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Braided surfaces and Seifert ribbons for closed braids

LEE RUDOLPH[(1)]

Abstract. A *positive band* in the braid group B_n is a conjugate of one of the standard generators; a negative band is the inverse of a positive band. Using the geometry of the configuration space, a theory of bands and *braided surfaces* is developed. Each representation of a braid as a product of bands yields a handle decomposition of a *Seifert ribbon* bounded by the corresponding closed braid; and up to isotopy all Seifert ribbons occur in this manner. Thus, *band representations* provide a convenient calculus for the study of ribbon surfaces. For instance, from a band representation, a Wirtinger presentation of the fundamental group of the complement of the associated Seifert ribbon in D^4 can be immediately read off, and we recover a result of T. Yajima (and D. Johnson) that every Wirtinger-presentable group appears as such a fundamental group. In fact, we show that every such group is the fundamental group of a Stein manifold, and so that there are finite homotopy types among the Stein manifolds which cannot (by work of Morgan) be realized as smooth affine algebraic varieties.

Contents

§0. Introduction

Stallings, reporting [S] on constructions of fibred knots and links, mentions (almost in passing) a construction which associates to any braid $\beta \in B_n$ a certain Seifert surface in S^3 bounded by the closed braid $\hat{\beta}$. Actually – and importantly – that construction begins *not* with a braid (an element of the group B_n) but with a

[1] Research partially supported by NSF grant MCS 76-08230

braid word (an expression of the braid as a word in the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n, and their inverses). Stallings describes the constructed Seifert surface as being plumbed together from $n-1$ simpler surfaces.[2] More naively, the surface is simply given as a handlebody: the union of n (2-dimensional) 0-handles connected by orientable 1-handles whose number and location are specified by the particular braid word.

The plumbing description, in Stallings's context of "homogeneous braids," is appropriate because it shows that the surface constructed from a homogeneous braid (word) is actually a fibre surface for the closed braid. In this paper I hope to show that the naive handlebody description, and a generalization of it which produces *Seifert ribbons*, can be appropriate in other contexts.

This work fits into a circle of ideas going back to Alexander, E. Artin, van Kampen, and Zariski. In 1923, without bringing the (then undiscovered) *braid groups* into it, Alexander [Al] showed that every (tame) link type contains representative *closed braids*. In other words: the construction that begins with a braid $\beta \in B_n$ and produces an oriented link $\hat{\beta} \subset S^3$ is perfectly general – every link type can be so produced. Artin introduced the braid groups B_n in 1925, giving algebraic structure to the geometric braids, and used that algebraic structure to describe (among other things) a class of group presentations which included presentations of precisely the link groups $\pi_1(S^3 - L)$. Meanwhile, Zariski [Z1, Z2] was investigating the groups $\pi_1(\mathbf{C}P^2 - \Gamma)$, where Γ was a (possibly singular) complex algebraic curve, and seems actually to have commissioned van Kampen to prove the now-famous "van Kampen's Theorem" [vK] precisely to get presentations of those groups – which are of course intimately related to the groups $\pi_1(\mathbf{C}^2 - \Gamma)$.

A *ribbon surface* in the 4-disk D^4 is a 2-manifold-with-boundary embedded in a certain restricted way (see §§1 and 2, below). A (non-singular) piece of algebraic curve is, as it turns out, always a ribbon surface (cf. [Mi]). In §2 I show how, from a braid β together with $\vec{b}$, an expression for β as a word in certain generators of B_n (the set of conjugates of the standard generators), one can construct a ribbon surface in D^4 bounded by (a link of the type of) $\hat{\beta}$; and in §3 I show that this construction is perfectly general, and produces representatives of each isotopy class of orientable ribbon surface. One may say that Alexander's theorem is the boundary of these results. (A modification of the construction produces, equally generally, "ribbon immersions" in S^3; and in particular all ordinary Seifert surfaces can be constructed from "embedded band representations" of braids.)

In §4 a presentation for the group $\pi_1(D^4 - S(\vec{b}))$, of the form called a *Wirtinger presentation*, is derived from $\vec{b}$. Every group that has a Wirtinger presentation at

[2] This kind of plumbing was first discovered by Murasugi [Mu].

all, has one of the sort that appears here (and from such a presentation $\vec{b}$ is immediately read off). Thus we recover Dennis Johnson's improvement [J] of T. Yajima's [Y] result that any group with a Wirtinger presentation can be realized as $\pi_1(S^4 - S)$ for some smooth orientable surface S (the improvement being in the ribbon-like nature of the surface, see below). Actually, I show somewhat more, and as an application show that each such group also appears as the fundamental group of a Stein manifold (in fact a complex surface in $\mathbf{C}^5$). John Morgan [Mo] has ruled out many groups, for instance $(x, y : [x, [x, [x, [x, y]]]] = 1) = G$ say, from being fundamental groups of (affine) smooth algebraic varieties; but G, as it happens, has a Wirtinger presentation.

Ribbon genus and related matters are attacked in §5. An appendix indicates how the work can be extended to (alternatively) ribbon surfaces with nodes in D^4, or surfaces immersed in S^3 with both ribbon and clasp singularities.

The long §1 lays the groundwork for the rest of the paper, relating information about the geometry of the *configuration space* (the space of which B_n is the fundamental group) to algebraic information about B_n and geometric information about surfaces in S^3 and D^4.

§1. Loops and disks in the configuration space: closed braids, braided surfaces, and band representations

Apparently it was only as late as 1962 that topologists first realized that "B_n may be considered as the fundamental group of the space ... of configurations of n undifferentiated points in the plane" (this "previously unnoted remark" being then made by Fox and Neuwirth [F–N, p. 119]).[(3)] In this section some further relations among the geometry of that space, the geometry of links and surfaces, and the algebra of the braid group, will be explored. Simply for convenience here, the plane $\mathbf{R}^2$ (in which the configurations of n points lie) will be identified with the complex line $\mathbf{C}$; for a further application of the theory, where the complex structure is really at the heart of things, see [Ru].

By identifying the complex n^{th} degree monic polynomial $\prod_{j=1}^{n}(w - w_j) = w^n + c_1 w^{n-1} + \cdots + c_{n-1}w + c_n$ with on the one hand the un-ordered n-tuple $\{w_1, \ldots, w_n\}$ of its roots, and on the other hand the ordered n-tuple $(c_1, \ldots, c_n)$ of its non-leading coefficients, we effect the well-known identification of $\mathbf{C}^n/\mathfrak{S}_n$ with $\mathbf{C}^n$. (The symmetric group $\mathfrak{S}_n$ on n letters acts on $\mathbf{C}^n$ by permuting the coordinates.) Now, $\mathbf{C}^n/\mathfrak{S}_n$ (being the quotient of $\mathbf{C}^n$ by a finite group of automorphisms) inherits from $\mathbf{C}^n$ a natural structure of (singular, affine) algebraic

[3] Magnus [M], in a review of [Bi], indicates that Hurwitz, [Hu], studying monodromy in 1891, had in fact noted this definition.

variety; its singular locus $\mathcal{S}(\mathbf{C}^n/\mathfrak{S}_n)$ is the quotient by $\mathfrak{S}_n$ of the multi-diagonal in $\mathbf{C}^n$, that is, it contains exactly those n-tuples $\{w_1, \ldots, w_n\}$ in which for some $j \neq k$, $w_j = w_k$. But, via the identification of $\mathbf{C}^n/\mathfrak{S}_n$ (the space of roots) with $\mathbf{C}^n$ (the space of coefficients), we also give $\mathbf{C}^n/\mathfrak{S}_n$ a non-singular structure, which is the normalization and the minimal resolution of the quotient structure. Let us denote $\mathbf{C}^n/\mathfrak{S}_n$ with this non-singular structure by E_n, and let Δ denote its subset which "is" the old singular locus. Then Δ is a hypersurface of the affine space E_n; when $n \geqq 3$, Δ is singular. (Algebraic geometers know Δ as the *discriminant locus.*) Still, a smooth map of a manifold into E_n may be perturbed arbitrarily slightly to make it transverse to Δ, since Δ is the image of a smooth manifold (any one of the hyperplanes $w_j = w_k$ back in the multidiagonal of the space of roots) by a smooth map. (Incidentally, this resolution shows that Δ is irreducible, so that its regular set $\mathcal{R}(\Delta)$ is connected, a fact we need later.) In particular, all the transversality we will need in the sequel is collected in the following lemma.

LEMMA 1.1. *Let M be a compact, smooth manifold-with-boundary of dimension no greater than 3. Then any smooth map $f: M \to E_n$ may be perturbed by an arbitrarily small homotopy to a smooth map which misses the singular locus $\mathcal{S}(\Delta)$ entirely (since $\mathcal{S}(\Delta)$ has real codimension 4) and which intersects the smooth, codimension-2 manifold $\mathcal{R}(\Delta)$ of regular points of Δ transversely. If $f \mid \partial M$ is already transverse to Δ in this sense then the homotopy need not alter $f \mid \partial M$.* □

The (open, dense) set $E_n - \Delta \subset E_n$ is the *configuration space* (of n "undifferentiated points in the plane"). The fundamental group $\pi_1(E_n - \Delta)$ (we will suppress basepoints whenever it is decent to do so) is called the *braid group B_n*. (Its structure will be recalled later. General reference: [Bi].) Since E_n is contractible, every loop $f: \partial D^2 \to E_n - \Delta$ extends to a map $f: D^2 \to E_n$ – we can assume f is smooth, and by Lemma 1.1, transverse to Δ. Now, what is called a *geometric braid* is nothing more nor less than a loop in $E_n - \Delta$. What then is such an extension to a map of a disk?

DEFINITION 1.2. A (smooth) *singular braided surface* in a bidisk $D = D_1^2 \times D_2^2 = \{(z, w) \in \mathbf{C}^2 : |z| \leqq r_1,\ |w| \leqq r_2\}$ is a (smooth) map of pairs $i: (S, \partial S) \to (D, \partial_1 D)$ (here $\partial_1 D$ denotes the solid torus $\partial D_1^2 \times D_2^2$ which is half of the boundary of D), such that

(1) $pr_1 \circ i: (S, \partial S) \to (D_1^2, \partial D_1^2)$ is a branched covering map (and an honest covering on the boundaries),

(2) S is so oriented that $pr_1 \circ i$, away from its finite set of branch points, is orientation preserving (with respect to the complex orientation of $D_1^2 \subset \mathbf{C}$).

From (1) we see that S is orientable, so (2) makes sense.

The degree n of the branched covering $pr_1 \circ i$ is the *degree* of the braided surface; all but finitely many points $z \in D_1^2$ have n distinct preimages in S.

By an *embedded braided surface* in D let us mean a singular braided surface for which i is a smooth embedding, or, by abuse of language, also the image $i(S) \subset D$ of such an i.

EXAMPLE 1.3. If Γ is a complex-analytic curve in a neighborhood of D (possibly analytically reducible, but without multiple components), so situated that $\Gamma \cap \partial D$ is the transverse intersection of $\mathcal{R}(\Gamma)$ and $\partial_1 D$, then the normalization of $\Gamma \cap D$ mapping into D is a singular braided surface; and if there are no singularities of the curve inside D then it is an embedded braided surface. (Such analytic curves motivated these investigations, but by no means exhaust the examples.)

Here is the connection between braided surfaces and the configuration space.

On the one hand, given a smooth map $f:(D_1^2, \partial D_1^2) \to (E_n, E_n - \Delta)$ for which $f^{-1}(\Delta)$ is a finite subset of Int D_1^2, one can create a singular braided surface $f^{\#}$ in $D_1^2 \times D_2^2$, where the second radius r_2 is any strict upper bound for the absolute value of all elements w_j in all n-tuples $\{w_1, \ldots, w_n\} = f(z)$ for $z \in D_1^2$. (Begin by considering the set $S'_f = \{(z, w) \in D : w \in f(z)\}$. Then there is a finite subset $X \subset S'_f$ so that $S'_f - X$ is a genuine n-sheeted covering space of $D_1^2 - pr_1(X)$, embedded as a submanifold of D, with pr_1 as covering projection. Just from the continuity of f it is easy to resolve the singularities of S'_f, yielding a surface-with-boundary S_f on which the map $f^{\#}$ is forced; and this is clearly the desired singular braided surface. Note that its degree is n.)

On the other hand, given a singular braided surface $i:(S, \partial S) \to (D, \partial_1 D)$, of degree n, there is a corresponding smooth map $i_{\#}:(D_1^2, \partial D_1^2) \to (E_n, E_n - \Delta)$: on the set of those $z \in D_1^2$ where $\{w : (z, w) \in i(S)\}$ has n distinct elements, one sets $i_{\#}(z) = \{w : (z, w) \in i(S)\}$; again the extension to all z in D_1 is forced.

Note also that if f, as above, is transverse to Δ, then $f^{\#}$ is an embedded braided surface, and is also "in general position" – meaning here that branch points of $pr_1 \circ f^{\#}$ are all "simple vertical tangents". And conversely, given i as above, $i_{\#}$ will be transverse to Δ only if S is in fact embedded and its vertical tangents are all simple. Of course, any embedded braided surface is arbitrarily close (isotopic through embedded braided surfaces) to an embedded braided surface in general position.

Recall that a surface embedded in $D^4 = \{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leqq 1\}$ is a *ribbon* *surface* if the restriction to the surface of $|z|^2+|w|^2$ is a Morse function, identically 1 on the boundary, which may have saddles as well as local minima, but which has no local maxima. Ribbon surfaces in D^4, and the related ribbon immersions in S^3 and $\mathbf{R}^3$, will be discussed in greater detail in §§2 and 3. Here we will make the connection to braided surfaces.

PROPOSITION 1.4. *If $S \subset D$ is an embedded braided surface then there is an isotopic deformation of D to D^4 (in $\mathbf{C}^2$) which carries S onto a ribbon surface.*

Of course D has corners and D^4 is smooth, but the isotopy will be smooth except near the corners of D, which without loss of generality are missed by S.

Proof. After a slight perturbation of S, perhaps, the function $L_0(z, w) = |z|^2$ will, when restricted to S, be a Morse function with n (the degree of S) minima, a saddle point for each branch point, and no local maxima, and it will be identically 1 on ∂S. Then for small $\varepsilon > 0$, $L_\varepsilon = |z|^2 + \varepsilon\,|w|^2$, when restricted to S, has the same properties, except that it is not quite constant on the boundary. A small isotopy of D, supported near its own boundary, will fix $L_\varepsilon \mid \partial S$. The rest is clear. □

Remark 1.5. A consequence of the construction in §3 is that a converse to this proposition holds – every (orientable!) ribbon surface is isotopic to an embedded braided surface. This is the exact analogue, for ribbon surfaces, of Alexander's theorem [Al] for links, that they all occur as closed braids. I don't know a more direct proof of this converse.

Next we will dip into the algebra of B_n for a while.

The *standard generators* of B_n are $\sigma_1, \ldots, \sigma_{n-1}$. (With respect to a basepoint $* \in E_n$, for instance $* = \{1, \ldots, n\}$, σ_j is represented by a loop which as a motion of the n points leaves all but j and $j+1$ fixed constantly, while exchanging j and $j+1$ by a *counterclockwise* 180° rotation [this is the East Coast convention!].) The *standard presentation* of B_n is $B_n = (\sigma_1, \ldots, \sigma_{n-1} : R_i \quad (i = 1, \ldots, n-2), \quad R_{ij} \quad (1 \leqq i < j-1 \leqq n-1))$, where $R_i : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $R_{ij} : \sigma_i\sigma_j = \sigma_j\sigma_i$ are the *standard relations*. All the standard generators belong to one conjugacy class: for R_i may be rewritten as $\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1} = (\sigma_i\sigma_{i+1})\sigma_i(\sigma_i\sigma_{i+1})^{-1}$ and so by induction each σ_i is conjugate to σ_1. Also, this class is not equal to its inverse, and in the infinite cyclic abelianization of B_n, each generator σ_i maps to 1.

For reasons which will be evident in the next section, I call any element of the conjugacy class of σ_1 a *positive band*. The inverse of a positive band (i.e., a conjugate of σ_1^{-1}) is a *negative band*. A *band* is a positive or negative band.

In any group, I will use the notation ${}^{a}b$ to denote the conjugate aba^{-1}, when convenient.

For $n>2$ there are infinitely many bands in B_n. (B_2 is infinite cyclic.) Intermediate between the set of $2(n-1)$ standard generators and their inverses, and the set of all bands, is a set of $(n-1)n$ *embedded bands*. The positive embedded bands are $\sigma_{i,j} = {}^{A(i,j-1)}\sigma_j$, where (just here) $A(i, j-1) = \sigma_i \cdots \sigma_{j-1}$, and $1 \leqq i \leqq j \leqq n-1$. (So $\sigma_{j,j} = \sigma_j$.)

NOTATION 1.6. An ordered k-tuple $\vec{b} = (b(1), \ldots, b(k))$ with each $b(i)$ a band in B_n (of either sign) is a *band representation* (*in* B_n) *of the braid* $\beta(\vec{b}) = b(1) \cdots b(k)$, which we call *the braid of* $\vec{b}$. The *length* $l(\vec{b})$ is k. Conventionally, the braid of the unique 0-tuple is the identity of B_n.

If each $b(i)$ is an embedded band we call $\vec{b}$ an *embedded band representation*. If each $b(i)$ is a standard generator or the inverse of a standard generator, we identify $\vec{b}$ with a *braid word* in the usual sense; in that case length is more usually called *letter length*.

Since every braid is the braid of some braid word, it makes sense to define the *rank of* β in B_n, written $rk_n(\beta)$ or $rk(\beta)$, to be the least k such that some band representation of β has length k. Only the identity has rank 0. Rank is constant on conjugacy classes, and is less than or equal to "least letter length" (the analogue of rank when only braid words and not all band representations are used) and greater than or equal to the absolute value of the exponent sum (an invariant of words in the free group on $\sigma_1, \ldots, \sigma_{n-1}$ which clearly passes on to B_n). A band representation in which each band is positive is a *quasipositive band representation* and its braid is a *quasipositive braid* (cf. [Ru]); the length of a quasipositive band representation equals the exponent sum and the rank, and equals the least letter length if and only if the braid of the representation is actually a positive braid in the usual sense.

Remark 1.7. The notion of band is algebraic, geometric in E_n, *and* (as we shall see) geometric in S^3. The notion of embedded band is not algebraic, and seems to be geometric only in the latter context. Thus the idea of "embedded rank" seems to be unnatural and will be ignored.

There are some natural operations that relate different band representations of the same braid β. (Perhaps some natural incidence structure, of the "building" sort, awaits discovery in the set of such representations.) Let $\vec{b} = (b(1), \ldots, b(k))$, $k \geqq 2$. If for some j between 1 and $k-1$ we have $b(j)b(j+1) = 1 \in B_n$, then $(b(1), \ldots, b(j-1), b(j+2), \ldots, b(k))$ is another band representation of the same braid, gotten by *elementary contraction at the* j^{th} *place*.If j is between 1 and $k+1$

$(k \geqq 0)$, and a is any band, then the *elementary expansion of* $\vec{b} = (b(1), \ldots, b(k))$ *by* a *at the* j^{th} *place* is the band representation of the same braid $\vec{b}' = (b'(1), \ldots, b'(k+2))$ with $b'(i) = b(i)$ $(i<j)$, $b'(j) = a$, $b'(j+1) = a^{-1}$, $b'(i) = b(i-2)$ $(i > j+1)$.

Now let $1 \leqq j < k = l(b)$. The effect of S_j, the *forward slide at the* j^{th} *place*, is to replace $\vec{b}$ with $S_j\vec{b} = (b'(1), \ldots, b'(k)) : b'(i) = b(i)$ if $i \neq j$, $j+1$; $b'(j) = {}^{b(j)}b(j+1)$; and $b'(j+1) = b(j)$. The effect of S_j^{-1}, the *backward slide at the* j^{th} *place*, is to replace $\vec{b}$ with $S_j^{-1}\vec{b} = (b'(1), \ldots, b'(k)) : b'(i) = i$ if $i \neq j$, $j+1$; $b'(j) = b(j+1)$; $b'(j+1) = {}^{b(j+1)^{-1}}b(j)$. It is easy to check that $\beta(\vec{b}) = \beta(S_j\vec{b}) = \beta(S_j^{-1}\vec{b})$ and that S_j and S_j^{-1} are, indeed, inverse to each other.

(After preparing this paper, the author became aware of Moishezon's work [Moi] on "braid monodromies" of complex plane curves. My slides are Moishezon's "elementary transformations"; because he is dealing purely with what I have called quasipositive band representations, he does not introduce expansions and contractions.)

For a fixed $k \geqq 2$, the $k-1$ slides $S_1, \ldots, S_{k-1}$ generate a group which acts on the set of all band representations (of various braids) of length k. It is readily checked that these slides satisfy the standard relations $R_i(S_1, \ldots, S_{k-1})$ and $R_{ij}(S_1, \ldots, S_{k-1})$, and therefore mediate an action of the braid group B_k on this set of length-k band representations. Let two band representations (necessarily of the same braid) which are in the same B_k-orbit be called *slide-equivalent*. This will be elucidated in the next section, and in Prop. 1.11.

EXAMPLE 1.8. Let (a, b) be a band representation of length 2. It is easily checked that $S_1^{2m}(a, b) = ({}^{(ab)^m}a, {}^{(ab)^{m-1}a}b)$, $S_1^{2m-1}(a, b) = ({}^{(ab)^{m-1}a}b, {}^{(ab)^{m-1}}a)$ for any $m \in \mathbf{Z}$.

Remark 1.9. It is tempting to conjecture that a single slide-equivalence class should fill out the set of band representations of β of a given length, at least when that length is the rank of β. This fails to be true. For instance, in B_3, $(\sigma_1, \sigma_2^2\sigma_1\sigma_2^{-2})$ and $(\sigma_2^{-1}\sigma_1\sigma_2, \sigma_2\sigma_1\sigma_2^{-1})$ have the same braid and (being quasipositive) are of minimal length for that braid, but they are not slide-equivalent. (Sketch of proof: For typographical convenience, let σ_1 and σ_2 be abbreviated to 1, 2, respectively. Using Example 1.8 it suffices to show that ${}^{2^{-1}}1$ cannot be written as ${}^{(1\cdot{}^{22}1)^m}1$ or as ${}^{(1\cdot{}^{22}1)^m 1}2 \equiv {}^{(1\cdot{}^{22}1)^m 2^{-1}}1$ for any integer m. Now, in any group, three elements u, v, x satisfy ${}^ux = {}^vx$ if and only if uv^{-1} commutes with x. So we have to show that $(1 \cdot {}^{22}1)^m$ and $(1 \cdot {}^{22}1)^m 2$ don't commute with 1 for any m. A straightforward but unilluminating computation in $SL(2, \mathbf{Z})$, using the well-known representation $\sigma_1 \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, suffices to verify this.) It happens that these two band representations are "conjugate" in the obvious sense (by σ_2^{-1}) but not all examples of this phenomenon arise so simply.

Remark 1.10. In §4 we will see an example of a braid β of rank 2, and a band representation $\vec{b}$ of β of length 4, which is not slide-equivalent to any elementary expansion of any band representation of the minimal length 2.

Our next task is to relate band representations to disks in E_n. Now we fix a basepoint $* \in E_n - \Delta$, and identify B_n with $\pi_1(E_n - \Delta, *)$. For each $k \geqq 1$, moreover, we fix a set P_k of k distinct interior points of D^2 – let us be definite and say $P_k = \{1/m - 1 \in \mathbf{C} : m = 1, \ldots, k\} \subset D^2 = \{z \in \mathbf{C} : |z| \leqq 1\}$. Let the basepoint of D^2 be $* = -\sqrt{-1}$. Then $\pi_1(D^2 - P_k; *)$ is the free group of rank k on free generators x_j $(j = 1, \ldots, k)$, where x_j is the class of a loop consisting of a straight line segment from $*$ to a point on the circle of radius $1/2k(k-1)$ centered at $1/j - 1$, followed by the circle traversed once counterclockwise, followed by the segment back to $*$. If h is a diffeomorphism of D^2 to itself which is the identity on ∂D^2 and which preserves P_k as a set, then the automorphism $h_* : \pi_1(D^2 - P_k; *) \to \pi_1(D^2 - P_k; *)$ satisfies $h_*([\partial D^2]) = [\partial D^2]$, where $[\partial D^2]$ is the homotopy class of the (counterclockwise oriented) boundary of D^2, namely, $x_1 x_2 \cdots x_k$. It is a fact (cf. [Bi]) that the group of all such automorphisms h_* is naturally isomorphic to the braid group B_k; a diffeomorphism which is supported in a $1/2k(k-1)$-neighborhood of the interval

$$\left[\frac{1}{m+1} - 1, \frac{1}{m} - 1\right] \subset D^2$$

and rotates the interval 180° counterclockwise will induce the automorphism Σ_m corresponding to σ_m.

PROPOSITION 1.11. (i) *Let $f : (D^2, \partial D^2, *) \to (E_n, E_n - \Delta, *)$ be smooth and transverse to Δ, and suppose that $f^{-1}(\Delta)$ contains precisely k points. Let $h : D^2 \to D^2$ be a diffeomorphism, fixing ∂D pointwise, such that $h(P_k) = f^{-1}(\Delta)$. Then the k-tuple $((f \circ h)_* x_1, \ldots, (f \circ h)_* x_k)$ is a band representation in B_n, and its braid is $\beta = f_*([\partial D^2])$. The band representations which correspond to different choices of h are slide-equivalent, and vice versa.*

(ii) *Conversely, given a band representation $\vec{b}$ of β, and a smooth map $f : (\partial D^2, *) \to (E_n - \Delta, *)$ such that $f(\partial D^2)$ (oriented counterclockwise) represents β, then there is a smooth extension of f over the whole disk D^2, $f : (D^2, \partial D^2, *) \to (E_n, E_n - \Delta, *)$, which is transverse to Δ, with $f^{-1}(\Delta) = P_k$, and such that the band $b(j)$ equals $f_* x_j$ $(j = 1, \ldots, k)$. Such an extension is unique up to homotopy. If f is an embedding on ∂D^2 the extension may be taken to be an embedding also, unique up to isotopy.*

Proof. If $k=1$, then (i) says that a loop (through $*$) which bounds a disk that meets Δ transversely in exactly one point represents a band in B_n; and the existence half of (ii) says that every band arises like this. Both statements are true: for, indeed, an obvious explicit loop representing σ_1 (as in [Bi, p. 18]) bounds an equally obvious disk of the sort required in (i), and (up to orientation) all loops which bound such disks are conjugate in B_n because (by transversality) the map $\pi_1(E_n-\Delta, *)\to\pi_1(E_n-\mathscr{R}(\Delta), *)$ induced by inclusion is an isomorphism and (as remarked before Lemma 1.1) $\mathscr{R}(\Delta)$ is a *connected* submanifold of E_n of codimension 2.

Now, to prove (i) for any k, note that since $(f\circ h)_*:\pi_1(D^2-P_k;*)\to\pi_1(E_n-\Delta;*)$ is a homomorphism, certainly the product $(f\circ h)_*x_1\cdots(f\circ h)_*x_k$ equals $(f\circ h)_*(x_1\cdots x_k)$ which is $f_*([\partial D^2])$ since h is the identity on ∂D^2; and by the case $k=1$, each braid $(f\circ h)_*x_j$ is indeed a band; so we do have a band representation of β. A different choice of h corresponds to composing the original $f\circ h$ on the right with a diffeomorphism of D^2 which fixes the boundary pointwise and P_k as a set, and therefore to composing the original $(f\circ h)_*$ on the right by an automorphism in the group generated by the Σ_j's. But one quickly sees that, on the level of band representations, Σ_m corresponds to the forward slide S_m. So (i) is proved for all k.

As to (ii), given b one readily constructs a map g from a bouquet of k disks $\bigvee_{j=1}^k(D_j^2, *)$, identified at a common boundary point $*$, into E_n so that each restriction $g\mid D_j^2$ is smooth and transverse to Δ, meeting it at a single point, and taking ∂D_j^2 to a loop in the class $b(j)$. Then there is a map $q:(\partial D^2, *)\to(\bigvee_{j=1}^k\partial D_j^2, *)$ with $(g\circ q)_*[\partial D^2]=\beta$; and $g\circ q$ is homotopic (rel. $*$) to the given map f *in the complement of* Δ. Using q to glue the annulus (which is the domain of the homotopy between f and $g\circ q$) to $\bigvee_{j=1}^k D_j^2$, one creates a disk D^2 and a continuous extension of f from ∂D^2 across D^2. This extension is smooth on the boundary and near the preimage of Δ, to which it is transverse; and a small perturbation will preserve those properties, while rendering the extension smooth everywhere. Two different extensions differ, up to homotopy, by an element of $\pi_2(E_n-\Delta)$ but according to [F–N] the space $E_n-\Delta$ is a $K(B_n, 1)$: so any two extensions of f are homotopic. Finally, if $n>2$ the assertions about embeddings and isotopies are easy by general position, the ambient dimension being then at least 6; while if $n=2$, B_2 is $\mathbf{Z}$ and what little there is to be said can be justified by *ad hoc* arguments. $\square$

The following proposition shows how any two band representations of a braid are related. The proof given is geometric; the algebraically-minded reader may supply an algebraic proof.

PROPOSITION 1.12. *Two band representations of β in B_n may always be joined by a finite chain in which adjacent band representations differ either by an elementary expansion or contraction or by a forward or backwards slide.*

Proof. Let $f: D^2 \to E_n$ be smooth and transverse to Δ. Then the natural (complex) orientations of D^2 and $\mathcal{R}(\Delta)$ give the finite set $f^{-1}(\Delta)$ an orientation – the sign of a point equals the sign of a corresponding band. Let $F: D^2 \times I \to E_n$ be a homotopy between two such maps $f_i = F(\cdot, i)$, $i = 0, 1$, with $F \mid \partial D^2 \times \{t\}$ independent of t, and F smooth and transverse to Δ in the interior of the solid cylinder $D^2 \times I$. Then the set $F^{-1}(\Delta)$ is a smooth 1-manifold-with-boundary in $D^2 \times I$, with $\partial(F^{-1}(\Delta)) = f_0^{-1}(\Delta) \cup f_1^{-1}(\Delta)$; and in fact $F^{-1}(\Delta)$ has a natural orientation for which, as a relative cycle, $\partial F^{-1}(\Delta) = -f_0^{-1}(\Delta) + f_1^{-1}(\Delta)$. After possibly a small perturbation we can assume that $pr_2 \mid F^{-1}(\Delta): F^{-1}(\Delta) \to I$ is a Morse function. For all but critical values $t_1, \ldots, t_N$, $F(\cdot, t): D^2 \to E_n$ gives a band representation of the braid $[f_0(\partial D^2)]$. The band representations just below and above a local minimum (resp., maximum) differ by an elementary expansion (resp., an elementary contraction). In an interval without critical points, F is an isotopy rel. Δ and the band representations at the ends of such an interval differ by a sequence of slides (slides really appear: it may not be possible, as it were, to choose a fixed normal form for the disks $D^2 \times \{t\}$ over the whole interval). $\square$

Note that $F^{-1}(\Delta) \subset D^2 \times I$ may well be knotted and linked. There is a homomorphism $\pi_1(D^2 \times I - F^{-1}(\Delta)) \to B_n$ which takes meridians to bands. In a picture, it can be helpful to label arcs of the diagram of $F^{-1}(\Delta)$ with names of bands.

EXAMPLE 1.13. Figure 1.1 shows the geometric equivalent of the following chain of band representations (as before, we simplify typography by writing i for σ_i):

$$\begin{aligned}(1, {}^{22}1) &\to (1, {}^{22}1, 2, 2^{-1}) \to (1, 2, {}^{2}1, 2^{-1}) \\ &\to (2, {}^{2^{-1}}1, {}^{2}1, 2^{-1}) \to (2, {}^{2^{-1}}1, {}^{21}2^{-1}, {}^{2}1) \\ &\to (2, {}^{2^{-1}1221}2^{-1}, {}^{2^{-1}}1, {}^{2}1) \to ({}^{2^{-1}}1, {}^{2}1).\end{aligned}$$

The last link in the chain depends on the calculation $2 \cdot {}^{2^{-1}1221}2^{-1} =$ identity in B_3.

The reader may like to check that if a and b are any two bands satisfying $aba = bab$ (for instance, σ_i and σ_{i+1}) then the following chain corresponds to an arc knotted in a trefoil: $(a) \to (a, b^{-1}, b) \to ({}^{a}b^{-1}, a, b) \to ({}^{a}b^{-1}, b, {}^{b^{-1}}a) \to ({}^{ab^{-1}a^{-1}}b, {}^{a}b^{-1}, {}^{b^{-1}}a) \to ({}^{ab^{-1}a^{-1}}b) = (a)$ again.

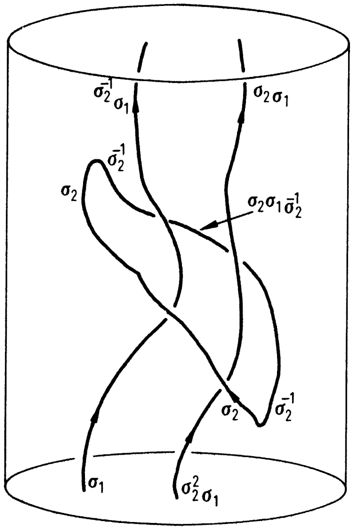

Figure 1.1

Because the relationship between different band representations of the same braid is of interest, the further study of the configurations $F^{-1}(\Delta)$ may be worth undertaking. In this regard one further construction may be mentioned here. For $n \geqq 3$ there is room in E_n to alter a homotopy F by surgeries, as follows. First, one may assume that $F(D^2 \times I)$ is an embedded 3-disk (i.e., F identifies only along intervals $\{z\} \times I$, $z \in \partial D^2$). Let L be any link in the interior of this 3-disk, disjoint from Δ. Then in E_n, L is the boundary of a collection of 2-disks which are pairwise disjoint and disjoint, except along L, from $F(D^2 \times I)$, and which are smoothly embedded transverse to Δ. Corresponding to any framing of any component L_i of L in the 3-disk there is an embedding of a bidisk $D_i^2 \times D^2$ in E_n so that $D_i^2 \times \{0\}$ is mapped to the 2-disk bounded by L_i and $\partial D_i^2 \times D^2$ with its product structure induces the given framing of L_i in the 3-disk, while $D_i^2 \times \partial D^2$ is transverse to Δ. Make a 3-manifold in E_n, with boundary equal to the 2-sphere $F(\partial(D^2 \times I))$, by removing the solid tori $\partial D_i^2 \times D^2$ from the 3-disk and replacing them with the solid tori $D_i^2 \times \partial D^2$; this 3-manifold is transverse to Δ and easily smoothed at its corners. In case L is a split link of trivial knots, each framed with ± 1, the new 3-manifold is again a 3-disk and a new homotopy has been created between the original pair of band representations f_0, f_1. It may be hoped that such surgeries, properly chosen, can replace general configurations $F^{-1}(\Delta)$ with ones that are special enough in some way to be more easily understood. For instance, crossings in a diagram of the link $F^{-1}(\Delta)$ can be reversed, at the expense (in general) of introducing new components (each component L_i of L will contribute a $(\pm 2, 2)$ torus link binding together the arcs whose crossing has switched sign). Remark 1.9 shows that what might be conceived to be the ultimate simplification is not always possible: we cannot assume that $F^{-1}(\Delta)$ is simply a braid (with respect to projection on I).

§2. Constructions of surfaces from band representations

The real content of this section, and the next, is in the pictures.

Figure 2.1 shows a surface of the type described in [S] (there named T_β) for the "homogeneous" braid word $\sigma_1\sigma_2^{-2}\sigma_1^3\sigma_2^{-1} \in B_3$. (Although the notation T_β would seem to suggest that the surface depends only on the braid, in fact the particular

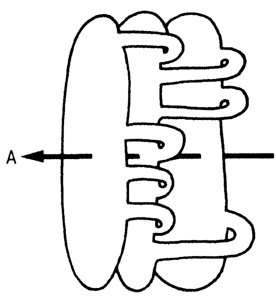

Figure 2.1

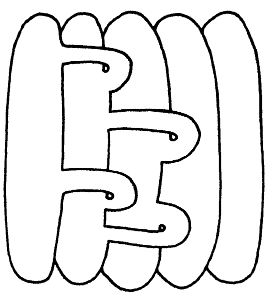

Figure 2.2

FOUR SURFACES $S(\vec{b})$

Figure 2.1. $\vec{b} = (\sigma_1, \sigma_2^{-1}, \sigma_2^{-1}, \sigma_1, \sigma_1, \sigma_1, \sigma_2^{-1})$ in B_3.
Figure 2.2. $\vec{b} = (\sigma_1, \sigma_2, \sigma_1, \sigma_2^{-1})$ in B_5.
Figure 2.3. $\vec{b} = (\sigma_{2,4}, \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{1,3})$ in B_5.
Figure 2.4. $\vec{b} = ({}^{\sigma_1\sigma_3}\sigma_2, {}^{\sigma_2^3}\sigma_1^{-1})$ in B_4.

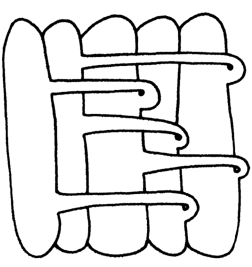

Figure 2.3

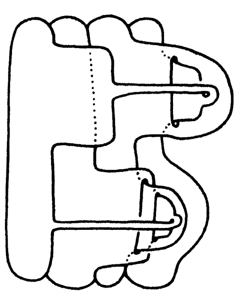

Figure 2.4

word is used to make the surface.) Instead of drawing the surface just as [S] would have it, with a twist (positive or negative according to the exponent of the corresponding letter in the braid word) to each band, I have preferred to give the bands half-curls: then, in Fox's expressive words [F, p. 151], "the resulting surface ... may be laid down flat on the table so that only one side of it is visible," whereas twists expose a bit of the back side.

Here is the procedure for making a surface according to a braid word (homogeneous or not): if the word represents an element of B_n and is of letter length k, the surface has an ordered handlebody decomposition $h_1^0 \cup \cdots \cup h_n^0 \cup h_1^1 \cup \cdots \cup h_k^1$; the 0-handles are embedded in $\mathbf{R}^3$ as planar cells, stacked in order in parallel planes; the 1-handles are attached (orientably) along the front edges of the 0-handles, in order; if the jth letter in the word is $\sigma_{i(j)}^{\varepsilon(j)}$, $\varepsilon(j) = \pm 1$, then the jth 1-handle connects $h_{i(j)}^0$ to $h_{i(j)+1}^0$; the half-curl is downwards (i.e., towards the next 1-handle) if $\varepsilon(j) = +1$ and upwards if $\varepsilon(j) = -1$. (The referee observes that this is really just Seifert's method of "Seifert circles" [F], applied to a natural oriented link diagram for the closure of the given braid word.)

Figure 2.2 illustrates the surface corresponding to the braid word $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$ considered as an element of B_5; just as including a braid in B_n (here, B_3) into the group B_{n+m} adds m trivial components to the link which is its closure, so does such an inclusion add disks to the constructed surface.

Somewhat more generally, if $\vec{b} = (b(1), \ldots, b(k))$ is an embedded band representation of $\beta = \beta(\vec{b})$ in B_n, then there is a Seifert surface for $\hat{\beta}$ made of n 0-handles connected by k 1-handles, where now the 1-handles may have to stretch across several intervening disks between their two ends.

