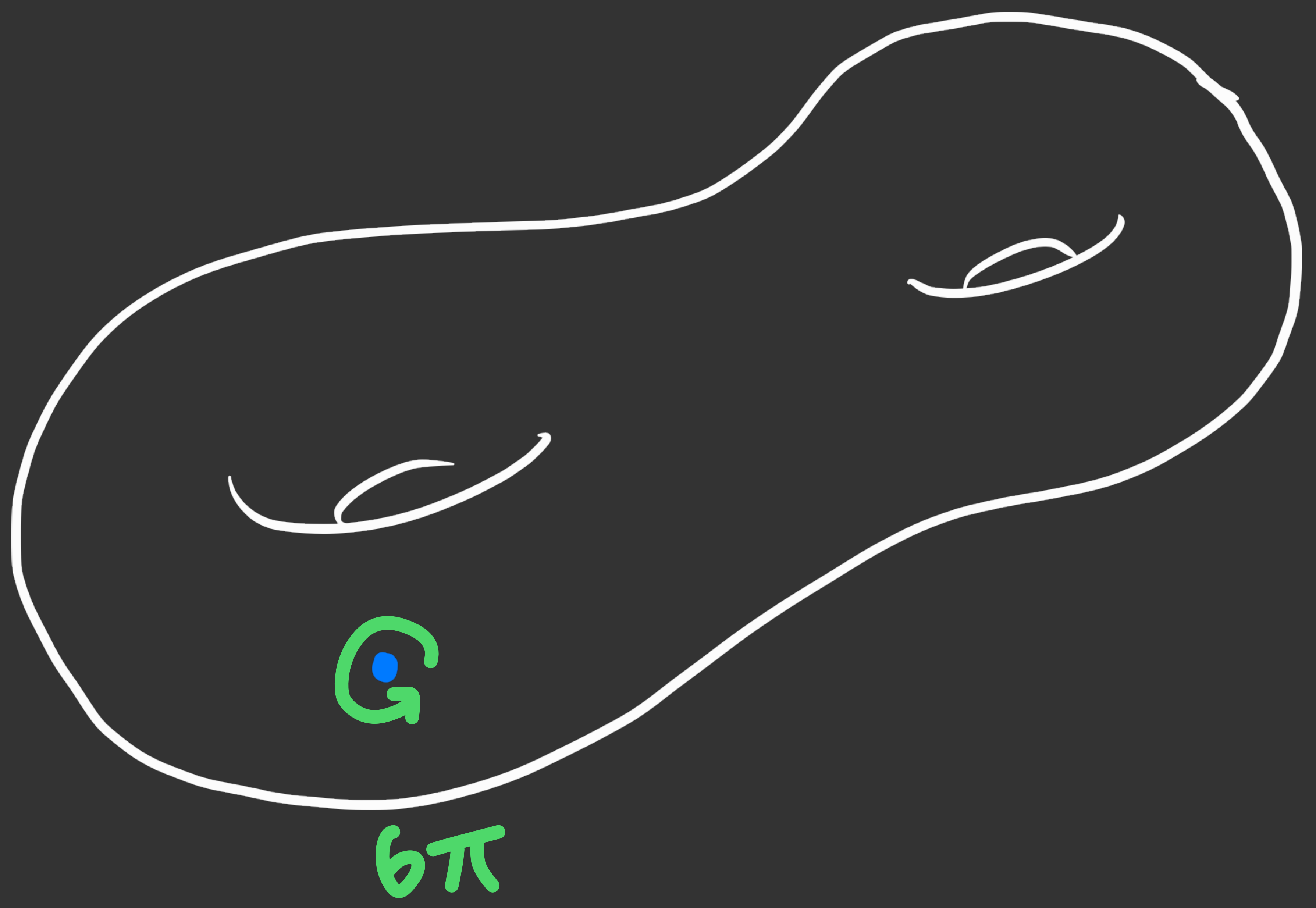

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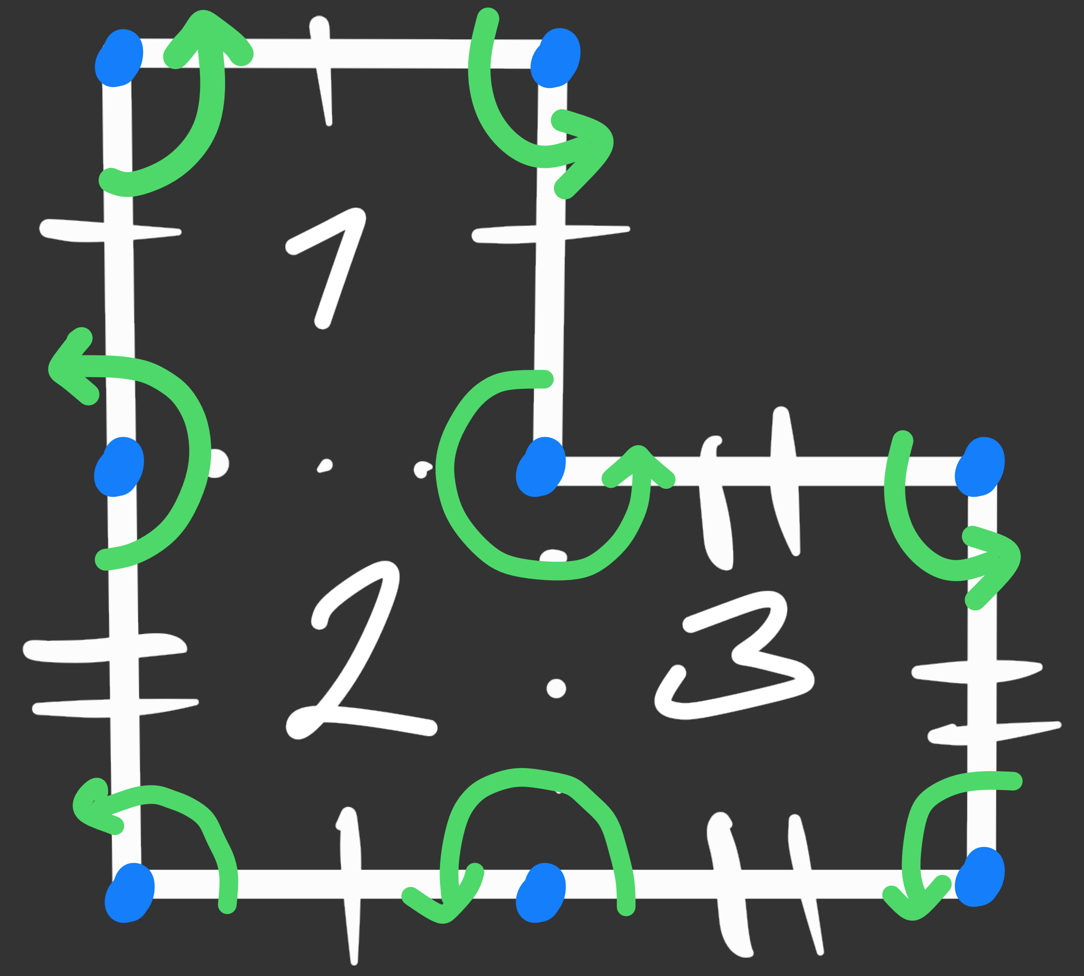

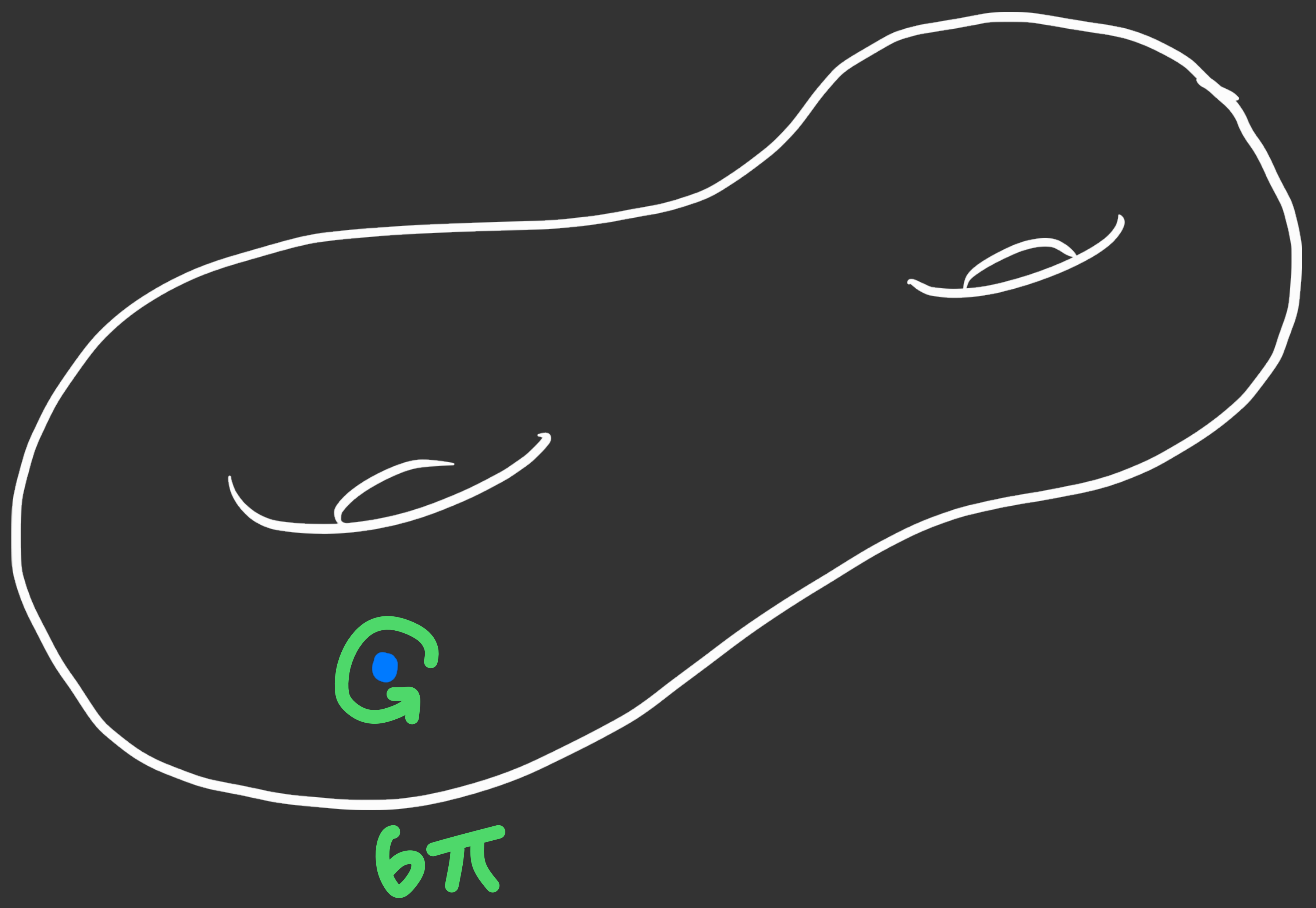

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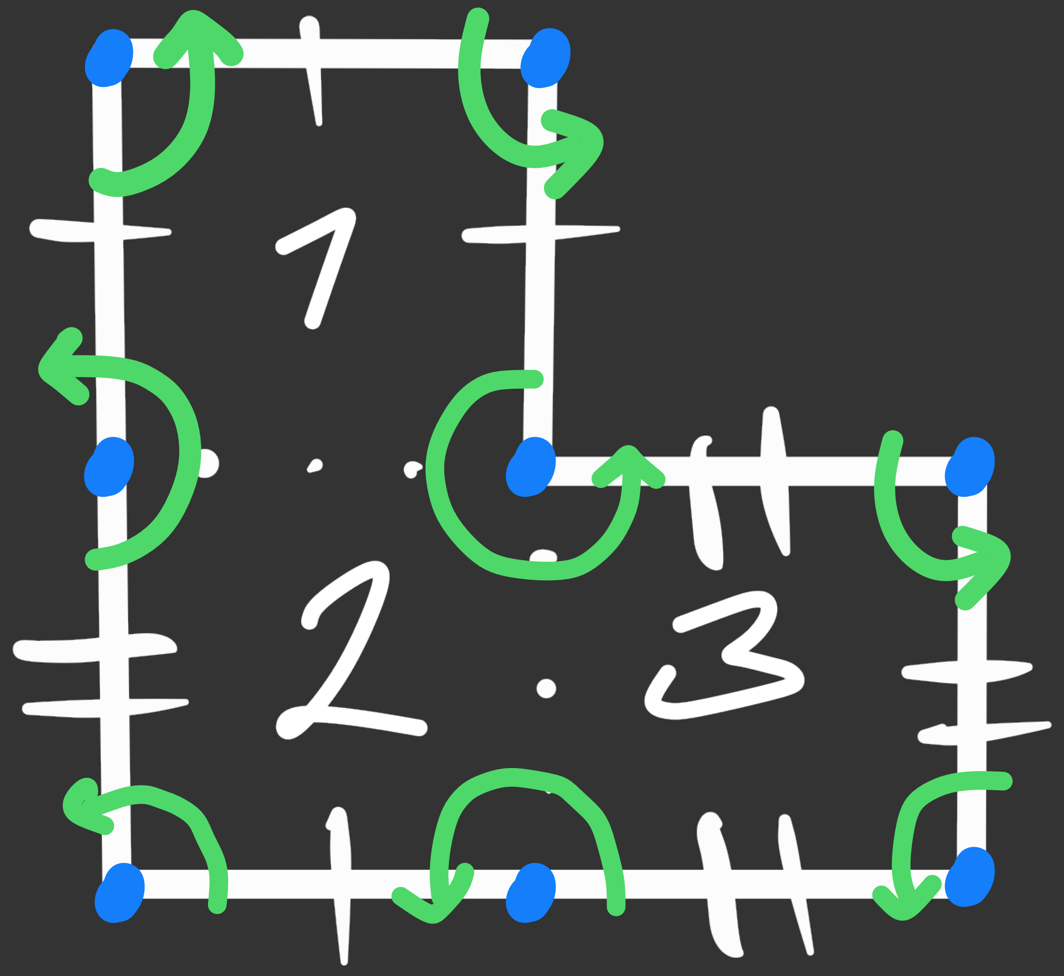

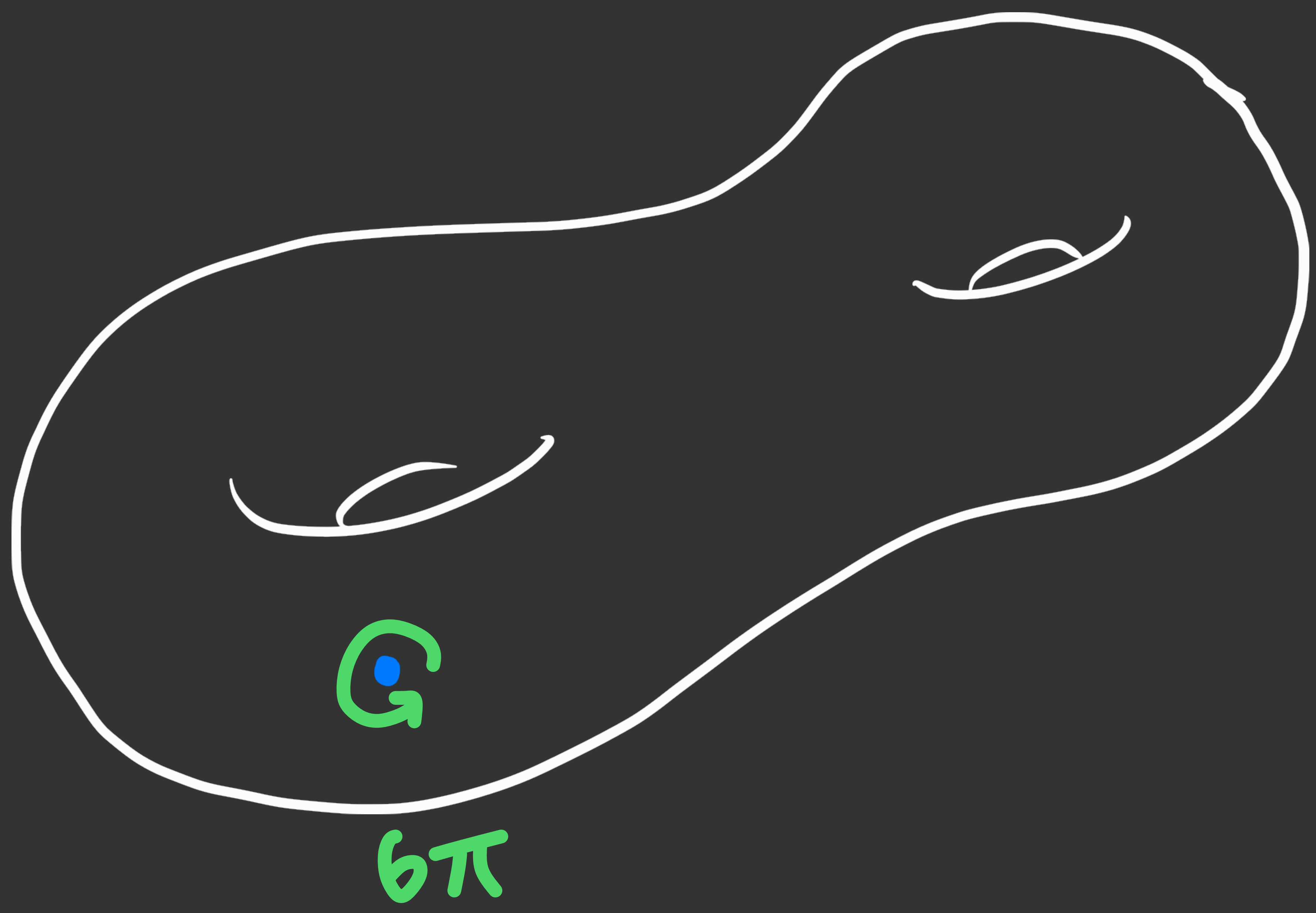

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In the 'stratum' $\mathcal{H}(2)$.

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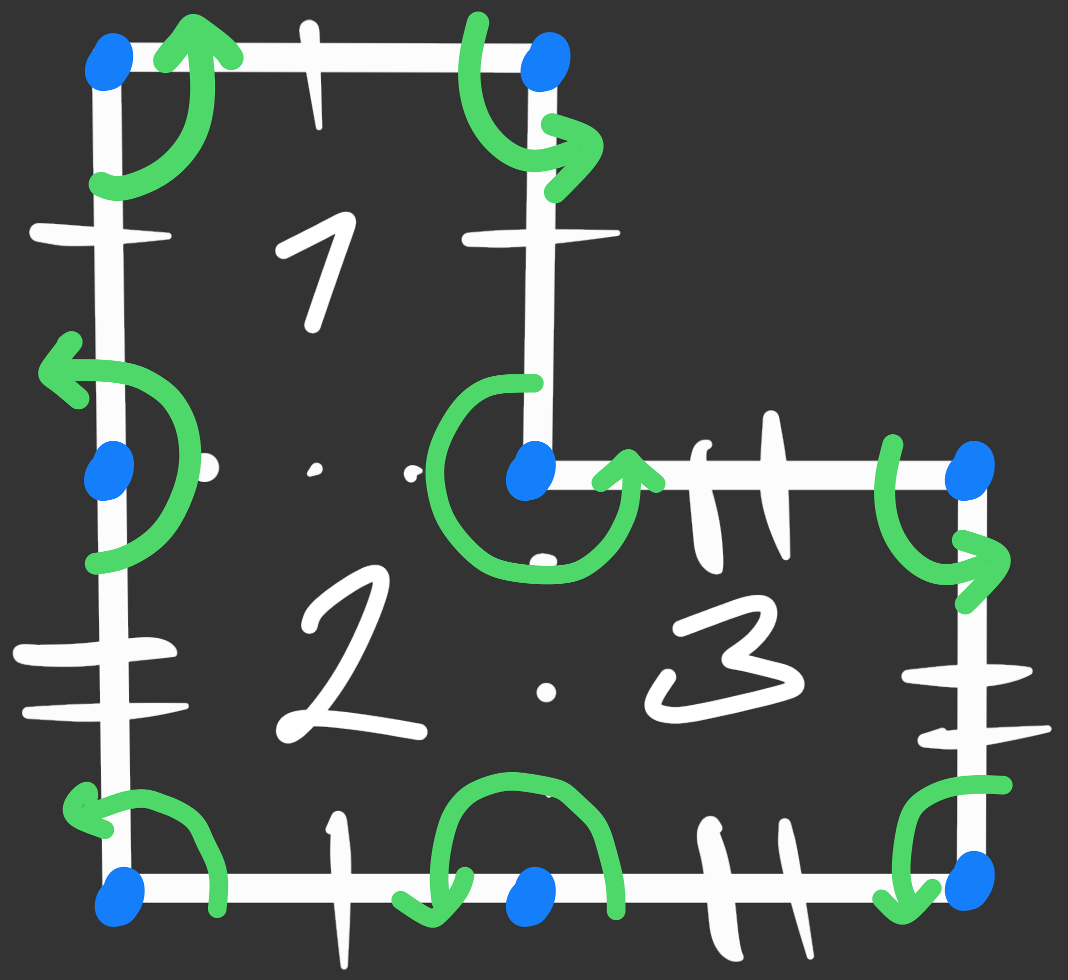

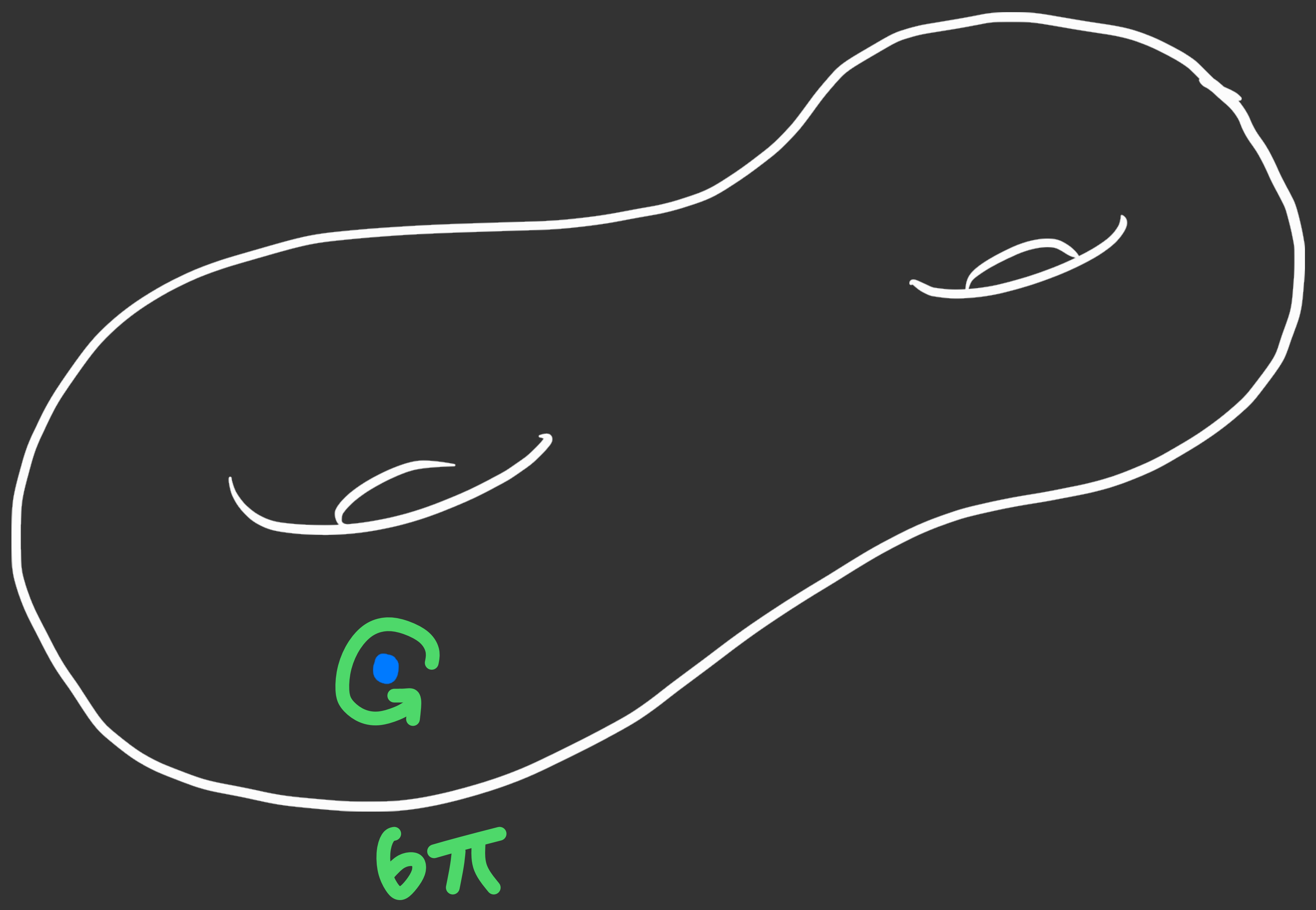

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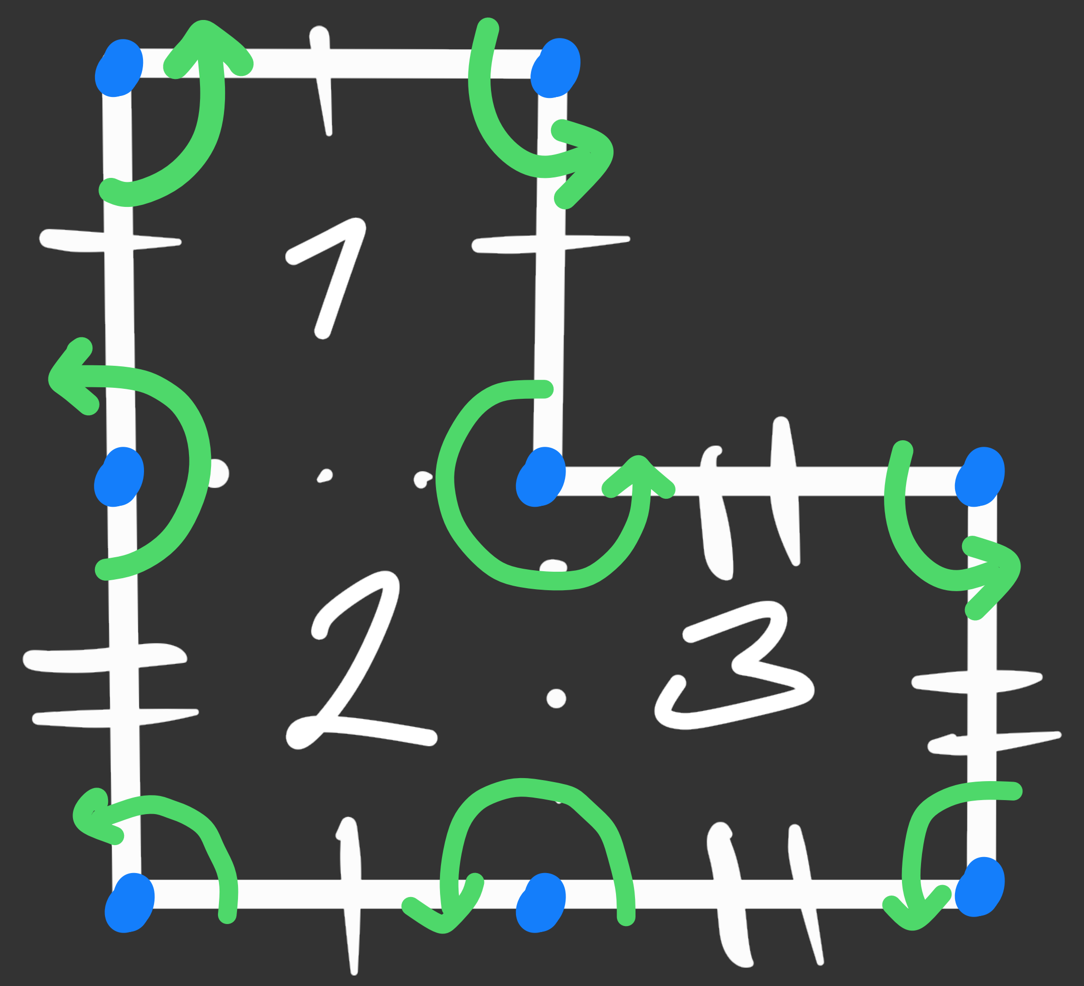

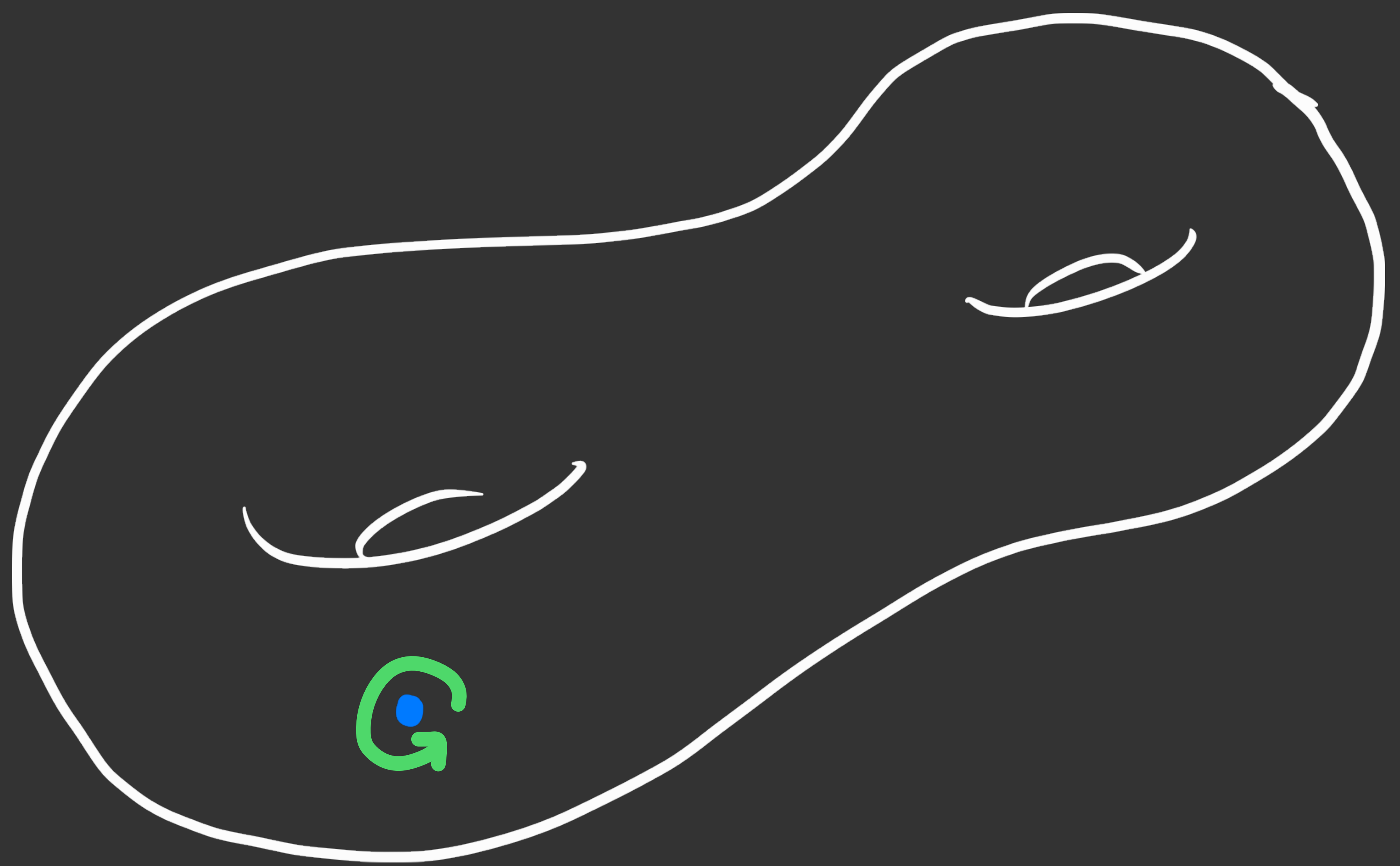

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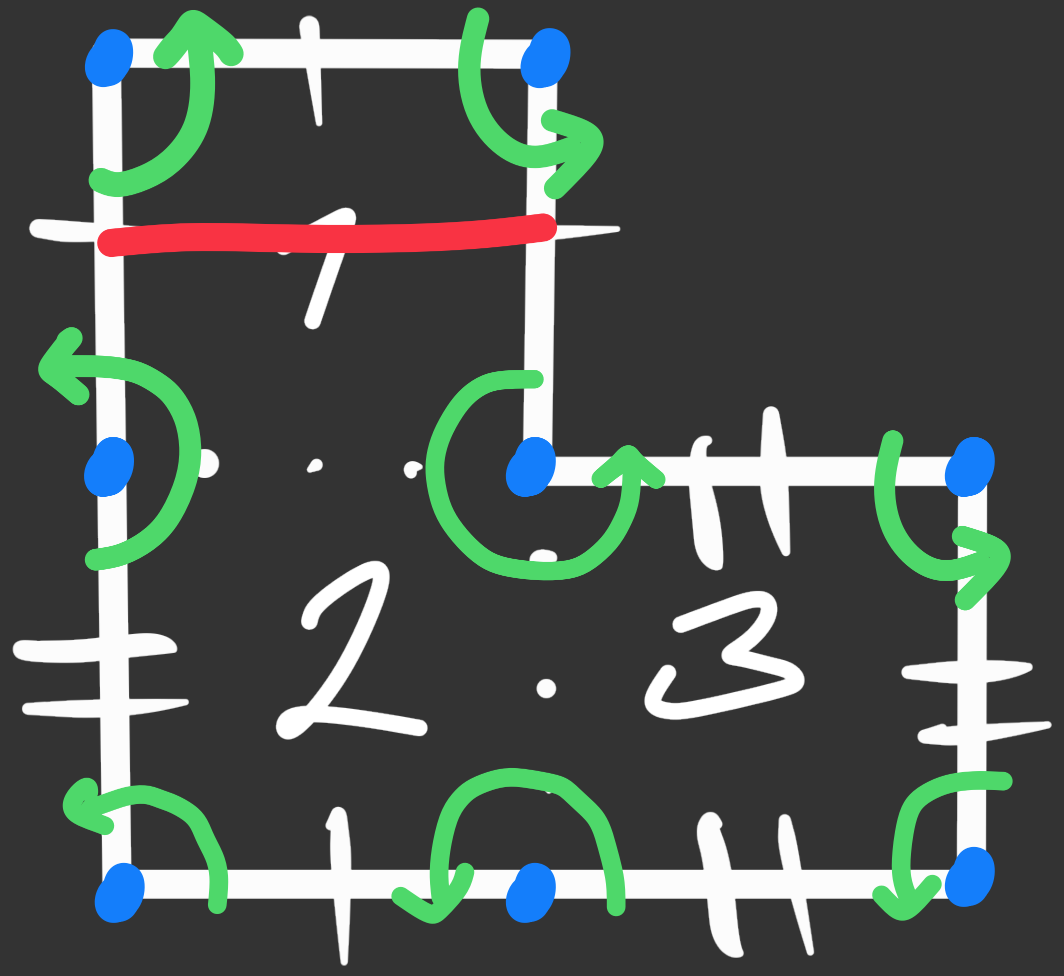

