GEODESICS AND VALUES OF QUADRATIC FORMS

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1. Introduction

Theorem 1.1. *Let p be a prime then the equation*

$$x^2 = -1$$

admits a solution in $\mathbb{F}_p$ iff $p = 2$ or $p - 1$ is a multiple of 4.

Theorem 1.2 (Fermat)**.** *Let p be a prime then the equation*

$$x^2 + y^2 = p$$

has a solution in integers iff $p = 2$ or $p - 1$ is a multiple of 4.

There are many proofs of these theorems but the approach initiated by Heath-Brown in [11] has inspired many admirers if not imitators see for example the very nice account of Elsholtz [7]. In some senses this manuscript is a companion to Elsholtz's where instead of looking at the number theory as combinatorics we work in an explicitly geometric context. As such we refer the reader to Elsholtz for historical perspective and the like.

1.1. **Involutions.**

The essential ingredients in the Heath-Brown paper are: a finite set X equipped with a pair of involutions such that:

- Any fixed point of the one of the involutions, should it exist, is a solution of the equation;
- The other involution has a unique fixed point which is easy to compute.

The existence of the unique fixed point of the second involution allows one to conclude that the X has an odd number of elements and so that any involution has a fixed point.

Probably the most elegant incarnation of this method is Dolan's proof [6].

1.2. **Arcs on a punctured surface.**

The starting point for this work was a series of observations concerning λ-lengths of simple arcs on a once punctured torus equipped with a hyperbolic structure for example $\mathbb{H}/\Gamma'$ where $\Gamma' < \mathrm{SL}(2,\mathbb{Z})$ is the commutator subgroup. The reader, unfamiliar with the relationship between hyperbolic geometry and number theory, should consult Springborn's articles [20, 21] where a dictionary between geometric and number theoretic notions is presented.

An important feature of a punctured surface is that it admits *uniform cusp regions* that is there is a neighborhood of each puncture, or more properly cusp, isometric to

$$\{z \in \mathbb{H}, \mathrm{Re} z \geq 1\}/\langle z \mapsto z+2\rangle$$

The torus $\mathbb{H}/\Gamma'$ can be obtained from a pair of ideal triangles by "gluing" and the sides of the ideal triangles form a triple of complete simple geodesics on the surface. The corners of the triangles glue up to a neighborhood of the puncture. We define an *arc* to be any complete geodesic on a punctured surface with both of its ends terminating at cusps. The three sides of the ideal triangle(s) above form a triple of disjoint simple arcs. Each of these arcs has infinite length but if we only consider the portion outside the uniform cusp region then its length is finite. The λ-length of the arc is exponential of half the length of this finite portion. Whilst this definition works well for simple arcs, since the portion of the arc outside the uniform cusp region is connected, more care is needed if the arc is not simple.

Since Penner first defined λ-length there has been much work on their applications

- Weil-Peterson volume of moduli space [17]
- cluster algebras [8].
- Conway-Coxeter frieze patterns [5].

Our approach depends on determining the number of arcs on a surface of λ-length p. Probably the easiest surface to count arcs on is $\mathbb{H}/\Gamma(2)$.

Theorem 1.3. *Let p be a prime number then the number of arcs of λ-length p on the surface $\mathbb{H}/\Gamma(2)$ is $3(p-1)$.*

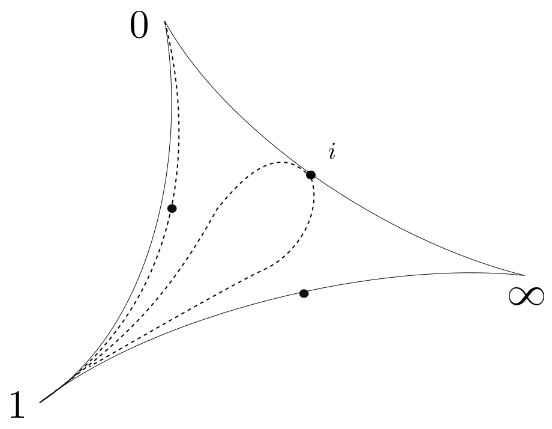

FIGURE 1. The surface $\mathbb{H}/\Gamma(2)$. The dotted lines are arcs one of which is a loop of λ-length 2, and the other has λ-length 1.

