Central limit theorem on CAT(0) spaces with contracting isometries

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Abstract

Let G be a group with a non-elementary action on a proper CAT(0) space X, and let μ be a measure on G such that the random walk $(Z_n)_n$ generated by μ has finite second moment on X. Let o be a basepoint in X, and assume that there exists a rank one isometry in G. We prove that in this context, $(Z_n o)_n$ satisfies a Central Limit Theorem, namely that the random variables $\frac{1}{\sqrt{n}}(d(Z_n o, o) - n\lambda)$ converge in law to a Gaussian distribution N_μ, for λ the (positive) drift of the random walk. The strategy relies on the use of hyperbolic models introduced by H. Petyt, A. Zalloum and D. Spriano in [PSZ22], which are analogues of curve graphs and cubical hyperplanes for the class of CAT(0) spaces. As a side result, we prove that the probability that the nth-step Z_n acts on X as a contracting isometry goes to 1 as n goes to infinity.

Contents

1 Introduction

Let G be a discrete group acting by isometries on a proper CAT(0) space X. Let μ be a Borel probability measure on G, which we always assume admissible, meaning that the support of μ generates G as a semigroup. Consider the sequence $\omega = (\omega_i)_i$, where the $\omega_i's$ are chosen independently according to the measure μ. The random walk $(Z_n(\omega))_n$ on G generated by μ is then defined by $Z_n(\omega) = \omega_1 \dots \omega_n$. Taking $o \in X$, we want to study the asymptotic behaviour of the random variables $(Z_n(\omega)o)_n$. To be more precise, we want to study limit laws of the random walk in a natural compactification of X. Even though these questions may be hard to solve for general metric spaces, the theory is very rich when X possesses nice linear or hyperbolic-like properties. In the fundamental paper of V. Kaimanovich [Kai00], the convergence of $(Z_n o)_o$ to a point of the visual boundary is proven for groups acting geometrically on proper hyperbolic spaces and several other classes of actions. More recently this result has been extended by J. Maher and G. Tiozzo in [MT18] for groups acting by isometries on non proper hyperbolic spaces. A major difficulty in the proof of the latter result was that in the non proper setting, the completion of a hyperbolic space by its Gromov boundary might be non compact. The results of Maher and Tiozzo will be fundamental in the sequel because we will deal with hyperbolic spaces without properness assumption. In non-specified CAT(0) space, Karlsson and Margulis proved in [KM99, Theorem 2.1] a first general result of convergence of the random walk, under the assumption that the escape rate $\lambda = \liminf \frac{d(Z_n o, o)}{n}$ is positive.

In [LeB22] we proved that if G acts on a CAT(0) space X with rank one isometries, then the random walk $(Z_n(\omega))_n$ almost surely converges to a point of the boundary of the visual compactification $\partial_\infty X$. A rank one element is an axial isometry whose axes do not bound any flat half plane. We give more details on this notion in Section 2, but a rank one element must be thought of as a contracting isometry with features that typically arise in hyperbolic settings. In this context, we also prove that the speed at which the random walk converges (the drift) is almost surely positive: there exists $\lambda > 0$ such that almost surely, $\lim_n \frac{d(Z_n o, o)}{n} = \lambda$. We review these results in Section 4. The present paper can be thought of as a continuation of [LeB22], and the goal of this work is study further limit laws of the random walk $(Z_n o)_n$, and more specifically central limit theorems for the random variables $(d(Z_n o, o))_n$.

In the case of a random product of matrices, a classical result of Furstenberg [Fur63] is the following. Take (M_n) a sequence of matrices in $GL_n(\mathbb{R})$, independent and identically distributed according to a probability measure μ whose support generates a noncompact subgroup of $GL_n(\mathbb{R})$ that does not preserve any proper linear subspace of $\mathbb{R}^n$. Assume that μ has finite first moment. Then there exists $\lambda > 0$ such that for all $v \in \mathbb{R}^n - \{0\}$,

$$\frac{1}{n} \log \|M_n \dots M_1 v\| \longrightarrow_n \lambda.$$

This result can be thought of as an analogue of a law of large numbers on the random walk $(M_n \dots M_1 v)_n$. In this context, central limit theorems and other limit laws were proven by Furstenberg-Kesten [FK60], Le Page [LP82] and Guivarc'h-Raugi [GR85]. These state that there exists $\sigma_\mu > 0$ such that for every $v \in \mathbb{R}^n$,

$$\frac{\log \|M_n \dots M_1 v\| - n\lambda}{\sqrt{n}} \underset{n}{\longrightarrow} \mathcal{N}(0, \sigma_\mu^2),$$

where $\mathcal{N}(0, \sigma^2)$ is a centred Gaussian law on $\mathbb{R}$. We recall that the convergence in law means that for any bounded continuous function $F : \mathbb{R} \to \mathbb{R}$, one has

$$\lim_{n\to\infty} \int_G F(\frac{\log \|M_n \dots M_1 v\| - n\lambda}{\sqrt{n}}) d\mu^{*n}(g) = \int_{\mathbb{R}} F(t) \frac{\exp(-t^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}} dt.$$

Those kinds of results were also obtained in negative curvature settings, for example in Gromov-hyperbolic groups [Bjö10]. However, the results stated thus far were obtained under rather strong moment conditions. Typically, μ was assumed to have a finite exponential moment, meaning that there exists $\alpha > 0$ such that $\int_G \exp(\alpha d(o, go)) d\mu(g) < \infty$.

More recently, Benoist and Quint have developed a new approach to this question and have proven central limit theorems in the linear context [BQ16b] and for hyperbolic groups [BQ16a]. They could weaken the moment condition and only assume that the measure μ has finite second moment $\int_G (\log \|gv\|)^2 d\mu(g) < \infty$. Namely, if μ is such a measure on a group G acting non elementarily on a proper hyperbolic space Y with basepoint o, then there exists $\lambda > 0$ such that the random variables $\frac{d(Z_n(\omega)o, o) - n\lambda}{\sqrt{n}}$ converge in law to a non-degenerate Gaussian distribution [BQ16a, Theorem 1.1].

Using this approach, C. Horbez proved central limit theorems for mapping class groups of closed connected orientable hyperbolic surfaces and on $\mathrm{Out}(F_N)$ [Hor18]. More recently, T. Fernós, J. Lécureux and F. Mathéus proved that if G is a group acting non-elementarily on a finite-dimensional CAT(0) cube complex, then we also have a central limit theorem for the random variables $(d(Z_n(\omega)o, o))_n$ [FLM21]. In both cases, the authors only assume a second moment condition.

The main result of this paper is to prove a similar result in the context of a group acting on a general CAT(0) space with a rank one isometry. We say that the group action $G \curvearrowright X$ is non-elementary if there are no fixed points in $\overline{X}$ nor a fixed pair of points in $\partial_\infty X$.

Theorem 1.1. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk. Then the random variables $\frac{1}{\sqrt{n}}(d(Z_n o, o) - n\lambda)$ converge in law to a non-degenerate Gaussian distribution N_μ.*

Our strategy also relies heavily on the approach developed by Benoist and Quint. To summarize, they give a condition under which a given random cocycle converges in law to a Gaussian distribution. However, in order to prove that we can apply this result, one needs to obtain good estimates on this cocycle. In the aforementioned papers [Hor18] and [FLM21], the authors use hyperbolic models, on which they could derive quantitative estimates. The strategy is then to study the behaviour of the random walk on a hyperbolic model, on which we know many properties thanks to Maher and Tiozzo [MT18]. Then, one must lift this information back on the original space X using contracting properties.

- For $\mathrm{Mod}(S)$, the hyperbolic model is the curve complex $C(S)$, and the lifting to $\mathcal{T}(S)$ is done in [Hor18, Section 3.4].
- For a CAT(0) cube complex, the hyperbolic model is the contact graph $\mathcal{C}X$, and the lifting is implemented in [FLM21, section 5].

In [PSZ22], H. Petyt, D. Spriano and A. Zallum introduced analogues of curve graphs and cubical hyperplanes for the class of CAT(0) spaces. Using a generalized notion of hyperplane, they build a family of hyperbolic metrics $(d_L)_L$ on X which conserve many of the geometric features of the original CAT(0) space. These spaces capture hyperbolic behaviours in X and behave very well under the isometric action of a group. Moreover, a rank one isometry of X acts on some hyperbolic model as a loxodromic isometry. The strategy we use is then to chose a good hyperbolic model $X_L = (X, d_L)$, and then to make use of the limit laws proven by Maher and Tiozzo in [MT18].

A different approach for the study of limit laws was implemented in [MS20], where the authors prove central limit theorems on acylindrically hyperbolic groups. Their strategy relies on a control of deviation inequalities, which encapsulate the way the random walk progresses in an “almost aligned” way, hence they avoid the use of boundaries and compactifications. While there is a slight overlap with the results stated here (especially [MS20, Theorem 13.4]), their setting is different from this paper because they study random walks on acylindrically hyperbolic groups with a word metric, while here we have no properness assumption on the action. Also, their assumptions on the measure μ are a bit more restrictive.

Another interesting question in the study of $(Z_n(\omega))_n$ is the proportion of steps that are "hyperbolic". In the context of random walks on hyperbolic spaces, Maher and Tiozzo show that the probability that a random walk of size n is a loxodromic isometry goes to 1 as n goes to infinity [MT18, Theorem 1.4]. For a non-elementary action on an irreducible CAT(0) cube complex, Fernós, Lécureux and Mathéus show that the proportion of steps Z_n that are contracting goes to 1 as n goes to infinity. They use this result to show that if a group G acts non elementarily and essentially on a (possibly reducible) finite-dimensional CAT(0) cube complex, then there exist regular elements, extending a result of Caprace and Sageev [CS11]. In our context, we also prove that "most" of the steps in the random walk are rank one. This result is not involved in the proof of Theorem 1.1, but is of independent interest.

Theorem 1.2 (Rank one elements in the random walk)**.** *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G, and assume that G contains a rank one element. Then*

$$\mathbb{P}(\omega \,:\, Z_n(\omega) \textit{ is a contracting isometry }) \underset{n\to\infty}{\to} 1.$$

While we were working on this project, Inhyeok Choi released a paper in which he states central limit theorems along with other limit laws in CAT(0) spaces, Teichmüller spaces and outer spaces [Cho22]. One of the main assumptions is still the presence of a pair of independent contracting isometries in the group, but the methods and the proofs are different. Indeed, the author uses a pivotal technique introduced by Gouëzel in [Gou21], while our paper uses hyperbolic models that depend on specific features of CAT(0) spaces. We think that this approach is natural from a geometric point of view, and we hope that the interplay between CAT(0) spaces and their underlying hyperbolic models will be useful in the study of still open questions about limit laws.

We believe our approach can be of use in order to determine if the boundary $\partial_\infty X$ endowed with the hitting measure is actually the Poisson boundary of (G, μ), extending a result of Karlsson and Margulis for cocompact actions [KM99, Corollary 6.2].

