CONSTRUCTION OF REAL ALGEBRAIC FUNCTIONS WITH PRESCRIBED PREIMAGES

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Abstract. We present real algebraic functions with prescribed preimages.

Smooth closed manifolds are, according to one of Nash's greatest studies, regarded as the zero sets of some real polynomials and smooth. Moreover, canonical projections of spheres naturally embedded in the 1-dimensional higher Euclidean spaces (*affine spaces*) and some natural functions on projective spaces, Lie groups and their quotient spaces are important examples of real algebraic functions. However, in general, it is very important to construct explicit examples. In addition, in considerable cases, the structures of the functions and maps are hard to understand.

We construct examples by answering to Sharko's problems and author's suitably revised ones in improved styles. They ask whether we can find nice smooth functions with prescribed preimages. We have previously given an answer with real algebraic functions and this result is one of key ingredients.

1. Introduction.

Among Nash's greatest work, [18] shows that every smooth closed manifold has the structure of a smooth real algebraic manifold and the zero set of some real polynomial. For surveys on related theory on real algebraic manifolds, see [11] for example. For some smooth closed manifolds, explicit real algebraic functions are well-known. Spheres, some projective spaces, some Lie groups and some of their quotient spaces are known to have some natural ones. On the other hand, in general, explicit construction and knowing preimages and more generally, their explicit global structures are hard.

1.1. **Terminologies and notation on polyhedra, manifolds, smooth maps and some real algebraic functions and maps.** We first introduce fundamental terminologies and notation on smooth and real algebraic manifolds. For a topological space X homeomorphic to a cell complex whose dimensions is finite, we can define the dimension $\dim X$ as an integer uniquely. A topological space homeomorphic to a topological manifold has the structure of a CW complex. A smooth manifold is known to have the structure of a canonically and uniquely obtained polyhedron. This is a so-called PL manifold. It is also well-known that a topological space having the structure of a polyhedron of dimension at most 2 has the structure of a polyhedron uniquely. For a topological manifold of dimension at most 3, this also holds. This is due to [17] for example.

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Let $\mathbb{R}^k$ denote the k-dimensional Euclidean space. This is a simplest smooth manifold. This is also regarded as a Riemannian manifold with the standard Euclidean metric. For a point $x \in \mathbb{R}^k$, we can define $||x|| \geq 0$ as the metric between x and the origin 0 under the standard Euclidean metric. This is also regarded as a simplest real algebraic manifold: the k-dimensional affine space.

$S^k := \{x \in \mathbb{R}^{k+1} \mid ||x|| = 1\}$ is the k-dimensional unit sphere. It is a smooth compact submanifold of $\mathbb{R}^{k+1}$ with no boundary and of dimension $k \geq 0$. It is connected for $k \geq 1$ and a discrete set with exactly two points for $k = 0$. It is also a smooth real algebraic submanifold defined by the zero set of the real polynomial $||x|| - 1 = \Sigma_{j=1}^{k+1} {x_j}^2 - 1$ where $x := (x_1, \cdots, x_{k+1})$. $D^k := \{x \in \mathbb{R}^k \mid ||x|| \leq 1\}$ is the k-dimensional unit disk. It is a smooth compact and connected submanifold of $\mathbb{R}^k$ and of dimension $k \geq 0$.

The topologies of these manifolds are easily understood.

For a differentiable map $c : X \to Y$ between differentiable manifolds, $x \in X$ is a *singular* point if the rank of the differential at x is smaller than the minimum between $\dim X$ and $\dim Y$. $c(x)$ is a *singular value* of c. Let $S(c)$ denote the *singular set* of c, defined as the set of all singular points. In our paper, we consider smooth maps as differentiable maps unless otherwise stated.

For a unit sphere, a canonical projection is defined by the restriction of a canonical projection defined by the map $\pi_{k,k_1} : \mathbb{R}^k \to \mathbb{R}^{k_1}$ mapping $x = (x_1, x_2) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} = \mathbb{R}^k$ to $x_1 \in \mathbb{R}^{k_1}$ to the unit sphere S^{k-1} where $k_1, k_2 > 0$ and $k = k_1 + k_2$.

More generally, so-called real projective spaces, complex ones and quaternion ones have natural real algebraic functions. Some Lie groups are embedded naturally in Euclidean spaces and admit functions represented by some real polynomials. In considerable cases, their quotient spaces have ones. We do not explain about them precisely. Related to such functions, see [19, 28] and see also [14] for example.

