PYTHAGOREAN TRIPLES

1. Introduction

A *Pythagorean triple* is a triple of integers (a, b, c) such that

$$a^2 + b^2 = c^2.$$

The most famous example is the so-called *egyptian triple* $(3, 4, 5)$ which is *a priori* the "smallest" Pythagorean triple. The set of Pythagorean triples is infinite, and it is a classical problem to find all Pythagorean triples. It is useful to define the notion of *primitive Pythagorean triple*, which is a Pythagorean triple (a, b, c) such that the three integers have no common divisor greater than 1. Evidently, any Pythagorean triple can be written as a multiple of a primitive Pythagorean triple. In fact, the set of primitive Pythagorean triples forms what is essentially (see below for a precise statement) a single orbit under the action of a group of transformations the orthognal group $O(2, 1; \mathbb{Z})$ on the Minkowski space $\mathbb{R}^{2,1}$. This is because every Pythagorean triple is an integer point on the *light cone* of Minkowski space:

$$\{(x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 0\}.$$

We note that the "smallest" integer point on the light cone is not $(3, 4, 5)$, but rather $(1, 0, 1)$ or $(0, 1, 1)$. Of course, these two points correspond to degenerate Pythagorean triangles i.e. triangles of area zero.

2. Euclid's parameterization

The Pythagorean triples that are relatively prime, called the *primitive triples*, have the elementary and beautiful characterization as integers due to Euclid:

$$(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$$

where m and n are coprime integers of opposite parity and $m > n > 0$. Another way to think of this is that c factors over the Gaussian integers $\mathbb{Z}[i]$ as

$$c = (m + ni)(m - ni),$$

where m and n are coprime integers of opposite parity, that is exactly one is odd and the other is even so it follows that:

- $c = m^2 + n^2$ is odd,
- a is the real part of the product
- b is the imaginary part of the product.

Note that a and b are, like m and n, are of opposite parity.

More generally, we have maps :

$$(x, y) \in \mathbb{R}^2 \mapsto z = x + iy \mapsto z^2 \mapsto (\Re z^2, \Im z^2, |z^2|) = (a, b, c) \in \mathcal{C}.$$

The composition $p : (x, y) \mapsto (a, b, c)$ is a surjection since the system of equations below always has a solution for $x, y \in \mathbb{R}_+$:

$$\begin{aligned} 2x^2 &= a + c, \\ 2y^2 &= c - a > 0. \end{aligned}$$

Further, the restriction of the map $p : (x, y) \mapsto (|a|, |b|, |c|)$ to the subset

$$\tilde{\mathcal{P}} = \{(m, n) \in \mathbb{Z}^2, \gcd(m, n) = 1, m + n \text{ odd}\}$$

is a surjection onto the set of primitive Pythagorean triples. Note that $\tilde{\mathcal{P}}$ is a subset of the set of *primitive elements* of the group $\mathbb{Z}^2$. The group of integer matrices $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ acts on $\mathbb{Z}^2$, it acts transitively on primitive elements of $\mathbb{Z}^2$, so does not preserve the set $\tilde{\mathcal{P}}$ however the principal congruence subgroup $\Gamma(2)$ does.

3. Hall matrices

The set of polynomials with integer coefficients $\mathbb{Z}[X, Y]$ forms a free $\mathbb{Z}$-module. There is a submodule freely generated by the polynomials

$$(X^2 - Y^2, 2XY, X^2 + Y^2).$$

The group $\Gamma(2)$ acts on this submodule by change of basis and we can compute the matrices of the generators of $\Gamma(2)$ with respect to this basis. Since the basis satisfies the relation:

$$-(X^2 + Y^2)^2 + (X^2 - Y^2)^2 + (2XY)^2 = 0,$$

these matrices are elements of $O(2, 1; \mathbb{Z})$.

4. Enumeration

The problem of finding all primitive Pythagorean Triples is equivalent to the problem of finding all the primitive elements (m, n) of the group $\mathbb{Z}^2$ that satisfy:

- $n \geq m \geq 0$,
- $m + n$ is odd.

There is a very efficient way to do this using the so-called *Stern-Brocot tree.*

4.1. **Stern-Brocot tree.** Recall that the *mediant* of a pair of fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be the fraction$\frac{a+c}{b+d}$. The Stern-Brocot sequence is constructed by a process of *mediant insertion*, starting from an initial pair of ractions $\frac{0}{1}$ and $\frac{1}{0}$.