The surface is $\mathbb{H}/\Gamma(2)$ is not compact and has three cusps which, as we shall see, are naturally associated with $0, 1, \infty \in \partial\mathbb{H}$. This is a manifestation of the fact that $\Gamma(2)$ acts on the extended rationals $\mathbb{Q} \cup \{\infty\}$. By *parity* we mean one of the three classes one obtains by taking the numerator and denominator of $\frac{p}{q}$ modulo 2 where, by convention, $\infty = \frac{1}{0}$. Actually $\mathbb{H}/\Gamma(2)$ is a degree 6 cover of the modular surface $\mathbb{H}/\Gamma$. Unfortunately this covering is ramified at 2 points and the projection of an arc on $\mathbb{H}/\Gamma(2)$ may or may not be an arc. The problem lies in that an arc may be invariant by some involution in the covering group and as a consequence connects the cusp on $\mathbb{H}/\Gamma$ to a ramification point. In fact

Theorem 1.4. *Let p be a prime number then the number of arcs of λ-length p on the modular surface $\mathbb{H}/\Gamma$ is*

- *p-1 if 4 does not divide p-1,*
- *p-2 if p=2 or 4 divides p-1.*

As we shall see in Section **??**, this result is indeed equivalent to Theorem 1.2.

Computing the -length of an arc α on $\mathbb{H}/\Gamma(2)$ is relatively simple. One considers a lift $\hat{\alpha}$ of the arc. This is a Poincaré geodesic joining a pair of distinct rationals $a/c, b/d \in \mathbb{Q}$ and the λ-length is just the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Evidently the arcs on $\mathbb{H}/\Gamma(2)$ fall into two families:

- arcs that join distinct cusps.
- arcs that have both ends at the same cusp

We will refer to the second kind of arc as a *loops* (see Figure 1). Amusingly one can characterise loops using λ-lengthis.

Lemma 1.5. *An arc on $\mathbb{H}/\Gamma(2)$ is a loop if and only if its λ-length is even.*

2. Klein four group and the Burnside Lemma

We give a proof of Theorem 1.1 using the Burnside Lemma.

Recall that if G is a group acting on a finite set X then the Burnside Lemma says

$$|G||X/G| = \sum_{g} |X^g| \tag{1}$$

where, as usual, X^g denotes the set of fixed points of the element g and X/G the orbit space.

Let $p \neq 2$, $X = \mathbb{F}_p^*$ and G be the group generated by the two involutions

$$\begin{aligned} x &\mapsto -x \\ x &\mapsto 1/x. \end{aligned}$$

The group G has exactly four elements namely:

- the trivial element which has $p-1$ fixed points
- $x \mapsto -x$ which has no fixed points
- $x \mapsto 1/x$ has exactly two fixed points namely 1 and -1.
- $g : x \mapsto -1/x$ is the remaining element and the theorem is equivalent to the existence of a fixed point for it.

Note that since $\mathbb{F}_p$ is a field $|X^g| = \sharp\{x^2 = -1,\, x \in \mathbb{F}_p^*\}$ is either 0 or 2. Now for our choice of X and G equation (1) yields

$$4|X/G| = (p-1) + 2 + |X^g|. \tag{2}$$

The LHS is always divisible by 4 so the RHS is too and it follows from this that

$$|X^g|. = \begin{cases} 0 & \text{if } (p-1) = 2 \mod 4 \\ 2 & \text{if } (p-1) = 0 \mod 4 \end{cases}$$

This proves Theorem 1.1.

2.1. **Extending.** Thus we have shown that -1 is a quadratic residue modulo p if p is of the form $4k+1$. It is natural to consider the other questions considered by Fermat: namely for which values of p are -2 and -3 residues?