It should be noted that while the surfaces constructed from braid words are all unknotted (that is, the fundamental group of the complement of the surface is free – as the referee remarks, this is always true for surfaces constructed by Seifert's procedure), this is not true of all surfaces constructed from embedded band representations; see Fig. 2.3, an annulus knotted in a trefoil, corresponding to the embedded band representation $(\sigma_{2,4}, \sigma_{1,2}, \sigma_{2,3}, \sigma_{3,4}, \sigma_{1,4})$ in B_5.

Now consider a general band representation $\vec{b}$. Make a choice, for each $j = 1, \ldots, k$, of a particular braid word $w(j)$ such that $b(j) = {}^{w(j)}\sigma_{i(j)}^{\pm 1}$. (One is actually also choosing $i(j)$.) Then, as in Fig. 2.4, where the process is applied to $({}^{\sigma_1\sigma_3}\sigma_2, {}^{\sigma_2^2}\sigma_1^{-1})$ with $w(1)$, $w(2)$ as written, a surface $h_1^0 \cup \cdots \cup h_n^0 \cup h_1^1 \cup \cdots \cup h_1^k$ whose boundary is the closure of $\beta(\vec{b})$ can be constructed; but now it is not embedded in $\mathbf{R}^3$, but rather immersed. The 0-handles have interpenetrated each other according to the braid words $w(j)w(j)^{-1}$. Each component of the singular set of the immersed surface is of the same type: an arc of transverse double-points, of which the preimage on the abstract surface consists of two arcs, one entirely interior to the surface and one with both its endpoints on the boundary (briefly, a *proper arc*). A surface with only such singularities is called a *ribbon surface* (in $\mathbf{R}^3$ or S^3), or a *ribbon immersion.* We have constructed a *Seifert ribbon* for the closed braid.

A band representation $\vec{b}$ of β in B_n thus gives various different Seifert ribbons for $\hat{\beta}$ (each one a ribbon immersion, conceivably an embedding, of the same abstract surface), differing according to specific ways of writing the bands. For our purposes there seems to be no need to distinguish these various ribbons, any one of which will therefore be denoted by $S(\vec{b})$.

Ribbon immersions in S^3 are related to the previously introduced ribbon surfaces in D^4 as follows (a detailed exposition has been written up by Joel Hass, [H]). Let $i: S \to S^3 = \partial D^4$ be a ribbon immersion. Then without changing i on ∂S, one may isotopically push i into D^4 so as to separate the double-arcs and produce an embedding $(S, \partial S) \subset (D^4, \partial D^4)$ which is a ribbon surface in the sense of §1; and every ribbon surface in D^4 arises in this way (from any one of many different ribbon immersions i).

We will also use the symbol $S(\vec{b})$ to denote such a pushed-in version of (any one of) the Seifert ribbons $S(\vec{b})$. On this interpretation, $S(\vec{b})$ is uniquely defined (up to isotopy), perhaps justifying the ambiguity in the other interpretation; we can see this by explicitly using the data of $\vec{b}$ alone (no choices of conjugators $w(j)$) to construct $S(\vec{b})$ in D^4. Figure 2.5 illustrates stages in such a construction of $S(\sigma_1, {}^{\sigma^3}\sigma_1)$. Figure 2.6 shows a Seifert ribbon $S(\vec{b})$ in S^3 adorned with representative level sets showing how to push the ribbon immersion into D^4.

This is the general construction: if $\vec{b} = (b(1), \ldots, b(k))$ is in B_n, of length k, think of an n-string (open) braid which changes in time, from the (constant) trivial braid at $t = 0$ to $\beta(\vec{b})$ at $t = 1$. In between there are k singular times, $0 < t_1 < \cdots < t_k < 1$; the interval $[0, 2\pi]$ which parametrizes the changing braid is also divided into subintervals, by values $0 = \theta_1 < \cdots < \theta_k < 2\pi$. Between $t = 0$ and $t = \frac{1}{2}(t_1 + t_2)$, the braid changes only in the θ-interval $\theta_1 < \theta < \theta_2$, in which before and after the singular time t_1 it moves by isotopies, passing at t_1 through a stage where a simple crossing (a point of order 4, like the center of an **X**) appears. Similarly, between $t = \frac{1}{2}(t_1 + t_2)$ and $t = \frac{1}{2}(t_2 + t_3)$, the braid changes only in the θ-interval $\theta_2 < \theta < \theta_3$, where it has a simple crossing when $t = t_2$; and so on. When this movie of a changing open braid is used to create a surface in the bidisk $D = \{(z, w): |z| \leqq 1, |w| \leqq R\}$, by letting $z = t \exp i\theta$, the surface evidently is a braided surface, isotopic (*vide* Prop. 1.4) to a ribbon surface in D^4; and the boundary is (of the link type of) $\beta(\vec{b})$. We will use $S(\vec{b})$ also for the braided surface just constructed.

Remark 2.1. Of course, according to §1, $\vec{b}$ dictates an embedding $(D^2, \partial D^2) \to (E_n, E_n - \Delta)$ transverse to Δ, and this embedding in turn gives a

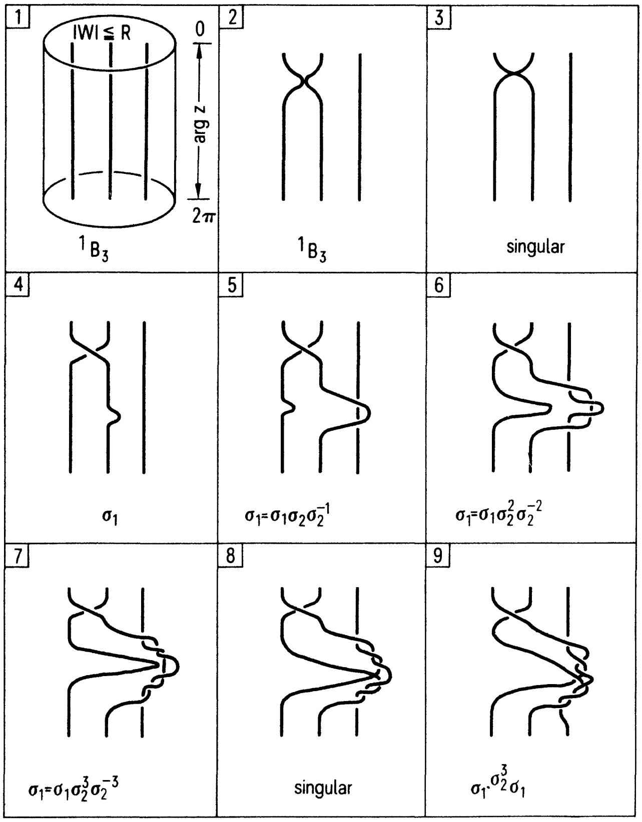

Figure 2.5. A movie of the construction of $S(\sigma_1, {}^{\sigma_2^3}\sigma_1)$

braided surface in D: it should be no surprise that this surface is none other than $S(\vec{b})$. It is hoped, however, that the pictorial approach taken has been of some help in understanding this situation.

Remark 2.2. The algebraic moves of §1 can now be interpreted geometrically. "Slides" on the level of band representations correspond to handle-slides of the surfaces $S(\vec{b}) \subset D^4$ with their ordered handlebody decompositions; thus, slide-equivalent band representations of β in B_n produce surfaces $S(\vec{b})$, $S(\vec{b}')$ which are isotopic in D^4 (but generally not through a level-preserving isotopy). An elementary expansion of $\vec{b}$ corresponds to adding a (hollow) handle to $S(\vec{b})$, either joining two components by a trivial tube $S^1 \times I$ or taking the connected sum of one component with a trivial torus $S^1 \times S^1$, in D^4 (the cases corresponding to whether or not the pair of inverse bands in question have a permutation that links

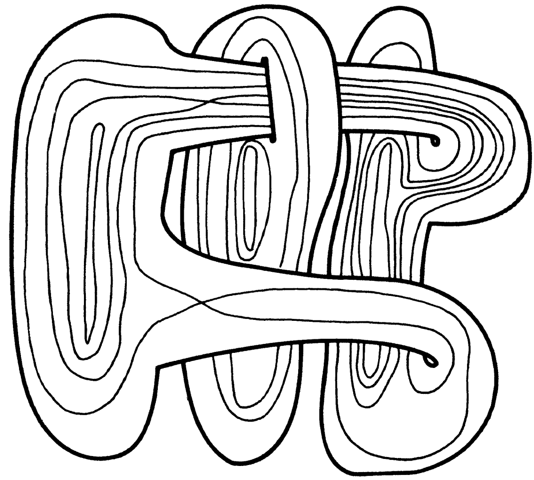

Figure 2.6.

previously disjoint cycles); an elementary contraction, when possible, corresponds to removing such a trivial tube or torus.

§3. The construction is general

We will show that every orientable ribbon surface is (isotopic to) some $S(\vec{b})$, $\vec{b}$ a band representation. We will do this on the interpretation of "ribbon surface" as "ribbon immersion in $\mathbf{R}^3 \subset S^3$"; the isotopy will be ambient isotopy. The proof is in two steps. First we show that every ribbon immersion "may be laid down flat on the table." Then we show how to move any such tabled surface around until it is an $S(\vec{b})$. As before, the text is secondary to the pictures.

Let us say that a surface immersed in $\mathbf{R}^3$ is *tabled* if it is oriented, and we have an oriented 2-plane (the table) so that orthogonal projection from the surface to the plane is an orientation-preserving immersion. In [F], Fox attributes to Seifert essentially the following procedure for finding a tabled surface in the isotopy class of a given embedded surface (oriented and without closed components, necessarily), S. There is a handlebody decomposition of S with n 0-handles h_i^0, k 1-handles h_j^1, and no 2-handles; and the 1-handles are attached orientably. (We might of course require that n be the number of components of S, but we don't have to; and when we come to the case of immersions this won't be possible.) Let $T \subset \mathbf{R}^3$ be an oriented 2-plane. By isotopy of S in $\mathbf{R}^3$, we may make each h_i^0 a 2-cell lying in a translate T_i of T, bearing the proper orientation there; and we can assume that the projections of these 0-handles into T are pairwise disjoint. Now by isotopy arrange that the core arcs of the h_j^1 project into T in general position, and with no points except their endpoints in the images of the h_i^0. Shrink each h_j^1 down to a narrow band around its core arc; then, without loss of generality, the projection of h_j^1 identifies some number of transverse arcs to points, and is otherwise an immersion, alternately preserving and reversing orientation in the regions between the transverse arcs – that is, h_j^1 is twisted (as seen from T). Also, of course, h_j^1 may be knotted, and the various 1-handles may link each other, too. As far as twisting goes, however, since the 0-handles already projected orientably and S is oriented, each h_j^1 has an even number of twists; and by further isotopy "these twists can be replaced by curls (just half as many curls as twists)" ([F, p. 151]). When the twists are all out, the surface is tabled.

Figure 3.1 illustrates this procedure as applied to a particular Seifert surface for the figure-8 knot, without regard to economy in the number of handles.

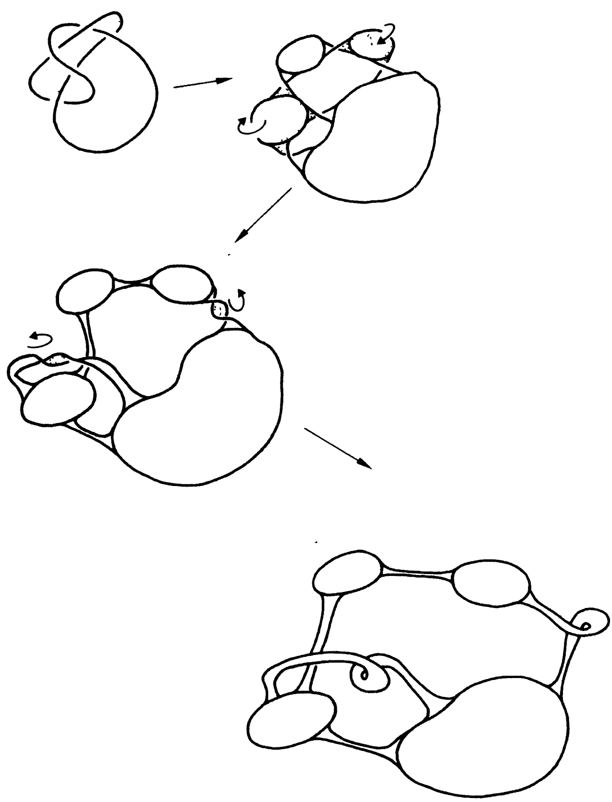

Figure 3.1. Tabling an embedded surface by twisting handles.

Now suppose that we begin with a surface which is not embedded, but is ribbon immersed by $i:S\to\mathbf{R}^3$, where on $i(S)$ the double-arcs are A_m ($m=1,\ldots,s$), and $i^{-1}(A_m)$ is the disjoint union of the proper arc A'_m and the arc $A''_m\subset\operatorname{Int} S$. It is easy to find a set A^*_m ($m=1,\ldots,s$) of proper arcs on S, pairwise disjoint and disjoint from all the A'_p, such that for each m, $A''_m\subset A^*_m$. Then there is a handlebody decomposition of S which includes among its 0-handles a neighborhood on S of each proper arc A'_m and A^*_m, and which has no 2-handles. (As always, S is oriented and without closed components.) It is now possible practically to mimic Seifert's procedure with $i(S)$, except of course that the 0-handles containing A^*_m and A'_m will not have disjoint images in T, and cannot both lie in planes parallel to T. Let us always take $A_m\subset i(S)$ to be actually a straight line segment, parallel to T; then of the two immersed 0-handles containing it, one can be taken to lie parallel to T, and the other to lie in another plane parallel to T except for a narrow tab which passes through A_m. (Note that to have both the 0-handles project orientably to T, one may have to "pivot" one of them about A_m.) Figure 3.2 illustrates this, for a particular ribbon immersion of a disk – the boundary being a stevedore's knot.

Returning to surfaces $S(\vec{b})$ for a moment, we see that they are of course tabled (as pictured in §2) – both from the point of view of the plane of the paper, and from the tilted plane perpendicular to the axis of the closed braid $\partial S(\vec{b})=\hat{\beta}$, in which perspective the 0-handles greatly overlap each other. So our second task is to take our ribbon surface, already assumed tabled, and isotope it until it has become an $S(\vec{b})$. First, skewer all the 0-handles; that is, pick an axis A perpendicular to T, and by isotopy of S through tabled surfaces arrange the 0-handles so that each one intersects A in its own plane (in the case of the 0-handles with tabs, let us make the intersection fall in the planar part, not in the tab). Now pick rectangular coordinates in T, and for reference a rectangle-with-rounded-corners R in T, its sides parallel to the axes, which we will call horizontal and vertical. Let one of the vertical sides of R be called its front edge. By further isotopy of S, we may so arrange the 0-handles so that each one projects either onto exactly R (if there is no double-arc in that 0-handle) or onto R suitably enlarged along the front edge (by a larger or smaller tab), and so that the double-arcs are vertical segments projecting outside R (past its front edge). Next we may arrange the 1-handles (if necessary, sliding their attaching maps along the boundaries of the 0-handles) so that: they attach only along the front edges of the projections (including front edges of tabs); so that their projections are neighborhoods of polygonal arcs composed solely of horizontal and vertical segments; and so that in the resulting "link diagram" of core arcs, each over-arc is a horizontal segment. (We always assume general position, so it also is assured that the $2k$ endpoints of the k 1-handles' core arcs have $2k$ distinct vertical coordinates; let also the

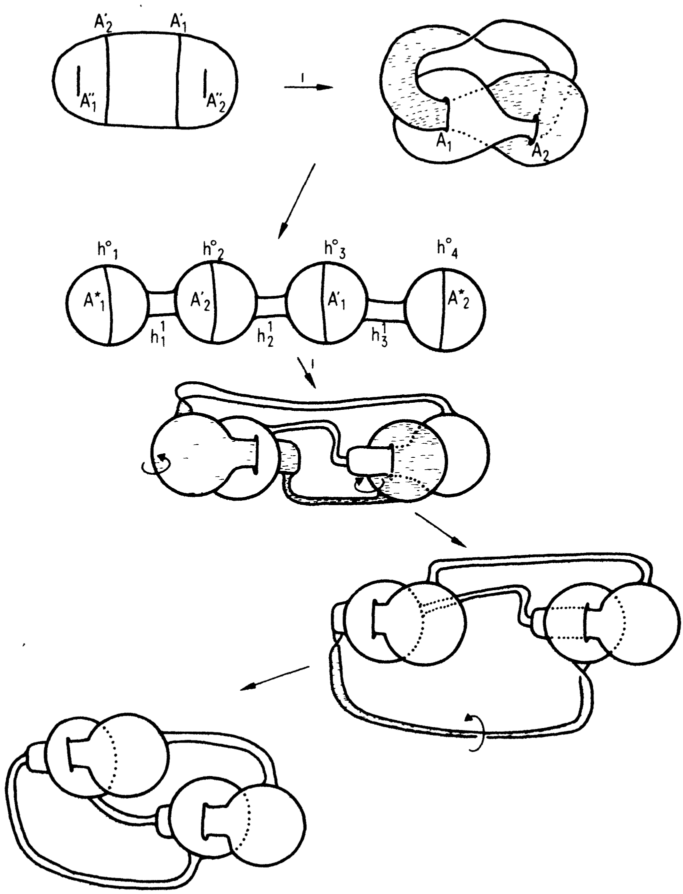

Figure 3.2. Tabling a ribbon immersed surface by twisting handles.

1-handles be sufficiently narrow that all attaching takes place inside $2k$ disjoint intervals.) This is illustrated in Fig. 3.3, continuing the example of the stevedore's knot.

We are nearly done now. One by one, vertical parts of the bands may be expanded into full-fledged 0-handles and these 0-handles slipped into the stack impaled by A – the adjacent horizontal segments, if they approach from the left, being given half-curls to allow the attachment to stay within the realm of tabled surfaces with all bands attached along front edges. When no vertical parts are left, the resulting surface is of the form $S(\vec{b})$, where $\beta(\vec{b})$ is a braid on some large number of strings (n plus the number of vertical segments).

Remark 3.1. Each band in such a band representation $\vec{b}$ is actually of the form ${}^{w}\sigma_j^{\pm 1}$ where for some $i \leqq j$ either $w = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}$ or $w = \sigma_i\sigma_{i+1} \cdots \sigma_{j-1}^{-1}$ – it is either embedded or has a single double-arc but goes directly from h_i^0 to h_{j+1}^0.

Figure 3.4 shows how this last part of the construction was used to make the surface in Fig. 2.3, beginning with an annulus knotted in a trefoil (already tabled); and Fig. 3.5 finishes the stevedore's knot with its ribbon disk.

Acknowledgement. The conviction that every ribbon surface should arise as $S(\vec{b})$ for some $\vec{b}$ came upon the author in 1978, after experimentation with cardboard models. It was some time before the idea of using, essentially, link diagrams with only vertical and horizontal segments in them, and every over-crossing horizontal, was incorporated into a proof. And it was only much later that the author remembered having first heard of such a construction at the October, 1977, topology conference in Blacksburg, Va. (at VPI&SU) from Herbert Lyon, in whose hands the construction was used to show that every (embedded, orientable) surface in S^3, without closed components, is a subsurface

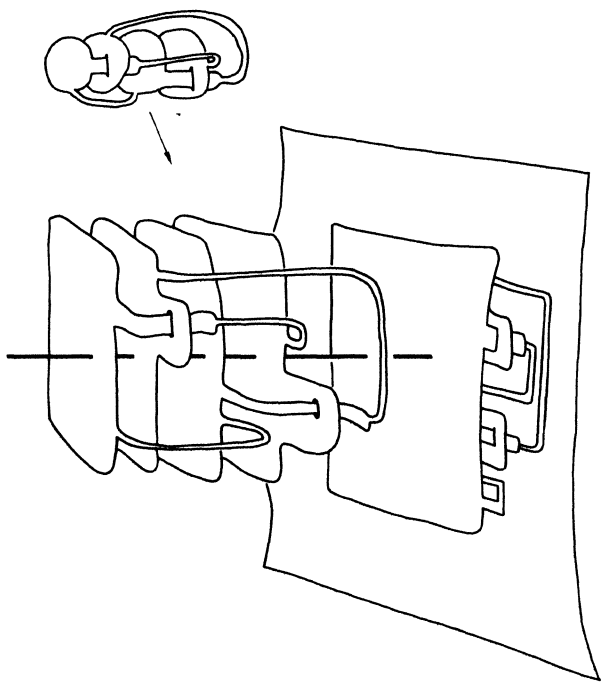

Figure 3.3. The stevedore's knot and its ribbon disk, continued.

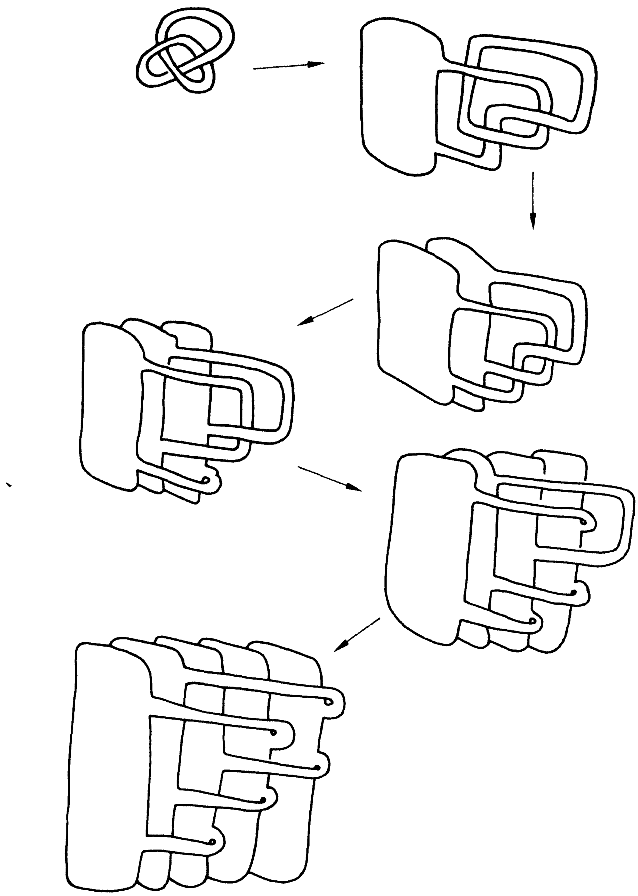

Figure 3.4. Thickening vertical parts of 1-handles into 0-handles.

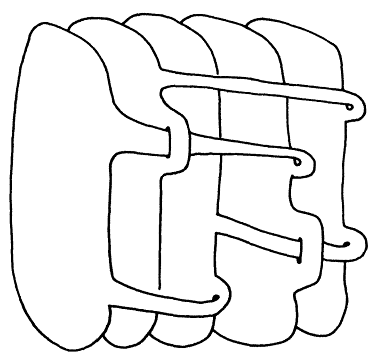

Figure 3.5. The ribbon disk bounded by a stevedore's knot, concluded.

of a fibre surface of some fibred knot, [L]. The unconscious memory of Professor Lyon's talk was undoubtedly an important ingredient in the genesis of the author's proof.

The fact that every ribbon surface appears as $S(\vec{b})$ for some $\vec{b}$ has some immediate consequences which may be noted here.

PROPOSITION 3.2. *Every (orientable) ribbon surface in the 4-disk is isotopic to a braided surface in the bidisk.* □

As stated in Remark 1.5, I don't know a more direct proof of this.

PROPOSITION 3.3 (Alexander). *Every link can be represented as a closed braid.* □

We might say that Proposition 3.3 is the boundary of Proposition 3.2; of course it relies on the existence of Seifert surfaces for every link.

Further consequences will be reserved to the next section.

§4. The fundamental group $\pi_1(D - S(\vec{b}))$

A *Wirtinger presentation* of a group G is a presentation $G = (x_1, \ldots, x_n : x_{i(r)} = {}^{w(r)}x_{j(r)},\ r = 1, \ldots, k)$, in which each $w(r)$ is a word in $x_1, \ldots, x_n$; a group with a Wirtinger presentation is a *Wirtinger group.* A *special* Wirtinger presentation is one in which each conjugator $w(r)$ is actually one of the generators, $x_{m(r)}$. It is clear that any Wirtinger group has a special Wirtinger presentation.

That any link group $\pi_1(S^3 - L)$ (for L tame) is Wirtinger is classical (presumably due to Wirtinger); and indeed that Wirtinger presentation which is written down in the usual way from inspection of a link diagram is special. Then Fox's method of cross-sections (for instance), or, indeed, Morse theory relative to the submanifold X, shows that any group $\pi_1(S^N - X)$, X a smooth orientable submanifold of codimension 2, is Wirtinger. Not every Wirtinger group appears as a link group in S^3. But the following is true.

PROPOSITION 4.1. (Yajima [Y], Johnson [J]). *If G is a Wirtinger group, then there is a smooth, orientable surface $S \subset S^4$ with $\pi_1(S^4 - S) \cong G$; and S may be taken to be the double of a ribbon surface in the 4-disk D^4.*

(The papers of Yajima and Johnson were pointed out to me by Professor Jonathan Simon; Johnson's proof, obtained independently of Yajima, introduces the ribbon refinement; the proof to be given here uses the formalism of band representations to render Johnson's construction by "band moves" even more perspicuous.)

Proof. First, let us derive a (Wirtinger) presentation for $\pi_1(D-S(\vec{b}))$ from $\vec{b}$. If $\vec{b}$ is in B_n, of length k, there will be n generators and k relations. Thinking of $S(\vec{b})$ as a closed braid changing in time from the (constant) trivial braid to $\hat{\beta}(\vec{b})$, one can identify the generators $x_1, \ldots, x_n$ as standard meridians at any one of the stages; the relations appear at the singular stages, and as in [F, p. 133] each relation takes the form "two meridians are equal" (not, of course, necessarily standard meridians). More explicitly: recall that there is a (faithful) representation of B_n as a group of left automorphisms of the free group $F_n = (x_1, \ldots, x_n : \ )$, given on generators by $\sigma_i x_i = {}^{x_i}x_{i+1}$, $\sigma_i x_{i+1} = x_i$, $\sigma_i x_j = x_j$ $(j \neq i,\ i+1)$; and that, if $\gamma \in B_n$ and the open geometric braid $K \subset D^2 \times I$ represents γ, then in terms of the standard meridians $x_1, \ldots, x_n$ of K in $D^2 \times \{0\}$, the standard meridians of K in $D^2 \times \{1\}$ (taken in the same order) are $\gamma x_1, \ldots, \gamma x_n$. (This is readily checked graphically; or see [Bi] where, however, right automorphisms are used.) Let $\vec{b} = (b(1), \ldots, b(k))$, $b(j) = {}^{w(j)}\sigma_{i(j)}^{\pm 1}$; then the j^{th} stage contributes the relation $w(j)x_{i(j)} = w(j)x_{i(j)+1}$. Noting that for any braid w and any x_i, wx_i is a conjugate of some x_j (true by inspection for w a generator, then generally true by induction), we see that this relation can be rewritten in Wirtinger form.

Consider two particular types of bands. An embedded band $\sigma_{i,j}^{\pm 1}$ contributes the relation $x_i = x_{j+1}$. A band of the form ${}^{w}\sigma_j^{\pm 1}$, with $w = (\prod_{m=i}^{j-2} \sigma_m)\sigma_{j-1}^{-1}$, contributes the relation $x_i = {}^{x_i}x_{j+1}$. According to Remark 3.1, every orientable ribbon surface $S \subset D^4$ can be constructed as $S(\vec{b})$ for some band representation with bands only of those two types; the corresponding presentation is special Wirtinger. Conversely, given any special Wirtinger presentation of a group G, after possibly adding new generators set equal to old ones, we can assume that each relation is of one of the two forms $x_i = x_{j+1}$, $x_i = {}^{x_i}x_{j+1}$; and it is easy to find a band representation with that as the corresponding presentation.

So every Wirtinger group appears as $\pi_1(D^4 - S(\vec{b}))$ for some $\vec{b}$. By Morse theory, because $S(\vec{b})$ is ribbon, the homomorphism $\pi_1(S^3 - \hat{\beta}(\vec{b})) \to \pi_1(D^4 - S(\vec{b}))$ is onto. Now, by van Kampen's theorem, the groups $\pi_1(D^4 - S(\vec{b}))$ and $\pi_1(S^4 - 2S(\vec{b}))$, where $2S(\vec{b}))$ is the double of $S(\vec{b})$ in S^4 (the double of D^4), are isomorphic. $\square$

EXAMPLE 4.2. Let $\vec{b} = (\sigma_1, {}^{\sigma_2^3}\sigma_1^{-1})$ in B_3; then $\hat{\beta}(\vec{b})$ is a square knot, $S(\vec{b})$ is a ribbon disk, and it is readily checked that $\pi_1(D - S(\vec{b})) = (x_1, x_2, x_3:$

$x_1=x_2,\ x_1={}^{x_2x_3x_2}x_3)=(x, y:xyx=yxy)$, the group of the trefoil knot. In fact, in this case the double of the disk pair $(S(\vec{b}), D)$ is the spun trefoil in S^4.

EXAMPLE 4.3. Here is the example, using a knotted 2-sphere, promised in Remark 1.10 to show the subtle structure of the set of band representations of a given braid. As before, let 1, 2, 3 abbreviate σ_1, σ_2, σ_3 respectively; and let $\bar{x}$ abbreviate x^{-1}. Then $\beta=\beta(3, {}^{\overline{222}}\bar{1})\in B_4$ closes to a split link with two unknotted components, and evidently $rk(\beta)=2$ since β is not the trivial braid. One calculates $\pi_1(D-S(3, {}^{\overline{222}}\bar{1}))=(x_2, x_3:\ )$. If b is any band in B_4, the elementary expansion $(3, {}^{\overline{222}}\bar{1}, b, \bar{b})$ gives a presentation $\pi_1(D-S(3, {}^{\overline{222}}\bar{1}, b, \bar{b}))=(x_2, x_3:r(x_2, x_3))$ with (at most) one new relation, since b and $\bar{b}$ give rise to the same relation according to Theorem 4.1. Then any band representation of β of length 4, which is slide equivalent to an elementary expansion of $(3, {}^{\overline{222}}\bar{1})$, gives a presentation of the same form.

On the other hand, consider the band representation $\vec{b}=({}^{2}3, {}^{\overline{22}}\bar{3}, {}^{3\bar{2}}1, {}^{1\bar{2}}\bar{1})$. Its conjugate ${}^{w}\vec{b}$, $w=3\bar{2}12333$, has braid equal to β (we use $\vec{b}$ for ease of computation). We calculate $\pi_1(D-S(\vec{b}))=(x_1,\ x_2,\ x_3,\ x_4:x_2=x_4,\ {}^{\overline{x_3x_2}}x_3=x_4,\ x_1={}^{x_3}x_4,\ {}^{x_1}x_2=x_3)=(x_2,\ x_3:x_2x_3x_2=x_3x_2x_3,\ [x_2^2, x_3]=1)$. This is the group of the 2-twist spun trefoil, and in fact when we cap off the two trivial components of the closed braid the 2-twist spun trefoil is the knotted 2-sphere we get. In any case, this is not a one-relator group, so the length-4 band representation ${}^{w}\vec{b}$ of $\beta(3, {}^{\overline{222}}\bar{1})$ cannot be slide equivalent to an elementary expansion of $(3, {}^{\overline{222}}\bar{1})$. (Presumably two elementary expansions, some sliding, and one elementary contradiction suffice to connect the two slide-equivalence classes, but I have not checked this.)

Remark 4.4. The Wirtinger presentation of $\pi_1(D-S(\vec{b}))$ derived in Proposition 4.1 evidently takes no notice of the signs of the bands in $\vec{b}$, so every Wirtinger group arises as $\pi_1(D-S(\vec{b}))$ for some quasipositive band representation $\vec{b}$. As shown in [Ru], if $\Gamma\subset D\subset \mathbf{C}^2$ is a piece of complex-analytic curve with $\partial\Gamma=\Gamma\cap\partial_1 D$ (transverse intersection), then $\partial\Gamma=\hat{\beta}$ is a closed quasipositive braid, and conversely every quasipositive braid arises in this manner. Examining the proof given there in the light of this paper, one sees that in fact (up to isotopy) such pieces of (non-singular) complex-analytic curves are precisely the surfaces $S(\vec{b})$ for $\vec{b}$ quasipositive.

We can use this to produce a Stein manifold $M\subset\mathbf{C}^N$ with fundamental group G, for any Wirtinger group G. In fact, find a quasipositive band representation $\vec{b}$ with $\pi_1(D-S(\vec{b}))=G$; realize $S(\vec{b})$ as a piece of complex-analytic curve (non-singular, and extendible to a slightly larger bidisk) in D. By the solvability of the Cousin problem for the bidisk (cf. [G$-$R]), there is a holomorphic function $f(z, w)$ in (a neighborhood of) D, of which the zero-set in D is precisely $S(\vec{b})$. Also, the 2-disk $\mathring{D}^2\subset\mathbf{C}$ may be embedded as a Stein submanifold of some $\mathbf{C}^N$

(and actually $N=2$ will do), by a proper analytic embedding $g:\mathring{D}^2 \to \mathbf{C}^N$. Then $(z, w) \mapsto (g(z), g(w), 1/f(z, w))$ is a proper analytic embedding of $\mathring{D} - S(\vec{b})$ onto a Stein submanifold $M \subset \mathbf{C}^{2N+1}$. (In fact, by Forster [Fr], if $N = 2$ or 3, M must be an analytic complete intersection, since it is parallelizable.)

SCHOLIUM. *There are finite homotopy types which can be realized as Stein manifolds but not as non-singular affine algebraic varieties.*

For John Morgan, using Hodge theory, has shown that, for instance, the group $G = (x, y : 1 = [x, [x, [x, [x, y]]]])$ is not the fundamental group of any non-singular algebraic variety (affine or not), [Mo]. Yet G has the Wirtinger presentation $G = (x, y, s, t, u, v, w : s = {}^y x, x = {}^s t, v = {}^x t, x = {}^v u, w = {}^x u, x = {}^w x)$. (To see this, one uses repeatedly that in any group ${}^c[a, b] = [{}^c a, {}^c b]$, and $[a, b] = 1$ iff $[a, b^{-1}] = 1$.)

Of course, there are infinite homotopy types among the Stein manifolds; for instance, any open subset of $\mathbf{C}$ (e.g., the complement of the integers) is a Stein manifold, [G−R]. (The analogue in $\mathbf{C}^n$, $n > 1$, is naturally quite false.)

The abelianization of a Wirtinger group is free abelian, so there are certainly finitely presented non-Wirtinger groups, and some of these appear as fundamental groups of Stein manifolds, indeed of algebraic varieties. Is it possible that every finitely presented group appears as the fundamental group of a Stein manifold? Given a finite presentation, it is easy to construct various (open) complex manifolds of complex dimension 2 with G as the fundamental group, but it is not at all clear how to make such a construction yield Stein manifolds.

§5. Rank and ribbon genus

Recall that $rk_n(\beta)$, for $\beta \in B_n$, is the least k such that some band representation of β in B_n has length k. Call such a shortest band representation *minimal in* B_n.

Recall also the definition of the *ribbon genus* of a knot or link, $L \subset S^3 = \partial D^4$. Every such L is, of course, the boundary of various connected orientable smooth surfaces $S \subset \partial D^4$. The ribbon genus $g_r(L)$ is the least integer that appears as the genus of such a surface which is ribbon embedded in D^4; clearly we have $g(L) \geqq g_r(L) \geqq g_s(L)$, where the (classical) *genus* $g(L)$ restricts the surfaces over which the minimum is taken to those actually in S^3, and the *slice genus* $g_s(L)$ makes no restrictions.

The genus is quite a classical invariant; ribbon genus and slice genus are of more recent interest; both have been under study by some quite high-powered methods, cf. Gilmer [G1, G2]. As band representations and ribbon surfaces are so closely related, there might be some hope that the more naive methods of this paper would be relevant to the study of g_r. This section presents some observations on the beginnings of such a program.

It should be noted that (because of the requirement that the surfaces involved be connected) g, g_r, and g_s all are most satisfactory when applied to links which can bound connected surfaces (without closed components) only: for instance, knots, or more generally links in which any two distinct components have non-zero linking number.

The following Proposition is an immediate consequence of the construction in §3, applied to any Seifert ribbon for L which happens to have genus $g_r(L)$.

PROPOSITION 5.1. *Let L be a link for which every Seifert ribbon is connected. Then for some n and some braid $\beta \in B_n$ with $\hat{\beta} = L$, we have* $g_r(L) = \frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$ *(where $c(\beta)$ is the number of cycles in the permutation of β; or equivalently the number of components of L).* $\square$

It would be nice if the quantity $\frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$ always computed $g_r(\hat{\beta})$. This, alas, is not the case, as Professor Andrew Casson pointed out to me. Example 5.3. is due to him. (Below, i abbreviates σ_i, and $\bar{i}$ abbreviates σ_i^{-1}.)

EXAMPLE 5.2. In B_4, consider $\beta = \overline{33}2\bar{3}211\bar{2}1\bar{2}$. Then $\hat{\beta}$ is a split link of two unknotted circles. (The reader familiar with Markov moves may verify this by first increasing the string index to five by the move $\beta \to \overline{343}2\bar{3}211\bar{2}1\bar{2}$; conjugating this braid to get $\bar{3}2\overline{343}211\bar{2}1\bar{2}$; reducing the string index to four by moving back to $\bar{3}2\overline{33}211\bar{2}1\bar{2}$; and then by a fairly straightforward series of conjugations and reductions in string index, proceeding to the identity in B_2, which certainly has the closure advertised. Alternatively, experiments with string, or pencil and eraser, may give the result more quickly.) So $\hat{\beta}$ bounds a pair of disjoint disks. If there were a band representation of β in B_4 of which the associated ribbon surface was a pair of disks – even ribbon disks – then it would have to have length 2, and (since β has exponent sum 0) it would have one positive and one negative band.

But in fact β is not the product of two bands in B_4. We check this as follows. There is a homomorphism ϕ of B_4 onto $SL(2, \mathbf{Z})$, given by

$$\phi(\sigma_1) = \phi(\sigma_3) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \phi(\sigma_2) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

The image of β is found to be $\begin{bmatrix} 49 & 30 \\ -18 & -11 \end{bmatrix}$. One also finds that the general form of the image of a band is $\begin{bmatrix} 1-ac & \pm a^2 \\ \mp c^2 & 1+ac \end{bmatrix}$, where a and c are coprime integers; the upper (lower) sign corresponding to a positive (negative) band. Then, up to conjugation in $SL(2,\mathbf{Z})$, a product of one positive and one negative band takes the form

$$\begin{bmatrix} 1-ac+c^2 & -1-ac+a^2 \\ -c^2 & 1+ac \end{bmatrix}.$$

This has trace $2+c^2$, so if it is conjugate to $\phi(\beta)$ we must have $c^2=36$. But let a unimodular integral matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ conjugate $\begin{bmatrix} 49 & 30 \\ -18 & -11 \end{bmatrix}$; the lower left hand corner of the conjugate is $-18w^2+60wz-30z^2$. Yet $-18w^2+60wz-30z^2=-36$ can have no integral solutions (z, w), for dividing it by 6 and taking both sides modulo 5 yields $2w^2 \equiv -1 \pmod 5$, which is impossible.

Although β is not a product of two bands in B_4, the product ${}^{\bar{3}}\bar{4} \cdot \beta \in B_5$ is a product of three bands in B_5, namely ${}^{\bar{3}}\bar{4} \cdot \beta = {}^{\overline{43}2\bar{3}}\bar{4} \cdot {}^{\overline{4332}}1 \cdot {}^{\bar{4}32}\bar{1}$. It would be interesting to know whether β *considered as an element of* B_5 is a product *of two bands in* B_5. If it were, we would have here an example in which the rank of a braid decreases when the braid is considered to lie in a braid group of larger string index.