Moreover, it seems natural to use these hyperbolic spaces to prove that if μ has finite first moment, then limit points of the random walk almost surely belong to the sublinear Morse boundary constructed by Qing and Rafi in [QR22]. Note that this question is linked to the previous one, because it is believed that the sublinear Morse boundary is often a good candidate for the Poisson boundary, especially for finitely supported measures, see for example [QR22, Theorem F] and [QRT20, Theorem B]. In both cases, the use of hyperbolic models seems useful because of the richness of the theory in this context.

In Section 2, we review basic definitions about random walks, rank one isometries and explain our setting. In Section 3, we explicit the construction and properties of the hyperbolic models (X, d_L), and give various geometric lemmas that will be useful afterwards. Section 4 is dedicated to presenting the works of Maher and Tiozzo in [MT18], and the first results in proper CAT(0) spaces that were found in [LeB22]. We explain the strategy developed by Benoist and Quint in Section 5, and give the proof of our main Theorem in Section 6.

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2 Background

2.1 Random walks and CAT(0) spaces

Let G be a discrete countable group and $\mu \in \mathrm{Prob}(G)$ a probability measure on G. Throughout the article we will assume that μ is admissible, i.e. $\mathrm{supp}(\mu)$ generates G as a semigroup. Let $(\Omega, \mathbb{P})$ be the probability space $(G^{\mathbb{N}}, \delta_e \times \mu^{\mathbb{N}^*})$.

The application

$$(n,\omega) \in \mathbb{N} \times \Omega \mapsto Z_n(\omega) = \omega_1\omega_2 \dots \omega_n,$$

where ω is chosen according to the law $\mathbb{P}$, defines the random walk on G generated by the measure μ.

Let now (X,d) be a proper CAT(0) metric space, on which G acts by isometries. If the reader wants a detailed introduction to CAT(0) spaces, the main references that we will use are [BH99] and [Bal95]. We recall that the boundary $\partial_\infty X$ of a CAT(0) space X is the set of equivalent classes of rays $\sigma : [0,\infty) \to X$, where two rays σ_1, σ_2 are equivalent if they are asymptotic, i.e. if $d(\sigma_1(t), \sigma_2(t))$ is bounded uniformly in t.

Given two points on the boundary ξ and η, if there exists a geodesic line $\sigma : \mathbb{R} \to X$ such that the geodesic ray $\sigma_{[0,\infty)}$ is in the class of ξ and the geodesic ray $t \in [0,\infty) \mapsto \sigma(-t)$ is in the class of η, we will say that the points ξ and η are joined by a geodesic line. The reader should be aware that in general, such a geodesic need not exist between any two points of the boundary, as can be seen in $\mathbb{R}^2$. A point ξ of the boundary that is called a *visibility point* if, for all $\eta \in \partial_\infty X - \{\xi\}$, there exists a geodesic from ξ to η. We will see in the next section a criterion to prove that a given boundary point is a visibility point.

An important feature in CAT(0) spaces is the existence of closest-point projections on complete convex subsets. More precisely, given a complete convex subset C in a CAT(0) space, there exists a map $\pi_C : X \to C$ such that $\pi_C(x)$ minimizes the distance $d(x, C)$:

Proposition 2.1 ([BH99, Lemma 2.4])**.** *The projection π_C onto a convex complete subset in a CAT(0) space satisfies the following properties:*

- *$\forall x \in X$, $\pi_C(x)$ is uniquely defined and $d(x, \pi_C(x)) = d(x, C) = \inf_{c\in C} d(x, c)$;*
- *if x' belongs to the geodesic segment $[x, \pi(x)]$, then $\pi_C(x') = \pi_C(x)$;*
- *π is a retraction of X onto C that does not increase the distances: for all $x, y \in X$, we have $d(\pi_C(x), \pi_C(y)) \leq d(x, y)$.*

It is immediate to see that the above properties can be applied to geodesic segments, which are convex and complete with the induced metric. When $\gamma : [a,b] \to X$ is a geodesic segment, we will write π_γ for the projection onto the image $[\gamma(a), \gamma(b)] \subseteq X$.

When X is a proper space, the space $\overline{X} = X \cup \partial X$ is a compactification of X, that is, $\overline{X}$ is compact and X is an open and dense subset of $\overline{X}$. We recall that the action of G on X extends to an action on $\partial_\infty X$ by homeomorphisms.

Another equivalent construction of the boundary can be done using horofunctions. If $x_n \to \xi \in \partial_\infty X$ and $x \in X$, we denote by $b_\xi^x : X \mapsto \mathbb{R}$ the horofunction given by

$$b_\xi^x(z) = \lim_n d(x_n, z) - d(x_n, x).$$

It is a standard result in CAT(0) geometry (see for example [Bal95, Proposition II.2.5]) that this limit exists and that given any basepoint x, a horofunction characterizes the boundary point ξ. When the context is clear. we will often omit the basepoint and just write b_ξ.

2.2 Rank one elements

Let $g \in G$. We say that g is a *semisimple* isometry if its displacement function $x \in X \mapsto \tau_g(x) = d(x, gx)$ has a minimum in X. If this minimum is non-zero, it is a standard result (see for example [Bal95, Proposition II.3.3]) that the set on which this minimum is obtained is of the form $C \times \mathbb{R}$, where C is a closed convex subset of X. On the set $\{c\} \times \mathbb{R}$ for $c \in C$, g acts as a translation, which is why g is called *axial* and the subset $\{c\} \times \mathbb{R}$ is called an *axis* of g. A *flat half-plane* in X is defined as a euclidean half plane isometrically embedded in X.

Definition 2.2. We say that a geodesic in X is *rank one* if it does not bound a flat half-plane. If g is an axial isometry of X, we say that g is rank one if no axis of g bounds a flat half-plane.

If G acts on X by isometries and possesses a rank one element $g \in G$ for this action, we may say that G is rank one. However, the theory of CAT(0) groups is not as clear as for Gromov hyperbolic groups. For example, there is no good (i.e. invariant under quasi isometry) notion of boundary of a CAT(0) group, as shown by Croke and Kleiner in [CK00]. To summarize, it is better to keep in mind that "rank one" is always attached to a given action $G \curvearrowright X$ on a CAT(0) space.

More information on rank one isometries and geodesics can be found in [Bal95, Section III. 3], and more recently in [CF10] and in [BF09].

Definition 2.3. We say that the action $G \curvearrowright X$ of a rank one group G on a CAT(0) space X is *non-elementary* if G neither fixes a point in $\partial_\infty X$ nor stabilizes a geodesic line in X.

To justify this definition, we use a result from Caprace and Fujiwara in [CF10]. What follows comes from the aforementioned paper.

Definition 2.4. Let g_1, $g_2 \in G$ be axial isometries of G, and fix $x_0 \in X$. The elements $g_1, g_2 \in G$ are called independent if the map

$$\mathbb{Z} \times \mathbb{Z} \to [0, \infty) : (m, n) \mapsto d(g_1^m x_0, g_2^n x_0) \tag{1}$$

is proper.

Remark 2.5. In particular, the fixed points of two independent axial elements form four distinct points of the visual boundary.

Let us end this section by stating two results about rank one isometries. The first one was proven by P-E. Caprace and K. Fujiwara in [CF10].

Proposition 2.6 ([CF10, Proposition 3.4])**.** *Let X be a proper CAT(0) space and let $G < \mathrm{Isom}(X)$. Assume that G contains a rank one element. Then exactly one of the following assertions holds:*

1. *G either fixes a point in $\partial_\infty X$ or stabilizes a geodesic line. In both cases, it possesses a subgroup of index at most 2 of infinite Abelianization. Furthermore, if X has a cocompact isometry group, then $\overline{G} < \mathrm{Isom}(X)$ is amenable.*
2. *G contains two independent rank one elements. In particular, $\overline{G}$ contains a discrete non-Abelian free subgroup.*

As a consequence, the action $G \curvearrowright X$ of a rank one group G on a CAT(0) space X is non-elementary if and only if alternative 2 of the previous Proposition holds.

Rank one isometries are especially interesting because they induce natural contracting properties on the space. These properties mimic how loxodromic isometries behave in the hyperbolic setting.

Definition 2.7. A geodesic σ in a CAT(0) space is said to be *C-contracting* with $C > 0$ if for every metric ball B disjoint from σ, the projection $\pi_\sigma(B)$ of the ball B onto σ has diameter at most C. An axial isometry is contracting if there exists $C > 0$ such that one of its axes is C-contracting.

It is clear that a contracting isometry is rank one. It turns out that the converse is true if X is a proper CAT(0) space, as was shown by M. Bestvina and K. Fujiwara in [BF09]. This result will allow us to use the hyperbolic models described in Section 3.

Theorem 2.8 ([BF09, Theorem 5.4])**.** *Let X be a proper CAT(0) space, $g : X \to X$ be an axial isometry and σ be an axis of g. Then there exists B such that σ is B-contracting if and only if σ does not bound a half-flat. In other words, g is contracting if and only if g is a rank one isometry.*

2.3 Gromov products

Let (X, d) be a metric space. One defines the Gromov product of $x, y \in X$ with respect to $o \in X$ as

$$(x|y)_o = \frac{1}{2}(d(x,o) + d(y,o) - d(x,y)).$$

The quantity $(x|y)_o$ must be thought of as representing the distance between o and the geodesic between x and y. This notion is particularly interesting because it does not require X to be actually geodesic, and in fact we often deal with only quasigeodesic spaces. Also, we can use Gromov products to characterize hyperbolic spaces. We recall that a metric space (X, d) is hyperbolic if there is $\delta > 0$ such that for all $x, y, z \in X$,

$$(x|z)_o \geq \min((x|y)_o, (y|z)_o) - \delta.$$

If the reader wants a detailed introduction to hyperbolic spaces, a standard reference is [BH99].

If (X, d) is a proper CAT(0) space, the Gromov product can be extended to the visual boundary $\partial_\infty X$ of X by the following formulas: for $x, y \in \partial_\infty X$, $o, m \in X$,

$$\begin{aligned} (m|x)_o &:= \frac{1}{2}(d(o,m) - b_x^o(m)); \\ (x|y)_o &:= -\frac{1}{2}\inf_{q \in X}(b_x(q) + b_y(q)). \end{aligned}$$

If there is a geodesic line γ between x and x', then by triangular inequality this infimum is actually a minimum, and is obtained at any point on γ. As a consequence, if (x_n) and (y_n) are sequences converging to $x, y \in \partial_\infty X$ respectfully, then $(x|y)_o = \lim_{n,m}(x_n|y_m)_o$.

3 Hyperbolic models for proper CAT(0) spaces

The goal of this section is to briefly present some ideas of [PSZ22], in which the authors build a way of attaching a family of hyperbolic metric spaces $X_L = (X, d_L)_L$ to a proper CAT(0) space. What is interesting about these spaces is that they convey much of the geometry of the original space, especially at infinity, and they behave very well under isometric actions. More specifically, rank one isometries will act on some well-chosen spaces as loxodromic isometries. This construction can be understood as the analogue (and generalization) of the curve graphs that exist in the context of CAT(0) cube complexes, see [Hag13] and [Gen19].