In fact -2 is a residue if p is 2 or of the form $8k+1$ or $8k+3$. Showing this in the spirit of Heath-Brown requires one to consider a group generated by the involutions

$$\begin{aligned} x &\mapsto -x \\ x &\mapsto 2/x. \end{aligned}$$

One immediately sees that things are more complicated as the second involution has fixed points if and only if 2 is a residue whereas $x \mapsto 1/x$ always had exactly two fixed points. Thus there are two cases:

- $p = 8k+1$ and both 2 and -2 are residues
- $p = 8k+3$ and -2 is a residue but 2 is not.

To prove this second assertion one must show that $x \mapsto 2/x$ has no fixed point so that, by Burnside, $x \mapsto -2/x$ has two fixed points both of which are square roots of -2. Thus one must show that the only solution of the associated diophantine equation

$$np = x^2 - 2y^2,$$

is the trivial solution $n = x = y = 0$. Now using the fact that $x^2 - 2y^2$ is the norm of $x + y\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ which is a euclidean ring for this norm, one reduces to considering just the solutions of

$$p = x^2 - 2y^2.$$

Finally, one concludes by showing that if x, y are integers then $x^2 - 2y^2$ never takes the value $3 \mod 8$.

2.2. **The case $p = 11$.** The first real case of interest in understanding $x \mapsto -2/x$ is that of $\mathbb{F}_{11}^*$. Evidently, $11 = 3^2 + 2 \times 1^2$ so that $\bar{3}$ and $-\bar{3} = \bar{8}$ rare the fixed points of the involution.

The reduction homomorphism $x \mapsto \bar{x}$ allows one to identify the elements of $\mathbb{F}_p$ with the equivalence classes that constitute the quotient $\mathbb{Z}/p\mathbb{Z}$. It is the usual to choose the integers $0, 1, 2 \dots p-1$ as representatives for the latter, however, we shall find it convenient to work with another set of representatives, the even integers $0, 2, 4 \dots 2p-2$. Using the euclidean algorithm to compute $\bar{x}^{-1} \in \mathbb{F}_{11}$ we have the following table:

x	12	2	14	4	16	6	18	8	20	10
$\bar{x}$	1	2	3	4	5	6	7	8	9	10
$\bar{x}^{-1}$	12	6	4	14	20	2	8	18	16	10
$-2\bar{x}^{-1}$	20	10	14	16	4	18	6	8	12	2

One notes that there are two fixed points of $-\bar{x} \mapsto -2\bar{x}^{-1}$ namely $\overline{14} = \bar{3}$ and $\bar{8}$.

3. Counting sums of squares

We count solutions for the diophantine problem $n = mc^2 + d^2$ in two ways:

- Firstly by showing that solutions are naturally associated to the Γ orbit of $i\sqrt{m}$;
- Secondly by counting arcs of λ-length n.

3.1. **From solutions to Γ orbits.** The transformation $z \mapsto z+1$ generates an infinite cyclic group acting on $\mathbb{H}$. The standard fundamental domain for this group is an infinite strip, which we will refer to as the *fundamental strip*, consisting of all the $z \in \mathbb{C}$ such that the real part is between 0 and 1.

Lemma 3.1. *Let $n \geq 2$ be an integer. The number of ways of writing n as a sum of squares*

$$n = c^2 + d^2$$

with c, d coprime integers is equal to the number of points of $\Gamma.\{i\}$, the $\mathrm{SL}(2,\mathbb{Z})$ orbit of i, in the fundamental strip such that the imaginary part (euclidean height) is $\frac{1}{n}$.

Note that we are counting $c^2 + d^2$ and $d^2 + c^2$ as *different* representations of n.

Proof. Suppose there is a point w verifying the hypotheses, in particular

$$w = \frac{ai+b}{ci+d}, \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Then one has:

$$\operatorname{Im} w = \operatorname{Im}\left(\frac{ai+b}{ci+d}\right) = \frac{\operatorname{Im} i}{c^2+d^2},$$

so $n = c^2 + d^2$ as claimed.