Now, β when considered as an element of B_5 has closure a split link of three unknotted components. If $rk_5(\hat{\beta})=2$, it is at least reasonable to suppose that among the minimal band representations of β in B_5, some at least correspond to the Seifert ribbon for $\hat{\beta}$ which consists of three unknotted disks. But if such a band representation of length 2 does exist, it still will not be possible to get from it to the band representation of length 4 given by $\beta = {}^{\bar{3}}4 \cdot {}^{\overline{43}2\bar{3}}\bar{4} \cdot {}^{\overline{4332}}1 \cdot {}^{\bar{4}32}\bar{1}$ simply by inserting a pair of cancelling bands and sliding: this can be shown by an argument like that in Example 4.3, comparing the fundamental groups of the complements of the surfaces corresponding to the different band representations.

EXAMPLE 5.3. (Casson). In B_3, let $\gamma=(1\bar{2})^5$. Then $\hat{\gamma}$ is a ribbon knot, but $rk_3(\gamma) \geqq 4$; so that $\frac{1}{2}(2-n+rk_n(\gamma)-c(\gamma))=\frac{1}{2}(-2+rk_3(\gamma)) \geqq 1 > 0 = g_r(\hat{\gamma})$.

To see that $\hat{\gamma}$ is ribbon, we consider $\gamma_1 = {}^{1\bar{2}1\bar{2}}\bar{3} \cdot \gamma \in B_4$, which has the same closure, and observe that the equation $\gamma_1 = {}^{12}\bar{3} \cdot \beta$ (where β is as in Example 5.2) displays $\hat{\gamma}_1$ as the boundary of a ribbon disk made from the two disks bounded by $\hat{\beta}$ and a single band joining them.

To see $rk_3(\gamma)>2$ (whence it must be at least 4), we represent B_3 in $SL(2,\mathbf{Z})$ and make an argument similar to that above; details are left to the reader.

In an earlier draft of this paper, the following hypotheses were put forth as conjectures.

Hypothesis I. *$rk_{n+1}(\beta) = rk_n(\beta)$ for all $\beta \in B_n \subset B_{n+1}$.*

Hypothesis II. *$rk_{n+1}(\beta\sigma_n^{\pm 1}) = rk_n(\beta) + 1$ for all $\beta \in B_n$.*

Hypothesis III. *If $\vec{b} = (b(1), \ldots, b(k))$ is a minimal band representation in B_{n+1} of $\beta \in B_n$, then $b(k) \neq \sigma_n^{\pm 1}$.*

Hypothesis IV. *If b is a minimal band representation in B_{n+1} of $\beta \in B_n$, then actually each band $b(j)$ belongs to B_n.*

The logical relationships of these hypotheses are as follows.

PROPOSITION 5.4. *Hypothesis* II $\Rightarrow$ *Hypothesis* I; *Hypothesis* IV $\Rightarrow$ *Hypothesis* III; (*Hypothesis* III *&* *Hypothesis* I) $\Rightarrow$ *Hypothesis* II.

Proof. Certainly IV implies III.

Observe that all band representations of a given braid have lengths of the same parity; that $rk_{n+1}(\beta) \leqq rk_n(\beta)$ for any $\beta \in B_n$; and that $rk_n(\beta\gamma) \leqq rk_n(\beta) + rk_n(\gamma)$ for any $\beta, \gamma \in B_n$.

Suppose β falsifies I. Then $rk_{n+1}(\beta) \leqq rk_n(\beta) - 2$. Then if $\vec{b}$ is a minimal band representation of β in B_{n+1}, $(\vec{b}, \sigma_n^{\pm 1})$ is a band representation of $\beta\sigma_n^{\pm 1}$ in B_{n+1}, so $rk_{n+1}(\beta\sigma_n^{\pm 1}) \leqq rk_n(\beta) - 1$, and β also falsifies II. Thus II implies I.

Now suppose III and I are both true. Let $\vec{b}$ be a minimal band representation of $\beta\sigma_n^{\varepsilon}$ in B_{n+1} ($\varepsilon = \pm 1$). Then $(\vec{b}, \sigma_n^{-\varepsilon})$ is a band representation of β in B_{n+1}; by III it is not minimal, so $rk_{n+1}(\beta) \leqq rk_{n+1}(\beta\sigma_n^{\varepsilon}) - 1$. But in any case $rk_{n+1}(\beta) \geqq rk_{n+1}(\beta\sigma_n^{\varepsilon}) - 1$; so in fact $rk_{n+1}(\beta\sigma_n^{\varepsilon}) - 1 = rk_{n+1}(\beta)$, and by I this equals $rk_n(\beta)$; so II is true. $\square$

However, Hypothesis II is not true.

To see this, recall Markov's Theorem (alluded to in Example 5.2), as proved in [Bi]: Let $\beta \in B_n$ and $\beta' \in B_{n'}$ be braids with closures of the same (oriented) link type. Then there is a finite sequence $\beta_1, \ldots, \beta_s$ of braids $\beta_i \in B_{n(i)}$, with $\beta_1 = \beta$, $\beta_s = \beta'$, such that for each $i = 2, \ldots, s$, one of the following holds – either

(M1) $n(i) = n(i-1)$ and β_i is conjugate to β_{i-1} in $B_{n(i)} = B_{n(i-1)}$, or

(M2) $n(i) = n(i-1) + 1$ and $\beta_i = \beta_{i-1}\sigma_{n(i-1)}^{\pm 1}$, or

(M2^{-1}) $n(i) = n(i-1) - 1$ and $\beta_{i-1} = \beta_i\sigma_{n(i)}^{\pm 1}$.

LEMMA 5.5. *Suppose Hypothesis* II *is true. Then if two braids* β, β' *differ by a Markov move* $(M1)$, $(M2)$, *or* $(M2^{-1})$, *they have the same difference between string index and rank.*

Proof. In (M1) string index is constant, and so is rank since it is a conjugacy-class invariant. In (M2), let $\beta \in B_n$, $\beta' = \beta\sigma_n^{\pm 1}$; then by Hypothesis II, the rank of β' is one greater than the rank of β, but so is its string index, and the difference is constant. Similarly for $(\mathrm{M}2^{-1})$. □

By Lemma 5.5 and Markov's Theorem, if Hypothesis II is true, then the difference of string index and rank would be an invariant of oriented link type; but Examples 5.2 and 5.3 show that it is not, so Hypothesis II is false.

Then, by Proposition 5.4, not both of Hypothesis III (or the stronger Hypothesis IV) and Hypothesis I can be true. I will still conjecture (weakly) that Hypothesis I is true.

I will conclude by asking various questions.

Is there an algorithm for calculating the rank of an arbitrary braid? Such an algorithm, with Proposition 5.1, would at least estimate the ribbon genus of a knot.

The rank of a quasipositive braid is, of course, its exponent sum. But is there an algorithm for determining whether a braid (which has not been given as the braid of a quasipositive band representation) is quasipositive? Is there an algorithm for determining whether a given knot or link has, among its various expressions as a closed braid, one which is quasipositive? Is there even a criterion which can rule out certain knots as possibly quasipositive? (No criterion based on a Seifert form for some Seifert surface can work – not, e.g., signatures or Alexander polynomials; [Ru2].)

Is there a way of determining (perhaps for a limited class of braids) whether the stable situation of Proposition 5.1 has been achieved? In particular, if $\beta \in B_n$ is quasipositive, is $g_r(\hat{\beta}) = \frac{1}{2}(2 - n + rk_n(\beta) - c(\beta))$? For the particular case that $\hat{\beta}$ is one of the quasipositive iterated torus links associated to singular points of complex plane curves, that this equality holds has been conjectured by Milnor. Also, if equality fails for some quasipositive braid, then (using results of [Ru]) one could represent some positive homology class in $H_2(\mathbf{C}P^2; \mathbf{Z})$ by a smooth manifold of genus strictly less than that of the homologous smooth algebraic curve – a situation which Thom has conjectured cannot occur.

Finally, note that if $g_s(L) = g$, then for some m (the number of local maxima of the radius-squared on some surface in D^4, with boundary L and genus $g_s(L)$) the link $L \cup mo$ consisting of L and m (split) unknots has $g_r(L \cup mo) = g$. If $L = \hat{\beta}$, $\beta \in B_n$, then $L \cup mo$ is the closure of β considered as an element of B_{n+m}.

Particularly in case Hypothesis I is true, can the method of band representations give any information about the slice genus?

Appendix. Clasps and nodes, überschneidungszahl, etc.

Here we sketch briefly how the various sections of the paper proper can be extended to a broader class of surfaces.

A.1. A *nodal braided surface* is a singular braided surface $i: S \to D$ for which i is an immersion in general position (that is, each singularity is a transverse doublepoint, briefly, a *node*). A nodal braided surface is itself "in general position" if no two nodes lie over any one point of D_1^2, and no node lies over a branch point (whence also neither tangent plane at a node is vertical). We will tacitly take all nodal braided surfaces to be in general position.

Though the disk $i_\#: D_1^2 \to E_n$ which corresponds to a nodal braided surface is not transverse to Δ (if there really are nodes), its intersections with Δ are either transverse or simply tangent.

An immersion $f: S \to D^4$ of an orientable surface is a ribbon immersion in D^4 provided that $L \circ f$ (where $L(z, w) = |z|^2 + |w|^2$) is Morse without local maxima. Such an immersion, if it is in general position, has only nodes as singularities, and no node is a critical point of $L \circ f$.

The analogue of Proposition 1.4 holds: any nodal braided surface in the bi-disk is isotopic to a ribbon immersed surface in the disk.

Let a *node* in B_n be the square of a band. A *nodal band representation* $\vec{\nu}$ is a k-tuple $(\nu(1), \ldots, \nu(k))$ in which each $\nu(i)$ is either a band or a node; as before, $l(\vec{\nu}) = k$ is the *length of* ν, $\beta(\vec{\nu}) = \prod_{i=1}^k \nu(i)$ is its *braid*. Let $\kappa(\vec{\nu})$ be the number of nodes in $\vec{\nu}$.

In analogy to Proposition 1.11, we see that to a nodal band representation $\vec{\nu}$ and a smooth map $f: \partial D^2 \to E_n - \Delta$ representing $\beta(\vec{\nu})$, there corresponds a smooth extension of f over D^2 with $l(\vec{\nu})$ intersections with Δ, of which $\kappa(\vec{\nu})$ are "node-like". A suitable converse holds.

Slides (as well as various species of expansion and contraction) can be defined as before. In particular one sees that any nodal band representation is slide-equivalent to (and thus has the same braid as) a nodal band representation with all the nodes at the end.

A.2. From a nodal band representation $\vec{\nu}$ a nodal braided surface $i: S \to D$, and hence a ribbon immersion $S \to D^4$, each with boundary (of the link type of) $\hat{\beta}(\vec{\nu})$, can be constructed; likewise, after a choice of conjugators $w(i)$ with

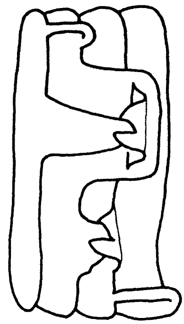

Figure A.1. A clasp/node surface derived from a nodal band representation.

$\nu(i) = {}^{w(i)}\sigma^{\pm s}_{j(i)}$, an immersion $S \to S^3$ with boundary $\hat{\beta}(\vec{\nu})$. All of these, as before, will be indiscriminately denoted by $S(\vec{\nu})$. The model in S^3 is no longer a ribbon immersion if $\kappa(\vec{\nu}) \neq 0$. It will have, besides ribbon singularities, so-called *clasp singularities.*

A component of the set of clasp singularities on the immersed surface is an arc of double-points A; the two inverse images A' and A'' each have one endpoint on the boundary of the abstract surface, and one in its interior; and the two sheets of the immersion are transverse along A. Figure A.1 shows how each node in the nodal band representation contributes one clasp (and, of course, possibly some ribbon singularities). Observe that the immersion isn't quite tabled – again, from each node there is a contribution of a single flap of the backside of the surface exposed to view.

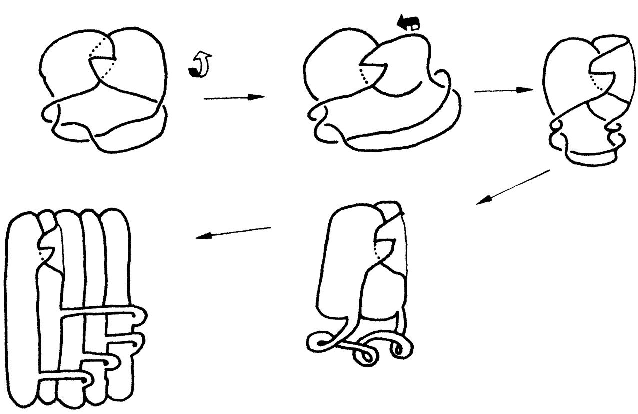

Figure A.2. Putting a clasp disk bounded by the stevedore's knot into the form $S(\vec{\nu})$.

A.3. Again, the construction is general; again, this is most easily seen in $\mathbf{R}^3$. One finds, in the isotopy class of the given clasp/ribbon surface, a surface which is "almost tabled" – tabled except for flaps such as those mentioned above.* As before, all the double-arcs of ribbon singularities can be made "vertical" segments in planes parallel to the table; and now all the double-arcs of clasp singularities are taken to be "horizontal." In the neighborhood of a clasp double-arc, two flaps (one tabled, the other not) interpenetrate each other, each attached to the front edge of one of the stacked 0-handles. Then one proceeds just as before. Figure A.2 illustrates this for a clasp disk bounded by the stevedore's knot.

A.4. If $\vec{\nu}$ is a nodal band representation in B_n of length k, the fundamental group $\pi_1(D-S(\vec{\nu}))$ can be presented with n generators and k relations; $\kappa(\vec{\nu})$ among the relations will be of the form "two (not necessarily standard) meridians commute," while the rest as before set two meridians equal. Analogues of all the results in §4 hold here.

A.5. If $K \subset S^3$ is a knot, define $\ddot{u}(K)$ [resp., $\ddot{u}_r(K)$; $\ddot{u}_s(K)$] to be the least integer k such that there is a ribbon immersion of a disk in D^4, bounded by K, with only one local minimum [resp., a ribbon immersion of a disk in D^4, bounded by K; an immersion of a disk in D^4, bounded by K] with exactly k singular points, each one a node. Then $\ddot{u}(K)$ is the ordinary *überschneidungszahl* of K, and may also be defined as the least number of self-crossings in a generic regular homotopy of K to an unknot; while $\ddot{u}_r$ and $\ddot{u}_s$ may be called the "ribbon überschneidungszahl" and "slice überschneidungszahl" respectively, for obvious reasons.

Now, if $S \subset M^4$ is any generically immersed surface in a 4-manifold, a surgery may be done on S inside M, replacing two 2-disks on S with a node as their intersection by an annulus with the same boundary, thereby increasing the genus of S (if it is connected) while decreasing the number of nodes by the same amount; and if S is ribbon-immersed in the 4-disk, such a surgery can be done within the class of ribbons, each annulus introducing two new saddles and no local extrema. Thus we have inequalities $\ddot{u}(K) \geqq \ddot{u}_r(K) \geqq g_r(K)$, $\ddot{u}_r(K) \geqq \ddot{u}_s(K) \geqq g_s(K)$, for any knot K.

PROPOSITION. *For any knot K, $\ddot{u}_r(K)$ is the least number k of self-crossings in a (generic) regular homotopy of K to a ribbon knot.*

Proof. The trace of a regular homotopy of K to K' is an annulus with singularities (generically, only nodes) in $S^3 \times I$. So $k \geqq \ddot{u}_r(K)$. To see that $k \leqq \ddot{u}_r(K)$, let S be a disk, ribbon immersed in D^4 with boundary K, with exactly

* Added in proof: using vertical double arcs, clasps too may be tabled completely.

$\ddot{u}_r(K)$ nodes. Then there is a nodal band representation $\vec{\nu}$ such that S is ambient isotopic to $S(\vec{\nu})$; if $\vec{\nu}$ is in B_n, then an Euler characteristic argument shows that $l(\vec{\nu})=\kappa(\vec{\nu})+n-1=\ddot{u}_r(K)+n-1$. After slides, if necessary, we may assume that all the nodes in $\vec{\nu}$ are collected at the end, $\vec{\nu}=(\vec{\nu}', \vec{\nu}'')$ where $\vec{\nu}'$ is an ordinary band representation of length $n-1$ in B_n, and $\vec{\nu}''$ is all nodes. The permutation of a node is trivial, so $\beta(\vec{\nu}')$ has the same permutation as $\beta(\vec{\nu})$, and $\hat{\beta}(\vec{\nu}')$ is a knot K', bounding a ribbon disk $S(\vec{\nu}')$. Now, evidently, $\vec{\nu}''$ may be understood as defining a regular homotopy of K to K' with $\ddot{u}_r(K)$ self-crossings. □

PROPOSITION. *If the knot K is the closure of a strictly positive braid $\beta\in B_n$, then $\ddot{u}(K)\leqq\frac{1}{2}(e(\beta)-n+1)$.*

Proof. Recall that the diagram $\mathbf{D}(\beta)$ of a positive braid β is the (finite) set of positive braid words with that braid. Call β *square-free* if no word in its diagram has two consecutive letters equal.

Suppose that for $k<e(\beta)$, $m\leqq n$, if $\gamma\in B_m$ is a strictly positive braid, $e(\gamma)=k$, then there is a regular homotopy of $\hat{\gamma}$ to an unknot with $\frac{1}{2}(e(\gamma)-m+1)$ self-crossings. If β_f is not square-free, let $\beta=\beta(\vec{b})$ where some two consecutive letters in $\vec{b}$ are equal, and let $\vec{b}_0$ be $\vec{b}$ with those two letters omitted; then there is a regular homotopy of K to $\hat{\beta}(\vec{b}_0)$ with one self-crossing, and this may be followed by the inductively-assumed homotopy of $\beta(\vec{b}_0)$ to the unknot, to produce the desired homotopy of K.

So let β be square-free. Each word in its diagram has at least one σ_{n-1} in it, for β is strictly positive. If some word in $\mathbf{D}(\beta)$ has σ_{n-1} in it exactly once, then that letter may be omitted to obtain a braid of smaller exponent sum (in one lower string index) with closure K, and the homotopy we seek exists by the inductive hypothesis. So we may assume each word in $\mathbf{D}(\beta)$ contains σ_{n-1} at least twice. Find $\vec{b}$ in $\mathbf{D}(\beta)$ with the fewest possible uses of σ_{n-1}, and, among those, with some two uses of σ_{n-1} separated by as few letters as possible, say $\vec{b}=\alpha\sigma_{n-1}\gamma_0\sigma_{n-1}\delta$, $\gamma_0\in B_{n-1}$. Then γ_0 is not empty (since σ_{n-1}^2 cannot appear in $\vec{b}$), and it certainly begins with σ_{n-2} (for a letter with a smaller subscript could be commuted forwards past σ_{n-1}, shortening γ_0), and likewise ends with σ_{n-2}. Write $\gamma_0=\sigma_{n-2}\gamma_1\rho_0$, where $\gamma_1\in B_{n-2}$ and ρ_0 is either empty (in which case so is γ_1) or begins and ends in σ_{n-2}. Continue this process iteratively as long as possible, writing $\gamma_i=\sigma_{n-2-i}\gamma_{i+1}\rho_i$, where $\gamma_{i+1}\in B_{n-2-i}$ and ρ_i is either empty (in which case so is γ_{i+1}) or begins and ends in σ_{n-2-i}. The process must stop eventually, and at that point we have $\vec{b}=\alpha\sigma_{n-1}\sigma_{n-2}\cdots\sigma_{n-2-i}\rho_{i+1}\cdots\rho_0\sigma_{n-1}\delta$. But ρ_{i+1} isn't empty (the process stops at the first empty remainder), so it begins with σ_{n-1-i}, and we have $\vec{b}=\alpha\cdots\sigma_{n-1-i}\sigma_{n-2-i}\sigma_{n-1-i}\cdots\delta$. Now apply the standard relation to rewrite $\sigma_{n-1-i}\sigma_{n-2-i}\sigma_{n-1-i}$ as $\sigma_{n-2-i}\sigma_{n-1-i}\sigma_{n-2-i}$, and commute the first of the new letters forward past everything till it passes σ_{n-1}. Now the two σ_{n-1}'s are separated by fewer letters than in $\vec{b}$, contrary to assumption; so no square-free word has σ_{n-1} in it twice.

We are done, once we start the induction. But the only strictly positive braid of exponent sum 1 is σ_1 in B_2, for which $\ddot{u}(\hat{\beta})=\frac{1}{2}(1-2+1)$. $\square$

Remark. Milnor [Mi] conjectured the proposition, with an equality, in the particular case of links of singularities; and Henry Pinkham has given an inductive argument (based on the structure of such links as iterated torus knots) proving the proposition, again in that case.

CONJECTURE. *If β is a strictly positive braid (coming, for instance, from the link of an irreducible singular point of a complex plane curve), then there are equalities $\ddot{u}(\hat{\beta})=\ddot{u}_r(\hat{\beta})=g(\hat{\beta})=g_r(\hat{\beta})$.*

This would follow from the discredited Hypothesis II of §5, and still seems a good bet.

Index of notation

Notations introduced in the paper (other than ephemera, used briefly in proof or exposition and then discarded) are listed with their page of definition. Standard symbols appear on the list if their use is somewhat idiosyncratic, or if they are very basic, or occasionally if their domain of standard use is remote from topology.

$b; b(i)$	a band; the i^{th} band in a band representation (6, 7)
$\vec{b}$	a band representation (7)
$\beta(\vec{b}); \hat{\beta}(\vec{b})$	the braid of a band representation (7); its closure
B_n	the braid group on n strings ($n-1$ generators) (4)
$c(\beta)$	the number of cycles in the permutation of β (27)
$D; D^4$	a bidisk $D^2\times D^2$ (4); a round 4-disk (5)
Δ	the "discriminant locus" (4)
$e(\beta)$	the exponent sum of β (7) (the image of β in $\mathbf{Z}=H^1(B_n)$)
E_n	the ambient space of Δ (4)
$E_n-\Delta$	the "configuration space" of n points in $\mathbf{R}^2$ (4)
g, g_r, g_s	genus, ribbon genus, slice genus of a link (26)
$rk_n(\beta)$	the rank of β in B_n (7, 26)
R_i, R_{ij}	"standard relations" in the braid group (6)

$\mathscr{R}$	regular locus of an algebraic set (4)
S_j, S_j^{-1}	forward and backwards slides of band representations at the jth place (8)
$\mathfrak{S}_n$	the symmetric group on n letters
$\mathscr{S}$	singular locus of an algebraic set (4)
$S(\vec{b})$	the Seifert ribbon corresponding to a band representation $\vec{b}$ (15)
σ_i	a "standard generator" of B_n (6)
$\sigma_{i,j}$	an "embedded band" (7)
T_β	Stallings's notation for a particular kind of $S(\vec{b})$ (13)
$ü, ü_r, ü_s$	the überschneidungszahl of a knot, and a ribbon and slice analogue thereof (33)

In any group, $[x, y]$ denotes $xyx^{-1}y^{-1}$, and ${}^xy$ denotes xyx^{-1}.

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A reciprocity law for K_2-traces

SHMUEL ROSSET and JOHN TATE

Suppose $E \subset F$ is a finite field extension and let

$\mathrm{Tr}: K_2(F) \to K_2(E)$

be the trace map (also called transfer, see [5, §14]). If $x, y \in F^*$ and $\{x, y\}$ is the corresponding symbol in $K_2(F)$ then we know, since $K_2(E)$ is generated by symbols, that $\mathrm{Tr}_{F/E}(\{x, y\})$ can be expressed as a sum of symbols. In this paper we give an algorithm for computing such an expression explicitly (cf. the proposition in §3). The algorithm is based on a reciprocity law (§2) and involves repeated polynomial division with remainder, like the Euclidean algorithm. The proof works not only for Milnor's K_2, but for functors sufficiently like K_2, which we define in §1 and call Milnor functors. This abstraction is useful for it yields as a corollary (§3) the fact that the canonical map from K_2 to any Milnor functor commutes with traces. Another corollary is that, if $(F:E) = n$, then $\mathrm{Tr}_{F/E}(\{x, y\})$ can be written as a sum of n symbols (or less). On the other hand this is also the best bound: in §4 we give an example, using division algebras, of a symbol whose trace is not a sum of less than n symbols.

One of us (S.R) would like to thank David Saltman for a conversation which helped realize the example in section 4.

1. Milnor functors

Let K be a fixed base field and let $\mathfrak{C}$ be the category of commutative finite dimensional k-algebras.

DEFINITION. A *Milnor functor over k* is a functor $M: \mathfrak{C} \to$ (Abelian groups) together with

(i) For each $A \in \mathfrak{C}$ a bilinear map $\varphi = \varphi_A : A^* \times A^* \to M(A)$;

(ii) For each extension $A \to B$ in $\mathfrak{C}$ such that B is a projective A-module, a homomorphism $\mathrm{Tr}_{B/A} : M(B) \to M(A)$; such that the following properties hold.

(φ) The maps φ are functorial, i.e., induce a morphism of functors from the functor $A \mapsto A^* \times A^*$ to the functor $A \mapsto M(A)$, and satisfy

$$\varphi_A(a, 1-a) = 0, \quad \text{if} \quad a \in A^* \quad \text{and} \quad 1-a \in A^*,$$

$$\varphi_A(a, -a) = 0, \quad \text{if} \quad a \in A^*.$$

(Tr) if $A \to B \to C$ are $\mathfrak{C}$-morphisms such that C is projective over B and B over A, then

$$\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B}$$

(Tr$-\varphi$) If $A \to B$ is a $\mathfrak{C}$-morphism with B projective as A-module, and if $x \in A^*$, $y \in B^*$ then

$$\mathrm{Tr}_{B/A}\varphi_B(x, y) = \varphi_A(x, N_{B/A}y),$$

where $N_{B/A} : B^* \to A^*$ is the usual norm:

$$N_{B/A}(y) = \det \quad (\text{multiplication by } y).$$

EXAMPLE 1. Milnor's K_2; see [5] and [6].

EXAMPLE 2. Assume that the characteristic of k does not divide a given integer n and let μ_n denote the sheaf on n-th roots of 1 on the étale site over Spec A; here A is a given element in Ob ($\mathfrak{C}$). By Kummer theory

$$H^1(\mathrm{Spec}\, A, \mu_n) = A^*/(A^*)^n.$$

The cup product

$$H^1(\mathrm{Spec}\, A, \mu_n) \times H^1(\mathrm{Spec}\, A, \mu_n) \to H^2(\mathrm{Spec}\, A, \mu_n^{\otimes 2}) = M(A)$$

provides us with a context satisfying (i) and (ii). We refer to Milne's book [4] for details. The existence of a trace can probably be extracted from [7, exp. XVII]. However, this Milnor functor can be expressed entirely in terms of Galois cohomology and the trace in terms of corestriction, as follows. For $A \in \mathfrak{C}$, $\alpha \in M(A)$, and $x \in \mathrm{Spec}\, A$, let $\alpha(x) \in M(A/x)$ be the image of α under the residue class map $A \to A/x$. then the map

$$\alpha \mapsto (\alpha(x))_{x \in \mathrm{Spec}\, A}$$

gives an isomorphism

$$M(A) \xrightarrow{\sim} \prod_{x \in \operatorname{Spec} A} M(A/x). \tag{*}$$

For each $x \in \operatorname{Spec} A$, A/x is a finite extension field of k. If E is a finite extension field of k, then

$$M(E) = H^2(\operatorname{Gal}(E_s/E), \mu_n(E_s) \otimes \mu_n(E_s)),$$

where E_s is a separable algebraic closure of E. The map φ_A is characterized in terms of the isomorphism (*) by

$$(\varphi_A(a, b))(x) = \varphi_{A/x}(a(x), b(x))$$

for each $x \in \operatorname{Spec} A$, where $a(x)$ (resp. $b(x)$) is the residue mod x of a (resp. b), and for a field E the map

$$\varphi_E : E^* \times E^* \to M(E)$$

is the Galois cohomology symbol (cf. [8]) characterized by $\varphi(a, b) = da \cup db$, where $d : E^* \to H^1(\operatorname{Gal}(E_s/E), \mu_n(E_s))$ is the connecting homomorphism in the exact cohomology sequence associated with

$$0 \to \mu_n(E_s) \to E_s^* \xrightarrow{n} E_s^* \to 0.$$

Let $A \to B$ be an extension in $\mathfrak{C}$ such that B is a projective A-module. Then for each $x \in \operatorname{Spec} A$ and each $y \in \operatorname{Spec} B$ lying over x, the local ring B_y is a free A_x-module; let $r(y/x)$ denote its rank. Let $E_x = A/x$ and let F_y be the field between E_x and B/y such that F_y/E_x is separable and $(B/y)/F_y$ purely inseparable. Then the ratio

$$q(y/x) \overset{\text{defn}}{=} \frac{r(y/x)}{[F_y : E_x]}$$

is an integer, and the M-trace from B to A is characterized in terms of the isomorphism (*) by

$$(\operatorname{Tr}_{B/A}\beta)(x) = \sum_{y|x} q(y/x) \operatorname{cor}_{F_y/E_x}(\beta(y)),$$

where cor is the corestriction in Galois cohomology, and we identify $M(B/y)$ with $M(F_y)$ via the isomorphism induced by the inclusion $F_y \hookrightarrow B/y$.

In case $E \in \mathfrak{C}$ is a field containing a primitive n-th root of unity ζ, we can identify $M(E)$ with the group $\mathrm{Br}_n(E)$ of elements of order n in the Brauer group of E in such a way that

$$(a, b)_M = \text{the Brauer class of } A_\zeta(a, b)$$

where $A_\zeta(a, b)$ denote the cyclic algebra generated over E by elements X and Y subject to the relations

$$X^N = a, \qquad Y^n = b, \qquad XY = \zeta YX;$$

(cf. [5], p. 143).

EXAMPLE 3. The dlog symbol, see [1]. If A is a k algebra in $\mathfrak{C}$ let $\Omega^1_{A/k}$ be the A-module of Kähler differentials of A over k, and let $\Omega^2_{A/k}$ be its second exterior power. Define

$$\mathrm{dlog}: A^* \to \Omega^1_{A/k}$$

by $\mathrm{dlog}(f) = f^{-1} \cdot df$. It is simple to verify that Ω^2 and $\mathrm{dlog} \wedge \mathrm{dlog}$ satisfy axioms (i), (ii) above. The existence of a good trace is a non-trivial fact [2].

2. Reciprocity

Let M be a Milnor functor over k. In this section we shall write the M-symbol $\varphi_E(x, y)$ by

$$(x, y)_E, \quad \text{or} \quad (x, y)$$

if E is evident.

Let K be a field of finite degree over k. For relatively prime non-zero polynomials $f(T)$, $g(T)$ in $K[T]$ we define a new kind of symbol (f/g). Its values are in the group $M(K)$ and it is defined by the following requirements.

1) It is additive in g, i.e. if g_1, g_2 are both prime to f then

$$\left(\frac{f}{g_1 g_2}\right) = \left(\frac{f}{g_1}\right) + \left(\frac{f}{g_2}\right)$$

2) It is 0 if g is a constant or $g = T$.

3) If g is *monic irreducible* $\neq T$ and x is a root of $g(T)$ then

$$\left(\frac{f}{g}\right)=\mathrm{Tr}_{K(x)/K}\,(x, f(x))_{K(x)}.$$

It is clear that, thus defined, the symbol (f/g) is additive in f, as well as in g, and it depends only on the residue class of f modulo (g). As function of g it depends only on the ideal generated by g in the ring $K[T, T^{-1}]$.

To formulate the reciprocity law satisfied by (f/g) we introduce some notation: if

$$p(T)=a_nT^n+a_{n-1}T^{n-1}+\cdots+a_mT^m$$

with $a_ma_n\neq 0$. let

$$p^*(T)=(a_mT^m)^{-1}p(T)$$
$$c(p)=(-1)^na_n.$$

Reciprocity law

$$\left(\frac{f}{g}\right)=\left(\frac{g^*}{f}\right)-(c(g^*), c(f)). \tag{**}$$

Proof. We first dispose of a few trivial cases. If g is a constant or T it is easily checked that both sides are 0, so we assume henceforth that $g(T)$ is monic irreducible $\neq T$. let x be a root of $g(T)$. If $f(T)$ is a constant c then the left side of (**) is

$$\begin{aligned}\mathrm{Tr}_{K(x)/K}\,(x, c)_{K(x)}&=(N_{K(x)/K}x, c)_k\\ &=((-1)^{\deg(g)}\cdot g(0), c)=-((-1)^{\deg(g)}\cdot g(0)^{-1}, c)\\ &=-(c(g^*), c(f))\end{aligned}$$

which is equal to the right hand side since $(g^*/f)=0$, by definition.

A similar computation using $(x, -x)=0$ works when $f(T)=T$ so we now assume that both f and g are monic irreducible, and not T.

Let x be a root of g and y a root of f. Let

$$A=K(x)\otimes_K K(y).$$

$K(x)$ and $K(y)$ are naturally imbedded in A and we identify them as such. Then the elements $x, y, x-y$ are invertible in A, indeed the norm

$$N_{A/K(x)}(x-y)=f(x)$$

is invertible, so $x-y$ is.

The identity

$$(x, x-y)=\left(y, \frac{y-x}{-x}\right)+(x, -1)$$

follows from a little computation with the relations $(u, 1-u)=(u, -u)=0$. We use it to compute the same thing in two ways

$$\begin{aligned}\mathrm{Tr}_{A/K}\,(x, x-y) &= \mathrm{Tr}_{K(x)/K}\,\mathrm{Tr}_{A/K(x)}\,(x, x-y)\\ &= \mathrm{Tr}_{K(x)/K}\,(x, N_{A/K(x)}(x-y))\\ &= \mathrm{Tr}_{K(x)/K}\,(x, f(x))=\left(\frac{f}{g}\right).\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}_{A/K}\left(y, \frac{y-x}{-x}\right) &= \mathrm{Tr}_{K(y)/K}\,\mathrm{Tr}_{A/K(y)}\left(y, \frac{y-x}{-x}\right)\\ &= \mathrm{Tr}_{K(y)/K}\left(y, \frac{N_{A/K(y)}(y-x)}{N_{A/K(y)}(-x)}\right)\\ &= \mathrm{Tr}_{K(y)/K}\left(y, \frac{g(y)}{g(0)}\right)\\ &= \mathrm{Tr}_{K(y)/K}\,(y, g^*(y))=\left(\frac{g^*}{f}\right).\end{aligned}$$

Finally

$$\begin{aligned}\mathrm{Tr}_{A/K}\,(x, -1) &= \mathrm{Tr}_{K(y)/K}\,\mathrm{Tr}_{A/K(y)}\,(x, -1)_A\\ &= \mathrm{Tr}_{K(y)/K}\,(N_{A/K(y)}x, -1)_{K(y)}\\ &= \mathrm{Tr}_{K(y)/K}\,(N_{K(x)/K}x, -1)_{K(y)}\\ &= (c(g^*)^{-1}, (-1)^{\deg(f)})=-(c(g^*), c(f)).\end{aligned}$$

Here we used the obvious fact that

$$N_{A/K(y)}(x)=N_{K(x)/K}(x).$$

This completes the proof of the reciprocity law.

3. Consequences

Let $E \subset F$ be a finite extension of fields finite over k, and let $x, y \in F^*$. Then

$$\mathrm{Tr}_{F/E}(x, y) = \left(\frac{f}{g}\right)$$

where $g(T) \in E[T]$ is the monic irreducible polynomial with root x and $f(T) \in E[T]$ is the polynomial of smallest degree such that $N_{F/E(x)}y = f(x)$.

PROPOSITION. *Let* $g_0, g_1, \ldots, g_m \neq 0$, $g_{m+1} = 0$ *be the sequence of polynomials defined by:*

$$g_0 = g, \qquad g_1 = f,$$

and for $i \geq 1$

$g_{i+1} =$ *the remainder of the division of* g_{i-1}^* *by* g_i,

as long as $g_i \neq 0$. *We have then*

$$1 \leq m \leq \deg g = [E(x):E] \leq [F:E]$$

and

$$\mathrm{Tr}_{F/E}(x, y) = -\sum_{i=1}^{m} (c(g_{i-1}^*), c(g_i)).$$

By the reciprocity law, we find by induction on j, using $(g_{i-1}^*/g_i) = (g_{i+1}/g_i)$:

$$\left(\frac{g_1}{g_0}\right) = -\sum_{i=1}^{j} (c(g_{i-1}^*), c(g_i)) + \left(\frac{g_{j-1}^*}{g_j}\right)$$

for $1 \leq j \leq m$. But the last non-zero polynomial g_m is a constant because it divides the relatively prime polynomials g_0 and g_1. Hence $(g_{m-1}^*/g_m) = 0$, and the proposition follows on putting $j = m$; We have $m \leq \deg g$ because the degrees of the polynomials in the sequence are strictly decreasing, and $m \geq 1$ because $f \neq 0$.

COROLLARY 1. *If* $[F:E] = r$ *and* $x, y \in F^*$, *then* $\mathrm{Tr}_{F/E}(x, y)$ *is a sum of at most* r *symbols.*

The sequence of polynomials in the proposition depends only on F, E, x, and y, not on the Milnor functor M. Thus the trace of a symbol $(x, y)_M$ has an expression as a sum of symbols which is *independent of the Milnor functor M;* on symbols, the trace is uniquely determined. Any morphism $M_1 \to M_2$ of Milnor functors which carries each symbol $(a, b) \in M_1(A)$ to the "same" symbol $(a, b) \in M_2(A)$ must therefore commute with $\mathrm{Tr}_{F/E}$ on symbols. In particular, letting $R_F : K_2(F) \to M(F)$ be the homomorphism (whose existence and unicity are guaranteed by Matsumoto's theorem) such that $R_F(\{a, b\}) = (a, b)_M$ for $a, b \in F^*$, and similarly R_E, we have

COROLLARY 2. *The diagram*

$$\begin{tikzcd} K_2 \arrow[r, "R_F"] \arrow[d, "\mathrm{Tr}_{F/E}"'] & M(F) \arrow[d, "\mathrm{Tr}_{F/E}"] \\ K_2(E) \arrow[r, "R_E"'] & M(E) \end{tikzcd}$$

is commutative.

4. An example

We have just proved that if $[F:E] = r$ and $x, y \in F^*$ then $\mathrm{Tr}_{F/E}\,(x, y)$ is a sum of r symbols. Yet it is known that in some cases, e.g. global or local fields, every element of K_2 (say) is a symbol [8, 3], so it is well to give an example where $\mathrm{Tr}\,(x, y)$ cannot be written as a sum of fewer than r symbols. For this it will suffice to work with the functor of Example of Section 1.

Let $n \geq 2$ and $r \geq 1$ be integers. Let k_0 be a field containing a primitive n-th root of unity, ζ. Let $u_1, v_1, \ldots, u_r, v_r$ be $2r$ independent variable over k_0 and let

$$F = k_0(u_1, v_1; u_2, v_2; \ldots; u_r, v_r)$$

be the field they generate. Let M be the Milnor functor of Example 2.