The **Stern–Brocot sequence** of order 0 is the sequence

$$\frac{0}{1}, \frac{1}{0},$$

and the Stern–Brocot sequence of order i is the sequence formed by inserting a mediant between each consecutive pair of values in the Stern–Brocot sequence of order $i - 1$. The Stern–Brocot sequence of order i consists of all values at the first i levels of the Stern–Brocot tree, together with the boundary values

$$\frac{0}{1} \quad \text{and} \quad \frac{1}{0},$$

in numerical order. So the level 1 sequence is obtained by inserting the mediant $\frac{1}{1}$ between the two fractions of the level 0 sequence:

$$\frac{0}{1}\frac{1}{1}\frac{1}{0}$$

and the next level sequence is

$$\frac{0}{1}\frac{1}{2}\frac{1}{1}\frac{2}{1}\frac{1}{0}.$$

The vertices of the tree are obtained by grouping triples of consecutive fractions of the sequence and two vertices are adjacent if they have a fraction in commun.

$$\frac{0}{1}\frac{1}{4}\frac{1}{3}\frac{2}{5}\frac{1}{2}\frac{3}{5}\frac{2}{3}\frac{3}{4}\frac{1}{1}\frac{4}{3}\frac{3}{2}\frac{5}{3}\frac{2}{1}\frac{5}{2}\frac{3}{1}\frac{4}{1}\frac{1}{0}$$

One can see the tree structure by placing the fractions on different lines according to their depth in the tree like this:

$$\frac{0}{1} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \frac{1}{0}$$
$$\frac{1}{1}$$
$$\frac{1}{2} \qquad \qquad \qquad \qquad \frac{2}{1}$$
$$\frac{1}{3} \qquad \quad \frac{2}{3} \qquad \quad \frac{3}{2} \qquad \quad \frac{3}{1}$$
$$\frac{1}{4} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{3}{4} \quad \frac{4}{3} \quad \frac{5}{3} \quad \frac{5}{2} \quad \frac{4}{1}$$

Obviously primitive elements $(m, n) \in \mathbb{Z}^2$ satisfying our conditions

- $n \geq m \geq 0$,
- $m + n$ is odd,

from the first condition one sees that are in 1-1 correspondence with a subset of the fractions $0 \leq \frac{m}{n} \leq 1$ so we only need half of the Stern-Brocot tree. The second condition means we have to do some further "pruning" of the tree.

$\frac{0}{1}$

$\frac{1}{2}$

$\frac{1}{3}$ $\frac{2}{3}$

$\frac{1}{4}$ $\frac{2}{5}$ $\frac{3}{5}$

$\frac{1}{5}$ $\frac{2}{7}$ $\frac{3}{8}$ $\frac{3}{7}$ $\frac{4}{7}$ $\frac{5}{8}$

$\frac{1}{6}$ $\frac{2}{9}$ $\frac{3}{11}$ $\frac{3}{10}$ $\frac{4}{11}$ $\frac{5}{13}$ $\frac{5}{12}$ $\frac{4}{9}$ $\frac{5}{9}$ $\frac{7}{12}$ $\frac{8}{13}$ $\frac{7}{11}$ $\frac{7}{10}$

Taking numerators and denominators of the fractions modulo 2, we obtain the following:

$\frac{0}{1}$

$\frac{1}{1}$

$\frac{1}{0}$ $\frac{0}{1}$

$\frac{1}{1}$ $\frac{0}{1}$ $\frac{1}{0}$ $\frac{1}{1}$

$\frac{1}{0}$ $\frac{0}{1}$ $\frac{1}{1}$ $\frac{1}{0}$ $\frac{0}{1}$ $\frac{1}{1}$ $\frac{1}{0}$

$\frac{1}{1}$ $\frac{0}{1}$ $\frac{1}{0}$ $\frac{1}{1}$ $\frac{0}{1}$ $\frac{1}{0}$ $\frac{1}{1}$ $\frac{0}{1}$ $\frac{1}{0}$ $\frac{1}{1}$ $\frac{0}{1}$ $\frac{1}{0}$ $\frac{1}{1}$ $\frac{0}{1}$

5. Alperin's approach

Alperin's approach [2] to enumerating primitive Pythagorean triples is based on a correspondence between the triples a subset of the nilpotent cone matrices $\mathcal{N}_2$. Given a triple (a, b, c), we can associate the traceless matrix

$$\tilde{X} = \begin{pmatrix} -b & a+c \\ a-c & b \end{pmatrix}.$$

The determinant of this matrix is

$$\det \tilde{X} = -a^2 - b^2 + c^2,$$

which vanishes since (a, b, c) is a Pythagorean triple and, by Cayley-Hamilton theorem, the matrix $\tilde{X}$ satisfies $\tilde{X}^2 = 0$. We assume that b is even so that a, c are odd.