Conversely if c, d are coprime integers such that $n = c^2 + d^2$ then there exists a, b such that

$$ad - bc = 1 \Rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

By applying a suitable iterate of the parabolic transformation $z \mapsto z+1$, if necessary one can choose w such that $0 \leq \operatorname{Re} w < 1$.

□

By exactly the same argument one has a slightly more general result:

Lemma 3.2. *Let $n \geq 2$ be an integer and $m < n$ a square free integer. The number of ways of writing n as a sum of squares*

$$n = mc^2 + d^2$$

with c, d coprime integers is equal to the number of points of $\Gamma.\{i\sqrt{m}\}$ the $\mathrm{SL}(2, \mathbb{Z})$ orbit of $i\sqrt{m}$, in the fundamental strip at euclidean height $\frac{\sqrt{m}}{n}$.

3.2. **From Γ orbits to arcs.** Suppose that n is can be written as a sum of squares $c^2 + d^2$ and w is the corresponding point in the fundamental strip then we can associate a Poincaré geodesic to w in a natural way, we simply take the vertical line that passes through w. This geodesic joins two points in the ideal boundary of $\mathbb{H}$ namely ∞ and $\frac{ac+bd}{n} \in \mathbb{Q}$. Penner's λ-length This geodesic projects to an arc on the surface $\mathbb{H}/\Gamma(2)$ and, using the definition above, its λ-length is n. If n is not even then this arc is not a loop and so joins a pair of distinct cusps ∞ to 1 say.

4. Inversions

We denote by $\mathbb{H}$ the Poincaré upper half plane and $\partial\mathbb{H}$ its ideal boundary ie $\mathbb{R} \cup \{\infty\}$. Recall that an *inversion* is an orientation reversing isometry of $\mathbb{H} \cup \partial\mathbb{H}$. A Poincaré geodesic is either a vertical line or a semicircle orthogonal to $\mathbb{R}$. In both cases it is uniquely determined by its endpoints in the ideal boundary. To each Poincaré geodesic is associated a unique inversion which fixes it pointwise. The inversion $\phi_h : z \mapsto -\bar{z}$ fixes 0 and ∞ and so the geodesic joining them. The group of isometries acts transitively on pairs of distinct points $a, b \in \partial\mathbb{H}$ and so there is an inversion that fixes the geodesic joining a, bwhich is in fact conjugate to ϕ_h. The inversion fixing $1, -1$ is easily seen to be $\phi_v : z \mapsto \frac{1}{\bar{z}}$.

Note that if a, b are coprime integers then:

- The image of $\frac{a}{b}$ under ϕ_h is $-\frac{a}{b}$ and the λ-length of the geodesic joining them is $2ab$.
- The image of $\frac{a}{b}$ under ϕ_v is $\frac{b}{a}$ and the λ-length of the geodesic joining them is $|a^2 - b^2|$.

It follows from these remarks that:

Lemma 4.1. *Let $p > 2$ be a prime then:*

- *There is no arc of λ-length p invariant under ϕ_h.*
- *There are exactly two arcs of λ-length p invariant under ϕ_v;*

Proof. The first part is easy because the λ-length of such an arc is an even integer. The second part follows from the fact that p factorises as

$$p = |(a - b)||(a + b)|,$$

so up to permutation and change of sign a, b are the integers $\frac{1}{2}(p \pm 1)$. $\square$

5. Congruence subgroups

The Hecke congruence subgroup $\Gamma_0(N)$ of level N is the subgroup of $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is a normal. It is a subgroup of $\Gamma_0(N)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \mathrm{mod}\ N.$$

For $N = 2$ this is generated by just two elements namely:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The product $P^{-1}Q$ is an element of order 2:

$$P^{-1}Q = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}.$$

So the quotient $\mathbb{H}/\Gamma_0(2)$ is a non-compact orbifold with two cusps and a single cone point. This orbifold admits a Klein four group as its group of orentations and the quotient by this group is a hyperbolic triangle with angles $0, \pi/2, \pi/4$.