LEMMA. *The element* $\beta = \sum_{i=1}^{r} (u_i, v_i)$ *in* $M(F)$ *is not a sum of fewer than r symbols.*

Proof. We use the identification $M(F) \xrightarrow{\sim} \mathrm{Br}_n\,(F)$ discussed at the end of

Example 2. For $1 \leq i \leq r$ let B_i by the cyclic algebra over F generated by elements X_i and Y_i subject to the relations

$$X_i^n = u_i, \qquad Y_i^n = v_i, \qquad X_i Y_i = \zeta Y_i X_i,$$

so that (u_i, v_i) is the Brauer class of B_i. Then β is the Brauer class of $B = \otimes_{i=1}^r B_i$, an algebra of dimension n^{2r} over F. We will show B is a division algebra. This will prove the lemma, for it shows that β cannot be the Brauer class of an algebra of dimension less than n^{2r}, and consequently cannot be a sum of fewer than r symbols.

If B were not a division algebra it would have zero divisors, and multiplying these zero divisors by a common denominator of their coefficients in F relative to the basis

$$\{X_1^{l_1} Y_1^{m_1} \cdots X_r^{l_r} Y_r^{m_r}\} \quad (0 \leq l_i, m_i < n)$$

for B over F, we would find zero divisors in the ring

$$R = k_0[u_1, v_1, \ldots, u_r, v_r][X_1, Y_1, \ldots, X_r, Y_r] = k_0[X_1, y_1, \ldots, X_r, Y_r].$$

But this ring has no zero divisors, for it has a basis over k_0 consisting of the monomials

$$X_1^{l_1} Y_1^{m_1} \cdots X_r^{l_r} Y_r^{m_r}$$

with l_i, m_i integers ≥ 0, and the product of two such monomials is a power of ζ times the monomial obtained by adding exponents. Hence, if we order the monomials by the lexicographical order of their exponent sequences, the product of two non-zero polynomials will contain the product of the highest terms in the two factors with a non-zero coefficient, so will not be 0. This proves the lemma.

Let σ be the automorphism of F which is identity on k_0 and acts on the variables by

$$\sigma u_i = u_{i+1}, \qquad 1 \leq i \leq r; \qquad u_{r+1} = u_1,$$
$$\sigma v_i = v_{i+1}, \qquad 1 \leq i \leq r; \qquad v_{r+1} = v_1.$$

Let G be the cyclic group of order r generated by σ, and let $E = F^G$.

PROPOSITION. *The image of* $\{u_1, v_1\}$ *under* $\mathrm{Tr}_{F/E} : K_2F \to K_2E$ *is not a sum of fewer than r symbols.*

Proof. We use the commutativity of

$$\begin{tikzcd} K_2(F)/nK_2(F) \arrow[r, "R_F"] \arrow[d, "\mathrm{Tr}"'] & \mathrm{Br}_n\,(F) \arrow[d, "\mathrm{Tr}"] \\ K_2(E)/nK_2(E) \arrow[r, "R_E"'] & \mathrm{Br}_n\,(E). \end{tikzcd}$$

and the rule

$$\mathrm{res}_{E/F}\,\mathrm{Tr}_{F/E}\,\alpha = \sum_{\tau\in G} \tau\alpha$$

for $\alpha \in \mathrm{Br}\,F$. If $\mathrm{Tr}\,\{u_1, v_1\}$ were a sum of $s < r$ symbols so also would be

$$\begin{aligned} \mathrm{res}\,R_E\,\mathrm{Tr}\,\{u_1, v_1\} &= \mathrm{res}\,\mathrm{Tr}\,R_F\{u_1, v_1\} = \mathrm{res}\,\mathrm{Tr}\,(u_1, v_1) \\ &= \sum_{\tau\in G} \tau(u_1, v_1) = \sum_{i=1}^{r} (u_i, v_i) = \beta, \end{aligned}$$

contradicting the lemma.

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Acyclic groups of automorphisms

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1. Introduction

A discrete group Γ is said to be acyclic if its Eilenberg–MacLane homology groups $H_i(\Gamma)$ with coefficients in the trivial Γ-module $\mathbf{Z}$ are zero for all $i>0$. In this paper we show that certain groups, such as the group $GL(V)$ of all continuous linear automorphisms of an infinite dimensional Hilbert space V, are acyclic. This is a folk theorem which was surely known long ago to experts in the field such as Quillen and Segal. However it seems worthwhile to publish a proof in view of the recent interest shown in such questions. For example, Herman pointed out in [He] that the group of diffeomorphisms of a compact manifold admits a canonical representation in $GL(V)$. Therefore, if $GL(V)$ had carried non-trivial cohomology, one might have been able to define non-trivial characteristic classes for groups of diffeomorphisms. See also section 2.6 in [Ma] and the concluding remark of [H2].

We will consider the following groups.

1. The group $\Sigma(X)$ of all permutations of an infinite set X.

2. The group $\mathcal{A}(\Omega)$ of measure preserving automorphisms of a Lebesgue measure space $(\Omega, \mathcal{B}, \mu)$ where μ is infinite and non-atomic. (As usual one identifies two automorphisms which agree μ-a.e.)

3. The group $GL(W)$ of all linear automorphisms of an infinite dimensional vector space W.

4. The group $GL(V)$ of all continuous linear automorphisms of an infinite dimensional Hilbert space V over the real, complexes or quaternions, as well as the group $U(V)$ of invertible isometries of V.

5. The group $GL(M)$ of invertible elements in a properly infinite von Neumann algebra M, and the subgroup $U(M)$ of unitary elements.

THEOREM. *The groups defined above are acyclic.*

The above list is by no means complete. One could add many "classical

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groups" in the sense of [H3], and also the group of continuous linear automorphisms of an infinite dimensional topological vector space E for suitable E. The Banach spaces c_0 and l_p, $1 \leqslant p < \infty$, are possible candidates: see proposition 2.a.2 in [LT]. However Douady [D] constructs a Banach space E for which the group of connected components of $GL(E)$ is isomorphic to $\mathbf{Z}$. It follows that $GL(E)$ is not perfect and hence not acyclic. Therefore the above theorem does not hold for $GL(E)$ where E is an arbitrary Banach space. See also [St]. For acyclic groups of a quite different nature from those of our list, see [BDH] and [BDM].

Here is one consequence of the theorem.

COROLLARY. *Let G be one of the groups above and let A be a finitely generated abelian group. Then any extension*

$$0 \to A \to \tilde{G} \to G \to 1$$

is trivial.

Proof. Any non-trivial normal subgroup of G is of uncountable index. (See Appendix 1.) In particular any homomorphism from G to Aut (A) is trivial and so G acts trivially on A in the above extension. Our main theorem implies that $H^2(G; A)$ is zero. Hence the extension is a semi-direct product. Again using the fact that the action of G on A is trivial, we see that the product is direct. ∎

A notable feature of the groups in 2, 4 and 5 is that they are contractible when given their natural topologies. (See [Ke] for $\mathscr{A}(\Omega)$, [DD] for $U(V)$ and $U(M)$ with the strong topology, [Ku] for $GL(V)$ and $U(V)$ with the uniform topology, and [BW] for $GL(M)$ and $U(M)$ with the uniform topology.) There are many other contractible groups of automorphisms which are acyclic when considered as discrete groups: for example, the group of compactly supported homeomorphisms of $\mathbf{R}^n$ [M], and the group of diffeomorphisms of $\mathbf{R}^n$ which are the identity near the origin [Se]. On the other hand, Sah pointed out that the universal cover $\widetilde{SL(2, \mathbf{R})}$ of $SL(2, \mathbf{R})$ is contractible as a topological group but is not acyclic as a discrete group [SW]. The main tool which we use in proving acyclicity is the infinite repetition argument of Mather [M] and Wagoner [W]. (See also [BDH] §4 and [Be] ch. 3.) There are several contractible groups which are more "flexible" than $\widetilde{SL(2, \mathbf{R})}$, but are still not large enough for this argument to be used. We have in mind groups such as $\mathscr{A}(\Omega)$, where Ω has finite measure, or the group of compactly supported homeomorphisms of $\mathbf{R}^n$ which preserve Lesbegue measure, for $n > 2$. These groups are known to be perfect [F1], [F2], and it would be interesting to know if they are acyclic. One could also consider much bigger groups such as the group of all homeomorphisms of a Hilbert cube or a Hilbert space. These are shown to be contractible in [Re]. The groups $GL(M)$ and $U(M)$, where M is a finite continuous von Neumann factor, are not contractible. They are discussed further in section 4.

The theorem is not hard to prove. We first show that the subgroup G_F of elements in G which are the identity on an appropriately defined "flag" F is acyclic. Then we show, using a technique due to Segal (§2 in [Se]), that this forces the whole group G to be acyclic. The first of these two steps uses the infinite repetition argument of [M] and [W] and, in the general case, an elegant algebraic trick due to Quillen [Q2]. The second step works essentially because the Tits building (or partially ordered set) formed by the flags is contractible. We give the proof for $GL(V)$ in full detail, and in section 4 sketch the modifications needed for the other groups.

We discuss in Appendix 1 the results about normal subgroups of G needed for the corollary above. Though these are old results, we indicate for $GL(W)$ and $GL(V)$ a proof much shorter than the originally published ones. Doing this, we again show that G is perfect, namely that $H_1(G)$ is trivial. This is what our main result and proof reduce to when cleared from homological machinery.

Finally Appendix 2 describes a result due to Quillen according to which the monoids (or semi-groups) related to our groups are contractible and hence acyclic.

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2. Subgroups of $GL(V)$

In this section and the next one, V denotes an infinite dimensional Hilbert space. Let F be a *flag* in V: we mean by this that F is a nested sequence $S_1 \supset S_2 \supset S_3 \supset \cdots$ of closed subspaces of $V = S_0$ such that S_{i-1}/S_i is isomorphic to V for each $i \geq 1$. Define

$$G_i = \{g \in GL(V) \mid g = \mathrm{id} \text{ on } S_i\}$$

and

$$G_i' = \{g \in G_i \mid g(S_i^\perp) = S_i^\perp\}$$

for each $i \geq 0$. Define also G_∞ to be the union of the G_i's and G_∞' that of the G_i''s. Then

$$\begin{array}{ccccccccc} 1 = G_0 & \subset & G_1 & \subset \cdots \subset & G_i & \subset \cdots \subset & G_\infty \\ \| & & \cup & & \cup & & \cup \\ G_0' & \subset & G_1' & \subset \cdots \subset & G_i' & \subset \cdots \subset & G_\infty'. \end{array}$$

For $g \in G_\infty$, observe that $g = \mathrm{id}$ on $S_\infty = \bigcap S_i$. For notational convenience, we assume $S_\infty = \{0\}$. (But proposition 1 as well as its consequences in section 3 and the variations of section 4 would obviously hold without this assumption.) The result of this section is:

PROPOSITION 1. *The groups G'_∞ and G_∞ are acyclic.*

We shall recall the following facts from §2 in [W]. A *flabby* group is a group Γ such that there exist homomorphisms

$$\mu : \Gamma \times \Gamma \to \Gamma \quad \text{(direct sum)}$$
$$\tau : \Gamma \to \Gamma \quad \text{(infinite repetition)}$$

with the following properties: For any finite subset $\Phi \subset \Gamma$, there are elements a, b, c in Γ satisfying

(1) $g \mu 1 = aga^{-1}$, $1 \mu g = bgb^{-1}$ where 1 is the identity element in Γ,

(2) $g \mu \tau(g) = c\tau(g)c^{-1}$

for all $g \in \Phi$.

LEMMA 2 (Wagoner). *A flabby group is acyclic.*

Sketch of proof. Any inner automorphism of Γ acts trivially on homology. By (1), this implies first that μ induces a (non associative) ring structure $\mu_* : H_*(\Gamma) \otimes H_*(\Gamma) \to H_*(\Gamma)$ on homology, with two-sided unit the number 1 in $H_0(\Gamma) = \mathbf{Z}$. By (2), this implies also that $\mu(\mathrm{id} \times \tau)\Delta$ and τ act the same way on homology, where $\Delta : \Gamma \to \Gamma \times \Gamma$ is the diagonal map.

Let i be an integer, $i > 0$, and assume inductively that $H_n(\Gamma)$ is trivial for $0 < n < i$ (this holds trivially if $i = 1$). Choose $z \in H_i(\Gamma)$. By the Künneth formula

$$\Delta_*(z) = z \otimes 1 + 1 \otimes z \in H_i(\Gamma) \otimes H_0(\Gamma) + H_0(\Gamma) \otimes H_i(\Gamma) = H_i(\Gamma \times \Gamma)$$

so that

$$(\mu(\mathrm{id} \times \tau)\Delta)_*(z) = \mu_*(z \otimes 1 + 1 \otimes \tau_*(z)) = z + \tau_*(z) \in H_i(\Gamma).$$

As this must coincide with $\tau_*(z)$ one has $z = 0$. Hence $H_i(\Gamma)$ is trivial. ■

LEMMA 3. *The group G'_∞ is flabby.*

Proof. Let T_0^0 be a Hilbert space isomorphic to V. For any pair (j, k) of positive integers, let T_j^k be a copy of T_0^0. We identify V and $T = \bigoplus_k \bigoplus_j T_j^k$ in such a way that

$$S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$$

(where $\bigoplus_k$ means $\bigoplus_{k=0}^{\infty}$). For each $j \geq 0$ define an isometry ρ_j from $\bigoplus_k T_j^k$ onto T_j^0 and an isometry (shift) σ_j from $\bigoplus_k T_j^k$ onto $\bigoplus_{k=1}^{\infty} T_j^k$ with $\sigma_j(T_j^k) = T_j^{k+1}$ for all $k \geq 0$. Denote by ρ the isometry $\bigoplus_j \rho_j$ from T onto $\bigoplus_j T_j^0$ and by σ the shift $\bigoplus_j \sigma_j$. Define the maps

$$\perp\!\!\!\perp : \begin{cases} GL(T) \times GL(T) \to GL(T) \\ (g, h) \to \rho g \rho^* + \sigma h \sigma^* \end{cases}$$

and

$$\tau : \begin{cases} GL(T) \to GL(T) \\ g \to \sum_k \sigma^k \rho g \rho^* \sigma^{*k} \end{cases}$$

(The series converges strongly, and ρ^* is the adjoint of ρ; in view of section 4, it is appropriate to define ρ^* by $\rho^*(\xi) = \eta$ if $\eta = \rho(\xi) \in \operatorname{Im}(\rho)$ and $\rho^*(\xi) = 0$ if $\xi \perp \operatorname{Im}(\rho)$.)

It is easy to check that $\perp\!\!\!\perp$ and τ are homomorphism because ρ and σ are isometries with orthogonal complementary ranges. Similarly $\perp\!\!\!\perp(\mathrm{id} \times \tau)\Delta = \tau$. For each $i \geq 0$ one has $\perp\!\!\!\perp(G_i' \times G_i') \subset G_i'$ and $\tau(G_i') \subset G_i'$ because $\rho_j \rho_j^* + \sigma_j \sigma_j^*$ coincides with the identity on $\bigoplus_k T_j^k$ for $j \geq i$. It follows that $\perp\!\!\!\perp$ and τ induce homomorphisms $G_\infty' \times G_\infty' \to G_\infty'$ and $G_\infty' \to G_\infty'$, denoted below by $\perp\!\!\!\perp$ and ρ again. Requirement (2) in the definition of a flabby group obviously holds (with $c = 1$).

Consider some integer $i \geq 0$. Let a_i be an invertible isometry of T which acts as $\bigoplus_{j=0}^{i-1} \rho_j$ on $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$, as the identity on $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$, and (thus) maps in some way $\bigoplus_k T_i^k$ onto

$$\left(\bigoplus_{k=1}^{\infty} \bigoplus_{j=0}^{i-1} T_j^k\right) \oplus \left(\bigoplus_k T_i^k\right).$$

One has $a_i \in G_{i+1}' \subset G_\infty'$ and $a_i g a_i^* = g \perp\!\!\!\perp 1$ for all $g \in G_i'$. Similarly, let b_i be an invertible isometry of T which acts as $\bigoplus_{j=0}^{i-1} \sigma_j$ on $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ and as the identity on $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$. Then $b_i \in G_{i+1}'$ and $b_i g b_i^* = 1 \perp\!\!\!\perp g$ for all $g \in G_i'$. It follows that requirement (1) above holds. ■

We know thus that G_∞' is acyclic. The reader who is interested in $U(V)$ and not in $GL(V)$ may skip the end of this section since $G_\infty \cap U(V) = G_\infty' \cap U(V)$.

Let us now recall what we need from a result due to Quillen (theorem 1′ of [Q2]). Let A be a $\mathbf{Q}$-algebra with unit, let Γ be the group of invertible (2×2)-matrices over A which have the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, let Γ' be the subgroup of Γ consisting of diagonal matrices and let $\pi:\Gamma\to\Gamma'$ be the homomorphism defined by

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If R is a $\mathbf{Z}[\Gamma]$-module, we denote by $H_i(\Gamma, R)$ the i^{th} Eilenberg–MacLane homology group of Γ with coefficients in R; moreover R is assumed to have the trivial $\mathbf{Z}[\Gamma]$-structure if there is no strong reason for any other one (such as $R=H_t(N;\mathbf{K})$ below).

LEMMA 4 (Quillen). *Let $\mathbf{K}$ be a field which is either finite or the rationals. Then π induces an isomorphism on $H_*(-;\mathbf{K})$.*

Proof. Let N be the subgroup of Γ consisting of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, which is isomorphic to the additive group of the algebra A. As N is torsion-free and abelian, $H_*(N;\mathbf{Z})$ is isomorphic to the additive group $\bigwedge_{\mathbf{Z}} N$. (This holds for finitely generated free abelian groups, as one checks knowing homology of compact tori; this holds in general because N and the inductive limit of finitely generated subgroups of N have the same homology.) It follows that $H_*(N;\mathbf{K})\approx(\bigwedge_{\mathbf{Z}} N)\otimes_{\mathbf{Z}}\mathbf{K}$ for any field $\mathbf{K}$. In particular $H_*(N;\mathbf{K})=H_0(N;\mathbf{K})=\mathbf{K}$ if $\mathbf{K}$ is finite (because N is divisible) and $H_*(N;\mathbf{Q})=\bigwedge_{\mathbf{Q}} A$. (This is a highly degenerate form of the results described in §8 of [Q2].)

Consider the Hochschild–Serre spectral sequence

$$E^2_{s,t}=H_s(\Gamma';H_t(N;\mathbf{K}))\Rightarrow H_{s+t}(\Gamma;\mathbf{K})$$

corresponding to the extension

$$0\to N\to\Gamma\to\Gamma'\to 1.$$

If $\mathbf{K}$ is a finite field, one has $H_t(N;\mathbf{K})=0$ for $t>0$ and $H_0(N;\mathbf{K})=\mathbf{K}$. The spectral sequence therefore degenerates, giving the desired result.

Suppose $\mathbf{K}=\mathbf{Q}$. Make $\mathbf{Q}^*$ act on Γ by

$$\lambda\cdot\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}a & \lambda b\\ 0 & 1\end{pmatrix}.$$

Thus $\lambda\in\mathbf{Q}^*$ acts on the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\lambda"] & \Gamma \arrow[r] \arrow[d, "\lambda"] & \Gamma' \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 0 \arrow[r] & N \arrow[r] & \Gamma \arrow[r] & \Gamma' \arrow[r] & 1 \end{tikzcd}$$

and consequently also on the spectral sequence. As $\lambda\in\mathbf{Q}^*$ acts on $H_t(N;\mathbf{Q})=\bigwedge^t_{\mathbf{Q}}(N\otimes_{\mathbf{Z}}\mathbf{Q})$ by multiplying by λ^t, and acts trivially on Γ', it follows that λ acts on $E^2_{s,t}$ by multiplying by λ^t. Assume $\lambda\neq\pm1$; as the differentials commute with the $\mathbf{Q}^*$-action and as $\lambda^t\neq\lambda^{t'}$ for $t\neq t'$, all differentials are zero. It follows that

$$E^2_{s,t}=E^\infty_{s,t}\quad\text{for all}\quad s,t\geqslant 0.$$

Now $\bigoplus_{s+t=n}E^\infty_{s,t}$ is the graded object associated to the natural filtration of $H_n(\Gamma;\mathbf{Q})$ for each integer $n\geqslant 1$. Since $\mathbf{Q}^*$ acts on Γ by inner automorphisms, the induced action on $H_n(\Gamma;\mathbf{Q})$ is trivial; thus $\mathbf{Q}^*$ acts trivially on each $E^\infty_{s,t}$. Hence $E^\infty_{s,t}=0$ for any (s,t) with $s\geqslant 0$ and $t>0$. This shows that $H_s(\Gamma';\mathbf{Q})=H_s(\Gamma;\mathbf{Q})$ for any $s\geqslant 0$. ■

COROLLARY 5 (a universal coefficient argument). *The homomorphism $\pi:\Gamma\to\Gamma'$ induces an isomorphism on $H_*(-)=H_*(-;\mathbf{Z})$.*

Proof. We know that π induces an isomorphism for $H_*(-;R)$ if R is the additive group of a finite field. Using direct products and extensions of the coefficients, one checks the same holds for R a finite abelian group. As homology commutes with inductive limits of coefficients, this holds also when $R=\mathbf{Q}/\mathbf{Z}$. Using the sequence

$$0\to\mathbf{Z}\to\mathbf{Q}\to\mathbf{Q}/\mathbf{Z}\to 0$$

and the fact that π_* is an isomorphism for $R=\mathbf{Q}$ and $R=\mathbf{Q}/\mathbf{Z}$, one proves the claim. ■

The proof of Proposition 1. We use again the notations defined earlier in this section, and we denote by $L(V)$ the algebra of all bounded operators on V. For each $i>0$ the spaces $S_i^{\perp}$ and S_i are both isomorphic to V. It follows that G_i is isomorphic to

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \; a, b \in L(V) \text{ with } a \text{ invertible}\right\}$$

and that G_i' consists of matrices in G_i with $b=0$. Quillen's argument shows that the inclusion of G_i' in G_i induces an isomorphism $H_*(G_i') \approx H_*(G_i)$. It follows that the inclusion of G_∞' in G_∞ induces also an isomorphism $H_*(G') \approx H_*(G)$, so that the proof of proposition 1 is complete. ■

Let us end this section by two observations. First the groups of our main theorem are not flabby. Consider for example $G = U(V)$ with V an infinite dimensional separable complex Hilbert space, and suppose there exists a "direct sum" homomorphism $\mu : G \times G \to G$ with property (1) preceding lemma 2; we shall reach a contradiction.

Choose an orthonormal basis $(e_j)_{j\in N}$ of V and a sequence $(\lambda_j)_{j\in N}$ of pairwise distinct numbers in the interval $]-\pi, \pi[$. Define $r \in G$ by $r(e_j) = \exp(i\lambda_j)e_j$ for $j \in N$. The centralizer of r in G is the abelian group T of unitary operators which are diagonal with respect to the chosen basis.

Consider the homomorphism $\mu_1 : G \to G$ given by $g \mapsto \mu(g, 1)$. By hypothesis $\mu_1(g)$ is conjugate to g. Therefore, μ_1 is injective and, because its image commutes with $\mu(1, r)$, the centralizer of $\mu(1, r)$ is not abelian. But there exists $b \in G$ with $\mu(1, r) = brb^{-1}$. Therefore the centralizer of $\mu(1, r)$ is the abelian group bTb^{-1}. This contradiction shows that G is not flabby.

The second observation is that there are plenty of (non trivial) G-modules R with non trivial $H_*(G, R)$ or $H^*(G, R)$. Consider for example a subgroup G_1 of G and a G_1-module R_1. Let $R = \mathrm{Hom}_{\mathbf{Z}G_1}(\mathbf{Z}G, R_1)$, where $\mathbf{Z}G$ is considered as a left $\mathbf{Z}G_1$-module and as a right $\mathbf{Z}G$-module; then R is naturally a G-module (namely a left $\mathbf{Z}G$-module). A standard result known as Shapiro's lemma states that $H^n(G_1, R_1)$ is naturally isomorphic with $H^n(G, R)$ for all $n \geq 0$; see for example §34.2 in [Bab]. Choose in particular a finite cyclic subgroup G_1 of G and let R_1 be a trivial G_1-module isomorphic to G_1 as abelian group. Then $H^n(G, R) \neq 0$ for all $n > 0$.

This is quite a general construction. Indeed, let Γ be any group with more than one element. One shows by induction from a (possibly infinite) cyclic subgroup of Γ that there exists a Γ-module M and an integer $n > 0$ with $H^n(\Gamma; M) \neq 0$.

3. The set of flags

Let Gr be the set of those closed subspaces S of V which are isomorphic to V/S. (Thus Gr is the set of points in a Grassmannian space.)

LEMMA 6. *Let $\{S_1, \ldots, S_p\}$ be a finite subset of Gr. There exist $S'_1, \ldots S'_p \in Gr$ with $S'_m \subset S_m$ $(1 \leqslant m \leqslant p)$ and $S'_m \perp S'_n$ $(1 \leqslant m < n \leqslant p)$.*

Proof. Any subspace of V whose codimension is strictly smaller than the dimension of V intersects non trivially any element of Gr. One may thus choose unit vectors as follows

$$v_{1,1} \in S_1,\ v_{2,1} \in S_2 \cap \{v_{1,1}\}^\perp, \ldots, v_{p,1} \in S_p \cap \{v_{1,1}, \ldots, v_{p-1,1}\}^\perp$$

and in general

$$\begin{aligned} &v_{1,i} \in S_1 \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i-1}, \ldots, v_{p,i-1}\}^\perp, \\ &\ldots, \\ &v_{p,i} \in S_p \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i}, \ldots, v_{p-1,i}\}^\perp. \end{aligned}$$

(The index i runs over $\mathbf{N}^*$ if V is separable and over some suitable infinite set if V is "larger".) Define S'_m to be the closed linear span of the $v_{m,i}$'s. Then $S'_1, \ldots, S'_p$ have the desired properties. ■

LEMMA 7. *Let $S_1, \ldots, S_p \in Gr$ and let $h_1, \ldots, h_p \in GL(V)$. There exist $S'_1, \ldots, S'_p \in Gr$ with $S'_m \subset S_m$ $(1 \leqslant m \leqslant p)$, $S'_m \perp S'_n$ and $h_m(S'_m) \perp h_n(S'_n)$ $(1 \leqslant m < n \leqslant p)$.*

Proof. By Lemma 6 there exist $S''_1, \ldots, S''_p \in Gr$ with $S''_m \subset S_m$ $(1 \leqslant m \leqslant p)$ and $S''_m \perp S''_n$ $(1 \leqslant m < n \leqslant p)$. Define $T_m = h_m(S''_m)$ $(1 \leqslant m \leqslant p)$. There exist also $T'_1, \ldots, T'_p \in Gr$ with $T'_m \subset T_m$ $(1 \leqslant m \leqslant p)$ and $T'_m \perp T'_n$ $(1 \leqslant m < n \leqslant p)$. Define $S'_m = h_m^{-1}(T'_m)$ $(1 \leqslant m \leqslant p)$. ■

Now consider the set $\mathfrak{F}$ of flags $F = \{S_1 \supset S_2 \supset \cdots\}$ with $\bigcap S_i = \{0\}$ as defined in section 2. Let $F = \{S_1 \supset S_2 \supset \cdots\}$, $F' = \{S'_1 \supset S'_2 \supset \cdots\}$ and $h \in GL(V)$. We write $F' \leqslant F$ if $S'_i \subset S_i$ for all i. If $S'_i \perp S_i$, we write $F' \perp F$. If in addition $S_1 \oplus S'_1 \in Gr$, the spaces $S_1 \oplus S'_1 \supset S_2 \oplus S'_2 \supset \cdots$ form a flag which we call $F' \oplus F$. Finally the flag $\{h(S_1) \supset h(S_2) \supset \cdots\}$ is called $h(F)$.

We may reformulate lemma 7 for flags.

LEMMA 8. *Let* $F_1, \ldots, F_p \in \mathfrak{F}$ *and let* $h_1, \ldots, h_p \in GL(V)$. *There exist* $F'_1, \ldots, F'_p \in \mathfrak{F}$ *with* $F'_m \leqslant F_m$ $(1 \leqslant m \leqslant p)$, $F'_m \perp F'_n$ *and* $h_m(F'_m) \perp h_n(F'_n)$ $(1 \leqslant m < n \leqslant p)$.

Proof. Let $F_m = \{S_{m,1} \supset S_{m,2} \supset \cdots\}$ and write $T_{m,i} = S_{m,i}^{\perp} \cap S_{m,i-1}$ where $S_{m,0} = V$ $(1 \leqslant m \leqslant p$ and $i \geqslant 1)$. Then $S_{m,i} = \bigoplus_{j=i+1}^{\infty} T_{m,j}$. The result now follows by applying lemma 7 to the spaces $T_{1,j}, \ldots, T_{p,j}$ for each $j \geqslant 1$. ■

We review now the Milnor construction for classifying space (see e.g. [Hu], chap. 4, §11). Given any (discrete) group Γ, let $E\Gamma$ be the simplicial complex whose p-simplices are the ordered subsets $(\gamma_0, \ldots, \gamma_p)$ of Γ. We denote by $|E\Gamma|$ the topological space obtained by realizing $E\Gamma$. It is well-known and easy to see that $|E\Gamma|$ is contractible (compare the proof of lemma 10 below). Moreover the group Γ acts freely on $|E\Gamma|$ by multiplication on the left. Thus the quotient space $B\Gamma = \Gamma \backslash |E\Gamma|$ is a model (the "infinite join" model) for the classifying space of the group Γ. In particular this means that the groups $H_i(\Gamma)$ $(i \in \mathbf{N})$ are just the integral homology groups of the space $B\Gamma$.

For the rest of this section, we will write G for $GL(V)$, E for $EGL(V)$ and B for $BGL(V)$. For each flag $F = \{S_1 \supset S_2 \supset \cdots\}$ in $\mathfrak{F}$, let G_F be the subgroup of G containing those operators which agree with the identity on S_i for i large enough, and let E_F be the subcomplex of E defined as follows: a k-simplex $(g_0, \ldots, g_k)$ of E is in E_F if $g_0, \ldots, g_k$ agree on S_i for i large enough. (For short, we will say that $g_0, \ldots, g_k$ agree on F.) Let $F, F' \in \mathfrak{F}$. If $F' \leqslant F$, observe that $G_F \subset G_{F'}$ and that E_F is a subcomplex of $E_{F'}$. If $F \perp F'$ and if $F \oplus F' \in \mathfrak{F}$, then $G_{F \oplus F'} = G_F \cap G_{F'}$.

LEMMA 9. *For any* $F \in \mathfrak{F}$, *the complex* E_F *is* G-*invariant and the quotient* $G \backslash |E_F|$ *is naturally isomorphic to* BG_F.

Proof. "Naturally" means that, if $F, F' \in \mathfrak{F}$ with $F' \leqslant F$, then the map $BG_F \to BG_{F'}$ induced by $G_F \hookrightarrow G_{F'}$ is just the inclusion of BG_F in $BG_{F'}$ (both are subspaces of B).

The space $|E_F|$ is not connected. Indeed two 0-simplices (g) and (g') define points lying in the same connected component if and only if there is a sequence of 1-simplices in E_F of the form

$$(g, g_1), (g_1, g_2), \ldots, (g_m, g').$$

This holds if and only if g and g' agree on F, namely if and only if g and g' belong to the same right coset of G_F in G. It follows that connected components of $|E_F|$

are parametrized by G/G_F. The coset G_F corresponds to $|E'_F|$, where E'_F is the subcomplex of E_F consisting of simplices $(g_0, \ldots, g_k)$ where $g_0, \ldots, g_k$ agree with the identity on F.

It is clear that E_F is G-invariant. It follows from the discussion above that $G\backslash|E_F|$ may be identified with $G_F\backslash|E'_F|$, which is nothing but the infinite join model BG_F for the classifying space of G_F. ■

Let E_* be the union of the E_F's over $F \in \mathfrak{F}$; it is a subcomplex of E which is invariant by G. Let $B_* = G\backslash|E_*|$; it is a subspace of B which is the union of the $G\backslash|E_F|$'s over F in $\mathfrak{F}$.

LEMMA 10. *The space E_* is contractible.*

Proof. Let $\sigma_1, \ldots, \sigma_p$ be simplices in E_*. Choose

$$F_1 = \{S_{1,1} \supset S_{1,2} \supset \cdots\}, \ldots, F_p = \{S_{p,1} \supset S_{p,2} \supset \cdots\}$$

in $\mathfrak{F}$ with $\sigma_m \in E_{F_m}$. There is an integer k such that the vertices in σ_m agree on $S_{m,k}$; denote by h_m their common restriction on $S_{m,k}$ $(1 \leq m \leq p)$. Let $F'_1, \ldots, F'_p$ be as in lemma 8: one has $\sigma_m \in E_{F'_m} (1 \leq m \leq p)$. Then the cone on $\sigma_1 \cup \cdots \cup \sigma_p$ with vertex h_0 is in E_*.

It follows that, for any finite subcomplex K of E_*, there exists a subcomplex L of E_* containing K and contracting to a point. Hence $|E_*|$ itself is c ·tractible (see e.g. corollary 7.6.24 in [Sp]). ■

LEMMA 11. *The inclusion $B_* = \bigcup_{F \in \mathfrak{F}} BG_F \to B = BG$ is a homotopy equivalence.*

Proof. Since the quotient maps $|E| \to B$ and $|E_*| \to B_*$ are covering maps, this follows immediately from the two previous lemmas. ■

The following lemma holds for $p = 1$ by section 2.

LEMMA 12. *Let $F_1, \ldots, F_p \in \mathfrak{F}$. Then $BG_{F_1} \cup \cdots \cup BG_{F_p}$ is contained in an acyclic subspace of B_*.*

Proof. Choose any flag $F_0 \in \mathfrak{F}$. By Lemma 8 there exist $F'_0, F'_1, \ldots . F'_p \in \mathfrak{F}$ with $F'_m \leq F_m$ $(0 \leq m \leq p)$ and $F'_m \perp F'_n$ $(0 \leq m < n \leq p)$; in particular $F'_1 \oplus \cdots \oplus F'_p$ is a flag in $\mathfrak{F}$. As $BG_{F_m} \subset BG_{F'_m}$ $(1 \leq m \leq p)$, it suffices to check that $BG_{F'_1} \cup \cdots \cup BG_{F'_p}$ is acyclic. Hence we may assume without loss of generality that $F_m \perp F_n$ $(1 \leq m < n \leq p)$ and that $F_1 \oplus \cdots \oplus F_p \in \mathfrak{F}$.

Let us assume as induction hypothesis that, in this situation, both

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_1 \oplus F_{p-1}} \cup \cdots \cup BG_{F_{p-2} \oplus F_{p-1}}$$

are acyclic. (When $p = 2$, the former works by proposition 1 and the latter is vacuous.)

Consider first the Mayer–Vietoris homology sequence of the subcomplexes

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-2} \oplus F_p} \quad \text{and} \quad BG_{F_{p-1} \oplus F_p}$$

of B_* with intersection

$$BG_{F_1 \oplus (F_{p-1} \oplus F_p)} \cup \cdots \cup BG_{F_{p-2} \oplus (F_{p-1} \oplus F_p)}.$$

By the induction hypothesis, two of any three consecutive terms in this sequence vanish. Hence all terms vanish and

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}$$

is acyclic.

Consider now the Mayer–Vietoris sequence of the subcomplexes

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}} \quad \text{and} \quad BG_{F_p}$$

of B_* with intersection

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-1} \oplus F_p}.$$

From the previous step and from the induction hypothesis it follows that

$$BG_{F_1} \cup \cdots \cup BG_{F_p}$$

is acyclic. ■

THEOREM 13. *The group G is acyclic.*

Proof. The homology of a complex is generated by that of its finite subcomplexes. Thus lemma 12 implies that B_* is an acyclic space, and lemma 11 that G is acyclic. ■

4. Variations

Unitary group $U(V)$ of an infinite dimensional Hilbert space V.

The proof that $U(V)$ is acyclic is much simpler than for $GL(V)$ since section 2 may be reduced to Lemmas 2 and 3. Section 3 is unchanged.

Symmetric group $\Sigma(X)$ of an infinite set X

Here a flag is a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of subsets of $X = S_0$ such that $S_{i-1} - S_i$ is equipotent with X for each $i \geq 1$ and such that $\bigcap S_i = \emptyset$. Define

$$\Sigma_i = \{g \in \Sigma(X) \mid g = \mathrm{id} \text{ on } S_i\}$$

for each $i \geq 0$ (no distinction here between Σ_i' and Σ_i) and $\Sigma_\infty = \bigcup_{i=0}^{\infty} \Sigma_i$. The argument of Lemma 3 shows that Σ_∞ is a flabby group. Read "disjoint union" instead of "direct sum", "injection" instead of "isometry". The adjoint ρ^* of an injection ρ is defined only on the image of ρ by $\rho^*\rho = \mathrm{id}$; then a formula like $\rho g \rho^* + \sigma h \sigma^*$ is clear because $\rho g \rho^*$ is a permutation of some subset of X and $\sigma h \sigma^*$ is a permutation of its complement. The group Σ_∞ is consequently acyclic.

Let Gr be the set of those subsets S of X equipotent with their complements $S^\perp = X - S$. For two subsets S_1, S_2 of X, read $S_1 \cap S_2 = \emptyset$ for $S_1 \perp S_2$. Lemmas 7 and 8 may then be repeated without change and all of section 3 with minor changes only. It follows that $\Sigma(X)$ is acyclic.

Automorphism group $\mathcal{A}(\Omega)$ of a Lebesgue space $(\Omega, \mathcal{B}, \mu)$

Let $(\Omega, \mathcal{B}, \mu)$ be a Lebesgue space where the measure μ is infinite and non atomic. A flag is now a nested sequence $F = \{S_1 \supset S_2 \supset \cdots\}$ of measurable subsets of $\Omega = S_0$ such that $S_{i-1} - S_i$ has infinite measure for each $i \geq 1$ and such that $\bigcap S_i$ has measure zero. Comments for $\sum(X)$ above apply to $\mathcal{A}(\Omega)$, with the understanding that everything in view is now measurable. Therefore $\mathcal{A}(\Omega)$ is also acyclic.

Let $(\tilde{\Omega}, \mathcal{B}, \mu)$ be a Lebesgue measure space. Let X be the set of atoms in $\tilde{\Omega}$, let $X = \coprod_j X_j$ be the partition of X according to the masses of the atoms, and let $\Omega = \tilde{\Omega} - X$. Then the sequence

$$1 \to \mathcal{A}(\Omega) \to \mathcal{A}(\tilde{\Omega}) \to \prod_j \sum (X_j) \to 1$$

is exact (and splits). Suppose $\mu(\Omega) = \infty$, and suppose that X is not empty. Then $\mathcal{A}(\tilde{\Omega})$ is clearly acyclic if and only if each X_j is either one point or an infinite set.

Automorphisms of an infinite dimensional vector space W over a (possibly skew) field $\mathbf{F}$

Case (i): Char $\mathbf{F}=0$.

A flag is in this case a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of subspaces of $W=S_0$ such that S_{i-1}/S_i is isomorphic to W for each $i \geqslant 1$ and such that $\bigcap S_i=\{0\}$. As in Lemma 3 we may identify W with $\bigoplus_k \bigoplus_j T_j^k$, where each $T_j^k \cong W$, in such a way that $S_i=\bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$ for all i. Then the subspace $R_i=\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ complements S_i.

Define

$$
\begin{aligned}
G_i^W &= \{g \in GL(W) \mid g=\text{id on } S_i\},\\
G_i^{W'} &= \{g \in G_i^W \mid g(R_i)=R_i\}.
\end{aligned}
$$

One checks as in Lemma 3 that $G_\infty^{W'}$ is flabby. When Char $\mathbf{F}=0$, Lemma 4 and 5 show that G_∞^W is acyclic.

