Review for "*Geometry of Fermat's sum of squares*" by Mcshane and Sergesciu.

General comments

We think that the article describes interest interplay between hyperbolic geometry and number theory, and would be a good contribution to the volume. However, we believe that the article is in need of revision, both for small corrections and alterations of expressions, and more generally, to include greater sign-posting and guidance for the reader. In particular, the readership for such a paper is likely to consist of both number theorists and hyperbolic geometers, and the two communities do not interact as much as they should.

Specific suggestions

We offer up the following list of suggestions. The authors need not completely comply as many of the recommendations are stylistic rather than corrective.

- The authors should correct and uniformize the use of "Burnside's lemma". For example, the first page of the introduction has multiple instances of "Burnsides's lemma" and "Burnsides lemma" and latter parts of the article contains "Burnside lemma". We advocate for "Burnside's lemma".
- The authors have utilized λ-lengths in a nonconventional matter, and we advise against this. There is already room for convention in that the original and the current standard definition of λ-lengths differ by a factor of $\sqrt{2}$. This is further exacerbated by the authors use of λ-lengths to refer to their squares. If the authors choose to stick to their current conventions, clear sign-posting (and internal consistency) is needed.
- Instances of names such as Poincaré and Bézout are given without accents in this article, and we are unsure if this is by choice (to increase the searchability of the names, for example), or if it is an artefact of tex-processing removing the accents. We prefer seeing the names with accents, but this is perhaps a matter of style.
- Pg. 1.
  - We believe that the second author's name is incorrect and should be Vlad Sergiescu
  - application of Burnsides's Lemma $\to$ application of Burnside's lemma
  - Amusingly Burnsides's Lemma $\to$ Amusingly, Burnside's lemma
  - Organisation, Remarks $\to$ Organisation and Remarks
  - recall the statement of Burnsides Lemma $\to$ recall the statement of Burnside's lemma

- Pg. 2.
  - The names appearing at the top of the page are inconsistent: one is a surname and one is a given name. We advise changing to McShane and Sergiescu
  - orientation preserving → orientation-preserving
  - Heath-Brown published a proof → Heath-Brown published a proof [5]
  - Like our proof it is based on the action of a Klein four group → Like our proof, it is based on an action of the Klein four group
  - an auxiliary equation namely → an auxiliary equation, namely
  - $p = 4xy + z^2 \rightarrow p = 4xy + z^2,$
  - the remark made in 1.1.2. is interesting, and we have not understood it yet, and hope to see a bit more explanation in the article
  - multiple instances of "tessalation" should be "tessellation"
  - of simple bicuspidal → of a simple biscuspidal
  - not strictly necessary → not strictly necessary,
  - proof in terms of λ lengths → proof in terms of λ lengths,
  - in the context we consider → in the context we consider,
- Pg 3.
  - Bezouts Theorem → Bézout's/Bezout's Theorem
  - please elucidate the use of Bézout's Theorem, and indeed, which result by Bézout is being applied
  - is transitive on $\mathbb{Q} \cup \infty$ → acts transitively on $\mathbb{Q} \cup \{\infty\}$
  - without using our notion of lengths for a bicuspidal geodesics → without λ-lengths
  - we recommend consistency with naming conventions. Specifically, Louis Funar is given name + surname, whereas Xu Binbin is the other way around.
  - 2. Burnside Lemma → 2. Burnside's Lemma
  - the Burnside Lemma → Burnside's lemma/Lemma
- Pg. 4.
  - the remaining element and the theorem is equivalent to the existence of a fixed point for it → the remaining element, and Theorem 1.1 is equivalent to the existence of a fixed point for g.
  - Note that since $\mathbb{F}_p$ is a field → Since $\mathbb{F}_p$ is a field,
  - We feel that the statement $|X^g| = \#\{x^2 = -1, x \in \mathbb{F} + p^*\}$ might benefit from a little more explanation
  - and it follows from this that → and hence
  - $|X^g|. = \rightarrow |X^g| =$, and in the same equation array, add a full stop after the second $\mod 4$.

- We are very interested in seeing an expanded explanation of the Note here (as we do not yet understand this point).
- we feel that the . in the action of $\mathrm{SL}(2,\mathbb{Z})$ might look better as a $\cdot$ (this comment applies to all future instances)
- "connexion" is bordering on becoming archaic and may be distacting for the reader
- Ford circle of infinite radius $\rightarrow$ Ford circle of infinite Euclidean radius
- extra full stop after Lemma 3.1.

• Pg. 5.

