TITLE QUASI-FUCHSIAN SPACE OF THE ONCE-PUNCTURED TORUS.

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Quasi-Fuchsian space of the once-punctured Torus

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Summary

In this thesis we look at two generator groups of Möbius transformations where the commutator of the generators is parabolic. In particular we are interested in quasi-Fuchsian groups Γ whose quotient surface Ω/Γ consists of two once-punctured tori. The set of all such quasi-Fuchsian groups, which is called the quasi-Fuchsian space of the once-punctured torus, is defined in chapter 1. In chapter 2 we introduce two sets of suitable coordinates for quasi-Fuchsian space. The first is the well known trace parameters and the other set of coordinates involves an appropriate normalisation of conjugacy classes of quasi-Fuchsian groups.

We study a special class of quasi-Fuchsian groups in chapter 3 which are those groups obtained by pairing four classical circles, each of which is tangent to its neighbours. We find the exact subset of quasi-Fuchsian space where these groups lie and investigate their limiting behaviour. In chapter 4 we return to the whole of quasi-Fuchsian space of the once-punctured torus and investigate what happens when we change the generators of such a group Γ. In particular we reduce the modulus of the traces of generators of Γ, and use this information to build up a substantial picture of quasi-Fuchsian space.

In the last chapter we look at the traces of elements of one particular group, which gives rise to the Diophantine equation $a^2+b^2+c^2 = 3abc$ studied by Markoff.

If we arrange a solution triple of natural numbers (a, b, c) in ascending order, so that $a \leq b \leq c$, it has long been conjectured that the largest number uniquely determines the triple. We finish by proving that if c is prime then this statement is true. We show this using only algebraic number theory, but we mention the geometric motivation which originally gave the ideas for the proof, and where it appears in the earlier chapters.

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Chapter 1

Introduction

1.1 Main Concepts

We begin with some elementary definitions from the theory of Kleinian groups. Further details can be found in [16].

Definition 1.1.1

We are interested in discrete torsion-free subgroups of $PSL(2,\mathbb{C})$, which act as groups of Möbius transformations on the Riemann sphere $\overline{\mathbb{C}}$.

The *set of discontinuity* Ω of a discrete group G is the set of points $z \in \overline{\mathbb{C}}$ where G acts discontinuously.

The complement $\Lambda = \overline{\mathbb{C}} \backslash \Omega$ is called the *limit set* of the group G.

A *Kleinian group* G is a discrete group with $\Omega \neq \emptyset$. Thus we can form the quotient space Ω/G, which is a two dimensional manifold with a natural complex structure, and is made up of a union of Riemann surfaces, which are its connected components.

A *Fuchsian group* G is a Kleinian group that leaves invariant some oriented circle. The quotient space Ω/G is made up of two surfaces which are mirror images of each other.

A *quasi-Fuchsian group* Γ is a Kleinian group which leaves invariant some oriented Jordan curve. The quotient space Ω/Γ also consists of two surfaces, which are homeomorphic, namely the quotients of the interior and exterior of the curve.

It was established by Bers and Maskit (see [2] and [20]) that any finitely generated quasi-Fuchsian group Γ can be obtained by a quasiconformal deformation of a Fuchsian group G, so that we have a quasiconformal homeomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\Gamma = fGf^{-1}$.

A *quasi-Fuchsian once-punctured torus group* is a quasi-Fuchsian group with a quotient space consisting of two tori, each with a point removed. Since there are no elliptic elements we can say that given two quasi-Fuchsian once-punctured torus groups G_1 and G_2 with quotient spaces Ω/G_1 and Ω/G_2, the spaces are conformally equivalent if and only if the groups are conjugate in $PSL(2,\mathbb{C})$.

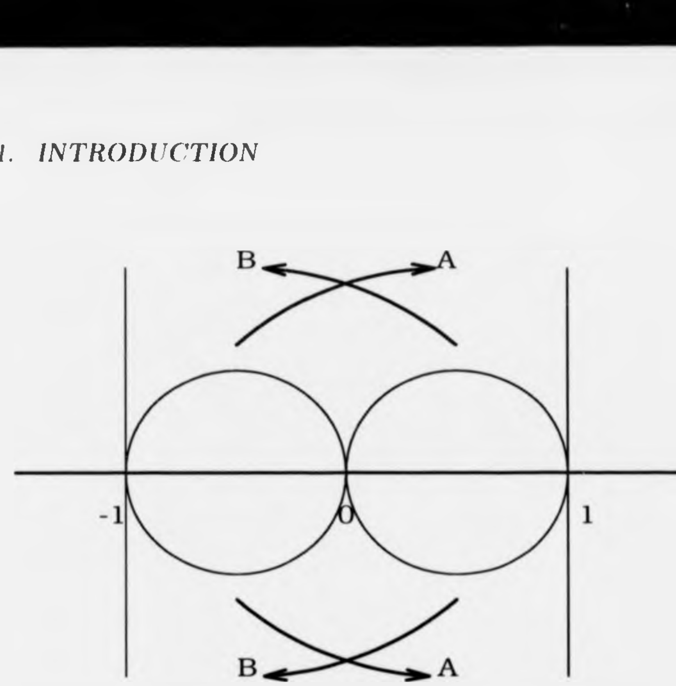

Figure 1.1: A Fuchsian once-punctured torus

Example 1.1.2

Consider the group G generated by the transformations

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

We have a Fuchsian group with limit set the real line, and by identifying the sides of the fundamental domain in figure 1.1, we obtain the quotient space with two punctured tori, one upstairs and one down.

Note that the commutator $B^{-1}A^{-1}BA$ is parabolic, with trace -2. In fact the only parabolic elements of G are the powers of $B^{-1}A^{-1}BA$, along with their conjugates in G. All other transformations of G are hyperbolic. Also G considered purely as an abstract group is the free group on the two symbols A and B, which is isomorphic to the fundamental group of each component of the quotient space.

Now when a quasiconformal deformation of G takes place, figure 1.1 still appears the same topologically, although our four circles now become Jordan curves.

In this thesis, we wish to look at the different conformal structures that quasi-Fuchsian once-punctured tori can possess. We do this by looking at their quasiconformal deformation space.

Definition 1.1.3

Let G be our group in Example 1.1.2, with quotient space $T_1 \cup T_2$. Consider the space of all quasiconformal homeomorphisms $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that fGf^{-1} is a quasi-Fuchsian group Γ. Then the quotient space of Γ, $\Omega/G = S_1 \cup S_2$, is also a quasi-Fuchsian once-punctured torus, and we can project $\overline{f}$ down to obtain $f : T_1 \cup T_2 \rightarrow S_1 \cup S_2$.

We define $\mathcal{H}$ to be the space of all such quasiconformal maps f.

The *space of quasiconformal deformations of the quasi-Fuchsian once-punctured torus* is the set of equivalence classes $[f]$ for $f \in \mathcal{H}$, where $f' \in [f]$ if the map $f'f^{-1} : S_1 \cup S_2 \rightarrow S'_1 \cup S'_2$ is homotopic to a conformal map.

It is possible to use other interpretations of the points in this space. Here the one we shall adopt involves considering quasi-Fuchsian groups with a fixed marking of generators.

In particular, define

$$\mathcal{R} = \{(\Gamma; X, Y) | \Gamma \subseteq PSL(2, \mathbb{C})\}$$

where Γ is a quasi-Fuchsian once-punctured torus group and X, Y are elements generating Γ.

We know there exists a quasiconformal homeomorphism $\overline{f} : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with $\Gamma = \overline{f} G \overline{f}^{-1}$ and

$$X = \overline{f} A \overline{f}^{-1} \quad , \quad Y = \overline{f} B \overline{f}^{-1}.$$

From this, we project $\overline{f}$ down to $f \in \mathcal{H}$ and thence to a point $[f]$ in the quasiconformal deformation space.

Which elements of $\mathcal{R}$ are mapped to the same point in this space? Suppose we have $(\Gamma'; X', Y') \in \mathcal{R}$, with a map $\overline{f'}$ satisfying the same conditions as above, and projecting to $f' \in [f]$.

Then $f'f^{-1} : S_1 \cup S_2 \rightarrow S'_1 \cup S'_2$ is homotopic to a conformal map c, which can be lifted to a conformal map C between the sets of discontinuity and then, using Marden's isomorphism theorem [17], we can extend it over the Riemann sphere, so that C is a Möbius transformation.

However, c and $f'f^{-1}$ must induce the same isomorphism of the fundamental groups of the quotient spaces, and as we can consider our marking to be a pair of generators for the fundamental groups, then on lifting c to C we must have:

$$\begin{aligned} \Gamma' &= C\Gamma C^{-1} \\ X' &= CXC^{-1} \\ Y' &= CYC^{-1} \end{aligned} \tag{1.1}$$

for $C \in PSL(2, \mathbb{C})$.

So if we regard the marked quasi-Fuchsian groups $(\Gamma; X, Y)$ and $(\Gamma'; X', Y')$ as equivalent if the relations in (1.1) hold, then the quotient $\mathcal{R}/PSL(2, \mathbb{C})$ acting by conjugation is in bijection with the space of quasiconformal deformations of the quasi-Fuchsian once-punctured torus. It is $\mathcal{R}/PSL(2, \mathbb{C})$ that will be our object of study for this thesis, since it provides a concrete model for examining the matrices in our quasi-Fuchsian groups.

We need to put a topology on $\mathcal{R}/PSL(2, \mathbb{C})$ and this is done by looking at points $(\Gamma; X, Y) \in \mathcal{R}$. We know that as an abstract group Γ is the free group on

X and Y, and we must have

$$\operatorname{tr}(Y^{-1}X^{-1}YX) = -2. \tag{1.2}$$

As X and Y are Möbius transformations with complex entries, we can consider the pair (X,Y) as belonging to the subset of $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$ whose matrices satisfy (1.2). Thus the natural topology on $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$ gives rise to the subset topology on $\mathcal{R}$ and hence to the quotient topology on $\mathcal{R}/PSL(2,\mathbb{C})$.

This determines the topology of algebraic convergence on $\mathcal{R}/PSL(2,\mathbb{C})$, i.e. the sequence of equivalence classes $[(\Gamma_n; X_n, Y_n)]$ tends to the limit $[(\Gamma; X, Y)]$ if there exist representative matrices with each entry of X_n tending to the entries in X, and similarly for Y_n and Y.

Definition 1.1.4

We define our ***quasi-Fuchsian space of the once-punctured torus*** (shortened to ***quasi-Fuchsian space*** or merely $\mathcal{Q}$) to be the set $\mathcal{R}/PSL(2,\mathbb{C})$ with this topology.

1.2 Parametrising Quasi-Fuchsian Space

Having seen above that we have different abstract models for the same quasi-Fuchsian space, the question that now arises is how do we introduce suitable coordinates? Problems of this type originate in the work of Fricke and Klein, and since then there has been much study carried out by mathematicians such as Fenchel and Nielsen, Ahlfors and Bers. In particular we have an embedding of quasi-Fuchsian space as a complex analytic set, which could lead to various methods of parametrisation. For our purposes, we need to consider what properties we would want our coordinates to possess. It seems reasonable that:

(1) They should be global coordinates.

(2) They should be complex numbers, in order to bring out the two complex dimensional analytic structure of quasi-Fuchsian space.

(3) They should be directly constructive, in that a point in quasi-Fuchsian space should lead immediately to a marked group of matrices.

(4) They should bring out the symmetry of the space.

(5) The coordinates should relate in some sense to the geometry of each quasi-Fuchsian once-punctured torus group, so that we can gain information about each group's limit set and set of discontinuity.

(6) We should be able to investigate the boundary of quasi-Fuchsian space when embedded into a suitable set, and hopefully interpret points on the boundary as appropriate Kleinian groups.

For instance, to illustrate one possible approach to this problem, we know that the simplest and most direct method available of constructing a quasi-Fuchsian once-punctured torus is to take four discs in the Riemann sphere, which are disjoint apart from four points of tangency where each disc meets its neighbours. In order to generate the group we then take two loxodromic transformations, each sending

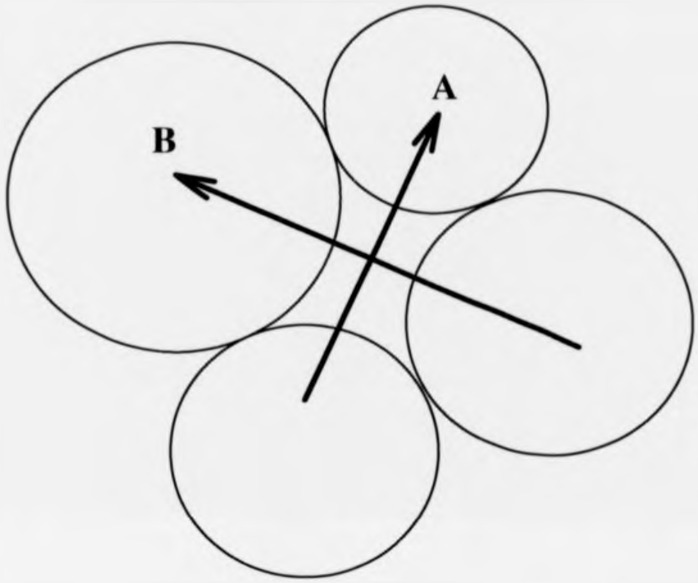

Figure 1.2: Two Loxodromic Transformations

the interior of a disc to the exterior of the disc opposite (as in figure 1.2). Then by ensuring the commutator of the transformations is parabolic we obtain a quasi-Fuchsian group with its limit set lying inside the discs and passing through the points of tangency.

One question about this construction that we shall examine later is can we obtain every quasi-Fuchsian once-punctured torus in this manner? It turns out that the answer is no, even if we require our group merely to contain two transformations pairing disjoint tangent discs, rather than to have them as the generators of our marked group.

In order to find suitable coordinates for quasi-Fuchsian space, there seem to be two possible approaches which might bear fruit. The first is to choose a particular marked group within each such conjugacy class by utilising a standard normalisation. The other is to use coordinates which are conjugation invariant so that no such normalisation is needed.

We will use both approaches simultaneously so that the advantages of each can be exploited, and we now go on to describe them in detail.

Chapter 2

Describing the Coordinates

2.1 Choosing a normalisation

In this chapter we introduce two sets of coordinates for quasi-Fuchsian space. First we choose a particular normalisation for any conjugacy class of marked quasi-Fuchsian once-punctured torus groups. Later we will describe the well known trace parameters for quasi-Fuchsian space.

Given a marked quasi-Fuchsian once-punctured torus group (Γ, A, B), we know the element $B^{-1}A^{-1}BA$ is parabolic and so has a unique fixed point in $\overline{\mathbb{C}}$, say z_1. The conjugate element $AB^{-1}A^{-1}B$ fixes only a point z_2, which is equal to $A(z_1)$.

Similarly, if z_3 and z_4 are defined by

$$BAB^{-1}A^{-1}(z_3) = z_3 = B(z_2)$$

and

$$A^{-1}BAB^{-1}(z_4) = z_4 = A^{-1}(z_3)$$

then we have $B^{-1}(z_4) = z_1$. Hence these four points lie in the limit set Λ (thus giving rise to an orientation of Λ as a Jordan curve), and each of them is moved onto its neighbour by one of the marked generators. The normalisation we adopt sends z_1 to 0, so that $B^{-1}A^{-1}BA$ is of the form

$$\begin{pmatrix} -1 & 0 \\ -k & -1 \end{pmatrix} \qquad \text{for some } k \in \mathbb{C}\backslash\{0\}.$$

Also we insist that z_3 moves to infinity, hence we assume

$$BAB^{-1}A^{-1} = \begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix} \qquad \text{for some } K \in \mathbb{C}\backslash\{0\}.$$

Finally we ensure that $k = K$ by conjugating each transformation above with the map $z \mapsto \lambda z$, for an appropriate choice of λ.

Hence we need to ask ourselves what is the most general form of elements $A, B \in PSL(2, \mathbb{C})$ such that the commutators are of the form above.

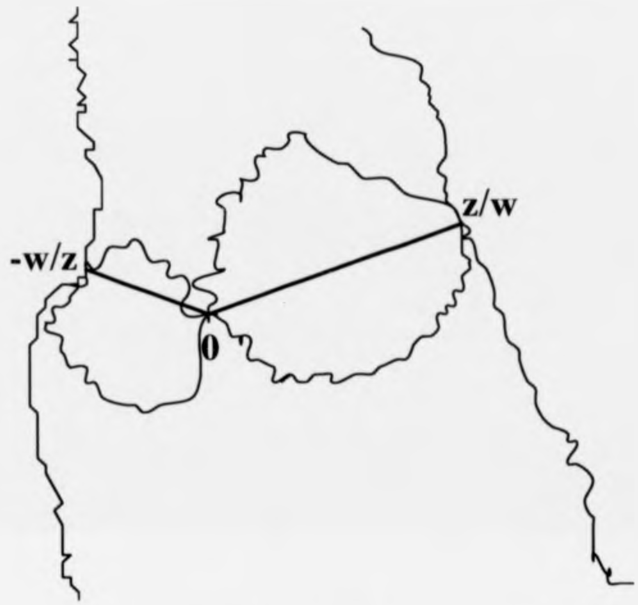

Figure 2.1: A quasiconformal deformation of figure 1.1

A long but straightforward calculation, which has been relegated to the appendix, shows that there exist complex parameters $z, w \in \mathbb{C}\backslash\{0\}$ such that for all marked groups normalised as above, we have matrices of the form

$$A(z,w) = \begin{pmatrix} \frac{1+z^2}{w} & z \\ z & w \end{pmatrix} \quad , \quad B(z,w) = \begin{pmatrix} \frac{1+w^2}{z} & -w \\ -w & z \end{pmatrix} \tag{2.1}$$

which generate each quasi-Fuchsian once-punctured torus group.

Thus our cycle of four parabolic fixed points starts at 0, which is moved by A to z/w, then by B to ∞, then onto $-w/z$ with A^{-1}, and finally B^{-1} returns the point to zero. Also the commutator $BAB^{-1}A^{-1}$ fixing infinity can be calculated to be the map

$$x \to x - 2\left(\frac{1+z^2+w^2}{zw}\right).$$

Hence our original picture in figure 1.1 has now been transformed by a quasiconformal map into something like figure 2.1, where our generators now pair up the four quasicircles. Here we can examine the question posed at the end of the last chapter, namely when can we take them to be classical circles?

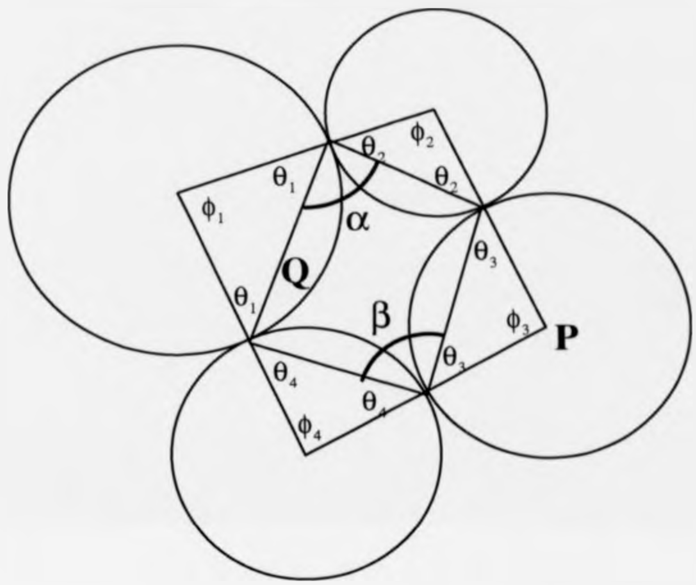

Figure 2.2: Four tangent circles

Definition 2.1.1

A *semi-Fuchsian* group is a marked quasi-Fuchsian once-punctured torus group $(\Gamma; A, B)$ where there exist four round discs, disjoint apart from each one having a point of tangency with its two neighbours, and the generators A and B each map the interior of one such disc onto the exterior of the disc opposite (as we have already seen in figure 1.2).

Proposition 2.1.2

Given four tangent circles, as above, then their points of tangency all lie on a circle.

Proof of 2.1.2

From figure 2.2, we have a quadrilateral **P** formed by joining the centres of the circles and a quadrilateral **Q** formed by joining the points of tangency. We will show that opposite angles of **Q** add up to 180 degrees, thus we have a cyclic quadrilateral by the result in good old fashioned Euclidean geometry.