In Lemmas 6 to 8, understand $S'_m \perp S'_n$ as $S'_m \cap S'_n=\{0\}$, and $v \in S \cap \{v_1, \ldots, v_m\}^\perp$ as $v \in S$ with v not in the linear span of $\{v_1, \ldots, v_m\}$. Then section 3 holds for $GL(W)$, which is consequently an acyclic group. All our arguments allow the field $\mathbf{F}$ to be non-commutative.

Case (ii): Char $\mathbf{F}=p>0$.

The arguments of section 2 show that $\tilde{H}_*(G_\infty^W; \mathbf{K})=0$ if Char $\mathbf{K} \neq$ Char $\mathbf{F}$ (where $\tilde{H}_*$ denotes reduced homology). It follows that $\tilde{H}_*(GL(W); \mathbf{K})=0$ when Char $\mathbf{K} \neq$ Char $\mathbf{F}$. Therefore, in order to show that $GL(W)$ is acyclic, it will suffice to prove that $\tilde{H}_*(GL(W); \mathbf{K})=0$ when $\mathbf{K}$ is the algebraic closure $\bar{\mathbf{k}}$ of the finite field $\mathbf{k}$ with p elements. To do this we need

LEMMA 14. *For each flag F and integer $d>0$ there is a subgroup G_F^d of $GL(W)$ which contains G_F and is such that $H_j(G_F^d; \bar{\mathbf{k}})=0$ for $0<j<d$.*

Proof. Quillen proves the following lemma in [Q2] §9.

LEMMA. *Let $\bar{\mathbf{k}}$ be an algebraically closed field and d an integer >0. Then there exists an order D in a number field of degree d over $\mathbf{Q}$ with the following properties: Given any D-module N, let the group of units D^* act on it by multiplication, and let the group homology $H_*(N, \bar{\mathbf{k}})$ be endowed with the induced action of D^*. Then for each t, $H_t(N, \bar{\mathbf{k}})$ is a direct sum of one-dimensional representations of D^* over $\bar{\mathbf{k}}$. Furthermore, $H_t(N, \bar{\mathbf{k}})$ does not contain the trivial representation for $0<t<d$.*

Let D be as in this lemma. The choice of a basis over $\mathbf{Z}$ for D gives rise to a ring homomorphism

$$\rho_0 : D \to M_d(\mathbf{Z}) \to M_d(\mathbf{F})$$

where $M_d(A)$ is the ring of d-by-d matrices over A and where $M_d(\mathbf{Z}) \to M_d(\mathbf{F})$ is reduction mod p. Let F be the flag $\{S_1 \supset S_2 \supset \cdots\}$. For each pair (j, k) of positive integers, let now T_j^k be a copy of $\mathbf{F}^d$. We identify W and $T = \bigoplus_k \bigoplus_j T_j^k$ in such a way that $S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$, and we denote by R_i "the" complement $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ of S_i. Define a ring homomorphism $\rho_i : D \to GL(W)$ by setting

$$\rho_i(\lambda) = \begin{cases} \rho_0(\lambda) & \text{in } T_j^k \text{ for } j \geqslant i, \text{ all } k \\ \mathrm{id} & \text{in the other } T_j^k. \end{cases}$$

Now put

$$G_i^d = \{g \in GL(W) \mid g = \rho_i(\lambda) \text{ in } S_i \text{ for some } \lambda \in D^*\}$$

and let $G_F^d = \bigcup_{i \geqslant 1} G_i^d$. Clearly $G_F \subset G_F^d$. We must show that $H_j(G_F^d; \bar{\mathbf{k}}) = 0$ for $0 < j < d$.

Let

$$G_i^{d'} = \{g \in G_i^d \mid g(R_i) = R_i\}.$$

and consider the induced D^*-action on the spectral sequence of the extension $0 \to N \to G_i^d \to G_i^{d'} \to 1$. It follows from the lemma that each E_{st}^r, $2 \leqslant r \leqslant \infty$, breaks up into a sum of one dimensional representations preserved by the differentials. Since D^* acts trivially on the abutment, the subspaces on which D^* acts trivially form a spectral sequence which converges to $H_*(G_i^d; \bar{\mathbf{k}})$. By the lemma, the terms E_{st}^2 of this sequence vanish when $0 < t < d$. Hence $H_j(G_i^d; \bar{\mathbf{k}}) \cong H_j(G_i^{d'}; \bar{\mathbf{k}})$ for $0 < j < d$.

Now note that $G_i^{d'}$ is the product of G_i' with $\rho_i(D^*)$. But $\rho_i(D^*)$ is isomorphic to a subgroup of the group of units of $D/pD \cong \mathbf{k}_d$, where $\mathbf{k}_d$ is the field of order p^d. Hence $\rho_i(D^*)$ has order prime to p. Therefore $\tilde{H}_*(\rho_i(D^*); \bar{\mathbf{k}}) = 0$ which implies that $H_*(G_i^{d'}; \bar{\mathbf{k}}) \cong H_*(G_i'; \bar{\mathbf{k}})$. Now consider the diagram

$$\begin{tikzcd} G_i^d \arrow[r, "\alpha_3"] & G_{i+1}^d \\ G_i^{d'} \arrow[u, hook, "\alpha_2"] & \\ G_i' \arrow[u, hook, "\alpha_1"] \arrow[uur, "\alpha_4"'] & \end{tikzcd}$$

We have seen that the inclusions α_1 and α_2 induce an isomorphism on $H_j(-;\bar{\mathbf{k}})$, $0<j<d$. Since α_4 factors through a group isomorphic to G'_∞, it induces the zero map on $\tilde{H}_j(-;\bar{\mathbf{k}})$. Hence α_3 must induce the zero map on $H_j(-;\bar{\mathbf{k}})$, $0<j<d$. This implies that

$$H_j(G^d_F;\bar{\mathbf{k}}) = \lim_i H_j(G^d_i;\bar{\mathbf{k}}) = 0, \qquad 0<j<d. \quad \blacksquare$$

To finish the proof of the theorem we must find an appropriate substitute for Lemma 12. If $F_1, \ldots, F_n$ are disjoint flags such that $F_1 \oplus \cdots \oplus F_n$ is also a flag, choose groups $G^d_{F_i}$ as above and, for each subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, set

$$G^d_{F_{i_1}\oplus\cdots\oplus F_{i_k}} = G^d_{F_{i_1}} \cap \cdots \cap G^d_{F_{i_k}}.$$

The proof of Lemma 14 shows that these groups G^d_F, for $F = F_{i_1} \oplus \cdots \oplus F_{i_k}$, are acyclic. The inductive argument of Lemma 12 then readily shows that

$$H_j(BG^d_{F_1} \cup \cdots \cup BG^d_{F_n};\bar{\mathbf{k}}) = 0 \qquad 0<j<d-2n.$$

Clearly, this suffices to show that the inclusion $B_* \hookrightarrow B$ annihilates $\tilde{H}_*(-;\bar{\mathbf{k}})$.

Properly infinite von Neumann algebras

Let M be a properly infinite von Neumann algebra, faithfully represented in $L(V)$ for some complex Hilbert space V. A flag is a nested sequence $\{S_1 \supset S_2 \supset \cdots\}$ of closed subspaces of $V = S_0$ with $\bigcap S_i = \{0\}$ such that the orthogonal projection P_i from V onto S_i is in M and such that $P_{i-1} - P_i$ is equivalent to the identity for each $i \geq 1$. It is easy to choose every operator appearing in sections 2 and 3 in the algebra M. Therefore the appropriately defined groups G'_∞ and G_∞ are acyclic, as well as $U(M)$ and $GL(M)$.

It is likely that the argument applies to a large class of infinite C^*-algebras. Let B be such an algebra, let $M(B)$ be its multiplier algebra, let $U(B)$ be the subgroup of the unitary group $U(M(B))$ consisting of those elements g for which $g-1 \in B$, and let $U(B)_0$ be the connected component of $U(B)$ with respect to the norm topology. There are many cases in which $U(B)_0$ is known to be contractible for the norm topology [Mi]; in these cases, $U(B)_0$ and the similarly defined "general linear group" $GL(B)_0$ should "often" be acyclic.

Finite von Neumann algebras

Let M be a finite continuous factor, and let $U(M)$ be the group of unitaries in M. When given the norm topology, $U(M)$ has a fundamental group isomorphic to the additive group of the real numbers: this was first proved in [AS], but it follows also essentially from Bott periodicity as formulated in theorem 1.11 of chapter III of [Ka]. Indeed

$$\pi_i(U(M)_{\mathrm{norm}}) \approx \begin{cases} \mathbf{R} \text{ if } i \text{ is odd, } i \geqslant 0 \\ 0 \text{ if } i \text{ is even, } i > 0 \end{cases}$$

(See III.7.7 in [Ka], or theorem 5 in [Br]; both state the analogous "stable fact", but the isomorphism holds also as above.) Let

$$0 \to \mathbf{R} \to \tilde{U}(M) \to U(M) \to 1$$

be the (topological) universal covering of $U(M)$. It is known that $U(M)$ is perfect (indeed simple up to centre [FH]). One may conjecture that $\tilde{U}(M)$ is also perfect, namely that the short exact sequence above is still a covering in the algebraic sense of [Ker], and thus that there exists a surjective homomorphism of $H_2(U(M))$ onto $\mathbf{R}$. In any event it seems very unlikely that the group $U(M)$ is acyclic.

Appendix 1. About normal subgroups

If X is an infinite countable set, $\Sigma(X)$ has exactly two non trivial normal subgroups: the group $\Sigma_f(X)$ of permutations of X with finite support and its derived group $A_f(X)$ of even permutations [SU]. If X is any infinite set, normal subgroups of $\Sigma(X)$ which are neither trivial nor $A_f(X)$ are in bijection (via supports) with infinite cardinals smaller than the cardinal of X [B].

If $(\Omega, \mathcal{B}, \mu)$ is a Lebesgue measure space with μ infinite and non atomic, $\mathcal{A}(\Omega)$ has exactly one non trivial normal subgroup consisting of those bi-measurable transformations α with support $\{\omega \in \Omega \mid \alpha(\omega) \neq \omega\}$ of finite measure [F1], [Ei].

If W is an infinite dimensional vector space over a field $\mathbf{F}$, normal subgroups of $GL(W)$ have been studied in [R]; we present hereafter part of these results with different proofs inspired by [And], [Ep] and [Hi].

LEMMA A1. *The group $GL(W)$ is perfect.*

Proof. If I is a set and if $(W_i)_{i \in I}$ is a family of copies of W, we write any element in $GL(\bigoplus W_i)$ as an $(I \times I)$-matrix with coefficients in $\mathrm{End}(W)$. If I is countable, we may identify $\bigoplus W_i$ and W.

In $GL(W\oplus W\oplus W)$ one has

$$\begin{pmatrix}1&x&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&-x&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-1\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&x\\0&1&0\\0&0&1\end{pmatrix}$$

for each $x\in\operatorname{End}(W)$. It follows that any element of the form $\begin{pmatrix}1&*\\0&1\end{pmatrix}$ in $GL(W\oplus W)$ is a product of two commutators. In $GL(\bigoplus_{i\in N}W_i)$, one may apply the infinite repetition argument used in section 2. We write $\gamma_1\sim\gamma_2$ if two elements γ_1,γ_2 in a group Γ are conjugate. For any $x\in GL(W)$ one has

$$\begin{pmatrix}x&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{pmatrix}\begin{pmatrix}1&&&&\\&x&&&\\&&1&&\\&&&x&\\&&&&\ddots\end{pmatrix}\sim\begin{pmatrix}1&&&&\\&x&&&\\&&1&&\\&&&x&\\&&&&\ddots\end{pmatrix}$$

in $GL(\bigoplus_{i\in N}W_i)$. It follows that any element of the form $\begin{pmatrix}*&0\\0&1\end{pmatrix}$ in $GL(W\oplus W)$ is a commutator.

Let $g\in GL(W)$. Choose sequences (u_i) and (v_i) of vectors in W as follows:

$$u_1\in W-\{0\}\qquad u_1'=g(u_1)\qquad v_1\in W-\operatorname{span}(u_1,u_1')$$

and in general

$$u_{i+1}\in W-\operatorname{span}\begin{pmatrix}u_1&v_1&g^{-1}(v_1)\\ \cdot&\cdot&\cdot\\ u_i&v_i&g^{-1}(v_i)\end{pmatrix}\qquad u_{i+1}'=g(u_{i+1})$$

$$v_{i+1}\in W-\operatorname{span}\begin{pmatrix}u_1&u_1'&v_1\\ \cdot&\cdot&\cdot\\ u_i&u_i'&v_i\\ u_{i+1}&&u_{i+1}'\end{pmatrix}.$$

(The index i runs over N^* if the dimension of W is countable and over some suitable set otherwise.) Define

$$U = \operatorname{span}(u_1, u_2, \ldots) \qquad V_1 = \operatorname{span}(v_1, v_3, \ldots)$$
$$V_2 = \operatorname{span}(v_2, v_4, \ldots) \qquad V = V_1 \oplus V_2.$$

It is easy to check that $U \cap V = \{0\}$ and $g(U) \cap V = \{0\}$. Thus there exists $t \in GL(W)$ with $tu_i' = u_i$ and $tv_{2i} = v_{2i}$ for each i. As $t = \mathrm{id}$ on V_2 one has $t \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$; as $tg = \mathrm{id}$ on U one has $tg \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$. It follows from the beginning of the proof that g is a product of commutators in $GL(W)$. ■

The proof above shows also the following *fragmentation lemma*: any element in $GL(W)$ may be written as a product of finitely many elements similar to $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ in $GL(W \oplus W)$. Indeed, it remains to be checked that $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ has this property, and this is clear if one looks at

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $GL(W \oplus W \oplus W)$.

Let $N_{\max}$ be the normal subgroup of $GL(W)$ containing those elements of the form $\lambda + X$ with λ a homothety and X an endomorphism of W with rank strictly smaller than the dimension of W. Let $g \in GL(W)$ with $g \notin N_{\max}$. Let us check that there exists a subspace V of W with V isomorphic to W/V and with $V \cap g(V) = \{0\}$.

One may choose a sequence (v_i) of vectors in W as follows:

$$v_1 \in W - \{0\} \quad \text{with} \quad g(v_1) \in W - \operatorname{span}(v_1)$$

and in general

$$v_{i+1} \in W - \operatorname{span}\begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \end{pmatrix} \quad \text{with} \quad g(v_{i+1}) \in W - \operatorname{span}\begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \\ v_{i+1} & \end{pmatrix}$$

Indeed, suppose one cannot find v_{i+1}. Let

$$F=\operatorname{span}\begin{pmatrix} v_1 \cdots v_i \\ g(v_1)\cdots g(v_i)\end{pmatrix}.$$

Then $v\in W-F$ implies $g(v)\in\operatorname{span}(F,v)$; for any $u\in F$, one has also $g(v+u)\in\operatorname{span}(F,v)$; hence $g(u)\in\operatorname{span}(F,v)$. It follows that F is invariant by g and that g induces a homothety on W/F. But this is ruled out by hypothesis.

Then $V=\operatorname{span}(v_1,v_2,\ldots)$ has the desired properties.

PROPOSITION A2. *Any non trivial normal subgroup of $GL(W)$ is contained in $N_{\max}$.*

Proof. Let N be a normal subgroup of $GL(W)$ and assume that $N\not\subset N_{\max}$. There exist $f\in N$ and a subspace V of W with V isomorphic to W/V and with $f(V)\cap V=\{0\}$. We may thus view N as a normal subgroup of $GL(W\oplus W)$ containing an element f of the form $\begin{pmatrix} * & * \\ * & 0\end{pmatrix}$.

By the fragmentation lemma, it is enough to check that N contains any element of the form $\begin{pmatrix} 1 & 0 \\ 0 & *\end{pmatrix}$. Consider $r,s\in GL(W)$ and define $g=\begin{pmatrix} 1 & 0 \\ 0 & r\end{pmatrix}$, $h=\begin{pmatrix} 1 & 0 \\ 0 & s\end{pmatrix}$. As N is normal, N contains $\hat{h}=hfh^{-1}f^{-1}$ and $g\hat{h}g^{-1}\hat{h}^{-1}$. By a straightforward matrix computation, the latter is of the form

$$g\hat{h}g^{-1}\hat{h}^{-1}=\begin{pmatrix} 1 & * \\ 0 & rsr^{-1}s^{-1}\end{pmatrix}.$$

As $GL(W)$ is perfect, it follows that, for any $k\in GL(W)$, there exists $z\in\operatorname{End}(W)$ with $\begin{pmatrix} 1 & z \\ 0 & k\end{pmatrix}\in N$.

Let now $a,b\in GL(W)$ with $a+b=1$. (One may define a as an infinite direct sum of automorphisms of a vector space of dimension two, each represented by $\begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}$, and similarly for b with $\begin{pmatrix} 0 & -1 \\ -1 & 1\end{pmatrix}$.) There exist $x,y\in\operatorname{End}(W)$ with

$$\begin{pmatrix} 1 & x \\ 0 & a^{-1}\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 1 & y \\ 0 & b^{-1}\end{pmatrix}$$

in N. Then

$$\begin{pmatrix} 1 & -z \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -xa \\ 0 & a\end{pmatrix}=\begin{pmatrix} 1 & z(a-1) \\ 0 & 1\end{pmatrix}\in N$$

and

$$\begin{pmatrix}1 & z(a-1)\\0 & 1\end{pmatrix}\begin{pmatrix}1 & z(b-1)\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & -z\\0 & 1\end{pmatrix}\in N.$$

It follows that

$$\begin{pmatrix}1 & z\\0 & k\end{pmatrix}\begin{pmatrix}1 & -z\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & k\end{pmatrix}\in N$$

the proof is complete. ■

It would be easy to prove by similar arguments all of theorem B (and thus also theorem A) in [R].

Let now V be an infinite dimensional Hilbert space over the reals, complexes or quaternions and $GL(V)$ be as in the introduction. Let $GE(V, C)$ be the normal subgroup of $GL(V)$ containing those elements of the form $\lambda + x$ with λ a homothety and X a compact operator (we assume V to be separable). It is quite easy to check that $GL(V)$ is perfect (see problems 191 and 192 in [Hal]). There is a fragmentation lemma which follows straightforwardly from polar decomposition and spectral theorem. Any $g \in GL(V)$ with $g \notin GE(V, C)$ is similar to an element of the form $\begin{pmatrix}* & *\\ * & 0\end{pmatrix}$ in $GL(V \oplus V)$: this is corollary 3.4 in [BP] or theorem 1 in [AnS]. Hence the proof above applies, and is very much simpler than that of [H1]. The subgroup of $GL(V)$ containing all bijective isometries of V can be handled either as in [H1] or as suggested in [H3], and we have proved the following result.

PROPOSITION A3. *Any non trivial normal subgroup of $GL(V)$ is contained in $GE(V, C)$. Any non trivial normal subgroup of $U(V)$ is contained in $UE(V, C) = U(V) \cap GE(V, C)$.*

For normal subgroups of $GL(M)$ and $U(M)$, when M is a properly infinite von Neumann algebra, see [H3] and papers reviewed there.

COROLLARY A4. *Let G be one of the groups described in the introduction and let N be a non trivial normal subgroup of G. Then N is of uncountable index in G.*

Let G be as above and let $N_{\max}$ be the maximal normal subgroup of G. There are cases for which we have information about the homology of $N_{\max}$: see works by Nakaoka and Priddy [P] if $G = \sum(X)$ and $N_{max} = \sum_f(X)$ with X infinite countable, the papers on group cohomology in [E] if $G = GL(W)$, or [BHS] if $G = GL(V)$. In each case our main theorem provides corresponding information about the homology of the quotient G/N_{max}.

Appendix 2. About monoids of monomorphisms

Each of the acyclic groups of automorphisms considered above is the group of units in a corresponding monoid (or semigroup) of monomorphisms. For example, $\sum(X)$ is the group of units in the monoid $M(X)$ formed by all injective maps from X to X. One can form the classifying space BM of a monoid in exactly the same way as that of a group; see [Se]. In particular, the Eilenberg–MacLane homology groups $H_i(M; \mathbf{Z})$ are just the integral homology groups of the space BM. Quillen pointed out in an unpublished version of [Q1] that the classifying spaces of monoids such as $M(X)$ are contractible. Of course, this implies that the monoids are acyclic.

Here is a sketch of his argument. Say two homomorphisms $f, g: M \to M$ are semi-conjugate if there is $m \in M$ such that $mf(n) = g(n)m$ for all $n \in M$. The argument is based on the fact that two homomorphisms which are semi-conjugate induce homotopic maps on BM; see [Q1] §1. Choose $p \in M(X)$ so that the image $p(X)$ of X under p is in Gr. Define $f: M(X) \to M(X)$ by $f(n)(x) = pnp^{-1}(x)$ if $x \in p(X)$ and by $f(n)(x) = x$ otherwise. Then f is semi-conjugate both to the identity homomorphism and to the trivial homomorphism which takes every $n \in M(X)$ to the identity element. It follows that $BM(X)$ is contractible.

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On the homology of Lie groups made discrete

J. MILNOR

§1. Introduction

Let G be an arbitrary Lie group and let G^δ denote the same group with the discrete topology. Then the natural homomorphism $G^\delta \to G$ gives rise to a continuous mapping $\eta: BG^\delta \to BG$ between classifying spaces. This paper is organized around the following conjecture which was suggested to the author by E. Friedlander, at least in the complex case. (Compare Quillen, p. 176.)

ISOMORPHISM CONJECTURE. *This canonical mapping $BG^\delta \to BG$ induces isomorphisms of homology and cohomology with* mod p *coefficients, or more generally with any finite coefficient group.*

Here the homology of BG^δ is just the usual Eilenberg–MacLane homology of the uncountably infinite discrete group G^δ. These homology groups are of interest in algebraic K-theory (see for example Quillen), in the study of bundles with flat connection (Milnor, 1958), in the theory of foliations (Haefliger, 1973), and also in the study of scissors congruence of polyhedra (Dupont and Sah). They are difficult to compute, and tend to be rather wild. For example if G is non-trivial and connected, then Sah and Wagoner show that $H_2(BG^\delta; \mathbf{Z})$ maps *onto* an uncountable rational vector space. (See also Harris.) The homology and cohomology groups of BG, on the other hand, are much better behaved and better understood. (Borel, 1953.)

In §2 we will see that this Isomorphism Conjecture is true whenever the component of the identity in G is solvable. If it is true for simply-connected simple groups, then it is true for all Lie groups. It is always true for 1-dimensional homology, and is true in a number of interesting special cases for 2-dimensional homology. (See §4.) For higher dimensional computations which tend to support the conjecture, see Karoubi, p. 256, Parry and Sah, as well as Thomason.

Another partial result is the following (§3). *If G has only finitely many components, then for any finite coefficient group A the homomorphism $H_*(BG^\delta; A) \to H_*(BG; A)$ is split surjective.* Thus we obtain a direct sum decomposition

$$H_i(BG^\delta; A) \cong H_i(BG; A) \oplus (\text{unknown group}),$$

where the unknown summand is of course conjectured to be zero. The proof is based on Becker and Gottlieb, and generalizes a theorem of Bott and Heitsch. *As an immediate corollary, it follows that the integral cohomology* $H^*(BG;\mathbf{Z})$ *injects into* $H^*(BG^\delta;\mathbf{Z})$.

An appendix discusses the analogous homomorphisms with rational coefficients, which behave very differently. For example *the homomorphism* $H_i(BG^\delta;\mathbf{Q})\to H_i(BG;\mathbf{Q})$ *is identically zero for* $i>0$ *whenever* G *is compact, or complex and semi-simple with finitely many components.* More generally, even when these homomorphisms are not identically zero, it is often possible to describe the precise kernel of the associated ring homomorphism $H^*(BG;\mathbf{Q})\to H^*(BG^\delta;\mathbf{Q})$.

The methods used in this note are all more or less well known. I am particularly grateful to J. F. Adams, E. Friedlander, A. Haefliger, and D. McDuff for pointing out some of the necessary tools to me, to A. Borel for pointing out an error in an earlier version, and to the Institut des Hautes Etudes Scientifiques for its hospitality.

§2. The solvable case

First some general definitions. We will always use singular homology theory with constant (i.e., untwisted) coefficients.

For any topological group G, *let* $\bar{G}$ *be the homotopy fiber of the map* $G^\delta\to G$. (Compare Thurston.) Thus $\bar{G}$ is the topological group consisting of all pairs (g, f) where g is a point of G^δ and f is a path from the identity element to the image of g in G. We will be particularly interested in the classifying space $B\bar{G}$. Mather calls the homology of $B\bar{G}$ the "local homology" of the topological group G, since it is completely determined by the germ of the group G at the identity element. (See also Haefliger 1978, which uses the notation $B\mathfrak{g}$ for our space $B\bar{G}$, and McDuff 1980, which uses the notation $\bar{B}G$.) If G is locally contractible, so that the identity component G_0 has a universal covering group U, note that the natural homomorphisms $U\to G_0\to G$ induce isomorphisms $\bar{U}\to\bar{G}_0\to\bar{G}$. *Hence the homology groups of* $B\bar{G}$ *depend only on the universal covering group of* G. In the case of a Lie group, it follows that they depend only on the Lie algebra of G.

LEMMA 1. *The Isomorphism Conjecture of* §1 *is true for a connected Lie group* G *if and only if the associated space* $B\bar{G}$ *has the* mod p *homology of a point, for every prime* p. *If it is true for a connected group* G, *then it is true for any Lie group* H, *connected or not, which is locally isomorphic to* G.

Proof. This follows easily from the mod p homology spectral sequences associated with the fibrations $B\bar{G}\to BG^\delta\to BG$ and $B\bar{G}\to BH^\delta\to BH$. (Note that

BG is simply-connected.) The passage from mod p coefficients to arbitrary finite coefficients can be carried out by induction on the order of the abelian coefficient group A, making use of the homology exact sequence associated with a coefficient sequence $A' \to A \to A/A'$, where A' is some non-trivial proper subgroup of A. Details will be omitted. ■

LEMMA 2. *If a discrete abelian group Γ is uniquely divisible, then its classifying space $B\Gamma$ has the* mod p *homology of a point.*

Proof. A "uniquely divisible" group is just one which is isomorphic to a vector space over the rational numbers $\mathbf{Q}$. First suppose that this vector space is 1-dimensional. Then Γ is a direct limit of free cyclic groups, hence its homology is trivial in all dimensions greater than one; and evidently the group

$$H_1(B\Gamma; \mathbf{Z}/p\mathbf{Z}) \cong H_1(B\Gamma; \mathbf{Z}) \otimes \mathbf{Z}/p\mathbf{Z} \cong \Gamma \otimes \mathbf{Z}/p\mathbf{Z}$$

is also zero. Next suppose that Γ is finite dimensional over $\mathbf{Q}$. Then the conclusion follows inductively, using the Künneth Theorem. Finally, the infinite dimensional case follows by a straightforward direct limit argument. ■

Combining these two results, we obtain the following.

LEMMA 3. *If the component of the identity of G is solvable, then the Isomorphism Conjecture is true for G.*

Proof by induction on the dimension. By Lemma 1 it suffices to consider the case of a simply-connected solvable group. In the 1-dimensional case, $G \cong \mathbf{R}$, the conclusion follows immediately, since $B\mathbf{R}$ is contractible, and $B\mathbf{R}^\delta$ has the mod p homology of a point by Lemma 2. In the case of a higher dimensional simply-connected solvable group, choose a homomorphism from G onto $\mathbf{R}$ with kernel N. Then the short exact sequence $N \to G \to \mathbf{R}$ gives rise to a Serre fibration $B\bar{N} \to B\bar{G} \to B\bar{\mathbf{R}}$. We may assume inductively that $B\bar{N}$ has the mod p homology of a point, and a spectral sequence computation shows that $B\bar{G}$ does also. ■

More generally, for any Lie group G, the associated Lie algebra $\mathfrak{g}$ has a maximal solvable ideal $\mathfrak{n}$, and the quotient $\mathfrak{g}/\mathfrak{n}$ splits as a direct product of simple Lie algebras $\mathfrak{s}_i$. Let S_i be corresponding simple Lie groups.

LEMMA 4. *If the Isomorphism Conjecture is true for each simple Lie group S_i, then it is true for G.*

The proof, based on the fibration $B\bar{N} \to B\bar{G} \to \prod B\bar{S}_i$, is easily supplied. ■

§3. The Gottlieb transfer

Let $\pi: E \to B$ be the projection map of a smooth fiber bundle, with a closed manifold as fiber. The *Gottlieb transfer* $\mathrm{tr}: H^iE \to H^iB$ can be defined intuitively as the cup product with the Euler characteristic along the fiber, followed by integration along the fiber. (For a precise definition see Gottlieb.) *Here, and throughout most of this section, some fixed coefficient group A is to be understood.* There is a completely analogous transfer homomorphism in homology. One basic property is that the composition

$$H_iB \xrightarrow{\mathrm{tr}} H_iE \xrightarrow{\pi_*} H_iB$$

is equal to multiplication by the Euler characteristic of the fiber.

Let G be any Lie group with finitely many components, and let K be a maximal compact subgroup. According to Mostow, the quotient space G/K is contractible, hence the natural map $BK \to BG$ is a homotopy equivalence. Let N be the normalizer of a maximal torus in K. According to Hopf and Samelson, the quotient manifold K/N has Euler characteristic $+1$. Note that there is a canonical fibration $\pi: BN \to BK$ with fiber K/N. Following Becker and Gottlieb, this implies the existence of a transfer homomorphism $\mathrm{tr}: H_iBK \to H_iBN$ such that the composition $H_iBK \to H_iBN \to H_iBK$ is just the identity map of H_iBK. *Therefore the natural homomorphism $\pi_*: H_iBN \to H_iBK$ is a split surjection.* A similar argument shows that the corresponding cohomology homomorphism $\pi^*: H^iBK \to H^iBN$ is a split injection.

Now let us assume that the coefficient group A is finite. Then $H_*BN^\delta \cong H_*BN$ by §2. We continue to assume that G has only finitely many components.

THEOREM 1. *The canonical homomorphism $\eta_*: H_iBG^\delta \to H_iBG$ is a split surjection. That is some direct summand of H_iBG^δ maps isomorphically onto H_iBG. Similarly, the cohomology homomorphism $\eta^*: H^iBG \to H^iBG^\delta$ is a split injection.*

Proof. This follows by inspection of the commutative diagram

$$\begin{tikzcd} H_iBN^\delta \arrow[rr] \arrow[d, "\cong"] & & H_iBG^\delta \arrow[d, "\eta_*"] \\ H_iBN \arrow[r, "\pi_*"] & H_iBK \arrow[r, "\cong"] & H_iBG, \end{tikzcd}$$

or the analogous cohomology diagram. ■

COROLLARY 1. *The homomorphism* $\eta^*: H^i(BG;\mathbf{Z}) \to H^i(BG^\delta;\mathbf{Z})$ *of integral cohomology is injective.*

Proof. This follows from the commutative diagram

$$\begin{tikzcd} H^i(BG;\mathbf{Z}) \arrow[r, "n"] & H^i(BG;\mathbf{Z}) \arrow[r] \arrow[d] & H^i(BG;\mathbf{Z}/n\mathbf{Z}) \arrow[d] \\ & H^i(BG^\delta;\mathbf{Z}) \arrow[r] & H^i(BG^\delta;\mathbf{Z}/n\mathbf{Z}), \end{tikzcd}$$

using the fact that $H^i(BG;\mathbf{Z})$ is finitely generated, so that the intersection of the subgroups $nH^i(BG;\mathbf{Z})$ is zero; and using the fact that the right hand vertical arrow is injective. ■

The corresponding statement in homology would be false. For example if G is the unitary group $U(n)$ or the special linear group $SL(n, \mathbf{C})$, then we will see in the Appendix that $\eta_*: H_i(BG^\delta;\mathbf{Z}) \to H_i(BG;\mathbf{Z})$ is identically zero for $i>0$. However we can prove the following weaker statement.

COROLLARY 2. *Every element of finite order n in* $H_i(BG;\mathbf{Z})$ *lifts to an element of order n in* $H_i(BG^\delta;\mathbf{Z})$.

Proof. This follows from the commutative diagram

$$\begin{tikzcd} H_{i+1}(BG^\delta;\mathbf{Z}/n\mathbf{Z}) \arrow[r] \arrow[d, "\text{onto}"] & H_i(BG^\delta;\mathbf{Z}) \arrow[r, "n"] \arrow[d] & {} \\ H_{i+1}(BG;\mathbf{Z}/n\mathbf{Z}) \arrow[r] & H_i(BG;\mathbf{Z}) \arrow[r, "n"] & {} \end{tikzcd}$$ ■

§4. Examples for H_2

Homology with *integer* coefficients is to be understood throughout this section. We will need the following observation to relate integer homology to mod p homology.

LEMMA 5. *A path-connected space X has the* mod p *homology of a point for every prime p if and only if the integer homology group* H_iX *is uniquely divisible for* $i>0$.

In particular, a connected group G satisfies the Isomorphism Conjecture if and only if the integer homology $H_i B\bar{G}$ is uniquely divisible for $i>0$.

Proof. This follows from the homology exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$. ■

Recall from §2 that it would suffice to prove the Isomorphism Conjecture for connected semi-simple groups.

LEMMA 6. *If G is connected and semi-simple, then $H_1B\bar{G}$ is zero, and there is a split exact sequence $0 \to H_2B\bar{G} \to H_2BG^\delta \to H_2BG \to 0$.*

Here H_2BG can be identified with the fundamental group π_1G, since G is connected. So the last statement means that H_2BG^δ splits as the direct sum of the finitely generated group π_1G, and a group $H_2B\bar{G}$ which is conjectured to be a rational vector space.

Proof. For the computation of $H_1B\bar{G}$, we may assume that G is simply-connected (compare §2), and hence that $H_2BG = 0$. Since G is perfect, the group $H_1BG^\delta \cong G/[G, G]$ is zero. The statement that $H_1B\bar{G} = 0$ then follows from the spectral sequence of the fibration $B\bar{G} \to BG^\delta \to BG$.

For any connected Lie group G, note that H_3BG is finite, since the rational cohomology of BG is a polynomial algebra on even dimensional generators (Borel, 1953). Therefore H_3BG^δ maps onto H_3BG by Corollary 2 of §3. If G is semi-simple, so that $H_1B\bar{G} = 0$, an elementary spectral sequence argument now yields the required short exact sequence; and it follows from Corollary 2 that this exact sequence splits. ■

LEMMA 7. *If G is a Chevalley group over the real or complex numbers, then $H_2B\bar{G}$ is uniquely divisible and uncountably infinite.*

For the proof, which is based on deep results of Steinberg, Moore and Matsumoto, the reader is referred to Sah and Wagoner, p. 623. ■

Note that any *complex* simply-connected simple Lie group is automatically a Chevalley group. In the complex case, the proof shows that $H_2B\bar{G}$ is naturally isomorphic to the group $K_2\mathbf{C}$ of algebraic K-theory, which is uniquely divisible by a theorem of Bass and Tate.

Typical examples of real Chevalley groups are special linear group $SL(n, \mathbf{R})$, the rotation groups $SO(n, n)$ and $SO(n, n+1)$, and the symplectic group consisting of automorphisms of a skew form on $\mathbf{R}^{2n}$. In the real case, $H_2B\bar{G}$ is isomorphic to the "real part" of $K_2\mathbf{C}$, that is the subspace fixed under the involution arising from complex conjugation.

For non-Chevalley groups, the known information is rather sparse. Alperin and Dennis have proved an analogous result for the stable special linear group over the quaternions. Their paper also contains an ingenious argument due to Mather, which proves the following. *If* $T \cong S^1$ *is a maximal torus in the* 3-*sphere group* $SU(2)$, *then* H_2BT^δ *maps onto* $H_2BSU(2)^\delta$. Since H_2BT^δ is known to be uniquely divisible, it follows that $H_2BSU(2)^\delta$ is at least divisible. I do not know how to prove the corresponding statement even for $SU(3)$. Alperin has shown that the successive homomorphisms

$$H_2BSU(3)^\delta \to H_2BSU(4)^\delta \to \cdots$$

are surjective (and bijective from $SU(6)$ on); but no more precise information about these groups seems to be available.

Appendix: Real or rational coefficients

The cohomology of BG^δ with real or rational coefficients behaves quite differently from cohomology with finite coefficients, and is somewhat better understood. In fact, there are two basic tools which help to make the real case tractable, namely the Chern–Weil theory of characteristic classes expressed in terms of curvature forms, and the van Est theory of continuous cohomology. One consequence of these theories is the following.

LEMMA 8. *If* G *is compact, then the canonical homomorphism* $H_iBG^\delta \to H_iBG$, *with real or rational coefficients, is zero for* $i > 0$.

If the integer homology $H_i(BG; \mathbf{Z})$ happens to be free abelian, then it follows easily that the corresponding homomorphism with integer coefficients is also zero. This is the case, for example, when G is the unitary group $U(n)$.

More generally, let G be any Lie group with finitely many components, and let K be a maximal compact subgroup.

LEMMA 9. *In this case, the homomorphism* $H_iBG^\delta \to H_iBG$ *is zero for* i *greater than the dimension of* G/K.

Here and elsewhere, real or rational coefficients are to be understood. Evidently this reduces to the previous statement if G itself is compact.

Here is a different generalization. Let G be any Lie group which contains a discrete cocompact subgroup Γ. Such a subgroup exists, for example, whenever G

is connected and semi-simple (see Borel and Harish-Chandra), or whenever G is simply-connected and nilpotent with rational structure constants (Mal'cev).

LEMMA 10. *Then the image of* $\eta_*: H_i BG^\delta \to H_i BG$ *is precisely equal to the image of the composition*

$$H_i\Gamma \to H_i BG^\delta \to H_i BG.$$

Similarly, the kernel of the ring homomorphism $\eta^*: H^*BG \to H^*BG^\delta$ is equal to the kernel of $H^*BG \to H^*B\Gamma$. Here are some examples. If G is compact, then we can take Γ to be trivial, and recover Lemma 8. If G is the group $PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm I\}$, then a maximal compact subgroup K is a circle, and G can be identified with the group of all orientation preserving isometries of the hyperbolic plane G/K. In this case we can take Γ to be the fundamental group of a closed surface $\Gamma \backslash G/K \simeq B\Gamma$. The cohomology $H^*BG \cong H^*BK$ is a polynomial ring on one 2-dimensional generator, and it follows from either Lemma 9 or 10 that the square of this generator maps to zero in H^4BG^δ. However, the image of the generator itself in H^2BG^δ is non-zero. (Compare Milnor 1958, as well as Wood.)

Another closely related result is the following.

LEMMA 11. *If G is complex and semi-simple, with finitely many components, then again the homomorphism* $H_iBG^\delta \to H_iBG$ *is zero for* $i > 0$.

For a real semi-simple connected Lie group, the kernel of the cohomology homomorphism η^* can be computed as follows. Let $h: G \to G_{\mathbf{C}}$ be a *complexification* of G. That is, let $G_{\mathbf{C}}$ be a connected complex Lie group whose Lie algebra is the complexification $\mathfrak{g} \otimes \mathbf{C}$ of the Lie algebra of G, and let h be a homomorphism which induces the embedding of $\mathfrak{g}$ into its complexification. Note that the kernel of h is necessarily discrete and central.