The group $\mathrm{SL}(2, \mathbb{Z})$ acts on the nilpotent cone $\mathcal{N}_2$ by conjugation and the subgroup $\Gamma(2)$ preserves the set of embedded Pythagorean triangles and this allows him to prove his main result:

Theorem 1. *The set of positive primitive Pythagorean triples has the structure of a complete, infinite, rooted ternary-tree.*

Lemma 2. *An integer matrix X satisfies $X^2 = 0$ if and only if X has the form*

$$X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} mn & -n^2 \\ m^2 & -mn \end{pmatrix} = \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} m & -n \end{pmatrix}$$

for integers x, y, and z such that $x^2 + yz = 0$.

$$\begin{pmatrix} -y & x+z \\ x-z & y \end{pmatrix}$$

such that $x^2 + y^2 - z^2 = 0$.

Magic Correspondence		
	Minkowski space $\mathbb{R}^{2,1}$	Traceless 2×2 matrices
Main object	$\mathbf{v} = (x, y, z)$	$\tilde{\mathbf{v}} = \sum v^i \sigma_i = \frac{1}{2} \begin{pmatrix} -y & x+z \\ x-z & y \end{pmatrix}$
Norm	$\\|\mathbf{v}\\| = -x^2 - y^2 + z^2$	$\\|\mathbf{v}\\| = 4 \det \tilde{\mathbf{v}}$
Action	$\mathbf{v}' = A\mathbf{v}, (A \in O(2, 1; \mathbb{Z}))$	$\tilde{\mathbf{v}}' = \tilde{A}\tilde{\mathbf{v}}\tilde{A}^*, \quad (\tilde{A} \in SL^{\pm}(2, \mathbb{Z}))$
Minkowski scalar product	$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T G \mathbf{w}$	$\mathbf{v} \cdot \mathbf{w} = -2\mathrm{Tr}\, \tilde{\mathbf{v}}\tilde{\mathbf{w}}$
The ith coefficient	$v^i = \mathbf{v} \cdot \mathbf{e}_i$	$v^i = -\det \sigma_i \cdot \mathrm{Tr}(\tilde{\mathbf{v}}\sigma_1)$

Table 1. Correspondence between Minkowski space and traceless 2×2 matrices.

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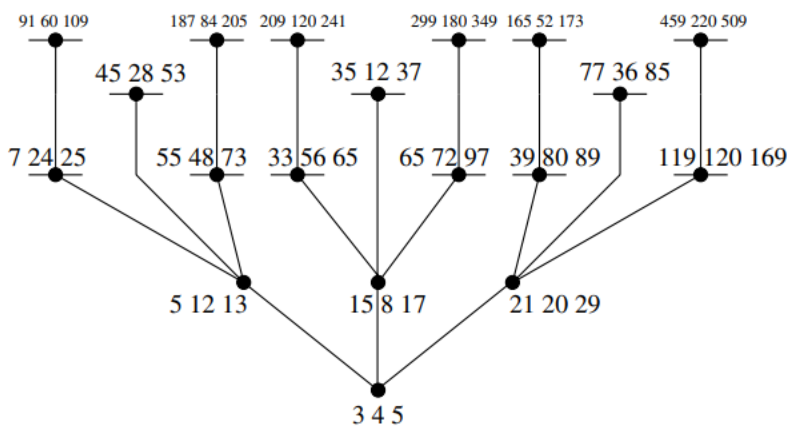

Figure 1. Alperin's tree of Pythagorean triples.

Hyperbolic Geometry	**Algebra/Number Theory**
horocycle	nonzero vector $(p, q) \in \mathbb{R}^2$
geodesic	indefinite binary quadratic form f
point	definite binary quadratic form f
signed distance between horocycles	$2 \log \left\| \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right\|$
signed distance between horocycle	$\log \left(\frac{f(p,q)}{\sqrt{\lvert \det f \rvert}} \right)$

TABLE 2. Correspondence between hyperbolic geometry and algebra/number theory.

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