1.2. **Graphs.** Graphs are important tools here. A graph is regarded as a 1-dimensional CW complex with the vertex set, defined as the set of all 0-dimensional cells and the edge set, defined as the set of all 1-dimensional cells. A vertex and an edge are elements of these sets respectively. The closure of an edge homeomorphic to a circle is called a *loop*. Hereafter, we do not consider such graphs. In other words, a graph is always a 1-dimensional polyhedron and may be a multi-graph or a graph with more than one edge between given two distinct vertices. An *isomorphism* from a graph K_1 onto K_2 is a PL or a piecewise smooth homeomorphism mapping the edge set and the vertex set of K_1 onto those of K_2.

1.3. **Reeb graphs.** We explain about *Reeb graphs*. They are graphs and also fundamental tools in our study. For a smooth function $c : X \to \mathbb{R}$, we can define an equivalence relation by the following rule. Two points x_1 and x_2 in X are defined to be equivalent if and only if they are in some same connected component of some preimage $c^{-1}(y)$.

We explain about a main ingredient of [22]. If a smooth function on a compact manifold has finitely many singular values, then the quotient space of it is homeomorphic to a graph. If it is one on a closed manifold, then we can define a graph whose vertex set consists of all points p whose preimages ${q_c}^{-1}(p)$ contain some singular points of c. Morse(-Bott) functions and smooth functions of some considerably wide classes satisfy this.

Definition 1. The graph, denoted by W_c, is called the *Reeb graph* of c.

We can also define the quotient map $q_c : x \to W_c$. We can define the map $\bar{c} : W_c \to \mathbb{R}$ enjoying the relation $c = \bar{c} \circ q_c$ uniquely.

Reeb graphs are classical and important tools and objects. They have information on the manifolds roughly and do not miss some important information. The Reeb graph of a function has been already defined in [20] for example.

1.4. **Our problems and our main results.**

Problem 1. For a graph, can we construct a nice smooth function whose Reeb graph is isomorphic to it? We do not fix the manifold beforehand?

[25] has asked this first and smooth functions on closed surfaces have been explicitly constructed for graphs satisfying some nice conditions. [13] generalizes this for arbitrary graphs. Later, for example, [12, 15] have set explicit problems and solved. Their studies are essentially on smooth functions on closed surfaces and Morse functions such that each connected component of each preimage containing no singular points is always a sphere for example. The following is a revised problem, introduced first by the author in [4].

Problem 2. In Problem 1, can we construct one whose singular points are mild in suitable senses, and whose preimages are as prescribed?

[5, 8, 9] have given answers. Some of [22] is on informal discussions on [5] by us.

Problem 3. Can we construct ones as real analytic or real algebraic ones?

[6] is regarded as the first affirmative answer. The class of graphs is restricted at present. In our study, we have the following answer as a new answer.

Main Theorem 1. *Let $l > 3$ and $m > 2$ be integers. Let G be a graph as follows.*

- *The vertex set is of size l and the j-th vertex is denoted by v_j for $1 \leq j \leq l$.*
- *The edge set is of size $l-1$ and the j-th edge connects the vertex v_j and v_{j+1} for $1 \leq j \leq l-1$.*

Let $\{F_j\}_{j=1}^{l-1}$ be a family of smooth manifolds satisfying the following conditions.

- *F_1 and F_{l-1} are the $(m-1)$-dimensional unit spheres S^{m-1}.*
- *The others are the unit spheres or represented as connected sums of finitely many manifolds diffeomorphic to the products $S^j \times S^{m-j-1}$ with integers $1 \leq j \leq m-2$ where the connected sum is taken in the smooth category.*
- *For adjacent integers j and $j+1$ where $1 \leq j \leq l-2$, either F_j or F_{j+1} is not diffeomorphic to the unit sphere.*

Then we have some m-dimensional real algebraic closed and connected manifold M and a real algebraic function $f : M \to \mathbb{R}$ enjoying the following properties.

1. *We have some suitable isomorphism $\phi : G \to W_f$ of the graphs and for the j-th edge e_j and each point p_{e_j} in the interior of the edge $\phi(e_j) \subset W_f$, the preimage ${q_f}^{-1}(p_{e_j})$ is diffeomorphic to F_j.*
2. *The singular set $S(f)$ is a finite set.*

In [6], essentially, connected components of preimages containing no singular points are spheres. This with additional Main Theorems, presented later, respects Problem 2 first.

We explain about the content of our paper. In the next section, we prove our Main Theorems. As important tools, we review [6] and so-called *fold* maps, generalizations of Morse functions, for example.

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Data availability.
Data essentially supporting our present study are all contained in the present paper.

2. On Main Theorems.

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2.1. **Additional several terminologies and notions.** A *diffeomorphism* means a smooth homeomorphism with no singular points. A *diffeomorphism on a smooth manifold* means a diffeomorphism from it to itself. The *diffeomorphism type* of a smooth manifold is defined as the equivalence class under the natural equivalence relation on the family of all smooth manifolds defined by the existence of diffeomorphisms.