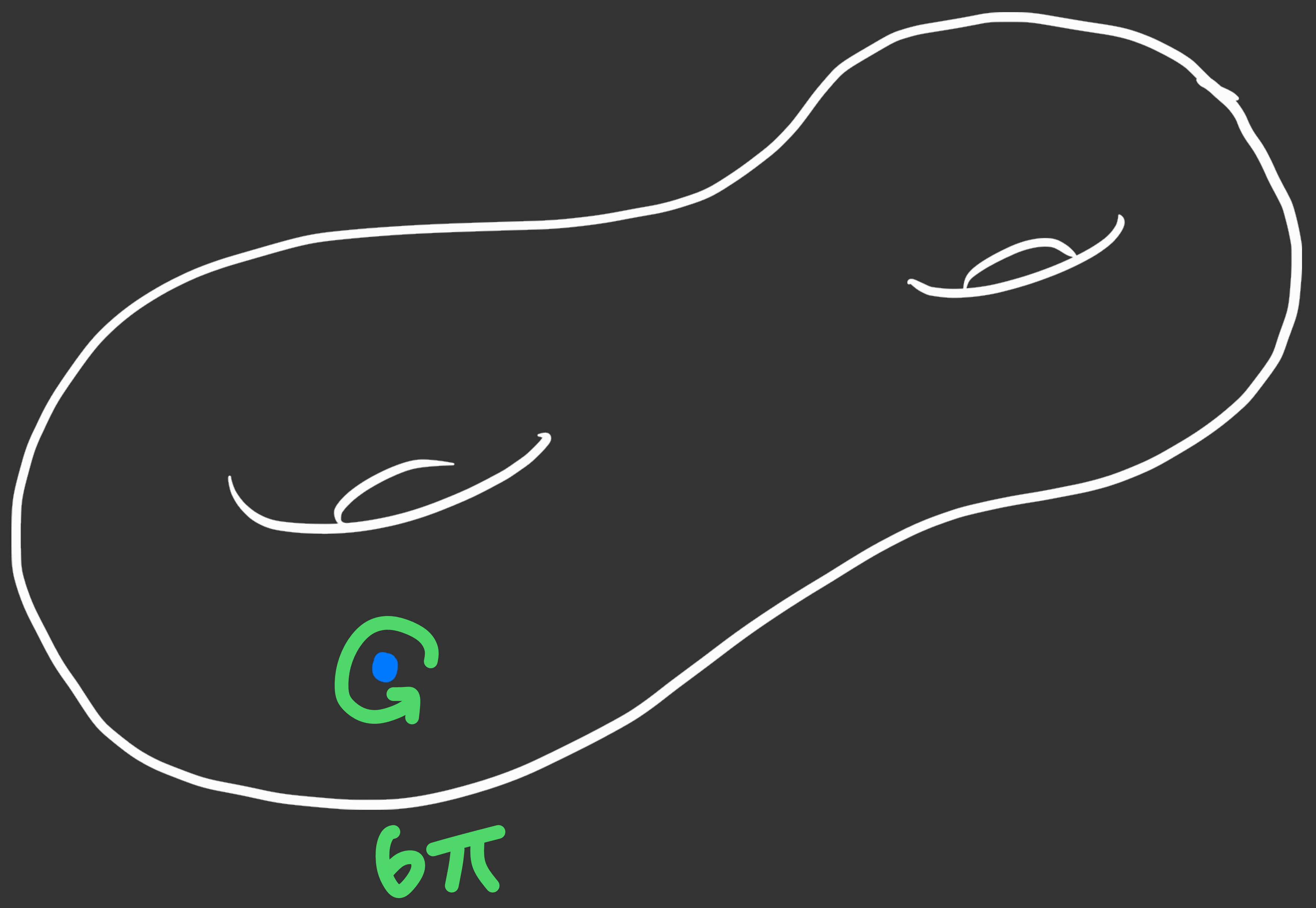

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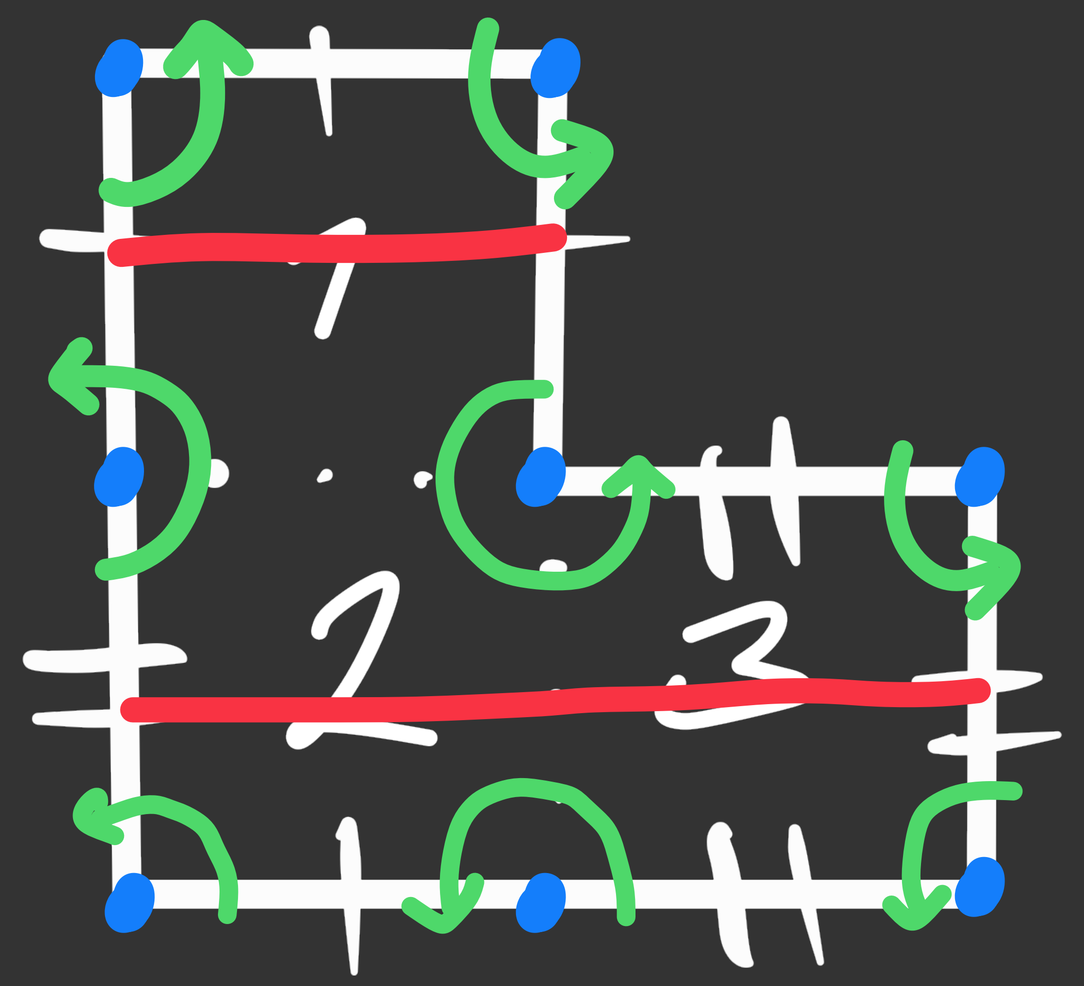

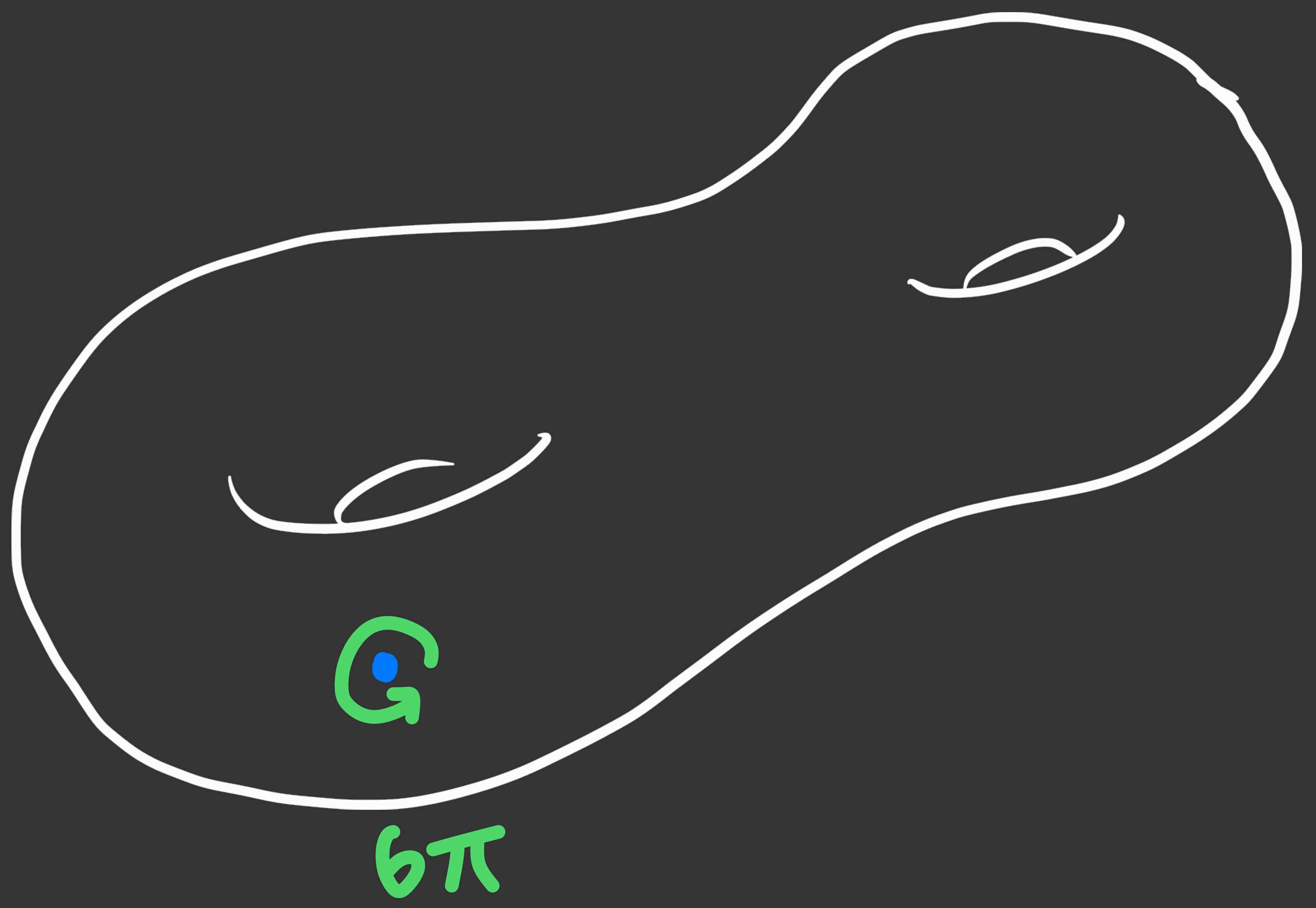

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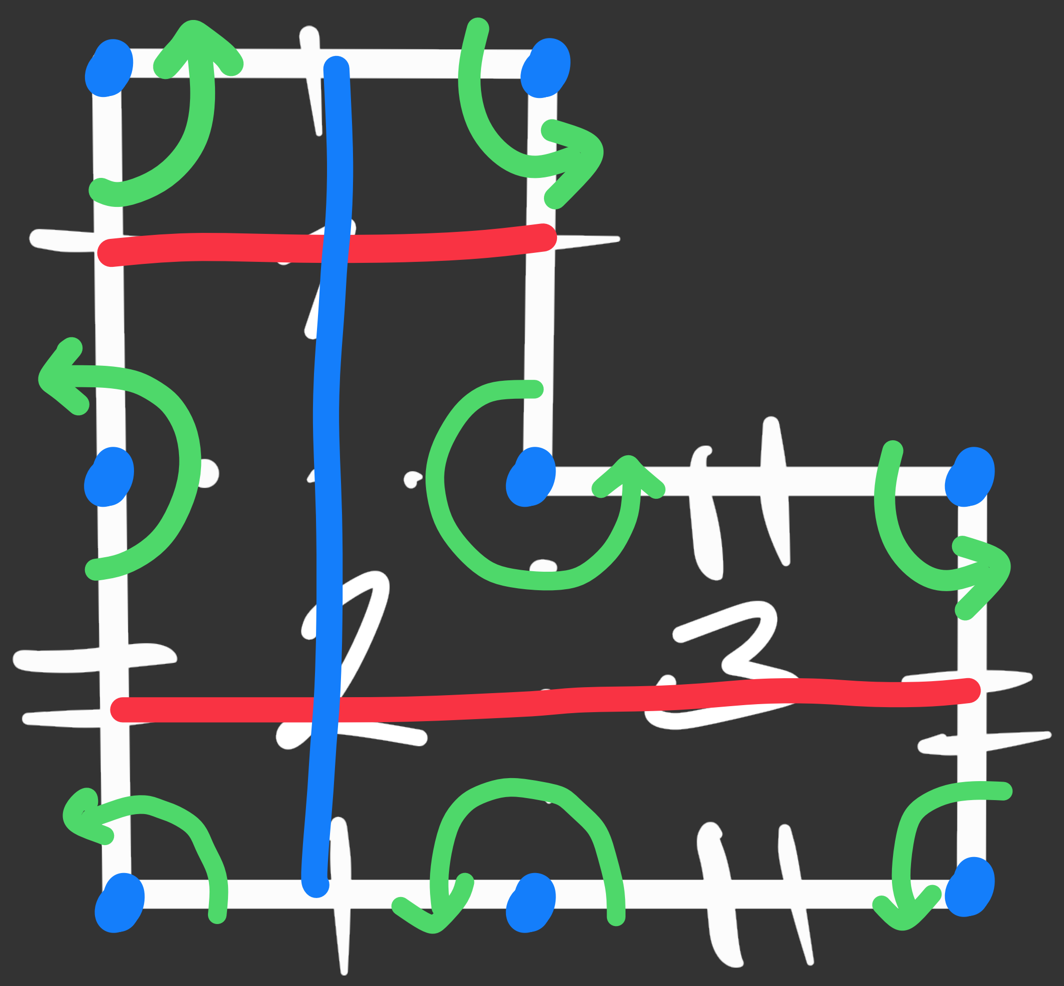

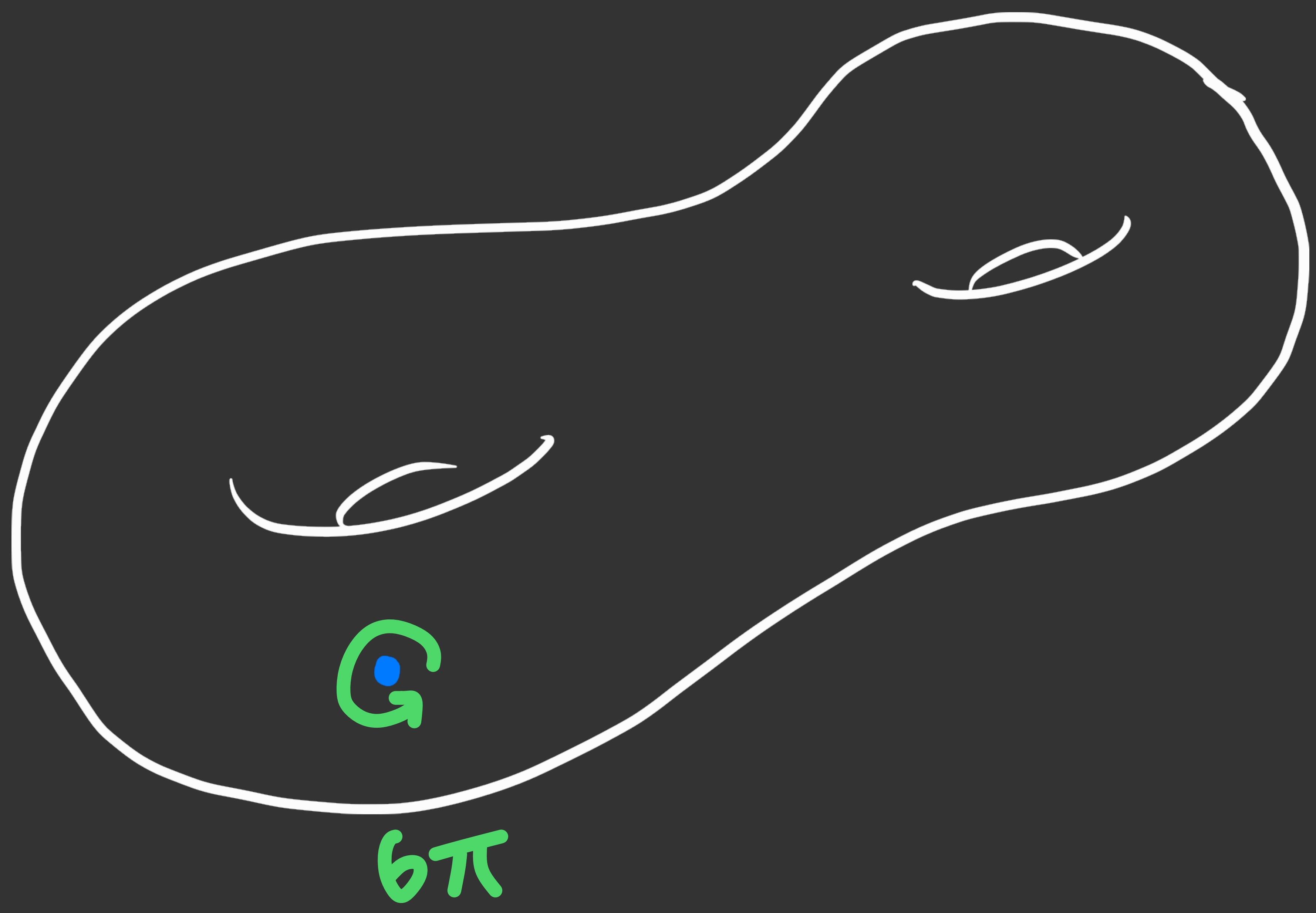

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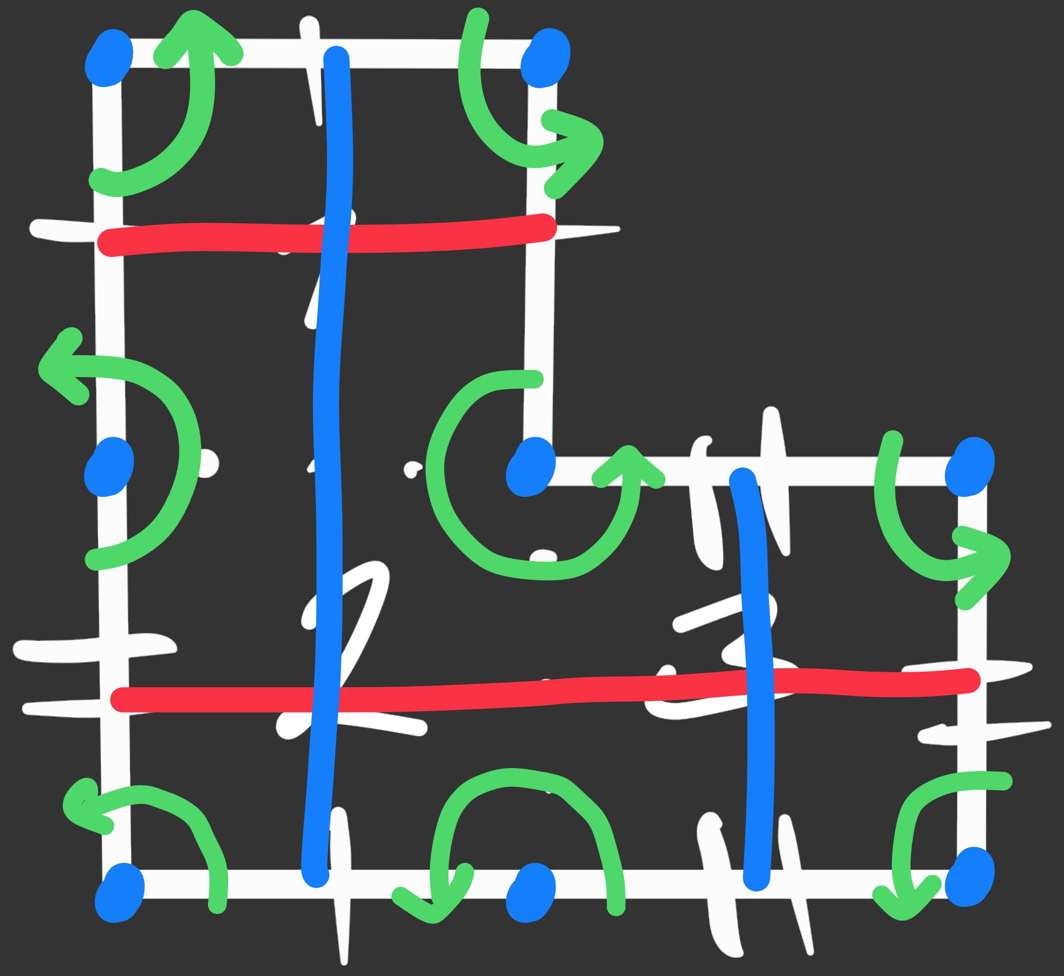

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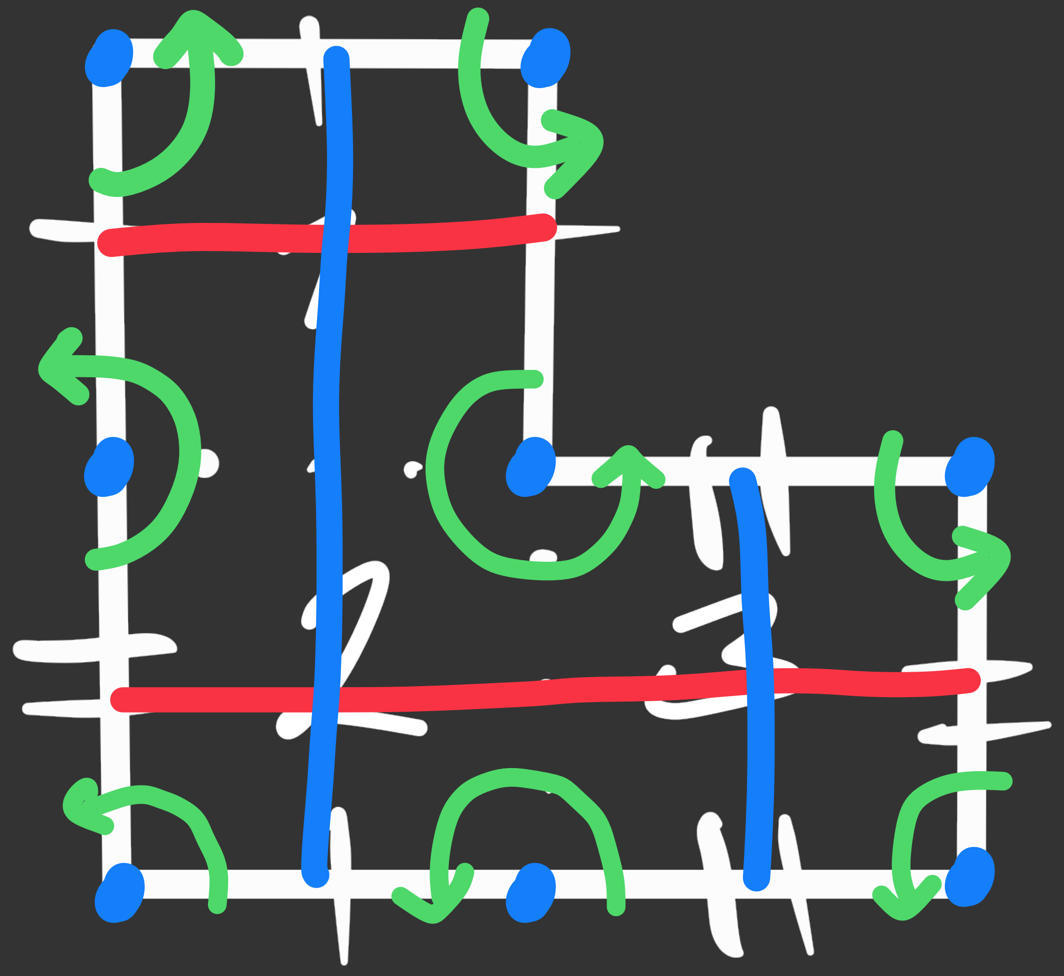

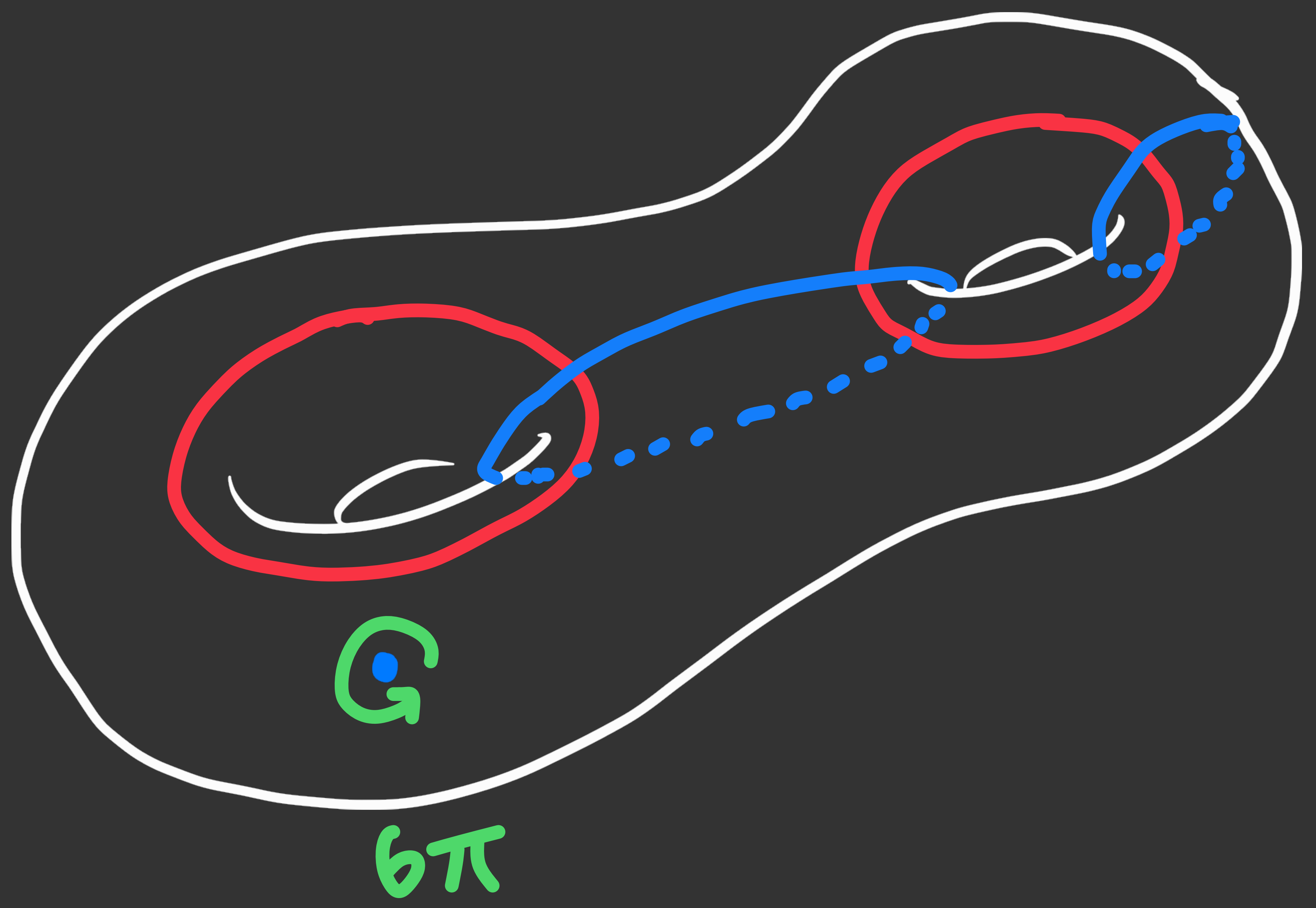

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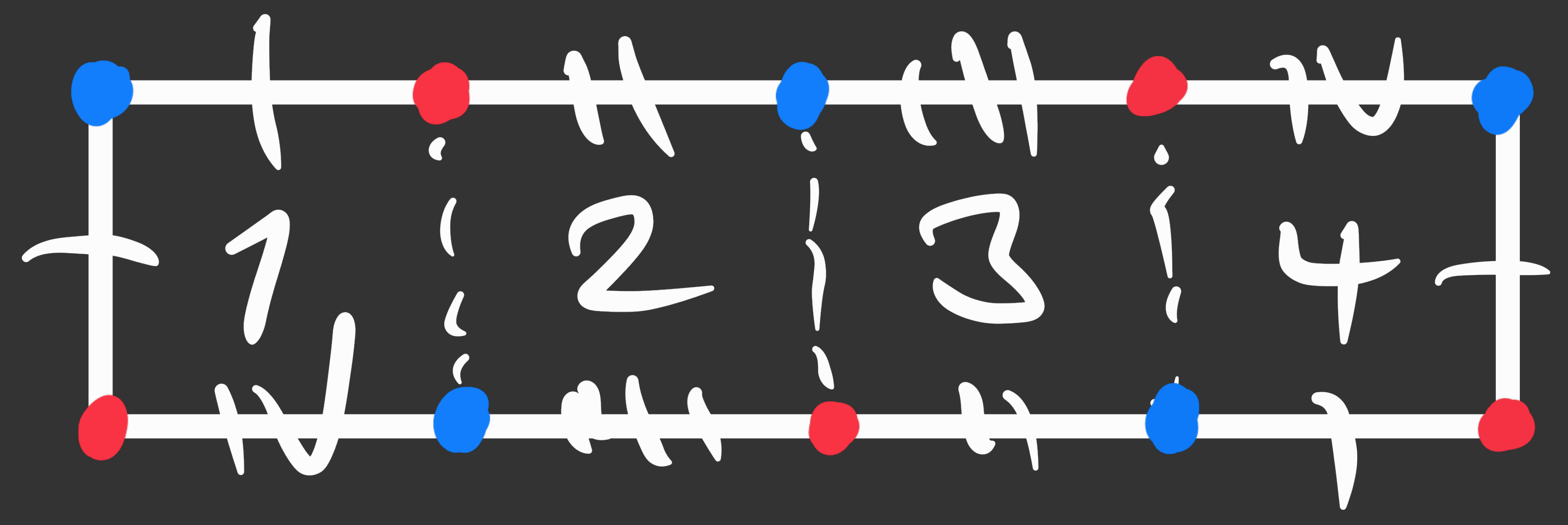

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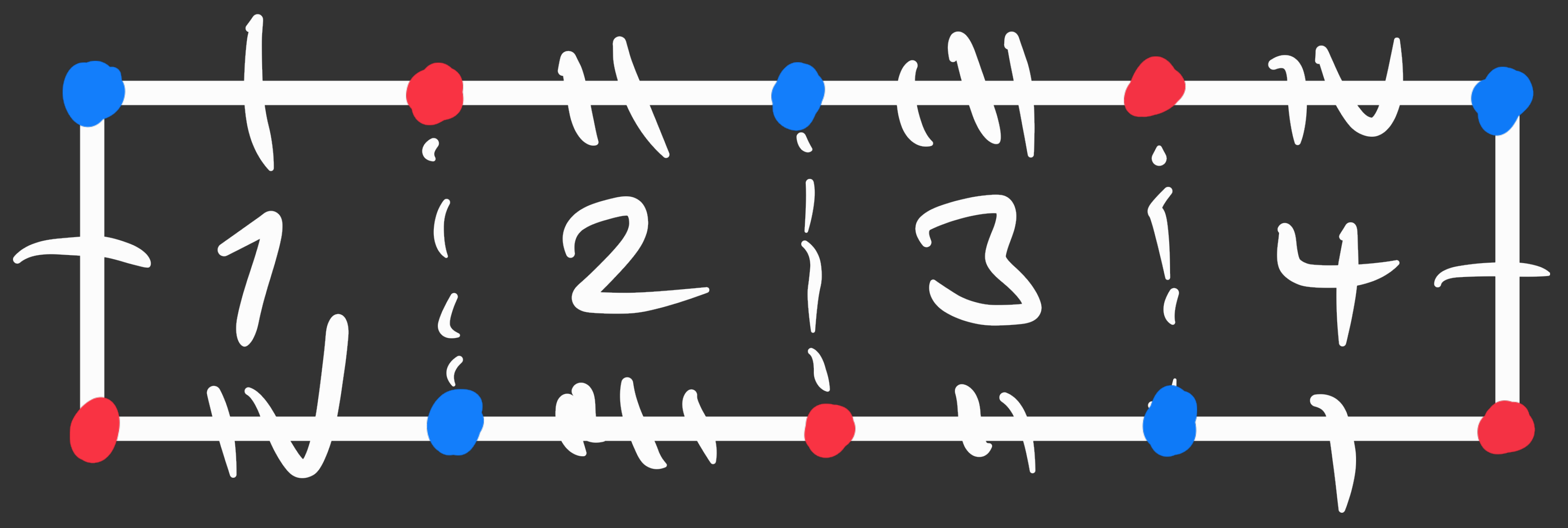