We have $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 360°.$ (Sum of angles of **P**)

Thus, as we know the sum of the angles in our four Euclidean triangles,

$$4.180^\circ - 2(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 360^\circ$$

so

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 180^\circ.$$

But

$$\theta_1 + \theta_2 + \alpha = \theta_3 + \theta_4 + \beta = 180^\circ$$

and hence

$$\alpha + \beta = 180^\circ.$$

We repeat the argument for the other two angles of **Q** and the result follows. □

Now suppose $(\Gamma; A, B)$ is semi-Fuchsian. We have the four points of tangency of the circles, which must be the four parabolic fixed points of the appropriate commutators. We then perform our normalisation with Möbius transformations, thus retaining four tangent circles and sending these points to $0, z/w, \infty$ and $-w/z$. By proposition 2.1.2 they must lie on a straight line through zero. But z/w and $-w/z$ are complex numbers with product -1 and so have arguments which are supplementary angles. Thus they can only lie in one of two places, namely either both are situated on the imaginary axis where they would have to be on the same side of zero but then the tangent circles would not be disjoint. So we can only have z/w, and hence $-w/z$, as real numbers.

So we find that inside the four real dimensions of the quasi-Fuchsian space of the once-punctured torus we have the space of semi-Fuchsian groups, with codimension one. A detailed examination of this space is the subject of the next chapter, although in passing we note that explicitly exhibiting a semi-Fuchsian group is easy, since once we find the four tangent circles we possess all the necessary ingredients for the Klein-Maskit combination theorems (see [21]). However, directly constructing groups which are not semi-Fuchsian (as in figure 2.1) is a far harder task. This can be thought of as being analogous to the examples of classical and non-classical Schottky space. For instance, see the papers [29] and [26].

2.2 Uniqueness and Symmetries of the Normalisation

Now that we know that every element $(\Gamma; A, B)$ in our quasi-Fuchsian space can be represented as a point $(z, w) \in \mathbb{C}^2$, we need to ask is this representation unique?

Firstly we note that replacing (z, w) with $(-z, -w)$ does not change A or B when considered as elements of $PSL(2, \mathbb{C})$.

Now suppose we have $(A(z,w), B(z,w))$ and $(A(z',w'), B(z',w'))$, as in equation (2.1), which are the generators of acceptable normalisations of $(\Gamma; A, B)$, with the normalisations obtained by conjugating Γ with C_1 and C_2 respectively. Now

C_1 sends the fixed point of the commutator $B^{-1}A^{-1}BA$ to zero, as does C_2, so defining C to be $C_1C_2^{-1}$ gives $C(0) = 0$. Similarly C must also fix infinity and hence be of the form

$$C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{for some } \lambda \in \mathbb{C}\backslash\{0\}.$$

Now if the commutators of the first normalisation are

$$\begin{pmatrix} -1 & 0 \\ -K & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & K \\ 0 & -1 \end{pmatrix}$$

and in the second case are

$$\begin{pmatrix} -1 & 0 \\ -K' & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & K' \\ 0 & -1 \end{pmatrix}$$

then C must conjugate the two matrices above to their respective partners below. Hence using the fact that any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ transforms to } \begin{pmatrix} a & b\lambda^2 \\ c\lambda^{-2} & d \end{pmatrix} \tag{2.2}$$

when conjugated by C, we have $K/\lambda^2 = K'$ and $K\lambda^2 = K'$.

Thus $\lambda^4 = 1$, and $\lambda = \pm 1$ merely gives the identity, whereas $\lambda = \pm i$ gives us the map $C : x \mapsto -x$.

How does this relate to our matrices $(A(z,w), B(z,w))$ and $(A(z',w'), B(z',w'))$, where (z,w) and (z',w') have been assumed to be representing the same point in quasi-Fuchsian space?

Conjugating $A(z,w)$ with C to obtain $A(z',w')$, we find

$$A(z',w') = \begin{pmatrix} \frac{1+z^2}{w} & -z \\ -z & w \end{pmatrix} = \begin{pmatrix} \frac{1+z'^2}{w'} & z' \\ z' & w' \end{pmatrix}$$

and similarly

$$B(z',w') = \begin{pmatrix} -\frac{1+w^2}{z} & -w \\ -w & -z \end{pmatrix} = \begin{pmatrix} \frac{1+w'^2}{z'} & -w' \\ -w' & z' \end{pmatrix}$$

as elements of $PSL(2,\mathbb{C})$.

Thus the moves $(z,w) \mapsto (-z,w)$ and $(z,w) \mapsto (z,-w)$ do not change our point in quasi-Fuchsian space. If we define

$$\mathcal{P} = (\mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0\})/\sim$$

where (z,w) is equivalent to $(z,-w), (-z,w)$ and $(-z,-w)$ then this gives rise to a unique point in quasi-Fuchsian space $\mathcal{Q}$ as an element $[(z,w)]$ in $\mathcal{P}$. Thus $\mathcal{Q}$ sits inside $\mathcal{P}$ as an open subset with boundary $\partial\mathcal{Q}$. The natural question to ask

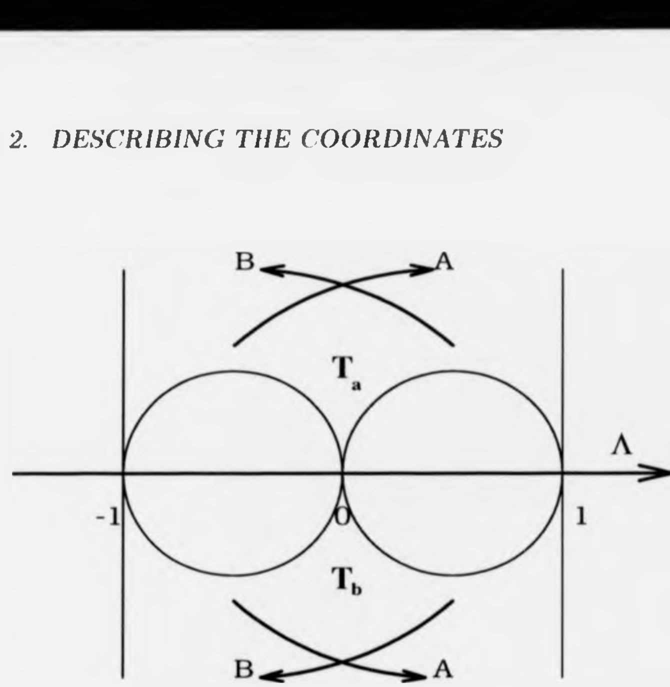

Figure 2.3: Two once-punctured tori

at this stage is exactly which set $\mathcal{Q}$ is. This is a difficult and open question but we look into it in Chapter 4 and build up a clearer picture.

Meanwhile, we can use our trick of conjugating our matrices by a suitable Möbius transformation and then finding new normalised matrices to build up some symmetries of (z,w) space. For instance we know that the quotient surface Ω/Γ of a quasi-Fuchsian once-punctured torus group Γ consists of a punctured torus lying above the limit set (which is an oriented Jordan curve) and another punctured torus below. How do we permute the torus on top with the torus on the bottom? We merely swap z and w.

Proposition 2.2.1

Given a point $(z,w) \in \mathcal{Q}$ with quotient surface T_a above the limit set and T_b below, then $(w,z) \in \mathcal{Q}$ has quotient surface T_b above the limit set and T_a below.

Proof of 2.2.1

Consider our group in figure 2.3. We have the cycle of four parabolic fixed points, namely 0 (which is the fixed point of $B^{-1}A^{-1}BA$) and then $A(0) = 1$, followed by $B(1) = \infty$ and then $A^{-1}(\infty) = -1$. Traversing these in order gives us an orientation for the limit set Λ, which is the real line, and we define the top torus T_a to be the one obtained from the quotient of the ordinary set on the left of Λ, and T_b to be obtained from the right.

Now given a point $(z,w) \in \mathcal{Q}$, we have matrices $A(z,w)$ and $B(z,w)$ which are the result of a quasiconformal conjugation. This homeomorphism is orientation preserving and so we get a picture as in figure 2.1, where the circles are now quasicircles, but we still retain the orientation of the limit set and hence we can identify the top and bottom tori.

Now conjugate the group with the map $C : x \mapsto -x$. This has the effect of turning figure 2.4 upside down.

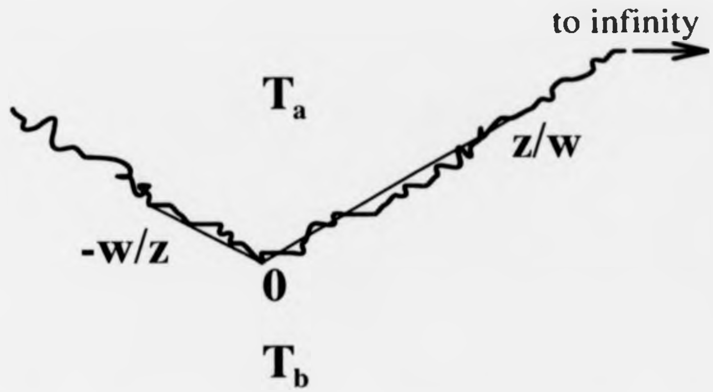

Figure 2.4: The limit set, with orientation

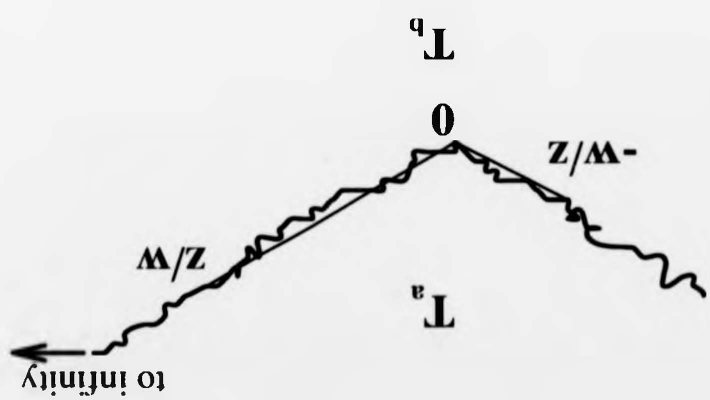

Figure 2.5: Figure 2.4 upside down

If we could find matrices $A(z,w)$, $B(z,w)$ with $A(z,w) = CB(z,w)C^{-1}$

$$\left(\begin{array}{lrcl} \text{so that} & A(0) & = & w/z \\ & A(-z/w) & = & \infty \end{array} \right)$$

and $\bar{B}(z,w) = CA(z,w)C^{-1}$

$$\left(\begin{array}{lrcl} \text{so that} & \bar{B}(0) & = & -z/w \\ & \bar{B}(w/z) & = & \infty \end{array} \right)$$

then we would have correctly normalised the marked group $(C\Gamma C^{-1}; \bar{A}, \bar{B})$ and will have reversed the orientation of the limit set. Thus T_a and T_b will have been swapped. But

$$CB(z,w)C^{-1} = \begin{pmatrix} \frac{1+w^2}{z} & w \\ w & z \end{pmatrix} = A(w,z)$$

and

$$CA(z,w)C^{-1} = \begin{pmatrix} \frac{1+z^2}{w} & -z \\ -z & w \end{pmatrix} = B(w,z)$$

using equations (2.1) and (2.2).

Thus $A(w,z) = \bar{A}(z,w)$ and $B(w,z) = \bar{B}(z,w)$. □

We can perform the same trick with other conjugations, such as $x \mapsto \bar{x}$.

Proposition 2.2.2

Replacing the point $(z,w) \in \mathcal{Q}$ with $(\bar{z},\bar{w})$ has the effect of replacing the quotient torus T_a above the limit set with $\overline{T_b}$ (this is the torus with the conjugate complex structure of the torus below) and T_b with $\overline{T_a}$.

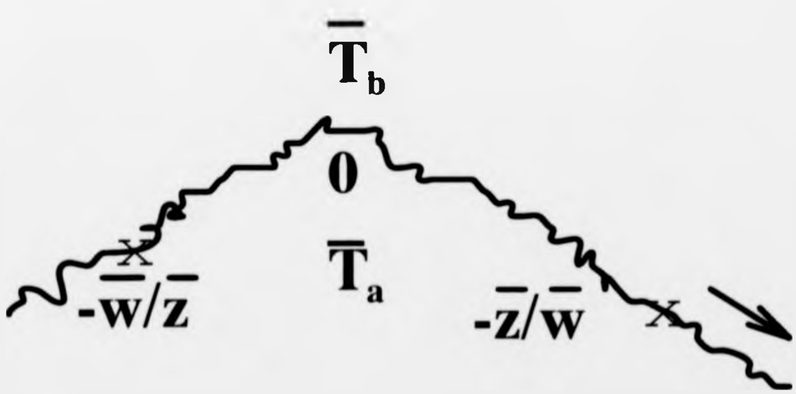

Figure 2.6: A conjugated figure 2.4

Proof of 2.2.2

Given the marked group $(\Gamma; A(z,w), B(z,w))$ we conjugate using the map $C : x \mapsto \bar{x}$ which has the effect of turning figure 2.4 into figure 2.6 above.

Thus now $\overline{T_b}$ is above the limit set, and $\overline{T_a}$ below. But we need to check what has happened to the generators.

$$CA(z,w)C^{-1} = \begin{pmatrix} \frac{1+\bar{z}^2}{\bar{w}} & \bar{z} \\ \bar{z} & \bar{w} \end{pmatrix} = A(\bar{z}, \bar{w})$$

and

$$CB(z,w)C^{-1} = \begin{pmatrix} \frac{1+\bar{w}^2}{\bar{z}} & -\bar{w} \\ -\bar{w} & \bar{z} \end{pmatrix} = B(\bar{z}, \bar{w})$$

which gives rise to the orientation of the limit set as in figure 2.6. □

Note We can see how this works in the special case of a Fuchsian group. We will show in the next section that the Fuchsian groups correspond precisely to where both z and w are real. In this case moving to $(\bar{z}, \bar{w})$ should not alter our quotient surfaces, and as we know that a Fuchsian group represents two surfaces which are mirror images of each other, we can see that taking the mirror image of each and then swapping them round leaves us where we started.

Also proposition 2.2.1 tells us that swapping z and w corresponds to swapping top and bottom tori. However, we would be wrong to assume that altering just one of z or w corresponds to altering the complex structure on only one torus. After all, if we start with z and w real and adjust only z say, but still keep it real, then we have moved from one Fuchsian group to another. But both quotient surfaces must have changed as they are reflections of each other.

We finish the section by looking at the hyperelliptic involution, the conformal automorphism possessed by every quasi-Fuchsian once-punctured torus.

Proposition 2.2.3

The Möbius map $H : x \mapsto -1/x$ of order two conjugates the normalised group $(\Gamma; A(z,w), B(z,w))$ to $(\Gamma; A^{-1}(z,w), B^{-1}(z,w))$. Thus $(\Gamma; A, B)$ and $(\Gamma; A^{-1}, B^{-1})$ are the same point in quasi-Fuchsian space.

Proof of 2.2.3

Taking $x \in \mathbb{C}$, we have

$$HA(z,w)H^{-1}(x) = H\left(\frac{\frac{1+z^2}{-wx} + z}{-\frac{z}{x} + w}\right) = H\left(\frac{zx - \frac{1+z^2}{w}}{wx - z}\right) = \frac{wx - z}{-zx + \frac{1+z^2}{w}} = A^{-1}(x).$$

Similarly

$$HB(z,w)H^{-1}(x) = \frac{zx + w}{wx + \frac{1+w^2}{z}} = B^{-1}(x).$$

As A and B are generators, we have $H\Gamma H^{-1} = \Gamma$, and we have an automorphism of Γ defined by conjugation of H. □

Corollary 2.2.4

The limit set $\Lambda(\Gamma)$ of the normalised group $(\Gamma; A(z,w), B(z,w))$ is invariant under $H : x \mapsto -1/x$.

Proof of 2.2.4

If $\Gamma = H\Gamma H^{-1}$ then

$$\Lambda(\Gamma) = \Lambda(H\Gamma H^{-1}) = H(\Lambda(\Gamma)). \qquad \square$$

Note that H permutes our cycle of four points $0, z/w, \infty$ and $-w/z$ to $\infty, -w/z, 0$ and z/w. Note also that the fixed points of $A(z,w)$ (and of $B(z,w)$) have product -1, and so are permuted under H.

2.3 The Trace Parameters

Our other set of coordinates for quasi-Fuchsian space involves looking at the traces of generators of our groups.

We have the following identity for all elements $A, B \in PSL(2,\mathbb{C})$,

$$\mathrm{tr}^2(A) + \mathrm{tr}^2(B) + \mathrm{tr}^2(AB) - 2 = \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(AB) + \mathrm{tr}(B^{-1}A^{-1}BA).$$

Thus given any marked quasi-Fuchsian once-punctured torus group $(\Gamma; A, B)$, set $a = \mathrm{tr}(A), b = \mathrm{tr}(B)$ and $c = \mathrm{tr}(AB)$ to obtain the trace equation

$$a^2 + b^2 + c^2 = abc. \tag{2.3}$$

Of course a conjugate marked group $(C\Gamma C^{-1}; CAC^{-1}, CBC^{-1})$ gives rise to the same triple (a, b, c) of complex numbers although we have the ambiguity that the trace of a matrix in $PSL(2,\mathbb{C})$ is only determined up to sign, so we may change the sign of any two of (a, b, c).

Now, in order to embed quasi-Fuchsian space $\mathcal{Q}$ in $\mathbb{C}^3$ using the trace parameters, we need to reverse the above argument and show that different marked groups $(\Gamma; A, B)$ and $(\Gamma'; A', B')$ with the same triple of traces (a, b, c) are indeed conjugate.

We do this by introducing the connection between the (z, w) parameters and the traces. Choosing $(\Gamma; A(z,w), B(z,w))$ to be the appropriate normalised group in the conjugacy class, we obtain from (2.1)

$$\begin{aligned} a &= \frac{1 + z^2 + w^2}{w} \\ b &= \frac{1 + z^2 + w^2}{z} \\ c &= \frac{1 + z^2 + w^2}{zw} \end{aligned} \tag{2.4}$$

which we notice can be easily inverted to give

$$z = \frac{a}{c}\,,\ w = \frac{b}{c}. \tag{2.5}$$

Thus our marked groups with the same trace parameters turn out to be the same point (z, w) in quasi-Fuchsian space (allowing for changes in sign) and so are conjugate.

We also see that the (z, w) parameters act as a projection of the manifold $a^2 + b^2 + c^2 = abc$ from $\mathbf{C}^3$ onto $\mathbf{C}^2$.

Knowing that the Fuchsian groups are precisely those with real trace parameters (a, b, c) leads us immediately to use equations (2.4) and (2.5) to deduce that this corresponds exactly to when z and w are both real numbers. Similarly the necessary condition that $z/w \in \mathbf{R}$ for a semi-Fuchsian group tells us that we must insist on the ratio of the traces a/b being real.

Proposition 2.3.1

We have the following symmetries of the trace relations (a, b, c):

(1) $(a, b, c) \mapsto (b, a, c)$ swaps the top and bottom tori.

(2) $(a, b, c) \mapsto (\bar{a}, \bar{b}, \bar{c})$ swaps the top and bottom tori, as well as replacing each surface with its complex conjugate.

Proof of 2.3.1

Use propositions 2.2.1, 2.2.2, and equation (2.5). □

Proposition 2.3.2

Given the normalised group $(\Gamma; A(z, w), B(z, w))$ possessing the trace parameters (a, b, c) then the limit set $\Lambda(\Gamma)$ is invariant under the translation $x \mapsto x + c$.

Proof of 2.3.2

Noting that the commutator $BAB^{-1}A^{-1}$ is the Möbius map

$$x \mapsto x - 2\left(\frac{1 + z^2 + w^2}{zw}\right),$$

we see from (2.4) that this simplifies to $x \mapsto x - 2c$, and we know that $\Lambda(\Gamma)$ is invariant under elements of Γ.

However, we can improve on this by adjoining the hyperelliptic involution H : $x \mapsto -1/x$ to Γ to form the discrete group Δ generated by $(A(z, w), B(z, w), H)$ which contains Γ as a subgroup of index two, and so has the same limit set.

Then we see that the element $BAH \in \Delta$ is of the form

$$\begin{pmatrix} \frac{1+w^2}{z} & -w \\ -w & z \end{pmatrix}\begin{pmatrix} \frac{1+z^2}{w} & z \\ z & w \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}.$$

In other words, we have gained the square root of $BAB^{-1}A^{-1}$, and thus $\Lambda(\Gamma)$ is invariant under $x \mapsto x + c$ (as well as $x \mapsto -1/x$ from corollary 2.2.4). □

2.4 Generator moves in Quasi-Fuchsian space

As we are looking at quasi-Fuchsian groups $(\Gamma; A, B)$ with a pair of marked generators singled out, we need to ask how do we change the generators while keeping the conformal structure of the once-punctured tori the same? Also, what effect do these moves have on the (z, w) and (a, b, c) parameters?

