DIOPHANTINE TRIPLES AND THE PTOLEMY RELATION

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Abstract. We discuss the construction of Diophantine triples from the Farey diagram and the action of the modular group on the set of such triples.

This is a collaboration with GitHub Copilot and ChatGPT.

1. Introduction

A *Diophantine triple* is a set of three distinct positive integers $\{a, b, c\}$ such that the product of any two integers from the set, increased by 1, results in a perfect square. In other words, for a Diophantine triple $\{a, b, c\}$, the following conditions must hold:

$$ab + 1 = \text{perfect square} \tag{1}$$
$$ac + 1 = \text{perfect square} \tag{2}$$
$$bc + 1 = \text{perfect square} \tag{3}$$

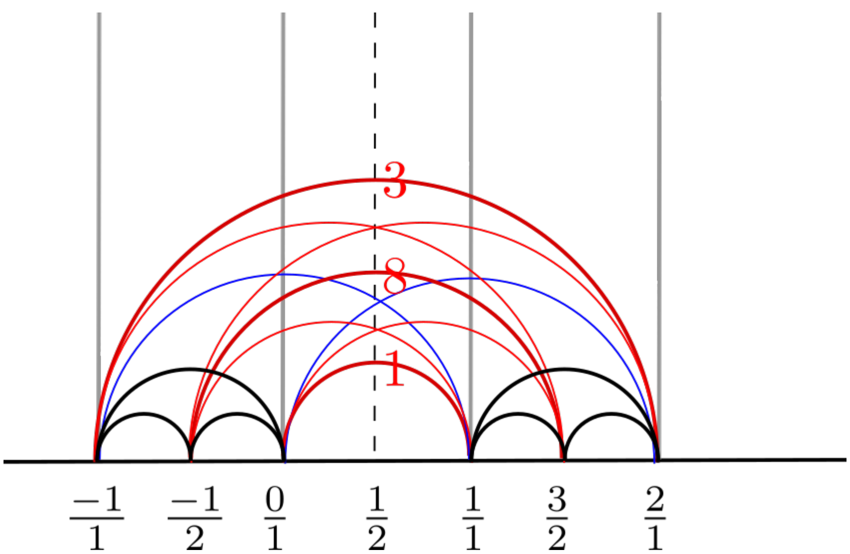

Figure 1. A pair of ideal triangles from the farey diagram that are swapped by the involution $z \mapsto 1 - \bar{z}$. The λ-length of the three central semicircles invariant under this involution are respectively $3, 8$ and 1.

For example, one well-known Diophantine triple is $\{1, 3, 8\}$, because:

2000 *Mathematics Subject Classification.* 57K20,14J50,11A41,11E25.

This work was partially funded by the Equipe Action ToFu part of Persyval-Lab.

$$1 + 3 \times 8 = 5^2 \tag{4}$$
$$1 + 3 \times 1 = 2^2 \tag{5}$$
$$1 + 8 \times 1 = 3^2 \tag{6}$$

There are also Diophantine quadruples and higher sets. A Diophantine set is a set of positive integers A with the property that the product of any two distinct elements of A increased by 1 is a perfect square There is a vast literature, dating back to Diophantus of Alexandria see the survey by Dujella [2] for an account. The most important result states that such sets A can have at most five elements, and there are only finitely many of them with five elements [1].

In this note we give a geometric construction of Diophantine triples using the Farey diagram and the notion of λ-length due to Penner [7]. The λ-length of each edge of an ideal triangle in the Farey diagram is 1. We begin with a pair of ideal triangles in the Farey diagram which are swapped by the involution $z \mapsto 1 - \bar{z}$. The relations (1), (2) and (3) arise naturally from the Ptolemy relation for an ideal quadrilateral in the Farey diagram.

1.1. **Thanks.** It is a pleasure to thank Louis Funar, Hidetoshi Masai, Robert Penner, Vlad Sergiescu, and Xu Binbin for discussions and suggestions over the years. I thank Allen Hatcher for his suggestions concerning Figure 2 the code for generating these figures can be found at [6].

Some of the text of this paper was suggested by GitHub Copilot[11, 12] and ChatGPT.