THEOREM 2. *With these hypotheses, the sequence of ring homomorphisms* $H^*BG_{\mathbf{C}} \to H^*BG \to H^*BG^\delta$ *is "exact", in the sense that the kernel of the second homomorphism is equal to the ideal generated by the positive dimensional elements in the image of the first.*

Remark. If we use real coefficients, then the image of $h^*: H^*BG_{\mathbf{C}} \to H^*BG$ can be identified with the image of the Chern–Weil homomorphism associated with G.

As an example, if $G = SL(2n, \mathbf{R})$, then we can take $G_{\mathbf{C}} = SL(2n, \mathbf{C})$. The cohomology ring H^*BG is a polynomial ring generated by the Pontrjagin classes $p_1, \ldots, p_n$, together with the Euler classs e, subject to the relation $e^2 = p_n$; and the image of h^* is equal to the subalgebra generated by the Pontrjagin classes. (See for example Milnor and Stasheff.) *Thus it follows that only the Euler class survives to H^*BG^δ (or to $H^*B\Gamma$ if Γ is a discrete cocompact subgroup).*

To begin the proofs, let us consider the *Chern–Weil homomorphism*

$$\theta: \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}'] \to H^*(BG; \mathbf{R})$$

associated with a Lie group G and its Lie algebra $\mathfrak{g}$. Here $\mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$ stands for the graded algebra consisting of all real valued polynomial functions on the vector space $\mathfrak{g}$ which are invariant under the adjoint action of G. Given such an invariant polynomial $f: \mathfrak{g} \to \mathbf{R}$, homogeneous of degree n, and given a smooth principal G-bundle over some manifold M, with a smooth G-invariant connection, the curvature 2-forms Ω of the connection give rise to a closed $2n$-form $f(\Omega)$, and hence to a characteristic cohomology class

$$(f(\Omega)) \in H^{2n}(M; \mathbf{R}).$$

This corresponds to the required class $\theta(f) \in H^{2n}(BG; \mathbf{R})$ under the canonical homomorphism $H^{2n}(BG; \mathbf{R}) \to H^{2n}(M; \mathbf{R})$. See Kobayashi and Nomizu or Spivak for details.

Chern–Weil Theorem. *If G is compact, then this homomorphism $\theta: \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}'] \to H^*(BG; \mathbf{R})$ is bijective.*

In particular, BG has only even dimensional cohomology with real coefficients. This theorem is proved in Cartan or Chern or Bott 1973.

Proof of Lemma 8. Any homology class in $H_{2n}(BG^\delta; \mathbf{Q})$ can be realized as the image of a homology class from some smooth open manifold which is mapped into BG^δ. To prove that its image in $H_{2n}(BG; \mathbf{Q})$ is zero, it evidently suffices to evaluate on an arbitrary real cohomology class in $H^*(BG; \mathbf{R}) \cong \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$. If $n > 0$, then choosing any homogeneous polynomial $f \in \mathrm{Inv}_G \mathbf{R}[\mathfrak{g}']$ of degree n, the characteristic class $(f(\Omega))$ of the induced bundle over M is zero since this induced bundle has curvature $\Omega = 0$. The conclusion follows. ■

In the case of a *complex* Lie group, there is an analogous homomorphism

$$\mathrm{Inv}_G \mathbf{C}[\mathfrak{g}'] \to H^*(BG; \mathbf{C}),$$

where now $\mathbf{C}[\mathfrak{g}']$ must be interpreted as the graded algebra consisting of all complex polynomial functions on the complex vector space $\mathfrak{g}$,

LEMMA 12. *If G is complex and semi-simple, with only finitely many connected components, then this complex Chern–Weil homomorphism* $\mathrm{Inv}_G\, \mathbf{C}[\mathfrak{g}'] \to H^*(BG; \mathbf{C})$ *is also bijective.*

Proof of Lemmas 12 *and* 11. Let $K \subset G$ be a maximal compact subgroup. (Compare Mostow.) Since K is essentially unique, it coincides with the compact real form of G, as constructed by Weyl. Hence the Lie algebra $\mathfrak{g}$ can be identified with the complexification $\mathfrak{k} \otimes \mathbf{C}$ of the Lie algebra of K. It is then not difficult to check that $\mathrm{Inv}_G\, \mathbf{C}[\mathfrak{g}']$ can be identified with $\mathrm{Inv}_K\, \mathbf{R}[\mathfrak{k}'] \otimes \mathbf{C}$, so that Lemma 12 follows from the Chern–Weil Theorem applied to K. Evidently Lemma 11 follows easily. ■

Next consider the following construction. Let G be any Lie group (with a finite or countably infinite number of components). Fixing some large integer N, let $E \to X$ be a smooth N-*universal* principal G-bundle. That is, we assume that the total space E is $(N-1)$-connected. Then the base space $X = E/G$ is a finite dimensional manifold such that the natural map $X \to BG$ induces isomorphisms of homology and cohomology in dimensions less than N. Let $A(E)$ be the de Rham complex of smooth differential forms on E, and let $\mathrm{Inv}_G\, A(E)$ be the subcomplex of G-invariant forms. We will be interested in the cohomology groups $H^n(\mathrm{Inv}_G\, A(E))$ in dimensions $n < N$.

If G has only finitely many components, then these groups $H^n(\mathrm{Inv}_G\, A(E))$ are isomorphic to the continuous (or the differentiable) Eilenberg–MacLane cohomology groups of G, as studied by van Est. (See for example Borel and Wallach, p. 279.) Furthermore $H^n(\mathrm{Inv}_G\, A(E))$ can also be identified with the group $H^n(\mathrm{Inv}_G\, A(G/K))$, where K is a maximal compact subgroup of G, or equivalently with the Lie algebra cohomology $H^n(\mathfrak{g}, K)$. *Thus this cohomology is zero in dimensions greater than the dimension of G/K.* (Compare van Est, Borel–Wallach, Dupont, or Haefliger 1973.) The following two lemmas are essentially due to van Est.

LEMMA 13. *The natural homomorphism* $\eta^* : H^n(BG; \mathbf{R}) \to H^n(BG^\delta; \mathbf{R})$ *factors through the group* $H^n(\mathrm{Inv}_G\, A(E))$, *providing that* $n < N$.

Clearly Lemma 9, with real coefficients, will follow as an immediate corollary once we have proved this statement; and the corresponding statement with rational coefficients will then also follow.

LEMMA 14. *If Γ is a discrete cocompact subgroup of G, then the composition $H^n(\mathrm{Inv}_G A(E)) \to H^n(BG^\delta; \mathbf{R}) \to H^n(B\Gamma; \mathbf{R})$ is injective for $n < N$.*

Proof of Lemmas 13 *and* 9. Evidently we can identify $H^n(BG; \mathbf{R})$ with the de Rham cohomology $H^n(A(E/G))$, which maps naturally to $H^n(\mathrm{Inv}_G A(E))$. On the other hand, if SE denotes the smooth singular complex of E, then G^δ operates freely and properly on SE, so the quotient complex SE/G^δ has the same cohomology groups as BG^δ in dimensions less than N. A canonical cochain homomorphism

$$\mathrm{Inv}_G A^n(E) \to C^n(SE/G^\delta; \mathbf{R})$$

is constructed by integrating G-invariant n-forms over smooth singular simplexes which are well defined up to right translation by G^δ. This cochain homomorphism induces the required homomorphism from $H^n(\mathrm{Inv}_G (A(E))$ to $H^n(BG^\delta; \mathbf{R})$. Further details will be left to the reader. ■

Proof of Lemmas 14 *and* 10. We can identify $H^n(B\Gamma; \mathbf{R})$ with the nth cohomology of the complex $\mathrm{Inv}_\Gamma A(E) \cong A(E/\Gamma)$ of Γ-invariant forms on E. Let α be a closed G-invariant n-form on E, and suppose that $\alpha = d\beta$ for some Γ-invariant $(n-1)$-form β. If we translate β by any element of the compact coset space $\Gamma\backslash G$, which acts on the right, then we obtain another $(n-1)$-form with coboundary α. Averaging these translates with respect to the Haar measure on this compact coset space, we obtain a G-invariant $(n-1)$-form with the same coboundary α. This proves Lemma 14; and Lemma 10 follows easily. ■

Proof of Theorem 2. Part of this Theorem, namely the statement that the composition $H^iBG_{\mathbf{C}} \to H^iBG \to H^iBG^\delta$ with real or rational coefficients is zero for $i > 0$, follows immediately from Lemma 11 together with the commutative diagram

$$\begin{array}{ccc} BG^\delta & \to & BG \\ \downarrow & & \downarrow \\ BG^\delta_{\mathbf{C}} & \to & BG_{\mathbf{C}}. \end{array}$$

Note, by Lemmas 13 and 14, that an element of $H^i(BG; \mathbf{R})$ maps to zero in $H^i(BG^\delta; \mathbf{R})$ if and only if it maps to zero in the group $H^i\,\mathrm{Inv}_G A(E) \cong H^i\,\mathrm{Inv}_G A(G/K)$. *Thus, to prove the Theorem, we must check that the sequence*

$$H^*BG_{\mathbf{C}} \to H^*BG \to H^*\,\mathrm{Inv}_G A(G/K),$$

with real coefficients, is "exact" in the sense of Theorem 2.

A standard elementary argument shows that the chain complex $\mathrm{Inv}_G A(G/K)$ can be identified with the complex $C^*(\mathfrak{g}, K) \cong \mathrm{Inv}_K \Lambda^*(\mathfrak{g}/\mathfrak{k})'$ consisting of all multi-linear skew forms on the vector space $\mathfrak{g}/\mathfrak{k}$ which are invariant under the adjoint action of K, provided with a suitable coboundary operator. If we pass to complex coefficients, then the cohomology of this complex can be computed in terms of the complexification $h: G \to G_{\mathbf{C}}$ as follows. Choose a maximal compact subgroup L of $G_{\mathbf{C}}$ with $h(K) \subset L$. Then G and L are both real forms of the complex Lie group $G_{\mathbf{C}}$. Hence the corresponding real Lie algebras $\mathfrak{g}$ and $\mathfrak{l}$ have isomorphic complexifications. It follows easily that $H^*(\mathfrak{g}, K) \otimes \mathbf{C}$ is isomorphic to $H^*(\mathfrak{l}, h(K)) \otimes \mathbf{C}$. This can be identified with the cohomology of the complex $\mathrm{Inv}_L A(L/h(K)) \otimes \mathbf{C}$, in fact, since L is compact and connected, it can simply be identified with $H^*(L/h(K); \mathbf{C})$.

Note also that $h(K)$ is the quotient of K by a finite central subgroup, so that the cohomology of $Bh(K)$, with real or rational coefficients is isomorphic to the cohomology of BK or of BG. *To simplify the notation, let us assume that $K \cong h(K)$, so that we may think of K as a subgroup of L.* The statement to be proved then reduces to the following.

LEMMA 15 (Cartan, p. 69). *Given compact connected Lie groups $K \subset L$, the sequence $H^*BL \to H^*BK \to H^*(L/K)$ of ring homomorphisms (with real or rational or complex coefficients) is "exact" in the sense of Theorem* 2.

Proof. The fibration sequence $L \to L/K \to BK$ gives rise to a cohomology spectral sequence; or alternatively to the statement that $H^*(L/K)$ is isomorphic to the cohomology of the complex $H^*BK \otimes H^*L$ under a coboundary operator d which has the following properties. The image $d(H^*BK \otimes 1)$ is zero; and furthermore, if $v \in H^*L$ is universally transgressive so that its transgression $\bar{v}$ is defined and lies in the image of $H^*BL \to H^*BK$, then $d(1 \otimes v) = \bar{v} \otimes 1$. (See Borel, 1953 p. 187.) Since H^*L is an exterior algebra generated by universally transgressive elements, it follows easily that the image of d intersected with $H^*BK \otimes 1$ is the ideal spanned by the $\bar{v}$. This proves the Lemma. ■

To prove the Theorem, we must identify the sequence $H^*BL \to H^*BK \to H^*(L/K)$, of Lemma 15, with the required sequence $H^*BG_{\mathbf{C}} \to H^*BG \to H^* \mathrm{Inv}_G\,(A(G/K))$, using complex coefficients. This can be done, making use of a purely algebraic construction of the last homomorphism. (See Haefliger 1973, p. 6.) Details will be omitted. ■

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Lifting idempotents and Clifford theory

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Let N be a normal subgroup of a finite group G and let R be a noetherian complete local commutative ring. Clifford theory deals with the relationship between RG-modules and RN-modules, using induction from N to G or restriction from G to N. Since Clifford's 1937 paper [1], the theory is well understood for irreducible representations (see also [2, §11C]). For an indecomposable RN-module W, several authors have proved a going-up theorem describing how $\mathrm{Ind}_N^G W$ decomposes (see [2, §19C]).

One purpose of this paper is to prove (in Section 2) a going-down theorem for indecomposable modules (analogous to Clifford's theorem), based on a refinement of the lifting idempotents theorem, presented in Section 1. The going-up and going-down theorems are actually equivalent in the sense that each can be derived as a corollary to the other one. One main assumption is necessary for the going-down theorem: the RG-module we start from must be projective relative to H. The whole procedure is presented in the more general context of Clifford systems. The paper concludes in Section 3 with another application of the lifting idempotents theorem, concerning the behaviour of indecomposable modules under ground ring extensions.

1. Lifting idempotents

THEOREM 1. *Let A be a ring and J a two-sided ideal contained in* Rad A. *Assume that A is complete in the J-adic topology (that is the natural map $A \to \varprojlim A/J^n$ is an isomorphism). Let Π be a finite group acting on A by automorphisms leaving J globally invariant. Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be a set of orthogonal idempotents of $\bar{A} = A/J$ satisfying $\sum_{i=1}^n \bar{e}_i = 1$. Assume the following three conditions:*

a) *The induced action of Π on $\bar{A}$ permutes the idempotents $\bar{e}_i$ transitively.*

b) *There exists $u \in A$ such that $Tr_\Omega(u) = 1$ where Ω is the stabilizer of $\bar{e}_1$ and $Tr_\Omega(u) = \sum_{\omega\in\Omega} \omega u$.*

c) *$\bar{u}$ commutes with each $\bar{e}_i$.*

Then $\{\bar{e}_1, \ldots, \bar{e}_n\}$ lifts to a set $\{e_1, \ldots, e_n\}$ of orthogonal idempotents of A which are permuted by Π transitively and such that $\sum_{i=1}^n e_i = 1$.

Remarks. 1) If A is the ring of endomorphisms of a representation V, we shall see that the condition b) corresponds to a condition of relative projectivity for V.

2) There are two situations where c) is always satisfied: either the idempotents $\bar{e}_i$ are central or the order $|\Omega|$ of Ω is invertible in A in which case one can choose u to be the central element $|\Omega|^{-1}$.

3) When Π acts regularly on the idempotents $\bar{e}_i$, that is when Ω is trivial, one can take $u=1$ so that b) and c) are trivially satisfied. This special case appears already in [3].

Proof. It suffices to prove the theorem when J is nilpotent because, since $A \cong \varprojlim A/J^n$, the lifted idempotents are constructed as limits of idempotents of A/J^n for $n \to \infty$.

For $\sigma \in \Pi$, write $\bar{e}_\sigma = \sigma\bar{e}_1$ so that $\bar{e}_\sigma = \bar{e}_\tau$ if and only if $\sigma\Omega = \tau\Omega$. Since Π acts transitively, every idempotent $\bar{e}_i$ can be written in that form.

We proceed by induction on the nilpotent index n of J. There is nothing to prove if $n=1$. If $n \geqslant 2$, let $I = J^{n-1}$ and write $\tilde{a}$ for the image of $a \in A$ modulo I. By induction, there exist idempotents $\tilde{e}_\sigma$ of A/I such that $\sigma\tilde{e}_\tau = \tilde{e}_{\sigma\tau}$ and $\sum_{\sigma \in \Pi/\Omega} \tilde{e}_\sigma = 1$. First lift arbitrarily the idempotents $\tilde{e}_\sigma$ to get orthogonal idempotents e_σ of A satisfying $\sum_{\sigma \in \Pi/\Omega} e_\sigma = 1$. This is well known to be possible (see [2, §6A]). Of course the notation implies that we keep the convention:

$$e_\sigma = e_\tau \quad \text{if and only if} \quad \sigma\Omega = \tau\Omega.$$

Since $\sigma\tilde{e}_\tau = \tilde{e}_{\sigma\tau}$, we have:

$$\sigma e_\tau = e_{\sigma\tau} + r_{\sigma,\tau} \quad \text{for some} \quad r_{\sigma,\tau} \in I.$$

We list several properties of the elements $r_{\sigma,\tau}$:

(1) If $\omega \in \Omega$, $r_{\sigma,\tau\omega} = r_{\sigma,\tau}$.

This follows from $e_{\eta\omega} = e_\eta$ for all $\eta \in \Pi$.

(2) $\sum_{\tau \in \Pi/\Omega} r_{\sigma,\tau} = 0$.

This follows when σ is applied to $1 = \sum_{\tau \in \Pi/\Omega} e_\tau$.

(3) $\eta r_{\sigma,\tau} = r_{\eta\sigma,\tau} - r_{\eta,\sigma\tau}$.

This is a consequence of $(\eta\sigma)e_\tau = \eta(\sigma e_\tau)$.

(4) $r_{\sigma,\tau} = e_{\sigma\tau} r_{\sigma,\tau} + r_{\sigma,\tau} e_{\sigma\tau}$.

This follows from the equality $\sigma e_\tau = (\sigma e_\tau)^2$ using also $I^2 = 0$. Multiplying (4) by e_η on the right or e_λ on the left (or both in the first case below), we get:

$$(5)\quad e_\lambda r_{\sigma,\tau} e_\eta = \begin{cases} 0 & \text{if } \lambda\Omega \neq \sigma\tau\Omega \neq \eta\Omega \\ r_{\sigma,\tau} e_\eta & \text{if } \lambda\Omega = \sigma\tau\Omega \neq \eta\Omega \\ e_\lambda r_{\sigma,\tau} & \text{if } \lambda\Omega \neq \sigma\tau\Omega = \eta\Omega \\ 0 & \text{if } \lambda\Omega = \sigma\tau\Omega = \eta\Omega \end{cases}$$

(6) If $\lambda\Omega \neq \eta\Omega$, $e_{\mu\lambda} r_{\mu,\eta} + r_{\mu,\lambda} e_{\mu\eta} = 0$.
This is a consequence of $(\mu e_\lambda) \cdot (\mu e_\eta) = 0$ using again $I^2 = 0$.

Now define: $f_\sigma = e_\sigma + \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} \cdot e_\lambda \cdot \lambda u$ where $u \in A$ satisfies hypotheses b) and c). By (1), we have:

(7) $f_{\sigma\omega} = f_\sigma$ if $\omega \in \Omega$.

(8) $\sum_{\sigma \in \Pi/\Omega} f_\sigma = 1$.

For

$$\sum_{\sigma \in \Pi/\Omega} f_\sigma = \sum_{\sigma \in \Pi/\Omega} e_\sigma + \sum_{\sigma \in \Pi/\Omega} \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} \cdot e_\lambda \cdot \lambda u$$

$$= 1 + \sum_{\lambda \in \Pi} \Big(\sum_{\sigma \in \Pi/\Omega} r_{\lambda,\lambda^{-1}\sigma} \Big) e_\lambda \cdot \lambda u = 1 \quad \text{by (2).}$$

(9) $f_\sigma f_\tau = 0$ if $\sigma\Omega \neq \tau\Omega$.

$$f_\sigma f_\tau = \sum_{\lambda \in \Pi} e_\sigma r_{\lambda,\lambda^{-1}\tau} e_\lambda \cdot \lambda u + \sum_{\lambda \in \Pi} r_{\lambda,\lambda^{-1}\sigma} e_\lambda \cdot \lambda u \cdot e_\tau.$$

By hypothesis c), $\lambda\bar{u} \cdot \bar{e}_\tau = \lambda(\bar{u} \cdot \bar{e}_{\lambda^{-1}\tau}) = \lambda(\bar{e}_{\lambda^{-1}\tau} \cdot \bar{u}) = \bar{e}_\tau \cdot \lambda\bar{u}$. Hence λu commutes with e_τ modulo J. Since $I \cdot J = J^{n-1} \cdot J = 0$, we have $r \cdot \lambda u \cdot e_\tau = r \cdot e_\tau \cdot \lambda u$ for all $r \in I$ and so we can permute λu and e_τ in the second sum. Therefore, the only non-zero terms appear for $\lambda \in \tau\Omega$. By (5), the same holds for the first sum. Consequently:

$$f_\sigma f_\tau = \sum_{\omega \in \Omega} (e_\sigma r_{\tau\omega,\omega^{-1}} + r_{\tau\omega,\omega^{-1}\tau^{-1}\sigma} e_{\tau\omega}) e_{\tau\omega} \cdot \tau\omega u.$$

Now apply (6) with $\eta = 1$, $\mu = \tau\omega$ and $\lambda = \omega^{-1}\tau^{-1}\sigma$, using also (1). The condition $\lambda\Omega \neq \eta\Omega$ is equivalent to $\sigma\Omega \neq \tau\Omega$. We get $f_\sigma f_\tau = 0$, as required.

Clearly (8) and (9) imply that f_σ is idempotent. There remains to prove the additional property we are looking for:

(10) $\tau f_\sigma = f_{\tau\sigma}$.

By (3), we have:

$$\tau f_\sigma = e_{\tau\sigma} + r_{\tau,\sigma} + \sum_{\lambda \in \Pi} (r_{\tau\lambda,\lambda^{-1}\sigma} - r_{\tau,\sigma}) \cdot (e_{\tau\lambda} + r_{\tau,\lambda}) \cdot \tau\lambda u.$$

Since $I^2 = 0$, we get:

$$\begin{aligned}\tau f_\sigma &= e_{\tau\sigma} + \sum_{\lambda \in \Pi} r_{\tau\lambda,\lambda^{-1}\sigma} \cdot e_{\tau\lambda} \cdot \tau\lambda u + r_{\tau,\sigma}\Big(1 - \sum_{\lambda \in \Pi} e_{\tau\lambda} \cdot \tau\lambda u\Big) \\ &= e_{\tau\sigma} + \sum_{\mu \in \Pi} r_{\mu,\mu^{-1}\tau\sigma} \cdot e_\mu \cdot \mu u + r_{\tau,\sigma}\Big(1 - \sum_{\mu \in \Pi/\Omega} e_\mu \cdot \sum_{\omega \in \Omega} \mu\omega u\Big) \\ &= f_{\tau\sigma} + r_{\tau,\sigma}\Big(1 - \sum_{\mu \in \Pi/\Omega} e_\mu \cdot \mu Tr_\Omega(u)\Big) = f_{\tau\sigma},\end{aligned}$$

using $Tr_\Omega(u) = 1$ and $\sum_{\mu \in \Pi/\Omega} e_\mu = 1$. ■

2. Clifford theory

Let N be a normal subgroup of a finite group G and $S = G/N$. Throughout this section, R denotes a noetherian local commutative ring which is complete in its natural topology of local ring. These assumptions are made in order to have the following properties:

(i) Every finitely generated RG-module is a direct sum of indecomposable submodules.

(ii) If M is an indecomposable RG-module, then $\mathrm{End}_{RG} M$ is a local ring. Hence Krull-Schmidt theorem holds for RG-modules.

In order to study the restriction to N of an indecomposable RG-module, we consider the more general case of an S-graded Clifford system $A = \oplus_{s \in S} A_s$ over R, in the sense of [2, §11C]. The case of group algebras corresponds to $A = RG$ and $A_1 = RN$. Recall that there exist units $a_s \in A_s$ such that $A_s = a_s A_1 = A_1 a_s$. Also $a_s a_t a_{st}^{-1} \in A_1$ because $A_s A_t = A_{st}$.

For the rest of this paper, all modules will be finitely generated left modules. For an A_1-module W, denote by W^A the induced module $\mathrm{Ind}_{A_1}^A W = A \otimes_{A_1} W$, while for an A-module V, we denote by V_{A_1} the restriction $\mathrm{Res}_{A_1}^A V$. If V is an A-module, then S acts on $\mathrm{End}_{A_1} V$ by $sf = a_s f a_s^{-1}$ and the set of fixed points is exactly $\mathrm{End}_A V$.

DEFINITIONS. 1) An A-module V is said to be *projective relative* to A_1 if V is a direct summand of a module induced from A_1 which actually can be chosen to be $(V_{A_1})^A$. This is equivalent to the existence of an endomorphism $u \in \mathrm{End}_{A_1} V$ such that $Tr_S(u) = 1$ where $Tr_S(u) = \sum_{s\in S} su$. The equivalence of these definitions is well known in the case of group algebras [2, §19A], but the proof can be carried over without change to the case of Clifford systems.

2) If W is an A_1-module, then $a_s \otimes W$ has a natural structure of A_1-module and is called a *conjugate* of W.

3) Let $M = \bigoplus_{i,j} M_{ij}$ be a decomposition of a module M into indecomposable summands such that $M_{ij} \cong M_{ik}$ for all i, j, k and $M_{ij} \not\cong M_{km}$ if $i \neq k$. Then $M_i = \bigoplus_j M_{ij}$ is called a *homogeneous component* of M. Contrary to the case of semi-simple modules, note that in general M_i is not uniquely determined by M.

Now we can state the going-down theorem analogous to Clifford's theorem:

THEOREM 2. *Let A be an S-graded Clifford system over R and V an indecomposable A-module. Assume that V is projective relative to A_1, that is there exists an indecomposable summand W of V_{A_1} such that V is a direct summand of W^A. Let $T = \{t \in S \mid a_t \otimes W \cong W\}$ be the inertial subgroup of W and let $\{s_1, \ldots, s_n\}$ be a set of coset representatives of T in S. Finally let $B = \bigoplus_{t\in T} A_t$ be the T-graded subalgebra of A. Then:*

(i) *V_{A_1} is isomorphic to a direct sum of conjugates of W.*

(ii) *$\{a_{s_i} \otimes W \mid i = 1, \ldots, n\}$ is a complete set of non-isomorphic conjugates of W and each appears with the same multiplicity in a decomposition of V_{A_1}.*

(iii) *There exists a decomposition $V_{A_1} = \bigoplus_{i=1}^n U_i$ into homogeneous components which are permuted transitively by $\{a_s \mid s \in S\}$ and such that $\{a_t \mid t \in T\}$ stabilizes U_1.*

(iv) *U_1 is an indecomposable B-module and V is isomorphic to U_1^A.*

Beside Theorem 1, the main ingredient for the proof of Theorem 2 is the following:

PROPOSITION 3. *Let A be an R-algebra, finitely generated as R-module, and M an A-module. Denote by a bar the reduction modulo the radical of $\mathrm{End}_A M$. Let $M = \bigoplus_{i=1}^n M_i$ (respectively $M = \bigoplus_{i=1}^n M_i'$) be any decomposition of M corresponding to idempotents $e_1, \ldots, e_n \in \mathrm{End}_A M$ (respectively $e_1', \ldots, e_n'$).*

(i) *The modules M_i are homogeneous components of M if and only if $\bar{e}_1, \ldots, \bar{e}_n$ are the primitive central idempotents of $\overline{\mathrm{End}_A M}$.*

(ii) *Assume the modules M_i and M_i' are homogeneous components of M, labelled in order to have $M_i \cong M_i'$ for all i. Then there exists $f \in \mathrm{Aut}_A M$ such that $f(M_i) = M_i'$ for all i and $\bar{f} = 1$.*

(iii) *Assume the modules M_i and M_i' are homogeneous components of M. Then $M_1 \cong M_1'$ if and only if $\bar{e}_1 = \bar{e}_1'$.*

Proof. (i) If the modules M_i are homogeneous components of M, write $M_i \cong m_i N_i$ with N_i indecomposable. Let $E_i = \mathrm{End}_A N_i$ and $D_i = \overline{\mathrm{End}_A N_i}$. By Fitting's theorem [2, §19C, lemma], there is a commutative diagram

$$\begin{tikzcd} E = \mathrm{End}_A M \arrow[dr] & \\ & \overline{\mathrm{End}_A M} \cong \prod_{i=1}^{n} M_{m_i}(D_i) \\ \prod_{i=1}^{n} M_{m_i}(E_i) \cong \prod_{i=1}^{n} \mathrm{End}_A M_i \arrow[uu] \arrow[ur] & \end{tikzcd}$$

Since e_i is the unit matrix of $M_{m_i}(E_i)$ (with zeros in all other components), $\bar{e}_i$ is the unit matrix of $M_{m_i}(D_i)$, i.e. $\bar{e}_i$ is a primitive central idempotent of $\overline{\mathrm{End}_A M}$.

If conversely $\bar{e}_i$ is primitive central, decompose it into primitive idempotents $\bar{e}_i = \bar{e}_{i1} + \cdots + \bar{e}_{im_i}$ and lift them to get $e_i = e_{i1} + \cdots + e_{im_i}$. Now $e_{ij}E \cong e_{ik}E$ because $\bar{e}_{ij}\bar{E} \cong \bar{e}_{ik}\bar{E}$. Therefore:

$$e_{ij}M \cong e_{ij}E \otimes_E M \cong e_{ik}E \otimes_E M \cong e_{ik}M.$$

So $e_iM = \bigoplus_{i=1}^{m_i} e_{ij}M$ is a homogeneous decomposition of e_iM into indecomposable summands. If some indecomposable summand of e_iM was isomorphic to a summand of e_kM for $k \neq i$, there would be less than n homogeneous components in M and so, by the first part of the proof, less than n primitive central idempotents in $\bar{E}$.

(ii) Consider again the commutative diagram

$$\begin{tikzcd} \mathrm{End}_A M \arrow[dr, "q"] & \\ & \overline{\mathrm{End}_A M} \\ \prod_{i=1}^{n} \mathrm{End}_A M_i \arrow[uu, "j"] \arrow[ur, "p"'] & \end{tikzcd}$$

We emphasize that not only q but also p is surjective. Choose an isomorphism $g_i : M_i \to M_i'$ for each i and define an automorphism g of M by $g|_{M_i} = g_i$. Since g is invertible, so is $q(g)$ and since p is onto, there exists $h \in \prod_{i=1}^{n} \mathrm{End}_A M_i$ such that $p(h) = q(g)^{-1}$. Clearly $f = g \cdot j(h)$ satisfies $f(M_i) = M_i'$ and $\bar{f} = 1$.

(iii) By (ii), if $M_1 \cong M_1'$, there exists $f \in \mathrm{Aut}_A M$ such that $f(M_1) = M_1'$, $f(\bigoplus_{i=2}^{n} M_i) = \bigoplus_{i=2}^{n} M_i'$ and $\bar{f} = 1$. It follows easily that $e_1' = fe_1f^{-1}$ and therefore $\bar{e}_1' = \bar{e}_1$.

Conversely suppose $\bar{e}_1' = \bar{e}_1$. By Krull–Schmidt theorem, $M_1' \cong M_i$ for some i. By the first part of this proof, $\bar{e}_1' = \bar{e}_i$. Hence $\bar{e}_i = \bar{e}_1$ and so $i = 1$. ■

Proof of Theorem 2. (i) Write $V_{A_1} = \bigoplus_{i=1}^{r} W_i$ with the W_i indecomposable. Since V is a direct summand of W^A, V_{A_1} is a summand of $(W^A)_{A_1} \cong \bigoplus_{s \in S} a_s \otimes W$. By Krull–Schmidt theorem, each W_i is isomorphic to some $a_s \otimes W$.

(ii) Changing notations write $V_{A_1}=\bigoplus_{i=1}^{n} m_iW_i$ where m_iW_i denotes the direct sum of m_i copies of W_i and $W_i \not\cong W_j$ if $i \neq j$. By (i), $W_i \cong a_s \otimes W$ for some s. Applying a_s to V, we get:

$$\bigoplus_{i=1}^{n} m_iW_i \cong V_{A_1}=(a_sV)_{A_1} \cong \bigoplus_{i=1}^{n} m_i(a_s \otimes W_i).$$

Comparing the multiplicities of W_i in both decompositions, we get $m_i = m_1$. The same argument applied with an arbitrary a_s shows that $a_s \otimes W$ must be isomorphic to some W_i. Therefore, by definition of T, $\{a_{s_i} \otimes W \mid i=1, \ldots, n\}$ is a complete set of non-isomorphic conjugates of W.

(iii) Let $E=\operatorname{End}_{A_1} V$ and $\bar{E}=E/\operatorname{rad}(E)$. The group S acts on E via $sf= a_sfa_s^{-1}$ and induces an action on $\bar{E}$ which necessarily permutes the primitive central idempotents of $\bar{E}$.

Let $V_{A_1}=\bigoplus_{i=1}^{n} U_i$ be a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents $e_1, \ldots, e_n$. Assume W is a summand of U_1. For $s \in S$, $V_{A_1}=\bigoplus_{i=1}^{n} a_sU_i$ is also a decomposition of V_{A_1} into homogeneous components, corresponding to idempotents $a_se_ia_s^{-1}=se_i$. By Proposition 3(i), $\{\bar{e}_1, \ldots, \bar{e}_n\}$ are the primitive central idempotents of $\bar{E}$. Since $a_sU_1 \cong a_s \otimes U_1 \cong U_i$ for some i, we have $s\bar{e}_1=\bar{e}_i$ by Proposition 3(iii). Moreover each U_i is isomorphic to some a_sU_1 by part (i) and (ii). This implies that S acts transitively on the set $\{\bar{e}_1, \ldots, \bar{e}_n\}$. Since W is a summand of U_1, T is the stabilizer of $\bar{e}_1$ (again by Proposition 3(iii)).

Now since V is projective relative to A_1, there exists $v \in \operatorname{End}_{A_1} V$ such that $Tr_S(v)=1$. Let $u=\sum_{i=1}^{n} r_iv$ where $r_1, \ldots, r_n$ are representatives of the cosets Tr. Then $Tr_T(u)=\sum_{t\in T} tu = Tr_S(v)=1$. Moreover $\bar{u}$ commutes with $\bar{e}_i$ for $\bar{e}_i$ is central. Therefore the hypotheses of Theorem 1 are satisfied. It follows that there exist orthogonal idempotents $f_1, \ldots, f_n$ of E (lifting $\bar{e}_1, \ldots, \bar{e}_n$) which are permuted transitively by S and such that T stabilizes f_1.

By Proposition 3(i), the modules $f_iV_{A_1}$ are homogeneous components of V_{A_1}. The equation $f_i=sf_1=a_sf_1a_s^{-1}$ means exactly that $a_s(f_1V_{A_1})=f_iV_{A_1}$. This completes the proof of part (iii).

(iv) Since $\{a_t \mid t\in T\}$ stabilizes $U_1=f_1V_{A_1}$, U_1 is a B-module. Now $V= \bigoplus_{i=1}^{n} a_{s_i}U_1$ which is the definition of an induced module. Finally U_1 is indecomposable otherwise V would be decomposable. ∎

Counter-example. Without the assumption of relative projectivity for V, Theorem 2 does not hold any more. Take K a field of characteristic 2, $G= C_4$, $N=C_2$ and $V=K[X]/(X-1)^3$ (the generator of C_4 acting by multiplication by X). Then: $\operatorname{Res}_N V=S_1\oplus S_2$ where $S_i=K[Y]/(Y-1)^i$ (the generator of C_2

acting by multiplication by Y). Since S_1 and S_2 do not have the same dimension, they cannot be conjugate. In fact, the two primitive central idempotents of $\overline{\mathrm{End}_{KN} V}$ are fixed under the action of $S = G/N$, and each of them can be lifted in four ways in $\mathrm{End}_{KN} V$. But no idempotent of $\mathrm{End}_{KN} V$ is fixed by S.

Now we can recall the going-up theorem, which we shall prove to be equivalent to Theorem 2.

THEOREM 4 (Conlon, Tucker, Ward [2, §19C]). *Let A be an S-graded Clifford system over R, W an indecomposable A_1-module, T the inertial subgroup of W and $B = \bigoplus_{t \in T} A_t$. If $W^B = \bigoplus_{i=1}^m Z_i$ is a decomposition of W^B into indecomposable B-modules, then each Z_i^A is an indecomposable A-module, that is $W^A = \bigoplus_{i=1}^m Z_i^A$ gives a decomposition of W^A into indecomposable A-modules.*

Proof. The notation $X \mid Y$ will mean: X is a direct summand of Y. Let Z be an indecomposable summand of W^B. Since T is the inertial subgroup of W, $(W^B)_{A_1} = |T| \cdot W$ and so Z_{A_1} is a multiple of W. Since $Z \mid (Z^A)_B$, there exists an indecomposable summand V of Z^A such that $Z \mid V_B$. Then $V \mid W^A$ and $W \mid V_{A_1}$. By Theorem 2, there exists an indecomposable B-module U such that $V \cong U^A$ and U_{A_1} is a multiple of W. Now $U \mid (Z^A)_B$ because $V \mid Z^A$ and $U \mid (U^A)_B = V_B$. But Z is the only indecomposable summand of $(Z^A)_B$ whose restriction to A_1 is a multiple of W, for $(Z^A)_{A_1} = \bigoplus_{i=1}^n a_{s_i} \otimes Z_{A_1}$ (where $\{s_1, \ldots, s_n\}$ is a set of coset representatives of T in S) and $a_{s_i} \otimes Z_{A_1}$ is a proper conjugate of Z_{A_1} (a multiple of a proper conjugate of W). It follows that $U \cong Z$ and so $Z^A \cong U^A \cong V$ is indecomposable. ∎

Equivalence of Theorems 2 *and* 4. If Theorem 4 is proved independently (e.g. by the proof of [2, §19C]), then Theorem 2 can be derived as corollary in the following way: Let V be an indecomposable A-module which is a summand of W^A for some indecomposable summand W of V_{A_1}. Let T be the inertial subgroup of W. By Theorem 4, there exists an indecomposable summand U of W^B such that $V = U^A$. Now $U_{A_1} \cong mW$ for some m because $(W^B)_{A_1} \cong |T|\, W$. Then clearly $V \cong \bigoplus_{i=1}^n a_{s_i} \otimes U$ and $V_{A_1} \cong \bigoplus_{i=1}^n m(a_{s_i} \otimes W)$ where $s_1, \ldots, s_n$ are coset representatives of T in S. This completes the proof of Theorem 2. ∎

3. Ground field extensions

Let K be a field and A a finite dimensional K-algebra. Let F be a finite Galois extension of K, with Galois group Π, and consider the F-algebra $F \otimes A$ (note that throughout this section $\otimes$ will always mean $\otimes_K$). Every element $\sigma \in \Pi$ induces a semi-linear automorphism $\sigma : F \otimes A \to F \otimes A$. If W is an $F \otimes A$-module, one can define a new $F \otimes A$-module structure on W by scalar extension via σ (or equivalently restriction via σ^{-1}). Explicitly the new structure is given by $a \cdot w = \sigma^{-1}(a)w$, $a \in F \otimes A$, $w \in W$. This module is called a Galois conjugate of W.

Now if V is a finitely generated indecomposable A-module, then $F \otimes V$ has a natural structure of $F \otimes A$-module. Moreover, Π acts on $F \otimes V$ via $\sigma(f \otimes v) = \sigma f \otimes v$, $\sigma \in \Pi$, $f \in F$, $v \in V$. This action is semi-linear with respect to $F \otimes A$, i.e. $\sigma(aw) = \sigma(a)\sigma(w)$, $\sigma \in \Pi$, $a \in F \otimes A$, $w \in F \otimes V$. If $F \otimes V = \bigoplus_{i=1}^{n} W_i$ is a decomposition of $F \otimes V$ into homogeneous components, then so is $F \otimes V = \bigoplus_{i=1}^{n} \sigma W_i$. One can readily check that σW_i is a Galois conjugate of W_i. By Krull–Schmidt theorem, $\sigma W_i \cong W_j$ for some j. Moreover, it is easy to see that for given i and j, there exists $\sigma \in \Pi$ such that $\sigma W_i \cong W_j$. The purpose of this section is to derive from Theorem 1 a stronger result, namely that for a suitable choice of the submodules W_i, one can replace this isomorphism by an equality:

PROPOSITION 5. *In the above notations, there exists a decomposition* $F \otimes V = \bigoplus_{i=1}^{n} W_i$ *of* $F \otimes V$ *into homogeneous components such that the modules* W_i *are permuted transitively under the natural action of* Π *on* $F \otimes V$.