The *diffeomorphism group* of a smooth manifold is the group of all diffeomorphisms on it. This is also a topological group and topologized with the so-called *Whitney C^{∞} topology*. More generally, Whitney C^{∞} topologies on the set of all smooth maps between two given smooth manifolds and subspaces of this space are important in the (singularity) theory of smooth maps for example.

A *smooth* bundle means a bundle whose fiber is a smooth manifold and whose structure group is regarded as (some subgroup) of the diffeomorphism group of the fiber.

We introduce *fold* maps. For the Whitney C^{∞} topology and fold maps for example, see also [3] as a book for elementary singularity theory of differentiable maps for example.

Definition 2. Let X and Y be smooth manifolds with no boundaries satisfying $\dim X \geq \dim Y$. A *fold* map $c : X \to Y$ is a smooth map such that at each singular point p, we have suitable integer $0 \leq i(p) \leq \frac{\dim X - \dim Y + 1}{2}$ and local coordinates around p and $c(p)$ and that there we have a local form $c(x_1, \cdots, x_{\dim X}) = (x_1, \cdots, x_{\dim Y-1}, \Sigma_{j=1}^{\dim X - \dim Y - i(p)+1} {x_{\dim Y - 1 + j}}^2 - \Sigma_{j=1}^{i(p)} {x_{\dim X - i(p) + j}}^2)$ around the singular point p and the singular value $c(p)$.

Proposition 1. *In the previous definition, $i(p)$ is chosen uniquely and defined as the* index *of p. The set of all singular points of c of a fixed index is a smooth regular submanifold of c and of dimension* $\dim Y - 1$. *If X is closed, then the submanifold is compact and with no boundary. The restriction to the submanifold is a smooth immersion.*

Definition 3. If in the definition of the fold map, $i(p) = 0$ always holds, then this is called a *special generic* map.

A Morse function is of course a fold map. In short, fold maps are locally projections or the product map of a Morse function and the identity map on some disk. For special generic maps, the Morse function is chosen as a so-called *height function* of a unit disk. A *height function* h of a unit sphere is a function of the form $h(x) = \pm||x||^2 + c$ where c is some real number.

We introduce very fundamental and explicit special generic maps. They are also keys in our main results.

Example 1.
1. Canonical projections of unit spheres are special generic. The restrictions to the singular sets, which are also regarded as unit spheres, are embeddings. The images are regarded as the unit disks whose dimensions are same as those of the Euclidean spaces of the targets.
2. Let $m > n$ be positive integers. Let M be an m-dimensional smooth manifold represented as a connected sum of $l > 0$ manifolds diffeomorphic to $S^{k_j} \times S^{m-k_j}$ for each integer $1 \leq j \leq l$ and some integer $1 \leq k_j \leq n-1$ where the connected sum is taken in the smooth category. We easily have a special generic map $f : M \to \mathbb{R}^n$ such that the restriction to the singular set $S(f)$ is an embedding and that the image is a smoothly embedded submanifold diffeomorphic to one represented as a boundary connected sum of $l > 0$ manifolds diffeomorphic to $S^{k_j} \times D^{n-k_j}$ for each integer $1 \leq j \leq l$. The boundary connected sum is, as before, taken in the smooth category.

For these maps, we also have the following two trivial smooth bundles. We consider a case where m and n are the dimensions of the manifolds of the domain and the target.

- We have some small collar neighborhood of the boundary of the image and the composition of the restriction of the map to the preimage with the canonical projection to the boundary gives a trivial smooth bundle whose fiber is diffeomorphic to a unit disk D^{m-n+1}.
- On the complementary set of the interior of the collar neighborhood, the restriction of the map gives a trivial smooth bundle whose fiber is diffeomorphic to a unit sphere S^{m-n}.

Moreover, they are glued by the product map of the diffeomorphism for the natural identification between the base spaces and the identity map on the fiber. Note that fibers are identified in some canonical way. For such special generic maps, see also the preprint [7, 10] of the author.

A *Morse-Bott* function is a smooth function on a manifold with no boundary at each singular point of which it is represented as the composition of some projection with a Morse function for suitable local coordinates. See [2].

As our main work, we construct real algebraic functions such that for each singular point, the singularity is as one of these cases (at least) topologically. We do not investigate these singularities in our main theorems. This presents another fundamental, important and difficult problem on singularities of polynomial maps or more generally, smooth maps.