2. Geometry of the Farey diagram

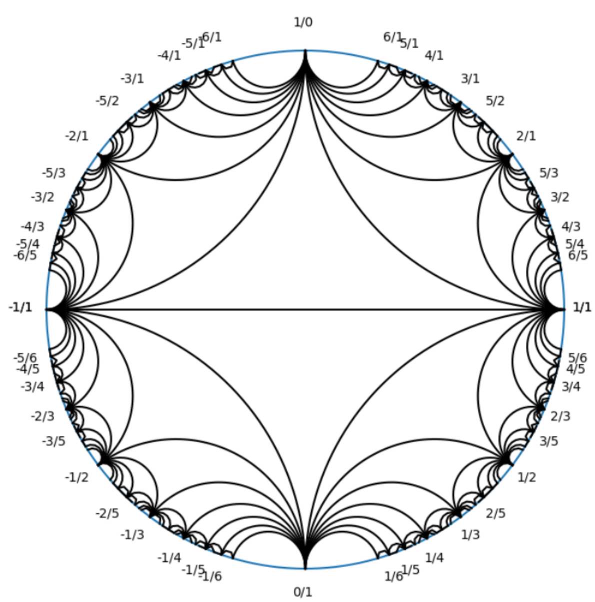

Figure 2. Farey diagram.

2.1. **Farey diagram and λ-lengths.** The Farey diagram (see Figure **??**) is a geometric representation of the relationships between rational numbers, closely associated with the

Farey sequence and the Stern–Brocot tree. The Farey diagram is often visualized as a set of points on the unit circle or as a triangulation of the hyperbolic plane. The diagram has applications in number theory [8], hyperbolic geometry, and the study of continuedfractions [9, 10].

It provides a visual and geometric interpretation of how rational numbers are related to each other, particularly their mediants and properties of the modular group.

2.2. **Construction.** The Farey diagram can be constructed by representing rational numbers as points on the real line viewed as the ideal boundary of the Poincaré half plane $\mathbb{H}$. Pairs of rational numbers $\frac{a}{c}, \frac{b}{d}$ are joined by a Poincaré geodesic (often referred to as an *arc*) if the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is ± 1 and such a pair are called *Farey neighbors.* The mediant of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ is given by:

$$\text{mediant}\left(\frac{a}{c}, \frac{b}{d}\right) = \frac{a+b}{c+d}$$

Note that the mediant of a pair of Farey neighbors is a Farey neighbor of each of them. A triple of Farey neighbors form the set of vertices of an ideal triangle in the Farey diagram: for example the triple $1 = 1/1, 2 = 2/1, 3/2$ in Figure 1 is an ideal triangle in the Farey diagram. More generally, if $b/d < a/c$ are Farey neighbors then the triple $b/d < (a+b)/(c+d) < a/c$ are the vertices of an ideal triangle in the Farey diagram.

2.3. **Symmetries.** The Farey diagram is invariant under the action of the modular group

$$\Gamma = \mathrm{PSL}(2, \mathbb{Z}) < \mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}.$$

An element of $\mathrm{PSL}(2, \mathbb{Z})$ acts on the Farey diagram by fractional linear transformations that is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The action is transitive on the set of Farey neighbors since the image of $\infty = 1/0$ is a/c and the image of $0 = 0/1$ is b/d. The Farey diagram is also invariant under the orientation reversing involutions

$$\begin{aligned} z &\mapsto -\bar{z} \\ z &\mapsto 1 - \bar{z}. \end{aligned}$$

Note that these involutions are not conjugate by an element of $\mathrm{PSL}(2, \mathbb{Z})$.

2.4. **λ-lengths.** Penner introduced the notion of λ-length to study the geometry of the decorated Teichmüller space of a punctured surface [7]. We give an equivalent formulation for arcs in the half plane $\mathbb{H}$ joining pairs of extended rationals $\frac{a}{c}, \frac{b}{d}$ and define the λ-length of an arc to be the absolute value of the determinant of the matrix:

$$\begin{pmatrix} a & b \\ c & d. \end{pmatrix}$$

Although we will not use it here, we note that this has a geometric interpretation: the λ-length is the exponential of half the length of the portion of the arc outside the Ford circles tangent at $a/c, b/d$.

The *Ptolemy relation* is a classical result from Euclidean geometry that relates the lengths of the sides and diagonals of a cyclic quadrilateral (a quadrilateral inscribed in a circle). It states that for any cyclic quadrilateral, the sum of the products of its two pairs of opposite sides is equal to the product of its diagonals. The relation provides an important bridge between geometry and algebraic structures, particularly when studying configurations of points and lengths. Penner proved a version of the Ptolemy relation for the λ-lengths of the sides of an ideal quadrilateral.