It is clear that we should quotient out the group of automorphisms of Γ, $\mathrm{Aut}(\Gamma)$ by the inner automorphisms $\mathrm{Aut}_0(\Gamma)$, since we do not distinguish between conjugacy classes of marked groups.

Also we quotient out by the move $h : (A, B) \mapsto (A^{-1}, B^{-1})$ as this is induced by the hyperelliptic involution, and so does not change our point in quasi-Fuchsian space.

We use the following proposition, see [13].

Proposition 2.4.1

The following motions generate the group of generator moves $\mathcal{M}$, where $\mathcal{M}$ is defined as $\mathrm{Aut}(\Gamma)/ < h, \mathrm{Aut}_0(\Gamma) >$.

$$\begin{array}{llll}
(1) & x : (A, B) \mapsto & (B, A^{-1}B^{-1}) & \text{(of order 3)} \\
(2) & y : (A, B) \mapsto & (B, A^{-1}) & \text{(of order 2)} \\
(3) & r : (A, B) \mapsto & (B, A) & \text{(of order 2)} \quad \square
\end{array}$$

We now convert these moves into transformations of (z, w) and (a, b, c).

$$\begin{array}{lrcl}
(1) & (a, b, c) & = & (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \\
 & & \overset{x}{\mapsto} & (\mathrm{tr}(B), \mathrm{tr}(A^{-1}B^{-1}), \mathrm{tr}(BAB^{-1})) \\
 & & = & (b, c, a)
\end{array}$$

which geometrically is a rotation about the origin, permuting the axes. This gives rise to the transformation $(z, w) \mapsto (\frac{w}{z}, \frac{1}{z})$ for $(z, w) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0\}$. We can picture this as being the projective transformation on the complex plane $(x, y, 1) \subset \mathbb{C}^3$ obtained from the rotation above.

$$\begin{array}{lrcl}
(2) & (a, b, c) & = & (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)) \\
 & & \overset{y}{\mapsto} & (\mathrm{tr}(B), \mathrm{tr}(A^{-1}), \mathrm{tr}(BA^{-1})) \\
 & & = & (b, a, ab - c)
\end{array}$$

using the identity

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(BA^{-1}) \qquad \text{for } A, B \in PSL(2, \mathbb{C}).$$

As for the action of y on (z, w), we obtain

$$\begin{aligned}
(z, w) &= \left(\frac{a}{c}, \frac{b}{c}\right) \mapsto \left(\frac{b}{ab - c}, \frac{a}{ab - c}\right) \\
&= \left(\frac{\frac{b}{c}}{a(\frac{b}{c}) - 1}, \frac{\frac{a}{c}}{a(\frac{b}{c}) - 1}\right) \\
&= \left(\frac{w}{z^2 + w^2}, \frac{z}{z^2 + w^2}\right).
\end{aligned}$$

which now admits a geometric interpretation as reflection in the complex circle $z^2 + w^2 = 1$, followed by reflection in the complex line $z = w$.

(3) The move r clearly induces the motions $(a, b, c) \mapsto (b, a, c)$ and $(z, w) \mapsto (w, z)$, and thus from proposition 2.2.1 swaps our top and bottom surfaces. Sometimes we will use only the orientation preserving generator moves, and we call this group $\mathcal{M}_+ =< x, y >$. But if we are not worried which surface ends up where, then it will be useful to us to use the full group $\mathcal{M} =< x, y, r >$, and then we possess all six permutations of the triple (a, b, c).

We finish the chapter by summarising the generator moves of $\mathcal{M}_+$, and their effects on each parameterisation. We also include here the *Dehn twists* about A and B, which are moves of infinite order generating the motions that fix these elements. Together the two Dehn twists can also be taken as generators for the group $\mathcal{M}_+$.

	Generator move	Parameter moves
Motion x	$(A, B) \mapsto (B, A^{-1}B^{-1})$	$(z, w) \mapsto (\frac{w}{z}, \frac{1}{z})$ $(a, b, c) \mapsto (b, c, a)$
Motion y	$(A, B) \mapsto (B, A^{-1})$	$(z, w) \mapsto (\frac{w}{z^2+w^2}, \frac{z}{z^2+w^2})$ $(a, b, c) \mapsto (b, a, ab - c)$
Dehn twist about $A = yx^2$	$(A, B) \mapsto (A, AB \text{ or } BA)$	$(z, w) \mapsto (\frac{zw}{1+z^2}, \frac{w}{1+z^2})$ $(a, b, c) \mapsto (a, c, ac - b)$
Dehn twist about $B = yx$	$(A, B) \mapsto (AB \text{ or } BA, B)$	$(z, w) \mapsto (\frac{z}{1+w^2}, \frac{zw}{1+w^2})$ $(a, b, c) \mapsto (c, b, bc - a)$

Chapter 3

The space of Semi-Fuchsian groups

3.1 Coordinates for Semi-Fuchsian groups

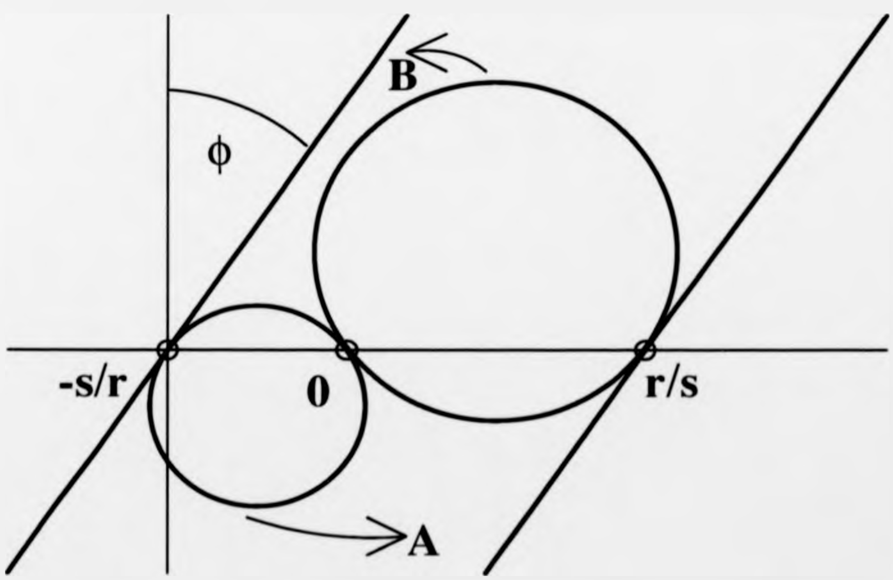

Figure 3.1: A normalised semi-Fuchsian group

Recall from chapter 2 that a semi-Fuchsian group is a quasi-Fuchsian once-punctured torus group which is obtained by the pairing of four tangent circles. We wish to find the precise subset of quasi-Fuchsian space $\mathcal{Q}$ where the semi-Fuchsian groups lie, which we shall denote by $\mathcal{S}$. We primarily use (z, w) coordinates for this purpose. We have already seen that given a marked semi-Fuchsian group $(\Gamma; A(z,w), B(z,w))$ which has been suitably normalised then z/w has to be real, and thus positive with no loss of generality. So our four tangent circles must look like the picture in figure 3.1, where we have set $z = re^{i\theta}$ and $w = se^{i\theta}$, with $r, s > 0$.

In order to draw the picture in figure 3.1, we can find the values of r/s and $-s/r$ straightaway from $(\Gamma; A(z,w), B(z,w))$. However, we need to know the angle ϕ as well. In fact the result could not be simpler.

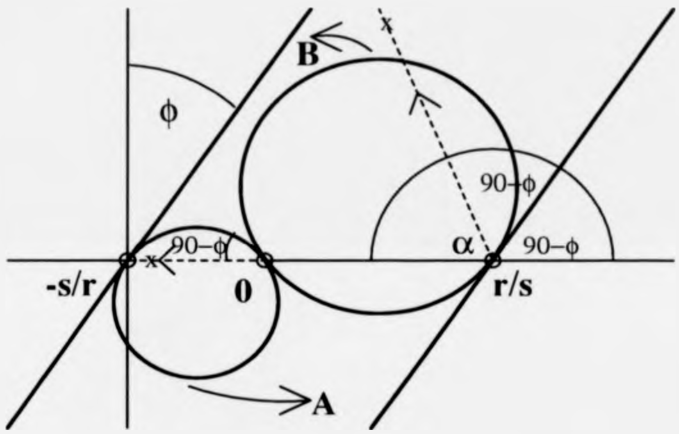

Figure 3.2: Figure 3.1 with more angles

Proposition 3.1.1

With notation as above, we have $\phi = \theta$.

Proof of 3.1.1

Choose a point $x \in \mathbf{R}$ with $-s/r < x < 0$ and look at its image under the map A. Then

$$A(x) = \frac{\left(\frac{1+z^2}{w}\right)x + z}{zx + w} = \frac{(e^{-2i\theta} + r^2)x + rs}{s(rx + s)}$$

and so

$$A(x) - \frac{r}{s} = \frac{e^{-2i\theta}x}{s(rx + s)}.$$

The real line is mapped by A, which is angle preserving, to a line through r/s. So noting that $r, s, x \in \mathbf{R}$ and $\frac{x}{s(rx+s)} < 0$ tells us that $\alpha = 2\theta = 2\phi$. □

Note: If we try to construct the group $(\Gamma; A(z,w), B(z,w))$ using only the information in the picture then we can read off θ and r/s $(= 1/\lambda$ say) immediately, but there exists a one real parameter family of groups with $(z,w) = (re^{i\theta}, \lambda re^{i\theta})$ all with the same figure 3.1 if we let r vary. We would need to know the image of one other point to define the group uniquely, for instance we could take $x = -s/2r$ in proposition 3.1.1 and then we can check that $A(x)$ is at a distance $1/rs$ from the point r/s, thus enabling us to find z and w.

In the arguments above, we were given a point (z,w) which we assumed already corresponded to a semi-Fuchsian group. Now suppose we are given an arbitrary point (z,w) in our parameter space $\mathcal{P}$, knowing only that z/w is real and greater than 0. We can certainly form matrices $A(z,w)$, $B(z,w)$ and generate the marked group $(\Gamma; A, B)$. We do of course have four tangent circles in our picture, but we wish to apply the Klein-Maskit combination theorems to show we will obtain a quasi-Fuchsian once-punctured torus group. This we can do if and only if the following three conditions hold.

(1) The four closed discs are disjoint apart from the four points of tangency.
(2) The transformations A and B are both loxodromic. (Throughout, we use the term loxodromic to include hyperbolic elements too.)
(3) The commutator $B^{-1}A^{-1}BA$ is parabolic.

Of course, (3) is always true as that is how we have set up the parameter space $\mathcal{P}$.

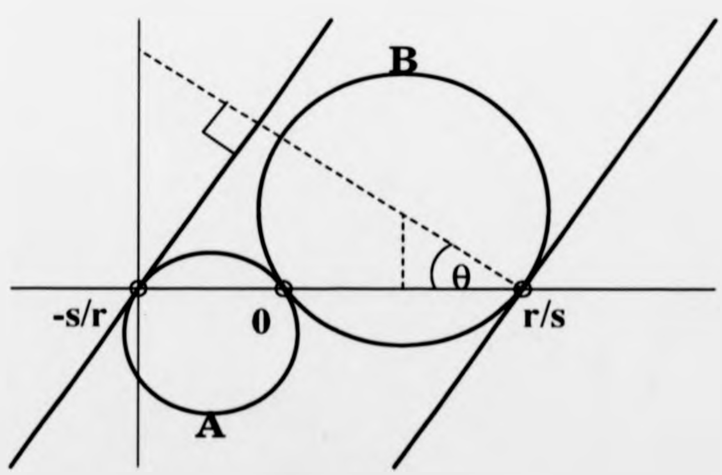

Figure 3.3: Condition (1) holds

Proposition 3.1.2

If we set $(z, w) = (re^{i\theta}, se^{i\theta})$ with $r, s > 0$ and $\theta \in (-\pi/2, \pi/2]$, then condition (1) holds precisely when $|\tan\theta| < \min(r/s, s/r)$.

Proof of 3.1.2

We require the radius of the circle **B** in figure 3.3 to be less then half the perpendicular distance between the two parallel lines.
Thus

$$\frac{r}{2s\cos\theta} < \frac{1}{2}\left(\frac{r}{s} + \frac{s}{r}\right)\cos\theta$$

giving

$$\begin{aligned} r^2 &< (r^2+s^2)\cos^2\theta \\ \tan^2\theta &< \frac{s^2}{r^2} \\ |\tan\theta| &< \frac{s}{r}. \end{aligned}$$

We repeat the argument for the circle **A**. □

Thus we can see that starting from $\theta = 0$, where we have the Fuchsian groups, we rotate the parallel lines and as we increase $|\theta|$ we restrict the ratio r/s to lie in the open interval $(|\tan\theta|, 1/|\tan\theta|)$. When we reach $\theta = \pi/4$ we can no longer have any semi-Fuchsian groups.

Finally we check the traces of the generating matrices A and B.

Proposition 3.1.3

The elements A and B are both loxodromic transformations when $r^2+s^2 \neq 1$. Where $r^2+s^2 = 1$ then $a = \mathrm{tr}(A)$, $b = \mathrm{tr}(B)$ are both real. In this case we have A and B both hyperbolic inside the region $|\tan\theta| < \min(r/s, s/r)$ mentioned above, and also A is parabolic when $|\tan\theta| = r/s$ and elliptic for $|\tan\theta| > r/s$. Similarly, B is parabolic for $|\tan\theta| = s/r$, $r^2+s^2 = 1$ and elliptic for $|\tan\theta| > s/r$, $r^2+s^2 = 1$.

Proof of 3.1.3

We know

$$A = e^{i\theta}\begin{pmatrix} \frac{r^2+e^{-2i\theta}}{s} & r \\ r & s \end{pmatrix}$$

so a is real when $\theta = 0$ or when $\mu = r^2 + s^2 + e^{-2i\theta}$ has argument $-\theta$. In figure 3.4 triangle **T** is isosceles, so the point μ lies on the line L only when $r^2+s^2 = 1$.

If this is the case, then

$$a = e^{+i\theta}\left(\frac{r^2+s^2+e^{-2i\theta}}{s}\right) = \frac{1}{s}(e^{i\theta}+e^{-i\theta}) = \frac{2\cos\theta}{s}.$$

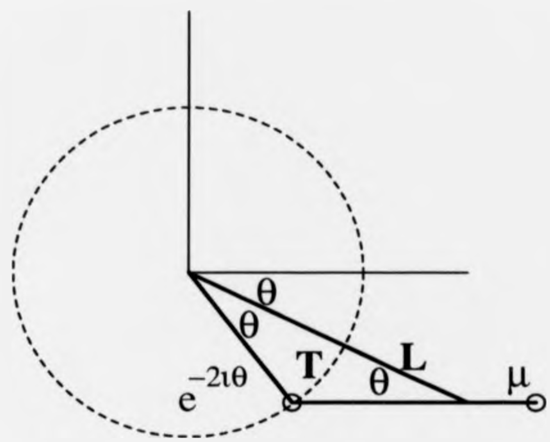

Figure 3.4: the unit circle

Then we take χ with $r = \sin\chi$ and $s = \cos\chi$, so that $|a| \leq 2$ if and only if $\cos\chi \geq |\cos\theta|$, which means $|\tan\theta| \leq \tan\chi = r/s$.

We repeat for the matrix B by swapping r and s. □

Thus we have now found that semi-Fuchsian space $\mathcal{S}$ is precisely the region $|\tan\theta| < \min(r/s, s/r)$, as it is here and only here where conditions (1), (2) and (3) all hold. By "squeezing" $\mathcal{S}$ into three real dimensions and taking the axes to be (r, s, θ) we can picture $\mathcal{S}$ as the interior of the roof-shaped region in figure 3.5.

Next we move on to investigate the boundary of $\mathcal{S}$ in parameter space $\mathcal{P}$, where $|\tan\theta| = r/s$ or s/r. The question we wish to consider is whether these boundary groups can still be quasi-Fuchsian once-punctured torus groups and so lie in $\mathcal{Q}$, or not, in which case we have reached the boundary of quasi-Fuchsian space.

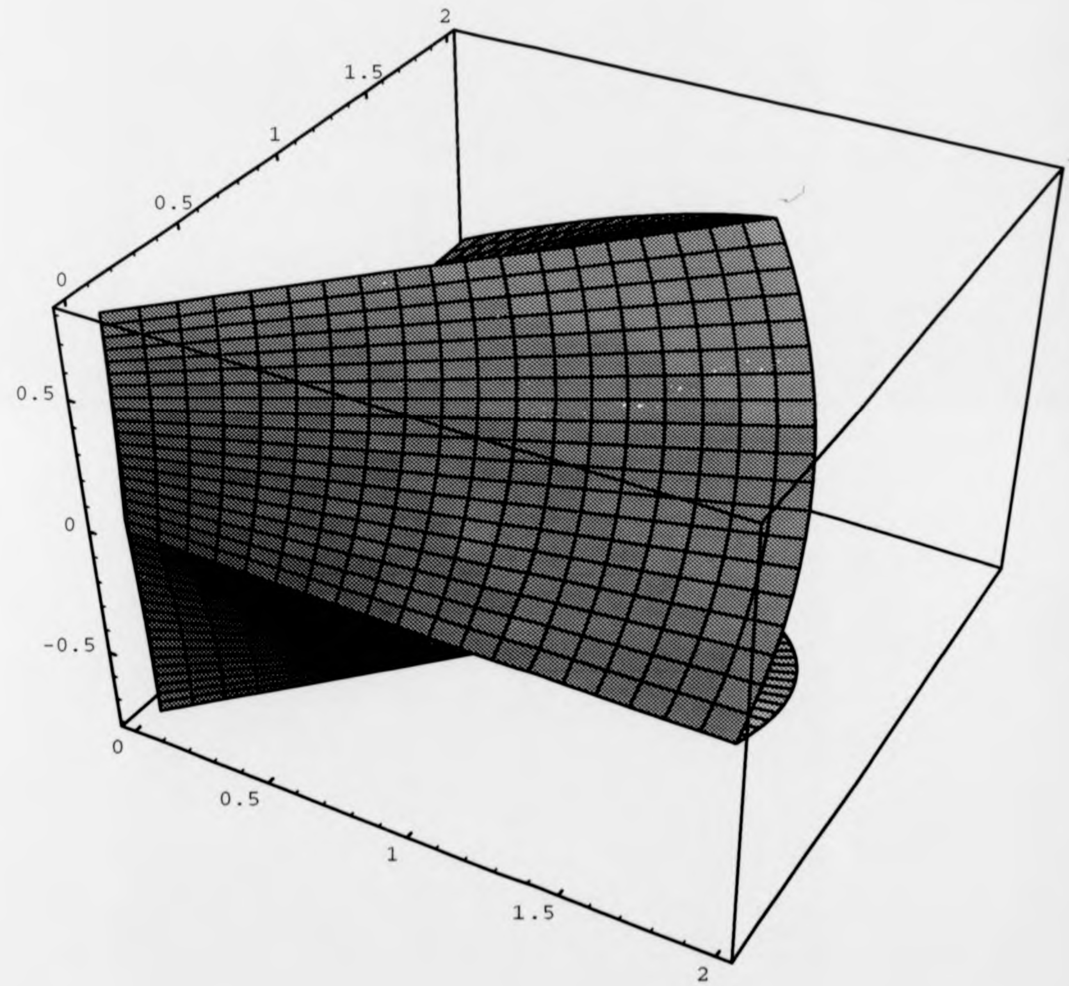

Figure 3.5: Semi-Fuchsian space

3.2 The Boundary of Semi-Fuchsian space

First, let us consider the locus $(z,w) = (re^{i\theta}, se^{i\theta})$ where $r^2 + s^2 = 1$. If we fix r and s, with $r > s$ say, and start to increase θ from 0 then this corresponds to rotating the parallel lines in figure 3.1 anticlockwise. We know from proposition 3.1.3 that the traces of A and B are both real and that they decrease as we increase θ. Then when $\tan\theta$ reaches s/r, the larger circle becomes tangent to the line through $-s/r$ at the point i. But now the trace of B is equal to 2, so B has become parabolic, with only one fixed point. Remembering from corollary 2.2.4 that the fixed points are invariant under $x \mapsto -1/x$, we must have B fixing i.

Heuristically, we can consider the simple closed curve representing B on each of our quotient surfaces. Downstairs we still expect to find a punctured torus, whereas on the surface upstairs the length of the curve B has shrunk to zero, or has been "pinched", and so topologically we would expect to have obtained a thrice-punctured sphere. If so, we would no longer have two invariant components of the ordinary set Ω, as the two quotient surfaces are not homeomorphic, so we would see an infinite number of simply connected noninvariant components, which are all conjugate and whose quotient surface is a thrice-punctured sphere.