3. On Main Theorems.

[6] presents most of our key tools. We review some important ingredients. We apply them. We also apply them in suitably improved ways in several scenes. At present we do not find essential errors in the (accepted) version on which a positive report for publication has been announced to have been sent. However we will not publish the paper revising the accepted version essentially.

Note that the work is motivated by [1] with [26, 27]. [1] studies *algebraic domains* collapsing to some graphs. An *algebraic domain* means an open set in an real affine space the boundary of whose closure is surrounded by smooth real algebraic hypersurfaces. Originally our main theorem of [6] respects these studies. The work essentially related to such work is regarded as one of important future problems. However, we do not investigate related problems in our paper. For the related studies, it seems that we need more sophisticated or advanced knowledge and arguments on real algebraic geometry.

3.1. **Proofs of Main Theorems.** We review [6] in a bit revised form. We show the following key theorem or key tool. Our proof is, in most arguments, regarded as a review of [6]. Some can be understood by our observation and we argue in this way.

Theorem 1 ((A bit revised version of) [6])**.** *Let $D \subset \mathbb{R}^n$ be a connected algebraic domain such that the closure $\overline{D}$ is compact and that the boundary $\partial \overline{D}$ of the closure consists of $k \geq 1$ smooth algebraic hypersurfaces. Let $\{S_j\}_{j=1}^k$ denote the family of these k hypersurfaces. Suppose the following conditions.*

- *S_j is a connected component of the zero set of some real polynomial $f_j(x_1, \cdots, x_n)$.*
- *$D \subset \bigcap_{j=1}^k \{(x_1, \cdots, x_n) \mid f_j(x_1, \cdots, x_n) > 0\}$. We have some open and connected neighborhood U of $\overline{D}$ and $U \bigcap \bigcap_{j=1}^k \{(x_1, \cdots, x_n) \mid f_j(x_1, \cdots, x_n) > 0\} = D$ and $U \bigcap \bigcap_{j=1}^k \{(x_1, \cdots, x_n) \mid f_j(x_1, \cdots, x_n) \geq 0\} = \overline{D}$ hold.*

Then for any integer $n' > n$, we have another connected algebraic domain D' whose closure $\overline{D'}$ is compact and whose boundary $\partial \overline{D'}$ is a smooth connected algebraic hypersurface enjoying the following properties.

1. *$\overline{D}$ is embedded in $\overline{D'}$ by the inclusion mapping $x \in \overline{D} \subset \mathbb{R}^n$ to $(x, 0) \in \overline{D'} \subset \mathbb{R}^{n'} = \mathbb{R}^n \times \mathbb{R}^{n'-n}$.*
2. *The boundary $\partial \overline{D'}$ is a connected component of the zero set of some real polynomial $f'(x_1, \cdots, x_n, \cdots, x_{n'})$. $D' \subset \{(x_1, \cdots, x_n) \mid f'(x_1, \cdots, x_n, \cdots, x_{n'}) > 0\}$. We have some open and connected neighborhood U' of $\overline{D'}$ and $U' \bigcap \{(x_1, \cdots, x_n, \cdots, x_{n'}) \mid f'(x_1, \cdots, x_n, \cdots, x_{n'}) > 0\} = D'$ and $U' \bigcap \{(x_1, \cdots, x_n, \cdots, x_{n'}) \mid f'(x_1, \cdots, x_n, \cdots, x_{n'}) \geq 0\} = \overline{D'}$ hold.*
3. *$\overline{D'}$ collapses to $\overline{D}$ by a smooth homotopy mapping each point $p \in \overline{D'}$ preserving the value p_1 in $p := (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^{n'-n} = \mathbb{R}^{n'}$.*
4. *The restriction of the canonical projection $\pi_{n',n}$ to the boundary $\partial \overline{D'}$ enjoys the following properties.*
   (a) *It is represented as the composition of some homeomorphism with another special generic map. Shortly, it is, at least* topologically*, a special generic map.*
   (b) *The manifold of the domain of the previously presented special generic map is diffeomorphic to that of the resulting map. This enjoys the properties on the bundles in Example 1.*
   (c) *The image of the resulting map is $\overline{D}$. The preimage of each point in the interior* Int $\overline{D}$ *does not contain any singular point of the resulting map here.*

A proof of Theorem 1. We use "$\prod$" for the products. For example, $\prod_{j=1}^k (f_j(x_1, \cdots, x_n))$ means the product of the k polynomials.