Lemma 2.1. If A, A' and B, B' denote the λ-lengths of opposite sides and D, D' the λ-lengths of the diagonals of an ideal of an ideal quadrilateral then the Ptolemy relation is:

$$A.A' + B.B' = D.D'.$$

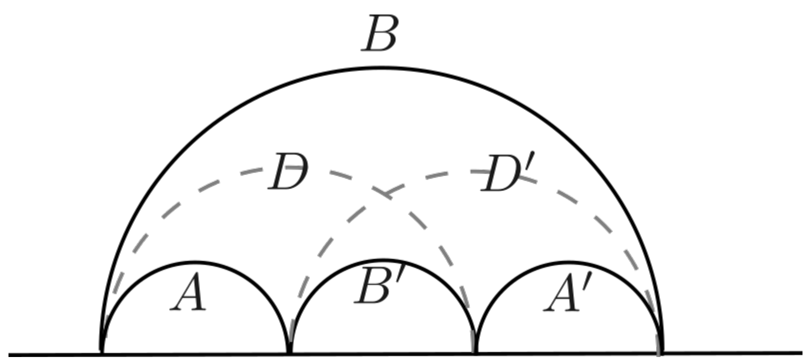

FIGURE 3. The Ptolemy relation for an ideal quadrilateral.

To illustrate this consider one of the ideal quadrilaterals in Figure 1, the quadrilateral with vertices $-1, -1/2, 3/2, 2$ say, then the λ-lengths of the sides are $B = 8, B' = 3$ and $A = A' = 1$ the diagonals have the same length $D = D' = 5$ and so the Ptolemy relation gives:

$$1 \times 1 + 3 \times 8 = 5 \times 5,$$

which is (4) above.

2.5. **Main result. Check this!!!!**

Theorem 2.2. If $a/c < (a+b)/(c+d) < b/d$ are a triple of Farey neighbors then

$$(2b-d)d, (2a-c)c, (2(a+b)-c-d)(c+d)$$

is a Diophantine triple.

This can be checked by direct calculation. For example for the pair $(2b-d)d, (2a-c)c$ Figure 4 have:

$$\begin{aligned} B &= (2b-d)d \\ B' &= (2a-c)c \\ D &= (2ad-cd-1) \\ D' &= (2bc-cd+1) \end{aligned}$$

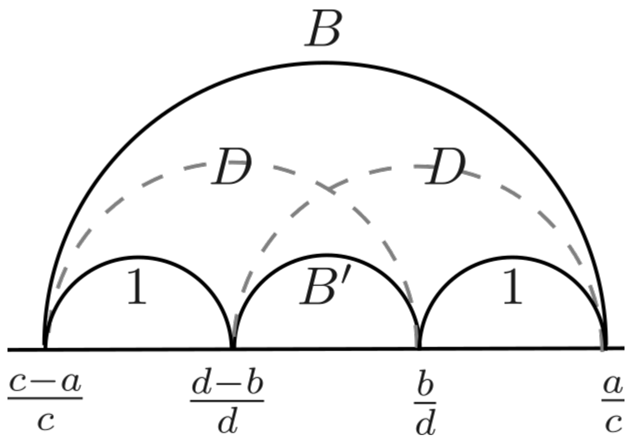

FIGURE 4. Proof that $1 + B \times B'$ is a square.

Note that $D - D' = 0$ since $ad - bc = 1$. Then

$$\begin{aligned} B \times B' &= (2b-d)d \times (2a-c)c \\ &= 4abcd + (cd)^2 + 2bc^2d + 2acd^2 \\ &= 4abcd + (cd)^2 - 1 + 2bc^2d + 2acd^2 + 2 - 1 \\ &= 4abcd + (cd)^2 - 1 + 2bc^2d + 2acd^2 - cd + bc + 2(bc - ad) - 1 \\ &= (2ad - cd - 1)(2bc - cd + 1) - 1 \\ &= D \times D' - 1 \end{aligned}$$

3. Concluding remarks

We have given a geometric construction of Diophantine triples.One hopes that this will lead to a better understanding of Diophantine sets and their relation to the modular group.

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