Proof. Let $E = \mathrm{End}_A V$ and $\bar{E} = E/\mathrm{Rad}\, E$. Since V is indecomposable, $\bar{E}$ is a division algebra containing K in its center. Now $F \otimes E = \mathrm{End}_{F \otimes A}(F \otimes V)$ and let $\overline{F \otimes E} = F \otimes E/\mathrm{Rad}\,(F \otimes E)$. Since F/K is separable, $\overline{F \otimes E} \cong F \otimes \bar{E}$. Let $F \otimes V = \bigoplus_{i=1}^{n} W_i$ be a decomposition of $F \otimes V$ into homogeneous components corresponding to idempotents $e_1, \ldots, e_n \in F \otimes E$. The decomposition $F \otimes V = \bigoplus_{i=1}^{n} \sigma W_i$ corresponds to the idempotents $\sigma e_1 \sigma^{-1}, \ldots, \sigma e_n \sigma^{-1}$ (where σ is viewed as a semi-linear automorphism of $F \otimes V$).

Now Π acts on $F \otimes E$ via $\sigma \cdot (f \otimes e) = \sigma f \otimes e$, $\sigma \in \Pi$, $f \in F$, $e \in E$. We claim that $\sigma z \sigma^{-1} = \sigma \cdot z$ for all $z \in F \otimes E$. Indeed, if $z = f \otimes e$, $f \in F$, $e \in E$, and if $g \otimes v \in F \otimes V$, then:

$$
\begin{aligned}
(\sigma z \sigma^{-1})(g \otimes v) &= \sigma(f \otimes e)(\sigma^{-1} g \otimes v) = \sigma(f \cdot \sigma^{-1} g) \otimes ev = \sigma f \cdot g \otimes ev \\
&= (\sigma f \otimes e)(g \otimes v) = (\sigma \cdot z)(g \otimes v).
\end{aligned}
$$

It follows that $\{\sigma \cdot e_1, \ldots, \sigma \cdot e_n\}$ are the idempotents corresponding to the decomposition $F \otimes V = \bigoplus_{i=1}^{n} \sigma W_i$. By Proposition 3(i), $\{\bar{e}_1, \ldots, \bar{e}_n\} = \{\overline{\sigma \cdot e_1}, \ldots, \overline{\sigma \cdot e_n}\}$ is the set of primitive central idempotents of $F \otimes \bar{E}$. Now Π acts transitively on $\{\bar{e}_1, \ldots, \bar{e}_n\}$ for if $\{\bar{e}_1, \ldots, \bar{e}_k\}$ is a Π-orbit, then $\bar{e} = \sum_{i=1}^{k} \bar{e}_k$ is an idempotent, invariant under Π, hence lies in $K \otimes \bar{E} = \bar{E}$. Since 1 is the only idempotent of $\bar{E}$, we get $\bar{e} = 1$ and so $k = n$.

Since F/K is separable, $Tr_{F/K}$ is surjective. Therefore there exists $x \in F$ such that $Tr_{F/K}(x) = \sum_{\sigma \in \Pi} \sigma x = 1$. In particular, if Ω denotes the stabilizer of $\bar{e}_1$ and $\sigma_1, \ldots, \sigma_n$ are coset representatives of Ω in Π, then $u = \sum_{i=1}^{n} \sigma_i x$ satisfies $\sum_{\omega \in \Omega} \omega u = 1$. Also $u \otimes \bar{1} \in F \otimes \bar{E}$ commutes with every $\bar{e}_i$. Therefore $u \otimes 1 \in F \otimes E$

satisfies the hypotheses of Theorem 1. Consequently $\{\bar{e}_1, \ldots, \bar{e}_n\}$ lifts to a set of orthogonal idempotents $f_1, \ldots, f_n$ of $F \otimes E$ which are permuted transitively by Π and such that $\sum_{i=1}^n f_i = 1$. By Proposition 3(i), the modules $W_i' = f_i(F \otimes V)$ are homogeneous components of $F \otimes V$. Finally, since $\sigma f_i = f_j$ for some j, we have:

$$\sigma W_i' = \sigma(f_i(F \otimes V)) = (\sigma f_i \sigma^{-1})(F \otimes V) = (\sigma \cdot f_i)(F \otimes V) = f_j(F \otimes V) = W_j'. \quad \blacksquare$$

Remarks. 1) If one replace homogeneous components of $F \otimes V$ by indecomposable summands, then one must consider sets of primitive idempotents $\{\bar{e}_1, \ldots, \bar{e}_n\}$ of $\bar{E}$ instead of primitive central idempotents of $\bar{E}$. If one can show that there exists such a set which is stable under the action of Π (this happens quite often), then the whole proof works without change, so that there exists a decomposition $F \otimes V = \bigoplus_{i=1}^n W_i$ into *indecomposable submodules* such that the modules W_i are permuted transitively under the natural action of Π on $F \otimes V$.

2) Proposition 5 holds more generally if one replaces the field K by a complete discrete valuation ring R and the extension F by an unramified Galois extension S (so that the Galois group of S/R is isomorphic to the Galois group of the residue field extension). Moreover, A must be an R-algebra which is finitely generated as R-module.

3) The similarity between restriction to a normal subgroup (Theorem 2) and ground field Galois extension (Proposition 5) extends a little further. If Ω denotes the stabilizer of the homogeneous component W_1 of $F \otimes V$ and if L is the fixed field of Ω, then W_1 is realizable over L, that is there exists an $L \otimes_K A$-module U such that $F \otimes_L U = W_1$. Moreover, by analogy with part (iv) of Theorem 2 (replacing group induction by scalar restriction), one can easily show that $V \cong \operatorname{Res}_K U$.

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Linking pairings on singular spaces

MARK GORESKY and PAUL SIEGEL

§1. Introduction

In [GM1] [GM2] [GM3] intersection homology groups $IH_i^{\bar{p}}(X)$ were defined for any n dimensional compact oriented pseudomanifold X and any perversity $\bar{p}$ between $\bar{0}=(0, 0, \ldots, 0)$ and $\bar{t}=(0, 1, 2, 3, \ldots)$. Questions concerning the torsion subgroups of the intersection homology groups have arisen in three contexts:

(A) Is the torsion in $IH_i^{\bar{p}}(X)$ dually paired with the torsion in $IH_j^{\bar{q}}(X)$ when $i+j=n$ and $\bar{p}+\bar{q}=\bar{t}$?

(B) Does the universal coefficient formula hold for $IH_i^{\bar{p}}(X)$?

(C) For a compact $4k$ dimensional pseudomanifold X with even dimensional strata, does the determinant of the intersection pairing on $IH_{2k}^{\bar{m}}(X)$ equal 1?

The answer to all these questions is "yes" if X is a manifold, but "no" if X is a general singular space. However, for singular spaces which are "locally $\bar{p}$-torsion free" the answer is "yes" to each of these questions:

DEFINITION. A pseudomanifold X is locally $\bar{p}$-torsion free if, for each stratum of X,

$$T^{\bar{p}}_{c-2-p(c)}(L)=0$$

where L denotes the link of that stratum, c denotes its codimension, and $T_i^{\bar{p}}(L)$ is the torsion subgroup of $IH_i^{\bar{p}}(L)$.

The answer to question (A) is:

THEOREM 4.4. *Suppose X is a compact oriented n dimensional pseudomanifold. Then there is a canonical torsion pairing*

$$T_i^{\bar{p}}(X)\times T^{\bar{q}}_{n-i-1}(X)\rightarrow \mathbf{Q}/\mathbf{Z} \qquad (*)$$

where $\bar{q}=\bar{t}-\bar{p}$. If X is also locally $\bar{p}$-torsion free then this pairing is nondegenerate.

Similarly, the answer to question (B) is:

THEOREM 8.1. *Suppose X is a locally $\bar{p}$-torsion free pseudomanifold. Let G be an abelian group. Then there is a natural exact sequence*

$$0 \to IH_i^{\bar{p}}(X) \otimes G \to IH_i^{\bar{p}}(X; G) \to \operatorname{Tor}(IH_{i-1}^{\bar{p}}(X), G) \to 0$$

For spaces which are not locally $\bar{p}$-torsion free there is a new torsion group $R_i^{\bar{p}}(X)$ which in some sense measures the degeneracy of the torsion pairing (*), i.e. there is a sequence

$$\cdots \to T_i^{\bar{p}}(X) \to \operatorname{Hom}(T_{n-i-1}^{\bar{q}}(X), \mathbf{Q}/\mathbf{Z}) \to R_i^{\bar{p}}(X) \to T_{i-1}^{\bar{p}}(X) \to \cdots$$

The group $R_i^{\bar{p}}(X)$ is a topological invariant of X and is the hypercohomology of a complex of sheaves which is supported on the singular set of X.

THEOREM 9.3. *For* any *compact oriented n dimensional pseudomanifold X there is a natural nondegenerate pairing*

$$R_i^{\bar{p}}(X) \otimes R_{n-i}^{\bar{q}}(X) \to \mathbf{Q}/\mathbf{Z}$$

where $\bar{q} = \bar{t} - \bar{p}$.

This pairing gives rise to a cobordism invariant characteristic class for certain singular spaces, which was first introduced in [S]:

Suppose X is a compact oriented $4k$ dimensional pseudomanifold with even dimensional strata (or, more generally suppose $IH_{l/2}^{\bar{m}}(L) = 0$ whenever L is the link of a stratum with odd codimensional $c = l + 1$). Then we have a nondegenerate rational pairing

$$I : IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \otimes IH_{2k}^{\bar{m}}(X; \mathbf{Q}) \to \mathbf{Q}$$

and a nondegenerate torsion pairing

$$K : R_{2k}^{\bar{m}}(X) \otimes R_{2k}^{\bar{m}}(X) \to \mathbf{Q}/\mathbf{Z}$$

THEOREM 11.3. *The Witt class (in $W(\mathbf{Q}/\mathbf{Z})$) of the pairing K is equal to the torsion part of the Witt class (in $W(\mathbf{Q})$) of the pairing I. This characteristic class is a cobordism invariant for cobordisms with even dimensional strata.*

This result suggests that the cobordism groups of the spaces (defined in §7.1) which satisfy Poincaré duality over the integers may coïncide with Mishchenko's higher Witt groups of $\mathbf{Z}$ (see [R]).

We are grateful to R. MacPherson for several valuable conversations and in particular for his suggestion that the "peripheral complex" $\underline{\underline{R}}_i^{\bar{p}}$ should be an interesting object to study. Many of the results in this paper have been worked out independently by P. Deligne.

§2. Notation

Our notation follows [GM2] and [GM3]. X will denote an n-dimensional compact oriented piecewise linear pseudomanifold with a P.L. stratification

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-2} = \Sigma = X_{n-1} \subset X_n = X$$

such that each point $x \in X_i - X_{i-1}$ has a neighborhood of the form $U = (i$-simplex$) \times$ cone (L) where L is the *link* of the stratum containing x.

The symbol $IH_i^{\bar{p}}(X)$ denotes the ith intersection homology group of X, with perversity $\bar{p} = (p_2, p_3, p_4, \ldots)$ where $p_c \leq p_{c+1} \leq p_c + 1$ and $p_2 = 0$. This group is canonically isomorphic to the hypercohomology group $\mathscr{H}^{-i}(\underline{\underline{IC}}_{\bar{p}}^{\cdot})$ of the complex of sheaves $\underline{\underline{IC}}_{\bar{p}}^{\cdot}$ which was constructed by Deligne [GM3]. It does not depend on the choice of P.L. structure or on the choice of stratification of X.

§3. Linking products in intersection homology

3.1. Let X be a compact n dimensional piecewise linear stratified pseudomanifold and suppose $\bar{p} + \bar{q} = \bar{r}$ are perversities as in [GM2] §1. We shall define a product

$$L : T_i^{\bar{p}}(X) \times T_j^{\bar{q}}(X) \to IH_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z}) \tag{1}$$

where $T_i^{\bar{p}}(X)$ denotes the torsion subgroup of $IH_i^{\bar{p}}(X)$. Let $\xi \in IC_i^{\bar{p}}(X)$ and $\eta \in IC_j^{\bar{q}}(X)$ be cycles which represent torsion classes $[\xi] \in T_i^{\bar{p}}(X)$ and $[\eta] \in T_j^{\bar{q}}(X)$. Then there are integers m_1 and m_2 and chains $\tilde{\xi} \in IC_{i+1}^{\bar{p}}(X)$, $\tilde{\eta} \in IC_{j+1}^{\bar{q}}(X)$ such that $\partial\tilde{\xi} = m_1\xi$ and $\partial\tilde{\eta} = m_2\eta$. We may choose ξ and $\tilde{\xi}$ so as to be dimensionally transverse to η and $\tilde{\eta}$ by [Mc].

Define $L([\xi], [\eta])$ to be the homology class of the intersection cycle $(1/m_1)\tilde{\xi} \cap \eta \in IC_{i+j-n-1}^{\bar{r}}(X; \mathbf{Q}/\mathbf{Z})$ (see [GM2] §2.1). It is easy to check that $L([\xi], [\eta])$ is well defined and is equal to the homology class of the cycle $(-1)^i(1/m_2)\xi \cap \tilde{\eta}$. Furthermore $L([\xi][\eta]) = (-1)^{(n-i)(n-j)}L([\eta], [\xi])$.

3.2. The torsion product for complementary dimensions $(i + j = n - 1)$ and perversities $(\bar{p} + \bar{q} = \bar{t} = (0, 1, 2, 3, \ldots))$ may also be constructed by the sheaf theoretic techniques of [GM3] from the intersection product, as follows: If $\underline{\underline{\mathbf{D}}}^{\bullet}_X$ denotes the dualizing complex on X, we have the product morphism

$$\underline{\underline{IC}}^{\bullet}_{\bar{p}} \otimes \underline{\underline{IC}}^{\bullet}_{\bar{q}} \to \underline{\underline{\mathbf{D}}}^{\bullet}_X[n]$$

and its adjoint

$$\underline{\underline{IC}}^{\bullet}_{\bar{p}} \to R\, \underline{\underline{\mathrm{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n] \tag{2}$$

Applying the hypercohomology functor $\mathcal{H}^{-i}$ and the universal coefficient theorem [B], we obtain a commuting diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
T^{\bar{p}}_i & \longrightarrow & \mathrm{Ext}\,(IH^{\bar{t}-\bar{p}}_{n-i-1}(X), \mathbf{Z}) \cong \mathrm{Hom}\,(T^{\bar{t}-\bar{p}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \\
\downarrow & & \downarrow \\
IH^{\bar{p}}_i(X) & \longrightarrow & \mathcal{H}^{-i}(X; R\, \underline{\underline{\mathrm{Hom}}}^{\bullet}\,(\underline{\underline{IC}}^{\bullet}_{\bar{t}-\bar{p}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n]) \\
\downarrow & & \downarrow \\
IH^{\bar{p}}_i/T^{\bar{p}}_i & \longrightarrow & \mathrm{Hom}\,(IH^{\bar{t}-\bar{p}}_{n-i}(X), \mathbf{Z}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The adjoint of the homomorphism on the top line is the desired product

$$L : T^{\bar{p}}_i \times T^{\bar{t}-\bar{p}}_{n-i-1} \to \mathbf{Q}/\mathbf{Z} \tag{3}$$

3.3. PROPOSITION. *The linking product* (3) *coincides with the augmented product* (1):

$$T^{\bar{p}}_i \times T^{\bar{q}}_j \to IH^{\bar{t}}_{i+j-n-1}(X; \mathbf{Q}/\mathbf{Z}) \to H_0(\text{point}, \mathbf{Q}/\mathbf{Z}) = \mathbf{Q}/\mathbf{Z} \tag{*}$$

when $j = n - i - 1$ *and* $\bar{q} = \bar{t} - \bar{p}$.

COROLLARY. *If the morphism* (2) *is a quasi-isomorphism, then the linking pairing* (*) *is nondegenerate.*

The proof of Prop. 3.3 is similar to the proof of Corollary 3.6 in [GM3] and will be omitted.

§4. Spaces for which the linking pairing is nondegenerate

4.1. DEFINITION. *A stratified pseudomanifold X is locally $\bar{p}$-torsion free if for each stratum of X we have*

$$T^{\bar{p}}_{q(c)}(L)=0 \tag{4}$$

where L is the link of the stratum, c is the codimension and $q(c)=c-2-p(c)$.

4.2. *Remark.* If X is a locally $\bar{p}$-torsion free space and L is the link of any stratum of X, then L is also a locally $\bar{p}$-torsion free space.

4.3. PROPOSITION. *X is locally $\bar{p}$-torsion free with respect to one stratification iff the same is true with respect to any refinement of that stratification.*

Proof. The link L' of a stratum in the refinement has the form of a join, $L'=S^k * L$ where L is the link of a stratum in the original stratification. We must verify that

$$T^{\bar{p}}_{r}(L')=0 \tag{4'}$$

where $r=l+k-1-p(l+k+1)$ and $l=\dim(L)$. For $k=0$, L' is the suspension of L and

$$IH^{\bar{p}}_i(\Sigma L)=\begin{cases} IH^{\bar{p}}_{i-1}(L) & \text{if} \quad i>l-p(l)-1 \\ 0 & \text{if} \quad i=l-p(l)-1 \\ IH^{\bar{p}}_i(L) & \text{if} \quad i<l-p(l)-1 \end{cases}$$

There are three possibilities: $p(l+2)=p(l)$, $p(l+2)=p(l)+1$, or $p(l+2)=p(l)+2$. In each case one calculates $T^{\bar{p}}_r(L)=0$ assuming (4) holds.

For $k>0$, equation (4′) may be verified by repeated application of the case $k=0$. Q.E.D.

4.4. THEOREM. *Suppose X is a compact n dimensional piecewise linear stratified pseudomanifold which is locally $\bar{p}$-torsion free. Then the morphism* (2) *is a quasi-isomorphism, so the linking pairing* (*) *is non-singular.*

Theorem 1 depends on a result from homological algebra which we now describe.

§5. Truncation of complexes

5.1. If $C^{\cdot}$ is a (cochain) complex of free abelian groups,

$$\cdots \xrightarrow{d} C^a \xrightarrow{d} C^{a+1} \xrightarrow{d} C^{a+2} \xrightarrow{d} \cdots$$

we denote by $\mathrm{Hom}^{\cdot}(C^{\cdot}, \mathbf{Z})$ the dual complex,

$$\mathrm{Hom}^b(C^{\cdot}, \mathbf{Z}) = \mathrm{Hom}(C^{-b}, \mathbf{Z}).$$

Deligne has defined truncation functors ([D1] [D2] [GM3])

$$(\tau_{\leq a} C^{\cdot})^n = \begin{cases} 0 & \text{if} \quad n > a \\ \ker d & \text{if} \quad n = a \\ C^n & \text{if} \quad n < a \end{cases}$$

$$(\tau^{\geq a} C^{\cdot})^n = \begin{cases} C^n & \text{if} \quad n > a \\ \operatorname{coker} d^{a-1} & \text{if} \quad n = a \\ 0 & \text{if} \quad n < a \end{cases}$$

It is easy to verify the following facts from homological algebra:

5.2. PROPOSITION. *Let $C^{\cdot}$ be a complex of free abelian groups. Then the following natural sequence is split exact*

$$0 \to \mathrm{Ext}(H^{-m+1}(C^{\cdot}), \mathbf{Z}) \to H^m(\mathrm{Hom}^{\cdot}(C^{\cdot}, \mathbf{Z})) \to \mathrm{Hom}(H^{-m}(C^{\cdot}), \mathbf{Z}) \to 0$$

Consequently,

$$H^a\,\mathrm{Hom}(\tau^{\geq b} C^{\cdot}, \mathbf{Z}) = \begin{cases} 0 & \text{if} \quad a \geq -b+2 \\ \mathrm{Ext}(H^b(C^{\cdot}), \mathbf{Z}) & \text{if} \quad a = -b+1 \\ H^a\,\mathrm{Hom}^{\cdot}(C^{\cdot}, \mathbf{Z}) & \text{if} \quad a \leq -b \end{cases}$$

$$H^a\,\mathrm{Hom}(\tau_{\leq b} C^{\cdot}, \mathbf{Z}) = \begin{cases} H^a(\mathrm{Hom}^{\cdot}(C^{\cdot}, \mathbf{Z})) & \text{if} \quad a \geq -b+1 \\ \mathrm{Hom}(H^b(C^{\cdot}), \mathbf{Z}) & \text{if} \quad a = -b \\ 0 & \text{if} \quad a \leq -b-1 \end{cases}$$

(The same result holds if we drop the hypothesis that $C^{\cdot}$ is free, and replace $\mathbf{Z}$ by its injective resolution $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$).

§6. Proof of Theorem 4.4

Let us assume by induction that the theorem has been proven for pseudomanifolds of dimension $\leq n-1$. Let X be a pseudomanifold of dimension n. The morphism (2) is clearly a quasi-isomorphism over the nonsingular part of X, so it suffices to verify that the complex of sheaves

$$\underline{\underline{S}}^{\cdot} = R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]$$

satisfies the axioms [AX1] of [GM3] §3.3, since these axioms uniquely determine $\underline{\underline{IC}}^{\cdot}_{\bar{p}}$. Specifically, we will verify the support axioms

$$[\mathrm{AX1}](c)\ \underline{\underline{H}}^m(\underline{\underline{S}}^{\cdot} \mid U_{k+1}) = 0 \qquad \text{for all } m > p(k) - n$$

$$[\mathrm{AX1}](d')\ \underline{\underline{H}}^m(j_k^!\underline{\underline{S}}^{\cdot} \mid U_{k+1}) = 0 \qquad \text{for all } m \leq p(k) - n + 1$$

where $j_k : Y = X_{n-k} - X_{n-k-1} \to U_{k+1} = X - X_{n-k-1}$ is the closed inclusion of the stratum with codimension k into U_{k+1}.

Verification of $[AX1](d')$. Let $\underline{\underline{\mathbf{D}}}^{\cdot}_Y$ and $\underline{\underline{\mathbf{D}}}^{\cdot}_{k+1}$ denote the dualizing complexes of Y and U_{k+1} respectively. Then

$$\begin{aligned} j_k^!\underline{\underline{S}}^{\cdot} &\cong \mathrm{dual}\ j_k^*\ \mathrm{dual}\ \underline{\underline{S}}^{\cdot} \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(j_k^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_Y)[n] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(j_k^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \underline{\underline{\mathbf{Z}}}_Y) \end{aligned}$$

These complexes are cohomologically locally constant on Y, so the stalk of the sheaf $R\,\underline{\underline{\mathrm{Hom}}}$ is the $R\,\mathrm{Hom}$ of the stalks. Let j_y denote the inclusion of a point $y \in Y$. Then the stalk cohomology at y is

$$\begin{aligned} \underline{\underline{H}}^m(j_k^!\underline{\underline{S}}^{\cdot})_y &= \mathrm{Hom}\,(j_y^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \mathbf{Z}) \\ &\cong \mathrm{Ext}\,(H^{-m+1}(j_y^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \mathbf{Z})) \oplus \mathrm{Hom}\,(H^{-m}(j_y^*\underline{\underline{IC}}^{\cdot}_{\bar{q}}[k-2n], \mathbf{Z}) \\ &= 0 \qquad \text{whenever } -m+k-2n > q(k)-n \text{ by } [\mathrm{AX1}](c) \text{ for } \underline{\underline{IC}}^{\cdot}_{\bar{q}}. \end{aligned}$$

This holds if $m \leq p(k) - n + 1$. (This verification did *not* use the assumption $T^{\bar{p}}_{q(k)}(L) = 0$).

Verification of $[AX1](c)$. We shall show the stalk cohomology over points $y \in Y$ satisfies the required vanishing condition

$$\begin{aligned} j_k^*\underline{\underline{S}}^{\cdot} \mid U_{k+1} &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(j_k^!\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_Y)[n] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(j_k^!\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{Z}}}_Y)[2n-c] \\ &\cong R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}\,(\tau^{\geq q(k-n)+1} j_k^* R i_* i^* \underline{\underline{IC}}^{\cdot}_{\bar{q}}[-1], \mathbf{Z})[2n-c] \end{aligned}$$

because of the distinguished triangle

$$\begin{tikzcd} Rj_{k*}j_k^!\underline{\underline{IC}}^{\cdot}_{\bar{q}} \mid U_{k+1} \arrow[r] & \underline{\underline{IC}}^{\cdot}_{\bar{q}} \mid U_{k+1} = \tau_{\le q(k)-n} Ri_* i^* \underline{\underline{IC}}^{\cdot}_{\bar{q}} \arrow[dl] \\ Ri_* i^* \underline{\underline{IC}}^{\cdot}_{\bar{q}} \mid U_{k+1} \arrow[u, "{[1]}"] & \end{tikzcd}$$

(Here, $i: U_k \to U_{k+1}$ is the inclusion.)

Thus the stalk at a point $y \in Y$ of $H^m(S^{\cdot})$ is

$$\underline{\underline{H}}^m(\underline{\underline{S}}^{\cdot}_y) = \begin{cases} H^{m+2n-c-1}(\operatorname{Hom}(j_y^* i_* i^* \underline{\underline{IC}}^{\cdot}_{\bar{q}}, \mathbf{Z})) & \text{if} \quad m \le p(k) - n \\ \operatorname{Ext}(H^{q-n+1}(j_y^* i_* i^* \underline{\underline{IC}}^{\cdot}_{\bar{q}}), \mathbf{Z}) & \text{if} \quad m = p(k) - n + 1 \\ 0 & \text{if} \quad m > p(k) - n + 1 \end{cases}$$

by Proposition 5.2.

Therefore axiom (c) will be satisfied iff the following group vanishes:

$$\operatorname{Ext}(H^{q-n+1}(j_y^* i_* i^* \underline{\underline{IC}}^{\cdot}_{q}), \mathbf{Z}).$$

This is isomorphic to $\operatorname{Ext}(IH^{\bar{q}}_{k-2-q(k)}(L), \mathbf{Z})$ by [GM3] §2.2. However the link L of the stratum Y is a $k-1$ dimensional pseudomanifold which is locally $\bar{q}$-torsion free (see Remark 4.2) so the theorem applies to L by induction and

$$T^{\bar{q}}_{c-2-q(c)}(L) \quad \text{is} \quad \mathbf{Q}/\mathbf{Z}\text{-dual to } T^{\bar{p}}_{q(c)}(L)$$

which is 0 by assumption.

§7. Spaces which satisfy Poincaré duality

In this section we describe a class of spaces such that the intersection homology group with middle perversity $\bar{m} = (0, 0, 1, 1, 2, 2, \ldots)$ satisfies Poincaré duality over the integers.

7.1. THEOREM. *Suppose the compact oriented n dimensional pseudomanifold X satisfies the following two conditions*:

(a) *For each stratum of odd codimension c,*

$$IH^{\bar{m}}_{(c-1)/2}(L; \mathbf{Z}) = 0$$

(b) *For each stratum of even codimension c,*

$T^{\bar{m}}_{c/2-1}(L)=0$

where L is the link of the stratum in question. Then there is a canonical split exact sequence

$$0 \to \operatorname{Hom}(T^{\bar{m}}_{i-1}(X), \mathbf{Q}/\mathbf{Z}) \to IH^{\bar{m}}_{n-i}(X;\mathbf{Z}) \to \operatorname{Hom}(IH^{\bar{m}}_i(X;\mathbf{Z}),\mathbf{Z}) \to 0$$

which is compatible with the intersection pairing and the linking pairing.

Proof. Conditions (a) and (b) guarantee (by Theorem 4.4) that the morphism induced by the product

$$\underline{\underline{IC}}^{\cdot}_{\bar{m}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n]$$

is a quasi-isomorphism, and condition (a) guarantees that the natural map $\underline{\underline{IC}}^{\cdot}_{\bar{m}} \to \underline{\underline{IC}}^{\cdot}_{\bar{n}}$ is a quasi-isomorphism (see the obstruction sequence argument in [S] or [GM3] §5.6). Together they imply that $\underline{\underline{IC}}_{\bar{m}}$ is self dual. The exact sequence is the universal coefficient theorem of [8].

7.2. *Remarks.* If X satisfies properties (a) or (b) above, with respect to one stratification then it satisfies the same properties with respect to any refinement of the stratification.

§8. Change of coefficients in intersection homology

In this section we will assume G is an abelian group. Recall that $IH^{\bar{p}}_i(X;G)$ is defined to be the ith homology group of the complex of chains $IC^{\bar{p}}_i(X;C)$ which consists of those $\xi \in C_i(X) \otimes G$ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable ([GM2] §6.3).

8.1. THEOREM. *Suppose X is a* P.L. *stratified pseudomanifold and for each stratum of X,*

$\operatorname{Tor}(IH^{\bar{p}}_{q(c)}(L), G)=0$

where L is the link of the stratum, c is its codimension, and $q(c)=c-2-p(c)$. Then *there is a canonical exact sequence*

$$0 \to IH_i^{\bar{p}}(X) \otimes G \to IH_i^{\bar{p}}(X; G) \to \operatorname{Tor}(IH_{i-1}^{\bar{p}}(X), G) \to 0$$

which is split.

Remark. If X is locally $\bar{p}$-torsion free then the hypothesis holds for any abelian group G.

8.2. *Proof.* $IH_i^{\bar{p}}(X; G)$ is the hypercohomology group $\mathcal{H}^{-i}(\underline{\underline{IC}}^{\cdot}_{\bar{p}}(\underline{\underline{G}}))$ of the complex of sheaves which is obtained by applying Deligne's construction to the constant sheaf G on $X-\Sigma$. We shall show that $\underline{\underline{IC}}^{\cdot}_{\bar{p}}(G)$ and $\underline{\underline{IC}}^{\cdot}_{\bar{p}} \otimes G$ are quasi-isomorphic under the hypotheses of the theorem. (The short exact sequence is then the statement of the universal coefficient theorem for the complex $\underline{\underline{IC}}^{\cdot}_{\bar{p}} \otimes G$).

The quasi-isomorphism is obtained by verifying the axioms [AX1] for the complex $\underline{\underline{IC}}^{\cdot}_{\bar{p}} \otimes G$. Since the verification is analogous to that in §6, we omit it here but remark that the relevant lemma from homological algebra is the following:

LEMMA. *Let $C^{\cdot}$ be a chain complex of free abelian groups. Then*

$$H^n\tau_{\leq a}(C^{\cdot} \otimes G) = \begin{cases} 0 & \textit{for} \quad n > a \\ H^n[(\tau_{\leq a}C^{\cdot}) \otimes G] \oplus \operatorname{Tor}(H^{n+1}(C^{\cdot}), G) & \textit{for} \quad n = a \\ H^n[(\tau_{\leq a}C^{\cdot}) \otimes G] & \textit{for} \quad n < a \end{cases}$$

§9. The peripheral complex $\underline{\underline{\mathbf{R}}}^{\cdot}_{\bar{p}}$

9.1. Let X be an n dimensional compact oriented pseudomanifold.

DEFINITION. $\underline{\underline{R}}^{\cdot}_{\bar{p}}$ is the (algebraic) mapping cone of the morphism (2). $R_i^{\bar{p}}(X)$ is the hypercohomology group $\mathcal{H}^{-i}(\underline{\underline{R}}^{\cdot}_{\bar{p}})$.

9.2. *Remarks.* (1) We have a distinguished triangle in $D^b(X)$,

$$\begin{tikzcd} \underline{\underline{IC}}^{\cdot}_{\bar{p}} \arrow[rr] & & R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n] \arrow[dl] \\ & \underline{\underline{\mathbf{R}}}^{\cdot}_{\bar{p}} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Thus, X is locally $\bar{p}$-torsion free if and only if $\underline{\underline{R}}^{\cdot}_{\bar{p}} \cong 0$.

(2) The cohomology sheaves associated to $\underline{\underline{R}}^{\bullet}_{\bar{p}}$ are supported on the singular set of X since the morphism (2) is a quasi-isomorphism over the nonsingular part of X.

(3) The hypercohomology groups $R_i^{\bar{p}}(X) = \mathcal{H}^{-i}(\underline{\underline{R}}^{\bullet}_{\bar{p}})$ are torsion groups since the morphism (2) becomes a quasi-isomorphism when both sides are tensored with the rationals (see [GM3] §5.3). Thus, the torsion sub-groups of the hypercohomology groups of the complexes in the above triangle can be identified as follows:

$$\cdots \longrightarrow T_i^{\bar{p}} \longrightarrow \operatorname{Hom}(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\alpha_i} R_i^{\bar{p}}(X) \xrightarrow{\beta_i} T^{\bar{p}}_{i-1} \longrightarrow \cdots$$

This sequence is exact except at $R_i^{\bar{p}}(X)$ (see diag. 11.3).

9.3. PROPOSITION. *There is a canonical nondegenerate pairing*

$$K : R_i^{\bar{p}}(X) \times R^{\bar{q}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}$$

such that $K(\alpha_i(a), b) = a \cdot \beta_{n-i}(b)$ *for all* $a \in \operatorname{Hom}(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z})$ *and all* $b \in R^{\bar{q}}_{n-i}$.

Proof. First we define K. The intersection product ([GM3] §5.2)

$$\underline{\underline{IC}}^{\bullet}_{\bar{p}} \otimes \underline{\underline{IC}}^{\bullet}_{\bar{q}} \to \underline{\underline{\mathbf{D}}}^{\bullet}_X[n]$$

induces a pair of adjoint morphisms

$$\phi_1 : \underline{\underline{IC}}^{\bullet}_{\bar{p}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{p}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n]$$
$$\phi_2 : \underline{\underline{IC}}^{\bullet}_{\bar{q}} \to R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{p}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n]$$

and $\underline{\underline{R}}^{\bullet}_{\bar{p}} =$ mapping cone (ϕ_1); $\underline{\underline{R}}^{\bullet}_{\bar{q}} =$ mapping cone (ϕ_2). Dualizing ϕ_2 gives rise to a pair of distinguished triangles

$$\begin{tikzcd}
 & \underline{\underline{R}}^{\bullet}_{\bar{p}} \arrow[dl, "{[1]}"'] \arrow[dd, dashed] & \\
\underline{\underline{IC}}^{\bullet}_{\bar{p}} \arrow[rr, "\phi_1"] \arrow[dd, "\text{biduality} \cong \text{isomorphism}"'] & & R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n] \arrow[ul] \arrow[dd, "\cong \text{identity}"] \\
 & R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{R}}^{\bullet}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n] \arrow[dl] & \\
R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}_{\bar{p}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X), \underline{\underline{\mathbf{D}}}^{\bullet}_X) \arrow[rr, "\phi_2^*"'] & & R\,\underline{\underline{\operatorname{Hom}}}^{\bullet}(\underline{\underline{IC}}^{\bullet}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\bullet}_X)[n] \arrow[ul, "{[1]}"']
\end{tikzcd}$$

From this diagram we obtain a quasi-isomorphism,

$$\underline{\underline{R}}^{\cdot}_{\bar{p}} \to R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{R}}^{\cdot}_{\bar{q}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)[n+1]$$

Applying hypercohomology and the universal coefficient theorem we obtain an isomorphism

$$\tilde{K}: R_i^{\bar{p}}(X) \to \mathrm{Hom}\,(R^{\bar{q}}_{n-i}(X), \mathbf{Q}/\mathbf{Z})$$

whose adjoint is the desired $K: R_i^{\bar{p}}(X) \otimes R^{\bar{q}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}$.

The compatibility of K with α and β is equivalent to the statement that the following diagram commutes:

$$\begin{tikzcd}
R_i^{\bar{p}}(X) \arrow[d, "\tilde{K}"'] & \mathrm{Hom}\,(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \arrow[l, "\alpha"'] \arrow[d, "\text{identity}"] \\
\mathrm{Hom}\,(R^{\bar{q}}_{n-i}(X), \mathbf{Q}/\mathbf{Z}) & \mathrm{Hom}\,(T^{\bar{q}}_{n-i-1}, \mathbf{Q}/\mathbf{Z}) \arrow[l, "\beta^*"']
\end{tikzcd}$$

However this diagram is simply the torsion in the hypercohomology of the right hand face of the preceding diagram.

9.4. EXAMPLE. If the singular set of X consists of a single stratum Σ of codimension c, then the stalk homology of $\underline{\underline{R}}^{\cdot}_{\bar{p}}$ is, for any $x \in \Sigma$,

$$\mathcal{H}^{-i}(\underline{\underline{R}}^{\cdot}_{\bar{p}})_x = \begin{cases} T_{c-2-p(c)}(L) & \text{if } i = n - p(c) - 1 \\ 0 & \text{if } i \neq n - p(c) - 1 \end{cases}$$

where L is the link of the stratum.

If X is obtained from an n-dimensional manifold M by attaching the cone on its boundary ∂M, then

$$R_i^{\bar{m}}(X) = \begin{cases} T_{[(n-1)/2]}(\partial M) & \text{if } i = [(n+1)/2] \\ 0 & \text{if } i \neq [(n+1)/2]^{\cdot} \end{cases}$$

where [ ] denotes the integer part. If $\dim(M) = 4k$ then the equivalence class in the Witt ring $W(\mathbf{Q}/\mathbf{Z})$ of the torsion pairing

$$T^{\bar{m}}_{2k-1}(\partial M) \times T^{\bar{m}}_{2k-1}(\partial M) \to \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* in [AHV].

§10. Spaces for which the peripheral complex is self dual

The canonical morphism $\underline{\underline{IC}}^{\cdot}_{\bar{p}} \to \underline{\underline{IC}}^{\cdot}_{\bar{q}}$ (where $\bar{p} \le \bar{q}$) induces canonical morphisms on the peripheral complexes, $\underline{\underline{R}}^{\cdot}_{\bar{p}} \to \underline{\underline{R}}^{\cdot}_{\bar{q}}$. The spaces for which $\underline{\underline{R}}^{\cdot}_{\bar{m}}$ is self dual over $\mathbf{Q}/\mathbf{Z}$ are the spaces such that $\underline{\underline{R}}^{\cdot}_{m} \to \underline{\underline{R}}^{\cdot}_{n}$ is a quasi-isomorphism.

10.1. THEOREM. *Suppose X is a compact oriented n dimensional pseudomanifold such that, for each stratum with* odd *codimension c,*

$$IH^{\bar{m}}_{(c-1)/2}(L;\mathbf{Z}) = 0$$

where L is the link of the stratum in question. Then $\underline{\underline{R}}^{\cdot}_{\bar{m}} \to \underline{\underline{R}}^{\cdot}_{\bar{n}}$ is a quasi-isormorphism so K induces a nondegenerate product

$$K : R^{\bar{m}}_i(X) \times R^{\bar{m}}_{n-i}(X) \to \mathbf{Q}/\mathbf{Z}.$$

Proof. The assumption implies $\underline{\underline{IC}}^{\cdot}_{\bar{m}} \to \underline{\underline{IC}}^{\cdot}_{\bar{n}}$ and $R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X) \to R\,\underline{\underline{\mathrm{Hom}}}^{\cdot}(\underline{\underline{IC}}^{\cdot}_{\bar{n}}, \underline{\underline{\mathbf{D}}}^{\cdot}_X)$ are quasi-isomorphisms (see [S] or [GM3] §5.6). Therefore the induced map $\underline{\underline{R}}^{\cdot}_{\bar{m}} \to \underline{\underline{R}}^{\cdot}_{\bar{n}}$ is also a quasi-isomorphism. Q.E.D.

