Short canonical decomposition for non-orientable surfaces

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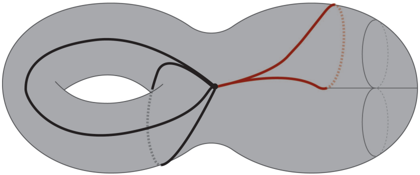
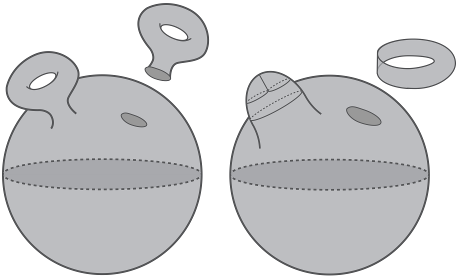


AMS-EMS-SMF 2022- Grenoble

Graphs embedded on surfaces /A Discrete Metric

- A **surface** is a topological space that locally looks like the plane.



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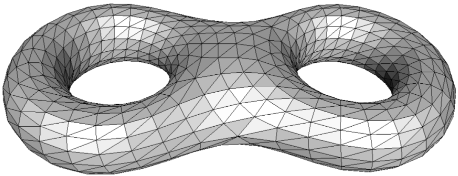

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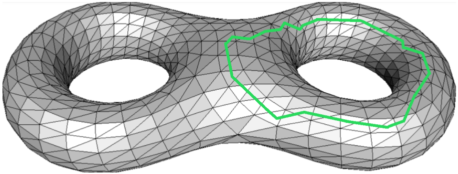
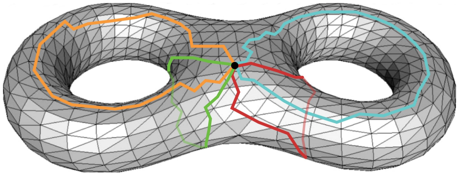
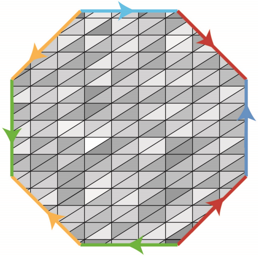


- This graph introduces a discrete metric to the surface.

Decompositions

- A **decomposition** for the embedded graph, is a second graph intersecting the first graph **transversely** and cuts it into a **disk**.





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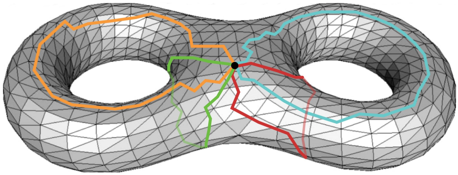

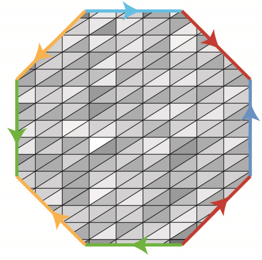

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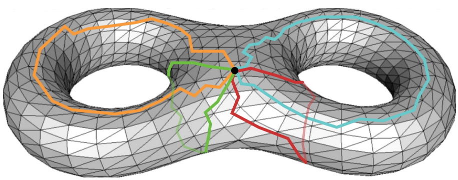

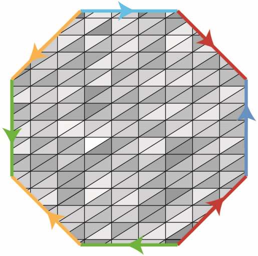

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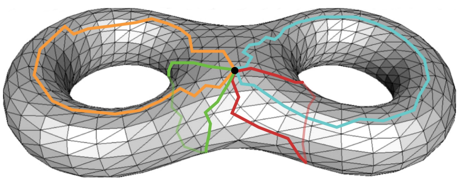

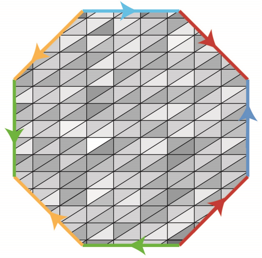

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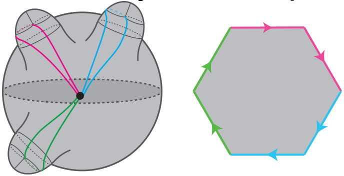
Theorem (Lazarus, Pocchiola, Vegter, Verroust '01)

*Given a graph cellularly embedded on an **orientable** surface of genus g, there exists an orientable canonical system of loops, so that **each** loop crosses **each** edge of the graph at most 4 times (total length $O(gn)$).*

Canonical decompositions for non-orientable surfaces

- What about non-orientable surfaces? Can I cut along **the non-orientable canonical system of loops**?