The space of all such marked groups is known as the Maskit embedding of the Teichmüller space of the once-punctured torus. This space is studied thoroughly in [15] where we have the following procedure:

(1) Marked groups $G_t =< S, T_t >$ are considered where S is parabolic, along with the commutator $T_t^{-1}S^{-1}T_tS$, but T_t is loxodromic.

(2) The groups are normalised, so that

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; T_t = \begin{pmatrix} t & -i \\ -i & 0 \end{pmatrix} \tag{3.1}$$

and the commutator

$$T^{-1}S^{-1}TS = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \qquad \text{for } t \in \mathbb{C}.$$

(3) The region $\mathbf{M} \subseteq \mathbb{C}$, where $t \in \mathbf{M}$ implies that the ordinary set $\Omega(G_t)$ is made up of a single invariant component $\Omega_0(G_t)$ whose quotient surface is a once-punctured torus and non-invariant components representing a thrice-punctured sphere, is studied and geometric coordinates for $\mathbf{M}$ are provided. Also the groups G_t with $t \in \partial\mathbf{M}$ are investigated.

The region $\mathbf{M}$, which is invariant under translations of the form $t \mapsto t + 2i$, is shown in figure 1 of [15].

It is also shown in [15] that groups G_t with $t \in \mathbf{M}$ have their limit set $\Lambda(G_t)$ consisting of the closure of a chain of tangent circles. The interiors of the circles are precisely the non-invariant components of the limit set, and the exterior of the whole chain is the invariant component $\Omega_0(G_t)$. Taking t to be real and greater than two, we obtain a fundamental domain for G_t in the form of figure 3.6.

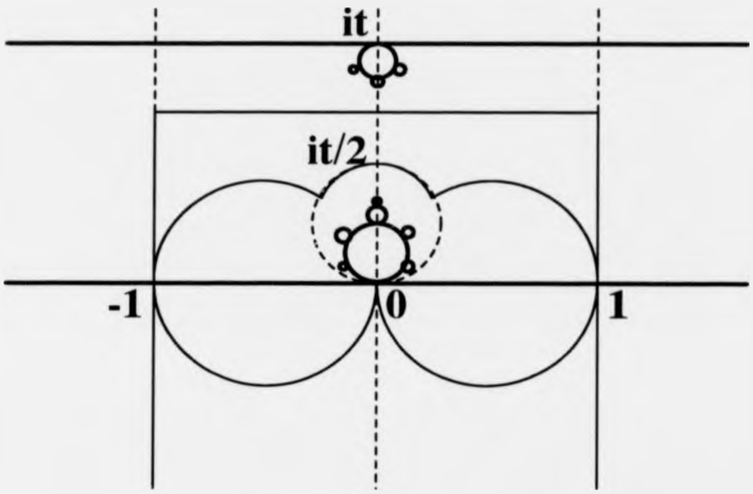

Figure 3.6: The group G_t, $t > 2$.

The fundamental domain for the ordinary set $\Omega(G_t)$ is shown in thin unbroken lines; there are two components, namely the fundamental domain for the thrice-punctured sphere down below, and for the invariant component above which projects down to a once-punctured torus Ω_0/G_t. The bold circles are in the limit set, which is made up of the closure of all images of the real line.

The groups G_t for $t \in \partial\mathbf{M}$ are all discrete with $\Lambda(G_t)$ made up of a circle packing, but fall into two distinct categories. The first case is when the invariant component $\Omega_0(G_t)$ degenerates into an infinite number of non-invariant components. This corresponds to having pinched a simple closed curve on the remaining once-punctured torus, so the quotient space $\Omega(G_t)/G_t$ is now made up of two thrice-punctured spheres.

In the other case we find that the invariant component $\Omega_0(G_t)$ disappears altogether, so that the quotient space $\Omega(G_t)/G_t$ has now lost its once-punctured torus, and consists only of a thrice-punctured sphere.

Now the idea is that we would expect to see the Maskit embedding of the once-punctured torus $\mathbf{M}$ present in the boundary of quasi-Fuchsian space $\partial\mathcal{Q}$. This we can show by comparing the traces (a, b, c) for a marked group $(\Gamma; A, B)$ in our parameter space $\mathcal{P}$ with the traces of T_t, S and T_tS in (3.1). If they were the same marked group up to conjugation, we would need

$$(a, b, c) = (t, 2, t - 2i) \qquad \text{for } t \in \mathbf{M}.$$

But as we have seen in section 2.3, this condition is also sufficient to ensure we merely have different normalisations of the same marked group. Thus figure 3.5 shows us the slice of the boundary ∂Q when embedded into (a,b,c) space corresponding to $b=2$, where we have made the generator B parabolic, and thus the curve representing B has been pinched on one of the punctured tori to obtain a thrice-punctured sphere.

Taking $t \in \mathbb{R}$ with $t > 2$ gives us a group which is the limit of semi-Fuchsian groups. In fact, if we consider the groups at the begining of the section, where we had the point $(re^{i\theta}, se^{i\theta})$ with $r^2+s^2=1$ and increased θ until $\tan\theta = s/r$ to create a new point of tangency between the four circles, then setting $t = 2/\tan\theta$ gives

$$\begin{aligned}
(a,b,c) &= \left(\frac{1+z^2+w^2}{w}, \frac{1+z^2+w^2}{z}, \frac{1+z^2+w^2}{zw}\right) \\
&= \left(\frac{2\cos\theta}{s}, \frac{2\cos\theta}{r}, \frac{2\cos\theta}{rse^{i\theta}}\right) \\
&= \left(\frac{2}{\tan\theta}, 2, 2\,\frac{\cos\theta - i\sin\theta}{\sin\theta}\right) \\
&= (t, 2, t-2i)
\end{aligned}$$

which is precisely the group in figure 3.6 under a different normalisation. Thus as we rotate our parallel lines in figure 3.1, both fixed points of B move closer and closer to i. However, our limit set is still a quasicircle passing through the fixed points but remaining inside the four discs. Then when the new point of tangency is created, the fixed points of B coalesce at i, and the limit set degenerates from a quasicircle into a chain of tangent circles (compare figures 3.7 and 3.8).

It would be nice to see where this chain of circles appears in the picture of our group on the boundary of semi-Fuchsian space. We know the existence of a matrix $M \in PSL(2,\mathbb{C})$ say, which conjugates the marked group $(G_t; T_t, S)$ into $(\Gamma; A, B)$ for $t > 2$. Thus the fixed point ∞ of S is moved to i, the fixed point of B, by M. Also -1, the fixed point of $S^{-1}T^{-1}ST$ is moved to 0 (the fixed point of $B^{-1}A^{-1}BA$), and noting that $S(-1)=1$, so that the commutator $T^{-1}STS^{-1}$ fixes 1, tells us that M moves 1 to the fixed point of $A^{-1}BAB^{-1}$, namely $-w/z$.

So the real line is moved by M to a circle through $0, i$ and $-w/z$. Inside this circle we have a fundamental domain for the thrice-punctured sphere (the thin unbroken lines in figure 3.8). By reversing the rôles of the matrices A and B, and by rotating our parallel lines in figure 3.1 clockwise, so that θ becomes negative, we obtain the following proposition.

Proposition 3.2.1

The boundary of semi-Fuchsian space $(z,w) = (re^{i\theta}, se^{i\theta})$ where $r^2+s^2=1$ and $|\tan\theta| = \min(r/s, s/r)$ corresponds to the following groups:

(1) If $s/r = \tan\theta$ then the curve representing B has been pinched on the top surface, to produce a thrice-punctured sphere.

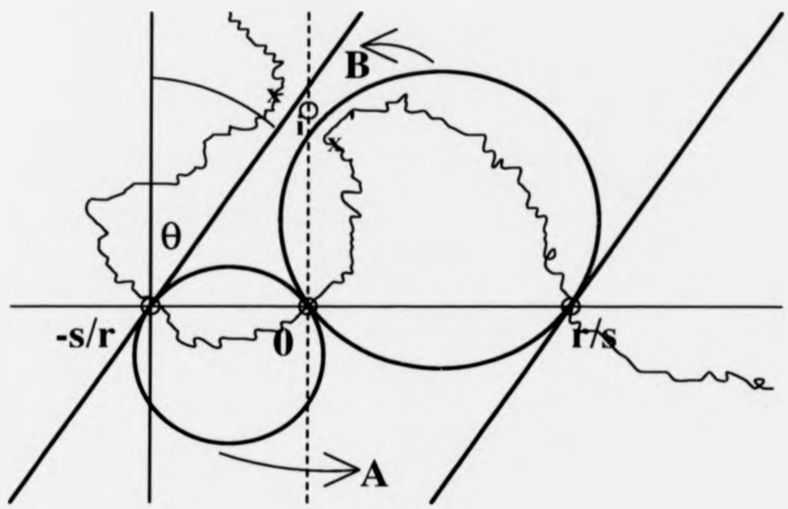

Figure 3.7: Limit set still a quasicircle

(2) If $r/s = \tan\theta$ then the curve A has been pinched above.
(3) If $s/r = -\tan\theta$ then B has been pinched below.
(4) If $r/s = -\tan\theta$ then A has been pinched below. □

Also we note that if $r = s = \sqrt{2}/2$ and $\theta = \pm\pi/4$ then we obtain thrice-punctured spheres on both top and bottom, with a limit set consisting of a packing of tangent circles. This corresponds to $t = 2$ in (3.1), and is none other than the Apollonian circle packing.

What we must now consider is the case when we have a group $(re^{i\theta}, se^{i\theta})$ on the boundary of semi-Fuchsian space $\partial\mathcal{S}$, but $r^2 + s^2 \neq 1$. Then we have A and B both loxodromic elements, but we will create a new point of tangency between two of our circles at i if we decide to take $\tan\theta = s/r$. As in figure 3.8, B still takes the circle through $0, r/s$ and i to the line through $-s/r$ and i, but i is no longer fixed. However, the idea to observe is that the region exterior to these two discs (i.e. the area containing $-i$) is still a fundamental domain for the cyclic group generated by the loxodromic element B.

Thus we can use the Klein-Maskit combination theorems as before with the above fundamental domain for $< B >$, and for $< A >$ we take the usual fundamental domain, which is exterior to the two other circles. We obtain a free group $(\Gamma; A, B)$ with a fundamental domain made up of the intersection of the two domains above.

Then to find out the quotient surface Ω/Γ we move the triangle T via A to

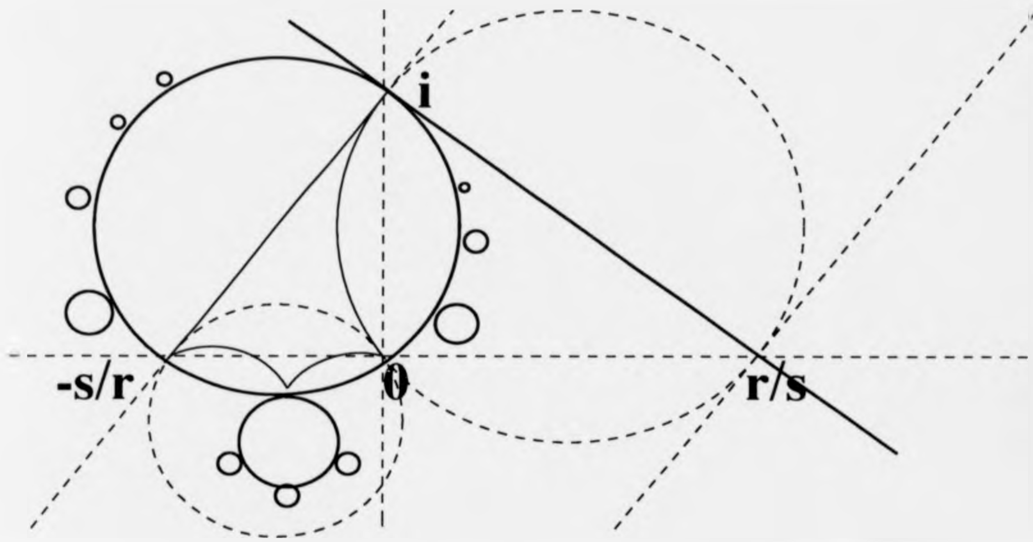

Figure 3.8: Limit set now a chain of circles

create a connected domain upstairs, consisting of the bold lines in figure 3.9. Performing the side pairings, we find a once-punctured torus below made up from a four sided domain, and a once-punctured torus with a six sided domain above. By definition this is a group in quasi-Fuchsian space $\mathcal{Q}$, and so we have left semi-Fuchsian space $\mathcal{S}$ only to stay in $\mathcal{Q}$.

This should not seem surprising in view of the fact that $\mathcal{S}$ is one real codimension smaller than $\mathcal{Q}$. However, an important point about the space $\mathcal{S}$ is that in the definition of a semi-Fuchsian group in 2.1.1 we specifically asked for the marked generators A and B to be the transformations pairing the four circles. Thus $\mathcal{S}$ is not invariant under the group of generator moves $\mathcal{M}$, and we would need to consider all images of $\mathcal{S}$ under $\mathcal{M}$ to obtain all marked groups in $\mathcal{Q}$ with four tangent circles paired up by some pair of generators. We would then pick up a countable number of copies of $\mathcal{S}$ in $\mathcal{Q}$, but they do not fit together in any nice way. The Fuchsian groups, for instance, are semi-Fuchsian regardless of which generators we take, yet a group in figure 3.9 on the boundary $\partial\mathcal{S}$ may or may not be semi-Fuchsian under some generator pair, depending on whether or not the orbit under $\mathcal{M}$ of the point $(z, w) \in \mathcal{Q}$ representing the group ever falls into $\mathcal{S}$.

In short, we have done all we can to study semi-Fuchsian space $\mathcal{S}$ explicitly, but it is just too small to give us significant information about the whole of quasi-Fuchsian space $\mathcal{Q}$. Therefore we turn our attention to all of $\mathcal{Q}$ from now on, and in particular we look closely at the action on $\mathcal{Q}$ of the group $\mathcal{M}$.

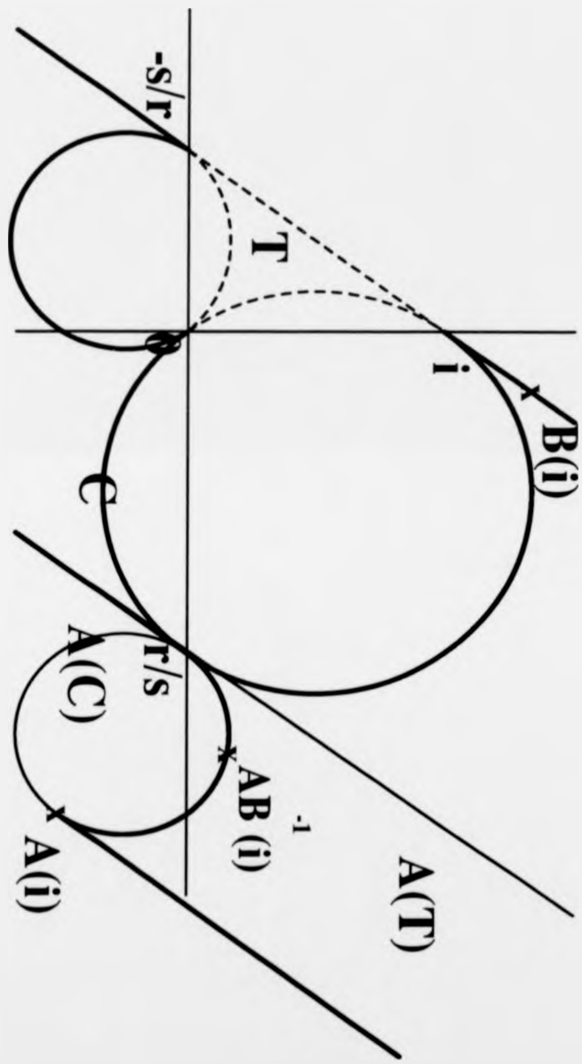

Figure 3.9: Changing the fundamental domain

Chapter 4

Changing Generators

4.1 Trace Reduction

In this chapter we are interested in finding out which trace parameters (a,b,c) give rise to a quasi-Fuchsian once-punctured torus group Γ. The first observation we can make is that the conformal structure of the quotient surface Ω/Γ is independent of any generators we may choose for Γ, and so we are searching for a region which is invariant under the generator moves. Thus what we would really like is a fundamental domain for the action of the group of generator moves $\mathcal{M}$ on quasi-Fuchsian space $\mathcal{Q}$. By the end of the chapter we will have obtained a substantial piece of such a fundamental domain, which will be made up of all quasi-Fuchsian once-punctured torus groups which are "well-behaved", in a definite sense.

First consider the submanifold of $\mathbb{C}^3$

$$\mathcal{Z} = \{(a,b,c) \in \mathbb{C}^3 |\, a^2 + b^2 + c^2 = abc\}.$$

The Euclidean topology on $\mathcal{Z}$ gives rise to the topology of algebraic convergence which we have already seen on $\mathcal{Q}$, so that if we have a sequence of triples $(a_n, b_n, c_n) \in \mathcal{Z}$ and then from these we form matrix triples (A_n, B_n, A_nB_n) with parabolic commutator, which are appropriately normalised and with the corresponding traces, then (a_n, b_n, c_n) tends to a limit (a,b,c) if and only if we have matrices A and B with $A_n \to A$ and $B_n \to B$ as elements of $PSL(2,\mathbb{C})$.

Using the fact that all norms on $\mathbb{C}^3$ give rise to the Euclidean topology, we take the max-norm

$$||(a,b,c)|| = \max(|a|,|b|,|c|) \qquad \text{for } (a,b,c) \in \mathbb{C}^3$$

and use this as a metric for $\mathcal{Z}$.

Note that if the orbit of any triple under $\mathcal{M}$ has infinitely many distinct triples contained in the set $B_K = \{(a,b,c) \in \mathcal{Z} |\max(|a|,|b|,|c|) \le K\}$ for some K, then as B_K is a compact set (being bounded and closed in $\mathcal{Z}$, which is closed in $\mathbb{C}^3$), we can find a convergent subsequence of distinct triples (a_n, b_n, c_n), all equivalent under $\mathcal{M}$. The group Γ represented by this orbit thus possesses a sequence of distinct elements which converge as matrices. Thus Γ cannot be a discrete subgroup of $PSL(2, \mathbf{C})$.

Now given an orbit of triples under the generating moves $\mathcal{M}$, we would like to have some way of finding a particular canonical triple in this orbit. The approach we adopt is a straightforward generalisation from Fuchsian group theory, where we have trace minimising algorithms for generators of groups. Dealing with complex numbers, we will attempt to reduce the moduli of the traces in a particular triple with the generator moves. For any triple $(a, b, c) \in \mathcal{Z}$ the choice of which generator move to make in order to reduce the element with the largest modulus is determined in the following way. First note that fixing any two elements in the triple yields a quadratic in the third. Thus we can replace any of the three elements with the other root of the corresponding quadratic equation (found by examining the sum of the roots). So we have moves of $\mathcal{M}$ which replace the triple (a, b, c) with its three neighbours

$$(a, b, ab - c)$$
$$(a, ac - b, c)$$
$$(bc - a, b, c).$$

We are really only interested in triples $(a, b, c) \in \mathcal{Z}$ up to permutation (and up to a pair of sign changes), and as the group $\mathcal{M}$ possesses all six permutations, we will order the elements of any triple so that $|a| \leq |b| \leq |c|$. Then we may move to a neighbouring triple, and in doing so we reorder if necessary with the appropriate move in $\mathcal{M}$. Thus starting with any triple, we obtain a binary tree of triples from its orbit under $\mathcal{M}$, where each vertex is connected by three edges.

Now suppose we have a triple $(a, b, c) \in \mathcal{Z}$, with $|a| \leq |b| \leq |c|$ and where a, b and c are large in modulus. Note that if $|a| > 2$ then

$$|bc - a| \geq |bc| - |a| \geq (|a| - 1)\,|c| > |c|$$

and so replacing a with $bc - a$ increases the maximum element of the triple. Similarly the condition $|a| > 2$ also implies that $|ac - b| > |c|$, so replacing b is no good either. The remaining possibility of removing c and hoping for a smaller alternative is covered in the following theorem, which underpins the whole of the remaining work.

Theorem 4.1.1

Given $(a, b, c) \in \mathcal{Z}$ with $3 < |a| \leq |b| < |c|$ then we have $|ab - c| < |b|$.