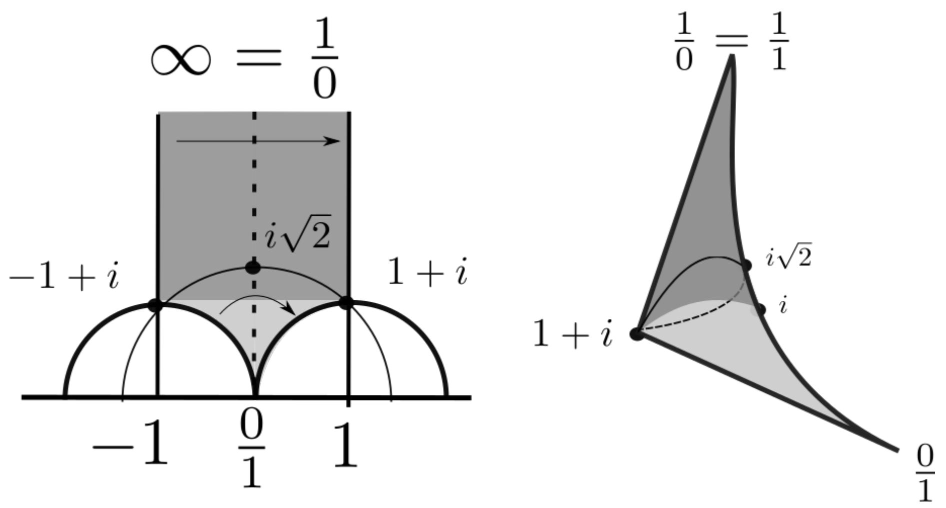

FIGURE 2. On the left a fundamental domain for $\Gamma_0^t(2)$ with side pairings. On the right the quotient surface $\mathbb{H}/\Gamma_0^t(2)$, the dark region is a cusp region.

The action of this group on $\mathbb{Q} \cup \{\infty\}$ is not transitive and there are two orbits. Now $\Gamma_0(2) < \Gamma(2)$ so each of these orbits is a union of $\Gamma(2)$-orbits. Since $\Gamma(2)$ preserves the parity of the numerator and denominator of a fraction there are exactly three $\Gamma(2)$ orbits corresponding to $\frac{0}{1} = 0$, $\frac{1}{1} = 1$, $\frac{1}{0} = \infty$. Now since P maps 0 to one:

$$\Gamma_0(2)\{0\} = \Gamma(2)\{0\} \cup \Gamma(2)\{1\}.$$

In the previous section we considered the action of the involution $x \mapsto -2/x$ on $\mathbb{F}_p^*$. It is natural to study the action of the corresponding involution of $\mathbb{H}$ that is $z \mapsto -2/z$ but unfortunately this does not normalise $\Gamma_0(2)$. However it does normalise the *anti Hecke congruence-group*:

$$\Gamma_0^t(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod N \right\} < \Gamma(2).$$

The Hecke congruence group and the anti Hecke group are isomorphic and in fact $z \mapsto -1/z$ conjugates them in Γ . We can determine the orbits of this group on $\mathbb{Q}$ using this conjugation and we have:

$$\Gamma_0^t(2)\{\infty\} = \Gamma(2)\{\infty\} \cup \Gamma(2)\{1\}.$$

We denote by F the set $\{z, \operatorname{Im} z > 1\}$ this is a *horoball in* $\mathbb{H}$ centered at ∞. The image of F under the $\mathrm{SL}(2,\mathbb{Z})$ action consists of F and infinitely many disjoint circles, the so-called *Ford circles*, each tangent to the real line at some rational m/n. We adopt the convention that F is also a Ford circle of infinite radius. If $G < \mathrm{SL}(2,\mathbb{Z})$ is any finite index subgroup then each Ford circle projects to a cusp region on $\mathbb{H}/G$ and we call this system the *canonical system of cusp regions*.