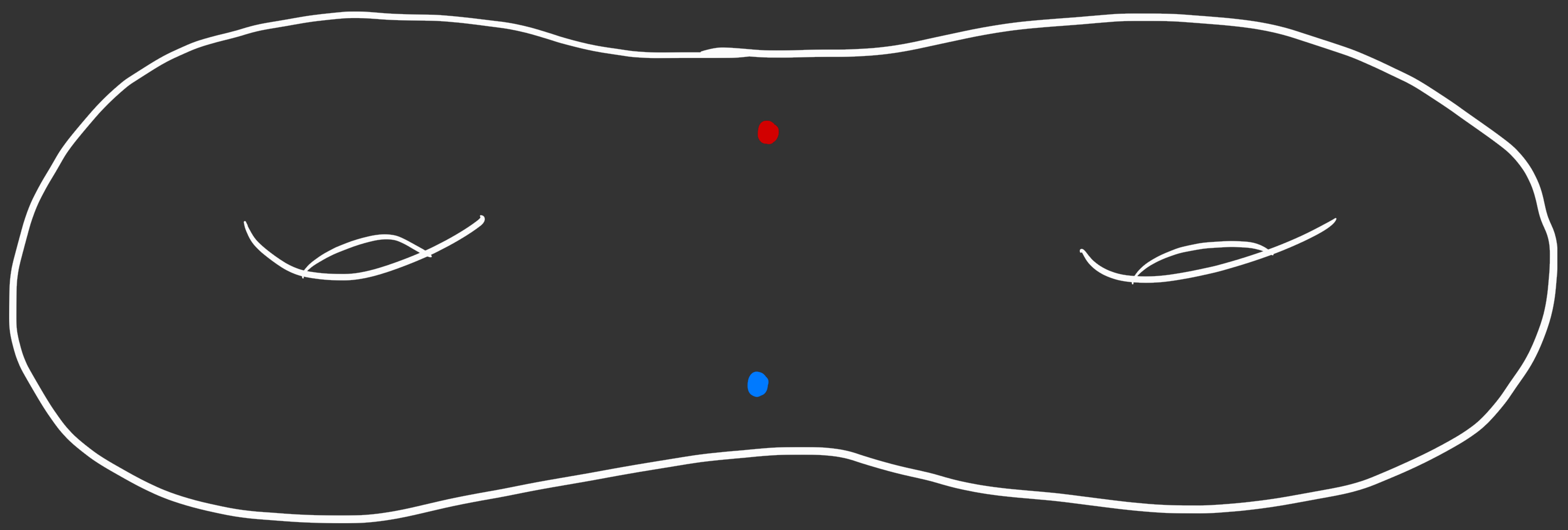

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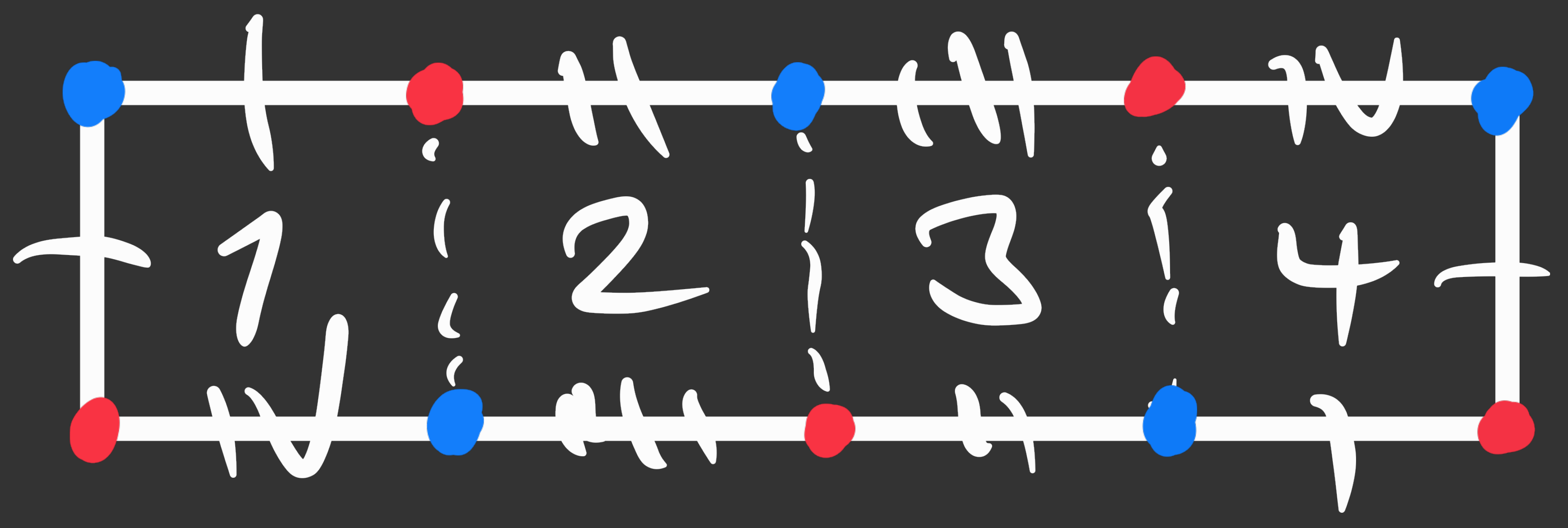

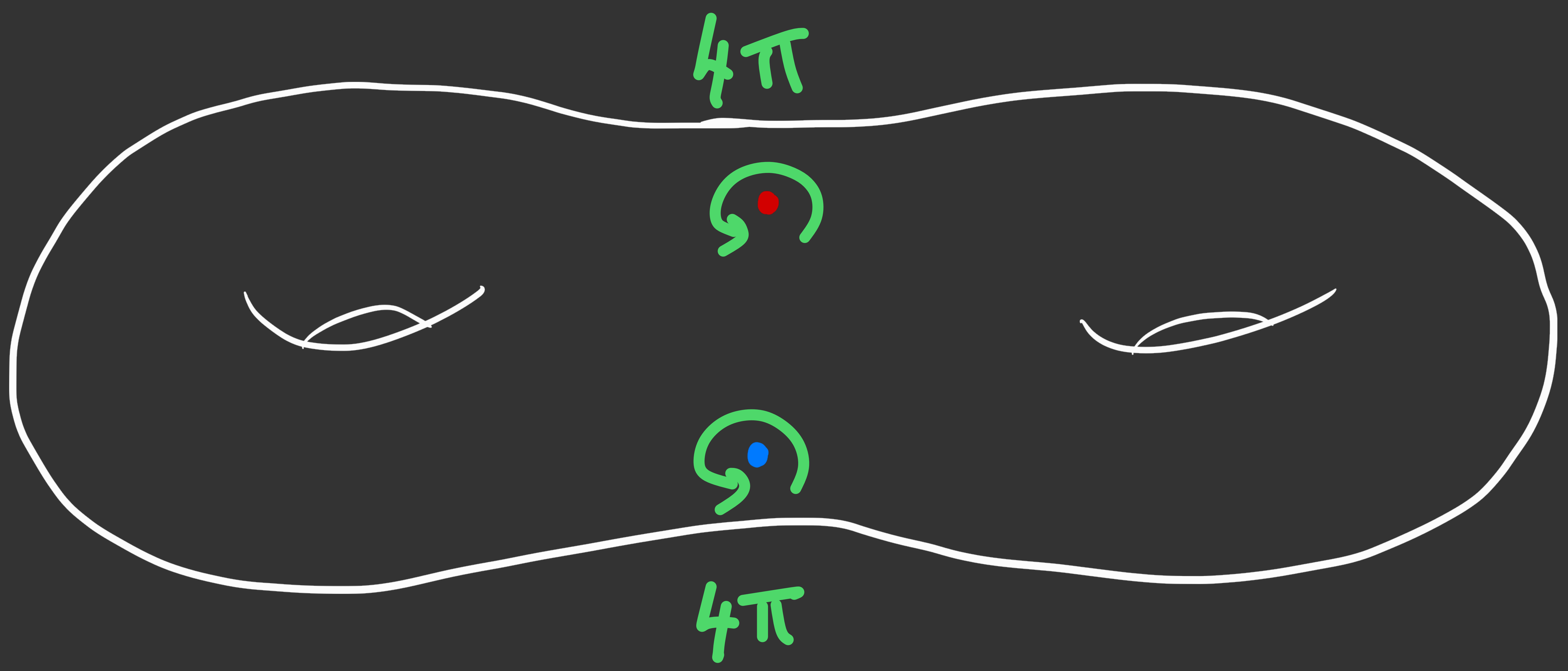

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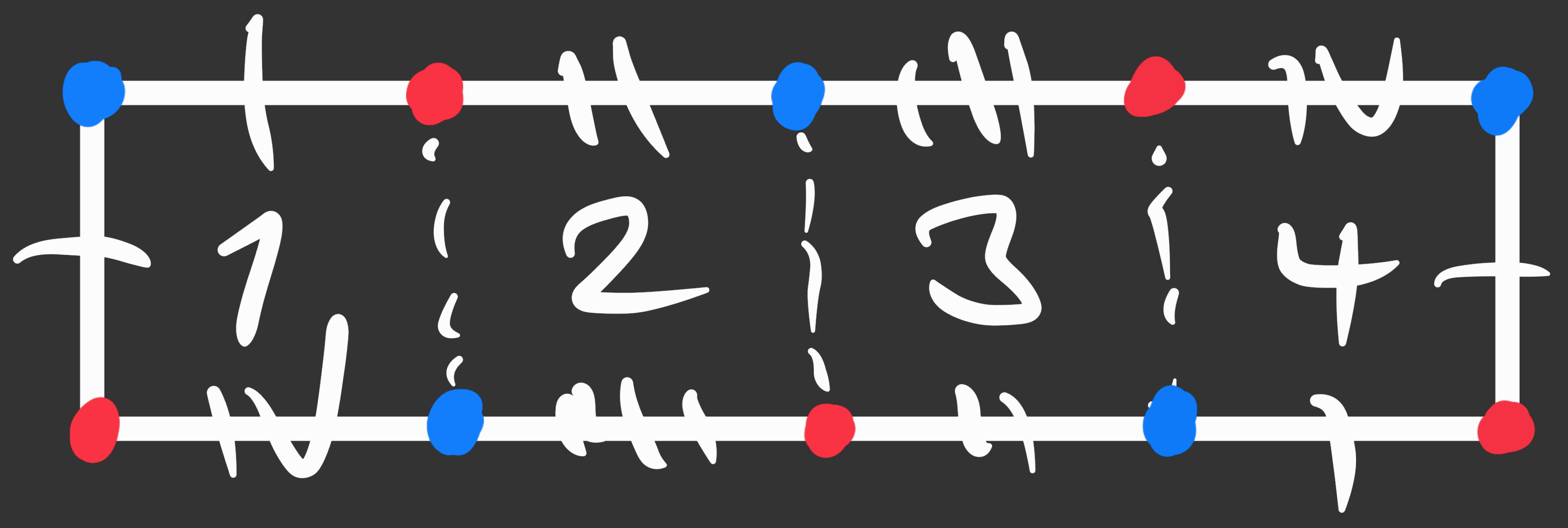

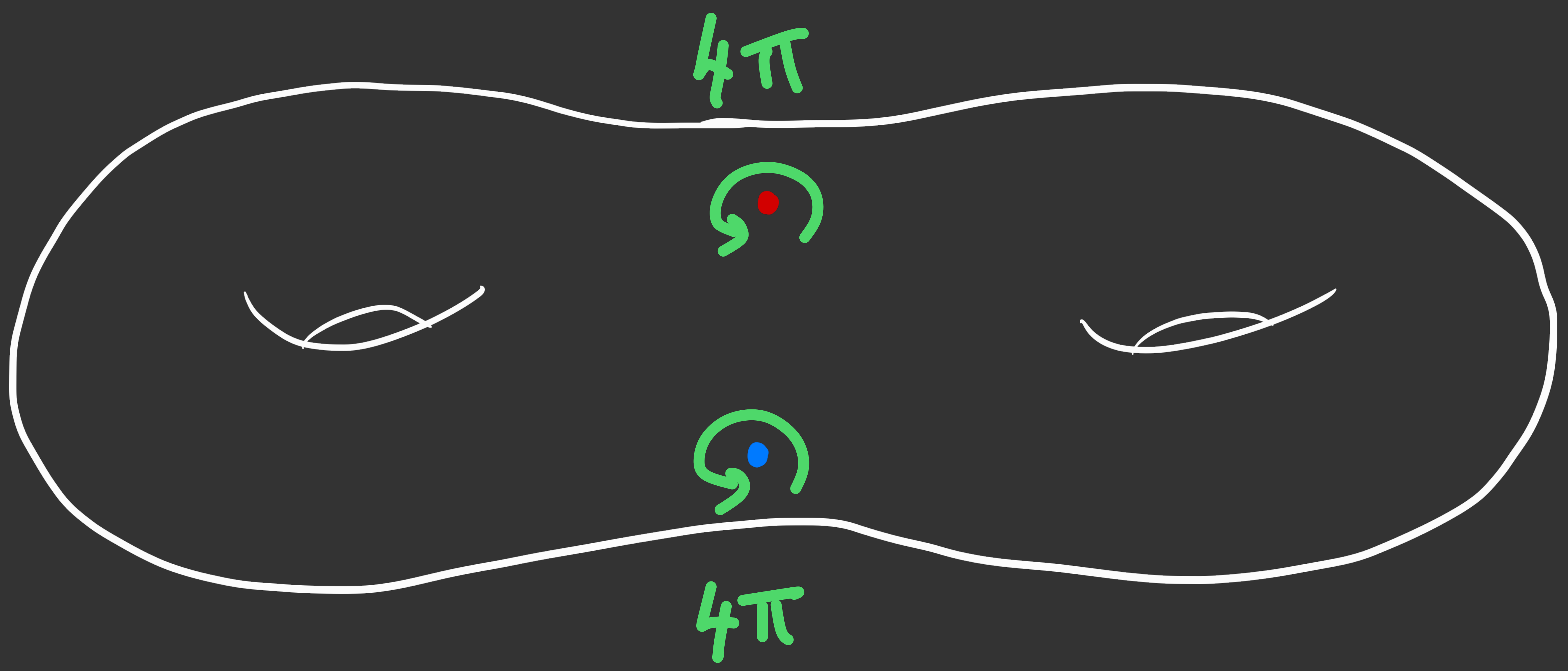

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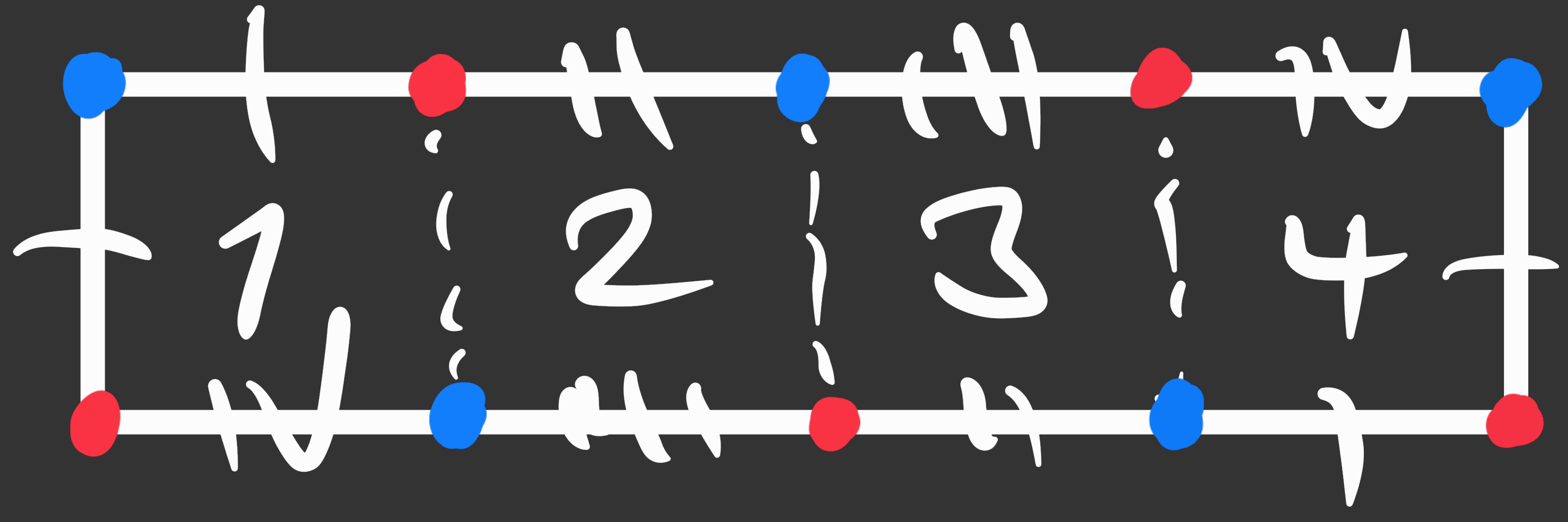

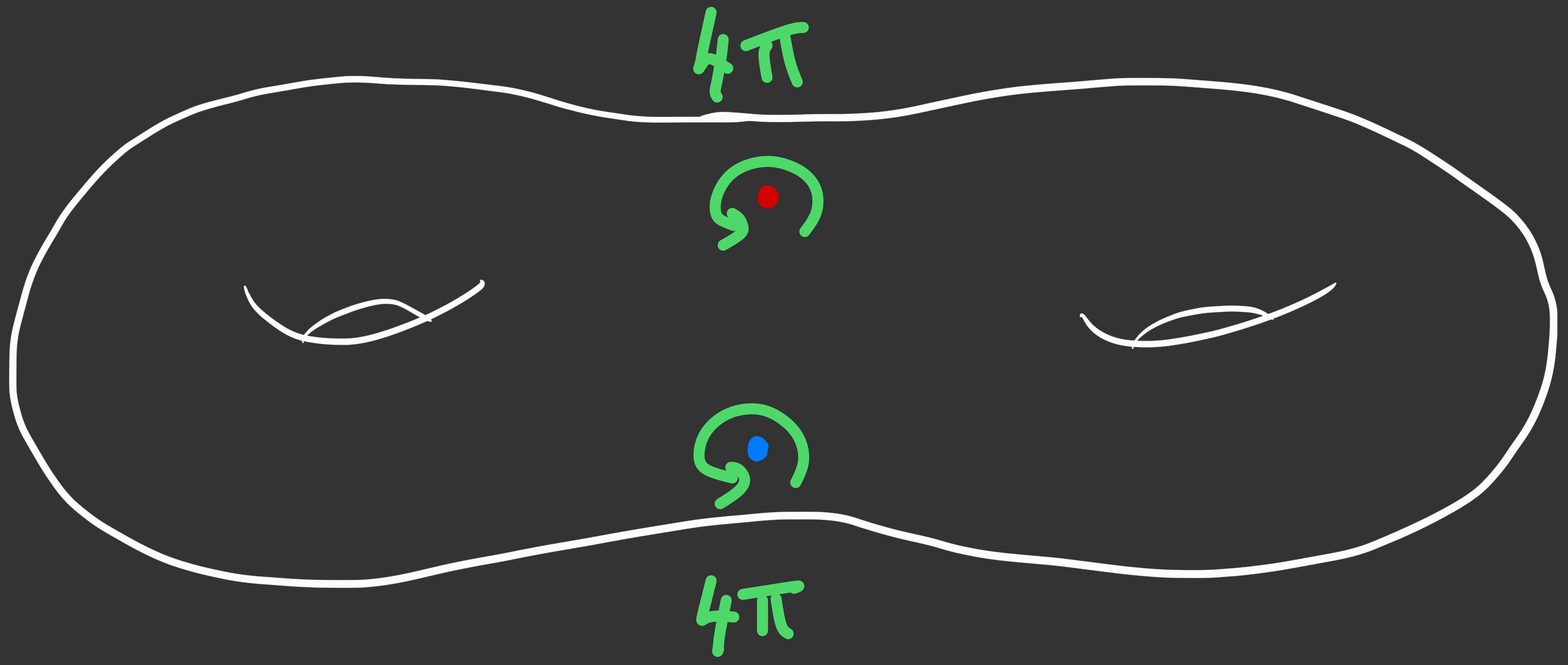

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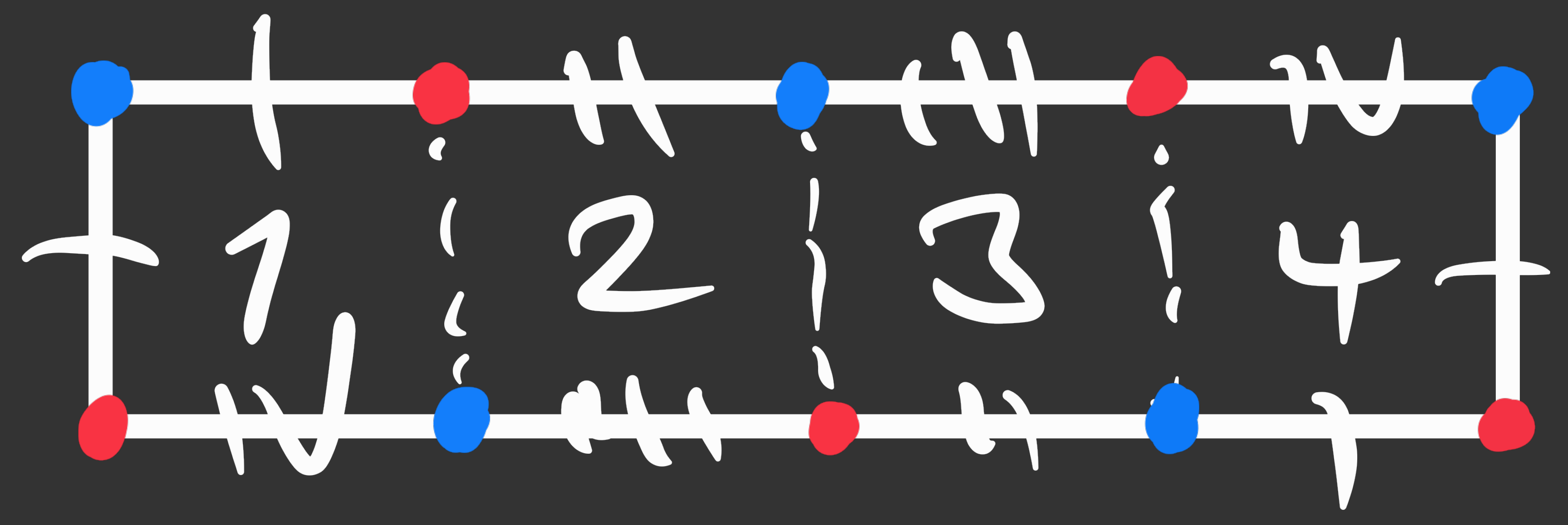

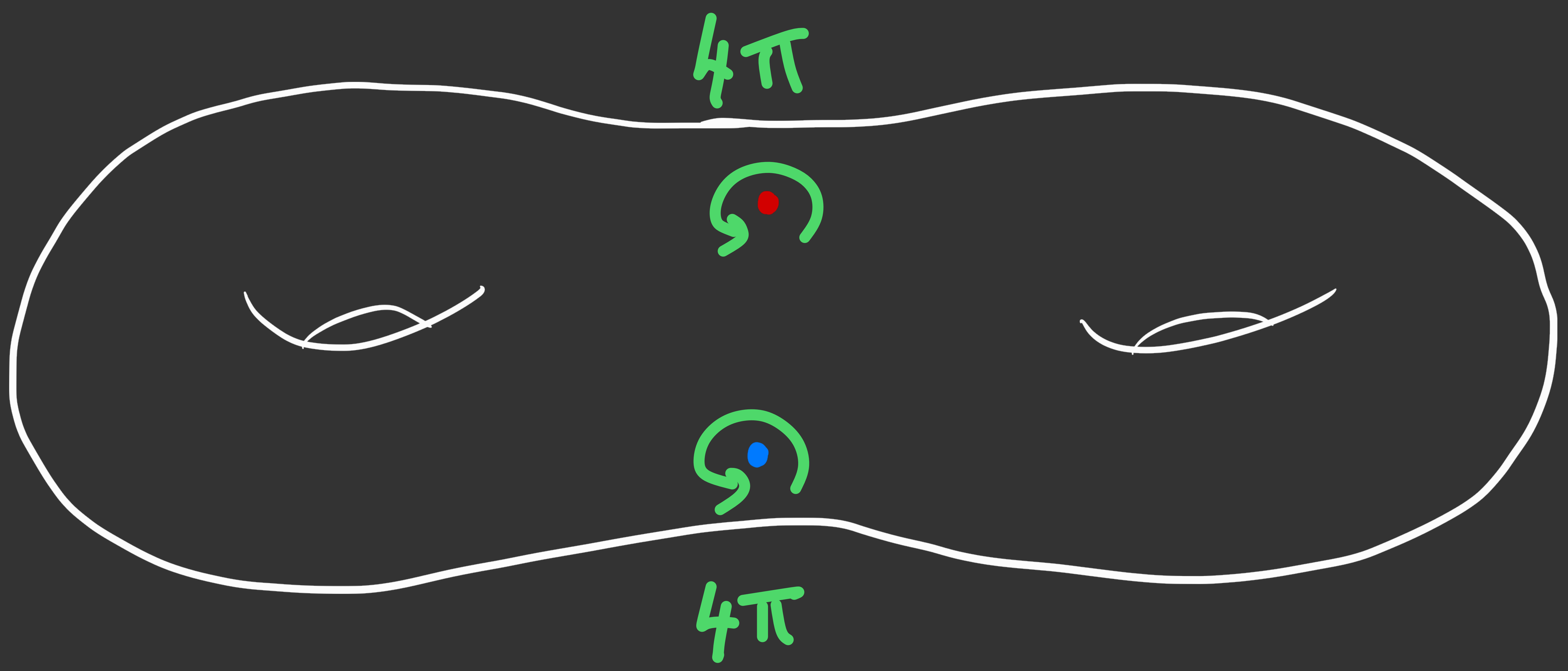

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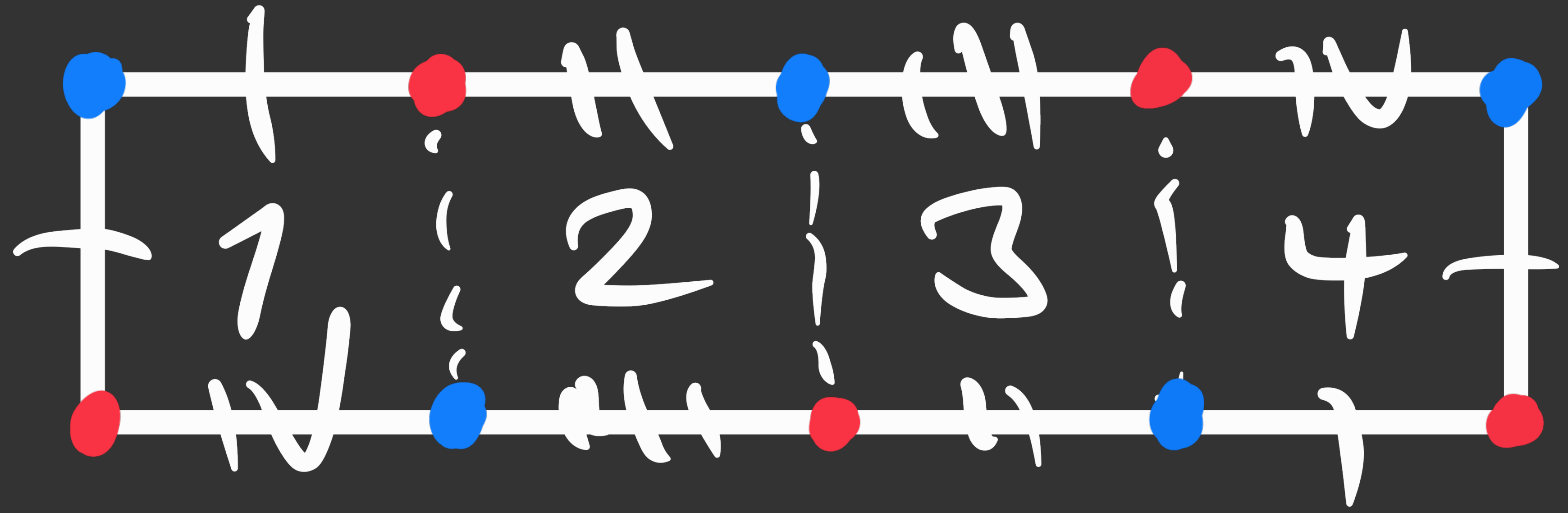

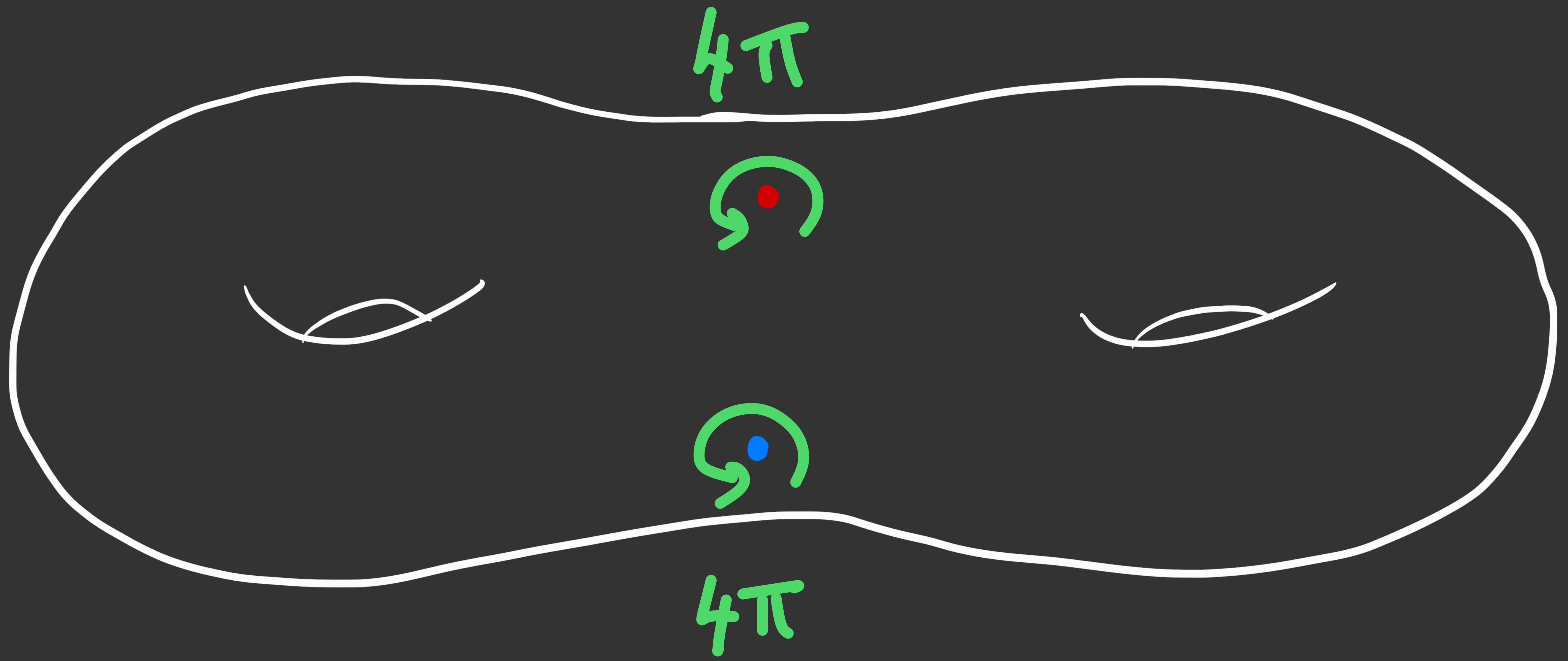

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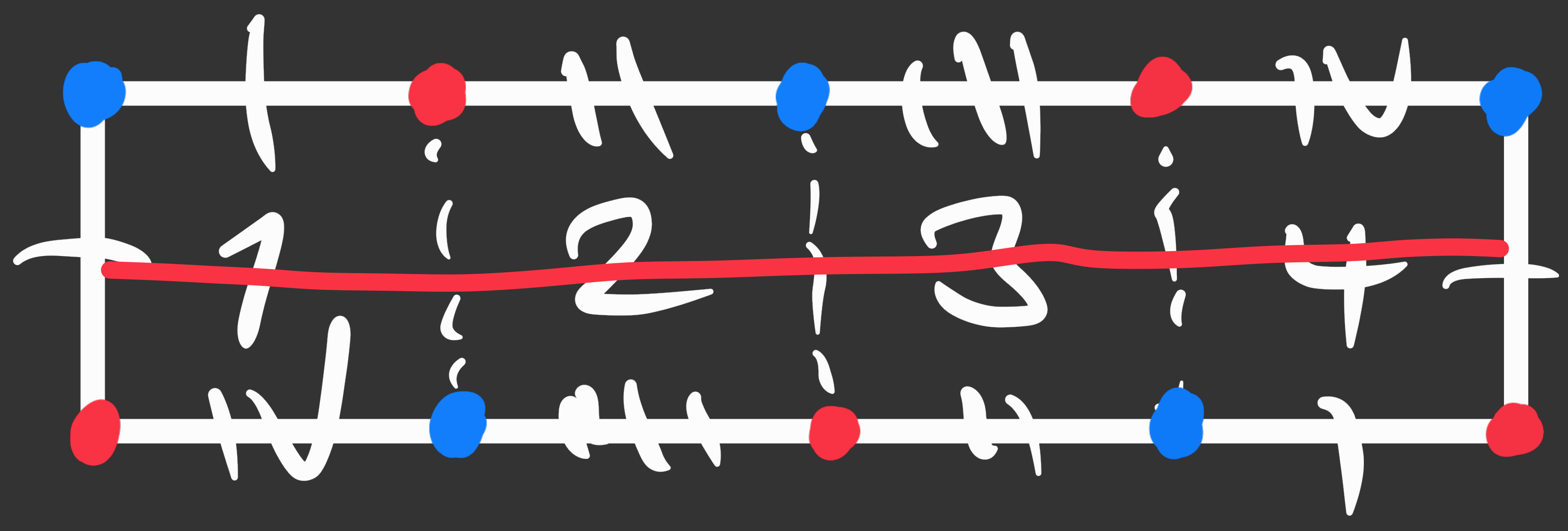