Proof of 4.1.1

Consider the quadratic equation

$$f(z) = z^2 - abz + a^2 + b^2$$

with roots c and $ab - c$. We use the principle of the argument to show that f has a root inside the circle $|z| = |b|$. This states that given functions $f(z), g(z)$ which are holomorphic on a region including the closed disc $|z| \leq R$, and if $0 < |g(z)| < |f(z)|$ for all z with $|z| = R$, then $f(z)$ and $f(z) - g(z)$ have the same number of zeros inside this disc.

So taking $f(z) = z^2 - abz + a^2 + b^2$ as above, and $g(z) = z^2 + a^2 + b^2$ with $R = |b|$, we obtain

$$|g(z)| \leq |a|^2 + 2|b|^2 \leq 3|b|^2 < |a||b|^2 = |f(z) - g(z)|$$

when $|z| = |b|$. Now $f(z) - g(z)$ has just the one root inside $|z| = |b|$, thus $f(z)$ does as well. □

Corollary 4.1.2

Every quasi-Fuchsian once-punctured torus group $\Gamma \subseteq PSL(2, \mathbf{C})$ possesses a generator A with $|\mathrm{tr}(A)| \leq 3$.

Proof of 4.1.2

Given a triple $(a, b, c) \in \mathcal{Z}$ representing Γ with $|a| \leq |b| \leq |c|$, assume $|b| \neq |c|$ and if $|a| > 3$ then apply theorem 4.1.1 repeatedly to reduce the largest modulus in the triple. If this process does not terminate then we obtain infinitely many distinct triples with a bound on their norms, and so by the comment at the beginning of the chapter Γ is not discrete. (In fact we shall see later that this process must terminate for any $(a, b, c) \in \mathcal{Z}$ with $2 < |a| \leq |b| \leq |c|$.)

To take care of the case $a \leq |b| = |c|$, note that

$$3|c|^2 \geq |a^2 + b^2 + c^2| = |abc| = |a|\,|c|^2$$

so here $|a| \leq 3$ already. □

We recall some facts in order to convert corollary 4.1.2 into a statement about the quotient 3-manifold $\mathbf{H}^3/\Gamma$.

The action of Γ can be extended from the Riemann sphere $\overline{\mathbf{C}}$ to

$$\mathbf{H}^3 = \{(x, y, z) \in \mathbf{R}^3 | z > 0\}$$

with the hyperbolic metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{z^2}$$

and then Γ acts as a group of isometries on $\mathbf{H}^3$. If Γ is a quasi-Fuchsian once-punctured torus group then we can form the quotient 3-manifold $\mathbf{H}^3/\Gamma$, which topologically is of the form $T \times (0, 1)$, with T a once-punctured torus.

Given a loxodromic element $A \in \Gamma$ then the unique geodesic line connecting the two fixed points is called the axis of A, and projects down to a closed geodesic inside the 3-manifold $\mathbb{H}^3/\Gamma$. We measure the length of this geodesic by conjugating A to take the form $x \mapsto \lambda^2 x$ for $\lambda \in \mathbb{C}, |\lambda| > 1$ (λ is called the multiplier of A) and checking the image of a point on the axis. We find that the length of the geodesic $l = 2\ln|\lambda|$.

Corollary 4.1.3

Every quasi-Fuchsian once-punctured torus group Γ has inside the 3-manifold $\mathbb{H}^3/\Gamma$ a closed geodesic with length l at most

$$\ln\left(\frac{11+3\sqrt{13}}{2}\right) = 2.38953...$$

Proof of 4.1.3

From corollary 4.1.2 there exists a generating loxodromic element A with $|\mathrm{tr}(A)| \leq 3$. As A is a generator, we obtain a closed geodesic inside $\mathbb{H}^3/\Gamma$ by projecting down its axis.

Then conjugating A to the form above, and setting $\lambda = re^{i\theta}$ with $r > 1$ gives

$$r - 1/r \leq \left|re^{i\theta} + \frac{1}{r}e^{-i\theta}\right| \leq 3.$$

Thus we must have $r^2 - 3r - 1 \leq 0$, so that r must lie in the interval $(1, 3/2+\sqrt{13}/2]$ and then $l \leq \ln(3+\sqrt{13})^2/4$. $\square$

Note: If we had only used Fuchsian once-punctured torus groups $G \subseteq PSL(2,\mathbb{R})$ then we could perform a similar argument with geodesics on the surface $\mathbb{H}^2/G$ to show that we always have a simple closed geodesic with hyperbolic length less than or equal to

$$\ln\left(\frac{7+3\sqrt{5}}{2}\right) = 1.92485...$$

Thus now we can turn our attention to triples $(a,b,c) \in \mathcal{Z}$ where at least one of the elements is small in modulus. Firstly we note that if any triple in the orbit of (a,b,c) under $\mathcal{M}$ contains a real number in the interval $[-2,2]$ then clearly $\Gamma((a,b,c))$ is not a quasi-Fuchsian once-punctured torus group, as we have a generator that is elliptic or parabolic. Also, the following unpublished result of Jørgensen [12] shows us that we can never have traces inside the unit disc.

Lemma 4.1.4

If $(a,b,c) \in \mathcal{Z}$ and $|a| < 1$, then the marked group $(\Gamma; A(z,w), B(z,w))$ with $z = a/c, w = b/c$ is not discrete.

Proof of 4.1.4

Adjoin the hyperelliptic involution H to Γ, so that the group $\Delta = <\Gamma, H>$ contains the parabolic element $x \mapsto x + c$, as in proposition 2.3.2. If Δ is discrete then form the conjugate group $C^{-1}\Delta C$ in $PSL(2, \mathbb{C})$, where $C(x) = cx$. This group now contains the element $x \mapsto x + 1$, and so we can apply Shimizu's Lemma.
But

$$C^{-1}AC = \begin{pmatrix} * & * \\ cz & * \end{pmatrix},$$

thus we conclude that $|cz| = |a| \geq 1$. $\square$

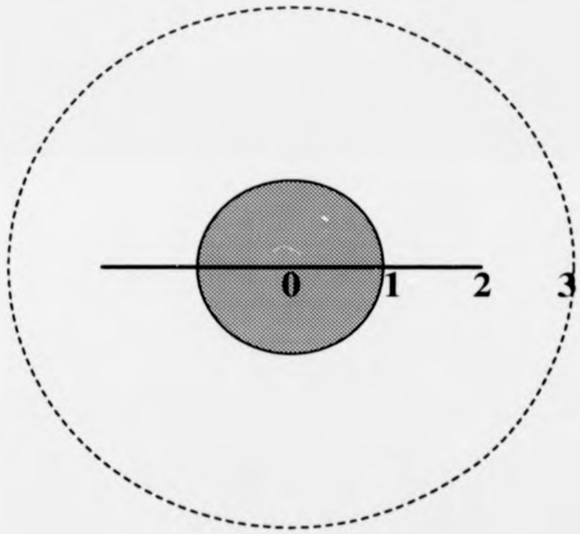

Figure 4.1: The Danger Area

Hence we have the "danger area" pictured in figure 4.1, from where the traces in the orbit of the triple (a, b, c) must stay away if (a, b, c) is to represent a quasi-Fuchsian once-punctured torus group. However, we know there exists a trace inside the closed disc of radius 3. Therefore in searching for a fundamental domain for the action of $\mathcal{M}$ on quasi-Fuchsian space $\mathcal{Q}$, it makes sense to try to find a fundamental domain containing triples with elements of small modulus, and preferably with elements whose moduli cannot be reduced any further. In order to suggest a suitable region, we first look in some detail at the Fuchsian once-punctured torus groups where (a, b, c) and (z, w) are real, and inspect the action of $\mathcal{M}$.

4.2 Fuchsian space

Fuchsian space, which we shall denote by $\mathcal{F}$, is of course easy to picture. Since we have seen in section 3.1 that every point $(z,w) \in \mathbf{R}\backslash\{0\} \times \mathbf{R}\backslash\{0\}$ gives rise to a Fuchsian once-punctured torus group, we obtain the four quadrants of $\mathbf{R}^2$ and thus by appropriate changes of sign we can take $\mathcal{F}$ to be the quadrant where $z, w > 0$. The orientation preserving group $\mathcal{M}_+$ is generated by

$$\begin{aligned} x : (z,w) &\mapsto (\frac{w}{z}, \frac{1}{z}) \\ y : (z,w) &\mapsto (\frac{w}{z^2+w^2}, \frac{z}{z^2+w^2}) \end{aligned}$$

and we have the presentation

$$\mathcal{M}_+ =< (x,y) | \, x^3 = y^2 = 1 >$$

so that $\mathcal{M}_+$ is in fact the modular group (see [14]).

We also have, of course, an alternative version of Fuchsian space using triples (a,b,c) of real numbers, namely

$$\mathcal{F} = \{(a,b,c) | \, a,b,c > 0, a^2 + b^2 + c^2 = abc\} \subseteq \mathbf{R}^3$$

with motions

$$\begin{aligned} x : (a,b,c) &\mapsto (b,c,a) \\ y : (a,b,c) &\mapsto (b,a,ab-c) \end{aligned}$$

generating $\mathcal{M}_+$, and we regard these two versions of Fuchsian space to be identical, swapping between the two coordinate systems at our convenience.

In the paper [13] a fundamental domain for $\mathcal{M}_+$ acting on $\mathcal{F}$ is shown to be

$$\mathcal{D} = \{(a,b,c) \in \mathcal{F} | \, a < c, b < c, c < \frac{ab}{2}\}$$

or, converting to (z,w) coordinates, we note that

$$c < \frac{ab}{2} \iff c^2 < abc - c^2 \iff \frac{a^2+b^2}{c^2} > 1$$

so we also have

$$\mathcal{D} = \{(z,w) \in \mathcal{F} | \, z < 1, w < 1, z^2 + w^2 > 1\}$$

and we can see (figure 4.2) that the closure of the fundamental domain $\overline{\mathcal{D}}$ has its sides paired by the elements x, x^2 and $y \in \mathcal{M}_+$.

Note: If we want a fundamental domain for the full group of generator moves $\mathcal{M}$, then we merely chop $\mathcal{D}$ in half along the line $z = w$ (or $a = b$).

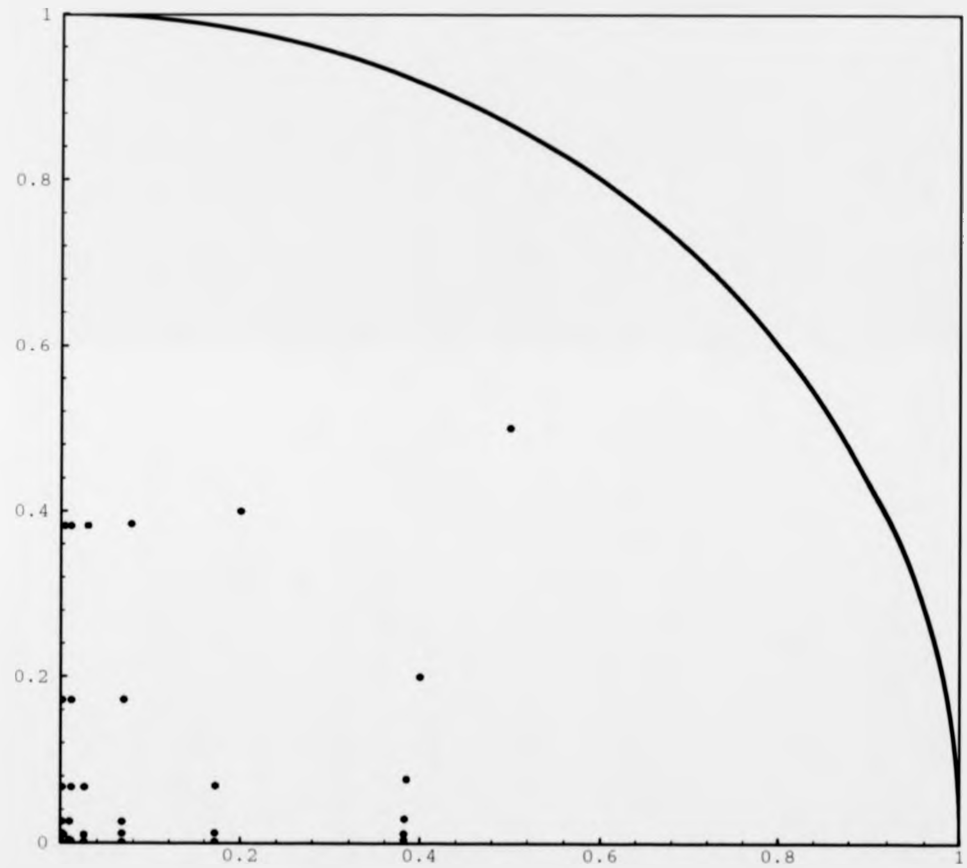

Figure 4.2: An orbit in Fuchsian space

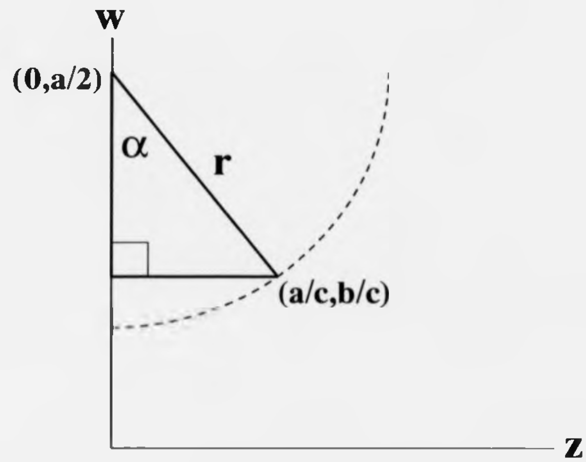

Figure 4.3: A point in Fuchsian space

We finish this section with some more information about how to picture the tessellation of $\mathcal{F}$ by images of $\mathcal{D}$, as these comments will have relevance to the final chapter where we consider triples (a, b, c) of positive integers.

First consider the orbit of the point $(1, 1)$, which is the vertex of the fundamental domain $\mathcal{D}$. A plot of its images under $\mathcal{M}$ with $z \leq 1$ and $w \leq 1$ (or $2 < a \leq c, 2 < b \leq c$) is shown in figure 4.2. Noting that the points seem to fall naturally into groups which could be joined up with a smooth curve leads us to try the following idea.

Fix a number $a > 2$, and consider all real triples (a, b, c) satisfying $a^2+b^2+c^2 = abc$. Then, to find the locus of this set in $\mathcal{F}$, we note that

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} + 1 = a\frac{b}{c}$$

so that

$$z^2 + w^2 + 1 = aw$$

or equivalently

$$z^2 + \left(w - \frac{a}{2}\right)^2 = \frac{a^2 - 4}{4} \tag{4.1}$$

which is a circle with centre $(0, \frac{a}{2})$ on the $w-$axis, and radius $\frac{\sqrt{a^2-4}}{2}$. Thus we have a foliation of $\mathcal{F}$ with semi-circles satisfying (4.1) and parametrised by a, whose radius tends to zero as we approach the point $(0, 1)$ on the $w-$axis. Incidentally, all these semi-circles are perpendicular to the unit circle $z^2 + w^2 = 1$.

If we have a marked Fuchsian group $(G; A, B)$ represented by (a, b, c) or (z, w), then recall from section 2.4 that the Dehn twist about A is the motion $yx^2 \in \mathcal{M}$.

which sends (a, b, c) to $(a, c, ac - b)$. So the motion fixes a, and hence each semi-circle satisfying (4.1) is invariant under the cyclic group generated by yx^2. Given one such point (a, b, c) lying on an invariant semi-circle, we look at the angle α it subtends at the centre, as in figure 4.3.

We have

$$\left(\frac{a}{2} - \frac{b}{c}\right)^2 + \frac{a^2}{c^2} = r^2$$

or

$$(ac - 2b)^2 + 4a^2 = c^2(a^2 - 4) \tag{4.2}$$

We could also derive equation (4.2) merely by rearranging $a^2 + b^2 + c^2 = abc$, but it will play an important rôle in chapter 5, and so rather than plucking this equation out of the air later on, we have given the original geometric motivation for its use.

The picture in figure 4.4 shows some of the images of the fundamental domain $\mathcal{D}$ under $\mathcal{M}_+$ inside the unit square $z \leq 1, w \leq 1$.

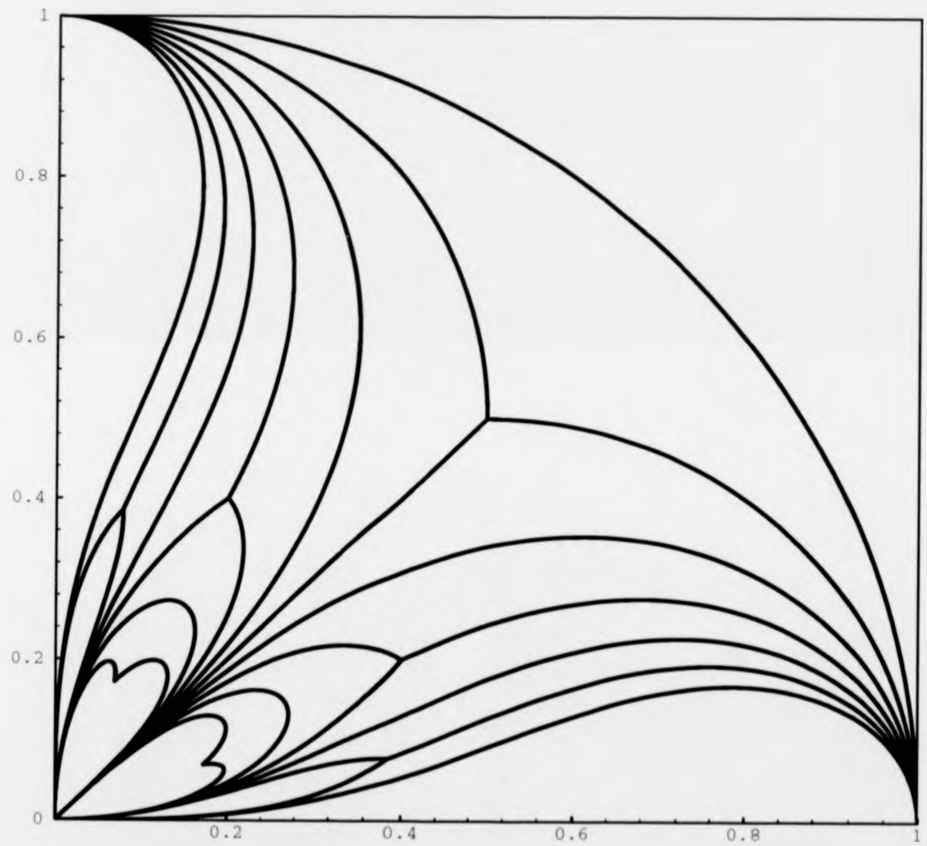

Figure 4.4: A fundamental domain in Fuchsian space

4.3 Well behaved Quasi-Fuchsian groups

Having seen a fundamental domain for the action of $\mathcal{M}_+$ on $\mathcal{F}$ we want to generalise this idea in the complex case, where $\mathcal{M}_+$ acts on

$$\mathcal{Z} = \{(a,b,c) \in \mathbb{C}^3 |\, a^2 + b^2 + c^2 = abc\}$$

or equivalently the parameter space $\mathcal{X} = \mathbb{C}\backslash\{0\} \times \mathbb{C}\backslash\{0\}$ for (z,w) coordinates.

A first attempt would be to take our real fundamental domain from the last section, and try to make it work for the complex case by sticking modulus signs around it, thus we now define

$$\mathcal{D} = \{(z,w) \in \mathcal{X} |\, |z| < 1, |w| < 1, |z^2 + w^2| > 1\}.$$

We have $\mathcal{D}$ as an open subset in $\mathcal{X}$ consisting of two connected components which are symmetric under the map $(z,w) \mapsto (z,-w)$, with the sides still paired by the elements x, x^2 and $y \in \mathcal{M}_+$, which looks promising. However, there is a problem in that not all points $(z,w) \in \mathcal{D}$ represent quasi-Fuchsian groups. For instance, taking $w = \lambda i$ with λ real and just less than one gives

$$|a| = \frac{|1 - \lambda^2 + z^2|}{\lambda} \leq \frac{1 - \lambda^2}{\lambda} + \frac{|z|^2}{\lambda}$$

which can be made arbitrarily small while (z,w) still remains in $\mathcal{D}$, and thus we are well inside the danger area in figure 4.1, containing indiscrete groups. In an attempt to surmount this problem, the idea is to refuse to consider all groups which contain any generator with a trace that is small in modulus. As for how small, we recall from corollary 4.1.2 that we can always find a trace with modulus three or less, whereas the danger area contains groups with a parabolic element having trace equal to two, and these cannot be quasi-Fuchsian groups.