Definition 3.1. Let X be a CAT(0) space, and let $\gamma : I \to X$ be a geodesic. Let π_γ be the projection onto the geodesic γ characterized by Proposition 2.1. Let $t \in I$ be such that $[t - \frac{1}{2}, t + \frac{1}{2}]$ belongs to I. Then the *curtain* dual to γ at t is

$$h = h_{\gamma,t} = \pi_\gamma^{-1}(\gamma([t - \frac{1}{2}, t + \frac{1}{2}])).$$

The *pole* of $h_{\gamma,t}$ is $\gamma([t-\frac{1}{2}, t+\frac{1}{2}])$. Borrowing from the vocabulary of hyperplanes, we will call $h^- = \pi_\gamma^{-1}(\gamma((-\infty, t-\frac{1}{2}) \cap I))$ and $h^+ = \pi_\gamma^{-1}(\gamma((t+\frac{1}{2}, +\infty) \cap I))$ the *halfspaces* determined by h. Note that $\{h^-, h, h^+\}$ is a partition of X. If $A \subseteq h^-$ and $B \subseteq h^+$ are subsets of X, we say that h *separates* A from B.

We will often denote a curtain by the letter h, even though one must keep in mind that $h = h_{\gamma,t}$ is characterized by a given geodesic $\gamma : I \to X$ and a point $t \in I$ (which defines a unique pole $P \subseteq \gamma$). Sometimes, we may also write $h = h_{\gamma,P}$ to emphasize on the pole P.

Remark 3.2. By Proposition 2.1, it is immediate that a curtains are closed subsets of X, and that they are thick: if h is a curtain, then $d(h^-, h^+) = 1$.

Curtains can fail to be convex: if $x, y \in h^-$, it may happen that there exists $z \in [x, y] \cap h^+$. Nonetheless, we have a weaker notion of convexity that the authors call star convexity:

Proposition 3.3 ([PS09, Lemma 2.6])**.** *Let h be a curtain dual to γ and $P \subseteq \gamma$ be its pole. For every $x \in h$, then $[x, \pi_P(x)] \subseteq h$.*

Definition 3.4. A family of curtains $\{h_i\}$ is said to be a chain if h_i separates h_{i-1} from h_{i+1} for every i. Chains can be used in order to define a metric on X by the following: for $x \neq y \in X$,

$$d_\infty(x, y) = 1 + \max\{\, |c| \,:\, c \text{ is a chain separating } x \text{ from } y\}.$$

One can check that this definition gives a metric. Let h be a curtain, we have seen that $d(h^-, h^+) = 1$, hence for any $x, y \in X$, $d_\infty(x, y) \leq \lceil d(x, y) \rceil$. Conversely, it turns out that d and d_∞ may differ by at most 1.

Lemma 3.5 ([PSZ22, Lemma 2.10])**.** *Let $x, y \in X$. Then there is a chain of curtains c dual to $[x, y]$ that realizes $d_\infty(x, y) = 1 + |c|$.*

We are now ready to refine the notion of separation in order to capture only some of the hyperbolic features of the space.

Definition 3.6 (L-separation)**.** Let $L \in \mathbb{N}^*$, we say that disjoint curtains are *L-separated* if every chain meeting both has cardinality at most L. A chain of pairwise L-separated curtains is called an *L-chain*.

The following geometric Lemma is a key ingredient for the proof that of Theorem 3.10, and will be used several times in the sequel. It means that L-separation induces good Morse properties. The picture one has to keep in mind is given by Figure 1.

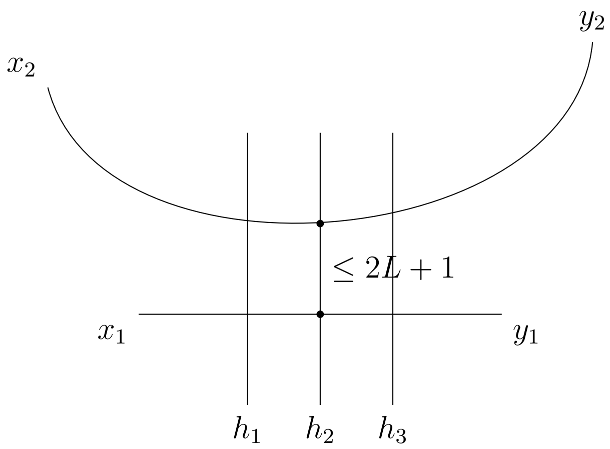

Figure 1: Illustration of Lemma 3.7.

Lemma 3.7 ([PSZ22, Lemma 2.14]). *Suppose that A, B are two sets which are separated by an L-chain $\{h_1, h_2, h_3\}$ all of whose elements are dual to a geodesic $\gamma = [x_1, y_1]$ with $x_1 \in A$ and $y_1 \in B$. Then for any $x_2 \in A$, $y_2 \in B$, if $p \in h_2 \cap [x_2, y_2]$, then $d(p, \pi_\gamma(p)) \leq 2L + 1$.*

The next Lemma states that if there is a L-chain separating two points x and y, we can find a (smaller) L-chain of curtains separating those, which is dual to the geodesic $[x, y]$ and whose size can be controlled. It will prove useful later on, especially when we want to use Lemma 3.7.

Lemma 3.8 ([PSZ22, Lemma 2.22]). *Let $L, n \in \mathbb{N}$, and let $\{h_1, \ldots, h_{(4L+10)n}\}$ be an L-chain separating A, $B \subseteq X$. Take $x \in A$, $y \in B$. Then A and B are separated by an L-chain of size $\geq n+1$ dual to $[x, y]$.*

We are now ready to define a family of metrics using L-separation.

Definition 3.9. Given distinct points $x \neq y \in X$, we define

$$d_L(x, y) = 1 + \max\{|c| : c \text{ is an } L\text{-chain separating } x \text{ from } y\}.$$

It turns out that for every L, d_L gives a metric on X [PSZ22, Lemma 2.17]. We will denote by $X_L = (X, d_L)$ the resulting metric space. With this definition in hand, Petyt, Spriano and Zalloum prove the that the metric spaces (X, d_L) are hyperbolic.

Theorem 3.10 ([PSZ22, Theorem 3.1]). *For any CAT(0) hyperbolic space X and any integer L, the space (X, d_L) is a quasi-geodesic hyperbolic space with hyperbolicity constants depending only on L. Moreover,* $\mathrm{Isom}(X)$ *acts by isometries on (X, d_L).*

We will then call (X, d_L) a hyperbolic model for the CAT(0) space X. Another useful fact about these spaces is that they behave well under isometries with "hyperbolic-like" properties.

Theorem 3.11 ([PSZ22, Theorem 4.9]). *Let g be a semisimple isometry of X. The following are equivalent:*

1. *g is a contracting isometry of the CAT(0) space X;*
2. *there exists $L \in \mathbb{N}$ such that g acts loxodromically on X_L.*

Another piece of information brought by this construction is the relation between the Gromov boundaries ∂X_L of the hyperbolic models $X_L = (X, d_L)$ and the visual boundary of the original CAT(0) space (X, d).

Definition 3.12. We say that a geodesic ray $\gamma : [0, \infty) \to X$ *crosses* a curtain h if there exists $t_0 \in [0, \infty)$ such that h separates $\gamma(0)$ from $\gamma([t_0, \infty))$. Alternatively, we may say that h separates $\gamma(0)$ from $\gamma(\infty)$. Similarly, we say that a geodesic line $\gamma : \mathbb{R} \to X$ crosses a hyperplane h if there exists $t_1, t_2 \in \mathbb{R}$ such that h separates $\gamma((-\infty, t_1])$ from $\gamma([t_2, \infty))$. We say that γ crosses a chain $c = \{h_i\}$ if it crosses each individual curtain h_i.

As a consequence of Lemma 3.7 and Lemma 3.8, if two geodesic rays with the same starting point cross an infinite L-chain c, then they are asymptotic, and hence equal.

Remark 3.13. Since curtains are not convex, it is not obvious that any geodesic ray γ meeting a given curtain h must cross it (γ could meet h infinitely often). However, by [PSZ22, Corollary 3.2] if γ is a geodesic ray that meets every element of an infinite L-chain $c = \{h_i\}_{i \in \mathbb{N}}$, then γ must cross c: for every i, there exists $t_i \in [0, \infty)$ such that h_i separates $\gamma(0)$ from $\gamma([t_0, \infty))$.

Given $o \in X$, we define $\mathcal{B}_L$ as the subspace of $\partial_\infty X$ consisting of all geodesic rays $\gamma : [0, \infty) \to X$ starting from o and such that there exists an infinite L-chain crossed by γ. In the case of the contact graph associated to a CAT(0) cube complex X, we had the existence of an $\mathrm{Isom}(X)$-equivariant embedding of the boundary of the contact graph into the Roller boundary $\partial_{\mathcal{R}} X$. The following result is the analogue in the context of CAT(0) spaces.

Theorem 3.14 ([PSZ22, Theorem 7.1]). *Let X be a proper CAT(0) space. Then, for every $L \in \mathbb{N}^*$, the identity map $\iota : X \longrightarrow X_L$ induces an* $\mathrm{Isom}(X)$*-equivariant homeomorphism $\partial_L : \mathcal{B}_L \longrightarrow \partial X_L$.*

Recall that the *support* of a measure m on a topological space Y is the smallest closed set C such that $m(Y \setminus C) = 0$. In other words $y \in \mathrm{supp}(m)$ if and only if for all U open containing y, $m(U) > 0$.

Definition 3.15. We say that the action by isometries of a group G on a hyperbolic space Y (not assumed to be proper) is *non-elementary* if there are two loxodromic isometries with disjoint fixed points on the Gromov boundary. A probability measure μ on G is said to be *non-elementary* if its support generates a group acting non-elementarily on Y.

In order to use the results concerning random walks in hyperbolic spaces, we must show that the action of a group G on a proper CAT(0) space with rank one isometries induces a non-elementary action on some hyperbolic model (X, d_L).

Proposition 3.16. *Let G be a group acting non-elementarily by isometries on a proper CAT(0) space (X, d), and assume that G possesses a rank one element for this action. Then there exists $L \in \mathbb{N}$ such that G acts on the hyperbolic space (X, d_L) non-elementarily by isometries.*

Proof. The action $G \curvearrowright (X, d)$ is non elementary and contains a rank one element, hence by Theorem 2.6 there exist two independent rank one isometries g, h in G. By Theorem 2.8, those rank one isometries are B-contracting for some B. Now, applying Theorem 3.11, there exists $L \in \mathbb{N}$ such that g and h act on (X, d_L) as loxodromic isometries. As g and H are independent, their fixed points form four distinct points of the visual boundary $\partial_\infty X$. Now seen in $X_L = (X, d_L)$, their fixed points sets must also form four distinct points of ∂X_L because of the homeomorphism $\partial_L : \mathcal{B}_L \longrightarrow \partial X_L$. This means that the action $G \curvearrowright X_L$ is non-elementary. $\square$

4 Random walks and hyperbolicity

The results of Section 3 allow us to read some information about the random walk in the hyperbolic models $X_L = (X, d_L)$, and then translate this information back to the original CAT(0) space. As the theory of random walks on hyperbolic spaces is well-studied, one may hope that this process is fruitful.

4.1 Random walks on hyperbolic spaces

In this section, we summarize what is known concerning random walks in hyperbolic spaces. Most of the work for the non-proper case was done by Maher and Tiozzo in [MT18]. The first result is the convergence of the random walk to the Gromov boundary.