We define a set in $D' \subset \mathbb{R}^{n'}$ by $D' := \{(x, y) \in D \times \mathbb{R}^{n'-n} \subset \mathbb{R}^n \times \mathbb{R}^{n'-n} = \mathbb{R}^n \mid \prod_{j=1}^k (f_j(x_1, \cdots, x_n)) - ||y||^2 = 0\}$ and we show that this is a desired algebraic domain. We also use the notation $y := (y_1, \cdots, y_{n'-n})$. Here we may replace $||y||^2$ by a general polynomial, represented in the form $\Sigma_{j=1}^{n'-n} a_j {y_j}^{b_j}$ where a_j and b_j are a positive integer and a positive even integer, respectively.

We show this is also a desired algebraic domain.

First by the definition, we have (1). The closure $\overline{D'}$ is easily known to be compact.

We consider the partial derivative of the function $\prod_{j=1}^{k}(f_j(x_1, \cdots, x_n)) - \Sigma_{j=1}^{n'-n} {y_j}^2$ for variants x_j and y_j. First we take a point $(x_0, y_0) \in \partial\overline{D'}$ in the boundary such that y_0 is not the origin. By the assumption $\prod_{j=1}^{k}(f_j(x_0)) > 0$. We use the notation $x_0 := (x_{0,1}, \cdots, x_{0,n})$ and $y_0 := (y_{0,1}, \cdots, y_{0,n'-n})$ for example as before. Here we consider the partial derivative of the function for a variant y_j. As a result we have $2y_{j_0} = 2y_{0,j_0} \neq 0$ for some variant y_{j_0}.

The differential of the restriction of the function defined canonically from the real polynomial $\prod_{j=1}^{k}(f_j(x)) - \Sigma_{j=1}^{n'-n} {y_j}^2$ at (x_0, y_0) is not of rank 0. This is not a singular point of this real polynomial function.

Second we take a point $(x_{\mathrm{O}}, y_{\mathrm{O}}) \in \partial\overline{D'}$ in the boundary such that y_{O} is the origin. Let $x_{\mathrm{O}} \in S_a$.

By the assumption, for the real polynomials $f_j(x)$, $f_j(x_{\mathrm{O}}) > 0$ for $j \neq a$. The real polynomial function defined canonically from the real polynomial f_a is assumed to have no singular points on the hypersurface S_a.

Here we consider the partial derivative of the function for each variant x_j. We have the value represented as the product of the partial derivative of the function $f_a(x)$ for x_a at $(x_{\mathrm{O}}, y_{\mathrm{O}})$ and the product of $l-1$ numbers defined as the values of the canonically defined real polynomial functions in the family $\{f_j\}_{j=1}^{l}$ at x_{O} except the number $j \neq a$. We have a real number which is not 0 for x_a by the fact that the function defined canonically from f_a is assumed to have no singular points on S_a.

The differential of the restriction of the function at $(x_{\mathrm{O}}, y_{\mathrm{O}}) \in \partial\overline{D'}$ is shown to be not of rank 0. This is not a singular point of the function. We have a result same as one in the case (x_0, y_0).

We can see D' is an algebraic domain whose boundary $\partial\overline{D'}$ is a smooth connected algebraic hypersurface. The implicit function theorem has been applied.

We also have (3) by the definition.

We prove (2) by using the assumption on the real polynomials. Let $f'(x, y) := \prod_{j=1}^{k}(f_j(x_1, \cdots, x_n)) - ||y||^2$ and we also show that this real polynomial is a desired one.

First we consider a point $p_1 \in D$ and a point $(p_1, q_1) \in \overline{D'}$ and take its sufficiently small open neighborhood U_{p_1,q_1} in $\mathbb{R}^{n'}$. By the definition and the assumption, $U_{p_1,q_1} \bigcap \{(x_1, \cdots, x_n, \cdots, x_{n'}) \mid f'(x_1, \cdots, x_n, \cdots, x_{n'}) > 0\} = U_{p_1,q_1} \bigcap D'$ and $U_{p_1,q_1} \bigcap \{(x_1, \cdots, x_n, \cdots, x_{n'}) \mid f'(x_1, \cdots, x_n, \cdots, x_{n'}) \geq 0\} = U_{p_1,q_1} \bigcap \overline{D'}$ hold.

Second we consider a point $p_2 \in \partial\overline{D}$ in the boundary $\partial\overline{D} \subset \overline{D}$ and a point $(p_2, q_2) \in \overline{D'}$. We take its sufficiently small open neighborhood U_{p_2,q_2} in $\mathbb{R}^{n'}$. By the definition and the assumption on the hypersurfaces S_j and the real polynomials, we have a similar observation.

This completes the proof of (2).

By observing the structures of the maps and the manifolds, we have (4).

This completes the proof.

□

A proof of Main Theorem 1. First we prepare an algebraic domain D_0 in $\mathbb{R}^3$ whose closure is a disk centered at the origin 0 and whose radius is $R > 0$.