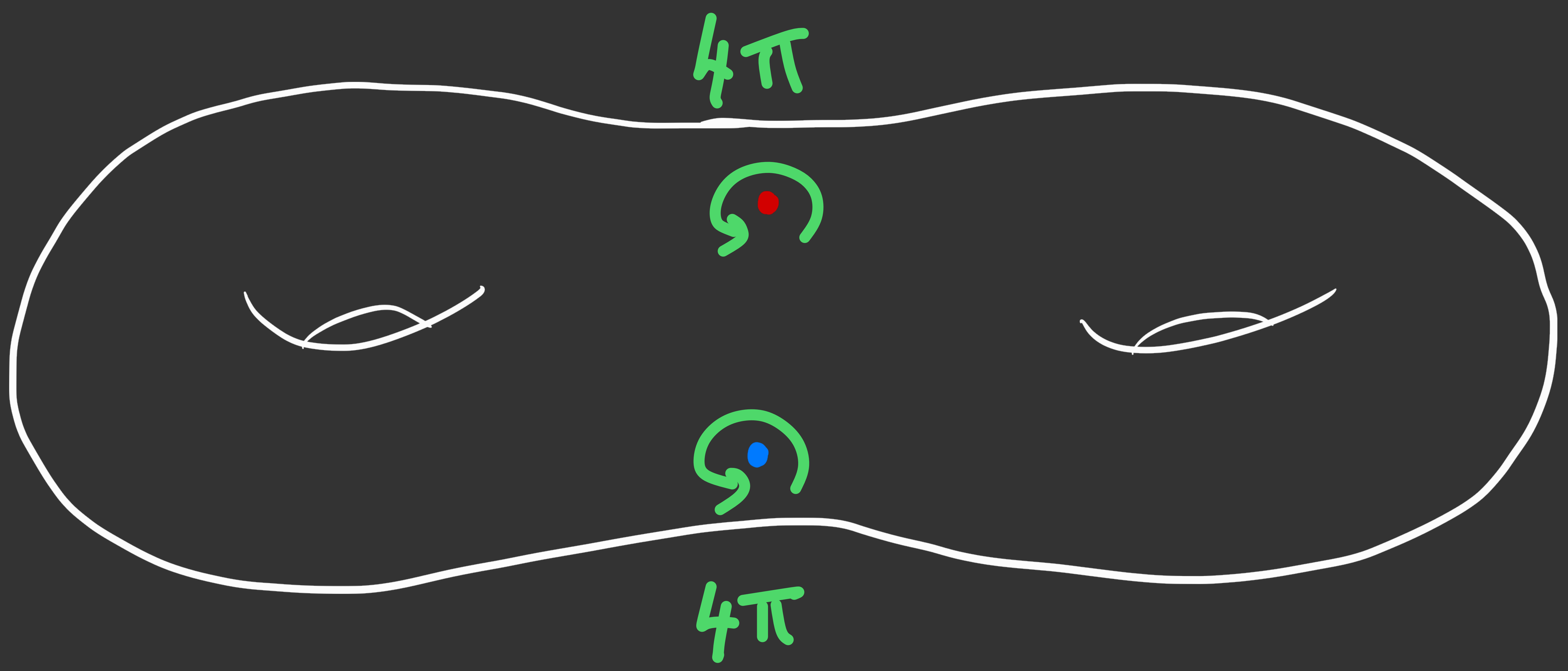

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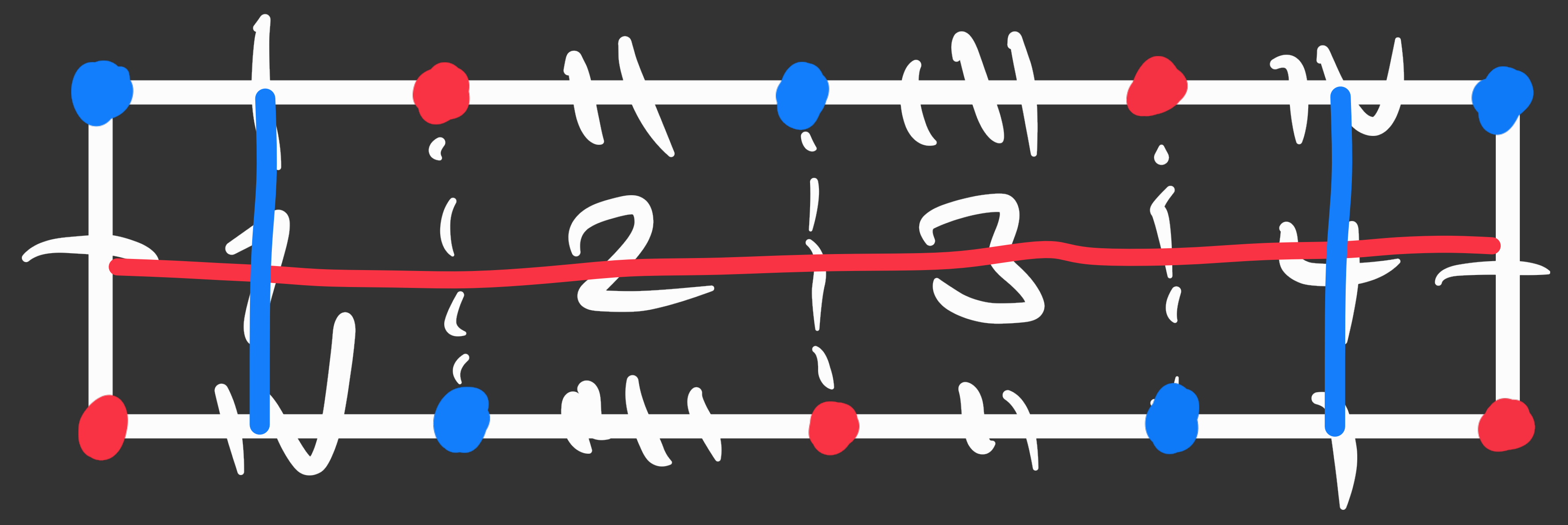

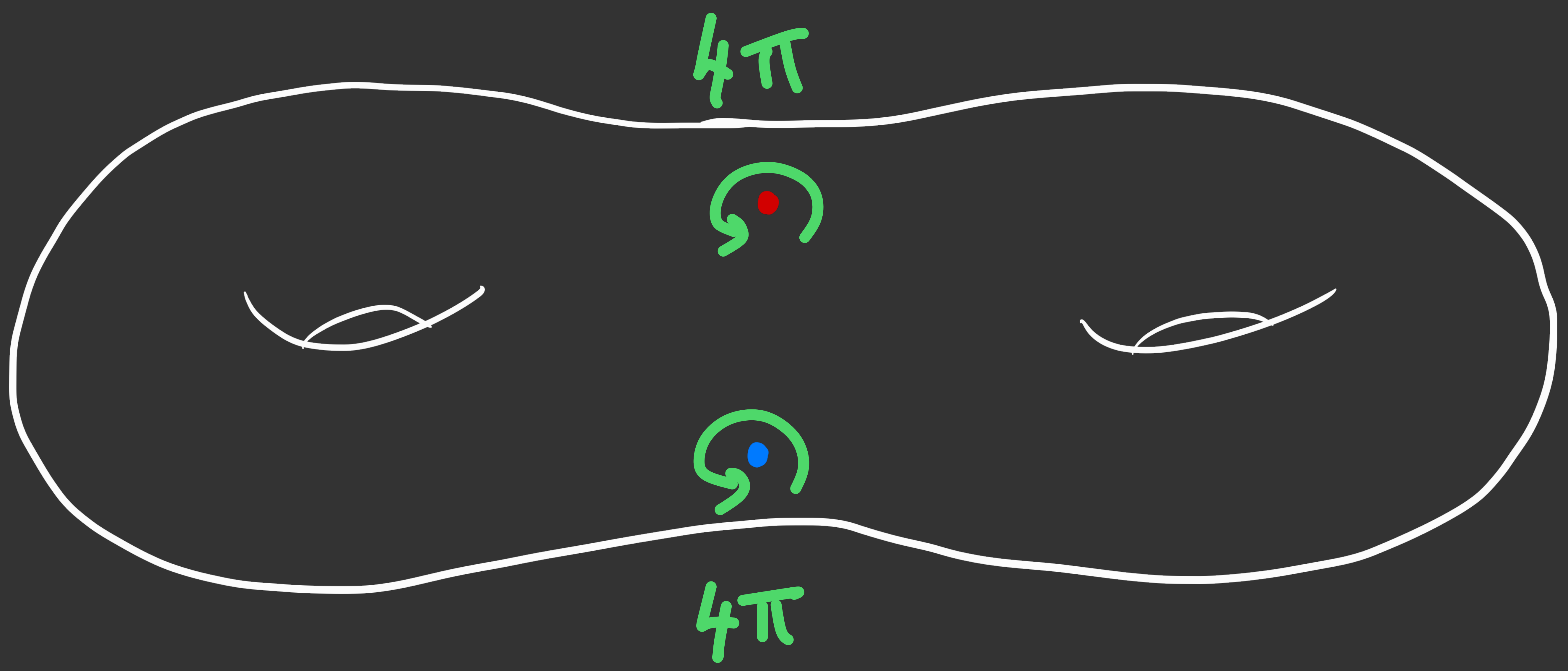

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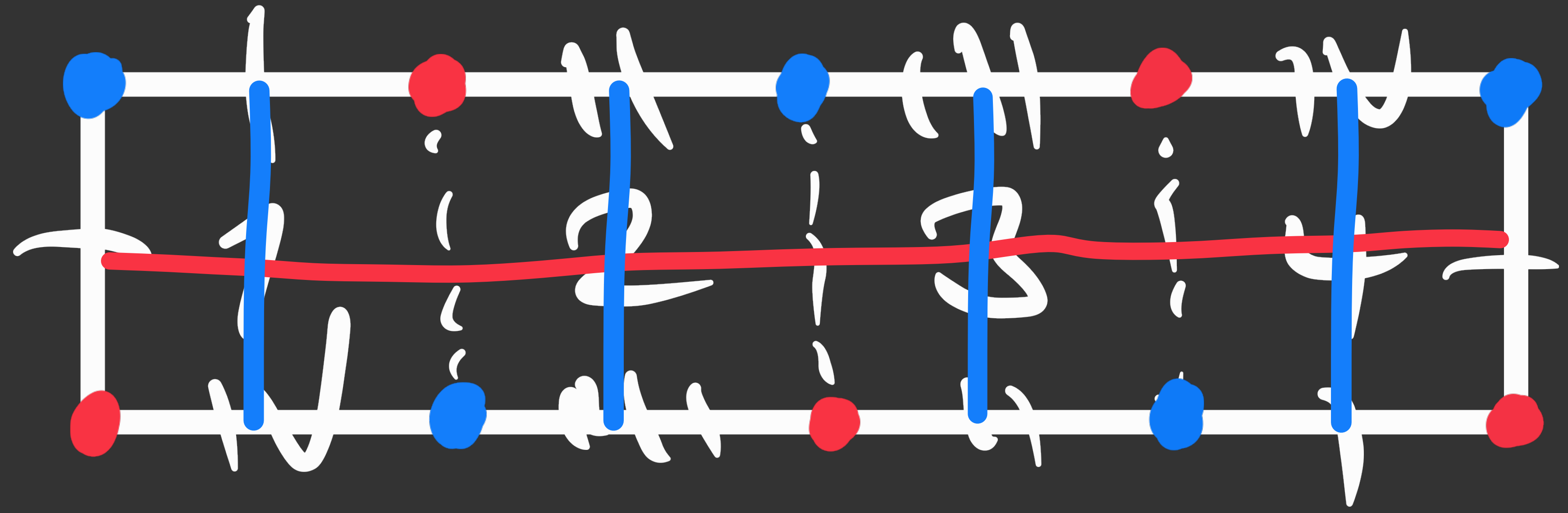

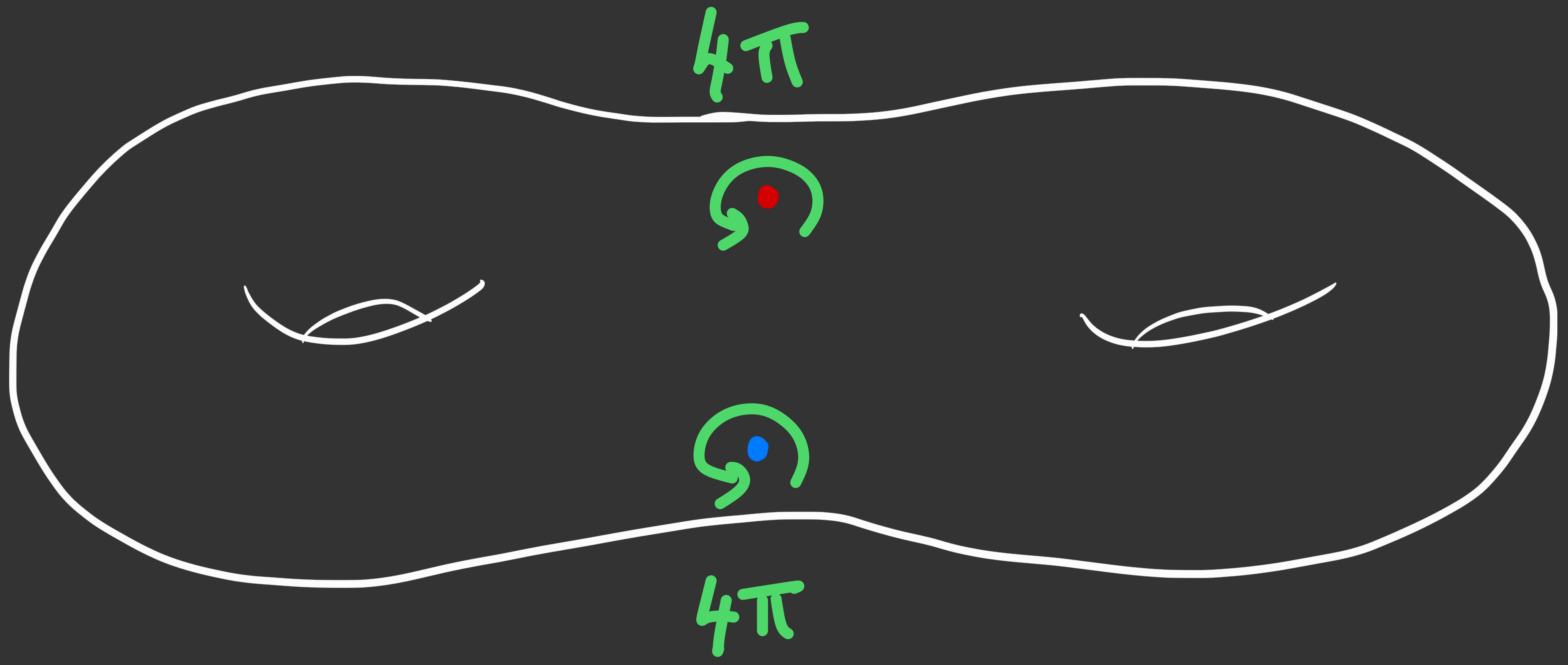

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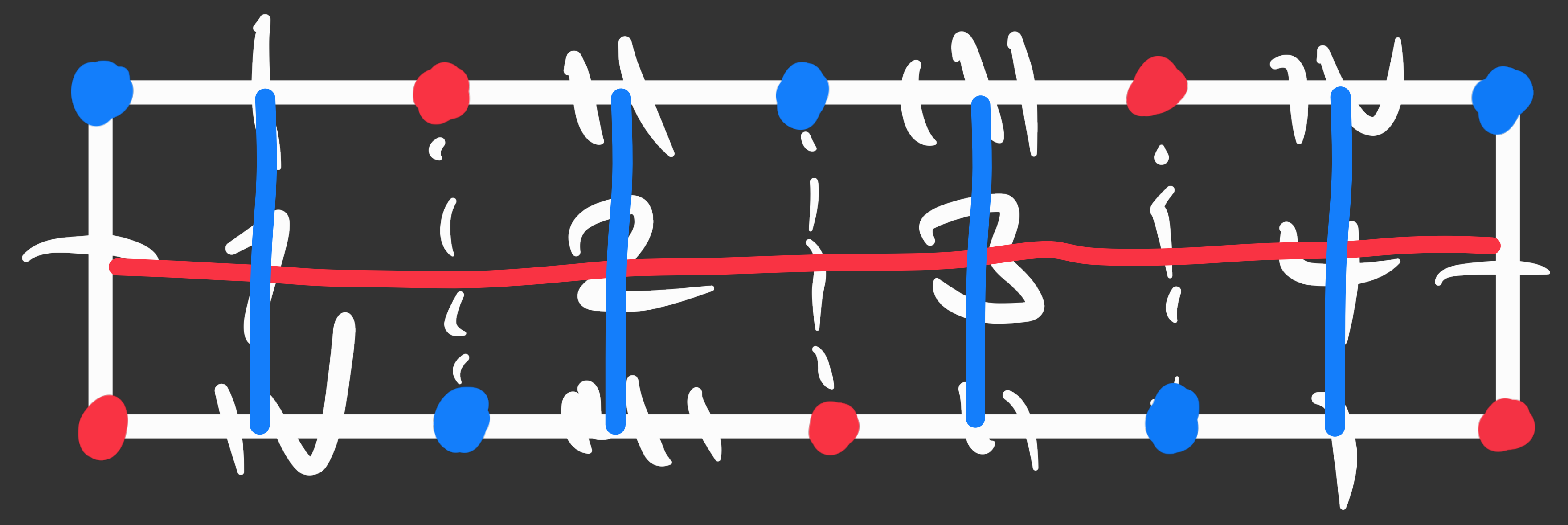

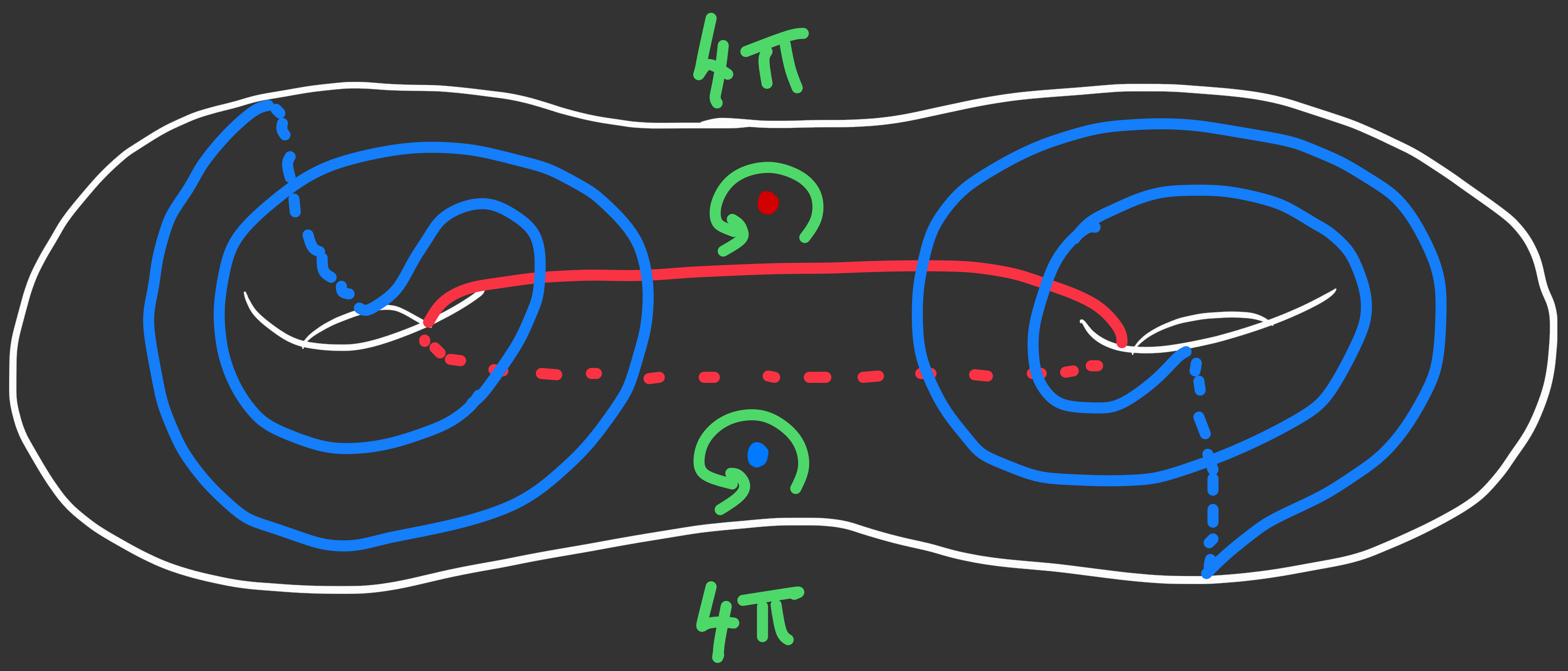

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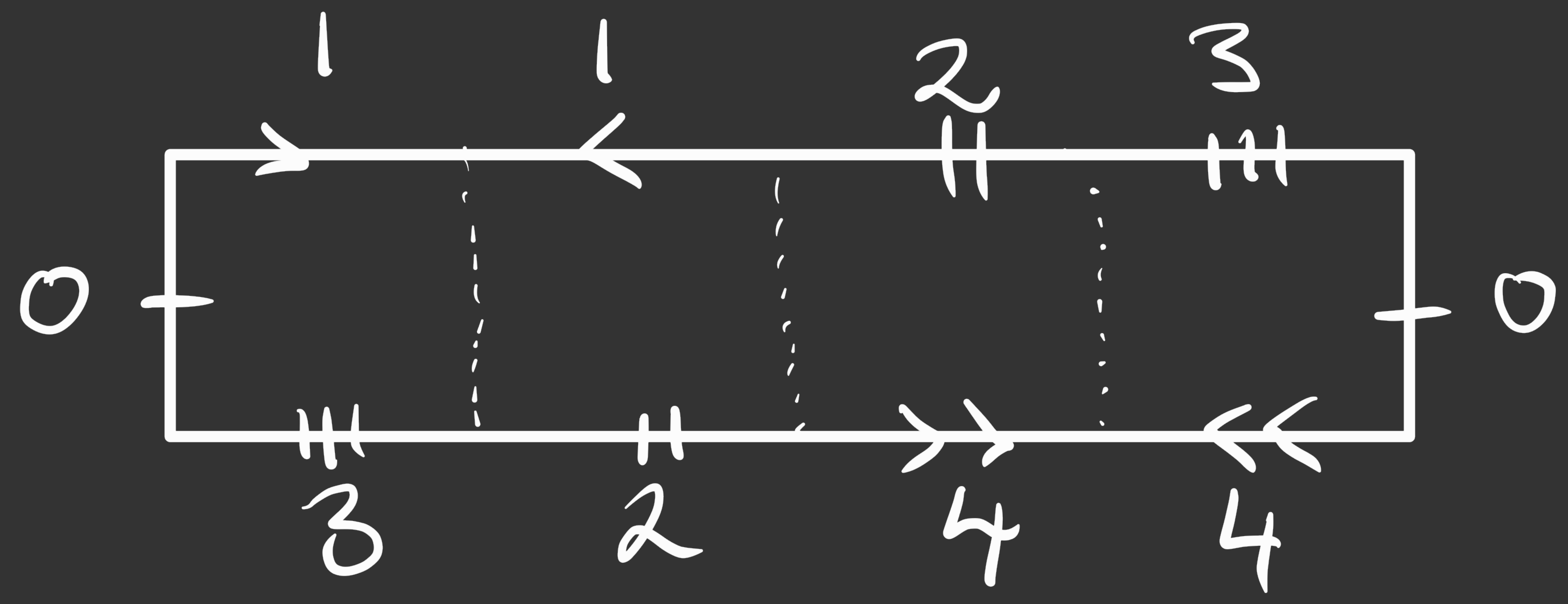
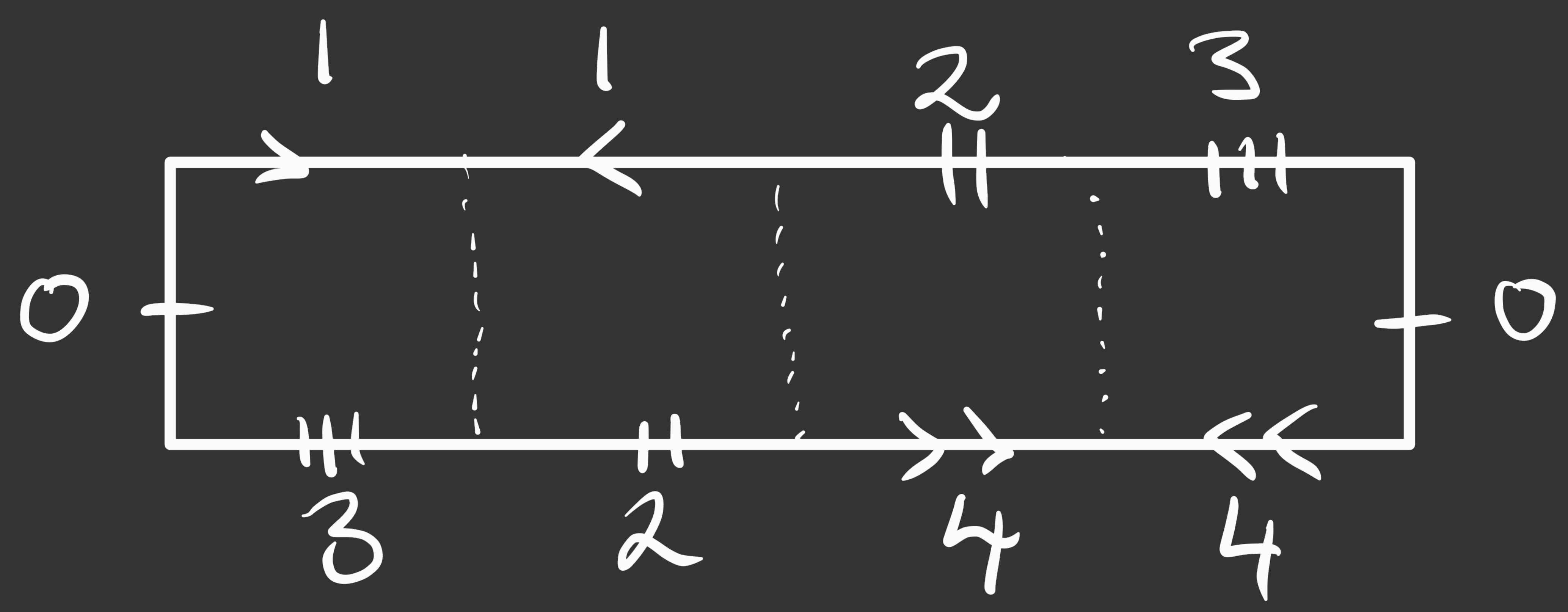


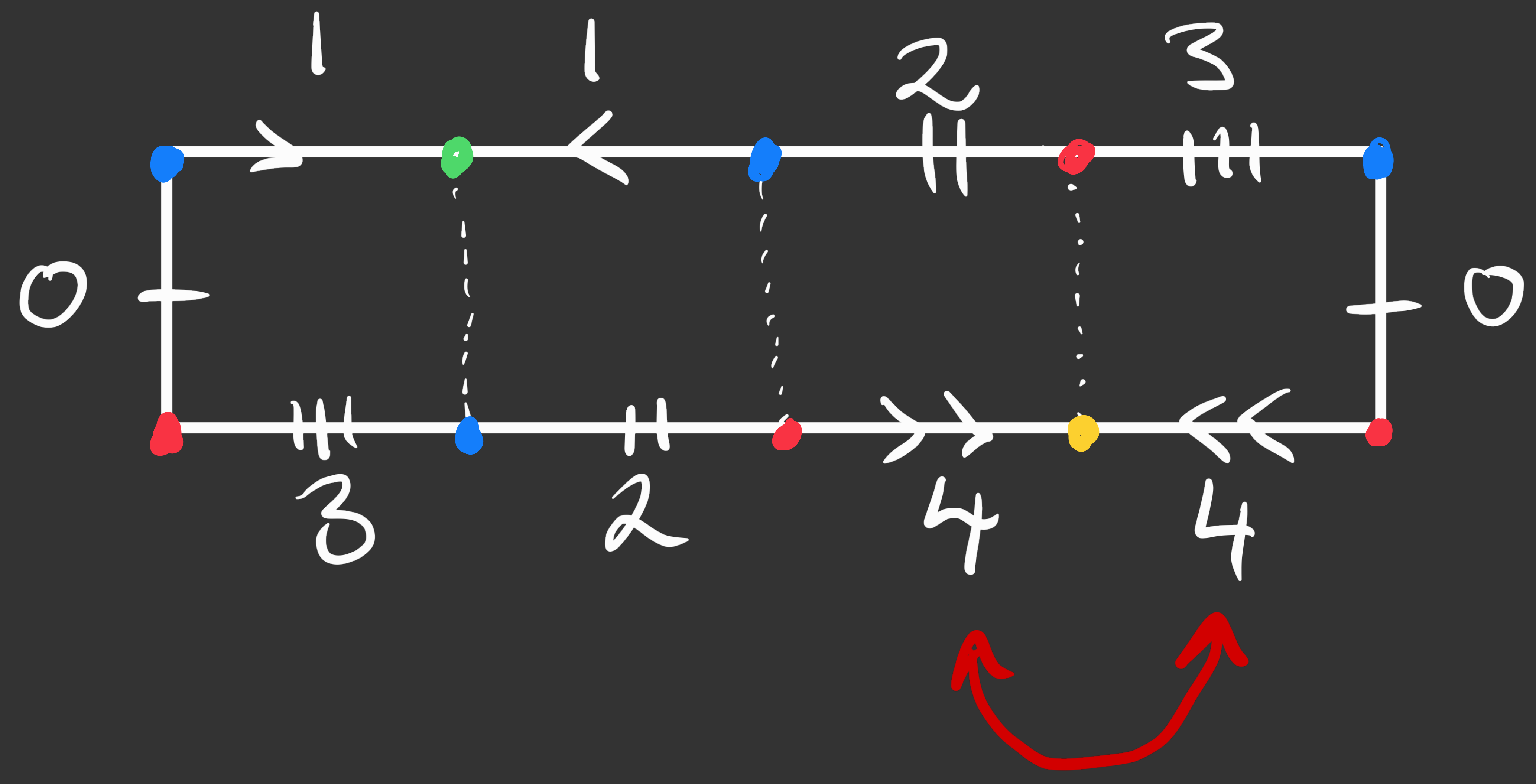
and another...



$-z+c$

'half-translations'

and another...



$-z + c$

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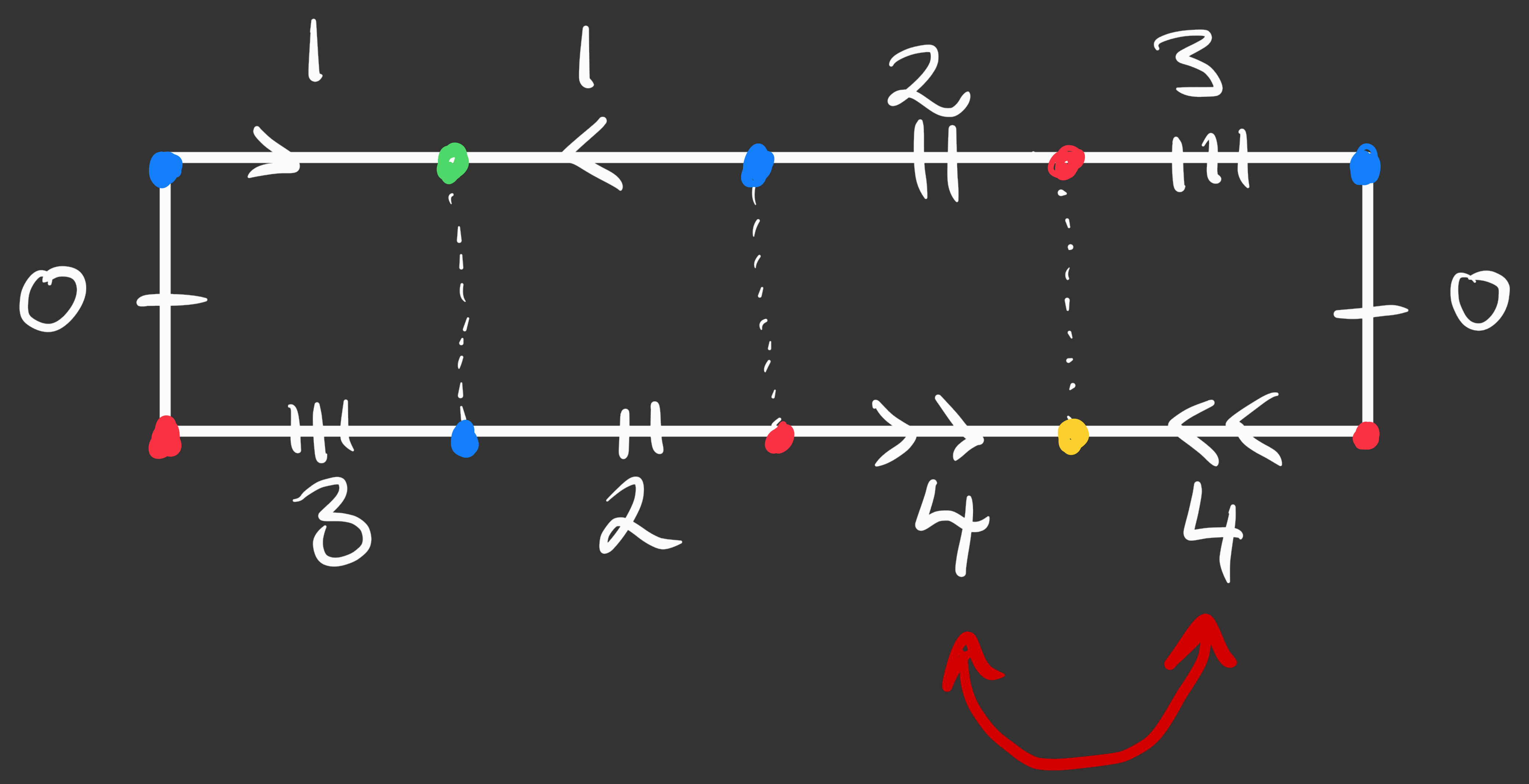

Two points of cone angle

$$3\pi = (1+2)\pi$$

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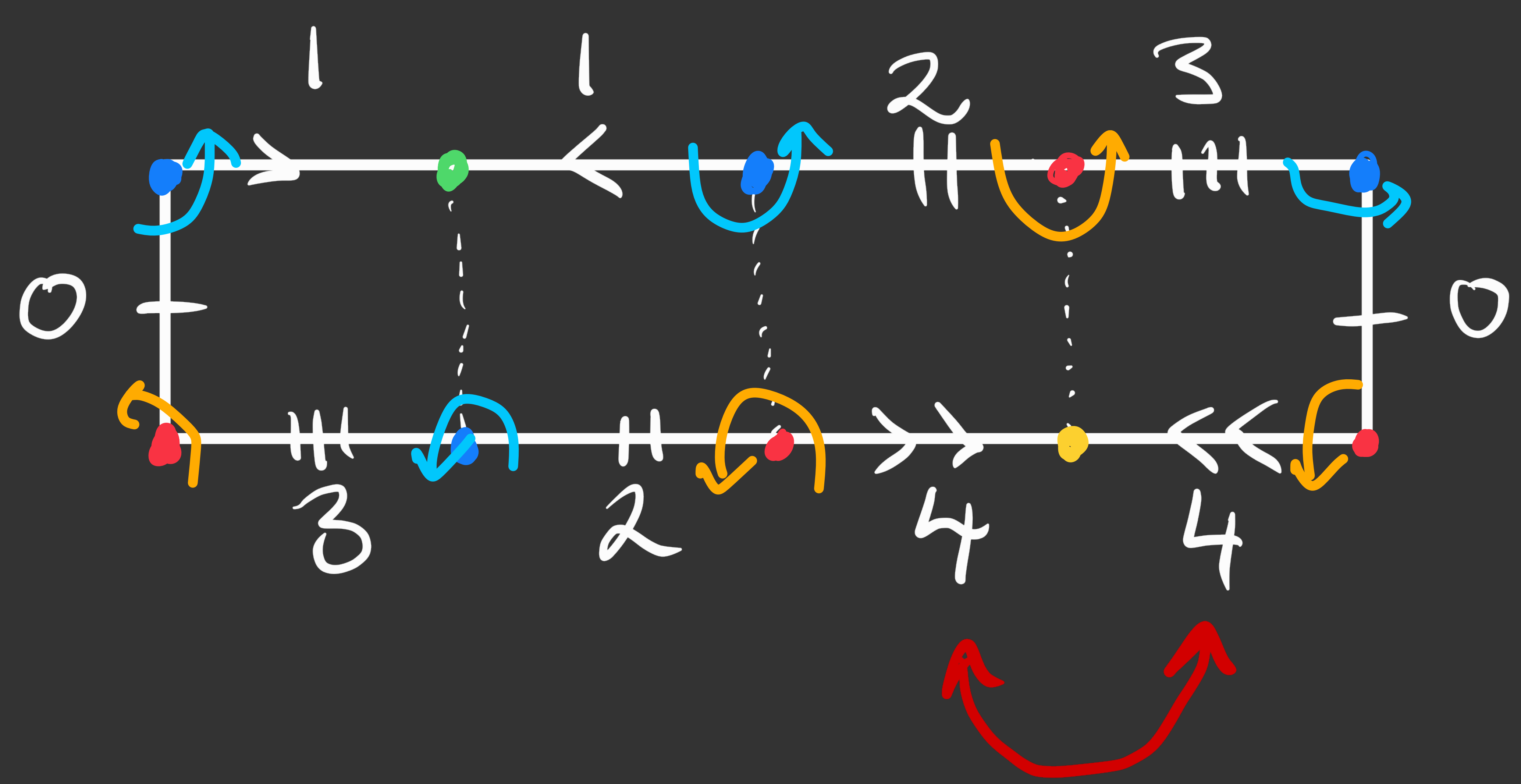
and another...