Thus we remove all groups having the trace of a generator in the closed disc of radius two, and so obtain the following definition.

Definition 4.3.1

If we denote

$$\mathcal{A}_2 = \{(z,w) \in \mathcal{X} |\, \left|\frac{1 + z^2 + w^2}{w}\right| \leq 2\}$$

then we define the *reduced parameter space* $\mathcal{X}^*$ to be

$$\mathcal{X}^* = \mathcal{X} \backslash \bigcup_{g \in \mathcal{M}} g(\mathcal{A}_2).$$

We have probably thrown away many genuine quasi-Fuchsian groups in this process. These groups would have a loxodromic element which has a small translation length, but which also twists points around its axis. The idea now is to show that what we have left does consist of other quasi-Fuchsian groups, which have all traces of generators greater than two in modulus, and so are well behaved.

Now we have our reduced parameter space $\mathcal{X}^*$, we define

$$\mathcal{D}^* = \mathcal{D} \cap \mathcal{X}^* = \{(z,w) \in \mathcal{X}^* \mid |z| < 1, |w| < 1, |z^2 + w^2| > 1\}$$

However, by having removed all images of $\mathcal{A}_2$ from $\mathcal{X}$, we might find that $\mathcal{X}^*$ is disconnected, and so splits into many components. We thus proceed carefully by first examining points in $\mathcal{D}^*$.

First note that $\mathcal{D}^*$ is certainly non-empty, as all Fuchsian groups can be made to lie either in $\mathcal{D}^*$, or on its boundary. Now consider $(z,w) \in \mathcal{D}\backslash\mathcal{A}_2$, assuming without loss of generality that $|z| \leq |w|$, so that we have $|z| \leq |w| < 1$ and $|z^2 + w^2| > 1$. The first condition tells us that in terms of (a,b,c) coordinates we have $2 < |a| \leq |b| \leq |c|$, thus we have already arranged our triple in order of modulus. Looking closely at the condition $|z^2 + w^2| > 1$, we can see that this is equivalent to

$$|a^2 + b^2| > |c|^2 \iff |ab - c| > |c|$$

which is exactly the case when we are unable to reduce the element of largest modulus in the triple. Hence the following theorem tells us that the process of trace reduction on a triple (namely replacing the element of largest modulus in a triple with the smaller alternative, and then reordering if necessary) must in this case have come to an end.

Theorem 4.3.2

Given $(a,b,c) \in \mathcal{Z}$ with $2 < |a| \leq |b| \leq |c|$ and $|a^2 + b^2| > |c|^2$ then we have $|a| \leq 3$, and a, b and c are the three numbers smallest in modulus amongst the orbit of (a,b,c) under $\mathcal{M}$.

Proof of 4.3.2

As we have $|a^2+b^2| > |c|^2$ implying $|ab-c| > |c|$, we obtain from theorem 4.1.1 that $|a| \leq 3$. Now consider the binary tree created by the orbit of (a,b,c) under $\mathcal{M}$, factored out by permutations of (a,b,c). Starting at the point (a,b,c), we can remove c to obtain an element larger in modulus, or remove a or b, which since $|a| > 2$ also increases the modulus. Thus all three neighbouring vertices of (a,b,c) increase the norm of the triple. From here we can move in two directions that again increase the norm (by replacing either of the two smallest elements of the triple, each of which still has modulus greater than two), or we can get rid of the largest element of the triple, which takes us back the way we came. Continuing this process inductively, we cover the whole tree. □

Corollary 4.3.3

The reduced set $\mathcal{D}^* = \mathcal{D} \cap \mathcal{X}^*$ is formed from $\mathcal{D}$ by removing only the two sets

$$\mathcal{A}_2 = \{(z,w) \in \mathcal{X} \mid \left|\frac{1 + z^2 + w^2}{w}\right| \leq 2\}$$

and

$$\mathcal{B}_2 = \{(z,w) \in \mathcal{X} \mid \left|\frac{1 + z^2 + w^2}{z}\right| \leq 2\}.$$

Proof of 4.3.3

Having removed $\mathcal{A}_2$ and $\mathcal{B}_2$ from $\mathcal{D} = \{|z| < 1, |w| < 1, |z^2 + w^2| > 1\}$, which corresponds to having removed those triples (a, b, c) with $|a|$ or $|b| \leq 2$, any remaining point $(z, w) \in \mathcal{D}$ has its triple (a, b, c) satisfying the conditions of theorem 4.3.2. Thus there are no numbers in its orbit under $\mathcal{M}$ falling inside the disc of radius two, and so no translates of $\mathcal{A}_2$ or $\mathcal{B}_2$ are removed from $\mathcal{D}$ when forming $\mathcal{D}^*$. □

We are now ready to prove that $\mathcal{D}^*$ is a fundamental domain for $\mathcal{M}_+$ acting on $\mathcal{X}^*$. We have to show two conditions are satisfied.

Proposition 4.3.4

We have

$$\mathcal{D}^* \cap g(\mathcal{D}^*) = \emptyset \qquad \text{for all } g \in \mathcal{M}_+ \backslash \{id\}.$$

Proof of 4.3.4

If we have $g \in \mathcal{M}_+$ such that $(z, w) \in \mathcal{D}^* \cap g(\mathcal{D}^*)$, then (z, w) and $(z', w') = g^{-1}(z, w)$ are then both in $\mathcal{D}^*$. But from theorem 4.3.2 we have our corresponding triples (a, b, c) and (a', b', c') having minimum norms amongst their respective orbits under $\mathcal{M}_+$. So the only possibility is that $(a, b, c) = (a', b', c')$, but we cannot have any fixed points of $M_+ \backslash \{id\}$ in $\mathcal{D}^*$. □

Theorem 4.3.5

For any $(z, w) \in \mathcal{X}^*$, we can find $g \in \mathcal{M}_+$ with $g(z, w) \in \overline{\mathcal{D}^*}$, the closure in $\mathcal{X}^*$ of $\mathcal{D}^*$.

Proof of 4.3.5

Take a point $(z, w) \in \mathcal{X}^*$ and assume that we can keep on applying trace reduction to the largest element of the triple (a, b, c). If at some stage this is not possible, then we obtain $2 < |a| \leq |b| \leq |c|$ and $|ab - c| \geq |c|$, so that $(z, w) \in \overline{\mathcal{D}^*}$.

Otherwise we perform the trace reduction process infinitely often to obtain a sequence of triples (a_n, b_n, c_n) in the orbit of (a, b, c) with the following properties:

(1) $2 < |a_n| \leq |b_n| \leq |c_n|$ for all n.

(2) $(a_{n+1}, b_{n+1}, c_{n+1}) \begin{array}{ll} = & (a_n, a_n b_n - c_n, b_n) \\ \text{or} & (a_n b_n - c_n, a_n, b_n). \end{array}$

We aim to show that no such triple (a, b, c) with these properties can exist, as promised in the statement of theorem 4.1.1.

Firstly we have a sequence of distinct triples contained in a compact subset of our space of all triples $\mathcal{Z}$. Take a convergent subsequence $(a_{n_j}, b_{n_j}, c_{n_j}) \rightarrow$

$(a', b', c') \in \mathcal{Z}$.
Then note that

$$2 < |b_{n_{j+1}}| \le |c_{n_{j+1}}| < |b_{n_j}| \le |c_{n_j}|$$

giving decreasing sequences bounded below, and so we have

$$|b_{n_j}|, |c_{n_j}| \to K = |b'| = |c'| \tag{4.3}$$

with $K \ge 2$.

If $|b_n| < |a_n b_n - c_n| \le |c_n|$ for any n, then the next attempt at trace reduction returns us to the number c_n, thus $|a_{n+1}b_{n+1} - c_{n+1}| \ge |c_{n+1}|$ and the process terminates. Hence we obtain

$$|b_{n_{j+1}}| \le |a_{n_j} b_{n_j} - c_{n_j}| \le |b_{n_j}|$$

and so we also have

$$|a_{n_j} b_{n_j} - c_{n_j}| \to K. \tag{4.4}$$

Now if we write $(z', w') = (a'/c', b'/c') \in \mathcal{X}$ in the usual way, then $|w'| = 1$ from (4.3) and $|z'^2 + w'^2| = 1$ from (4.4). Thus $|a'| = |(1 + z'^2 + w'^2)/w'| \le 2$. If we do not have equality then

$$2 < |a_{n_j}| \to |a'| < 2$$

which is absurd.

Otherwise, if $|a'| = 2$ then note $|a_{n_j}|$ cannot be eventually constant, thus we have $|a'| = |b'| = |c'| = 2$ and $z'^2 + w'^2 = 1$. So we find $(z', w') = (e^{i\pi/6}, e^{-i\pi/6})$ and $(a', b', c') = (\sqrt{3} + i, \sqrt{3} - i, 2)$.

Then take a triple (a_n, b_n, c_n) less than distance ϵ from $(\sqrt{3}+i, \sqrt{3}-i, 2)$. Considering $(a_{n+1}, b_{n+1}, c_{n+1})$ and two of its neighbours, (a_n, b_n, c_n) and $(a_{n+2}, b_{n+2}, c_{n+2})$, we conclude that (a', b', c') must have at least two neighbours with all elements having modulus close to two.

But $2(\sqrt{3} + i) - (\sqrt{3} - i) = \sqrt{3} + 3i$ and $2(\sqrt{3} - i) - (\sqrt{3} + i) = \sqrt{3} - 3i$ have modulus $\sqrt{12}$, so contradicting the existence of the only other possibility. □

Summarising the arguments in this section, we find that a quasi-Fuchsian once-punctured torus group Γ in $\mathcal{Q}$ either has a generator with the modulus of its trace less than or equal to two, in which case we find a short closed simple geodesic in the 3-manifold $\mathbb{H}^3/\Gamma$, or we can choose a marking for Γ consisting of the two generators with the smallest trace moduli, and then the (z, w) parameter for our group Γ lies in the closure of our fundamental domain $\mathcal{D}^*$.

However, one question we would like to answer in the affirmative is whether or not all points $(z, w) \in \overline{\mathcal{D}^*}$ represent quasi-Fuchsian once-punctured torus groups. This would seem reasonable as we know the traces of all generators for groups in $\overline{\mathcal{D}^*}$ do not accumulate and stay bounded away from ± 2, but in general we would need more information to show the groups are discrete.

But in this case if we assume an appropriate conjecture then we can obtain our result with little extra effort.

Definition 4.3.6

A *cusp group* is a marked group on the boundary of quasi-Fuchsian space $\partial\mathcal{Q}$ where there exists a generator which is parabolic.

Thus there must be a triple of the form $(\pm 2, b, c)$ in an orbit of points in $\mathcal{Z}$ that represent a cusp group.

Conjecture 4.3.7

Cusp groups are dense in the boundary of quasi-Fuchsian space $\partial\mathcal{Q}$.

This seems reasonable, in view of the recent result that cusp groups are dense in the boundary of the Bers embedding of the Teichmüller space of a Fuchsian group [22]. Since the theorem of simultaneous uniformisation by Bers tells us that quasi-Fuchsian space $\mathcal{Q}$ is biholomorphically equivalent to $T_{1,1} \times T_{1,1}$, where $T_{1,1}$ is the Teichmüller space of the Fuchsian once-punctured torus, then by fixing the conformal structure on the downstairs torus while letting the top torus vary, we obtain a copy of $T_{1,1}$ inside $\mathcal{Q}$, called a Bers slice.

Now in the Bers embedding, we regard $T_{1,1}$ as a subset of $B(L,G)$, where $B(L,G)$ is the space of bounded quadratic differentials in the lower half plane L for a fixed Fuchsian once-punctured torus group G. This is because taking $\phi \in B(L,G)$ and solving the Schwarzian differential equation for ϕ yields the meromorphic function $w_\phi : L \to \overline{\mathbb{C}}$, with which we can form the homomorphism $\chi_\phi : G \to G$ defined by $\chi_\phi(G) w_\phi = w_\phi(G)$. Then $T_{1,1}$ is the set of $\phi \in B(L,G)$ such that we can extend $w_\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ to a quasiconformal homeomorphism of the Riemann sphere compatible with G. Thus $G' = \chi_\phi(G) = w_\phi G {w_\phi}^{-1}$ is a quasi-Fuchsian group, with the bottom quotient surface conformally equivalent to L/G, but the top quotient surface is no longer U/G.

Now if we take $\phi \in \partial T_{1,1} \subseteq B(L,G)$, the paper of Bers [2] tells us that the homomorphism $\chi_\phi : G \to G'$ is an isomorphism, and also that given any $g \in G$ the map $\mathcal{T} : B(L,G) \to \mathbb{C}$ given by $\mathcal{T}(\phi) = \mathrm{tr}(\chi_\phi(g))^2$ is holomorphic.

So given $\phi_i \to \phi \in B(L,G)$ with $\phi \in \partial T_{1,1}$ and ϕ_i a sequence in $T_{1,1}$, we know that $\chi_{\phi_i}(G)$ are groups with a marking of generators, and so are points in quasi-Fuchsian space $\mathcal{Q}$. Also $\chi_\phi(G)$ is in our parameter space $\mathcal{P}$ since it is isomorphic to G, and taking $g \in G$ to be the parabolic commutator, we have the sequence

$$t_i = \mathrm{tr}(\chi_{\phi_i}(g))^2 = 4$$

for all i, and $\phi_i \to \phi$ implies

$$t_i \to t = \mathrm{tr}(\chi_\phi(g))^2 = 4$$

by holomorphicity of the map t.

Then taking the marked generators of $\chi_{\phi_i}(G) \in \mathcal{Q}$ and examining the limit of their traces in the same way gives $\chi_{\phi_i}(G) \to \chi_\phi(G)$ in the topology of parameter space $\mathcal{P}$, thus we conclude that $\phi \in \partial T_{1,1}$ implies the group $\chi_\phi(G) \in \partial\mathcal{Q}$, and also if we have $\phi, \phi_i \in T_{1,1} \cup \partial T_{1,1}$ with $\phi_i \to \phi$ then $\chi_\phi(G), \chi_{\phi_i}(G) \in \mathcal{Q} \cup \partial\mathcal{Q}$ with $\chi_{\phi_i}(G) \to \chi_\phi(G)$.

Theorem 4.3.8

Assuming 4.3.7, all groups in the closure in $\mathcal{X}^*$ of our fundamental domain

$$\overline{\mathcal{D}^*} = \{(z,w) \in \mathcal{X}^* \mid |z| \leq 1, |w| \leq 1, |z^2 + w^2| \geq 1\}$$

are quasi-Fuchsian groups.

Proof of 4.3.8

We note that $\overline{\mathcal{D}^*}$ falls into four connected components, each of which is equivalent under the maps $(z,w) \mapsto (\pm z, \pm w)$, and so each component represents the same marked groups.

Fix a point in $\overline{\mathcal{D}^*}$ already known to be quasi-Fuchsian (say a Fuchsian group) and join it by a path lying inside this component of $\overline{\mathcal{D}^*}$ to any other point. If this point is not in $\mathcal{Q}$ then there must exist a point (z,w) on the path with $(z,w) \in \overline{\mathcal{D}^*} \cap \partial\mathcal{Q}$. Now no point in $\mathcal{X}^*$ can be a cusp group, as these have a trace equal to two and so have been removed when passing from $\mathcal{X}$ to $\mathcal{X}^*$. Thus (z,w) is not a cusp group, and we can take a set open in $\mathcal{X}$ about (z,w) completely contained in $\mathcal{X}^*$ (to see this, think of the regions $\mathcal{D}, x(\mathcal{D}), x^2(\mathcal{D})$ and $y(\mathcal{D})$ stuck together with the side pairings to give an open set in $\mathcal{X}$ containing $\overline{\mathcal{D}^*}$, and then we remove eight closed sets when we move to our reduced parameter space, thus still retaining an open set in $\mathcal{X}$ containing $\overline{\mathcal{D}^*}$). Thus no sequence of cusp groups can tend to (z,w). □

4.4 Groups on the Boundary of Quasi-Fuchsian space

We finish this chapter with a list of the types of group appearing on $\partial\mathcal{Q}$, classified according to their quotient surfaces.

Any group in the whole parameter space $\mathcal{P}$ is generated by two elements having parabolic commutator. If Γ lies on the boundary $\partial\mathcal{Q}$ then a theorem of Chuckrow [6] tells us that Γ is isomorphic to groups in $\mathcal{Q}$, thus is the free group on two generators, and results of Jørgensen [11] say that Γ is discrete.

First we ask whether Γ is geometrically finite. If so then Γ must be Kleinian, for if $\Omega(\Gamma) = \emptyset$ then $\mathbb{H}^3/\Gamma$ has finite volume, and so Mostow rigidity (or Marden's isomorphism theorem) implies that Γ is determined uniquely up to conjugation. But there are plenty of non-conjugate groups which are free on two generators.

In the geometrically finite case, Γ must contain accidental parabolics, otherwise we are inside $\mathcal{Q}$ rather than on the boundary. If Γ has one accidental parabolic, then we have an invariant component of $\Omega(\Gamma)$ with quotient surface a once-punctured torus, and infinitely many conjugate components with quotient surface a thrice-punctured sphere, as in figure 3.6. Otherwise Γ has two accidental parabolic generators, giving rise to $\Omega(\Gamma)$ a circle packing of two conjugacy classes of components, each of which yields a thrice-punctured sphere. An example would be the point $(z, w) = (\sqrt{2}e^{i\pi/4}/2, \sqrt{2}e^{-i\pi/4}/2)$ or $(a, b, c) = (2, 2, 2-2i)$ in proposition 3.2.1.

If Γ is geometrically infinite then stranger things happen, and in fact we can think of these groups as having quotient surfaces where components have vanished completely.

First we ask whether Γ is Kleinian. If so, we can examine the quotient surface $\Omega(\Gamma)$ and again ask whether we have accidental parabolics.

We can have only one conjugacy class of parabolic generators, in which case Ω is made up of a circle packing, but with only one conjugacy class of components. Thus we have a quotient surface of one thrice-punctured sphere, with the other component having disappeared. Examples of these groups can be seen on the boundary of the Maskit embedding of the Teichmüller space of the once-punctured torus; they are represented by the points on the boundary between the rational pleating curves on figure 1 in [15], but explicitly writing down the traces is a hard problem.

If there are no accidental parabolics then the paper of Bers [2] tells us that we have a degenerate group, where $\Omega(\Gamma)$ is connected and simply connected. Again the quotient surface consists of only one component, which is a once-punctured torus.

"Most" groups on ∂Q are of this form, yet not a single constructive example of a degenerate group is known. We note that if Γ is a geometrically finite group then the multipliers of conjugacy classes of loxodromic elements in Γ do not accumulate (see [17]), and so we cannot have infinitely many traces of generators of Γ appearing in a bounded region of $\mathbb{C}$. However this might not be the case for geometrically infinite groups, so there would be interest in finding an orbit of a triple (a, b, c) which does not accumulate in $\mathbb{C}^3$ (otherwise we have an indiscrete group), but where the individual traces do accumulate in $\mathbb{C}$.

Finally, the other possibility is that Γ is geometrically infinite but not Kleinian, so that $\mathbb{H}^3/\Gamma$ has infinite volume and $\Omega(\Gamma) = \emptyset$. Now both quotient surfaces have disappeared.

Chapter 5

Uniqueness of the Prime Markoff numbers

5.1 Introduction and History

For the last chapter we specialise in looking at triples (a, b, c) of natural numbers which solve our trace equation $a^2 + b^2 + c^2 = abc$. We shall see in lemma 5.2.1 that these triples are precisely the orbit of the triple $(3, 3, 3)$ under the group of generating moves $\mathcal{M}$.

First we remove the factors of three that occur, using the following lemma.

Lemma 5.1.1

If $(a, b, c) \in \mathbb{N}^3$ satisfies

$$a^2 + b^2 + c^2 = abc$$

then $a = b = c = 0 \bmod 3$, and redefining (a, b, c) by cancelling out the common factor gives

$$a^2 + b^2 + c^2 = 3abc$$

Proof of 5.1.1

Suppose one of (a, b, c), say a, is $0 \bmod 3$. Then $b^2 + c^2 = 0 \bmod 3$, but $b \neq 0 \bmod 3$ implies $b^2 = 1 \bmod 3$ and then we have no solutions. So b, and thus c, are divisible by three.