We consider an increasing sequence $\{t_1\}_{j=2}^{l-1} \subset [-R, R]$ of real numbers of length $l-2$. We choose R as a sufficiently large number. Put $t_1 := -R$ and $t_l := R$. Let $r_j := \frac{t_j + t_{j+1}}{2}$ for each integer $1 \leq j \leq l-1$.

Suppose that F_j is a manifold represented as a connected sum of manifolds in the family containing exactly $k_{j,j'} \geq 0$ copies of $S^{j'} \times S^{m-j'-1}$ for each integer $1 \leq j' \leq \frac{m-1}{2}$. We also regard that $k_{j,j'} = 0$ always holds in the case F_j is the unit sphere.

We consider the following procedure for each integer $0 \leq i \leq m-3$ inductively starting from $i = 0$ and $D := D_0$.

- We check whether $i + 1 \leq \frac{m-1}{2}$ or not.
- We do the following procedure if and only if $i + 1 \leq \frac{m-1}{2}$ holds. For each integer $1 \leq j \leq l-1$, we choose and remove exactly $k_{j,i+1}$ mutually disjoint algebraic domains whose closures are disks centered at points of the form $(r_j, \cdots)$ and whose radii are $\frac{t_{j+1} - t_j}{2}$. We can also choose them in such a way that the closures are mutually disjoint and we do.
- We can apply Theorem 1 by putting D as the resulting algebraic domain, $n := \dim D$ and $n' := n + 1$ and defining $D_{i+1} := D'$ and we do this and we do.
- If $i < m-3$, then we put $D := D_{i+1}$, define i as $i+1$ instead and go to the first step again. If $i = m-3$, then we finish the procedure.

After this procedure, we consider the restriction of the canonical projection $\pi_{m+1,m}$ to the boundary $M := \partial \overline{D'}$, which is an m-dimensional closed and connected manifold, in Theorem 1.

By composing the canonical projection $\pi_{m,1}$, we have a real algebraic function $f : M \to \mathbb{R}$.

We observe the structures of the obtained function f and the manifold M.

By the construction, the preimages containing no singular points of the function are regarded as ones diffeomorphic to the manifolds of the domains of some special generic maps into $\mathbb{R}^{m-1}$. We investigate the special generic maps into $\mathbb{R}^{m-1}$.

We investigate the preimage $f^{-1}(r_j)$. The image of the map is regarded to be diffeomorphic to the $(m-1)$-dimensional unit disk in the case F_j is the unit sphere. The image is represented as a boundary connected sum of manifolds in the family containing exactly $k_{j,j'} \geq 0$ copies of $S^{j'} \times D^{m-j'-1}$ for each integer $1 \leq j' \leq \frac{m-1}{2}$ in the case F_j is not a sphere. The boundary connected sum is taken in the smooth category of course.

By the construction, the maps into $\mathbb{R}^{m-1}$ are represented as some natural smooth real algebraic embeddings into $\mathbb{R}^m$ with canonical projections. This restricts the structures of the maps and the topologies and the differentiable structures of the manifolds of the domains.

In the case F_j is the unit sphere, the preimage $f^{-1}(r_j)$ is also diffeomorphic to the unit sphere and F_j. In the case F_j is not a sphere, the preimage $f^{-1}(r_j)$ is also diffeomorphic to F_j. Furthermore, the maps are, at least topologically, special generic maps into $\mathbb{R}^{m-1}$ presented in Example 1.

For related arguments, see [23] for example. [23] studies so-called *lifts* of a special generic map or a problem whether a special generic map is represented as the composition of a smooth immersion or embedding with a canonical projection. Such problems are important and difficult ones related to both singularity theory and differential topology.

We can naturally consider a suitable isomorphism $\phi : G \to W_f$ and we can also see that the manifolds of the preimages are as desired.

We can also see that singular set is finite. In fact, they correspond to the points regarded as the "poles" of the spheres of the boundaries of the disks chosen in the previous procedure. By the assumption that for adjacent manifolds F_j and F_{j+1}, either F_j or F_{j+1} is not a sphere, we have exactly l singular points. The Reeb graph of the function is isomorphic to G. The function is, at least topologically, a Morse function.

This completes the proof. □

The *degree* of a vertex of a graph means the number of edges containing it.

FIGURE 1 of [6] shows two explicit cases for the paper.

We discuss the lower figure. Let $l > 1$ be an integer. The lower figure shows a graph with exactly 2 vertices of degree 1, exactly 2 vertices of degree $l+1$ and exactly $l+2$ edges. Furthermore, we can explain about the graph as follows.