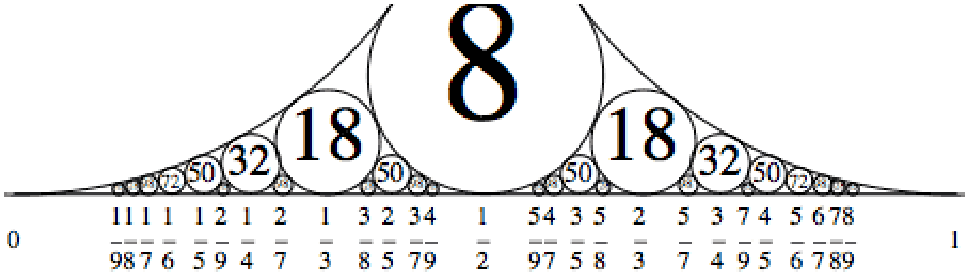

FIGURE 3. Ford circles with tangent points and curvatures. Recall that the curvature of a euclidean circle is the reciprocal of its radius.

The following is well known and is easily checked:

Lemma 5.1. *The Ford circle tangent to the real line at m/n has Euclidean diameter $1/n^2$.*

Corollary 5.2. *The λ-length of the arc joining $a/c, b/d \in \mathbb{Q}$ is the absolute value of the determinant of the associated matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. There exists $b' \in \mathbb{Z}$ and a matrix $A' \in \mathrm{SL}(2, \mathbb{Z})$ such that the product $A'A$ is an upper triangular matrix:

$$A'A = \begin{pmatrix} 1 & b' \\ 0 & \det A \end{pmatrix}.$$

The image of a/c under the Mobius transformation associated to A' is infinity and the image of b/d is $b'/\det A$. The Ford circle at ∞ is F and the diameter of the circle tangent at $b'/\det A$ is $(\det A)^2$.

$\square$

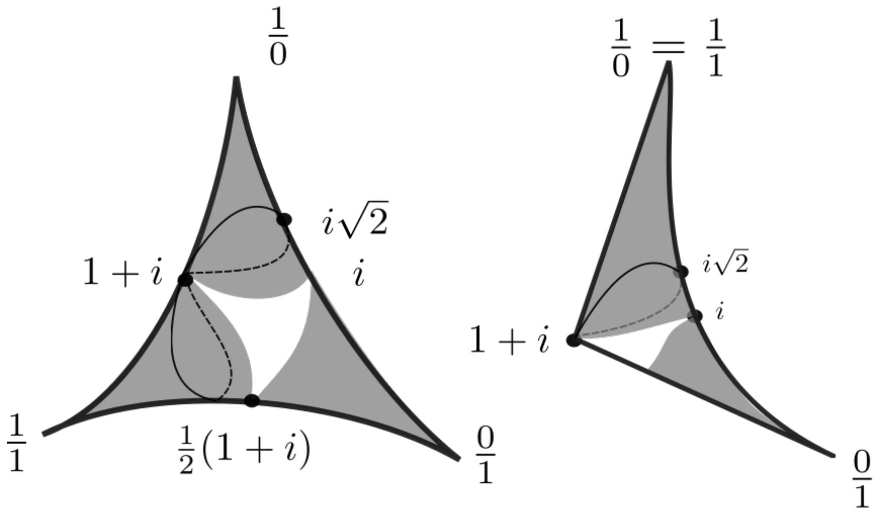

FIGURE 4. On the left $\mathbb{H}/\Gamma(2)$ with the cusp regions inherited from the Ford circles $\mathbb{H}$. On the right $\mathbb{H}/\Gamma_0^t(2)$ with the unmodified cusp regions.

5.1. **Cusp regions on $\mathbb{H}/\Gamma_0^t(2)$.** The canonical system on $\mathbb{H}/\Gamma(2)$ consists of three cusp regions one for each of the three cusps $0, 1, \infty$. The map $z \mapsto z/z+1$ fixes 0 and normalises $\Gamma(2)$, so induces an automorphism, in fact an involution of $\mathbb{H}/\Gamma(2)$ which fixes the cusp labeled 0. The quotient of $\mathbb{H}/\Gamma(2)$ by the involution is naturally identified with the surface $\mathbb{H}/\Gamma_0^t(2)$ inherits a system of cusp regions from $\mathbb{H}/\Gamma(2)$ via the quotient map. The involution $z \mapsto -2/z$ normalises $\Gamma_0^t(2)$ so induces an automorphism of $\mathbb{H}/\Gamma_0^t(2)$ which fixes the points labelled $1+i$ and $i\sqrt{2}$, swaps the cusps labelled $\frac{1}{0}$ and $\frac{0}{1}$ but which does not swap the cusp regions inherited from $\mathbb{H}/\Gamma(2)$. In fact a computation shows that the cusp region for $\frac{1}{0}$ has area 2 whilst the cusp region for $\frac{0}{1}$ has area 1. We remedy this by choosing a pair of cusp regions which are tangent at the fixed point of the automorphism and which both have area $\sqrt{2}$. To do this