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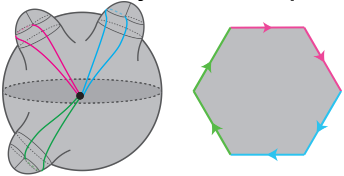

Theorem (F., Hubard, de Mesmay)

Given a graph cellularly embedded on a non-orientable surface, there exists a ***non-orientable canonical system of loops*** *such that* ***each*** *loop in the system crosses* ***each*** *edge of the graph at most in 30 points (total length $O(gn)$).*

- Previous best bound for the total length is $O(g^2n)$ (Lazarus '14).

Other cutting shapes?

A more general question on finding short decompositions is the following open problem.

Negami's conjecture '01

Let G_1 and G_2 be two graphs cellularly embedded on a surface S of genus g. Is there a homeomorphism $h : S \to S$ such that each edge of $h(G_1)$ crosses each edge of G_2 at most a constant number of times?

Best known bound:

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Two reasons to decompose a surface

1. Combinatorial group theory: How to change bases?

Given an orientable surface S, and a family of simple curves inducing a presentation of the fundamental group:

$$\pi_1(S) =< a, b, c, d \mid abcd\overline{abcd} = 1 >$$

How do I go from this presentation to my "favorite" presentation?

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- Here, one can take $a_1 = \overline{dc}a$, $b_1 = bcd$, $a_2 = \overline{c}$ and $b_2 = \overline{d}$. How much can we bound the length of these words?

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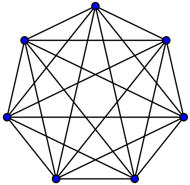
2. Visualisation: How to represent an embedded graph?



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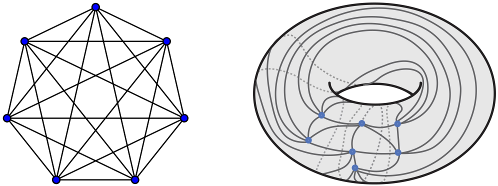

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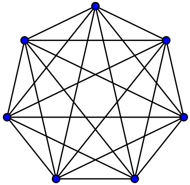

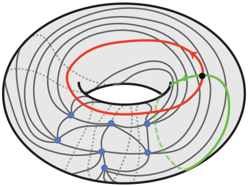

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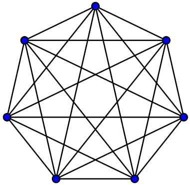

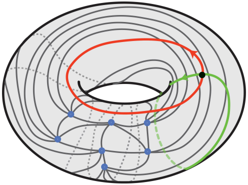

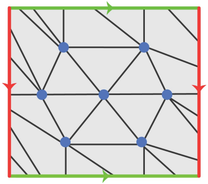

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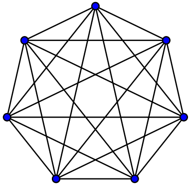
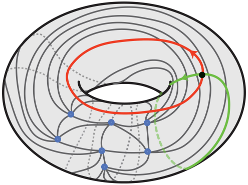
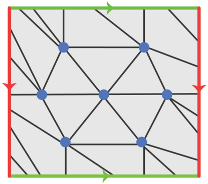


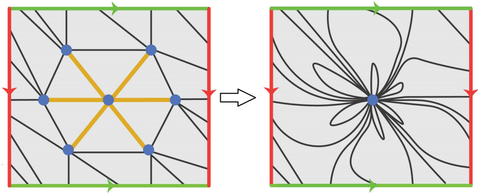




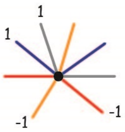
- Why non-orientable surfaces?
  they are more flexible; a graph with n edges might need $O(n)$ handles to be embedded while **one** cross-cap is enough.

Reduction to the one-vertex case

- By contracting a **spanning tree**, our problem reduces to the case of one-vertex graphs.

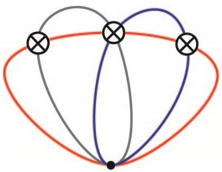


- An embedding for a one-vertex graph, is entirely described by the cyclic ordering of the edges around the vertex, and, in the non-orientable case, the sidedness of the curves, **an embedding scheme**.



Cross-cap drawing

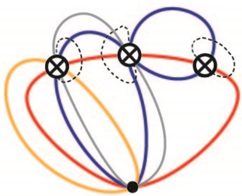
- **Cross-cap drawings**, a planar drawing in which the cross-caps are localized.



A different approach

Theorem (Schaefer-Štefankovič '15)

*A graph G embeddable on a non-orientable surface admits a cross-cap drawing in which each edge enters each cross-cap at most **twice**.*



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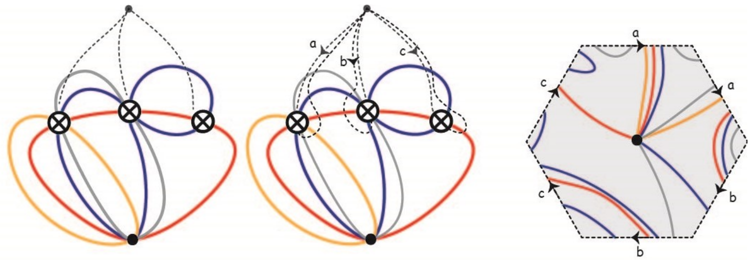