If no number is divisible by three then $a^2+b^2+c^2 = 0 \bmod 3$ but $abc \neq 0 \bmod 3$, so there are no solutions here. □

The Diophantine equation

$$a^2 + b^2 + c^2 = 3abc$$

possesses an interesting history, having appeared in many seemingly unrelated areas of mathematics. It was first examined in detail by A. A. Markoff (see [18] and [19]) in 1879 and 1880, when studying the theory of quadratic forms. He showed that given any solution (a, b, c) in positive integers of the above equation, we can create a binary tree of solution triples, just as we have done earlier under the orbit of $\mathcal{M}$. Conversely, he also showed that all solutions of the equation must appear in this tree somewhere and so was able to find, by direct calculation, a range of examples.

Here one question emerged which the evidence uncovered so far seems to support. Namely, if we arrange all solution triples of positive integers (a, b, c) in ascending order then is it true that the largest number c determines the triple uniquely? In other words if we pick triples (a, b, c) and (a', b', c') appearing in the tree and find that $c = c'$ then can we conclude that $a = a'$ and $b = b'$? If not, the number c then appears in different places in the tree as the largest element of a triple.

More information about the equation was discovered in 1913 by Frobenius [9] who wrote about the results of Markoff and also mentioned the above conjecture. The book by Cassels [5], published in 1957, provides a comprehensive introduction to these results and also brought the conjecture to the attention of many mathematicians. He states "No one has shown that there cannot be two distinct solutions (a, b, c), (a', b', c) occuring in different parts of the tree. No case of this is known, and it seems improbable. However, there is no ambiguity in practice. "

Later, with the advent of computers, Rosen and Patterson [23] in 1971 provided some numerical evidence by performing direct calculations and showing there were no duplications when the largest number c was below 10^{30}. This was pushed higher using modular arithmetic to 10^{105} by Borosh [3], but, as he noted in the paper, "it should be pointed out that there is actually no theoretical background to support the conjecture."

Then, in 1976, the problem seemed solved, by the paper of Rosenburger [24]. He used ideas on the traces of matrices in $SL(2, \mathbb{Z})$ to prove the Markoff conjecture. However, a mistake was found in one line when reviewed by Bumby [4], and with that the whole proof came crashing down. Since then, Zagier ([28] in 1982) has produced results on the size of growth of these solutions but we are aware of no other results on the uniqueness question.

For the last result in this thesis, we show that if the the largest number c is prime then we can conclude uniqueness. The proof was motivated by work on quasi-Fuchsian space in the previous chapters and we shall try to point out the connections as they appear, but the technical machinery of the proof uses only algebraic number theory.

Finally we mention other branches of mathematics where this equation is important. Just as this approach originated in hyperbolic geometry, so it crops up in the theory of quadratic forms and in Diophantine approximation (see [5] [18] [19] and [9]), in word reduction problems for free groups [7], in the Atiyah-Singer theorem on four-dimensional manifolds [10], also when looking at closed geodesics on a Riemann surface [1] and in algebraic geometry, such as when looking at exceptional bundles on the projective plane [25]. In all the above cases, knowing the truth of the Markoff conjecture would provide information in those subjects.

5.2 Basic Results and Motivation

As we have seen, the material being discussed owes its origins to A. A. Markoff, therefore we introduce some terminology borrowed from various authors, all with the intention of naming as much as possible after the founder.

So the equation

$$a^2 + b^2 + c^2 = 3abc \tag{5.1}$$

which we have already seen many times before (once we have remembered to divide by three) will henceforth be known as the Markoff equation. We refer to triples (a, b, c) of positive integers which solve (5.1) as Markoff triples, and any natural number appearing in such a triple is a Markoff number.

We have already seen the Markoff moves in chapters 2 and 4, where we replace any element of a triple (a, b, c) with an alternative solution, called a neighbouring triple. These are now of the form

$$\begin{aligned} &(3bc - a, b, c) \\ &(a, 3ac - b, c) \\ &(a, b, 3ab - c). \end{aligned} \tag{5.2}$$

Next (to remove permutations of the same solution) we know we will often want to consider Markoff triples (a, b, c) which have been written in ascending order, so that $1 \le a \le b \le c$, and we introduce the notation ${}^{a}c^{b}$ for when this is the case. We are particularly interested in the maximum element of any triple and we adapt the comment before theorem 4.1.1 to show that the two neighbours of ${}^{a}c^{b}$ where a is replaced by $3bc - a$ and where b is replaced by $3ac - b$ have maximum elements larger than c. However by replacing c with $3ab - c$, it is shown in [5] that we obtain a neighbouring triple with a lower maximum element, apart from when $c = 1$.

Starting with the solution ${}^{1}1^{1}$ we apply the the transformations (5.2) to obtain ${}^{1}2^{1}$, then ${}^{1}5^{2}$ and from then on we create two new solutions from each previous triple, giving rise to a binary tree (see figure 5.1) called the Markoff tree.

Lemma 5.2.1

(A) An ordered triple solves the Markoff equation if and only if it appears in the Markoff tree.

(B) The elements of a Markoff triple (a, b, c) are pairwise coprime.

(C) All Markoff numbers are 1 or 2 mod 4.

Proof of 5.2.1

(A) Given a Markoff triple ${}^{a}c^{b}$ solving (5.1) we choose its neighbour with maximum element less than c. We continue this process inductively, however, it must terminate at the stage when the maximum element can be reduced no further. We find that this can only happen at ${}^{1}1^{1}$ which is at the top of the Markoff tree.

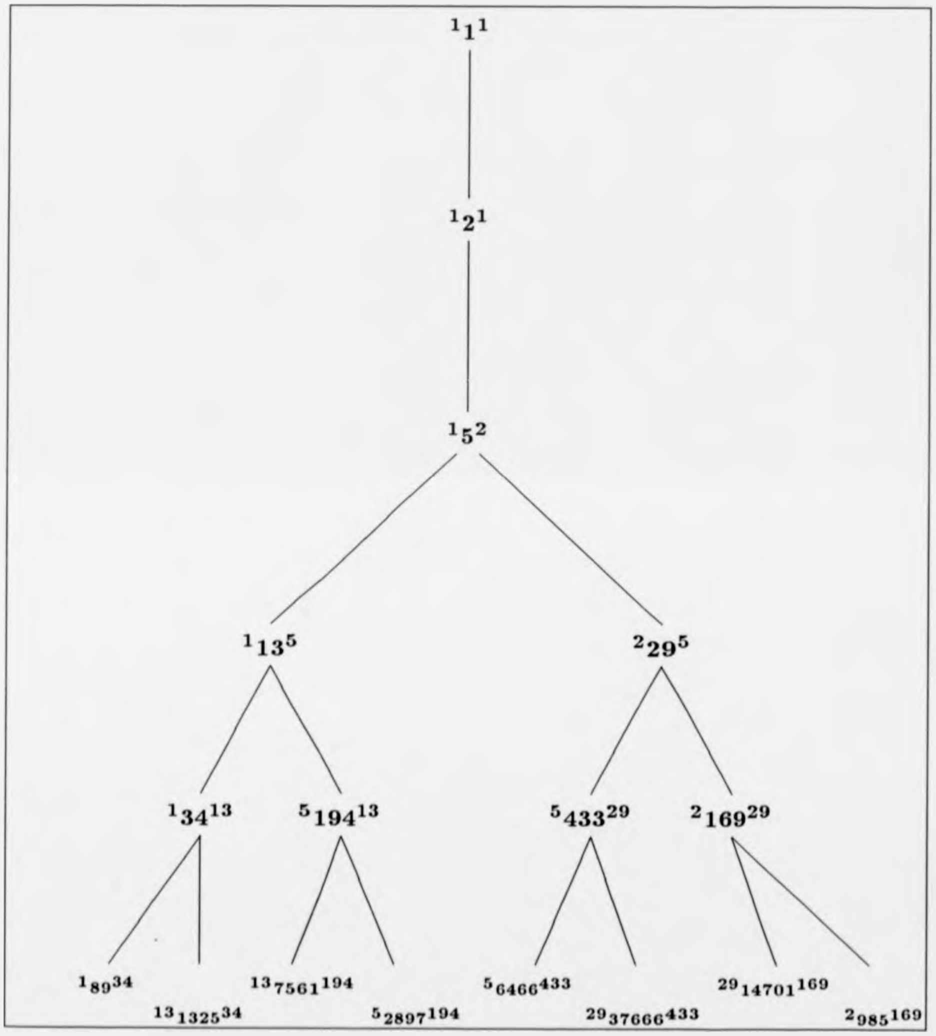

Figure 5.1: The Markoff Tree

(B) The transformation rules (5.2) preserve common factors between two numbers in a triple, but from (A) we can transform any Markoff triple into ${}^{1}1^{1}$.

(C) Either we can view the tree mod 4, terminating a branch when a triple is repeated, or use a simple proof of Frobenius:

We have $a^2 + b^2 = c(3ab - c)$ so c divides the sum of two squares of coprime integers. Thus c can only be a number with prime factors all 1 mod 4, or be double this. □

The proof of the Markoff conjecture (stating that any Markoff number c appears only once in the tree as the maximal element of a triple) for prime Markoff numbers in this chapter is based on the following idea. Fix any Markoff number c and wait for its first appearance, which will be as the largest element in a triple ${}^{a}c^{b}$. Then there are branches of the tree descending from ${}^{a}c^{b}$ which will retain c as a smaller element in each triple. We form the ${}^{a}c^{b}$-chain, which is defined to be all numbers appearing in any Markoff triple with c which is descended from ${}^{a}c^{b}$.

Now if c were to falsify the conjecture and appear elsewhere in the tree as ${}^{a'}c^{b'}$ then we would similarly get a ${}^{a'}c^{b'}$-chain of numbers and so we would obtain one chain for each occurrence of c as a maximal element. However, we now recall the pictures of Fuchsian space in section 4.2. Remember that fixing an element of a triple gave us the locus of a circle in (z, w) space, on which the (z, w) coordinates of all these triples must lie. In fact, we have seen these points in figure 4.2.

Now each ${}^{a}c^{b}$-chain is made up of numbers satisfying a second order recurrence relation (as we shall see in lemma 5.3.2). Then thinking about how the angle in the circle varies as in figure 4.3, we obtain a Diophantine equation (D_c) related to Pell's equation, where the solutions are precisely all the Markoff numbers that appear in a triple with c. So this solution set is the union of all the chains of c, but cannot distinguish between different chains. By relating the two different descriptions we reduce the conjecture to the problem of a particular factorisation in an appropriate real quadratic number field depending on c. Then when c is prime we use standard theorems in algebraic number theory to perform this factorisation, which shows uniqueness.

Section 5.3 defines and describes these chains and section 5.4 looks at the equation (D_c), which is a generalisation of Pell's equation. Then we examine the appropriate quadratic number fields and hence obtain our result.

5.3 The ${}^{a}c^{b}$-Chain

When our Markoff number c appears as the largest element of a triple ${}^{a}c^{b}$, we know we can derive two new solutions by removing either a or b, letting c take its place, and making the new maximal element (either $3bc - a$ or $3ac - b$) appear where c was. At the next stage we have four derived solutions, two of which lose c altogether, and the other two retain c as smallest element. We can keep c in a sequence of triples by following the two branches descending from these two solutions (see figure 5.2).

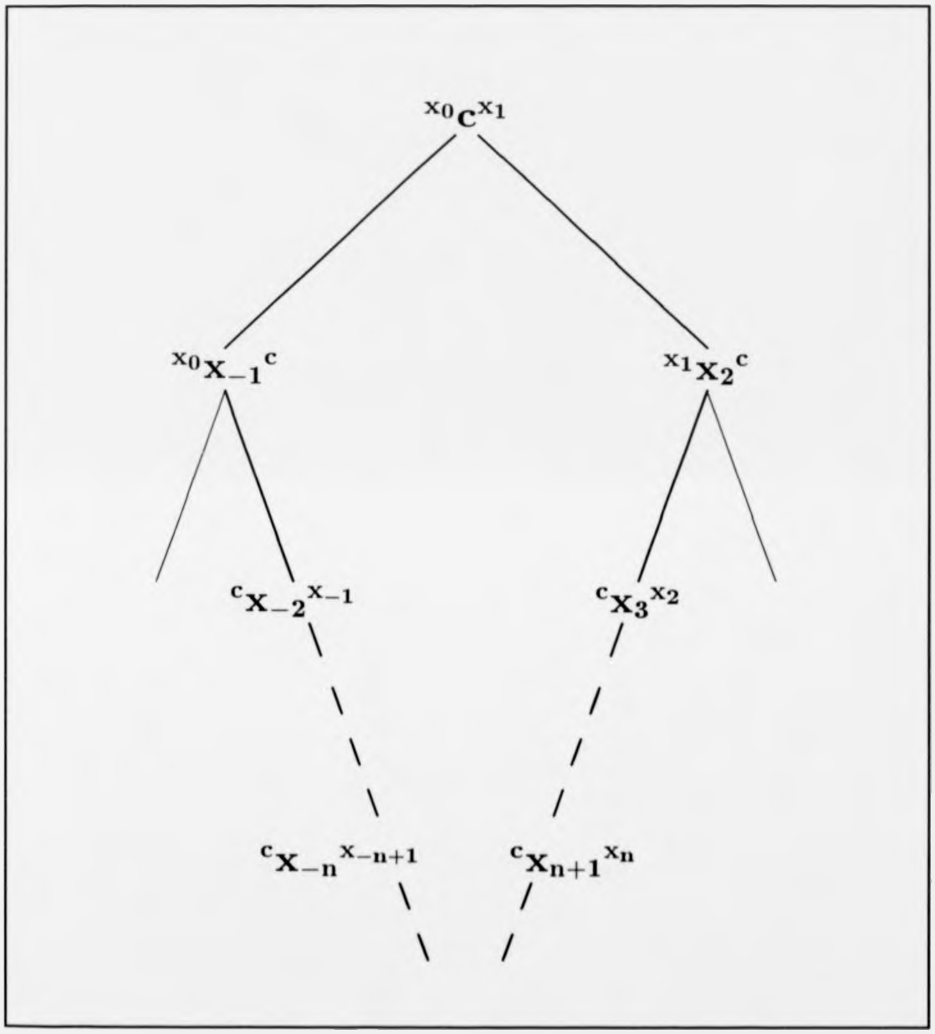

Figure 5.2: The ${}^a c^b$-chain

Note the convention in placing the two derived solutions relative to their upstairs neighbour: if we retain the smallest element then we continue in the same direction as before. Note also that this discussion is not true for the singular solutions ${}^{1}1^{1}$ and ${}^{1}2^{1}$ (but then we already know that 1 and 2 appear only once as the largest element).

Now we define the ${}^{a}c^{b}$-chain $\chi({}^{a}c^{b})$:

Definition 5.3.1

$\chi({}^{a}c^{b})=\{x \in \mathbb{N} | x$ appears with c in a Markoff triple that has been derived from the triple ${}^{a}c^{b}\}$

$\chi(c)$ is the union of all $\chi({}^{a}c^{b})$ for which c is the maximal element.

We have $a, b \in \chi({}^{a}c^{b})$ but $c \notin \chi({}^{a}c^{b})$ (unless $c = 1$). Also $\chi(c)$ is just the set of Markoff numbers appearing in any triple with c.

In fact we can describe $\chi({}^{a}c^{b})$ more succinctly by noting:

Lemma 5.3.2

$$\chi({}^{a}c^{b}) = \{\ldots, x_{-1}, x_0, x_1, x_2, \ldots\}$$

where (x_n) is the doubly infinite sequence defined by the second order recurrence relation

$$x_{n+1} = 3cx_n - x_{n-1} \tag{5.3}$$

and the initial conditions $x_0 = a, x_1 = b$.

Proof of 5.3.2

The recurrence relation (5.3) is just simply the transformation of the triple (c, x_{n-1}, x_n) into (c, x_n, x_{n+1}) as we descend the ${}^{a}c^{b}$-chain. The symmetry of (5.3) between $n - 1$ and $n + 1$ also means the relation works backwards, and we can check that allowing for order we have:

$$\ldots \longrightarrow {}^{c}x_{-2}{}^{x_{-1}} \longrightarrow {}^{x_{-1}}x_0{}^{c} \longrightarrow {}^{x_0}c^{x_1} \longrightarrow {}^{x_1}x_2{}^{c} \longrightarrow {}^{c}x_3{}^{x_2} \longrightarrow \ldots$$

5.4 The Diophantine Equation (D_c)

We continue the idea of considering a fixed Markoff number c, and we let a and b be the other two elements of any Markoff triple containing c (so now c is not necessarily the maximum element).

We have:

$$a^2 + b^2 + c^2 = 3abc$$

so

$$4a^2 - 12abc + 4b^2 = -4c^2$$

and then

$$(2b - 3ac)^2 - (9c^2 - 4)a^2 = -4c^2.$$

Note that this has been seen before as equation (4.2), with c and a swapped and the factor of three taken out.

Now if we set $X = |2b - 3ac|$ and $Y = a$ then

$$X^2 - (9c^2 - 4)Y^2 = -4c^2. \qquad (D_c)$$

Theorem 5.4.1

Any Markoff numbers a and b appearing in a Markoff triple with c give rise to a solution (X, Y) of (D_c) in positive integers. Conversely, any such solution of (D_c) allows us to find a solution triple of the Markoff equation which includes c.

Proof of 5.4.1

We only need to show the converse. Given (X, Y) we would like to set $a = Y$ and $b = (X + 3Yc)/2$ to form the triple (a, b, c). This we can do, provided $X + 3Yc$ is even. But $X^2 - (9c^2 - 4)Y^2$ is even so we have:

Case (1): c=1 mod 4

Then X^2 and Y^2, hence X and Y, must be both even or both odd. Either way, $X + 3Yc$ is even.

Case (2): c=2 mod 4

Here X^2 is even and thus so is $X + 3Yc$.

Now we can reverse the derivation of (D_c) and show (a, b, c) satisfies the Markoff equation. □

Corollary 5.4.2

The set $\{Y \mid (X, Y) \text{ solves } (D_c) \text{ with } X \text{ and } Y \text{ positive integers}\}$ equals the set $\chi(c)$. □

Corollary 5.4.3

If (X, Y) solves (D_c) then X and Y are coprime, apart from a possible factor of 2.

Proof of 5.4.3

If p divides X and Y then it also divides both a and b, which appear together in a triple. □

We set $N(c) = 9c^2 - 4$ and note that the equation $X^2 - NY^2 = -4c^2$ is a generalisation of Pell's equation $X^2 - NY^2 = \pm 1$. We now review the chief features of Pell's equation so that we can apply them in order to describe the solutions of (D_c).

First we obtain information from the continued fraction expansion of $\sqrt{N}$:

Theorem 5.4.4

(A) Given a positive integer N which is not a perfect square, the continued fraction expansion of $\sqrt{N}$ is of the form

$$\sqrt{N} = [q_0, \overline{q_1, q_2, \ldots, q_2, q_1, 2q_0}]$$

where q_0 is the integer below $\sqrt{N}$ and the bar denotes a periodic expansion.

We set $X_0/Y_0 = [q_0, q_1, q_2, \ldots, q_2, q_1]$.

(B) If the symmetric part $q_1, q_2, \ldots, q_2, q_1$ has an odd number of terms then $X = X_0$ and $Y = Y_0$ is the smallest solution of $X^2 - NY^2 = +1$. All solutions of this equation in positive integers are given by

$$(X_0 + \sqrt{N}Y_0)^n, \qquad n \in \mathbb{N}$$

Also the equation $X^2 - NY^2 = -1$ is insoluble.

(C) If the symmetric part has an even number of terms then $X = X_0$ and $Y = Y_0$ is the smallest solution of $X^2 - NY^2 = -1$. All solutions in positive integers to this equation are given by

$$(X_0 + \sqrt{N}Y_0)^n, \qquad \text{for } n \text{ odd}$$

and to $X^2 - NY^2 = +1$ by

$$(X_0 + \sqrt{N}Y_0)^n, \qquad \text{for } n \text{ even.}$$

(D) If we write $X_n + \sqrt{N}Y_n = (X_0 + \sqrt{N}Y_0)^n$ then we have the recurrence relations

$$\begin{aligned} X_{n+1} &= 2X_0X_n - X_{n-1} \\ Y_{n+1} &= 2X_0Y_n - Y_{n-1} \end{aligned} \tag{5.4}$$

Proof of 5.4.4

(A), (B), (C) can be found in many classical number theory texts, e.g. Davenport [8]. We will look at a generalisation of the proof of (B) in section 5.5.