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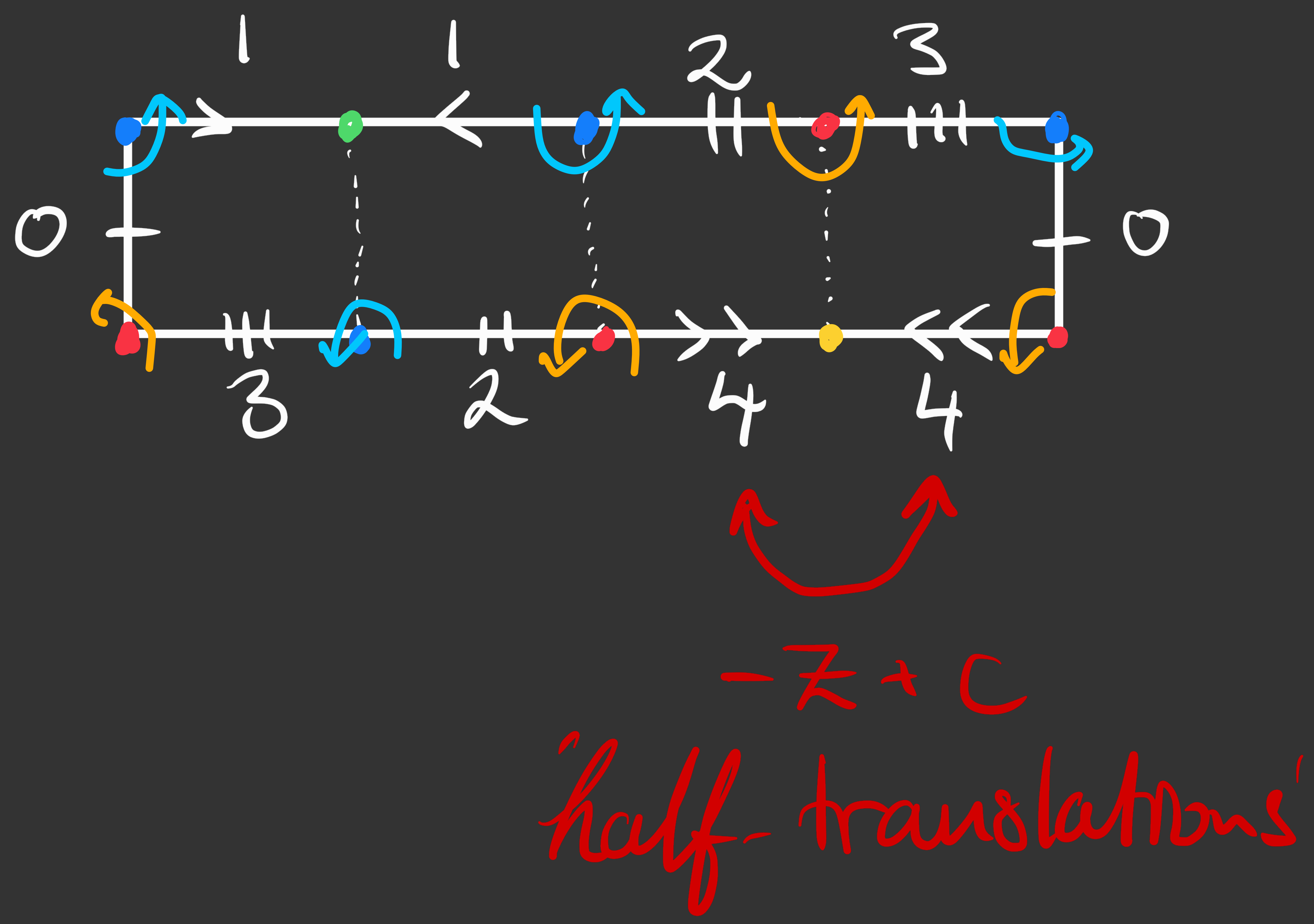

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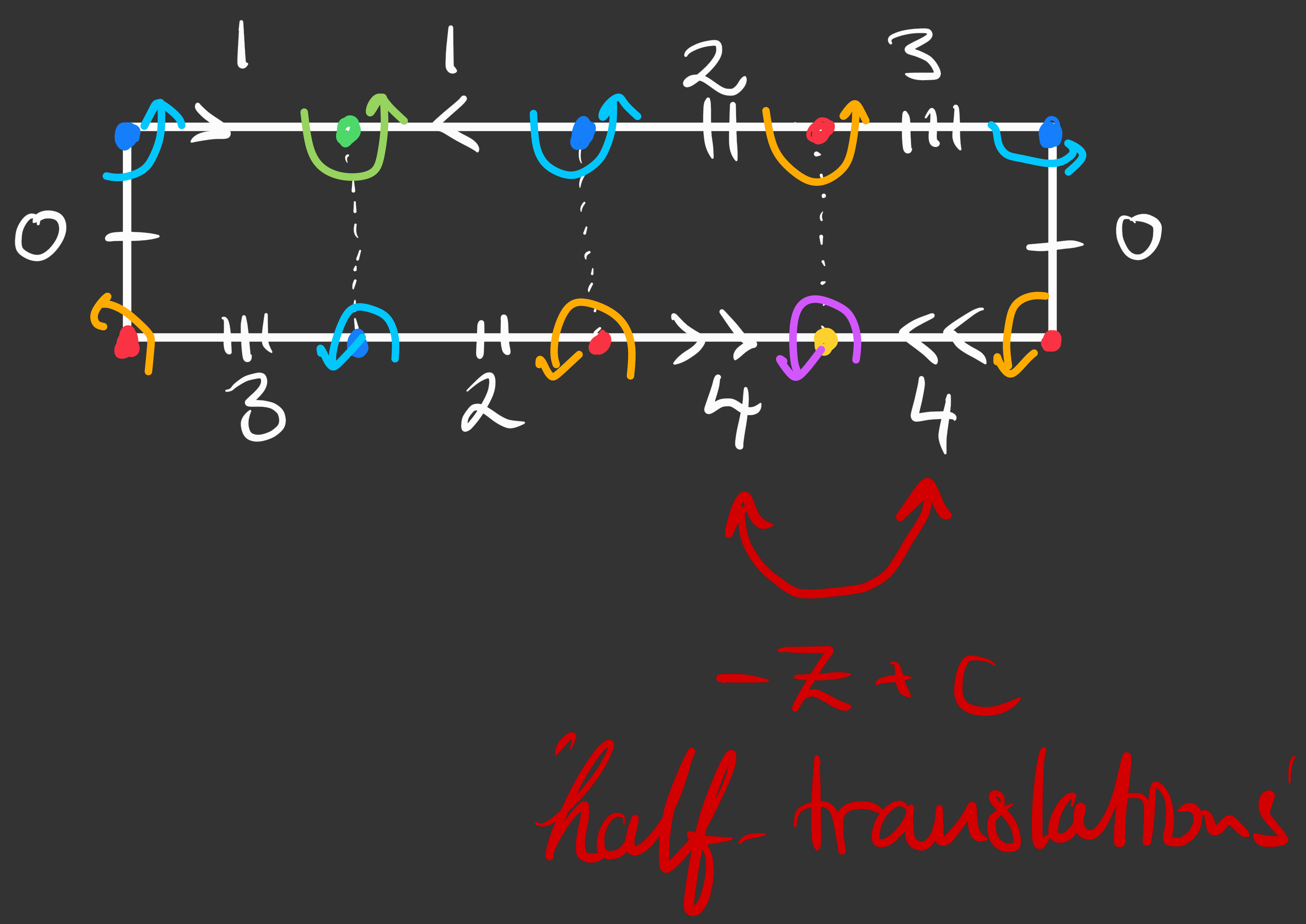

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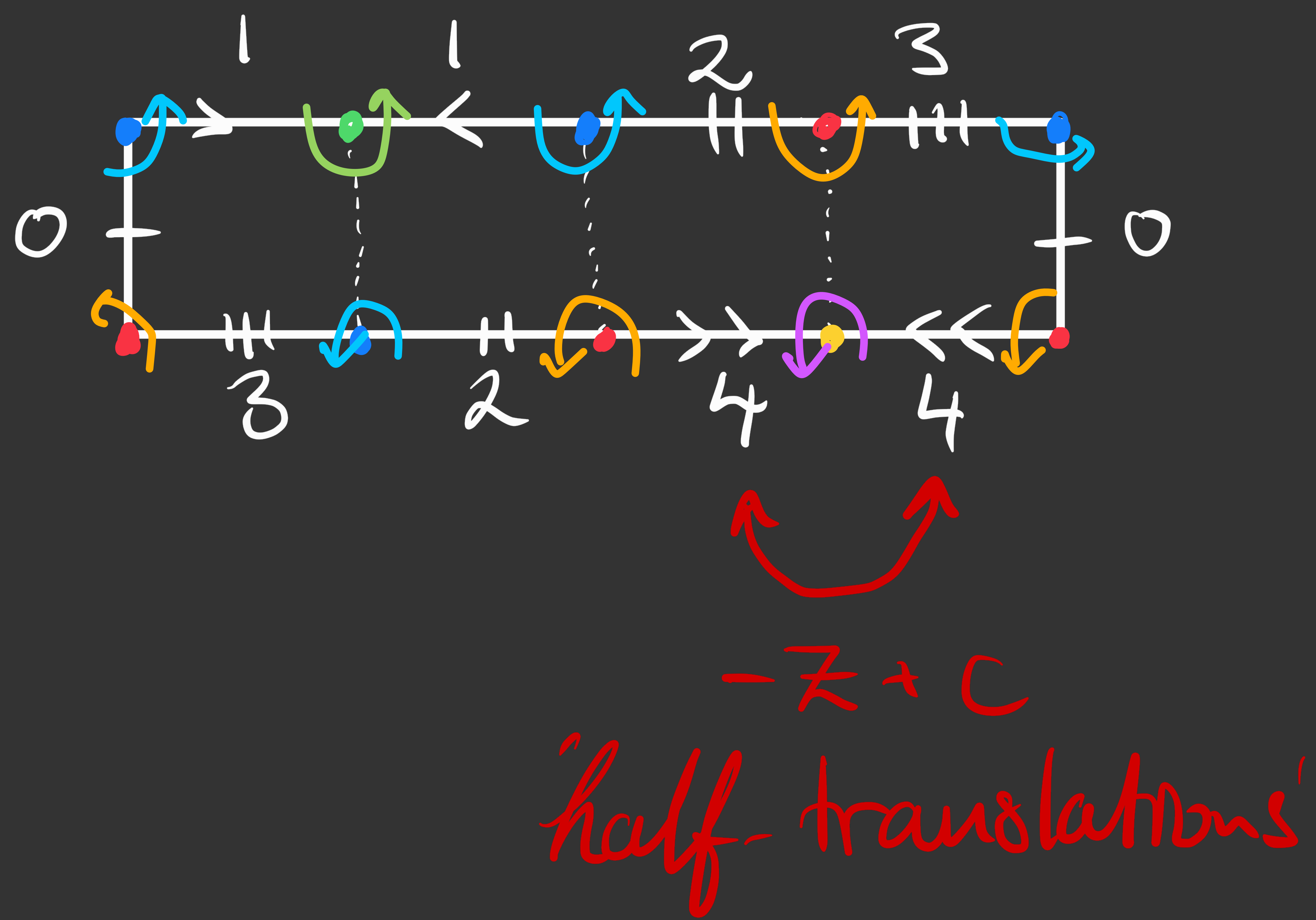

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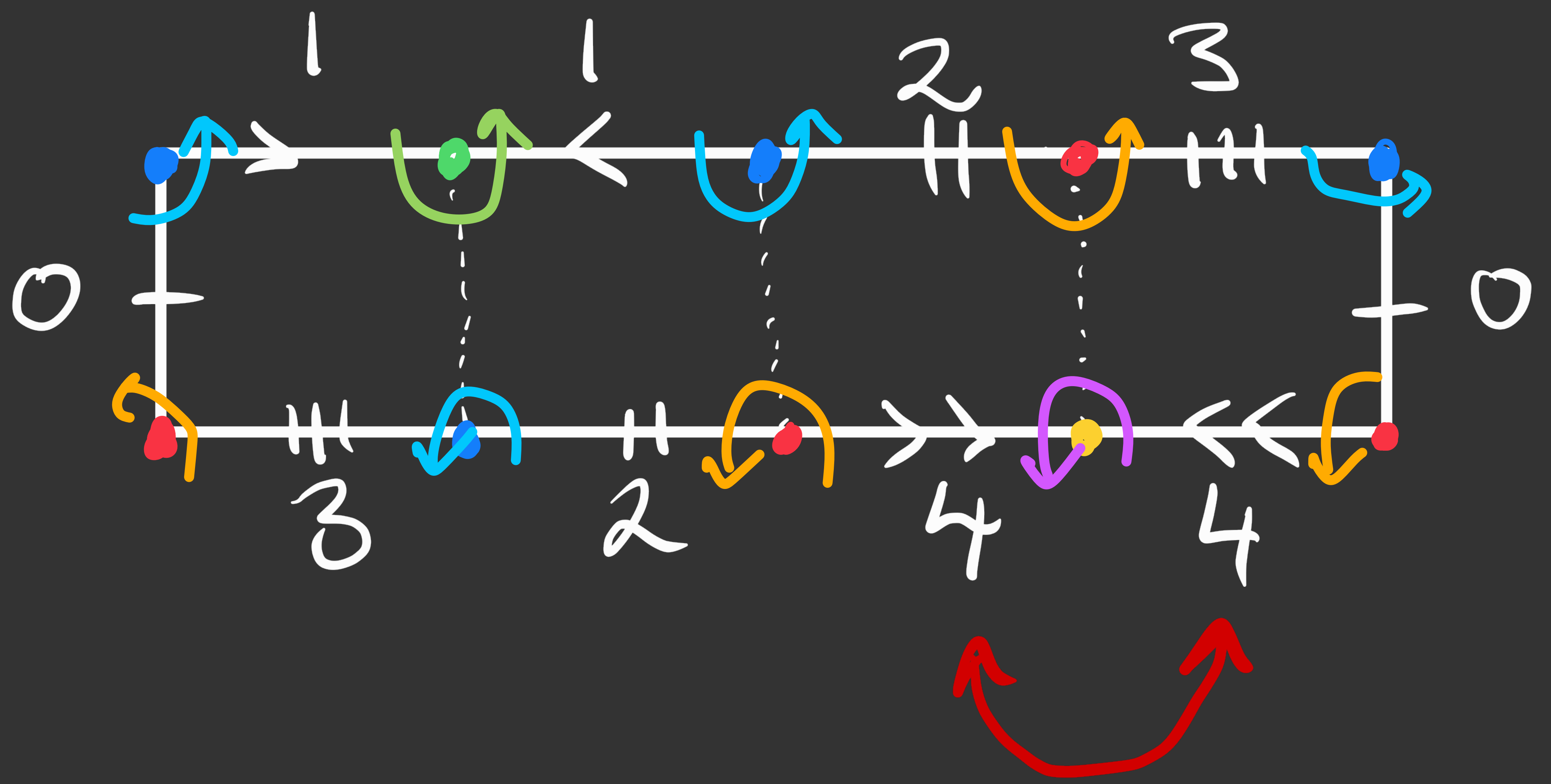
So in $Q(1,1,-1,-1)$

and another...



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Two points of cone angle

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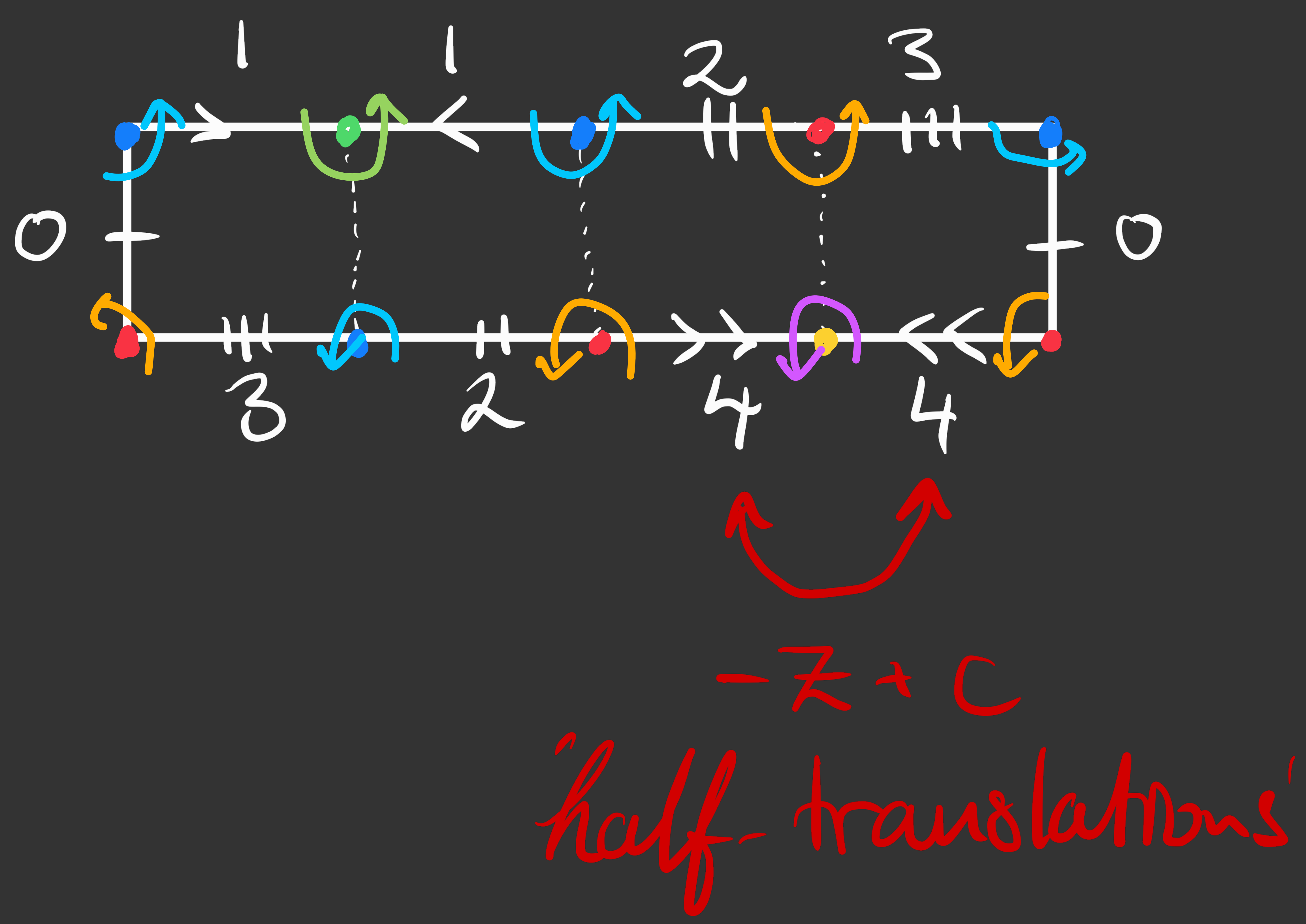

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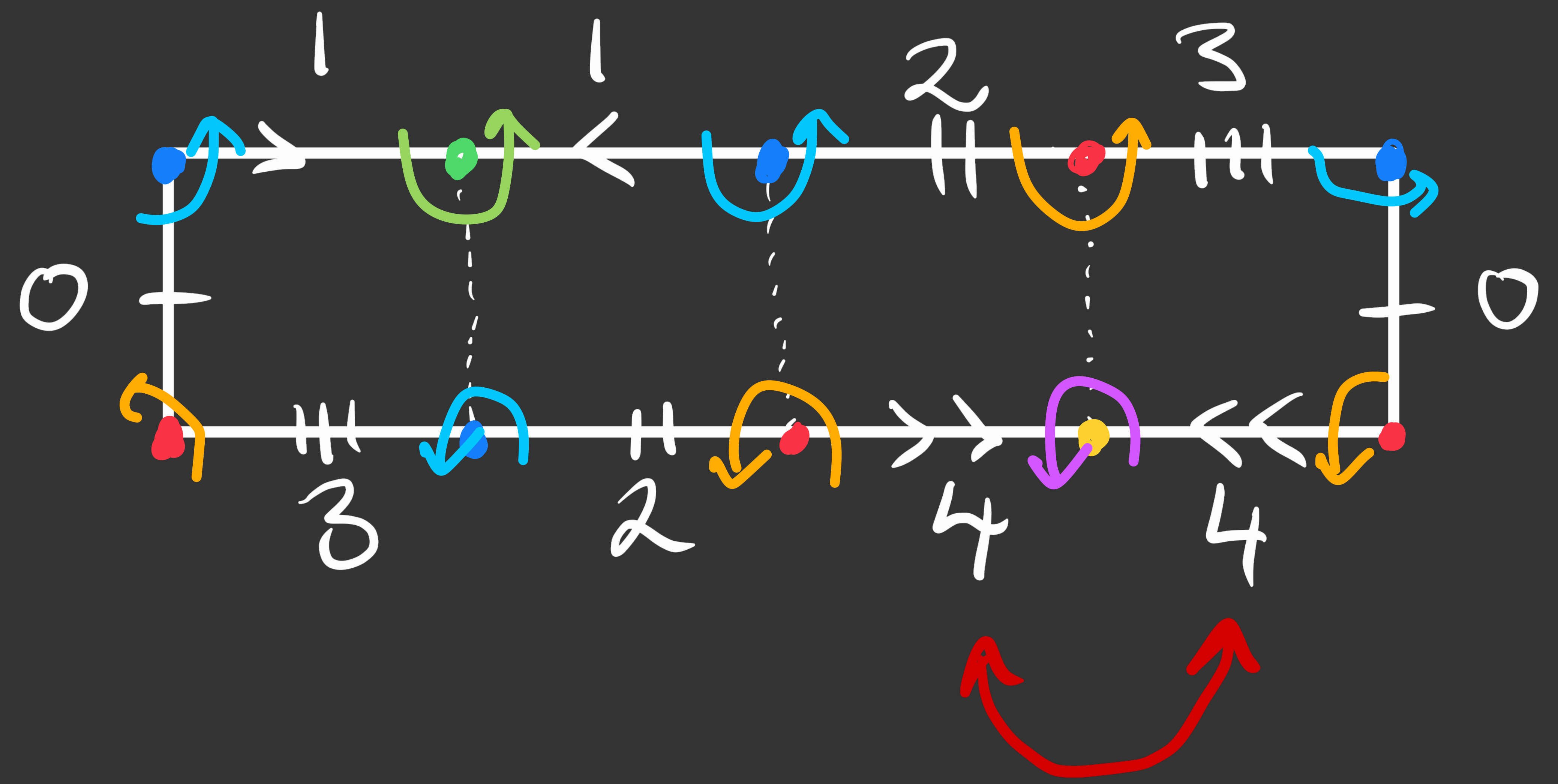

$-z+c$

'half-translations'

Two points of cone angle $3\pi = (1+2)\pi$

Two points of cone angle $\pi = (-1+2)\pi$.

So in $Q(1,1,-1,-1)$

A pillowcase cover requires at least

$$n_{min} = 2 \times \text{genus} + \#\text{singularities} - 2$$

many squares

$[1,1]$-pillowcase covers

[1, 1]-pillowcase covers

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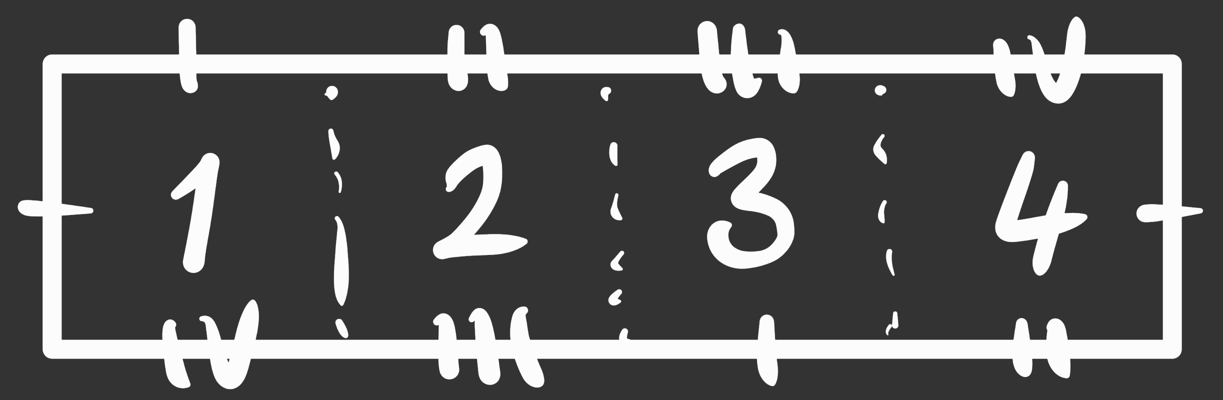
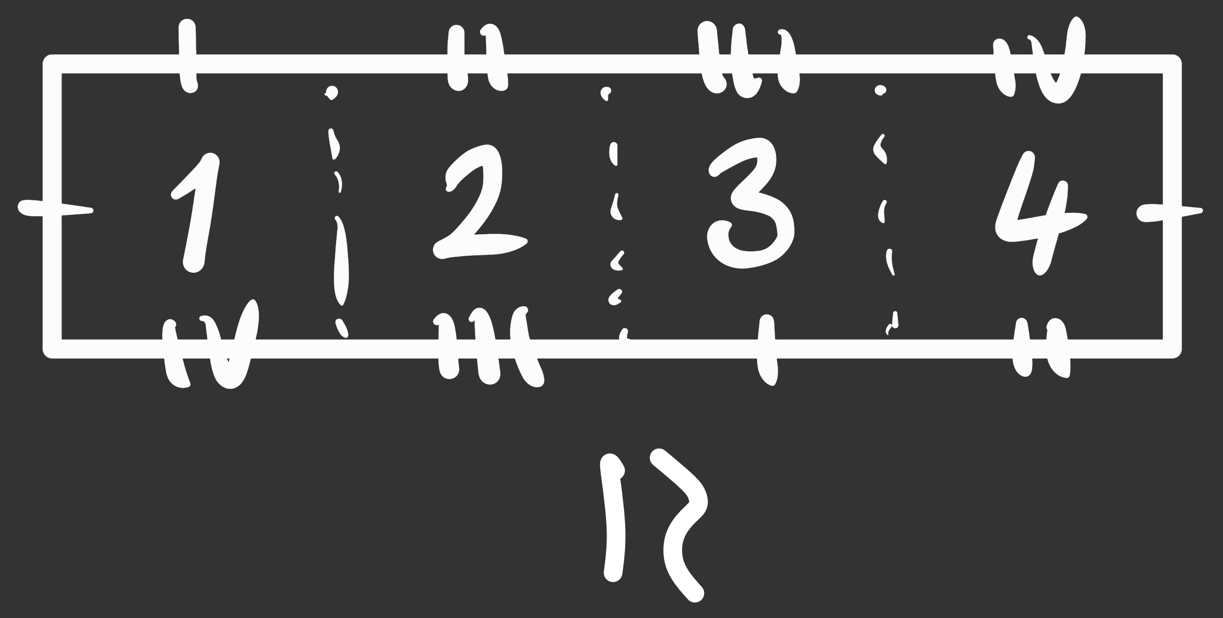