Theorem 4.1 ([MT18, Theorem 1.1])**.** *Let G be a countable group of isometries of a separable hyperbolic space Y. Let μ be a non-elementary probability distribution on G, and $o \in Y$ a basepoint. Then the random walk $(Z_n(\omega)o)_n$ induced by μ*

converges to a point $z^+(\omega) \in \partial_\infty X$, and the resulting hitting measure is the unique μ-stationary measure on $\partial_\infty X$.

Now assume that the measure μ has finite first moment $\int d(go, o) d\mu(g) < \infty$. Let us define the drift (or escape rate) of the random walk.

Definition 4.2. The *drift* of the random walk $(Z_n o)_n$ on a hyperbolic space (Y, d) is defined as

$$\lambda := \inf_n \int_\Omega d(Z_n(\omega)o, o) d\mathbb{P}(\omega) = \inf_n \int_G d(go, o) d\mu^{*n}(g).$$

By a classical application of Kingmann subadditive Theorem, if μ has finite first moment the drift can also be defined as

$$\lambda = \lim_n \frac{1}{n} d(Z_n(\omega)o, o), \tag{2}$$

and the above limit is essentially a constant (up to measure 0, does not depend on ω).

Maher and Tiozzo prove that in their context, the drift is almost surely positive. This can be seen as a law of large numbers.

Theorem 4.3 ([MT18, Theorem 1.2 and 1.3])**.** *Let G be a countable group of isometries of a separable hyperbolic space (Y, d_Y). Let μ be a non-elementary probability distribution on G, and $o \in Y$ a basepoint. Assume that μ has finite first moment. Then the drift $\lambda := \lim_n \frac{1}{n} d(Z_n o, o)$ is well-defined and almost surely positive.*

Moreover, for every $\lambda' < \lambda$, there exists $\kappa > 0$ such that

$$\mathbb{P}\big(\omega \in \Omega \,:\, d_Y(Z_n(\omega)o, o) \leq \lambda' n\big) < e^{-\kappa n}. \tag{3}$$

Another piece of information that can be given about the random walk is the proportion of hyperbolic isometries in the random variables $(Z_n)_n$. Recall that the translation length of an isometry in a hyperbolic space is defined as $|g| := \lim_n \frac{1}{n} d(g^n o, o)$, which does not depend on the basepoint o.

Theorem 4.4 ([MT18, Theorem 1.4])**.** *Let G be a countable group of isometries of a separable hyperbolic space Y. Let μ be a non-elementary probability distribution on G, and $o \in Y$ is a basepoint. Then the translation length $|Z_n(\omega)|$ grows almost surely at least linearly in n: there exists $K > 0$ such that*

$$\mathbb{P}(\omega \,:\, |Z_n(\omega)| \leq Kn) \underset{n\to\infty}{\longrightarrow} 0.$$

The above result thus implies that the probability that $Z_n(\omega)$ is not a loxodromic isometry goes to zero as n goes to infinity.

4.2 First results for random walks in CAT(0) spaces

In CAT(0) spaces, many of the previous theorems hold if we assume that G induces "hyperbolic-like" properties. Namely, if X is a proper CAT(0) space, we will assume that G contains rank one isometries of X. The first result deals with stationary measures on $\overline{X}$. Recall that a measure $\nu \in \mathrm{Prob}(\overline{X})$ is called stationary if $\mu * \nu = \nu$.

Theorem 4.5 ([LeB22, Theorem 1.1])**.** *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G, and assume that G contains a rank one element. Then there exists a unique μ-stationary measure $\nu \in$ Prob$(\overline{X})$.*

The convergence of the random walk to the boundary can then be established in this setting. It is the analogue of Theorem 4.1.

Theorem 4.6 ([LeB22, Theorem 1.2])**.** *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G, and assume that G contains a rank one element. Then for every $x \in X$, and for $\mathbb{P}$-almost every $\omega \in \Omega$, the random walk $(Z_n(\omega)x)_n$ converges almost surely to a boundary point $z^+(\omega) \in \partial_\infty X$. Moreover, $z^+(\omega)$ is distributed according to the stationary measure ν.*

Interestingly, we can prove that the limit points are almost surely rank one, meaning that for almost any pair of limit points $\xi, \eta \in \partial_\infty X$, there exists a rank one geodesic in X joining ξ to η ([LeB22, Corollary 1.3]). This feature suggests the use of hyperbolic models. First, we establish a result concerning the proportion of rank one elements in the random walk.

Theorem 4.7. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G, and assume that G contains a rank one element. Then*

$$\mathbb{P}(\omega \,:\, Z_n(\omega) \text{ is a contracting isometry }) \underset{n\to\infty}{\to} 1.$$

Proof. Because of Proposition 3.16, we can then apply the results of Maher and Tiozzo. In particular, by Theorem 4.1, the random walk $(Z_n o)_n$ in (X_L, d_L) converges almost surely to a point of the Gromov boundary ∂X_L, and by Theorem 4.4, the translation length $|Z_n(\omega)|_L$ of $(Z_n(\omega))_n$ grows almost surely at least linearly in n. In particular, the probability that $Z_n(\omega)$ is a loxodromic element of X_L goes to 1 as n goes to ∞. But thanks to Theorem 3.11, an isometry g of the CAT(0) space X is contracting if and only if there is an L such that g acts as a loxodromic isometry on X_L. The previous now implies that the probability that $Z_n(\omega)$ is a contracting isometry of X goes to 1 as n goes to ∞. $\square$

The analogue of Theorem 4.3 also holds in the context of CAT(0) spaces with rank one isometries.

Theorem 4.8 ([LeB22, Theorem 1.4])**.** *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G with finite first moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Then the drift λ is almost surely positive:*

$$\lim_{n\to\infty} \frac{1}{n} d(Z_n o, o) = \lambda > 0.$$

Actually H. Izeki worked on the drift-free case in [Ize22]. The author proves a strengthening of Theorem 4.8, in that it is valid even for finite dimensional, non proper CAT(0) spaces, and without the assumption that there are rank one isometries. The counterpart is that one needs to assume that μ has finite second moment. Namely, Izeki proves that in this context, either the drift λ is strictly positive, or there is a G-invariant flat subspace in X [Ize22, Theorem A]. However, for our purpose, we will only need Theorem 4.8.

In the proof of Theorem 4.8, we actually show that the displacement $d(Z_n(\omega)x, x)$ is almost surely well approximated by the Busemann functions $b_\xi(Z_n(\omega)x)$. This result will be used later when we give geometric estimates for the action.

Proposition 4.9 ([LeB22, Proposition 5.2])**.** *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G with finite first moment, and assume that G contains a rank one element. Let $x \in X$ be a basepoint. Then for ν-almost every $\xi \in \partial X$, and $\mathbb{P}$-almost every $\omega \in \Omega$, there exists $C > 0$ such that for all $n \geq 0$ we have*

$$|b_\xi(Z_n(\omega)x) - d(Z_n(\omega)x, x)| < C. \tag{4}$$

5 Central Limit Theorems and general strategy

In order to prove our main result, we use a strategy that is largely inspired by the works of Benoist and Quint on linear spaces and hyperbolic spaces, see [BQ16a] and [BQ16b]. They developed a method for proving central limit theorems for cocycles, relying on results due to Brown in the case of martingales [Bro71].

5.1 Centerable cocycle

Let G be a discrete group, Z a compact G-space and c a cocycle $c : G \times Z \to \mathbb{R}$, meaning that $c(g_1 g_2, x) = c(g_1, g_2 x) + c(g_2, x)$, and assume that c is continuous. Let μ be a probability measure on G.

Definition 5.1. Let c be a continuous cocycle $c : G \times Z \to \mathbb{R}$. We say that c has *constant drift* c_μ if $c_\mu = \int_G c(g, x) d\mu(g)$ does not depend on $x \in Z$. We say that c is *centerable* if there exists a bounded measurable map $\psi : Z \to \mathbb{R}$ and a cocycle $c_0 : G \times Z \to \mathbb{R}$ with constant drift $c_{0,\mu} = \int_G c_0(g, x) d\mu(g)$ such that

$$c(g, x) = c_0(g, x) + \psi(x) - \psi(gx). \tag{5}$$

We say that c and c_0 are cohomologous. In this case, the *average* of c is defined to be $c_{0,\mu}$.

Remark 5.2. Let $\nu \in \mathrm{Prob}(Z)$ be a μ-stationary measure, and let $c : G \times Z \to \mathbb{R}$ be a centerable continuous cocycle: for $g, \in G, x \in Z$, $c(g, x) = c_0(g, x) + \psi(x) - \psi(gx)$ with c_0 having constant drift and ψ bounded measurable. Then

$$\begin{aligned} & \int_{G\times Z} c_0(g, x) d\mu(g) d\nu(x) + \int_Z \psi(x) d\nu(x) - \int_{G\times Z} \psi(gx) d\mu(g) d\nu(x) \\ = \; & \int_G c_0(g, x) d\mu(g) + \int_Z \psi(x) d\nu(x) - \int_{G\times Z} \psi(gx) d\mu(g) d\nu(x) \\ = \; & \int_G c_0(g, x) d\mu(g) + \int_Z \psi(x) d\nu(x) - \int_Z \psi(x) d\nu(x) \text{ by stationarity} \\ = \; & c_{0,\mu} \text{ because } c_0 \text{ has constant drift.} \end{aligned}$$

Hence the average of c is given by $c_{0,\mu} = \int c(g, x) d\mu(g) d\nu(x)$, which explains the terminology and shows that it does not depend on the choices of c_0 and ψ.

The reason why we study limit laws on cocycles is the following result. This version is borrowed from Benoist and Quint, who improved previous results from Brown about central limit theorems for martingales [Bro71].

Theorem 5.3 ([BQ16b, Theorem 3.4])**.** *Let G be a locally compact group acting by homeomorphisms on a compact metrizable space Z. Let $c : G \times Z \to \mathbb{R}$ be a continuous cocycle such that $\int_G \sup_{x\in Z} |c(g, x)|^2 d\mu(g) < \infty$. Let μ be a Borel probability measure on G. Assume that c is centerable with average λ_c and that there exists a unique μ-stationary probability measure ν on Z.*

Then the random variables $\frac{1}{\sqrt{n}}(c(Z_n, x) - n\lambda_c)$ converge in law to a Gaussian law N_μ. In other words, for any bounded continuous function F on $\mathbb{R}$, one has

$$\int_G F\big(\frac{c(g, x) - n\lambda_c}{\sqrt{n}}\big) d(\mu^{*n})(g) \longrightarrow \int_{\mathbb{R}} F(t) dN_\mu(t).$$

Moreover, if we write $c(g,z) = c_0(g,z) + \psi(z) - \psi(gz)$ with ψ bounded and c_0 with constant drift c_μ, then the covariance 2-tensor of the limit law is

$$\int_{G\times Z} (c_0(g,z) - c_\mu)^2 d\mu(g) d\nu(z).$$

5.2 Busemann cocycle and strategy

Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in \mathrm{Prob}(G)$ be an admissible probability measure on G with finite first moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Theorems 4.6 and 4.8 ensure that the random walk $(Z_n(\omega)o)_n$ converges to a point of the boundary and that the drift $\lambda = \lim_n \frac{1}{n} d(Z_n(\omega)o, o)$ is well-defined and almost surely positive.