- *mid point* $\rightarrow$ midpoint
- Following Penner [6], see also [9] for a more recent account — this phrase has already made an appearance in 1.1.3.
- we define the λ-length of the arc to be the exponential of this one half of this length. — we suspect that this sentence was meant to be removed, it is slightly ungrammatical, contradicts the authors' definition of λ length, and writes λ-length with a hyphenation

• Pg. 6.

- "Further if $a/b = 1/0$" — we suspect this should be a/c instead, also in the proof
- The arcs of λ-length 1 — be consistent about hyphenation for λ length (or λ-length; note that we personally prefer the hyphenated version). Also note that justification should be given for why the λ-length 1 edges are those in the Farey diagram, perhaps Lemma 3.1.(2) will suffice
- remind readers that $F = \{z, \Im(z) > 1\}$, and perhaps explain why this permutation happens

• Pg. 7.

- §3.3 would benefit from much more signposting, a little indication of how this might be used later might go a long way.
- geometers might benefit enormously from a little explanation on $\Gamma(2)$. E.g.: it is the kernel of the canonical homomorphism from $\mathrm{SL}(2,\mathbb{Z}) \rightarrow \mathrm{SL}(2,\mathbb{Z}/2\mathbb{Z})$, especially given that the article appears to be fairly self-contained

• Pg. 8.

- Figure 4. Three punctured sphere $\rightarrow$ Figure 4. A depiction of the three punctured sphere $\mathbb{H}^2/\Gamma(2)$
- Pairs of these cusp regions are tangent $\rightarrow$ The boundary horocycles or any pair of these cusp regions are tangent
- It might be worth drawing these three horocycles on Figure 4.

- Pg. 9.
  - It follows that, since $\to$ Since
  - giving a source for $\Gamma(2)$ being normal in $\mathrm{GL}(2,\mathbb{Z})$ might be nice
  - G_4 makes its first appearance on this page, and we are expected to understand claims made about it and to "Recall" it
  - $U \circ V(z) = -1/z$ and so fixes i is awkwardly phrased
- Pg. 10.
  - the two dot points on the top of the page is close to a rehash of information discussed in words on the bottom of Pg. 9. There should be a better way to organize this information
  - automorphism induced by U ob $\mathbb{H}/\Gamma(2)$ $\to$ automorphism of $\mathbb{H}/\Gamma(2)$ induced by U
  - is the proof of the first part of Lemma 4.2. given?
  - For any integer n G_4 permutes $\to$ For any integer n, G_4 permutes
  - $k/n+it \to \frac{k}{n}+it$ for slightly more clarity on what's on the denominator
  - the quotient surface $\mathbb{H}/\Gamma(2)$ a familiy $\to$ the quotient surface $\mathbb{H}/\Gamma(2)$ is a family
  - sub families $\to$ sub-families
  - start of subsection 5.1 would also benefit from a bit more explanation
- Pg. 11.
  - Theore 1.2 follows from: $\to$ Theorem 1.2 follows from Lemma 3.3 and the following result:
- Pg. 12.
  - The first point is rather easy (the automorphism... in their complement) — we believe that readers with background mainly in number theory might benefit from a little explanation of this
  - is there a reference for the λ-length being $|a^2 - b^2|$
  - to be a lift of one of our arcs $\to$ to be a lift of one of our arcs, then by Lemma 3.2
  - $5^2 \times 2 = |(1+2i)^2(1+i)|^2 \to 5^2 \times 2 = |(1+2i)^4(1+i)|^2$
  - Despite this multiplication by 2 seems to have nice geometry interpretation,... — this statement deserves some sign-posting. The relationship to the clarifying and explanatory sentences after this one can be easy to misinterpret for a first-time reader still trying to make sense of the details.
  - We feel that a blunt statement about the aim of subsection 6.3 would be very useful for the reader, as it is easy to become mired in the details and to wonder if this is a part of proving Fermat's theorem.
- Pg. 13.
  - the Burnside Lemma $\to$ Burnside's Lemma

- quadrilaterial is invariant under $z \mapsto -1/z$ — it might be helpful to remind the reader that i is the midpoint of the selected pair of arcs

- Pg. 14.
  - the 1 in Figure 6 should be i
  - We feel that it would be expositionally clarifying to write down some algebraic identities corresponding to the two cases. E.g.: $(ad-bc)^2+(bd+ac)^2 = (a^2+b^2)(c^2+d^2)$ and $(b^2-a^2)^2+(2ab)^2 = (a^2+b^2)^2$. If not too taxing, perhaps the authors might also consider what Brahmagupta's identity geometrically corresponds to.

- Pg. 15.
  - We believe that the title of Springborn's paper should be italicized.