[1, 1]-pillowcase covers



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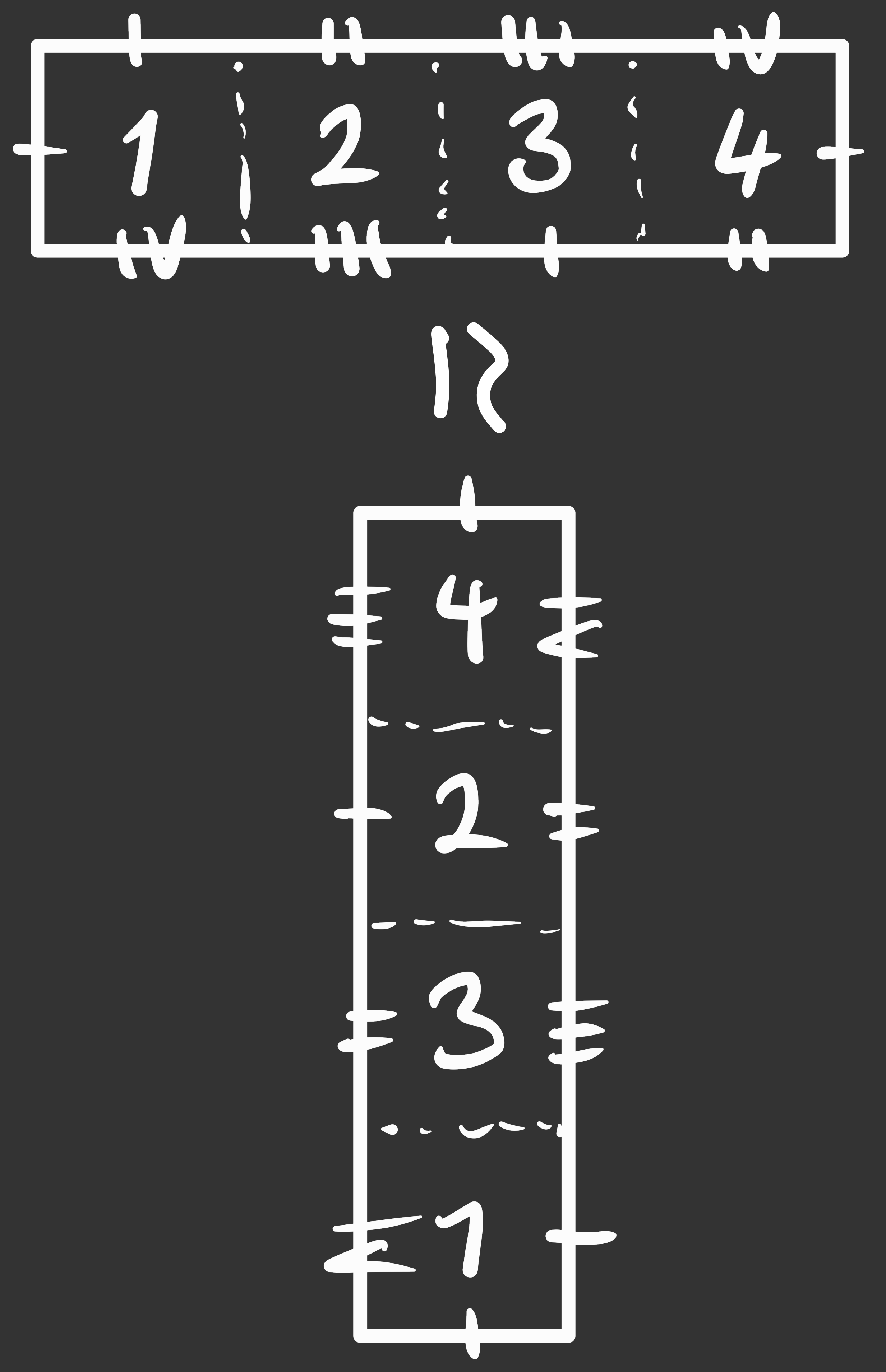
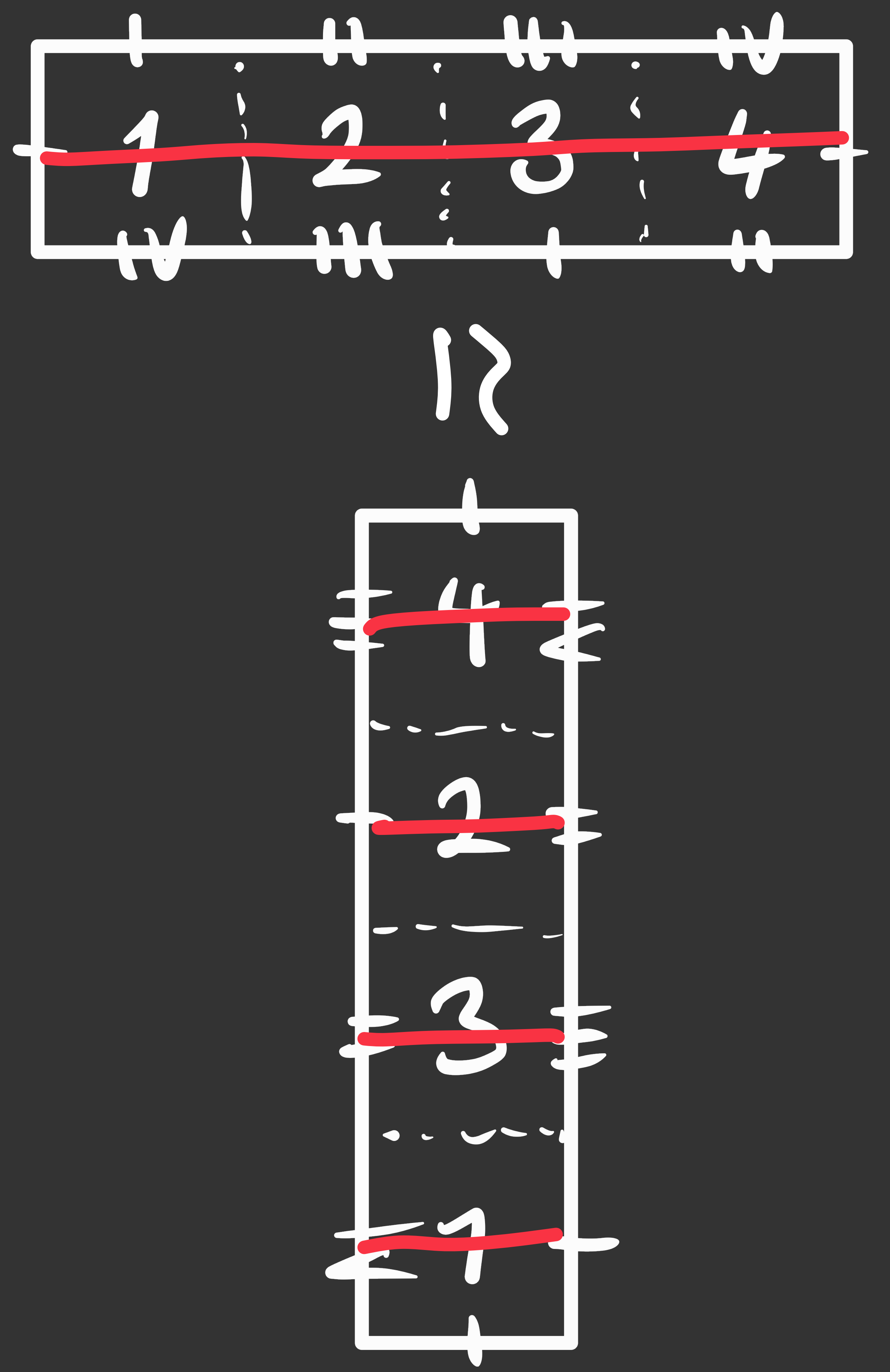
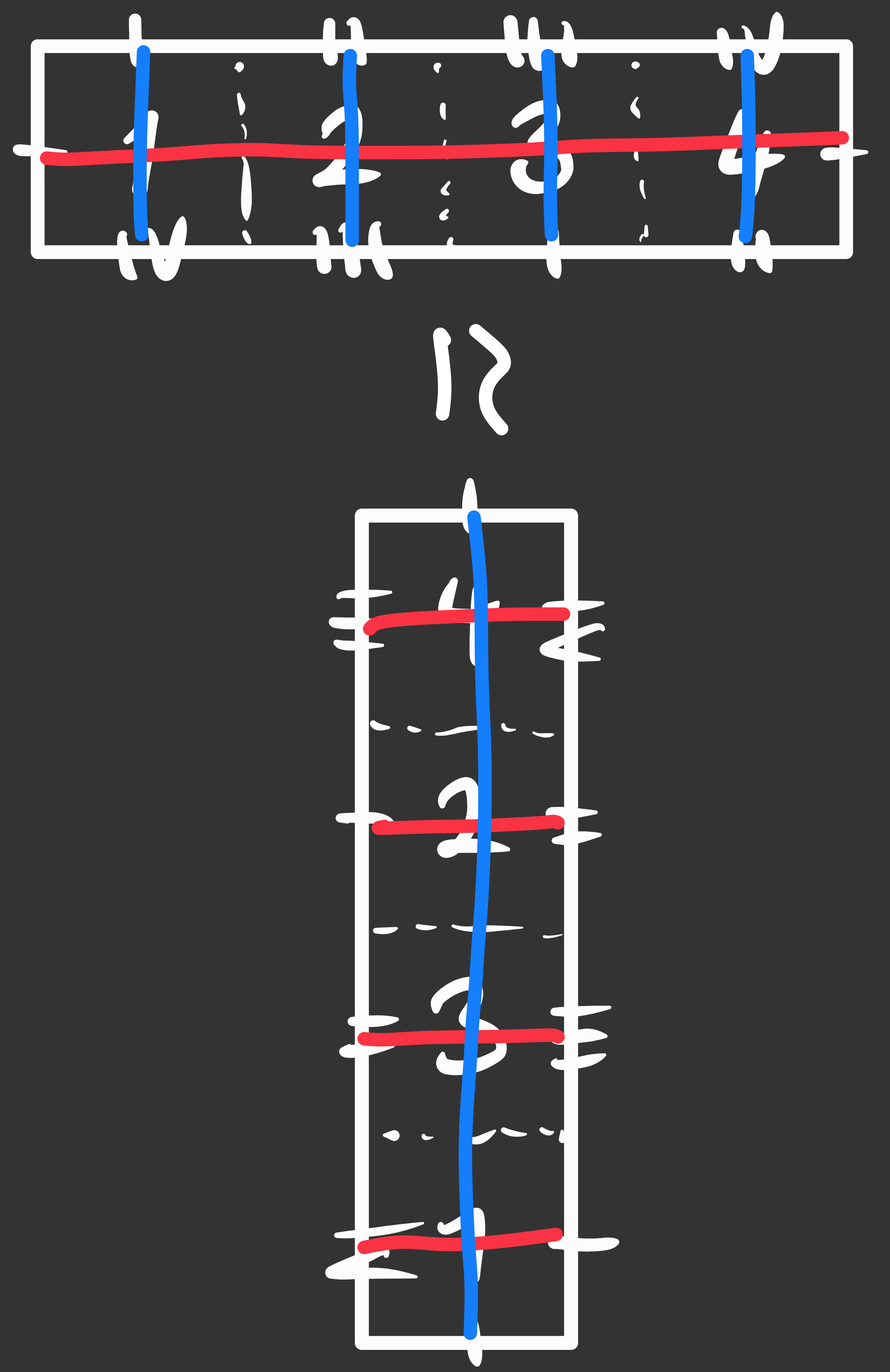


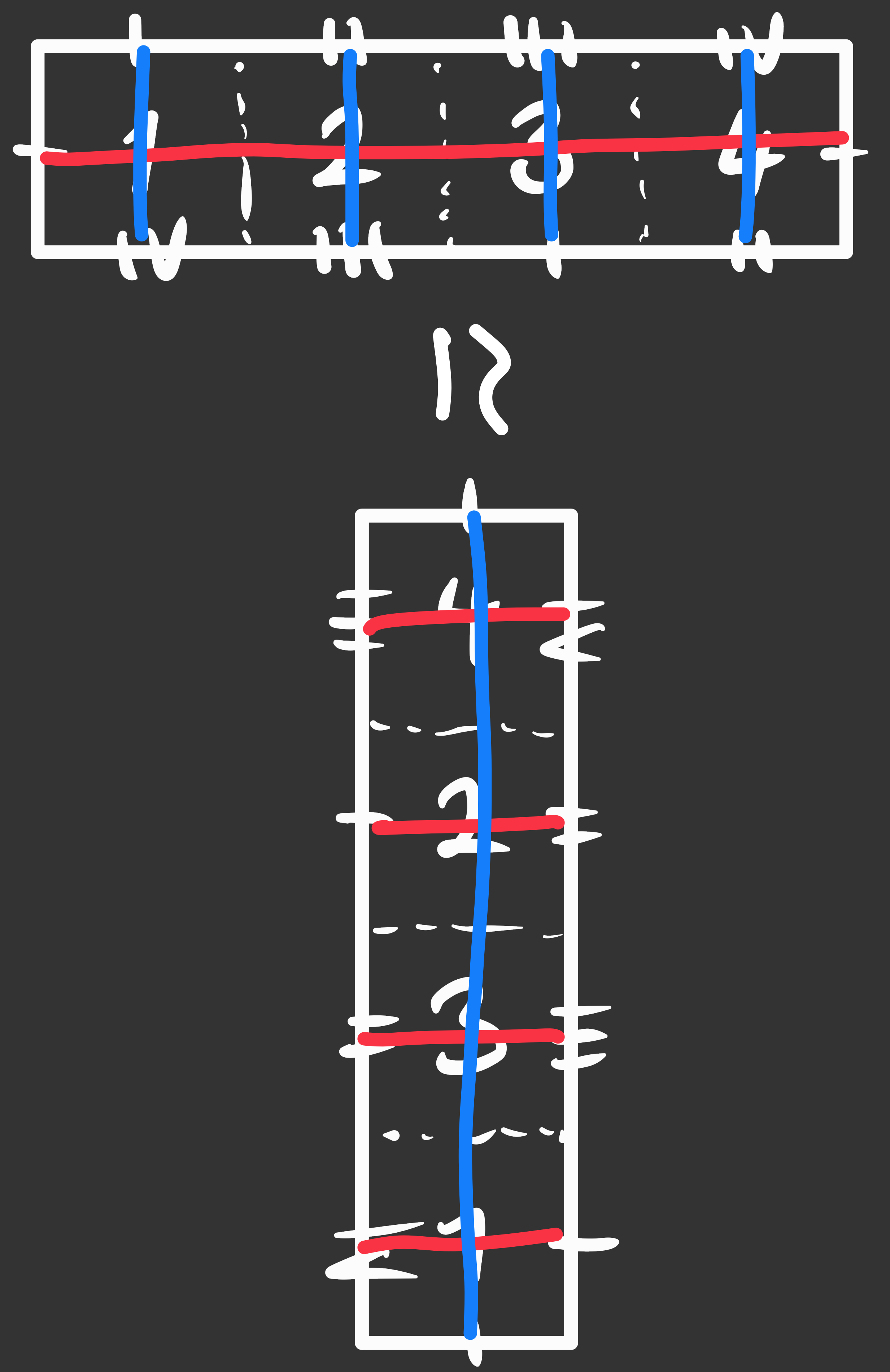
[1, 1]-pillowcase covers

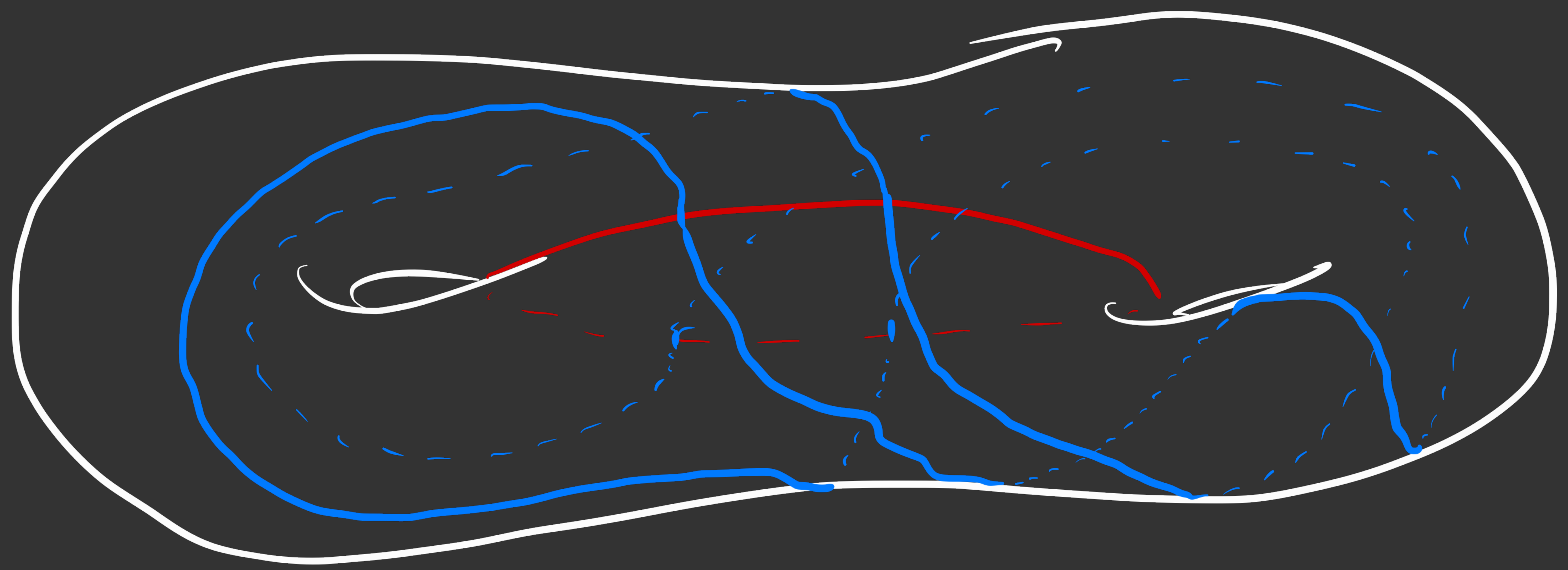


[1,1]-pillowcase covers



[1, 1]-pillowcase covers





$[1,1]$-pillowcase covers

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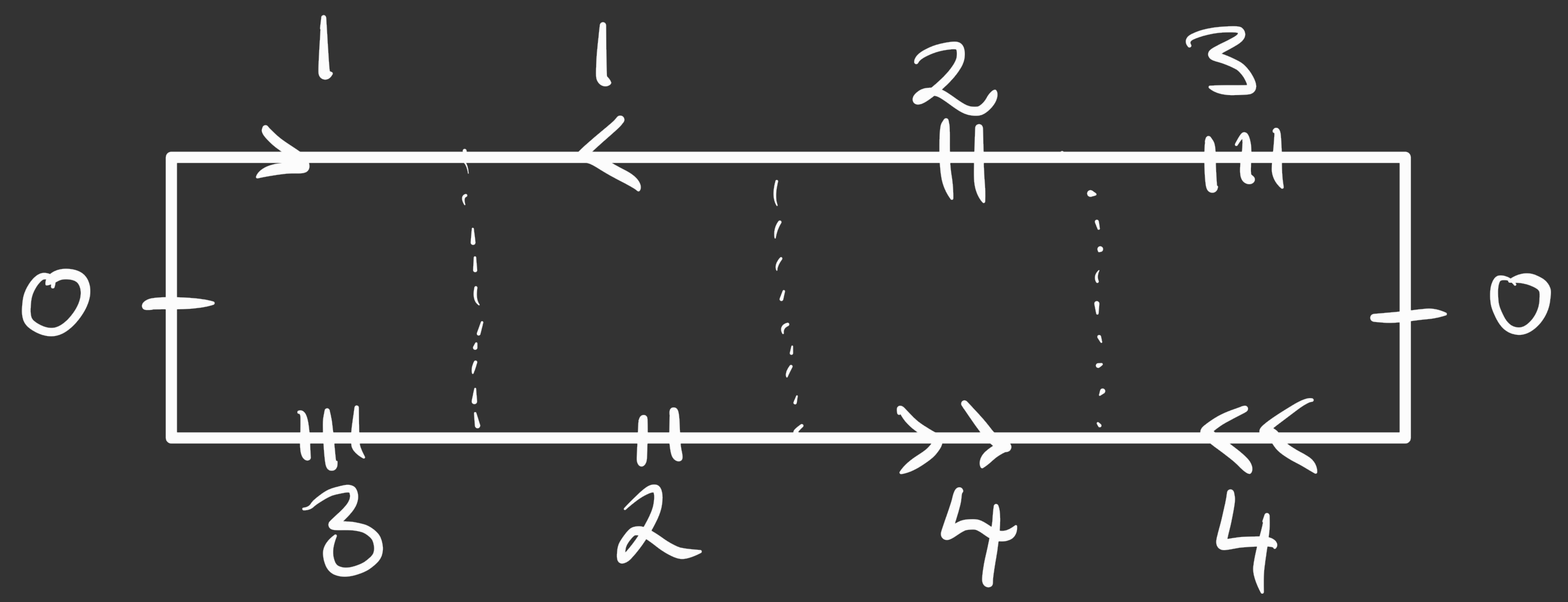
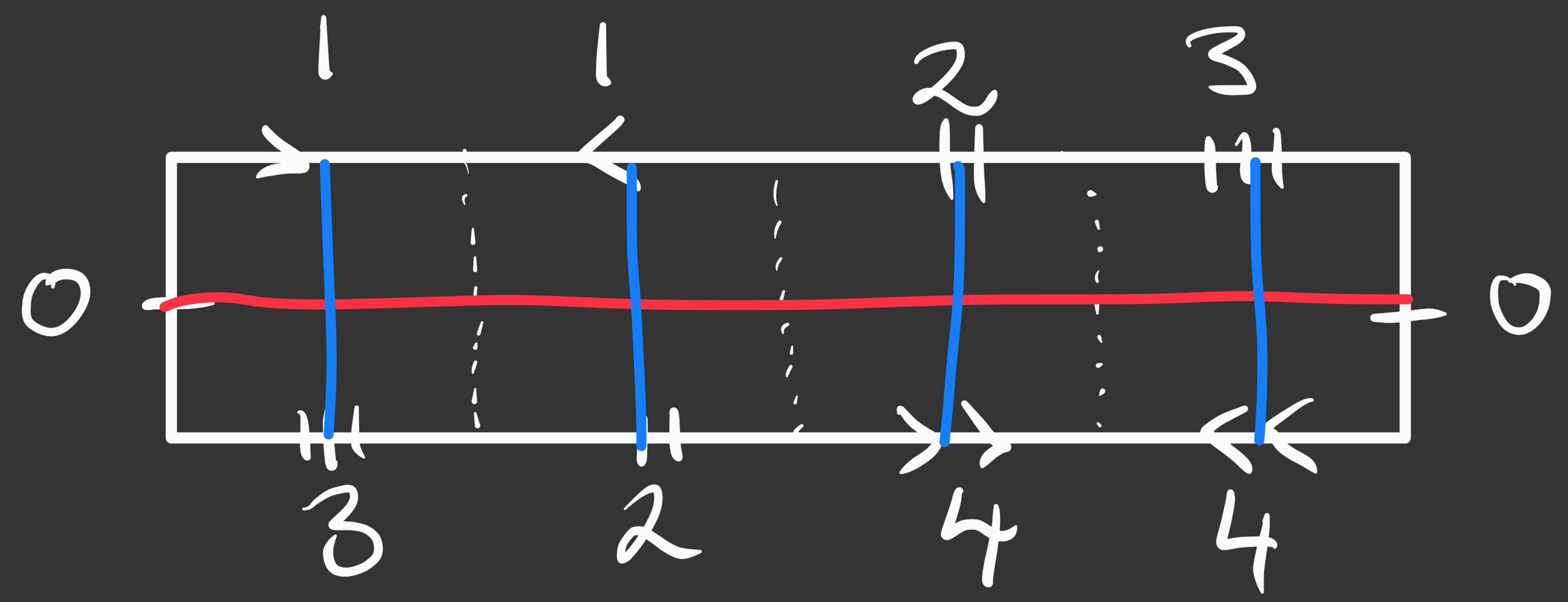


[1,1]-pillowcase covers



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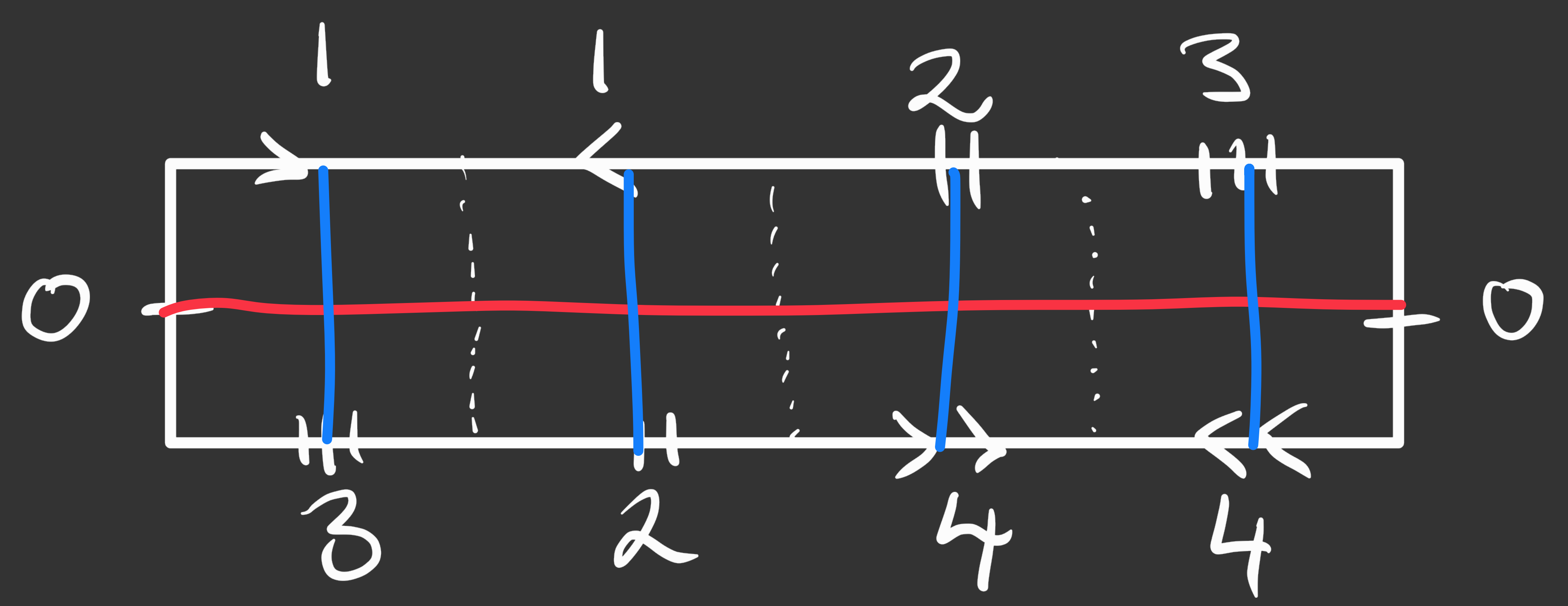

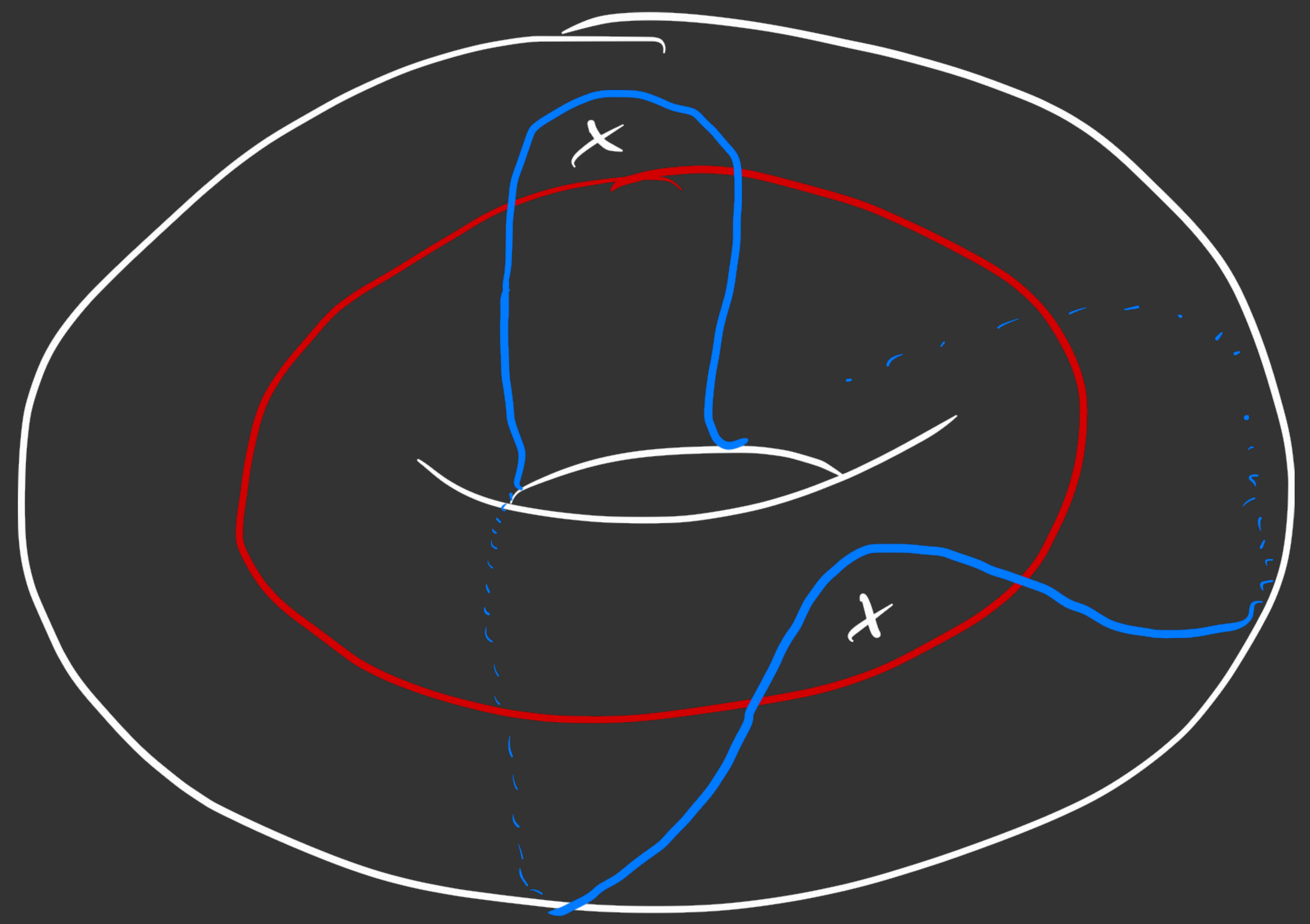

Motivations

Motivation 1: Filling pairs

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A pair of curves that decompose the surface into disks

(filling pair)

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Useful in the combinatorial and topological
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Exactly the dual curves of [1,1]-pillowcase covers

Motivation 2: The moduli space of quadratic differentials

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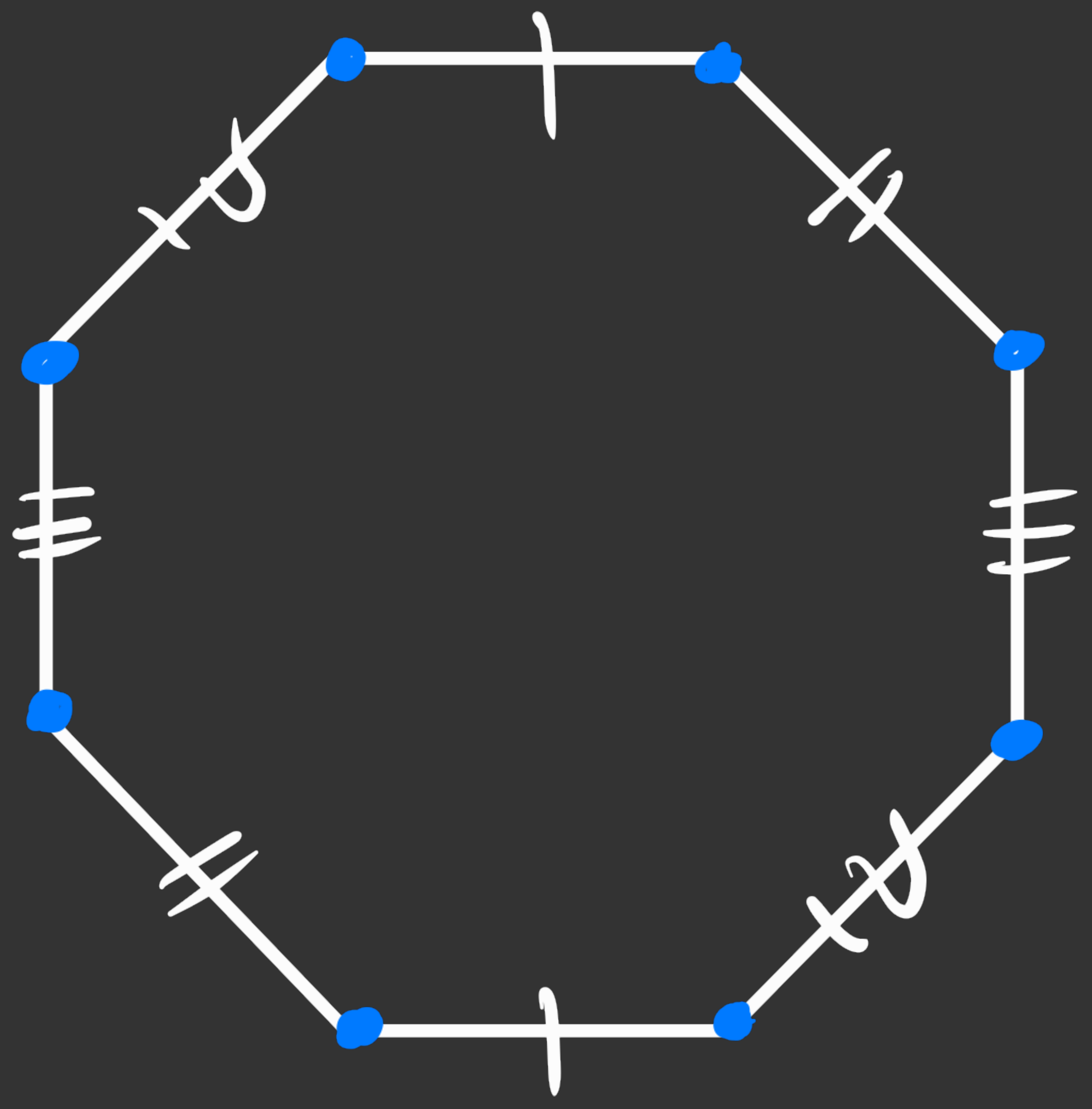
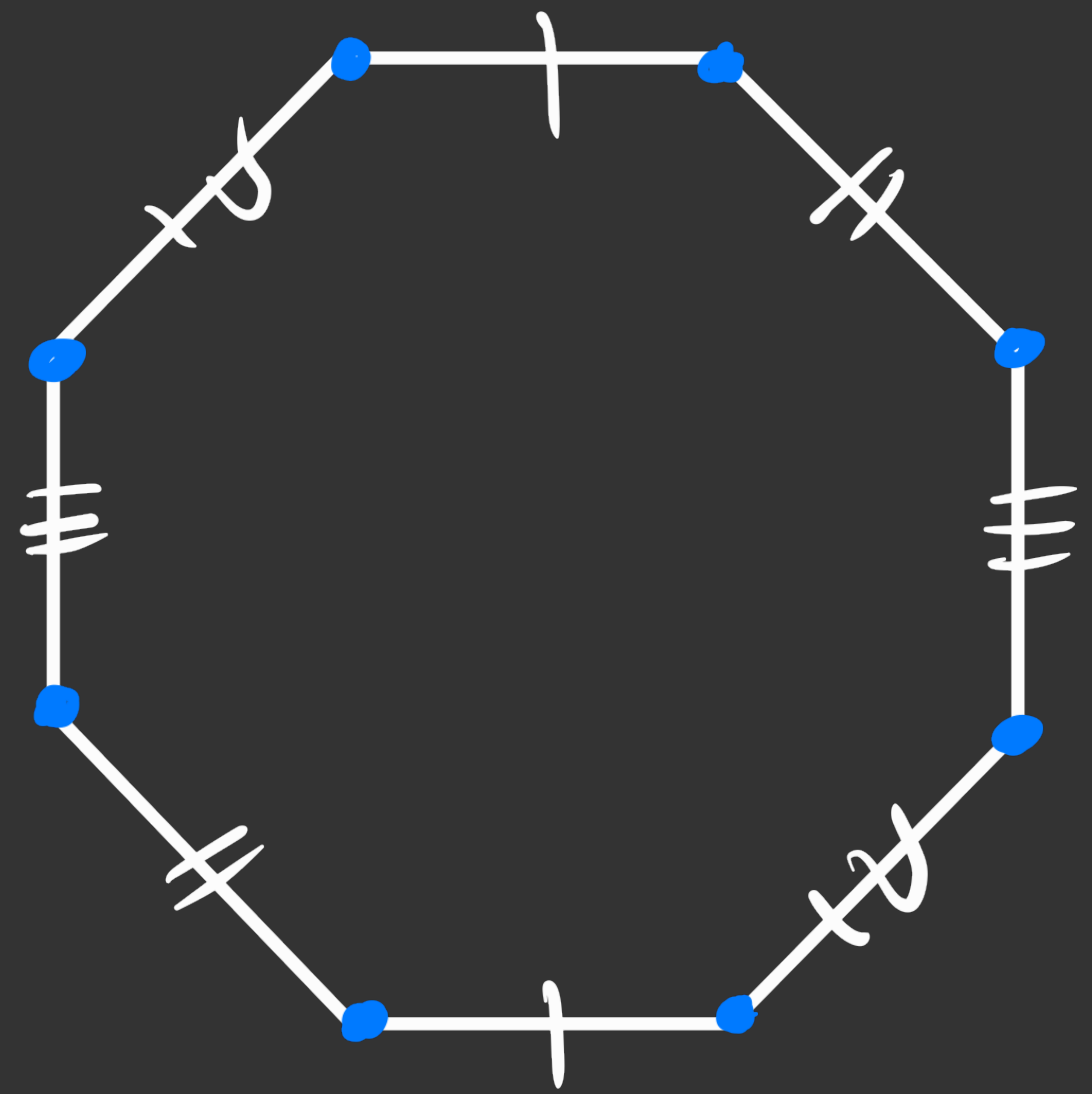
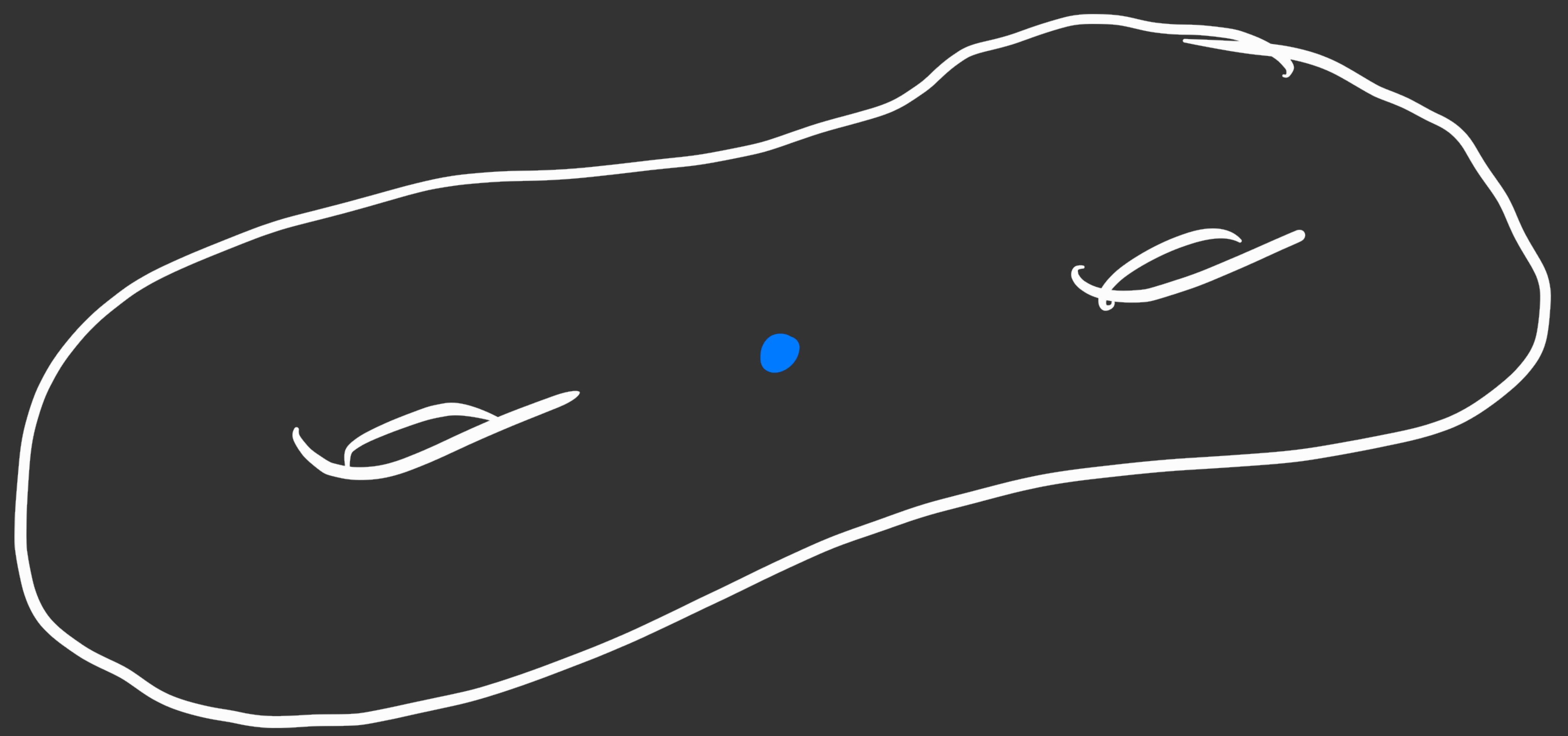