We denote by $\check{\mu}$ the probability measure on G defined by $\check{\mu}(g) = \mu(g^{-1})$. Let $(\check{Z}_n)_n$ be the right random walk associated to $\check{\mu}$. Since μ is admissible and has finite first moment, so does $\check{\mu}$. We can then apply Theorems 4.5, 4.6 and 4.8 to $\check{\mu}$. We will denote by $\check{\nu}$ the unique $\check{\mu}$-stationary measure on $\overline{X}$, and by $\check{\lambda}$ the positive drift of the random walk $(\check{Z}_n o)_n$.

Remark 5.4. One can check that

$$\begin{aligned} \check{\lambda} &= \inf_n \frac{1}{n} \int d(go, o) d\check{\mu}^{*n}(g) \\ &= \inf_n \frac{1}{n} \int d(o, g^{-1}o) d\check{\mu}^{*n}(g) \\ &= \inf_n \frac{1}{n} \int d(o, go) d\mu^{*n}(g), \end{aligned}$$

hence $\lambda = \check{\lambda}$.

In our context, the continuous cocycle that we consider is the Busemann cocycle on the visual compactification of the CAT(0) space X: for $x \in \overline{X}$, $g \in G$ and $o \in X$ a basepoint,

$$\beta(g,x) = b_x(g^{-1}o).$$

It is straightforward to show that β is continuous. Observe that for all $g_1, g_2 \in G$, $x \in Y$, horofunctions satisfy a cocycle relation:

$$\begin{aligned} b_\xi(g_1 g_2 o) &= \lim_{x_n \to \xi} d(g_1 g_2, x_n) - d(x_n, x) \\ &= \lim_{x_n \to \xi} d(g_2, g_1^{-1} x_n) - d(g_1 o, x_n) + d(g_1 o, x_n) - d(x_n, o) \\ &= \lim_{x_n \to \xi} d(g_2 x, g_1^{-1} x_n) - d(o, g_1^{-1} x_n) + d(g_1 x, x_n) - d(x_n, o) \\ &= b_{g_1^{-1}\xi}(g_2 o) + b_\xi(g_1 o). \end{aligned} \tag{6}$$

By (6), β satisfies the cocycle relation $\beta(g_1g_2, x) = \beta(g_1, g_2x) + \beta(g_2, x)$. Thanks to Proposition 4.9, for every $o \in X$, for ν-almost every $x \in \partial X$, and $\mathbb{P}$-almost every $\omega \in \Omega$, there exists $C > 0$ such that for all $n \geq 0$ we have

$$|\beta(Z_n(\omega)^{-1}, x) - d(Z_n(\omega)o, o)| < C. \tag{7}$$

Equation (7) shows that the cocycle $\beta(Z_n(\omega), x)$ "behaves" like $d(Z_n(\omega)o, o)$. Thus it makes sense to try and apply Theorem 5.3 to the Busemann cocycle $\beta(g, x)$.

Henceforth, we will assume that μ is an admissible probability measure on G with finite second moment $\int_G d(go, o)^2 d\mu(g) < \infty$.

The following proposition summarizes some properties of the Busemann cocycle. It shows that obtaining a central limit theorem on β will imply our main result.

Proposition 5.5. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk, and $\beta : G \times \overline{X} \to \mathbb{R}$ be the Busemann cocycle $\beta(g, x) = b_x(g^{-1}o)$. Then*

1. *$\int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\mu(g) < \infty$ and $\int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\check{\mu}(g) < \infty$;*
2. *For ν-almost every $\xi \in \partial_\infty X$, $\lambda = \lim_n \frac{1}{n}\beta(Z_n(\omega), \xi)$ $\mathbb{P}$-almost surely;*
3. *$\mathbb{P}$-almost surely, $\lambda = \int_{G \times \overline{X}} \beta(g, x) d\mu(g) d\nu(x) = \int_{G \times \overline{X}} \beta(g, x) d\check{\mu}(g) d\check{\nu}(x)$.*

Proof. As a consequence of Proposition 4.9, equation (7) gives that for ν-almost every $x \in \partial X$, and $\mathbb{P}$-almost every $\omega \in \Omega$, there exists $C > 0$ such that for all $n \geq 0$ we have

$$|\beta(Z_n(\omega)^{-1}, x) - d(Z_n(\omega)o, o)| < C.$$

Because the action is isometric and μ has finite second moment $\int_G d(go, o)^2 d\mu(g) < \infty$, we obtain

$$\int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\mu(g) < \infty.$$

With the same argument:

$$\int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\check{\mu}(g) < \infty.$$

Now thanks to Theorem 4.8, the drift of the random walk $\lambda = \lim_n \frac{1}{n} d(Z_n(\omega)o, o)$ is almost surely positive. Together with equation 7, this gives that for ν-almost every $\xi \in \overline{X}$,

$$\lambda = \lim_n \frac{1}{n} \beta(Z_n(\omega), \xi) \text{ } \mathbb{P}\text{-almost surely.}$$

The ideas in the proof of 3 are classical. We give the details for the convenience of the reader.

Let $T : (\Omega \times \overline{X}, \mathbb{P} \times \check{\nu}) \to (\Omega \times \overline{X}, \mathbb{P} \times \check{\nu})$ be defined by $T(\omega, \xi) \mapsto (S\omega, \omega_0^{-1}\xi)$, with $S((\omega_i)_{i\in\mathbb{N}}) = (\omega_{i+1})_{i\in\mathbb{N}}$ the usual shift on Ω. By [LeB22, Proposition 5.4], T preserves the measure $\mathbb{P} \times \check{\nu}$ and is an ergodic transformation. Define $H : \Omega \times \overline{X} \to \mathbb{R}$ by

$$H(\omega, \xi) = h_\xi(\omega_0 o) = \beta(\omega_0^{-1}, \xi).$$

By 1, it is clear that $\int |H(\omega, \xi)| d\mathbb{P}(\omega) d\check{\nu}(\xi) < \infty$.

By cocycle relation (6) one gets that

$$h_\xi(Z_n o) = \sum_{k=1}^{n} h_{Z_k^{-1}\xi}(\omega_k o) = \sum_{k=1}^{n} H(T^k(\omega, \xi)). \tag{8}$$

Then $\beta(Z_n(\omega)^{-1}, \xi) = \sum_{k=1}^{n} H(T^k(\omega, \xi))$, and by 2,

$$\lambda = \lim_n \frac{1}{n} \sum_{k=1}^{n} H(T^k(\omega, \xi)). \tag{9}$$

Now, by Birkhoff ergodic theorem, one obtains that almost surely,

$$\begin{aligned} \lambda &= \int_{\Omega\times\overline{X}} H(\omega, \xi) d\mathbb{P}(\omega) d\check{\nu}(x). \\ &= \int_{\Omega\times\overline{X}} h_\xi(\omega_0 o) d\mathbb{P}(\omega) d\check{\nu}(x) \\ &= \int_{G\times\overline{X}} \beta(g^{-1}, \xi) d\mu(g) d\check{\nu}(x) \\ &= \int_{G\times\overline{X}} \beta(g, \xi) d\check{\mu}(g) d\check{\nu}(x) \end{aligned} \tag{10}$$

The previous computations can be done similarly for μ and ν, hence we also have that

$$\lambda = \int_{G\times\overline{X}} \beta(g, x) d\mu(g) d\nu(x).$$

$\square$

In order to apply Theorem 5.3 on the Busemann cocycle β, it remains to show that β is centerable. If this is the case, by 3 and Remark 5.2, its average must be the positive drift λ. In other words, we need to show that there exists a bounded measurable function $\psi : \overline{X} \to \mathbb{R}$ such that the cohomological equation

$$\beta(g, x) = \beta_0(g, x) + \psi(x) - \psi(gx). \tag{11}$$

is verified. Then, proving the Central Limit Theorem in our context amounts to finding such a ψ that is well defined and bounded. This will be done by using a hyperbolic model that can give nice estimates on the random walk.

6 Proof of the Central Limit Theorem

6.1 Geometric estimates

In this section, we prove our main Theorem, following the strategy explained in Section 5. First, we will provide geometric estimates on the random walk that will be used later on. This is where we use the specific contraction properties provided by the curtains and the hyperbolic models exposed in Section 3. The goal is ultimately to prove that the candidate ψ for the cohomological equation is bounded.

Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Recall that B_L is defined to be the subspace of $\partial_\infty X$ consisting of all geodesic rays $\gamma : [0, \infty) \to X$ starting from o and such that there exists an infinite L-chain crossed by γ. By Theorem 3.14, there exists an $\mathrm{Isom}(X)$-equivariant embedding $\mathcal{I} : \partial X_L \to \partial_\infty X$, whose image lies in $\mathcal{B}_L$.

Proposition 6.1. *Let (g_n) be a sequence of isometries of G, and let $o \in X$, $x, y \in \partial_\infty X$. Assume that there exists $\lambda, \epsilon, A > 0$ such that:*

(i) $\{g_n o\}_n$ converges in $(\overline{X_L}, d_L)$ to a point of the boundary $z_L \in \partial X_L$, whose image in $\partial_\infty X$ by the embedding $\mathcal{I}$ is not y;

(ii) $d_L(g_n o, o) \geq An$;

(iii) $|b_x(g_n^{-1} o) - n\lambda| \leq \varepsilon n$;

(iv) $|b_y(g_n o) - n\lambda| \leq \varepsilon n$;

(v) $|d(g_n o, o) - n\lambda| \leq \varepsilon n$.

Then, one obtains:

1. $(g_n x|g_n o)_o \geq (\lambda - \varepsilon)n$*;*
2. $(y|g_n o)_o \leq \varepsilon n$*.*

If moreover $A \geq 2(4L+10)\varepsilon$*, then we have:*

3. $(y|g_n x)_o \leq \varepsilon n + (2L+1)$*.*

The proof of points 1 and 2 is straightforward, so we begin by these.

Proof of estimates 1 and 2. A simple computation gives that

$$(g_n x|g_n o)_o = \frac{1}{2}(b_x(g_n^{-1}o) + d(g_n o, o))$$

Then using assumptions (iii) and (v) gives immediately that $(g_n x|g_n o) \geq (\lambda - \epsilon)n$, which proves 1.

Now, by definition,

$$(y|g_n o) = \frac{1}{2}(d(g_n o, o) - b_y(g_n o))$$

Then by assumptions (iv) and (v), we obtain 2. □

The proof of point 3 is the hard part. We prove it in two steps. First, we show that under the assumptions, for n large enough, there exist at least three L-separated curtains dual to $[o, g_n o]$ separating $\{g_n o, g_n x\}$ on the one side and $\{o, y\}$ on the other, see Figure 2. Then we show that the presence of these hyperplanes implies the result.

