Stage M2R

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my webpage

Bounds and statistics for closed geodesics

Context

Consider an orientable surface Σ with negative Euler characteristic, a minimal set of generators of the fundamental group, and a constant curvature -1 metric on Σ. Each free homotopy class C of closed oriented curves on S determines three numbers: 1. the word length (that is, the minimal number of letters needed to express γ as a cyclic word in the generators and their inverses), 1. the minimal geometric self-intersection number 1. the geometric length $\ell(\gamma)$ ie the length of the unique closed geodesic in the class γ.

These three numbers can be explicitly computed (or approximated) using a computer so we can do experiments and check results algorithmically.

From a theoretical point of view :

- The relation between word length and geometric length was studied by Milnor in the 60s then Policott and Sharp from a statistical point of view in the 90s.
- The relation between intersection number and geometric length was studied by Basmajian about 10 years ago then by Chas and her coathors from a statistical point of view.

The principal (non statistical) results are easy to state precisely. Let γ be a cyclically reduced word in the fundamental group and $\ell(\gamma)$ the length of the unique closed geodesic representing this conjucacy class then :

Milnor proved that there is a constant K which depends on the metric such that

$$(1/K)|\gamma| < \ell(\gamma) < K|\gamma|.$$

where $|\gamma|$ is the word length and $\ell(\gamma)$ the length of the curve for the *choice* of metric.

Likewise Basmajian proved that there is a constant K which depends on the metric such that

$$(1/K)i(\gamma, \gamma) < \ell(\gamma) < Ki(\gamma, \gamma).$$

where $i(\gamma, \gamma)$ is the geometric intersection number.

The way that the factor K varies carries important geometric information.

Details of the stage

We will review the basic geometric constructions and in particular the work of Milnor which relates combinatorial group theory to Riemannian geometry. Then we will study Basmajian'method for obtaining inequalities relating intersection number to length. We will look at the applications of this to giving effective bounds in Scott's Theorem on simple lifts. Finally we will study the methods required by Lalley and Chas to obtain the statistical relation between length and intersection number.

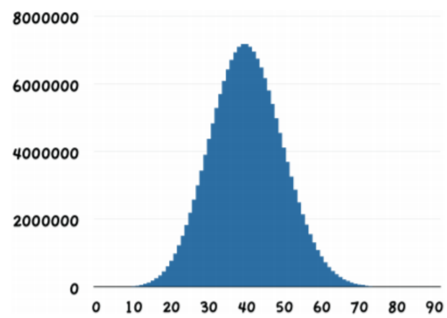

Figure 1. Histogram of all (about 175,000,000) non-power, free homotopy classes of word length $L = 20$ in the punctured torus, organized by self-intersection number. The mean of the self-intersection number is $400/9 \sim 45$.

Figure 1: img

References

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