Motivation 2: The moduli space of quadratic differentials





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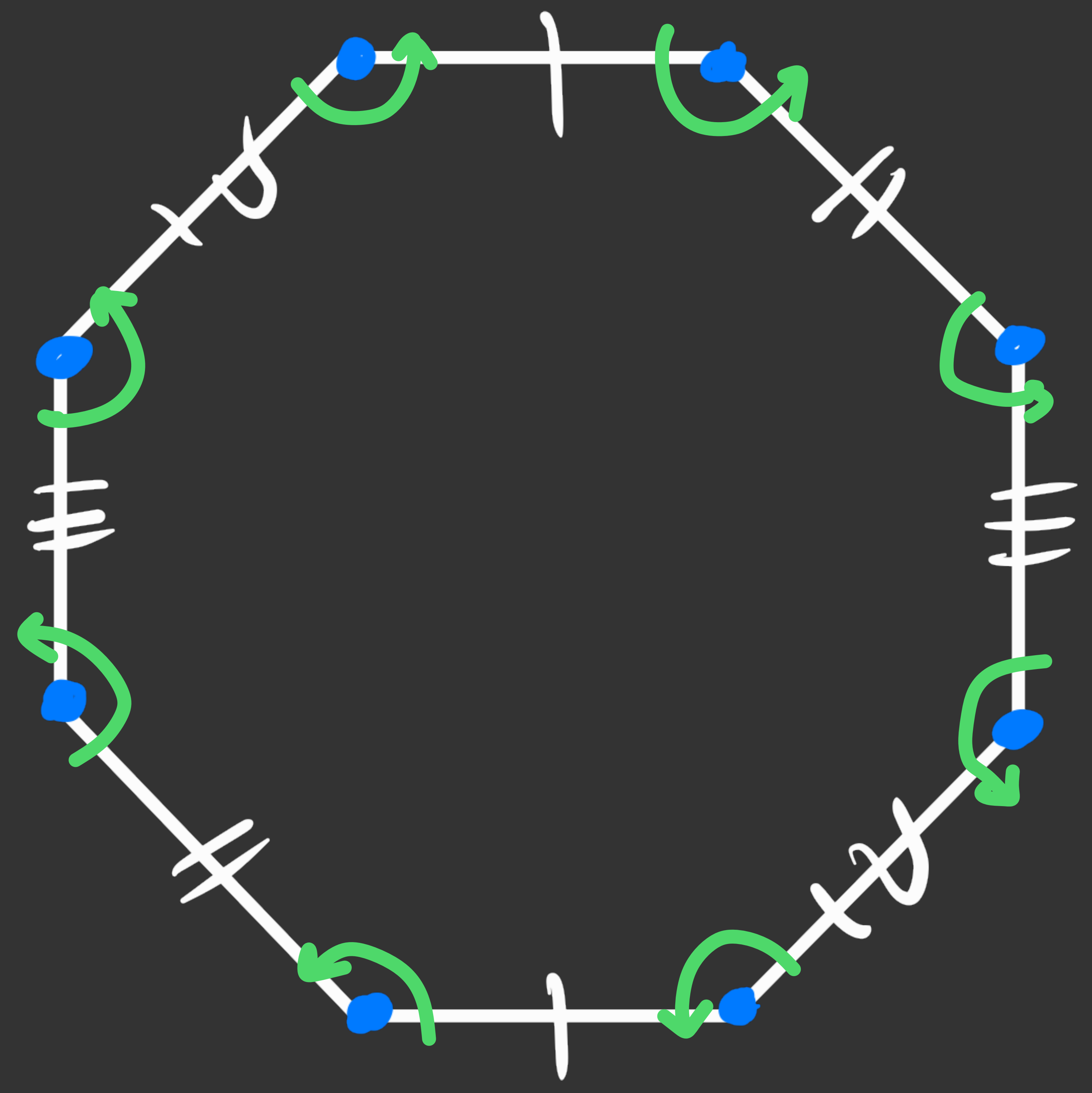

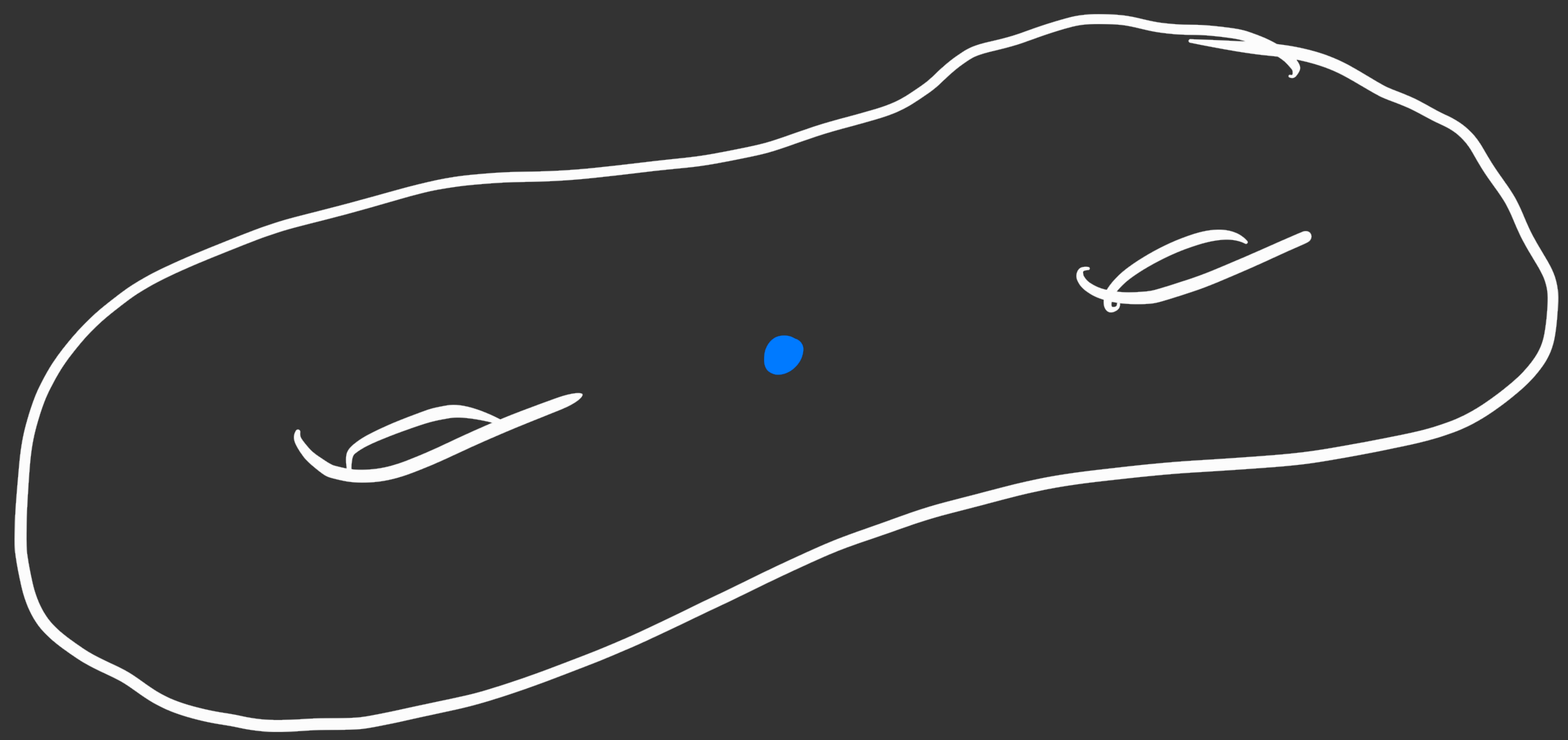

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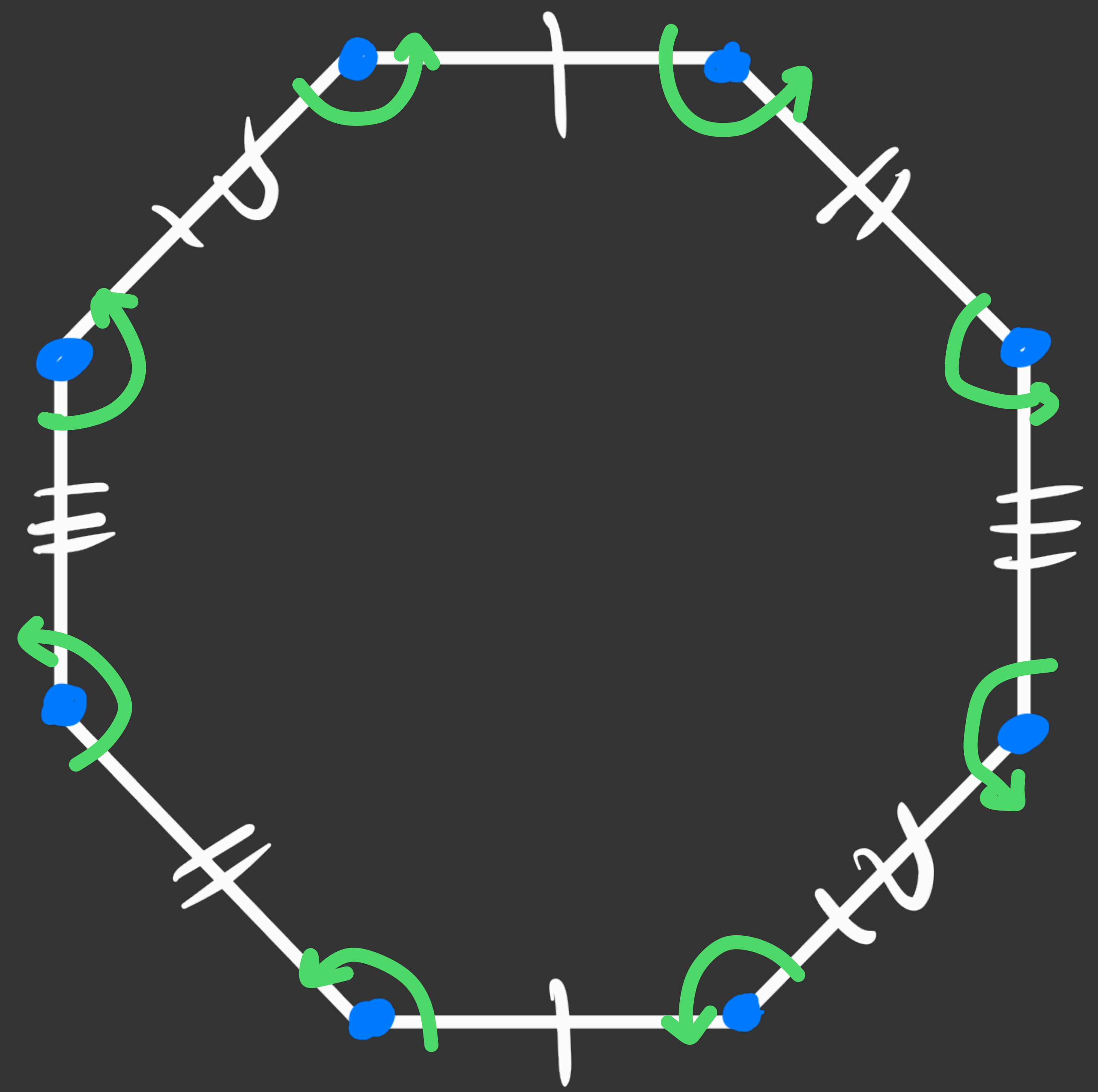

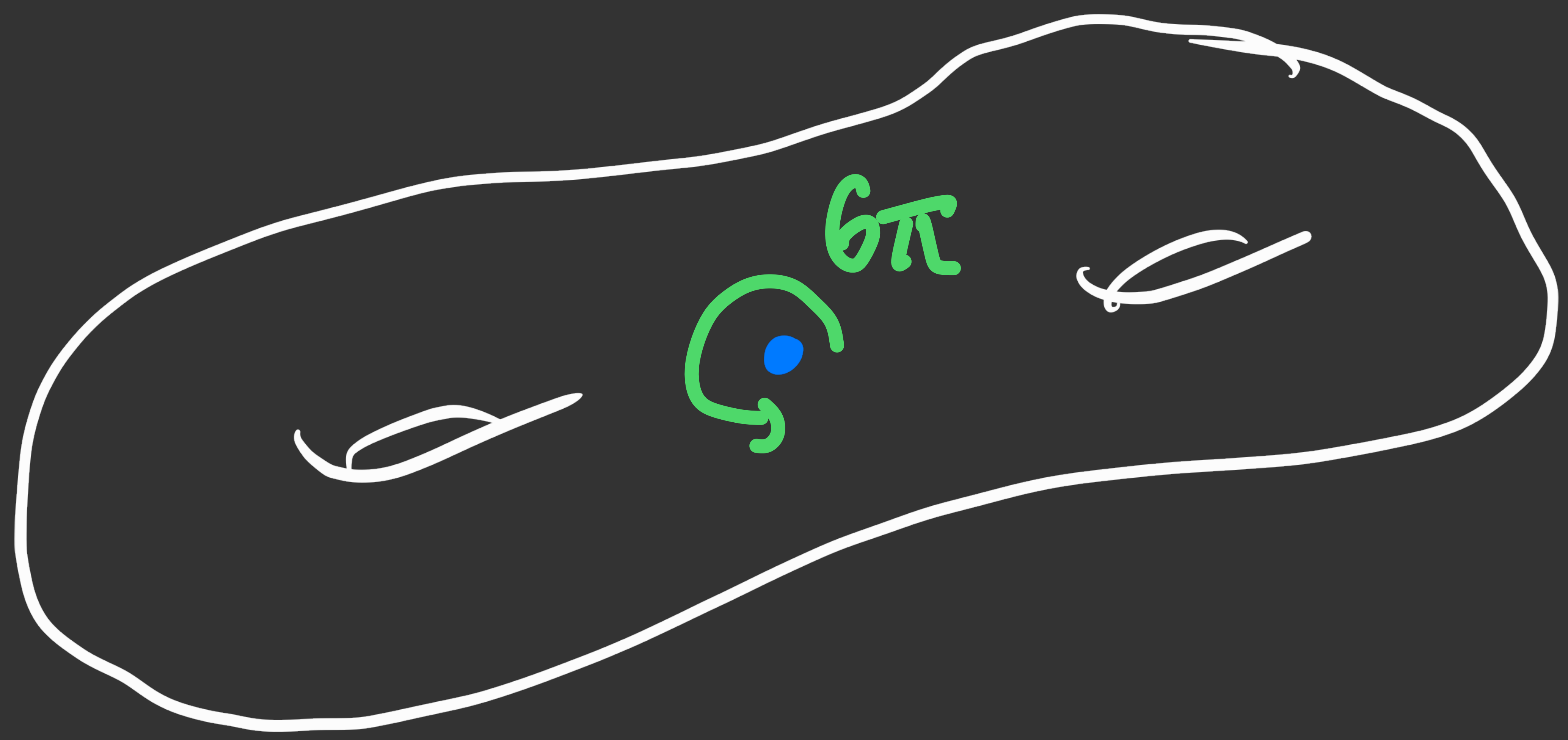

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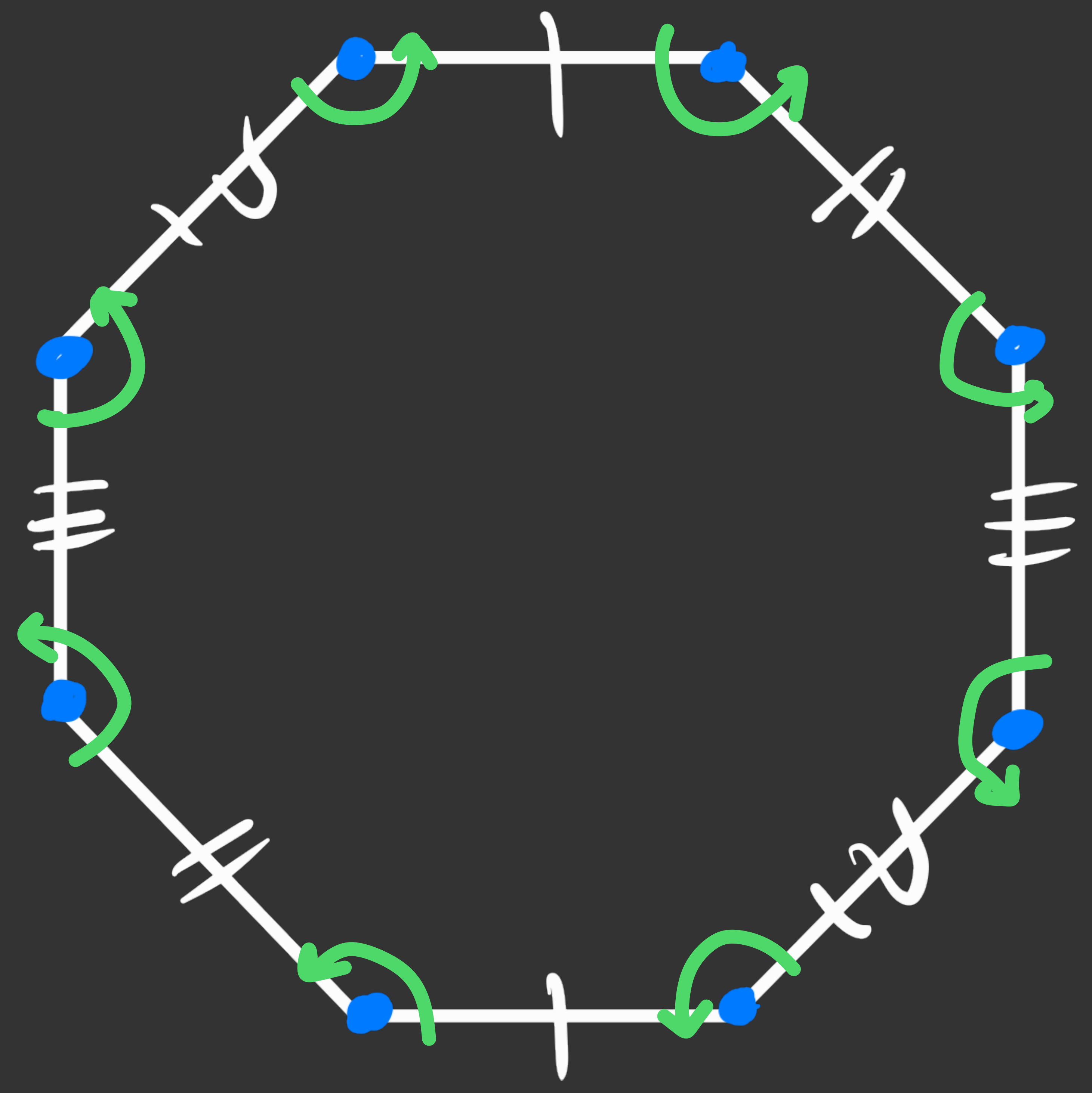
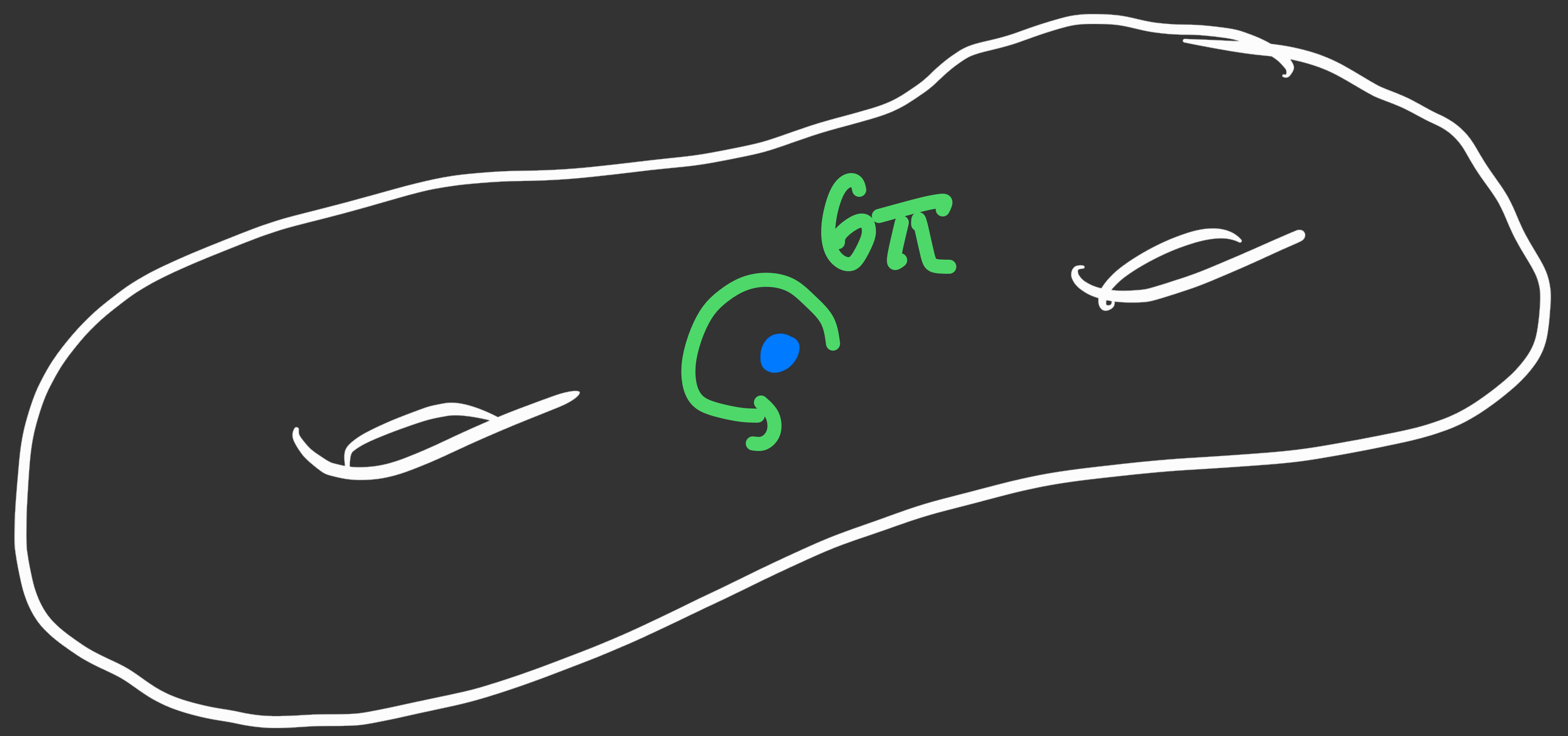
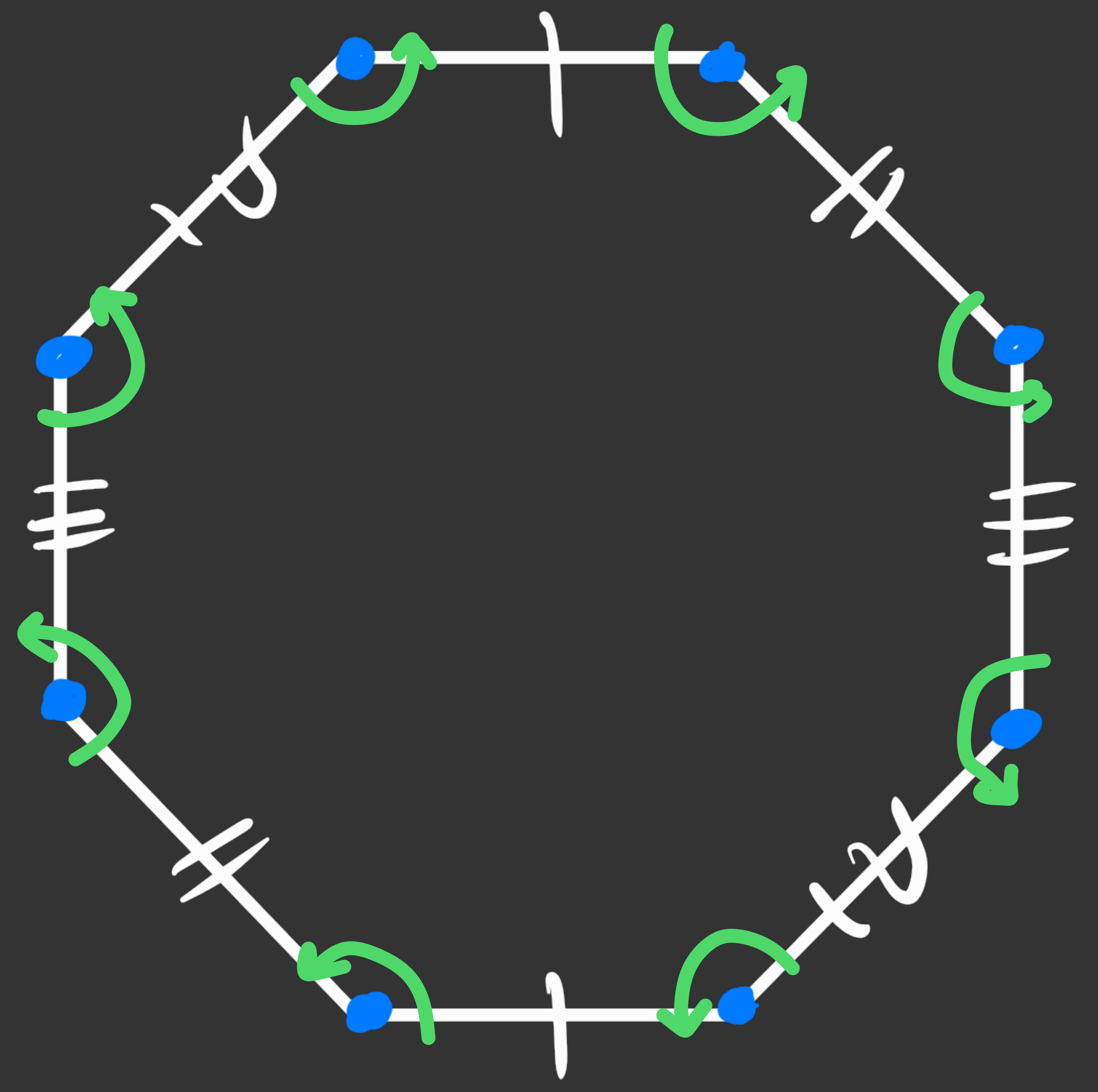
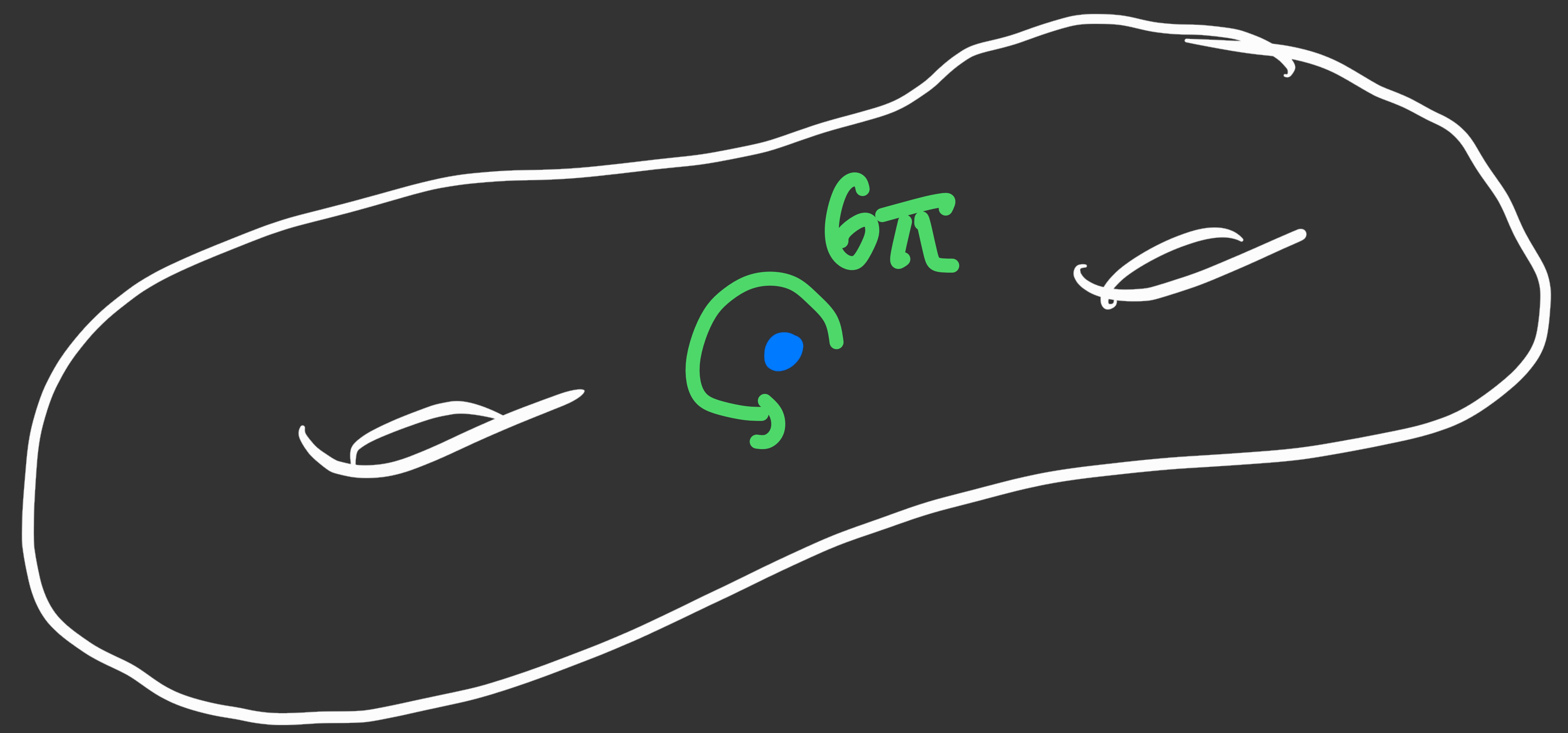




$$6\pi = (2+1)\,2\pi$$

Motivation 2: The moduli space of quadratic differentials





$$6\pi = (2+1)\,2\pi$$

In the stratum $\mathcal{H}(2)$.

$\mathcal{H} \sqcup \mathcal{Q}$

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$$\mathcal{H} = \bigsqcup_{\substack{(k_1, \dots, k_n) \\ k_i \geq 1 \\ \sum k_i = 2g-2}} \mathcal{H}(k_1, \dots, k_n)$$

$$\mathcal{H} \sqcup Q$$

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$$Q = \bigsqcup_{\substack{(k_1,\dots,k_n) \\ k_i \geq 1 \text{ or } k_i = -1 \\ \sum k_i = 4g-4}} Q(k_1,\dots,k_n)$$

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strata can be disconnected.

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Kontsevich-Zorich

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Kontsevich-Zorich

Lanneau, Chen-Möller

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Lanneau, Chen-Möller

strata can be disconnected.

- hyperelliptic components

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Kontsevich-Zorich

Lanneau, Chen-Möller

strata can be disconnected.

- hyperelliptic components
- non-hyperelliptic components

$$\mathcal{H} \sqcup \mathcal{Q}$$

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Kontsevich-Zorich

Lanneau,
Chen-Möller

strata can be disconnected.

- hyperelliptic components ← Focus later on
- non-hyperelliptic components

Motivation 3: Ratio-optimising pseudo-Anosovs

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Important in the study of the systole map

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Important in the study of the systole map

Aougab-Taylor '17

- Can construct ratio-optimising pseudo-Anosovs stabilising the Teichmüller disk of a quadratic differential.

What is known ?

Old Question:

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Given any connected component $\mathcal{C}$ of $\mathcal{H}$ or $\mathcal{Q}$, can you build a $[1,1]$-pillowcase cover in $\mathcal{C}$ using only d_{min} squares?

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Given any connected component $\mathcal{C}$ of $\mathcal{H}$ or $\mathcal{Q}$, can you build a $[1,1]$-pillowcase cover in $\mathcal{C}$ using only d_{min} squares?

If not, what is the minimum required?

Theorem (J, '21 + '22)

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Let $\mathcal{C}_1$ be any non-hyperelliptic connected component of $\mathcal{H}$ or $\mathcal{Q}$ (with no poles)

Theorem (J, '21 + '22)

Let $\mathcal{C}_1$ be any non-hyperelliptic connected component of $\mathcal{H}$ or $\mathcal{Q}$ (with no poles) then there exists a $[1,1]$-pillowcase cover in $\mathcal{C}_1$ built using n_{min} squares.

<u>Theorem</u> (J, '21 + '22)

Theorem (J, '21 + '22)

Let $\mathcal{C}_2$ be any hyperelliptic component of $\mathcal{H}$ or $\mathcal{Q}$ then any $[1,1]$-pillowcase cover in $\mathcal{C}_2$ requires strictly more than n_{min} squares.