- The first two vertices are denoted by v_{l} and v_{r}, respectively.
- The other two vertices are denoted by v_1 and v_2, respectively.
- Two of the edges are denoted by e_{l} and e_{r}, respectively. e_{l} connects v_{l} and v_1. e_{r} connects v_{r} and v_2.
- The remaining l edges are denoted by e_j where $1 \leq j \leq l$ is an integer. They connect v_1 and v_2.

More precisely, these two graphs show two simplest examples of the so-called *Poincaré-Reeb graphs.* A *Poincaré-Reeb* graph is defined for a pair of an algebraic domain in a real affine space and a canonical projection of the real affine space to the 1-dimensional real affine space. This is defined as a graph to which the algebraic domain naturally collapses.

Main Theorem 2. *Let $m > 1$ be an integer. Let G be a graph just before where we abuse the notation.*

Let F_{l} and F_{r} be the unit spheres S^{m-1}. Let $\{F_j\}_{j=1}^{l}$ be a family of smooth manifolds each of which is the unit sphere or represented as a connected sum of finitely many manifolds diffeomorphic to the products $S^j \times S^{m-j-1}$ with integers $1 \leq j \leq m-2$. The connected sum is taken in the smooth category.

Then we have some m-dimensional real algebraic closed and connected manifold M and a real algebraic function $f : M \to \mathbb{R}$ enjoying the following properties.

1. *We have a suitable isomorphism $\phi : G \to W_f$ of the graphs and we have the following three.*
   (a) *For the edge e_{l} of the graph G and each point $p_{e_{\mathrm{l}}}$ in the interior of the edge $\phi(e_{e_{\mathrm{l}}}) \subset W_f$, the preimage ${q_f}^{-1}(p_{e_{\mathrm{l}}})$ is diffeomorphic to F_{l}.*
   (b) *For the edge e_{r} of the graph G and each point $p_{e_{\mathrm{r}}}$ in the interior of the edge $\phi(e_{e_{\mathrm{r}}}) \subset W_f$, the preimage ${q_f}^{-1}(p_{e_{\mathrm{r}}})$ is diffeomorphic to F_{r}.*
   (c) *For the edge e_j of the graph G and each point p_{e_j} in the interior of the edge $\phi(e_j) \subset W_f$, the preimage ${q_f}^{-1}(p_{e_j})$ is diffeomorphic to F_j for each integer $1 \leq j \leq l$.*

(2) *The singular set $S(f)$ is a finite set.*

Proof. It is almost same as the proof of Main Theorem 1. The only one different part is, to consider a similar induction starting from an algebraic domain D_0 in $\mathbb{R}^2$ whose closure is a disk centered at the origin 0 and whose radius is $R > 0$.

As before, we consider two distinct real numbers $t_2 < t_3$ in $[-R, R]$. We choose R as a sufficiently large number. Put $t_1 := -R$ and $t_4 := R$. We define the number r_j similarly. In the first step, we consider a similar procedure. Except this, we consider the same induction.

Except this, we can prove similarly. This completes the proof.

□

By the construction and the structures of the maps and the manifolds, we also have the following. We can know this by an argument same as that for the preimage $f^{-1}(r_j)$ in the proof of Main Theorem 1.

Main Theorem 3. *The manifold M in Main Theorems 1 and 2 is one in Example 1 as a smooth manifold.*

In the construction, the following is also important. For this, see also Remark 1, presented in the end.

Main Theorem 4. *The function f in Main Theorems 1 and 2 is represented as the composition of a real algebraic map which is, at least topologically, regarded as a special generic map into $\mathbb{R}^m$ with a canonical projection to the 1-dimensional real affine space.*

3.2. **Previously obtained answers to Problem 2.** We present some previously obtained answers to Problem 2 of the author. The following is one of results closely related to our new results.

Theorem 2 ([4])**.** *Let a graph G whose vertex set and edge set are finite be given. Let l be a map from the edge set into the set of all non-negative integers. Then we have some 3-dimensional closed and orientable manifold M and a smooth function $f : M \to \mathbb{R}$ enjoying the following properties.*

1. *There exists an isomorphism $\phi : G \to W_f$ from G to the Reeb graph W_f.*
2. *For each edge e of G, consider the element $\phi(e)$ of the Reeb graph of W_f and choose an arbitrary point p_e in the interior of $\phi(e)$. The preimage ${q_f}^{-1}(p_e)$ is a closed, connected and orientable surface of genus $l(e)$.*
3. *If f does not have a local maximum or local minimum at a singular point of f, then it is locally a Morse function.*
4. *If f has a local maximum or local minimum at a singular point of f, then it is locally either of the following four.*
   (a) *A Morse function. This occurs if and only if for the singular point p_s, $q_f(p_s)$ is a vertex contained in the exactly one edge e_s of the graph and $l(e_s) = 0$ holds.*
   (b) *A Morse-Bott function which is not Morse.*
   (c) *A function which is not Morse-Bott and represented as the composition of a Morse function onto the interior of the 1-dimensional unit disk with a height function.*
   (d) *A function which is not as the presented previous three cases and which is represented as the composition of a fold map onto the interior of the*

2-dimensional unit disk with a height function. This occurs if and only if for the singular point p_s, $q_f(p_s)$ is a vertex contained in the exactly one edge e_s of the graph and $l(e_s) \neq 0, 1$ holds.