By assumption (ii), there exists an L-chain of size $S(n)$ that separates o and $g_n o$. By Proposition 3.8, there exists an L-chain dual to $[o, g_n o]$ of size greater than or equal to $\lfloor \frac{S(n)}{4L+1} \rfloor$ that separates o and $g_n o$. Denote by $c_n = \{h_i^n\}_{i=1}^{S'(n)}$ a maximal L-chain dual to $[o, g_n o]$, separating o and $g_n o$, and orient the half-spaces so that $o \in h_i^-$ for all i. When the context is clear, we might omit the dependence in n for ease of notations, and just write $\{h_i\}_{i=1}^{S'(n)}$ for a maximal L-chain dual to $[o, g_n o]$. Recall that $S(n) \geq An$, hence c_n must be of length $S'(n) \geq A'n$, where $A' = \frac{A}{4L+1}$.

Lemma 6.2. *Under the assumptions of Proposition 6.1, there exists a constant* C *such that for all* $n \in \mathbb{N}$*, the number of* L*-separated hyperplanes in* c_n *that do not separate* $\{o, y\}$ *and* $\{g_n o\}$ *is less than* C*.*

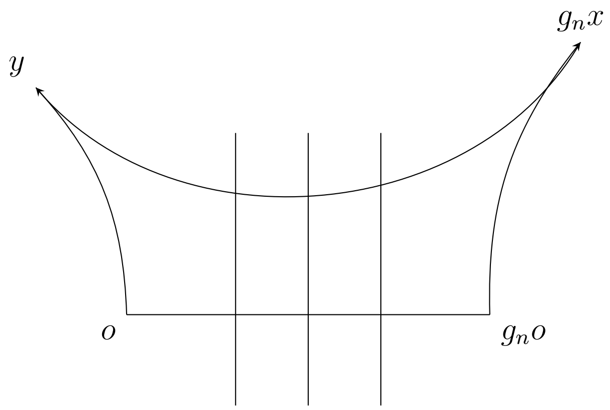

Figure 2: A "hyperbolic-like" 4 points inequality in Proposition 6.1.

Proof of Lemma 6.2. By assumption, $\{g_n o\}_n$ converges in $(\overline{X_L}, d_L)$ to a point of the boundary $z_L \in \partial X_L$. By Theorem 3.14, there exists an $\mathrm{Isom}(X)$-equivariant embedding $\mathcal{I} : \partial X_L \to \partial_\infty X$ that extends the canonical inclusion $X_L \to X$, and whose image lies in $\mathcal{B}_L$. Denote by $z := \mathcal{I}(z_L)$ the image in $\partial_\infty X$ of the limit point z_L by this embedding.

Denote by $\beta : [0, \infty) \to X$ a geodesic ray joining o to z. Since $z \in \mathcal{B}_L$, there exists an infinite L-chain $c = \{k_i\}_{i \in \mathbb{N}}$ that separate o from z. Note that because of Lemma 3.8 and Remark 3.13, we can assume that c is a chain of curtains which is dual to the geodesic ray β. Since $\{g_n o\}_n$ converges in $(\overline{X_L}, d_L)$ to $z_L \in \partial X_L$, and z is the image of z_L by the equivariant embedding $\mathcal{I}$, it implies that $\{g_n o\}_n$ converges to z in X. The fact that $z \in \mathcal{B}_L$ implies that for all $i \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, k_i separates o from $g_n o$. Now, we denote by $\gamma : [0, \infty) \to X$ the geodesic ray that represents $y \in \partial_\infty X$. See figure 3.

Due to Remark 3.13, meeting c infinitely often is equivalent to crossing it, then since $y \neq z$, there exists $p \in \mathbb{N}$ such that $\gamma \subseteq k_p^-$. Now consider n_0 such that for $n \geq n_0$, $g_n o \in k_{p+2}^+$. Fix $n \geq n_0$. Recall that c_n is a maximal L-chain dual to $[o, g_n o]$ separating o and $g_n o$.

Denote by $r \in \beta$ a point in the pole of k_{p+1}, and denote by $r' = r'(n)$ the projection of r onto the geodesic $[o, g_n o]$. Then by Lemma 3.7,

$$d(o, r'(n)) \leq d(o, r) + 2L + 1.$$

Due to the thickness of the curtains (Remark 3.2), the number of curtains in c_n that separate o and $r'(n)$ is $\leq d(o, r) + 2L + 2$. We emphasize that this number does not depend on $n \geq n_0$, because for all $n \geq n_0$, $g_n \in k_{p+2}^+$ and the previous equation holds.

Recall that $\gamma \subseteq k_p^-$, so in particular $\gamma \subseteq k_{p+1}^-$. Then by star convexity of the curtains (Lemma 3.3), every curtain in c_n whose pole belongs to $[r'(n), g_n o]$

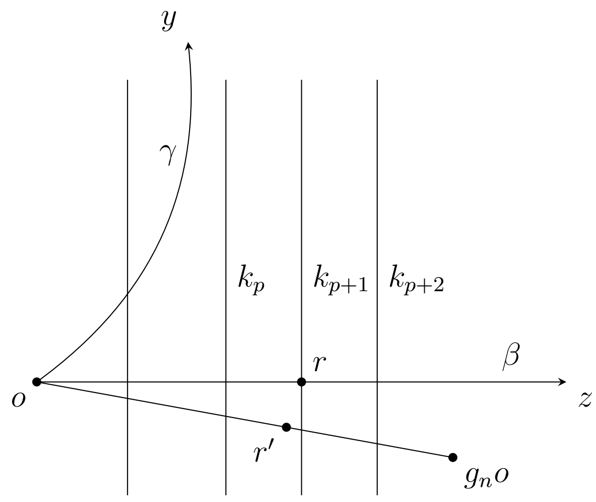

Figure 3: Illustration of Lemma 6.2.

separates $\{o, y\}$ from $g_n o$. Then by the previous argument, the number of curtains that do not separate $\{o, y\}$ from $\{g_n o\}$ is less than $d(o, r'(n))$. In particular, the number of curtains that do not separate $\{o, y\}$ from $\{g_n o\}$ is less than $d(o, r) + 2L + 2$. Since this quantity does not depend on n, we have proven the Lemma. $\square$

Now, for a fixed n, let us give an estimate for the number of curtains in $c_n = \{h_1^n, \ldots, h_{S'(n)}^n\}$ that separate o and $g_n x$. When a given n is fixed, we omit the dependence in n and just write $c_n = \{h_1, \ldots, h_{S'(n)}\}$ to ease the notations. Let $\gamma_n : [0, \infty) \to X$ be the geodesic ray joining o and $g_n x$. Let us take $k_0 = k_0(n)$ (depending on n) large enough so that for all $k \geq k_0$,

$$|(g_n o|g_n x)_o - (g_n o|\gamma_n(k)_o)| \leq 1.$$

Lemma 6.3. *Under the assumptions of Proposition 6.1, the number of L-separated hyperplanes in c_n that separate $\{o\}$ and $\{g_n o, \gamma_n(k_0)\}$ is unbounded in n. More precisely, for all $M \in \mathbb{N}$, there exists n_0 such that for all $n \geq n_0$, the number of L-separated hyperplanes in c_n that separate $\{o\}$ and $\{g_n o, \gamma_n(k)\}$ is greater than M for all $k \geq k_0$.*

Proof of Lemma 6.3. Let $k \geq k_0$. Suppose that the number of curtains in $c_n = \{h_1, \ldots, h_{S'(n)}\}$ separating o and $\gamma_n(k)$ is less than or equal to $p \in [0, S'(n) - 4]$. Then $\{h_{p+2}, \ldots, h_{S'(n)}\}$ is an L-chain separating $\{o, \gamma_n(k)\}$ and $\{g_n o\}$. We then denote by $r(n)$ a point on $h_{p+3} \cap [\gamma_n(k), g_n o]$ and by $r'(n)$ the projection of $r(n)$ onto $[o, g_n o]$, see Figure 4. By hypothesis on k,

$$\begin{aligned} 2((g_n x|g_n o)_o - 1) &\leq 2(\gamma_n(k)|g_n o)_o \\ &= d(\gamma_n(k), o) + d(g_n o, o) - d(g_n o, \gamma_n(k)). \end{aligned}$$

Now by the bottleneck Lemma 3.7 and the triangular inequality,

$$\begin{aligned}
2(\gamma_n(k)|g_n o)_o &= d(\gamma_n(k), o) + d(g_n o, o) - (d(g_n o, r(n)) + d(r(n), \gamma_n(k))) \\
&\leq d(\gamma_n(k), o) + d(g_n o, o) - (d(g_n o, r'(n)) - (2L+1) + d(r(n), \gamma_n(k))) \\
&\leq d(r(n), o) + d(g_n o, o) - d(g_n o, r'(n)) + 2L + 1 \\
&\leq d(r'(n), o) + 2L + 1 + d(g_n o, o) - d(g_n o, r'(n)) + 2L + 1 \\
&\leq 2d(r'(n), o) + 2(2L+1).
\end{aligned}$$

Because the pole of a curtain is of diameter 1, $d(o, r'(n)) \leq d(g_n o, o) - (S'(n) - (p+1))$. However, by assumptions (ii) and (v) of Lemma 6.1, one gets that $d(g_n o, o) \leq (\lambda + \varepsilon)n$ and $S(n) \geq An$. Recall that by Lemma 3.8, this means that $S'(n) \geq A'n$, where $A' = \frac{A}{4L+1}$. Combining this with the previous result yields

$$\begin{aligned}
& (g_n x|g_n o)_o - 1 \leq d(o, r'(n)) + 2L + 1 \\
\Rightarrow\ & (\lambda - \varepsilon)n - 1 \leq (\lambda + \varepsilon)n - (A'n - (p+1)) + 2L + 1 \text{ by Lemma 6.1, 1} \\
\Rightarrow\ & 0 \leq (2\varepsilon - A')n + 2L + p + 2.
\end{aligned}$$

If $A' > 2\varepsilon$, there exists n_0 large enough such that for all $n \geq n_0$, the above inequality gives a contradiction. As a consequence, if $A' > 2\varepsilon$, or equivalently if $A > 2(4L+1)\varepsilon$, there exists n_0 such that for all $n \geq n_0$, the number of curtains in c_n separating o and $\{\gamma_n(k), g_n o\}$ is greater than p. □

We can now conclude the proof of Lemma 6.1.