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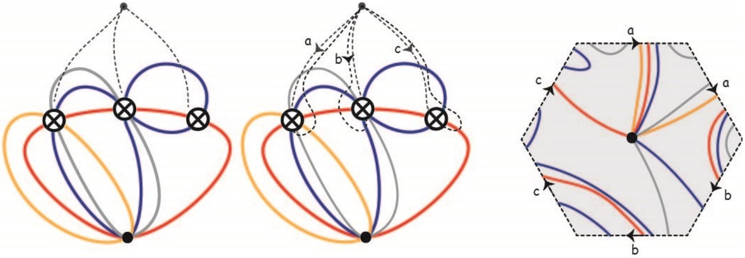
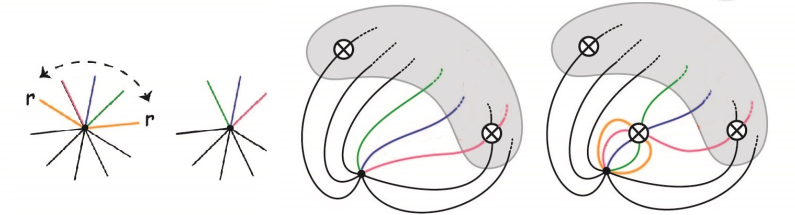


- If we can control the diameter of this cross-cap drawing, we can control the length of the canonical system of loops.

Sketch of the proof

- The proof is by induction on the number of edges.



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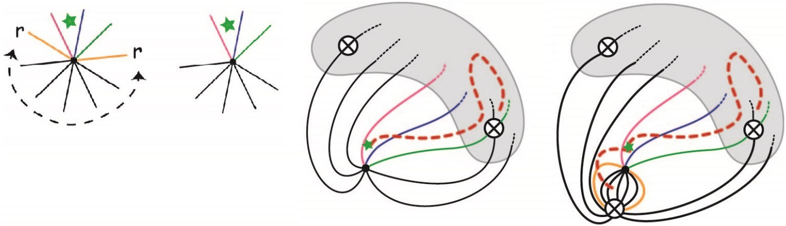

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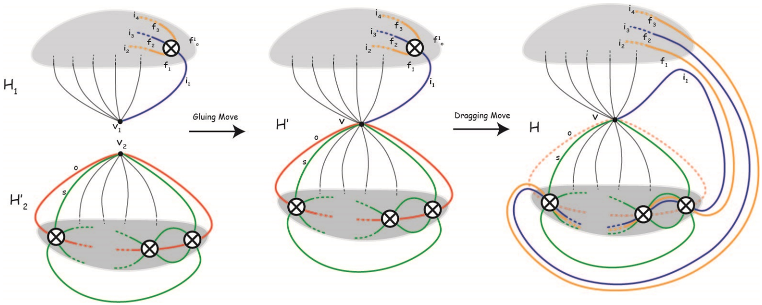

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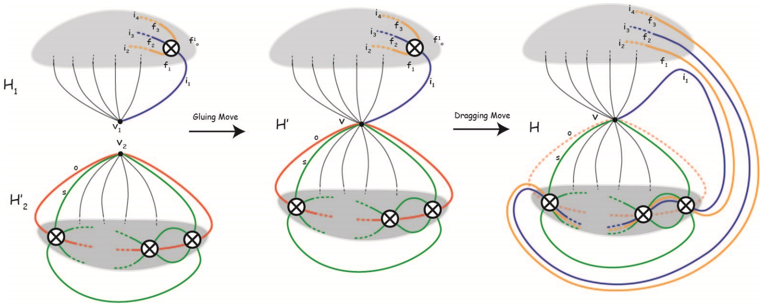
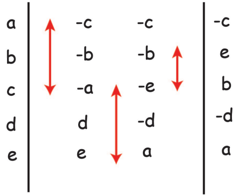


- To avoid cascading, we make sure to deal with all the separating loops at once.

A nice relation

- The **signed reversal distance** between two signed permutations is the minimum number of reversals to go from one to the other.
- Very important in **computational biology**, computable in **polynomial time** [Hannenhalli-Pevzner '99].
- Strong similarities with crosscap drawings, which we leverage in our proof.



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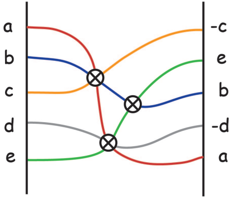

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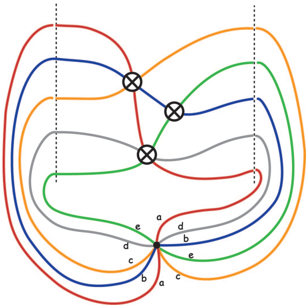

Open Questions

- Negami's conjecture remains open

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- On cross-cap drawings of ***loopless*** graphs

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