We present another previous result in a weaker form. This is also closely related to our new results.

Theorem 3 ([9]). *Let $m \geq 2$ be an integer. Let a graph G whose vertex set and whose edge set of G are finite be given. Let the following two maps be given.*

- *A continuous function $g : G \to \mathbb{R}$ enjoying the following properties.*
  - *The restriction of g to the each edge of the graph, which is a 1-dimensional cell, is injective.*
  - *g has a local maximum or a local minimum only at a vertex contained in exactly one edge in the graph.*
- *A map l on the edge set of G into the set of all diffeomorphism types of smooth manifolds as in the following.*
  - *$(m-1)$-dimensional unit spheres.*
  - *Manifolds represented as connected sums of finitely many manifolds of the forms $S^j \times S^{m-j-1}$ with $1 \leq j \leq m-2$ taken in the smooth category.*

  Furthermore, $l(e_v)$ is the diffeomorphism type of the unit sphere if the edge e_v contains a vertex v where g has a local maximum or a local minimum $g(v)$.

Then we have some m-dimensional closed and orientable manifold M and a Morse function $f : M \to \mathbb{R}$ enjoying the following properties.

1. *There exists an isomorphism $\phi : G \to W_f$ from G to the Reeb graph W_f.*
2. *For each element e of the edge set of G, consider the element $\phi(e)$ of the edge set of W_f and choose an arbitrary point p_e in the interior of $\phi(e)$, the diffeomorphism type of preimage ${q_f}^{-1}(p_e)$ is $l(e)$.*
3. *$\bar{f}(\phi(v)) = g(v)$ for each vertex v of G.*

Compare these theorems to our main results. These results are considered in the smooth category. We do not discuss in the real algebraic or the real analytic category in these studies. For example, [22] constructs global smooth functions by using so-called *bump functions* or ones closely related to them. [5] also use such functions. These functions are not real analytic.

It seems to be very difficult whether we can extend our new results to some important situations as in the smooth cases.

Last, we present a short comment mainly on Main Theorem 4. It is also one on Theorem 3 and Main Theorems.

Remark 1. For example, in Theorem 3, consider a case such that the degree of each vertex is 1 or 3 and that the diffeomorphism types are those of unit spheres. In this case, we can have a Morse function f such that at distinct singular points of f the (singular) values are always distinct. We consider a smooth embedding into a sufficiently high dimensional Euclidean space $e : M \to \mathbb{R}^{n_0}$ and the map $(e, f) : M \to \mathbb{R}^{n_0} \times \mathbb{R} = \mathbb{R}^{n_0+1}$, which is regarded as a smooth embedding. By the celebrated theory of Nash, presented as Theorem 3 of [11] for example, together with some theory from singularity theory of smooth maps for example, we can smoothly isotope it to a smooth real algebraic embedding by some small perturbation. By composing the canonical projection to the second 1-dimensional real affine space, we also have a Morse function regarded as one smoothly isotopic to the original one. This is regarded as an real algebraic function.

We cannot apply such arguments in general. For example, in Theorem 3, we cannot have Morse functions such that at distinct singular points the values are distinct in general.

Instead, we have a new method and have a real algebraic function with a nice representation as in Main Theorem 4.

For construction of Morse functions here in the smooth category, see also [15], having motivated us to present [4].

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https://sites.google.com/view/suzukimasahiko70/home.
This conference celebrates Masahiko Suzuki's 70th birthday and we would like to celebrate again. The talk is given by Osamu Saeki. Their study gives an explicit smooth map on the 3-dimensional real projective space into $\mathbb{R}^2$ as the restriction of a suitably chosen complex linear function on the 3-dimensional complex space to the intersection of the 5-dimensional unit sphere with the zero set of the polynomial function defined canonically from the complex polynomial $f(z_1, z_2, z_3) = {z_1}^2 + {z_2}^2 + {z_3}^2$. This comes from Milnor's celebrating theory [16]. The more we discuss the problems, the more we know. This leads us to [6], followed by the present paper.

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