Proof of estimate 3. Recall that we denote by $\gamma : [0, \infty) \to X$ the geodesic ray that represents $y \in \partial_\infty X$ such that $\gamma(0) = o$ and by $\gamma_n : [0, \infty) \to X$ the geodesic ray joining o and $g_n x$. Combining Lemma 6.2 and Lemma 6.3, we get that if $A > 2(4L+10)\varepsilon$, there exists n_0, k_0 such that for all $n \geq n_0$ and all $k \geq k_0$, c_n contains at least 3 pairwise L-separated curtains that separate $\{o, \gamma(k)\}$ on the one side and $\{g_n o, \gamma_n(k)\}$ on the other. Call these hyperplanes $\{h_1, h_2, h_3\}$ and arrange the order so that $h_i \subseteq h_{i+1}^-$. Denote by $m_k(n) \in h_2$ some point on the geodesic segment joining $\gamma(k)$ to $\gamma_n(k)$, and $m'_k(n)$ belonging to the geodesic segment $[o, g_n o]$ such that $d(m_k(n), m'_k(n)) \leq 2L+1$, see Figure 5

Then we have

$$\begin{aligned}
2(\gamma(k)|\gamma_n(k))_o &= d(\gamma(k), o) + d(o, \gamma_n(k)) - d(\gamma(k), \gamma_n(k)) \\
&\leq d(\gamma(k), o) + d(o, m'_k(n)) + d(m'_k(n), m_k(n)) \\
&\quad + d(m_k(n), \gamma_n(k)) - d(\gamma(k), \gamma_n(k)) \text{ by the triangular inequality} \\
&\leq d(\gamma(k), o) + d(o, m'_k(n)) - d(\gamma(k), m_k(n)) + 2L + 1
\end{aligned}$$

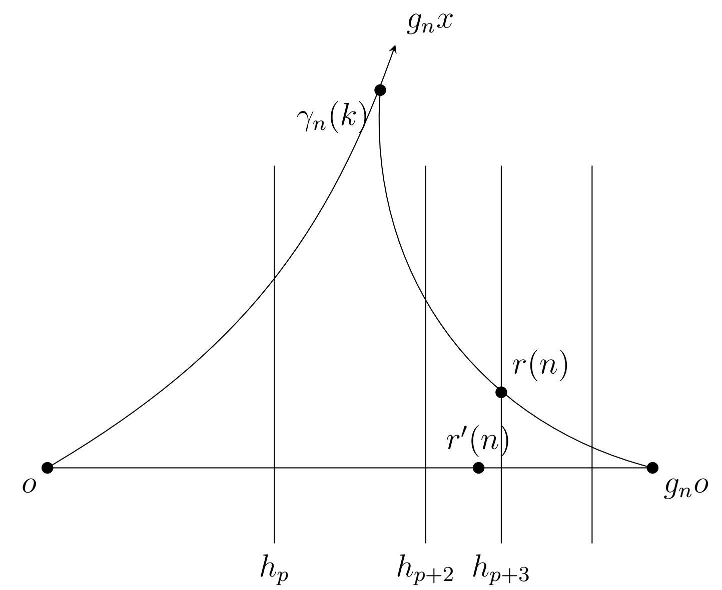

Figure 4: Illustration of Lemma 6.3.

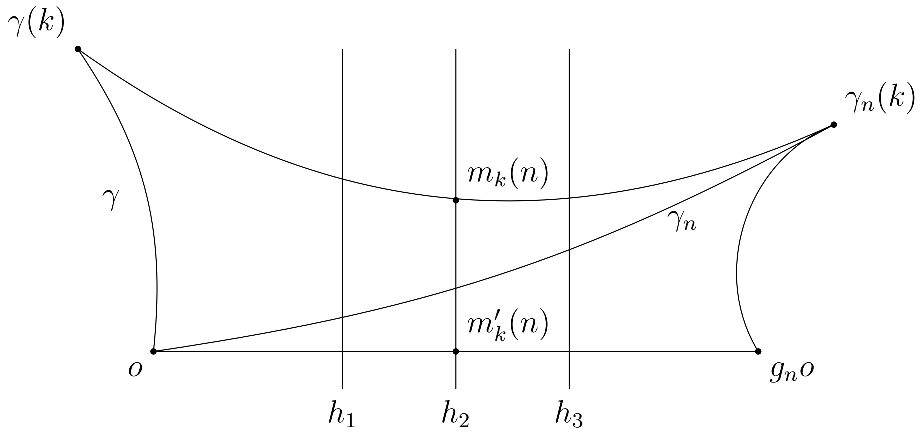

Figure 5: Proof of Proposition 6.1

by Lemma 3.7. Since m'_k is on $[o, g_n o]$, $d(o, m'_k) = d(o, g_n o) - d(g_n o, m'_k(n))$. We then have:

$$\begin{array}{rcl} 2(\gamma(k)|\gamma_n(k))_o & \leq & d(\gamma(k), o) + d(o, g_n o) - d(g_n o, m'k(n)) - d(\gamma(k), m_k(n)) + 2L + 1 \\ & = & d(\gamma(k), o) + d(o, g_n o) - (d(g_n o, m'_k(n)) + d(\gamma(k), m_k(n))) + 2L + 1. \end{array}$$

Now observe that

$$\begin{array}{rcl} d(\gamma(k), g_n o) & \leq & d(g_n o, m'_k(n)) + d(\gamma(k), m_k(n)) + d(m_k, m'_k) \\ & \leq & d(g_n o, m'_k(n)) + d(\gamma(k), m_k(n)) + 2L + 1 \text{ by Lemma 3.7,} \end{array}$$

hence $d(\gamma(k), g_n o) - (2L + 1) \leq d(g_n o, m'_k(n)) + d(\gamma(k), m_k(n))$. Then

$$\begin{array}{rcl} 2(\gamma(k)|\gamma_n(k))_o & \leq & d(\gamma(k), o) + d(o, g_n o) - (d(\gamma(k), g_n o) - (2L + 1)) + 2L + 1 \\ & = & d(\gamma(k), o) + d(o, g_n o) - d(\gamma(k), g_n o) + 2(2L + 1) \\ & = & 2(\gamma(k)|g_n o)_o + 2(2L + 1). \end{array}$$

As $k \to \infty$, one obtains that $(g_n x|y)_o \leq (g_n o|y)_o + (2L + 1)$, and the result follows from 2. $\square$

6.2 Proof of the Central Limit Theorem

In this section, we prove the main result of the paper. Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in \text{Prob}(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk provided by Theorem 4.8. We assume the action on X to be non elementary and rank one, hence due to Proposition 3.16, there exists a number $L \geq 0$ such that G acts by isometries on $X_L = (X, d_L)$ non elementarily. Then one can consider the random walk $(Z_n(\omega)o)_n$ as a random walk on (X, d_L), which we will write $(Z_n\tilde{o})_n$ when the context is not clear. The model (X, d_L) is hyperbolic, so we can apply the results of Maher and Tiozzo [MT18] summarized in Section 4. In particular, due to Theorem 4.1, the random walk $(Z_n\tilde{o})_n$ in X_L converges to a point of the Gromov boundary ∂X_L of (X, d_L).

Moreover, since we assume μ to have finite first moment (for the action on the CAT(0) space X), and since $d(x, y) \geq d_L(x, y)$ for all $x, y \in X$, the measure μ is also of finite first moment for the action on the hyperbolic model (X, d_L). In particular, the drift $\tilde{\lambda}$ of the random walk $(Z_n\tilde{o})_n$ is almost surely positive. In other words, we have that $\mathbb{P}$-almost surely,

$$\lim_{n\to\infty} \frac{1}{n} d(Z_n(\omega)\tilde{o}, \tilde{o}) = \tilde{\lambda} > 0.$$

Due to Theorem 4.5, there exists a unique μ-stationary probability measure ν on $\overline{X}$. If we define $\check{\mu} \in \mathrm{Prob}(G)$ by $\check{\mu}(g) = \mu(g^{-1})$, $\check{\mu}$ is still admissible and of finite second moment. We denote by $\check{\nu}$ the unique $\check{\mu}$-stationary measure on $\overline{X}$.

We recall that the Busemann cocycle $\beta : G \times \overline{X} \to \mathbb{R}$ is defined by:

$$\beta(g, x) = b_x(g^{-1}o).$$

Our goal is to apply Theorem 5.3 to the Busemann cocycle β. The results of Section 5 show that proving a central limit theorem for the random walk $(Z_n(\omega)o)_n$ amounts to proving that β is centerable. As in the works of [BQ16a], [Hor18] and [FLM21], the natural candidate to solving the cohomological equation (11) is the function:

$$\psi(x) = -2 \int_{\overline{X}} (x|y)_o d\check{\nu}(y).$$

Proposition 6.4. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Then the Borel map $\psi(x) = \int_{\overline{X}} (x|y)_o \, d\check{\nu}(y)$ is well-defined and essentially bounded. Thus, the cocycle $\beta(g, x) = h_x(g^{-1}o)$ is centerable.*

In order to show that ψ is well-defined and bounded, we need the following statement, which resembles [BQ16a, Proposition 4.2].

Proposition 6.5. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in$ Prob(G) be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint for the random walk $(Z_n(\omega)o)_n$. Let λ be the (positive) drift of the random walk, and ν a μ-stationary measure on $\overline{X}$. Assume that there exists $a > 0$ and $(C_n)_n \in \ell^1(\mathbb{N})$ such that for almost every $x, y \in \overline{X}$, we have, for every n:*

1. $\mathbb{P}((Z_n o | Z_n x)_o \leq an) \leq C_n$;
2. $\mathbb{P}((Z_n o | y)_o \geq an) \leq C_n$;
3. $\mathbb{P}((Z_n x | y)_o \geq an) \leq C_n$.

Then one has:

$$\sup_{x \in \overline{X}} \int_{\overline{X}} (x|y)_o d\nu(y) < \infty.$$

Proof. Suppose that there exist $a > 0$, $(C_n)_n \in \ell^1(\mathbb{N})$ such that for almost every $x, y \in \overline{X}$, we have estimates 1 to 3. We get:

$$\begin{aligned}
\nu(\{x \in X | (x|y) \geq an\}) &= \int_{\overline{X}} \mu^{*n}(\{g \in G \,|\, (gx|y)_o \geq an\}) d\nu(x) \text{ by } \mu\text{-stationarity} \\
&\leq \int_{\overline{X}} C_n d\nu(x) = C_n \text{ by estimate 3.}
\end{aligned}$$

Then, define $A_{n,y} := \{x \in \overline{X} \,|\, (x|y)_o \geq an\}$, so that by splitting along the subsets $A_{n-1,y} - A_{n,y}$, one gets

$$\begin{aligned}
\int_{\overline{X}} (x|y)_o d\nu(x) &\leq \sum_{n \geq 1} an(\nu(A_{n-1,y}) - \nu(A_{n,y})) \\
&\leq \sum_{n \geq 1} an(C_{n-1} - C_n) \\
&= a + \sum_{n \geq 1} aC_n(n + 1 - n) < \infty.
\end{aligned}$$

□

We want to show that estimates from Proposition 6.5 hold. As we will see, estimates 1 are quite straightforward to check using the positivity of the drift. Most of the work concerns estimate 3.

Combining Proposition 5.5 with Theorem 4.8 and [BQ16b, Proposition 3.2], one obtains the following:

Proposition 6.6. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk. Then, for every $\varepsilon > 0$, there exists $(C_n)_n \in \ell^1(\mathbb{N})$ such that for any $x \in \overline{X}$,*

$$\mathbb{P}(|\beta(Z_n, x) - n\lambda| \geq \varepsilon n) \leq C_n; \tag{12}$$

$$\mathbb{P}(|\beta(Z_n^{-1}, x) - n\lambda| \geq \varepsilon n) \leq C_n; \tag{13}$$

$$\mathbb{P}(|d(Z_n o, o) - n\lambda| \geq \varepsilon n) \leq C_n. \tag{14}$$

Proof. Recall that by Proposition 5.5, β is a continuous cocycle such that

$$\int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\mu(g) < \infty \text{ and } \int_G \sup_{x \in \overline{X}} |\beta(g, x)|^2 d\check{\mu}(g) < \infty.$$

Moreover,

$$\lambda = \int_{G\times \overline{X}} \beta(g,x)d\mu(g)d\nu(x) = \int_{G\times \overline{X}} \beta(g,x)d\check{\mu}(g)d\check{\nu}(x).$$

We can then apply [BQ16b, Proposition 3.2]: for every $\varepsilon > 0$, there exists a sequence $(C_n) \in \ell^1(\mathbb{N})$ such that for every $x \in \overline{X}$,

$$\mathbb{P}\big(\omega \in \Omega \,:\, \big|\frac{\beta(Z_n(\omega),x)}{n} - \lambda\big| \geq \epsilon\big) \leq C_n.$$

The same goes for $\check{\mu}$ and $\check{\nu}$, which gives estimates (12) and (13).