(D) For any $n \in \mathbb{Z}$ we have

$$\begin{aligned} & 2X_0X_n - X_{n-1} + \sqrt{N}(2X_0Y_n - Y_{n-1}) \\ = \; & 2X_0(X_n + \sqrt{N}Y_n) - (X_{n-1} + \sqrt{N}Y_{n-1}) \\ = \; & (X_{n-1} + \sqrt{N}Y_{n-1})(2X_0(X_0 + \sqrt{N}Y_0) - 1) \\ = \; & (X_{n-1} + \sqrt{N}Y_{n-1})(X_0 + \sqrt{N}Y_0)^2 \qquad (\text{using } X_0^2 - NY_0^2 = 1) \\ = \; & X_{n+1} + \sqrt{N}Y_{n+1} \quad \square \end{aligned}$$

So what happens when $N = 9c^2 - 4$? In fact we can describe the continued fraction of $\sqrt{N}$ completely. We get similar results for $c = 1 \bmod 4$ and $c = 2 \bmod 4$. However, as Markoff numbers 2 mod 4 are not in general prime, we cease to pay attention to them from now on.

Lemma 5.4.5

If $c \in \mathbb{N}, c \neq 1$ but $c = 1 \bmod 4$ then $\sqrt{(9c^2-4)}$ has continued fraction expansion

$$[3c-1, \overline{1, (3c-3)/2, 2, (3c-3)/2, 1, 6c-2}]$$

Proof of 5.4.5

We basically do this by direct expansion.
Certainly $(3c-1)^2 \leq 9c^2 - 4 < (3c)^2$ so this gives us our first term. From then on we have:

Integer below the remainder	Reciprocal of new remainder
$3c-1$	$\dfrac{\sqrt{9c^2-4}+3c-1}{6c-5}$
1	$\dfrac{\sqrt{9c^2-4}+3c-4}{4}$
$\dfrac{3c-3}{2}$	$\dfrac{\sqrt{9c^2-4}+3c-2}{3c-2}$
2	$\dfrac{\sqrt{9c^2-4}+3c-2}{4}$
$\dfrac{3c-3}{2}$	$\dfrac{\sqrt{9c^2-4}+3c-4}{6c-5}$
1	$\sqrt{9c^2-4}+3c-1$
$6c-2$	$\dfrac{\sqrt{9c^2-4}+3c-1}{6c-5}$

and we are back where we started. □

Corollary 5.4.6

If $N = 9c^2 - 4$ ($c = 1 \bmod 4, c \neq 1$) then $X^2 - NY^2 = -1$ has no integer solutions. □

Examples:
(1) If $c = 5$ so $N = 13.17 = 221$ then $\sqrt{221} = [14, \overline{1, 6, 2, 6, 1, 28}]$
(2) If $c = 9$ so $N = 25.29 = 725$ then $\sqrt{725} = [26, \overline{1, 12, 2, 12, 1, 52}]$

How do we use this to describe solutions to (D_c) when c is our Markoff number? We already know that solutions exist, and if we have $U^2 - NV^2 = -4c^2$ then

$$(U + \sqrt{N}V)(X_0 + \sqrt{N}Y_0)^n (U - \sqrt{N}V)(X_0 - \sqrt{N}Y_0)^n = -4c^2 \qquad \text{for } n \in \mathbb{Z}$$

So defining $(U_n + \sqrt{N}V_n) = (U + \sqrt{N}V)(X_0 + \sqrt{N}Y_0)^n$ gives us a whole sequence of solutions. But in the proof of theorem 5.4.4(D) we can multiply both sides of the equation by $U + \sqrt{N}V$ and obtain identical recurrence relations for U_n and V_n:

$$\begin{aligned} U_{n+1} &= 2X_0U_n - U_{n-1} \\ V_{n+1} &= 2X_0V_n - V_{n-1} \end{aligned} \tag{5.5}$$

In particular V_n will give us Markoff numbers appearing in a triple with c, by Theorem 5.4.1 Also we have seen something very similar to (5.5) before, namely equation (5.3). If it were the case that $2X_0 = 3c$ then they would be the same recurrence relation. Then the sequence (V_n) would give us exactly a ${}^a c^b$-chain of solutions and so each appearance of c as a maximal element gives us a fundamental solution to (D_c) with which we can generate each chain. Although this cannot hold as $3c$ is odd, it turns out that we merely have to widen our outlook to take account of the equations we are working with, and so we look at real quadratic number fields in the next section.

5.5 Quadratic Number Fields and their Ring of Integers

When we are trying to solve (D_c), of course we are really working in the real quadratic field $\mathbb{Q}(\sqrt{N})$. This has norm $\|(X + \sqrt{N}Y)\| = X^2 - NY^2$ and we are creating new solutions from old ones by multiplying by units of the field, i.e. those elements with norm ± 1. What we need to do is to be able to factorise (D_c) in this field, but we do not use $\mathbb{Z}(\sqrt{N})$ for this purpose. Instead we need the algebraic integers $\mathcal{O}(\sqrt{N})$ of $\mathbb{Q}(\sqrt{N})$, which are those elements θ such that there exists a monic polynomial $p(t)$ with integer coefficients satisfying $p(\theta) = 0$.

Since $c = 1 \bmod 4$, so is $N = 9c^2 - 4$ and assuming N is squarefree (if not then see the last section), we have $\mathcal{O}(\sqrt{N}) = \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{N}]$. In other words we obtain numbers of the form $x + \sqrt{N}y = \frac{X}{2} + \sqrt{N}\frac{Y}{2}$ where $X, Y \in \mathbb{Z}$ and are either both even or both odd.

We need to know all the units of $\mathcal{O}(\sqrt{N})$, having already obtained some from theorem 5.4.4. We notice $(\frac{3}{2}c)^2 - (\frac{1}{2}\sqrt{N})^2 = 1$ so the unit $\frac{3}{2}c + \frac{1}{2}\sqrt{N}$ fits our description at the end of the last chapter where $2X_0 = 3c$. In fact $\frac{3}{2}c + \frac{1}{2}\sqrt{N}$ is the fundamental unit of $\mathcal{O}(\sqrt{N})$and all other units are obtained from it.

Lemma 5.5.1

The units of $\mathcal{O}(\sqrt{N})$are

$$\pm\left(\frac{3c}{2}+\frac{\sqrt{N}}{2}\right)^n \qquad \text{for } n \in \mathbb{Z}$$

Proof of 5.5.1

Suppose $u+v\sqrt{N}, U+V\sqrt{N} \in \mathcal{O}(\sqrt{N})$ and $u^2-Nv^2=U^2-NV^2=1$. We can assume

$$0 < U - V\sqrt{N} < u - v\sqrt{N} < 1 < u + v\sqrt{N} < U + V\sqrt{N}$$

with u, v, U, V all positive, and so the last inequality gives $\sqrt{N}(v-V) < U-u$

If $V < v$ then $u < U$ and so $u - U < 0 < \sqrt{N}(v-V)$. Hence $0 < u - \sqrt{N}v < U - \sqrt{N}V$ which is not true. So we have $v < V$ and $u < U$, i.e. both terms of the smaller unit are less than each corresponding term of the larger unit. As $\mathcal{O}(\sqrt{N})$ is discrete in $\mathbb{Q}(\sqrt{N})$ we can take a fundamental solution $u_0 + \sqrt{N}v_0$, where u_0 and v_0 each take on the minimal positive value over all units. In our case this must be $\frac{3}{2}c + \frac{1}{2}\sqrt{N}$ as $v_0 = \frac{1}{2}$ is the smallest possible term.

Now suppose we have $U^2 - NV^2 = 1$ for positive U, V and

$$(u_0+\sqrt{N}v_0)^n < U + \sqrt{N}V < (u_0+\sqrt{N}v_0)^{n+1} \qquad \text{for some } n \in \mathbb{N}$$

Then multiplying throughout by $(u_0 - \sqrt{N}v_0)^n$, we have

$$1 < (U+\sqrt{N}V)(u_0-\sqrt{N}v_0)^n < u_0 + \sqrt{N}v_0$$

so the middle expression gives us a unit with positive terms smaller than the fundamental unit, which cannot happen.

Finally we need to check that we have not now obtained any units with norm -1. These would be of the form $\frac{X}{2} + \frac{Y}{2}\sqrt{N}$ for X, Y odd integers. But then $(\frac{X}{2} + \frac{Y}{2}\sqrt{N})^3$ also has norm -1, with first term

$$\frac{X^3+3NY^2X}{8} = \frac{4X(NY^2-1)}{8}$$

which is an integer. This gives a solution to $X^2 - NY^2 = -1$, contradicting corollary 5.4.6. □

Looking back at (D_c) we see we can absorb the factor of 4 to get factorisations in $\mathcal{O}(\sqrt{N})$ of $-c^2$:

$$(x+\sqrt{N}y)(x-\sqrt{N}y) = -c^2 \qquad (A_c)$$

For each such factorisation we can multiply by all positive units, obtaining elements $x_n + \sqrt{N}y_n$. But these are related by the recurrence relation (5.5), which is now the same as (5.3) due to lemma (5.5.1). Hence this operation merely reproduces all the elements in a ${}^{a}c^{b}$-chain from any given one.

Also, given any other factorisation of $-c^2$ which does not differ merely by units, we obtain a new chain of solutions to (D_c), and hence a new appearance of c as the maximal element of some other triple. So we have

Theorem 5.5.2

If c $(= 1 \bmod 4)$ is a Markoff number then the number of Markoff triples having c as maximal element is precisely the number of factorisations in $\mathcal{O}(\sqrt{N})$ of $-c^2$ in the form $(x + \sqrt{N}y)(x - \sqrt{N}y)$ up to the presence of units.

We now suppose that $c = p$, a prime number. We might at first hope that $\mathcal{O}(\sqrt{N})$is sometimes a unique factorisation domain, but in fact the existence of solutions to (A_c) directly disproves this.

For if p were prime in $\mathcal{O}(\sqrt{N})$ then a factorisation $-p^2 = (x+\sqrt{N}y)(x-\sqrt{N}y)$ would imply that p and (say) $x+\sqrt{N}y$ were associates. But p has norm p^2, $x+\sqrt{N}y$ has norm $-p^2$ and there are no units of norm -1.

Alternatively if p splits into primes in $\mathcal{O}(\sqrt{N})$ as (say) $(u + \sqrt{N}v)(u - \sqrt{N}v)$ then $-p$ must factor as $-(u + \sqrt{N}v)(u - \sqrt{N}v)$. So

$$-(u + \sqrt{N}v)^2(u - \sqrt{N}v)^2 = (x + \sqrt{N}y)(x - \sqrt{N}y)$$

and by uniqueness of factorisation

$$(u + \sqrt{N}v)^2 = x + \sqrt{N}y \qquad \text{(up to units)}$$

But again we have associate elements with norms of p^2 and $-p^2$.

However we can still proceed, using unique factorisation of ideals. Instead of considering x as the product of elements ab in a ring R, we can think of a product of principal ideals $< x >=< a >< b >$. This removes the ambiguity of multiplication by units. We recall:

Definition 5.5.3

An ideal $\boldsymbol{A}$ of a ring R is prime if whenever $\boldsymbol{B}$ and $\boldsymbol{C}$ are ideals of R with $\boldsymbol{BC} \subseteq \boldsymbol{A}$ then $\boldsymbol{B} \subseteq \boldsymbol{A}$ or $\boldsymbol{C} \subseteq \boldsymbol{A}$

Then if R is the ring of algebraic integers of a number field, say $\mathcal{O}(\sqrt{N})$, we have

Lemma 5.5.4

Every non-zero ideal of R can be written as a product of prime ideals, uniquely up to the order of factors.

Proof of 5.5.4

See [27], page 117. □

Of course these prime ideals might not be principal, which is precisely how we obtain non-unique factorisation of elements.

But if p is a prime number in $\mathbb{Z}$, how can we factorise $< p >$ in our number field? We can apply a theorem of Dedekind to the case which concerns us.

We have our ring of integers $\mathcal{O}(\sqrt{N})$ $(= \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{N}])$ where we say $N = 4s + 1$. The minimum polynomial $f(t)$ of $\frac{1}{2} + \frac{1}{2}\sqrt{N}$ is

$$\left(t - \frac{1+\sqrt{N}}{2}\right)\left(t - \frac{1-\sqrt{N}}{2}\right) = t^2 - t - s$$

Theorem 5.5.5

Given a rational prime p, we look at $f(t) = t^2 - t - s$ over $\mathbb{Z}_p$.

(A) If $f(t)$ is irreducible over $\mathbb{Z}_p$ then $< p >$ is a prime ideal in $\mathcal{O}(\sqrt{N})$.

(B) If $f(t)$ factors as $(t-\alpha)(t+\alpha-1)$ over $\mathbb{Z}_p$ then $< p >$ factors into prime ideals

$$< p >= \left\langle p, \frac{1+\sqrt{N}}{2} - \alpha \right\rangle \left\langle p, \frac{1-\sqrt{N}}{2} - \alpha \right\rangle = \mathcal{P}_+\mathcal{P}_-$$

Proof of 5.5.5

See [27], pages 186-187. □

When our prime p is c itself, we find that (B) is always the case.

Lemma 5.5.6

If c is prime then $< c >$ factors inside $\mathcal{O}(\sqrt{N})$ into prime ideals $\mathcal{P}_+\mathcal{P}_-$.

Proof of 5.5.6

Either we note that $f(t)$ factorises if and only if there is an integer n with $\alpha^2 - \alpha = s + nc$, which is equivalent to $(2\alpha - 1)^2 = N + 4nc$, so we show that N is a quadratic residue mod $4c$.

Or, if $< c >$ were prime then $< c^2 >$ can only factor as $< c >< c >$. But the existence of equation (A_c) gives

$$< x + \sqrt{N}y >< x - \sqrt{N}y >=< c >< c >$$

So $x + \sqrt{N}y$ and c are associates, with norms $-c^2$ and c^2. We obtain a contradiction as before. □

Finally we have

Theorem 5.5.7

Prime Markoff numbers appear only once as the maximal element of a Markoff triple.

Proof of 5.5.7

We factor $< c^2 >$ into prime ideals as $\mathcal{P}_+^2\mathcal{P}_-^2$. Then $(\mathcal{P}_+\mathcal{P}_-)(\mathcal{P}_+\mathcal{P}_-)$ gives us the factorisation $< c >< c >$, which corresponds to solutions of $x^2 - Ny^2 = c^2$, i.e. $x = c, y = 0$ and all its associates. As in Lemma 5.5.6 we do not get solutions to $x^2 - Ny^2 = -c^2$.

The only other possible factorisation is $(\mathcal{P}_+^2)(\mathcal{P}_-^2)$ which gives rise to the solution of the form

$$(x + \sqrt{N}y)(x - \sqrt{N}y) = -c^2$$

which generates the only ${}^ac^b$-chain. We now just have to finish off the case when N is not squarefree in the next section. □

5.6 The Last Case

If c is prime but N is of the form r^2d, where r will be odd and $d = 1 \bmod 4$, then our ring of integers of $\mathbb{Q}(\sqrt{N})$ is now $\mathcal{O}(\sqrt{d})$, so we must factor accordingly. We still have the factorisation of the ideal $< c^2 >$ as before. However, any solution $x^2 - Ny^2 = -c^2$ (with x,y both integers or both half-integers) gives a factorisation

$$(x + r\sqrt{d}y)(x - r\sqrt{d}y) = -c^2$$

and vice-versa, so we are only interested in elements of $\mathcal{O}(\sqrt{d})$ with the second term divisible by r. We still have units in $\mathcal{O}(\sqrt{d})$ of the form $\pm(\frac{3}{2}c + \frac{r}{2}\sqrt{d})^n$ which generate a ${}^ac^b$-chain. However what we now cannot conclude is that these are the full set, only that they make up all the units with second term divisible by r.

So the only possibility of generating more chains, and hence other appearances of c as a maximal element, is by multiplying a solution of (A_c) by some other unit of $\mathcal{O}(\sqrt{d})$. But if we had, say

$$(x + r\sqrt{d}y)(\alpha + \gamma\sqrt{d}) = u + r\sqrt{d}v$$

where $x + r\sqrt{d}y, u + r\sqrt{d}v$ have norm $-c^2$ in $\mathcal{O}(\sqrt{d})$ then $\alpha ry + \gamma x = rv$, so $r|\gamma x$. Now if x and r have a common (odd) prime factor p then p also divides c (as $x^2 - r^2dy^2 = -c^2$), giving a solution of $X^2 - NY^2 = -4c^2$ in integers with $p|X$ and $p|c$. By theorem 5.4.1, this would give rise to a Markoff triple which would not be pairwise coprime. So $r|\gamma$ and our unit was one we had already used.

Finally we might have units of norm -1 in $\mathcal{O}(\sqrt{d})$which would turn an element of norm c^2 into one of norm $-c^2$ and hence give us a possible solution. But our elements of norm c^2 are precisely the associates of c, so we would have

$$c(\alpha + \gamma\sqrt{d}) = u + r\sqrt{d}v$$

where $\alpha + \gamma\sqrt{d}$ is our unit of norm -1. We have $c\gamma = rv$, but as c and r are coprime (since c and N are), we realise that c divides both u and v. This again would give rise to a Markoff triple, whose existence would be contradicted by corollary 5.4.3.

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Appendix

As claimed in equation (2.1), we need to show

Theorem

If $A, B \in PSL(2, \mathbb{C})$ and

$$B^{-1}A^{-1}BA = \begin{pmatrix} -1 & 0 \\ -k & -1 \end{pmatrix} \quad , \quad BAB^{-1}A^{-1} = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$$

with $k \in \mathbb{C}\backslash\{0\}$ then we can choose $z, w \in \mathbb{C}$ such that

$$A = \begin{pmatrix} \frac{1+z^2}{w} & z \\ z & w \end{pmatrix} \quad , \quad B = \begin{pmatrix} \frac{1+w^2}{z} & -w \\ -w & z \end{pmatrix}.$$

Proof of Theorem

We do this in two stages, by setting $X = BA, Y = B^{-1}A^{-1}$ and by first proving the lemma.

Lemma

If $X, Y \in PSL(2, \mathbb{C})$ with

$$YX = \begin{pmatrix} -1 & 0 \\ -k & -1 \end{pmatrix} \quad , \quad XY = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}$$

then X and Y are of the form

$$X = \begin{pmatrix} e & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix}$$

for $d, e \in \mathbb{C}\backslash\{0\}$.

Proof of Lemma

Write

$$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad X = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then the zero coefficients give

$$\begin{aligned} af + bh &= 0 \\ ag + ch &= 0. \end{aligned}$$

First assume $h \neq 0$. Then we have $f \neq 0$ (since $f = 0$ implies that $b = 0$, otherwise X is a singular matrix, but then looking at XY tells us that $k = 0$.)

So writing $a/h = -b/f = -c/g = \lambda$ gives

$$Y = \begin{pmatrix} \lambda h & -\lambda f \\ -\lambda g & d \end{pmatrix}$$

and equating the two -1 coefficients in YX gives $\lambda he = dh$, so $d = \lambda e$. But then Y is just the inverse of X in $PSL(2, \mathbb{C})$.

Thus we can only have $h = 0$, and $a = 0$, giving $c = -1/b$ and $g = -1/f$. Again, equating the -1 coefficients in YX gives $b = f$, and comparing the $-k$ and k coefficients in YX and XY respectively tells us that

$$b(d+e) = \frac{d+e}{b}.$$

If $d = -e$ then k would be 0, so we can take $b = 1$ and the lemma is proved. □

Now we repeat the process, setting

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad , \quad A = \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix}$$

with

$$BA = X = \begin{pmatrix} e & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad AB = Y^{-1} = \begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix}.$$

Again the zero coefficients tell us that

$$\begin{aligned} \gamma\zeta + \delta\theta &= 0 \\ \eta\beta + \delta\theta &= 0 \end{aligned}$$

so $\gamma\zeta = \eta\beta$.

If say $\beta = 0$, then taking $\theta = 0$ and $\gamma = 0$ gives us the "singular solution" $z = i, w = 0$, and we argue the same way for any other combination of zeros.

Otherwise we write $\beta = \mu\gamma, \zeta = \mu\eta$, and then comparing the 1 coefficients in $X$and Y^{-1} tells us

$$\mu(\alpha\eta + \gamma\theta) = 1 = \alpha\eta + \gamma\theta$$

and $\mu = 1$. We can now set $z = \zeta = \eta$ and $w = -\beta = -\gamma$. Then by choosing $\det(A) = \det(B) = 1$, we are left with the four equations

$$\begin{aligned} \delta\theta &= wz \\ \alpha\delta - w^2 &= 1 \\ \epsilon\theta - z^2 &= 1 \\ z(\alpha + \delta) &= w(\epsilon + \theta) \end{aligned}$$

and we then eliminate first θ, then α and finally ϵ to obtain $\delta^2 = z^2$. We take $\delta = z$, so $\theta = w$ and $\alpha = (1 + w^2)/z, \epsilon = (1 + z^2)/w$. □

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