- the cusp region for $1/0$ shrinks by a factor of $\sqrt{2}$
- whilst the other cusp region for $0/1$ expands by $\sqrt{2}$.

The lifts of this modified pair of cusp regions to $\mathbb{H}$ form a family of circles each of which, like the Ford circles, is tangent to the real line at a rational $\frac{m}{n} \in \mathbb{Q}$. However, the diameter of the circle tangent at m/n is no longer $\frac{1}{n^2}$ as in Lemma 5.1

- $\sqrt{2} \times 1/n^2$ if m is even.
- $1/\sqrt{2} \times 1/n^2$ if m is odd.

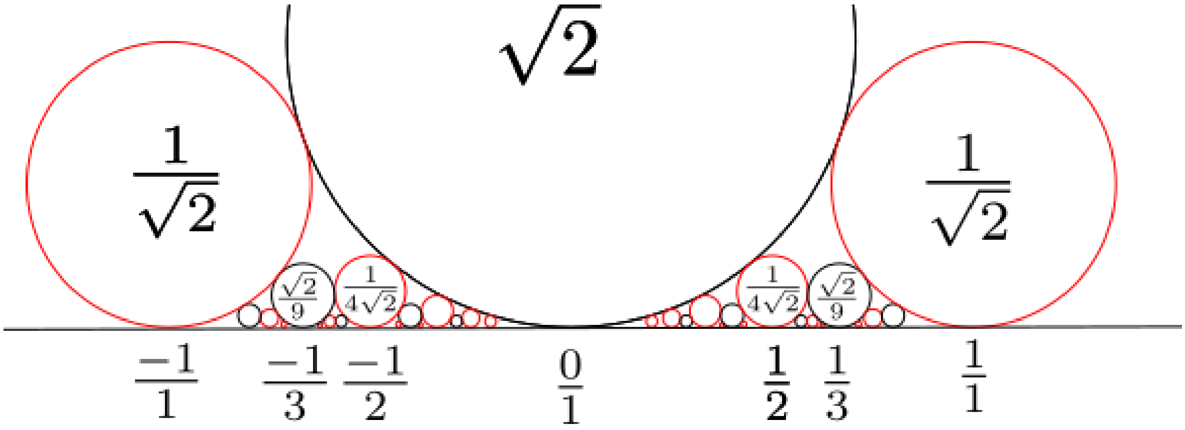

FIGURE 5. Modified Farey circles

5.2. **Arcs on** $\mathbb{H}/\Gamma_0^t(2)$**.** The surface has two cusps and so there are two kind of arc

- arcs that join distinct cusps $0/1$ and $1/0$
- arcs that have both ends at the same either cusps $0/1$ or $1/0$.

Lemma 5.3. *Arcs of the first kind, that is those which join distinct cusps* $0/1$ *and* $1/0$*, have the same* λ*-length for the inherited cusp regions and our modified cusp regions.*

Since the automorphism swaps cusps only arcs of the first kind can be invariant for it. Now any arc of the first kind lifts to a vertical line ending at some rational $\frac{m}{n} \in \mathbb{Q}$. It follows that for each p prime there are exactly $p-1$ arcs of the first kind, namely the projections to $\mathbb{H}/\Gamma_0^t(2)$ of the Poincaré geodesics $\infty, \frac{2k}{p}$ with $k = 1, 2 \ldots p-1$ and each of these has λ-length p for our choice of cusp regions.

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