Estimate (14) is then a straightforward consequence of Proposition 4.9. $\square$

The following Lemma will also be important in the proof of Proposition 6.4.

Lemma 6.7. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk.*

Then there exists $L > 0$, $\lambda_L > 0$ such that almost surely, $\liminf_n \frac{d_L(Z_n o,o)}{n} = \lambda_L$. *Moreover, there exists $A > 0$ and $(C_n) \in \ell^1(\mathbb{N})$ such that*

$$\mathbb{P}\big(d_L(Z_n o, o) < An\big) \leq C_n.$$

Proof. The action $G \curvearrowright (X,d)$ is non elementary and contains a rank one element, hence by Proposition 3.16, there exists L such that the action $G \curvearrowright (X,d_L)$ is non-elementary as the loxodromic isometries g and h are independent. We can then apply Theorem 4.3, which gives the Lemma. $\square$

Let us now complete the proof of Proposition 6.4.

Proof of Proposition 6.4. By assumptions, we can apply Theorem 4.1: there exists $L > 0$ such that $(Z_n(\omega)o)_n$ converges in (X_L, d_L) to a point z_L of the boundary. By Theorem 4.5, there is a unique μ-stationary measure ν on $\partial_\infty X$, and this measure is non-atomic.

Fix A as in Lemma 6.7, and $(C_n)_n \in \ell^1(\mathbb{N})$ such that

$$\mathbb{P}\big(d_L(Z_n o, o) < An\big) < C_n.$$

Now take $0 < \varepsilon < \min(\frac{A}{2(4L+10)}, \lambda/2)$. Due to Proposition 6.6, there exists a sequence $C'_n \in \ell^1(\mathbb{N})$ such that

$$\begin{aligned}
&\mathbb{P}(|\beta(Z_n,x) - n\lambda| \geq \varepsilon n) \leq C'_n\\
&\mathbb{P}(|\beta(Z_n^{-1},x) - n\lambda| \geq \varepsilon n) \leq C'_n\\
&\mathbb{P}(|d(Z_n o,o) - n\lambda| \geq \varepsilon n) \leq C'_n.
\end{aligned}$$

We can assume that $C_n = C'_n$ for all n. Then for ν-almost every $x, y \in \partial_\infty X$, we have the quantitative assumptions in Proposition 6.1: with

(i) $\{Z_n o\}_n$ converges in $(\overline{X_L}, d_L)$ to a point of the boundary $z_L \in \partial X_L$, whose image in $\partial_\infty X$ by the embedding $\mathcal{I}$ is not y;

(ii) $\mathbb{P}\big(d_L(Z_n o, o) \geq An\big) \geq 1 - C_n$;

(iii) $\mathbb{P}\big(|b_x(Z_n^{-1}o) - n\lambda| \leq \varepsilon n\big) \geq 1 - C_n$;

(iv) $\mathbb{P}\big(|b_y(Z_n o) - n\lambda| \leq \varepsilon n\big) \geq 1 - C_n$;

(v) $\mathbb{P}\big(|d(gZ_n o, o) - n\lambda| \leq \varepsilon n\big) \geq 1 - C_n$.

As a consequence, one obtains that for ν-almost every $x, y \in \partial_\infty X$, the probability that these estimates are not satisfied is bounded above by $4C_n$. Now choosing $a \in (\varepsilon, \lambda - \varepsilon)$, we get that for n large enough,

1. $\mathbb{P}\big((g_n x|g_n o)_o \geq an\big) \geq 1 - 4C_n$;
2. $\mathbb{P}\big((y|g_n o)_o \leq an\big) \geq 1 - 4C_n$;
3. $\mathbb{P}\big((y|g_n x)_o \leq an\big) \geq 1 - 4C_n$.

Since the sequence $(4C_n)_n$ is still summable, we can apply Proposition 6.5. Since ψ is measurable by Fubini, the proof is complete. $\square$

We can now state the following.

Theorem 6.8. *Let G be a discrete group and $G \curvearrowright X$ a non-elementary action by isometries on a proper CAT(0) space X. Let $\mu \in Prob(G)$ be an admissible probability measure on G with finite second moment, and assume that G contains a rank one element. Let $o \in X$ be a basepoint of the random walk. Let λ be the (positive) drift of the random walk. Then the random variables $(\frac{1}{\sqrt{n}}(d(Z_n o, o) - n\lambda))_n$ converge in law to a Gaussian distribution N_μ. Furthermore, the variance of N_μ is given by*

$$\int_{G \times \partial_\infty X} (b_x(g^{-1}o) - \psi(x) + \psi(gx) - \lambda)^2 d\mu(g) d\nu(x). \tag{15}$$

Proof. By Proposition 6.4, the cocycle β is centerable, with average λ. Since the measure ν is the unique μ-stationary measure on $\overline{X}$, we can then apply Theorem 5.3: the random variables $(\frac{1}{\sqrt{n}}(\beta(Z_n(\omega), x) - n\lambda))_n$ converge to a Gaussian law N_μ. But thanks to Proposition 4.9, this is equivalent to the convergence of the random variables $(\frac{1}{\sqrt{n}} d(Z_n(\omega)o, o) - n\lambda)_n$ to a Gaussian law. Moreover, by Theorem 5.3 and Proposition 5.5, the covariance 2-tensor of the limit law is given by

$$\int_{G\times\partial_\infty X}(\beta_0(g,z)-\lambda)^2 d\mu(g)d\nu(z),$$

where $\beta_0(g,x)=\beta(g,x)-\psi(x)+\psi(gx)$. This yields the result. $\square$

In order to prove Theorem 1.1, it only remains to prove that the limit law is non-degenerate. This is what we do in the next Proposition.

Proposition 6.9. *With the same assumptions and notations as in Theorem 6.8,*

$$\int_{G\times\partial_\infty X}(\beta_0(g,z)-\lambda)^2 d\mu(g)d\nu(z)>0.$$

In particular, the limit law N_μ *of the random variables* $(\frac{1}{\sqrt{n}}(d(Z_n o,o)-n\lambda))_n$ *is non-degenerate.*

Proof. Let g be a rank one isometry in G. Recall that g has an axis $\sigma\subseteq X$ on which g acts as a translation, and let ξ^+,ξ^- be its attracting and repelling fixed points in $\partial_\infty X$ respectively. We let $l(g)=\lim_n \frac{d(g^n o,o)}{n}$ be the translation length of g in (X,d). Observe that

$$l(g)=\lim_n \frac{b_{\xi^+}(g^{-n}o)}{n}. \tag{16}$$

Indeed, if o belongs to σ, then $b_{\xi^+}(g^{-n}o)=d(g^n o,o)$ and equation (16) is true. If o does not belong to σ, take $o'\in\sigma$, and by triangular inequality,

$$|b_{\xi^+}(g^{-1}o)-b^{o'}_{\xi^+}(g^{-1}o')|\leq 2d(o,o'),$$

where $b^{o'}_{\xi^+}$ is the horofunction with basepoint o'. Since $l(g)=\lim_n \frac{1}{n}b^{o'}_{\xi^+}(g^{-n}o')$, we obtain that $l(g)=\lim_n \frac{1}{n}b_{\xi^+}(g^{-n}o)$.

Suppose by contradiction that $\int_{G\times\partial_\infty X}(\beta_0(h,z)-\lambda)^2 d\mu(h)d\nu(z)=0$. This means that for almost every $\xi\in\operatorname{supp}(\nu)$ and almost every $h\in\operatorname{supp}(\mu)$,

$$b_\xi(h^{-1}o)-\lambda=\psi(\xi)-\psi(h\xi).$$

Since ψ is bounded, we get that for almost every $\xi\in\operatorname{supp}(\nu)$ and almost every $h\in\operatorname{supp}(\mu)$, $|b_\xi(h^{-1}o)-\lambda|\leq 2\|\psi\|$. Now consider the random walk generated by μ^{*n}, for $n\geq 1$. Observe that μ^{*n} is still admissible of finite second moment and has the same stationary measure ν on $\partial_\infty X$. We can then apply Theorems 4.8 and 6.8, but here the drift of the random walk generated by μ^{*n} is $n\lambda$. By the previous argument, for almost every $\xi\in\operatorname{supp}(\nu)$ and almost every $h\in\operatorname{supp}(\mu^{*n})$, $|b_\xi(h^{-1}o)-n\lambda|\leq 2\|\psi\|$.

Now since ξ^+ is the attractive fixed point of g, then $\xi^+ \in \operatorname{supp}(\nu)$. It is a consequence of the fact that for ν-almost every ξ in $\overline{X}$, $g^n\xi \to \xi^+$, but we give a proof for completeness. The isometry g is rank one, hence it acts on $\partial_\infty X$ with North-South dynamics [Ham09, Lemma 4.4]. This means that for every neighbourhood U of ξ^+, V of ξ^- in $\partial_\infty X$, there exists k such that for all $n \geq k$, $g^n(\partial_\infty X - V) \subseteq U$ and $g^{-n}(\partial_\infty X - U) \subseteq V$. Because ν is non-atomic, there exists a neighbourhood V of ξ^- such that $\nu(\partial_\infty X - V) > 0$. Take U a neighbourhood of ξ^+, and k large enough so that for all $n \geq k$, $g^n(\partial_\infty X - V) \subseteq U$. Since μ is admissible, there exists $p \in \mathbb{N}$ such that $g^k \in \operatorname{supp}(\mu^{*p})$. By μ^{*p}-stationarity,

$$\sum_{h \in G} \nu(h^{-1}U)\mu^{*p}(h) = \nu(U).$$

In particular, by North-South dynamics,

$$\nu(U) \geq \nu(g^{-k}U)\mu^{*p}(g^k) \geq \nu(\partial_\infty X - V)\mu^{*p}(g^k) > 0.$$

This is true for every neighbourhood U of ξ^+, hence $\xi^+ \in \operatorname{supp}(\nu)$.

Because μ is admissible, there exists m such that $\mu^{*m}(g) > 0$. Then, for all $n \geq 1$, $|b_{\xi^+}(g^{-n}o) - nm\lambda| \leq 2\|\psi\|$. By equation (16), we obtain that

$$\lim_n \frac{b_{\xi^+}(g^{-n}o)}{n} = l(g) = m\lambda.$$

But there also exists $q \in \mathbb{N}^*$ such that $1 \in \operatorname{supp}(\mu^{*q})$, hence $g \in \operatorname{supp}(\mu^{*(m+q)})$ and by the same argument, $l(g) = (m+q)\lambda$. Since by Theorem 4.8, λ is positive, we get a contradiction. □

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