Theorem (J, '21 + '22)

Let $\mathcal{C}_2$ be any hyperelliptic component of $\mathcal{H}$ or $\mathcal{Q}$ then any $[1,1]$-pillowcase cover in $\mathcal{C}_2$ requires strictly more than n_{min} squares. There exist $[1,1]$-pillowcase covers in $\mathcal{C}_2$ realising the bound.

Caveat: All of the dual curves produced are non-separating.

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This restriction is forced by the techniques used in the proof.

Question: How many squares are required to build a $[1,1]$-pillowcase cover in a given component $\mathcal{C}$ so that one or both of the dual curves are separating?

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NB: $\mathcal{C}$ must be in $\mathcal{Q}$.

Theorem (J, '22)

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The following is true for hyperelliptic components:

Theorem (J, '22)

The following is true for hyperelliptic components:

Connected component	Minimum number of squares required to produce a $[1,1]$-pillowcase cover whose cylinders are		
	both non-sep.	one sep. one non-sep.	both sep.
$\mathcal{H}^{hyp}(2g-2), g \geq 2$	$4g-4$	n/a	n/a
$\mathcal{H}^{hyp}(g-1, g-1), g \geq 2$	$4g-2$	n/a	n/a
$\mathcal{Q}^{hyp}(4j+2, 4k+2), k \geq j \geq 0$	$4j+4k+4$	$\max\{8j+6, 8k+4\}$	$16k-8j+8$
$\mathcal{Q}^{hyp}(4j+2, 2k-1, 2k-1), j \geq 1, k \geq 0, j \geq k$	$4j+4k+2$	$8j+4$	$16j-8k+12$
$\mathcal{Q}^{hyp}(4j+2, 2k-1, 2k-1), k > j \geq 0$	$4j+4k+2$	$8k$	$16k-8j$
$\mathcal{Q}^{hyp}(2j-1, 2j-1, 2k-1, 2k-1), k \geq 1, j \geq 0, k \geq j$	$4j+4k$	$\max\{8j+2, 8k\}$	$16k-8j+4$
$\mathcal{Q}^{hyp}(2, -1, -1)$	3	4	12

Proof Ideas

i. Hyperelliptic pillowcase covers are double covers of spheres.

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ii. The dual curves of a hyperelliptic $[1,1]$-pillowcase cover are lifts of filling arc and curve systems on the sphere.

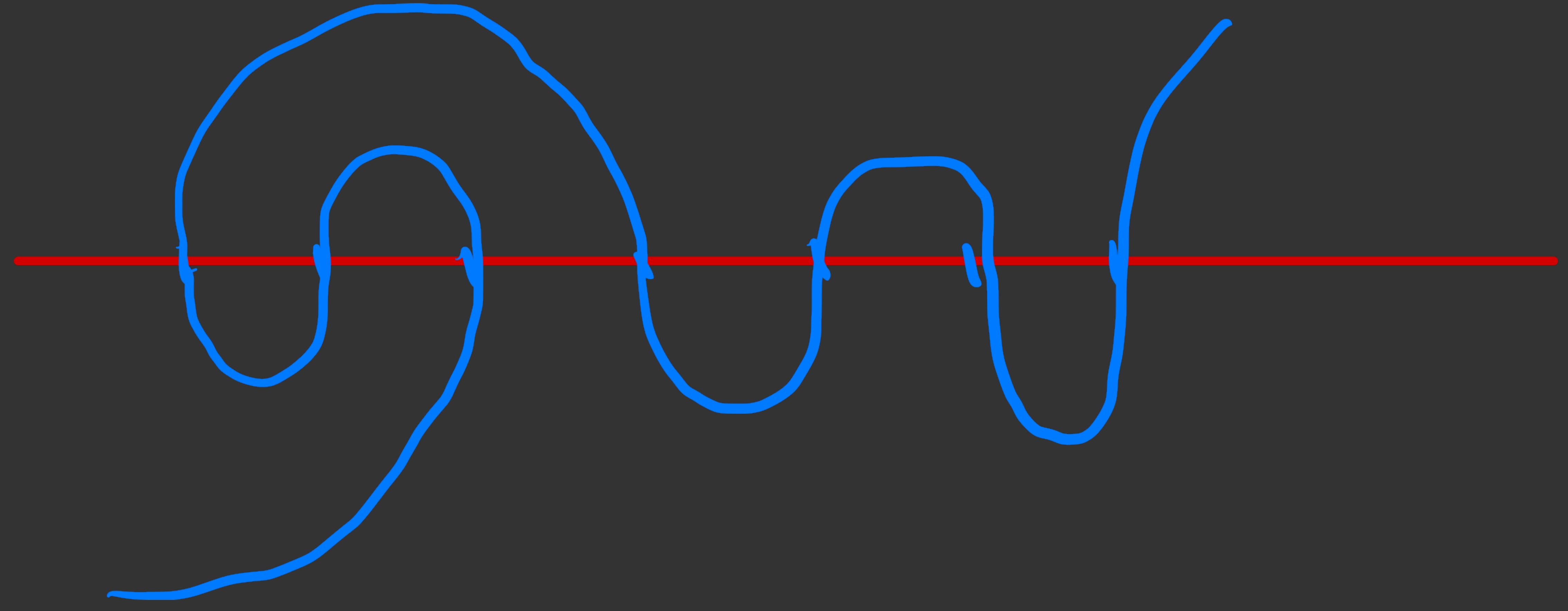
i. Hyperelliptic pillowcase covers are double covers of spheres.

ii. The dual curves of a hyperelliptic [1,1]-pillowcase cover are lifts of filling arc and curve systems on the sphere.

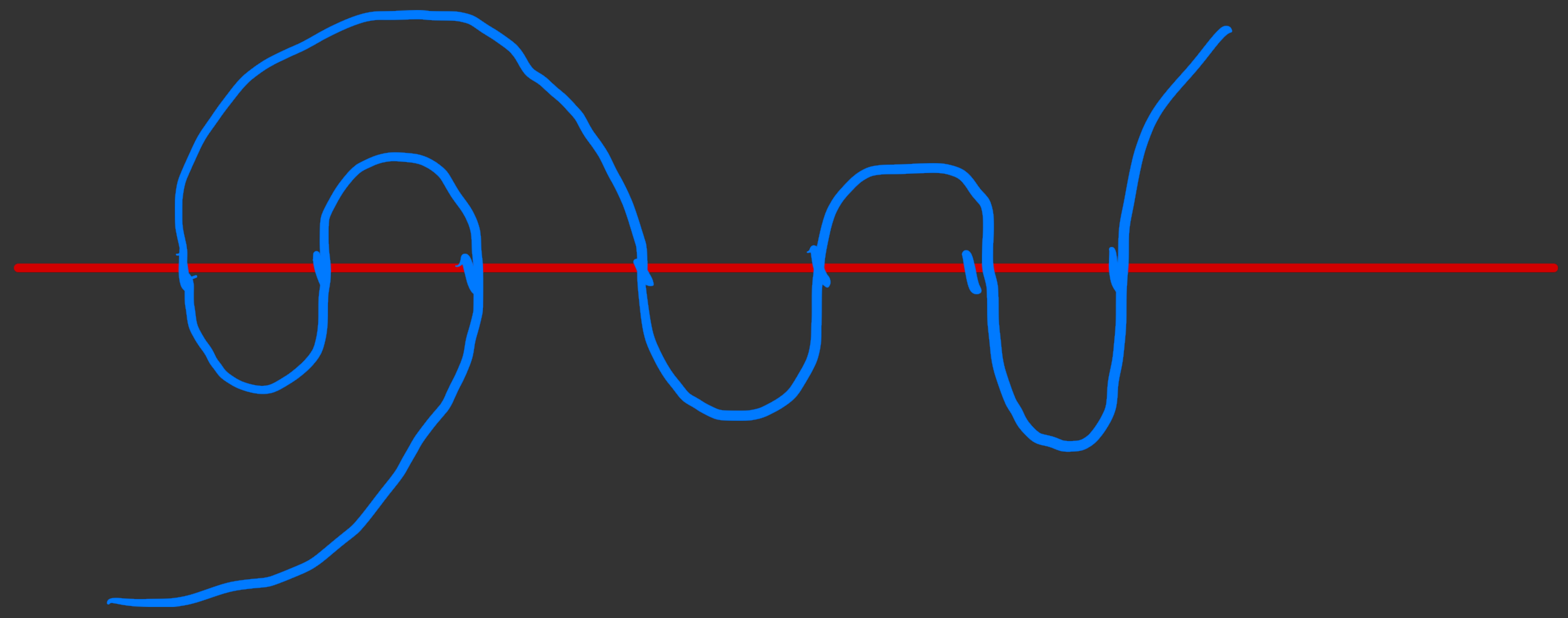
iii. These arc and curve systems are 'meanders'

iv. minimal constructions of specific meanders can be lifted to minimal hyperelliptic [1, 1]-pillowcase covers.

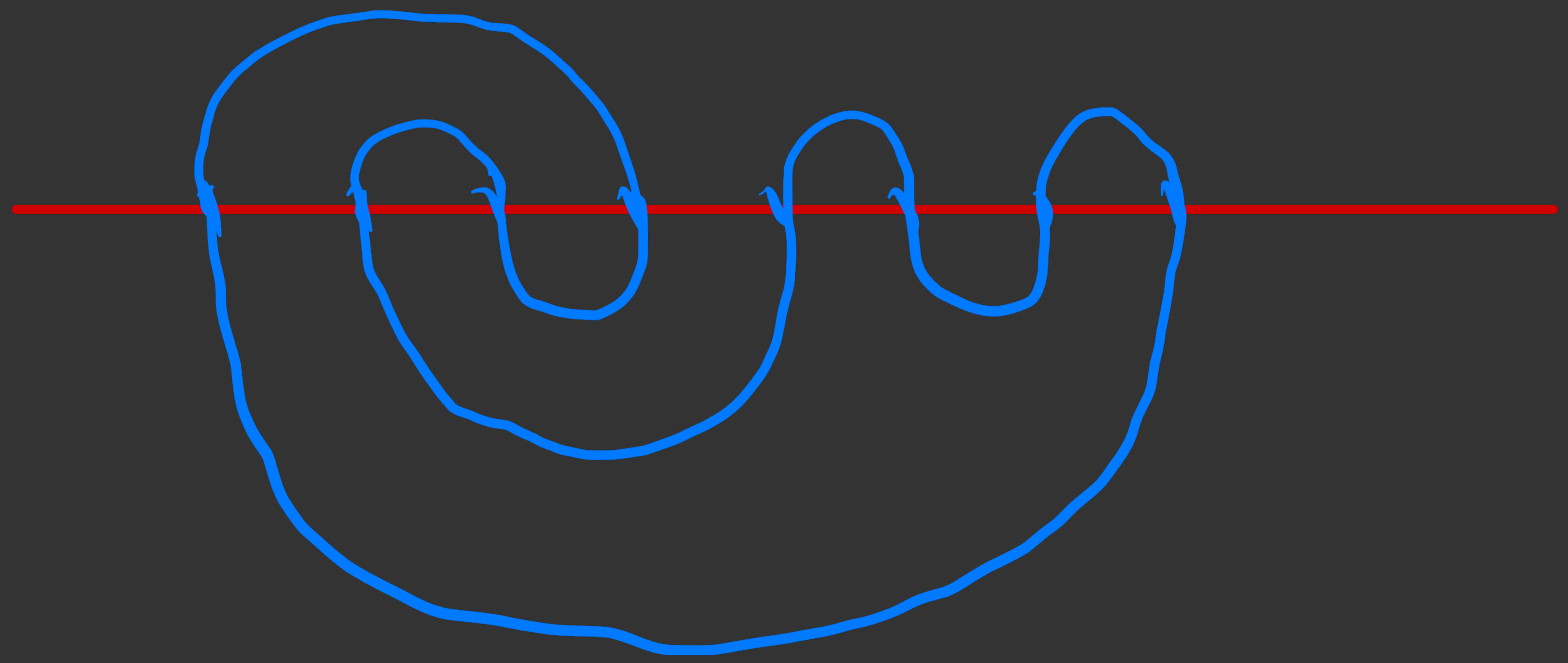
Meanders



open meander



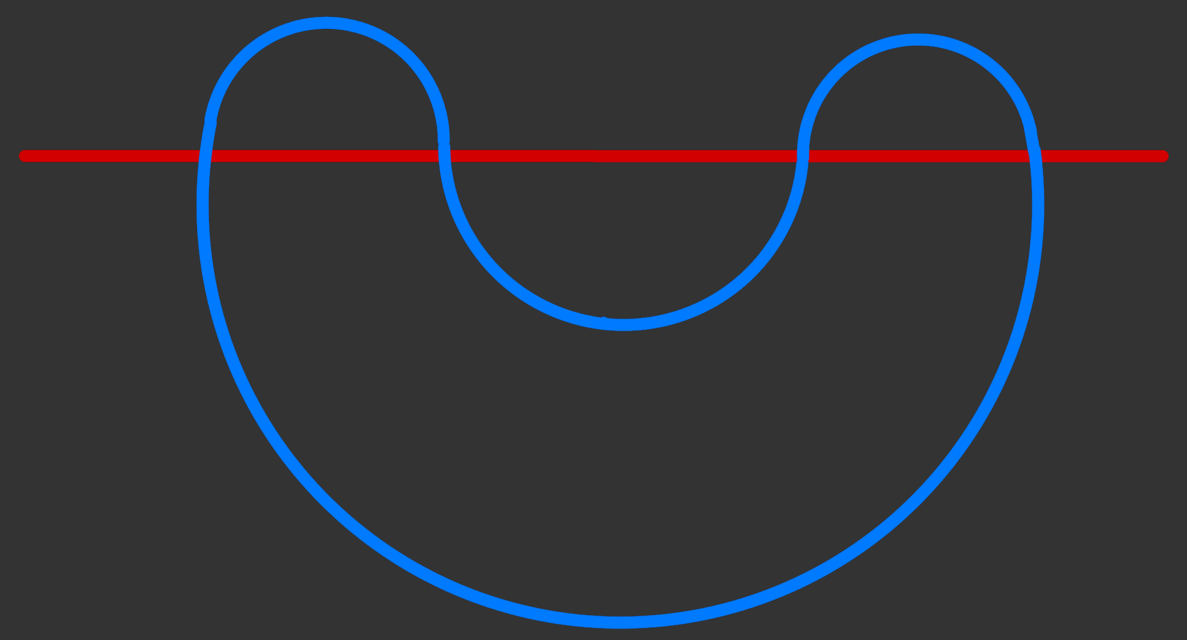
open meander



closed meander

Lifting

Lifting



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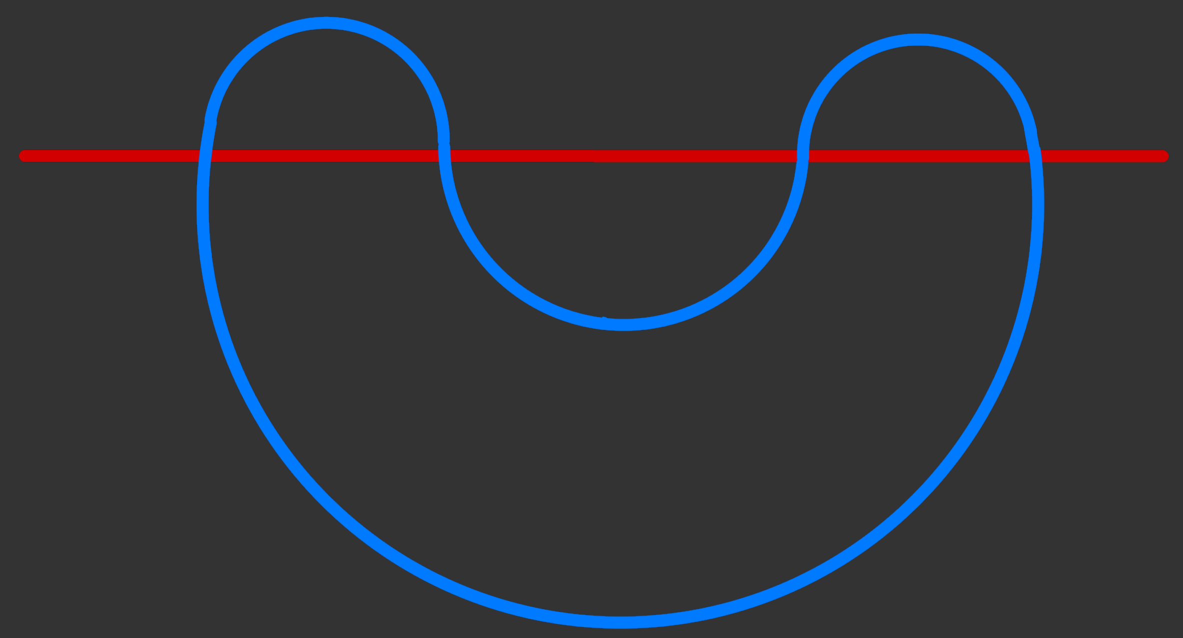

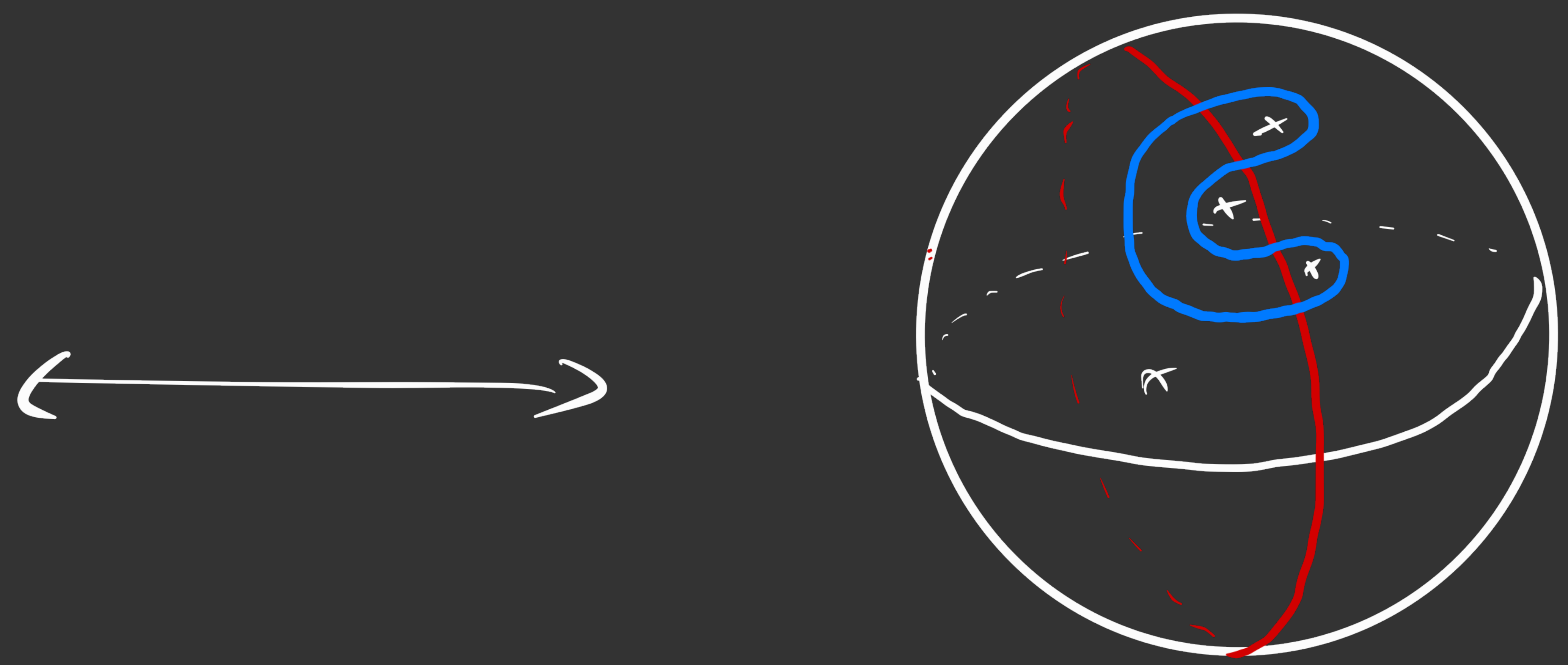

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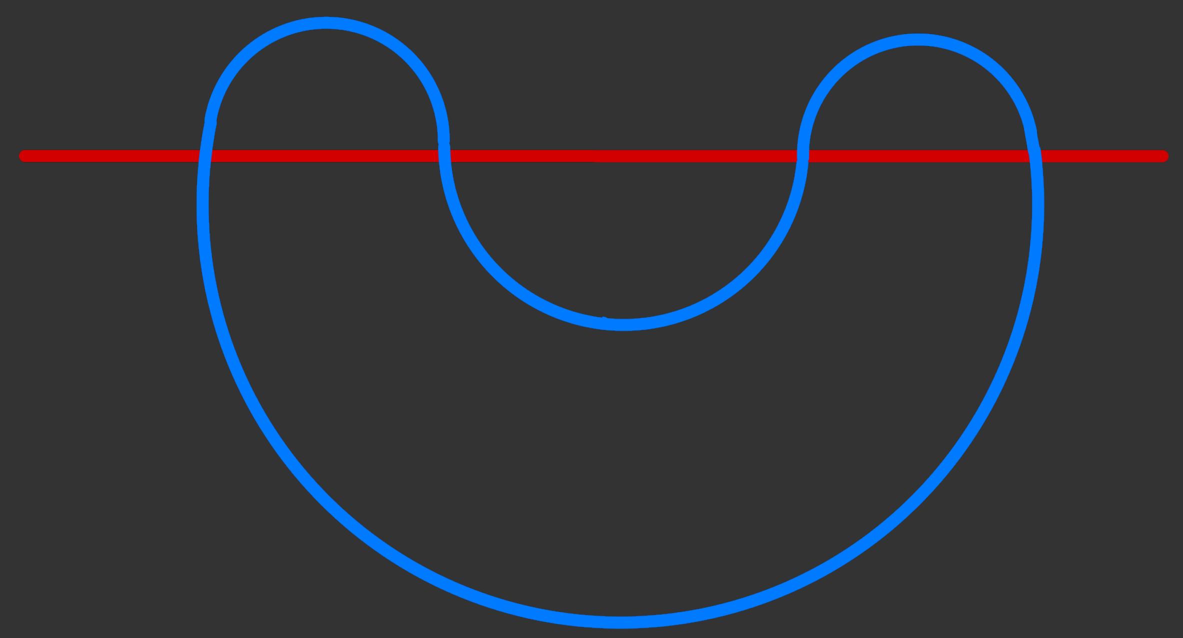
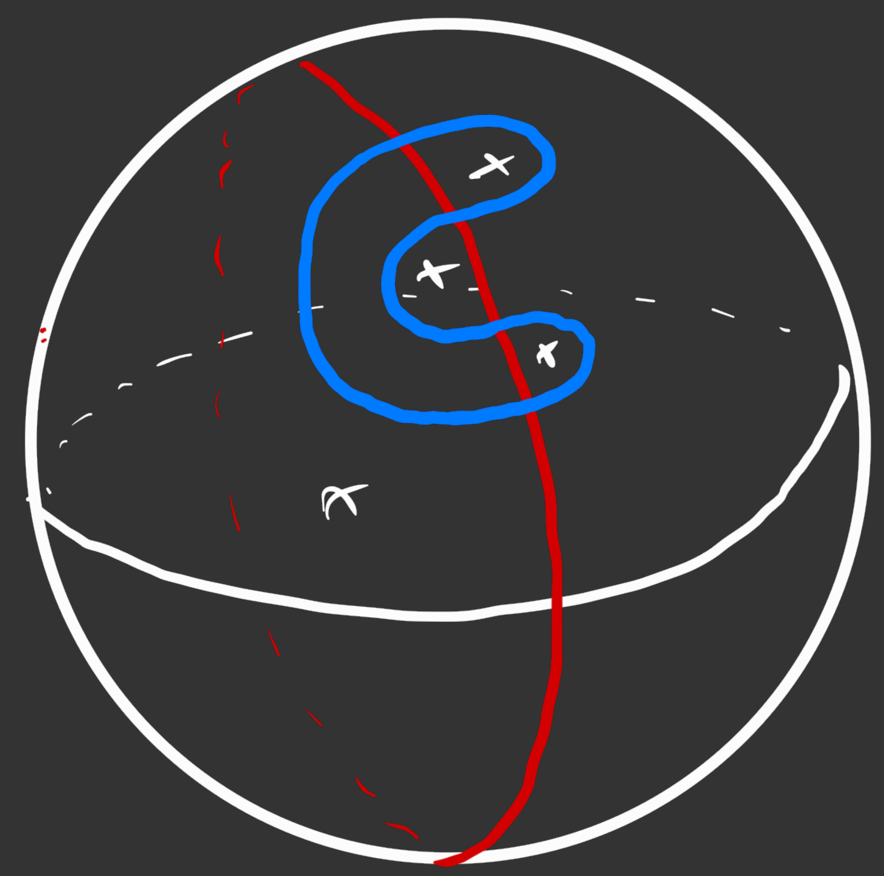
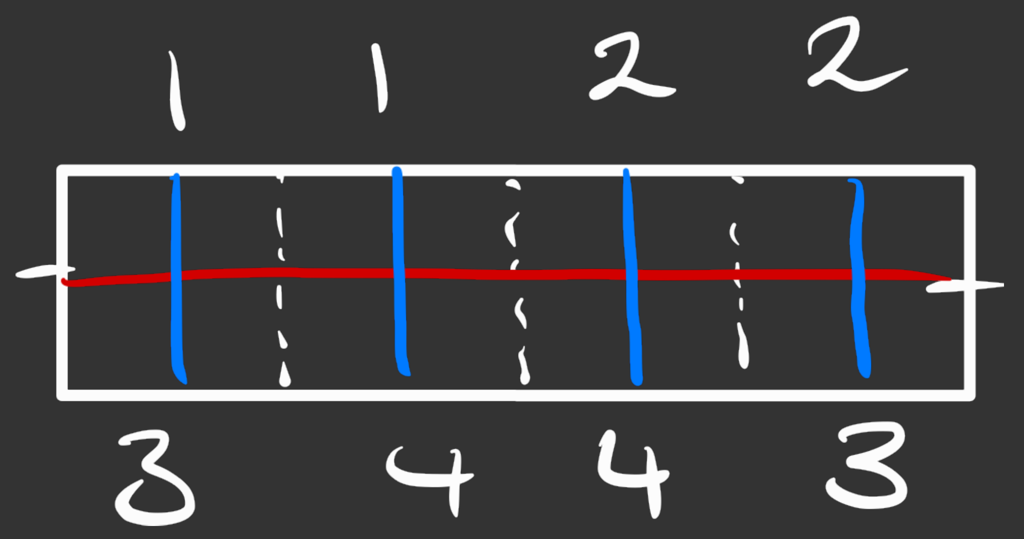


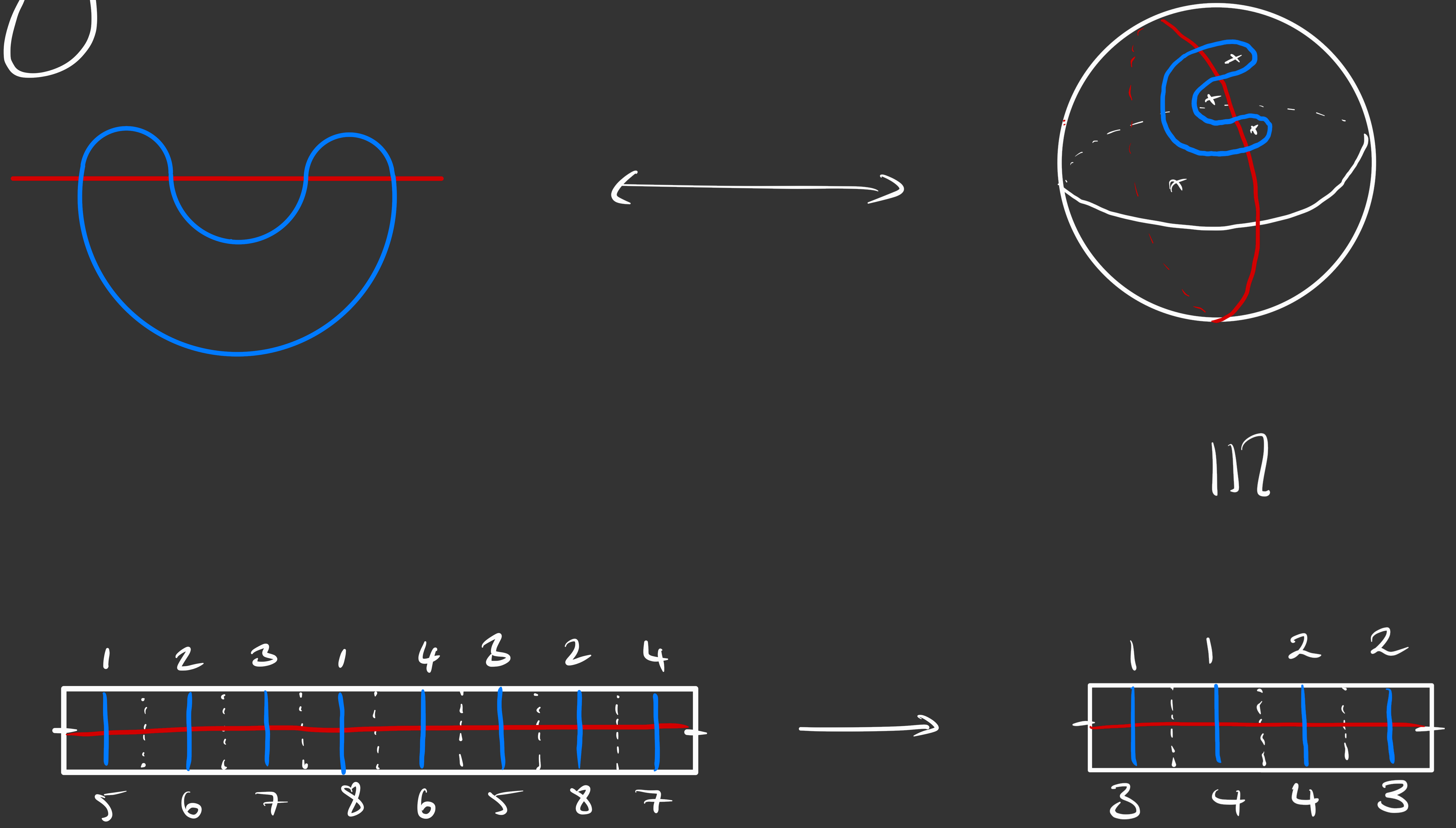
$\longleftrightarrow$

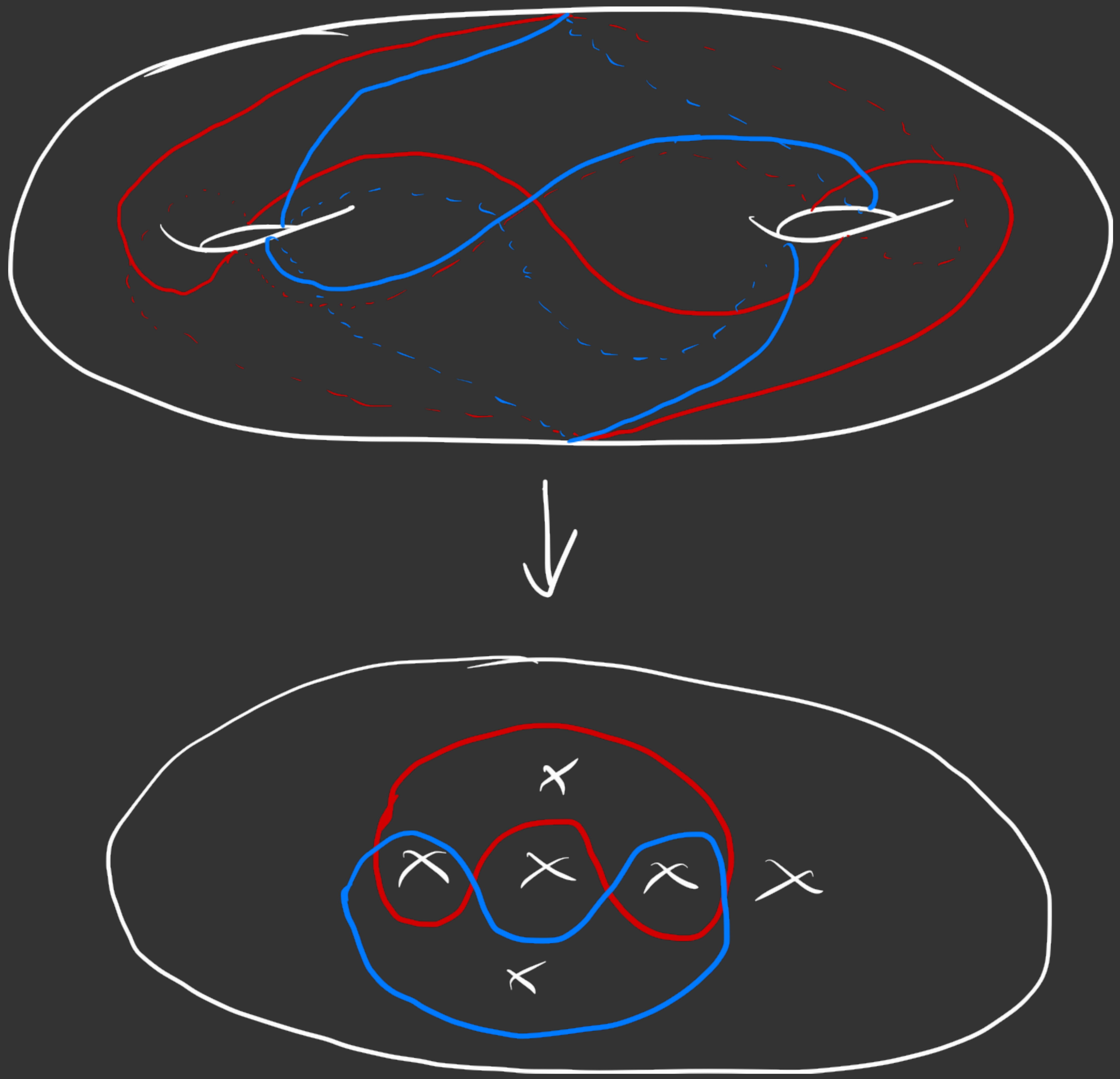


$\cong$



Lifting





Applications:

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Separating filling curves $\xleftrightarrow{\text{Aougab-Taylor}}$ Ratio-optimising pseudo-Anosovs in the Johnson filtration

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$$\mathrm{MCG}(S) \geq \mathrm{Torelli}(S) \geq \mathrm{JohnKer}(S) \geq \cdots$$

Open questions:

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- Given $\mathcal{G} \leq MCG(S)$ and $\mathcal{C} \subseteq \mathcal{HQ}$, can you tell if $\mathcal{G}$ contains ratio-optimisers stabilising some $q \in \mathcal{C}$?