10.2 DEFINITION. If X is a $4k$ dimensional space which satisfies the hypotheses of Theorem 9.1 then the equivalence class in $W(\mathbf{Q}/\mathbf{Z})$ of the pairing

$$K : R^{\bar{m}}_{2k}(X) \times R^{\bar{m}}_{2k}(X) \to \mathbf{Q}/\mathbf{Z}$$

is called the *peripheral invariant* of X. (Here $W(\mathbf{Q}/\mathbf{Z})$ is the Witt ring of $\mathbf{Q}/\mathbf{Z}$ and it consists of certain equivalence classes of symmetric $\mathbf{Q}/\mathbf{Z}$-valued pairings on finite abelian groups [MH]).

§11. Relation between the Witt class and the peripheral invariant

11.1. DEFINITION. An oriented pseudomanifold X is a rational Witt space if, for each stratum of X with odd codimension c,

$$IH^{\bar{m}}_{(c-1)/2}(L;\mathbf{Q}) = 0$$

where L is the link of that stratum. If $\dim(X) = 4k$ define $w(X) \in W(\mathbf{Q})$ to be the equivalence class (in the Witt ring of $\mathbf{Q}$) of the nondegenerate symmetric intersection pairing

$$IH^{\bar{m}}_{2k}(X;\ \mathbf{Q}) \times IH^{\bar{m}}_{2k}(X; \mathbf{Q}) \to \mathbf{Q}.$$

We recall the following fact from [S]:

THEOREM. *If X is a rational Witt space then $w(X)$ is a cobordism invariant (for cobordisms which are also rational Witt spaces). The association $X \mapsto w(x)$ determines an isomorphism.*

$$\Omega^{\text{Witt}}_{4k} \cong W(\mathbf{Q})$$
$$\Omega^{\text{Witt}}_{j} = 0 \quad \textit{if} \quad j \not\equiv 0 \pmod 4.$$

11.2. The structure of $W(\mathbf{Q})$ is given by the following split exact sequence [MH]:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(\mathbf{Z}) & \longrightarrow & W(\mathbf{Q}) & \xrightarrow{\delta} & W(\mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0 \\ & & \wr\Vert & & & & \wr\Vert & & \\ & & \mathbf{Z} & & & & \bigoplus_{p \text{ prime}} W(\mathbf{Z}/p\mathbf{Z}) & & \end{array}$$

11.3. THEOREM. *Suppose X is a $4k$ dimensional oriented pseudomanifold which satisfies the hypothesis of* §9.1, *i.e.,*

$$IH^{\bar{m}}_{(c-1)/2}(L; \mathbf{Z}) = 0$$

whenever L is the link of a stratum with odd codimension, c. Then X is a rational Witt space, and $\delta w(X)$ is equal to the peripheral invariant of X.

Proof. Consider the exact sequence on hypercohomology which is associated to the distinguished triangle of §9.2:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & T^{\bar{m}}_{2k}(X) \arrow[r] \arrow[d] & \operatorname{Hom}(T^{\bar{n}}_{2k-1}, \mathbf{Q}/\mathbf{Z}) \arrow[r, "\alpha"] \arrow[d] & R^{\bar{m}}_{2k} \arrow[r, "\beta"] \arrow[d, "\cong"] & T^{\bar{m}}_{2k-1}(X) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & IH^{\bar{m}}_{2k}(X) \arrow[r] \arrow[d] & IH^{2k}_{\bar{n}}(X) \arrow[r] \arrow[d] & R^{\bar{m}}_{2k} \arrow[r] \arrow[d] & IH^{\bar{m}}_{2k-1}(X) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & IH^{\bar{m}}_{2k}/T^{\bar{m}}_{2k} \arrow[r, "\theta"] \arrow[d] & \operatorname{Hom}(IH^{\bar{n}}_{2k}, \mathbf{Z}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & IH^{\bar{m}}_{2k-1}/T^{\bar{m}}_{2k-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

By [AHV] (Lemma 1.4), $\delta w(X)$ coincides with the Witt class of the induced pairing on $K = \operatorname{coker}(\theta)$. However, $K \cong \ker \beta/\operatorname{Im} \alpha$ since we may view the diagram above as a short exact sequence of chain complexes with the middle complex acyclic. But we have already shown (§9.4) that $(\ker \beta) = (\operatorname{Im} \alpha)^{\perp}$ so by [AHV] (Lemma 1.3), the Witt class of the pairing on $R^{\bar{m}}_{2k}(X)$ also coincides with the Witt class of the pairing on $\ker \beta/\operatorname{Im} \alpha$ Q.E.D.

Remark. The diagram and preceding argument may be found in [BM] in the case that X has isolated singularities.

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Poincaré duality groups of dimension two, II

BENO ECKMANN and PETER LINNELL

1. Introduction

A Poincaré duality group of dimension n, in short a PD^n-group, is a group G acting on $\mathbf{Z}$ such that one has natural isomorphisms

$$H^k(G; A) \cong H_{n-k}(G; \mathbf{Z} \otimes A)$$

for all integers k and all $\mathbf{Z}G$-modules A (where $\mathbf{Z} \otimes A$ is the tensor product over $\mathbf{Z}$ with diagonal G-action). G is called orientable or not according to whether or not $\mathbf{Z}$ is trivial as a $\mathbf{Z}G$-module. All "surface groups", i.e., fundamental groups of closed surfaces of genus ≥ 1 are well-known to be PD^2-groups. In Eckmann–Müller [4] it was proved that a PD^2-group with *positive* first Betti number β_1 is isomorphic to a surface group. The purpose of the present paper is to show that the condition on β_1 is automatically fulfilled:

THEOREM 1. *The first Betti number β_1 of a PD^2-group is positive.*

As a consequence we thus have a complete classification of PD^2-groups.

THEOREM 2. *A group G is a PD^2-group if and only if it is isomorphic to a surface group.*

For notations and properties concerning PD^n-groups, not explicitly mentioned here, we refer to [4] where also several (algebraic and topological) consequences are discussed.

2. Finitely generated projective $\mathbf{Z}G$-modules

For the proof of Theorem 1 we need the following fact, which may be of interest in connection with the conjectures of Bass (4.4 and 4.5 of [2]).

If B is an abelian group, we let rank B denote the dimension of the $\mathbf{Q}$-vector space $B \otimes \mathbf{Q}$.

PROPOSITION 3. *Let G be a PD^2-group, $M \neq 0$ a finitely generated projective $\mathbf{Z}G$-module, and $\mathbf{Z}$ the trivial $\mathbf{Z}G$-module. Then* $\operatorname{rank}(\mathbf{Z} \otimes_G M) \neq 0$.

Proof. Let r_M denote the Hattori-Stallings trace of the identity endomorphism of M as defined, e.g., in [1] and [2]. It is a finite linear combination with integral coefficients of the conjugacy classes τ in G,

$$r_M = \sum_{\tau} r_M(\tau)\tau.$$

For $x \in G$ let $r_M(x)$ be the coefficient of the conjugacy class of x. Suppose that $r_M(x) \neq 0$ for an element $x \in G$, $x \neq 1$. Then there exists, by Proposition 6.2 of [2], a prime p and an integer $n > 0$ such that x is conjugate to x^{p^n}. It follows (see the remark on p. 12 of [2]) that x is contained in a subgroup $H \cong \mathbf{Z}[1/p]$ of G. By Strebel's theorem [5] all subgroups of infinite index in G are of cohomological dimension 1 and thus free. Therefore H has finite index in G; since G is finitely generated so is H and we have a contradiction. Hence $r_M(x) = 0$ for all $x \in G \setminus 1$ and it follows that $r_M(1) = \operatorname{rank}(\mathbf{Z} \otimes_G M)$.

We now consider the nonzero finitely generated projective $\mathbf{C}G$-module $M \otimes \mathbf{C}$. We have $r_M(1) = r_{M \otimes \mathbf{C}}(1)$ which is positive by Kaplansky's theorem (see [1], Theorem 8.9), and the result follows.

3. Proof of Theorem 1. Euler characteristic

The completion of the proof is now in the same spirit as [3]. We first note that we can restrict attention to orientable PD^2-groups. Indeed (see [4], p. 511), if G is non-orientable and G_1 the orientable subgroup of index 2 in G then $\beta_1(G_1) > 0$ implies $\beta_1(G) > 0$.

So let G be an orientable PD^2-group, and

$$0 \to P \to \mathbf{Z}G^d \to \mathbf{Z}G \xrightarrow{\varepsilon} \mathbf{Z} \tag{1}$$

a projective resolution of the trivial $\mathbf{Z}G$-module $\mathbf{Z}$. Since PD^n-groups are of type (FP), the module P is finitely generated projective. Since $H^0(G; \mathbf{Z}G) = H^1(G; \mathbf{Z}G) = 0$ and $H^2(G; \mathbf{Z}G) = \mathbf{Z}$ with trivial G-action for any orientable

PD^2-group, applying $\mathrm{Hom}_G(-, \mathbf{Z}G)$ to (1) yields an exact sequence

$$\mathbf{Z} \overset{\gamma}{\twoheadleftarrow} P^* \leftarrow \mathbf{Z}G^d \leftarrow \mathbf{Z}G \leftarrow 0 \tag{2}$$

where $P^* = \mathrm{Hom}_G(P, \mathbf{Z}G)$ is finitely generated projective. Let IG be the kernel of ε (the augmentation ideal) and L the kernel of γ. Applying Schanuel's lemma to (1) and (2) gives

$$P^* \oplus IG \cong \mathbf{Z}G \oplus L.$$

There is a surjection $\mathbf{Z}G^d \twoheadrightarrow L$, and we obtain a surjection $\mathbf{Z}G^{d+1} \twoheadrightarrow P^* \oplus IG$ and hence a surjection $\mathbf{Z}G^{d+1} \twoheadrightarrow P^*$, with kernel $K \neq 0$. Obviously K is a finitely generated projective $\mathbf{Z}G$-module, and we see from Proposition 3 that $\mathrm{rank}\,(\mathbf{Z} \otimes_G K) \neq 0$. It follows that $\mathrm{rank}\,(\mathbf{Z} \otimes_G P^*) \leq d$.

The Euler characteristic $\chi(G)$ of G can be obtained by applying $\mathbf{Z} \otimes_G -$ to the resolution (2) and taking the alternating sum of the ranks:

$$\chi(G) = \mathrm{rank}\,(\mathbf{Z} \otimes_G P^*) - d + 1 \leq 1.$$

On the other hand $\chi(G) = \beta_0 - \beta_1 + \beta_2 = 2 - \beta_1$ since the Betti numbers β_0 and β_2 of an orientable PD^2-group are $= 1$. Thus $2 - \beta_1 \leq 1$, i.e., $\beta_1 > 0$.

4. Poincaré 2-complexes

As a corollary of the above group-theoretic results the topological application mentioned in [4], Section 2 can be given an improved version.

We recall that a Poincaré n-complex is a CW-complex dominated by a finite complex and fulfilling Poincaré duality of formal dimension n for arbitrary local coefficients. By results of Wall [6] a Poincaré 2-complex X with finite fundamental group $\pi_1(X)$ is homotopy equivalent to the 2-sphere or to the real projective plane; if $\pi_1(X)$ is infinite, then X is aspherical, i.e., an Eilenberg-MacLane complex $K(G, 1)$ for $G = \pi_1(X)$. In the latter case G is a PD^2-group, and thus by our Theorem 2 isomorphic to $\pi_1(Y)$ where Y is a closed surface of genus ≥ 1. The isomorphism $\pi_1(X) \cong \pi_1(Y)$ yields a homotopy equivalence between X and Y. In summary we have

THEOREM 4. *A CW-complex is a Poincaré 2-complex if and only if it is homotopy equivalent to a closed surface of genus ≥ 0.*

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Nilpotent completions and Lie rings associated to link groups

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§1. Introduction

The nilpotent completion and the Lie ring associated to a group with finitely generated abelianization are nilpotent invariants derived from its lower central series. In classical link theory, several authors have studied those for a link group, the fundamental group of the complement of a link, since it is much more practical rather than studying a group itself.

On the other hand, Sullivan gave a cohomological and infinitesimal method to compute these invariants when the group is the fundamental group of a polyhedron. Thus as he suggested in [17], Problem 5, it seems interesting to apply his theory to the link theory. In this paper, we are concerned with this vague question. Of course it is hopeless to expect complete algebraic characterization of these invariants for link groups, however it is possible to obtain some general results from infinitesimally computable cases. Such computations are attained in § 4 and § 5.

In § 4, we construct a minimal model for a polyhedron which is cohomologically equivalent to a bouquet of circles. We establish, as Corollary 6.3, the equivalence between the freeness of the nilpotent completion of its fundamental group and the vanishing of every Massey product on H^1. Now, a link complement can never be cohomologically equivalent to a bouquet of circles since H^2 is non trivial. However, to apply this construction, we do not need trivial H^2, but we do need just non-existence of decomposable elements in H^2, and it still has some significance in the link theory. Actually, Milnor [10] proved that the nilpotent completion of a link group is isomorphic to that of a free group iff all the $\bar{\mu}$-invariants vanish, and Porter [15] succeeded in expressing the $\bar{\mu}$-invariant in terms of the Massey product. In particular, we get the equivalence which is eventually a special case of Corollary 6.3.

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In § 5, we explicitly construct a family of minimal models for polyhedra which are cohomologically equivalent to the product of a bouquet of circles with a circle. In the special case where the polyhedron is the complement of a link, this cohomological condition is a condition on the linking numbers. Our construction asserts that the structure of the Lie ring associated to the fundamental group of such a polyhedron is very simple while the nilpotent completion is not. Corollary 6.4, which has been conjectured by K. Murasugi, came up as an application of the construction.

Besides these, several corollaries of the constructions are established in § 6. We review nilpotent completions and Lie rings associated to groups in § 2, and Sullivan's theory in § 3.

The content of § 4 is from my thesis supervised by Professor John Morgan. I would like to express my great appreciation for his constant encouragement.

§ 2. Nilpotent completions and Lie rings

Let G be a group and let $G=G_0\supset G_1\supset G_2\supset\cdots$ be the lower central series of G where $G_p=[G, G_{p-1}]$ for $p\geqslant 1$. Here are two invariants of G which come from the lower central series. The first one is the nilpotent completion of G. It is the tower of nilpotent groups:

$$\cdots\to G/G_2\to G/G_1\to\{e\}.$$

We will simply denote it by Nil(G). Nil(G) is said to be isomorphic to Nil(H) of a group H up to the pth stage if there is an isomorphism: $G/G_p\to H/H_p$. Then it induces an isomorphism: $G/G_q\to H/H_q$ for each $q\leqslant p$ and we get isomorphic towers up to the pth stage. We might say that Nil(G) is isomorphic to Nil(H) if these are isomorphic up to any stage.

Now, each G/G_p is a nilpotent group of index p and a central extension of G/G_{p-1} by the abelian group G_{p-1}/G_p. The second invariant is formed by these abelian groups. Let $\mathscr{L}_p(G)=G_{p-1}/G_p$ and $\mathscr{L}(G)=\oplus_{p\geqslant 1}\mathscr{L}_p(G)$. Then, the commutator operation determines a well defined bilinear mapping, $[\ ,\ ]:\mathscr{L}_p(G)\otimes\mathscr{L}_q(G)\to\mathscr{L}_{p+q}(G)$ such that

(1) $[\alpha,\beta]=-[\beta,\alpha]$ and

(2) $[[\alpha,\beta]\gamma]+[[\beta,\gamma]\alpha]+[[\gamma,\alpha]\beta]=0$.

Hence $\mathscr{L}(G)$ admits a graded Lie ring structure generated by $\mathscr{L}_1(G)$. See [7].

Both concepts have rational versions. Say, the rational nilpotent completion of G, which will be denoted by $\mathbb{Q}$-nil(G), is the tower of $\mathbb{Q}$-nilpotent groups:

$$\cdots\to G/G_2\otimes\mathbb{Q}\to G/G_1\otimes\mathbb{Q}\to\{e\},$$

where each $G/G_p \otimes \mathbb{Q}$ stands for the Malcev completion [8] of the nilpotent group G/G_p. Also taking tensor product by $\mathbb{Q}$ in usual sense, we get a graded Lie algebra $\mathscr{L}(G) \otimes \mathbb{Q}$ associated to G.

The first important result concerning the structure of $\mathscr{L}(G)$ may be one for a free group by Witt [18]. See also [7].

PROPOSITION 2.1 (Witt). *Let F_n be a free group of rank n. Then $\mathscr{L}(F_n)$ is a free Lie ring generated by n elements. Here, free means that there are no relations except those generated by* (1) *and* (2). *Furthermore, $\mathscr{L}_p(F_n)$ is a free abelian group of rank*

$$W(n, p) = \frac{1}{p} \sum_{d|p} \mu(d) n^{p/d},$$

where $\mu(d)$ is the Mobius function.

$W(n, p)$ is called the Witt number.

In general, the nilpotent completion is a stronger invariant than the associated Lie ring, however,

LEMMA 2.2. *For any $p \geqslant 1$, if $\mathscr{L}(G)$ is isomorphic to $\mathscr{L}(F_n)$ up to the* pth *stage, then* $\mathrm{Nil}(G)$ *is isomorphic to* $\mathrm{Nil}(F_n)$ *up to the* pth *stage.*

Proof. Since G/G_p is a nilpotent group for any $p \geqslant 1$, it is generated by n elements (see [7], Lemma 5.9) and hence we have an epimorphism $\phi: F_n \to G/G_p$ which induces an isomorphism: $\mathscr{L}(F_n) \to \mathscr{L}(G/G_p)$ up to the pth stage. Looking at the commutative diagram for $q \leqslant p$,

$$\begin{array}{ccccccccc}
1 \to & \mathscr{L}_q(F_n) & \to & F_n/(F_n)_q & \to & F_n/(F_n)_{q-1} & \to 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & \mathscr{L}_q(G/G_p) & \to & G/G_q & \to & G/G_{q-1} & \to 1,
\end{array}$$

we notice that ϕ induces an isomorphism: $F_n/(F_n)_q \to G/G_q$ until q becomes p by the five lemma and the induction on q.

LEMMA 2.3. *For any $p \geqslant 1$, if $\mathscr{L}(G) \otimes \mathbb{Q}$ is isomorphic to $\mathscr{L}(F_n) \otimes \mathbb{Q}$ up to the* pth *stage, then* $\mathbb{Q}$-$\mathrm{nil}(G)$ *is isomorphic to* $\mathbb{Q}$-$\mathrm{nil}(F_n)$ *up to the* pth *stage.*

Proof. Since for any $p \geqslant 1$, there is a homomorphism $\phi: F_n \to G/G_p$ which induces an isomorphism: $\mathscr{L}(F_n) \otimes \mathbb{Q} \to \mathscr{L}(G/G_p) \otimes \mathbb{Q}$ up to the pth stage, the same argument can be applied.

The next lemma will be used in § 5.

LEMMA 2.4. *Suppose that* $\mathcal{L}_1(G)$ *is generated by* $g_1, \ldots, g_{n+r}$ *such that*
(1) $[g_i, g_j] \neq 0$ *in* $\mathcal{L}_2(G)$ *for all* $i, j \leqslant n$ *and*
(2) $[g_i, g_{n+j}] = 0$ *in* $\mathcal{L}_2(G)$ *for all* $i, j \geqslant 1$.
Then $\mathcal{L}_p(G)$ *is generated by at most* $W(n, p)$ *elements. If* $\mathcal{L}_p(G)$ *is a free abelian group of rank* $W(n, p)$ *for all* $p \geqslant 2$, *then* $\mathcal{L}(G)$ *is isomorphic to* $\mathcal{L}(F_n \times \mathbb{Z}^r)$, *where* $\mathbb{Z}^r$ *is a free abelian group of rank* r

Proof. Let $h_1, \ldots, h_n$ be a basis of $\mathcal{L}_1(F_n)$ and define the homomorphism $\phi: \mathcal{L}_1(F_n) \to \mathcal{L}_1(G)$ by $\phi(h_i) = g_i$. Then ϕ naturally induces a homomorphism $\phi_p: \mathcal{L}_p(F_n) \to \mathcal{L}_p(G)$ for each p. What we want to show then is that ϕ_p is onto for $p \geqslant 2$.

Now, any element of $\mathcal{L}_p(G)$ can be written down as a linear combination of simple p-fold brackets of $g_1, \ldots, g_{n+r}$, like $[[\cdots[[g_{i_1} g_{i_2}]g_{i_3}]\cdots]g_{i_p}]$. Let us simply denote it by $(g_{i_1} \cdots g_{i_p})$. Suppose that $i_1, \ldots, i_p \leqslant n$, then this is the image of $(h_{i_1} \cdots h_{i_p})$ by ϕ_p. If $i_q > n$ for some $q < p$, then $(g_{i_1} \cdots g_{i_q}) = 0$ by induction hypothesis and therefore $(g_{i_1} \cdots g_{i_p}) = 0$. When $i_p > n$, We have the Jacobi identity,

$$(g_{i_1} \cdots g_{i_p}) = -((g_{i_{p-1}} g_{i_p})(g_{i_1} \cdots g_{i_{p-2}})) - ((g_{i_p}(g_{i_1} \cdots g_{i_{p-2}})g_{i_{p-1}}).$$

The both terms of the right side are zero in $\mathcal{L}_p(G)$, and we get $(g_{i_1} \cdots g_{i_p}) = 0$. Hence ϕ_p is an epimorphism for $p \geqslant 2$. If $\mathcal{L}_p(G)$ is a free abelian group of rank $W(n, p)$ for all $p \geqslant 2$, ϕ_p must be an isomorphism and we are done.

The rational version of this is also established.

LEMMA 2.5. *Suppose that* $\mathcal{L}_1(G) \otimes \mathbb{Q}$ *is generated by* $g_1, \ldots, g_{n+r}$ *such that*
(1) $[g_i, g_j] \neq 0$ *in* $\mathcal{L}_2(G) \otimes \mathbb{Q}$ *for all* $i, j \leqslant n$ *and*
(2) $[g_i, g_{n+j}] = 0$ *in* $\mathcal{L}_2(G) \otimes \mathbb{Q}$ *for all* $i, j \geqslant 1$.
Then $\mathcal{L}_p(G) \otimes \mathbb{Q}$ *is generated by at most* $W(n, p)$ *elements. If* $\dim \mathcal{L}_p(G) \otimes \mathbb{Q} = W(n, p)$ *for all* $p \geqslant 2$, *then* $\mathcal{L}(G) \otimes \mathbb{Q}$ *is isomorphic to* $\mathcal{L}(F_n \times \mathbb{Z}^r) \otimes \mathbb{Q}$.

§ 3. Differential graded algebras

A differential graded algebra $\mathcal{A}$ is a graded vector space $A = \bigoplus_{p \geqslant 0} A^p$ over a field (always $\mathbb{Q}$ in this paper) with differential $d: A^p \to A^{p+1}$ and associative multiplication $\wedge: A^p \otimes A^q \to A^{p+q}$ so that
(1) $d^2 = 0$,

(2) $d(x\wedge y)=dx\wedge y+(-1)^{\deg x}x\wedge dy$ and
(3) $x\wedge y=(-1)^{\deg x\deg y}y\wedge x$.

A d.g.a. is minimal if d is decomposable. This means that the image of any element by d can be written down as a sum of decomposable elements. A Hirsch extension of a d.g.a. $\mathscr{A}$ is an inclusion $\mathscr{A}\to\mathscr{B}$ of a d.g.a.'s which, when we ignore the differentials, is isomorphic to $\mathscr{A}\to\mathscr{A}\otimes\Lambda(V)^p$ and where the differential of $\mathscr{B}$ sends $V\to A^{p+1}$. The integer p is the degree of the extension. From now on, we consider a series of Hirsch extensions:

$$\mathbb{Q}\subset\mathscr{A}_1\subset\mathscr{A}_2\subset\cdots\subset\bigcup_{p\geq 1}\mathscr{A}_p=\mathscr{A}$$

of degree 1. We should point out here that a d.g.a. generated by elements of degree 1 is always minimal, so is $\mathscr{A}$. The series is called canonical if $\mathscr{A}_1$ is generated by all closed 1-forms of $\mathscr{A}$ and $\mathscr{A}_{p+1}$ is generated by $\mathscr{A}_p$ and all 1-forms x such that $dx\in\mathscr{A}_p$ for each p. The following lemma, which is an immediate consequence of the definition, characterizes a canonical series.

LEMMA 3.1. *If* $\mathscr{A}:\mathbb{Q}\subset\mathscr{A}_1\subset\mathscr{A}_2\subset\cdots$ *is canonical, then*
(1) $H^1(\mathscr{A}_p)\to H^1(\mathscr{A}_{p+1})$ *is an isomorphism for all* p *and hence* $H^1(\mathscr{A})=H^1(\mathscr{A}_1)$, *and*
(2) $H^2(\mathscr{A}_p)\to H^2(\mathscr{A}_{p+1})$ *is a monomorphism if we restrict it to the image of* $H^2(\mathscr{A}_{p-1})$.

Let us now consider the 1-minimal model of Sullivan. Let X be a polyhedron and let $\varepsilon(X)$ be $\mathbb{Q}$-polynomial forms on X. The 1-minimal model for X is a minimal d.g.a. $\mathscr{M}_X$ with a mapping $\rho:\mathscr{M}_X\to\varepsilon(X)$ of d.g.a.'s such that $\rho^*:H(\mathscr{M}_X)\to H(\varepsilon(X))$ is an isomorphism in degree 1 and injective in degree 2. We can find for instance in [11] § 5 how to construct $\mathscr{M}_X$ and its several properties. It turns out to be generated by elements of degree 1 and to have a canonical series:

$$\mathscr{M}_X:\mathbb{Q}\subset\mathscr{M}_1\subset\mathscr{M}_2\subset\cdots$$

Dualizing the part of degree 1, we get a tower of $\mathbb{Q}$-Lie algebras:

$$0\leftarrow\mathfrak{G}_1\leftarrow\mathfrak{G}_2\leftarrow\cdots$$

Each of the Lie algebras $\mathfrak{G}_i$ is nilpotent and hence the Campbell-Hausdorff formula defines a group structure C.H.$(\mathfrak{G}_i)$ on each $\mathfrak{G}_i$.

THEOREM (Sullivan). *If* X *is arcwise connected and* $H^1(\varepsilon(X))$ *is finite* *dimensional, then the tower of nilpotent groups:*

$$\{e\} \leftarrow \text{C.H.}(\mathfrak{G}_1) \leftarrow \text{C.H.}(\mathfrak{G}_2) \leftarrow \cdots$$

is isomorphic to $\mathbb{Q}$-nil$(\pi_1(X))$.

Thus knowing the rational nilpotent completion of $\pi_1(X)$ is equivalent to knowing the 1-minimal model for X. The proof of this theorem can be found in [2].

Let $\mathcal{M}_X : \mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ be the 1-minimal model for a polyhedron X and suppose that $\mathcal{M}_p$ is isomorphic to $\mathcal{M}_{p-1} \otimes \Lambda(V_p)$ as a vector space. Sullivan's theorem implies

$$\dim V_p = \operatorname{rank} \mathcal{L}_p(\pi_1(X)).$$

When $\dim H^1(\varepsilon(X)) = n$, the number above is bounded by the Witt number $W(n, p)$. Since $\mathcal{L}(\pi_1(X)) \otimes \mathbb{Q}$ is free up to the pth stage if and only if $\dim \mathcal{L}_q(\pi_1(X)) \otimes \mathbb{Q} = W(n, q)$ for all $q \leq p$, we have by Lemma 2.3 that

LEMMA 3.2. *If* $\dim V_q = W(n, q)$ *for all* $q \leq p$, *then* $\mathbb{Q}$-nil$(\pi_1(X))$ *is isomorphic to* $\mathbb{Q}$-nil(F_n) *up to the* pth *stage.*

This can be also proved by constructing isomorphisms for extensions of each stage.

LEMMA 3.3. *If* $\mathcal{L}_1(\pi_1(X))$ *is a free abelian group of rank n, then* $\dim V_q = W(n, q)$ *for all* $q \leq p$ *iff* Nil$(\pi_1(X))$ *is isomorphic to* Nil(F_n) *up to the* pth *stage.*

Proof. Since $\mathcal{L}_1(\pi_1(X))$ is generated by n elements, $\mathcal{L}_q(\pi_1(X))$ is generated by at most $W(n, q)$ elements by Proposition 2.1. Thus if $\dim V_q = W(n, q)$ for each $q \leq p$ and hence $\dim \mathcal{L}_q(\pi_1(X)) \otimes \mathbb{Q} = W(n, q)$, then $\mathcal{L}_q(\pi_1(X))$ must be a free abelian group of rank $W(n, q)$, which means that $\mathcal{L}(\pi_1(X))$ is isomorphic to $\mathcal{L}(F_n)$ up to the pth stage. The result follows from Lemma 2.2.

The next lemma will be used in § 5.

LEMMA 3.4. *Suppose that* $\mathcal{L}_1(\pi_1(X))$ *admits a system of generators as in Lemma* 2.5. *Then if* $\dim V_p = W(n, p)$ *for all* $p \geq 2$, *then* $\mathcal{L}(G) \otimes \mathbb{Q}$ *is isomorphic to* $\mathcal{L}(F_n \times \mathbb{Z}^r) \otimes \mathbb{Q}$.

Proof. Since $\dim V_p = \dim \mathcal{L}_p(\pi_1(X)) \otimes \mathbb{Q}$, this is a corollary of Lemma 2.5.

LEMMA 3.5. *Suppose that $\mathscr{L}_1(\pi_1(X))$ is a free abelian group of rank $n+r$ and admits a system of generators as in Lemma 2.4. Then if* $\dim V_p = W(n, p)$ *for all* $p \geq 2$, *then $\mathscr{L}(\pi_1(X))$ is isomorphic to $\mathscr{L}(F_n \times \mathbb{Z}^r)$.*

Proof. Since $\dim V_p = W(n, p)$ means that $\mathscr{L}_p(\pi_1(X))$ is a free abelian group of rank $W(n, p)$ in this case, this is a corollary of Lemma 2.4.

§ 4. The 1-minimal model for $S^1 \vee \cdots \vee S^1$

Our goal of this section is to construct the 1-minimal model for a cohomology bouquet of n circles.

Let A_p be the vector space over $\mathbb{Q}$ generated by the n^p elements, $x_{i_1 \cdots i_p}$'s where $i_1 \cdots i_p$ ranges over all sequences of integers of length p such that $1 \leq i_j \leq n$ for all $1 \leq j \leq p$. Consider the exterior algebra of the direct sum $A = \bigoplus_{p \geq 1} A_p$. We define the differential d by

$$dx_{i_1 \cdots i_p} = \sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p},$$

on a basis of A and extend it linearly to all of A and then extend it to all of $\Lambda(A)$ by the Leibnitz rule. Then

LEMMA 4.1. $d^2 = 0$

Proof. It suffices to check this for a generator.

$$\begin{aligned} d(dx_{i_1 \cdots i_p}) &= d\left(\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p}\right) \\ &= \sum_{k=1}^{p-1} \sum_{m=1}^{k-1} x_{i_1 \cdots i_m} \wedge x_{i_{m+1} \cdots i_k} \wedge x_{i_{k+1} \cdots i_p} \\ &\quad - \sum_{k=1}^{p-1} \sum_{m=k+1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_m} \wedge x_{i_{m+1} \cdots i_p} \\ &= \left(\sum_{m=1}^{p-1} \sum_{k=1}^{m-1} - \sum_{k=1}^{p-1} \sum_{m=k+1}^{p-1}\right) x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_m} \wedge x_{i_{m+1} \cdots i_p} \\ &= 0. \end{aligned}$$

Let I_p be the subspace of A_p $(p \geq 2)$ inductively defined by $\{u \in A_p;\ du = 0$ in $\Lambda^2(A_1 \oplus A_2/I_2 \oplus \cdots \oplus A_{p-1}/I_{p-1})\}$, denote A_p/I_p by $\bar{A}_p$ and also denote $A_1 \oplus$

$\bar{A}_2 \oplus \cdots \oplus \bar{A}_p$ by $\bar{\bar{A}}_p$. Then, $\mathcal{M}_p = \Lambda(\bar{\bar{A}}_p)$ with the induced differential (we use the same symbol d) produces a series of Hirsch extensions of minimal d.g.a.'s:

$$\mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \bigcup_{p \geq 1} \mathcal{M}_p = \mathcal{M}$$

of degree 1. Our first claim is

LEMMA 4.2. *The inclusion induces an isomorphism*: $H^1(\mathcal{M}_{p-1}) \to H^1(\mathcal{M}_p)$ *for all* $p > 1$.

Proof. We use the induction on p. Suppose that this is true for $p-1$, which means that any closed 1-form of $\mathcal{M}_{p-1}$ is contained in $\mathcal{M}_1$. Now, $\mathcal{M}_p = \mathcal{M}_{p-1} \otimes \Lambda(\bar{A}_p)$ as a vector space and since I_p is nothing but the kernel of $d|_{A_p}: A_p \to \Lambda^2(\bar{\bar{A}}_{p-1})$, the induced differential $d|_{\bar{A}_p}: \bar{A}_p \to \Lambda^2(\bar{\bar{A}}_{p-1})$ is injective. The image of this is contained in $\oplus_{i+j=p} \bar{A}_i \wedge \bar{A}_j$, however the image of $\Lambda^1(\bar{\bar{A}}_{p-1})$ by d is contained in $\oplus_{i+j<p} \bar{A}_i \wedge \bar{A}_j$, and they have no common points except zero. In other words, the Hirsch extension $\mathcal{M}_{p-1} \subset \mathcal{M}_p$ does not produce new closed 1-forms.

Let W_p be the image of A_p by d in $\Lambda^2(\bar{\bar{A}}_{p-1})$. That is to say, W_p is a subspace of $\Lambda^2(\bar{\bar{A}}_{p-1})$ generated by the closed 2-forms, $\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p}$'s. Since the subspace of exact forms in $\Lambda^2(\bar{\bar{A}}_{p-1})$ is contained in $\oplus_{i+j<p} \bar{A}_i \wedge \bar{A}_j$, W_p can be identified with a subspace of $H^2(\mathcal{M}_{p-1})$, and also since W_p is the image of A_p by d, it is mapped to 0 in $H^2(\mathcal{M}_p)$ by the inclusion.

LEMMA 4.3. $\dim \bar{A}_p \geq W(n, p)$.

Proof. Define the multiplication $\cdot$ on A by

$$x_{i_1 \cdots i_p} \cdot x_{j_1 \cdots j_q} = x_{i_1 \cdots i_p j_1 \cdots j_q}.$$

Then, A becomes an associative but not commutative graded algebra. The usual bracket operation on A:

$$[x, y] = x \cdot y - y \cdot x,$$

defines a graded Lie algebra structure on A. Let L be the graded Lie subalgebra of A generated by $x_1, \ldots, x_n$. Then, L_p, the intersection of L and A_p, is the set of Lie elements of degree p.

The symmetric group $\mathfrak{S}_p$, which consists of the permutations of integers $1, \ldots, p$, naturally acts on A_p by $\sigma x_{i_1 \cdots i_p} = x_{i_{\sigma(1)} \cdots i_{\sigma(p)}}$ for $\sigma \in \mathfrak{S}_p$. This extends to the action of the group ring $\mathbb{Q}[\mathfrak{S}_p]$ on A_p. We now define a specific element $\Omega_p \in \mathbb{Q}[\mathfrak{S}_p]$ in terms of the cyclic permutations $\sigma_j = (12 \cdots j)$ for $j = 2, \ldots, p$, by

$$\Omega_p = (1-\sigma_2)(1-\sigma_3^2) \cdots (1-\sigma_p^{p-1}).$$

Ω_p then determines a linear mapping: $A_p \to A_p$.

It is known that $\mathcal{L}_p(F_n)$ is generated by simple brackets $(g_{i_1} \cdots g_{i_p})$, where $g_1, \ldots, g_n$ are generators of F_n, and the mapping:

$$\mathcal{L}_p(F_n) \otimes \mathbb{Q} \to L_p \subset A_p$$

$$(g_{i_1} \cdots g_{i_p}) \qquad \Omega_p x_{i_1 \cdots i_p}$$

is an isomorphism. See for instance [7], Theorem 5.12. In particular the linear mapping Ω_p maps A_p onto L_p and we have

(1) $\operatorname{rank} \Omega_p = \dim L_p = W(n, p)$.

Take an element $u = \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p} x_{i_1 \cdots i_p}$ of A_p where the summation ranges over all sequences of length p, and let us compute a necessary condition for $du = 0$ in $\Lambda^2(\bar{\bar{A}}_{p-1})$, i.e. $u \in I_p$. Suppose that $du = 0$ there. Then since

$$\begin{aligned} du &= \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p}\, dx_{i_1 \cdots i_p} \\ &= \sum_{i_1 \cdots i_p} a^{i_1 \cdots i_p}(x_{i_1} \wedge x_{i_2 \cdots i_p} + \cdots + x_{i_1 \cdots i_{p-1}} \wedge x_{i_p}) \\ &= \sum_{i_p} \Bigl(\sum_{i_1 \cdots i_{p-1}} ((1-\sigma_p^{p-1}) a^{i_1 \cdots i_p}) x_{i_1 \cdots i_{p-1}} \Bigr) \wedge x_{i_p} \\ &\quad + \Bigl(\text{the terms contained in } \bigoplus_{i,j \geq 2} \bar{A}_i \wedge \bar{A}_j \Bigr), \end{aligned}$$

if we let

$$u_{i_p} = \sum_{i_1 \cdots i_{p-1}} ((1-\sigma_p^{p-1}) a^{i_1 \cdots i_p}) x_{i_1 \cdots i_{p-1}},$$

u_{i_p} must be an element of I_{p-1} for all $1 \leq i_p \leq n$. Repeating the same procedure $p-1$ times, we eventually obtain the condition that $\Omega_p a^{i_1 \cdots i_p} = (1-\sigma_2)(1-\sigma_3^2) \cdots (1-\sigma_p^{p-1}) a^{i_1 \cdots i_p} = 0$ for all sequences $i_1 \cdots i_p$.

We now think of the conjugate element $\bar{\Omega}_p$ of $\Omega_p \in \mathbb{Q}[\mathfrak{S}_p]$ by the conjugation $\sigma \leftrightarrow \sigma^{-1}$. Again $\bar{\Omega}_p$ determines a linear mapping: $A_p \to A_p$ which can be identified with the induced mapping of Ω_p on the dual space $A_p^* = \mathrm{Hom}(A_p, \mathbb{Q})$, and we have

$$\begin{aligned}\bar{\Omega}_p u &= \sum_{i_1\cdots i_p} a^{i_1\cdots i_p}\bar{\Omega}_p x_{i_1\cdots i_p}\\ &= \sum_{i_1\cdots i_p} (\Omega_p a^{i_1\cdots i_p}) x_{i_1\cdots i_p}.\end{aligned}$$

Suppose that $d\bar{\Omega}_p u = 0$ in $\Lambda^2(\bar{\bar{A}}_{p-1})$, then recalling the formula $\Omega_p^2 = p\Omega_p$ in $\mathbb{Q}[\mathfrak{S}_p]$ (see [7], p. 365) and the necessary condition above, we get

$$\Omega_p(\Omega_p a^{i_1\cdots i_p}) = p\Omega_p a^{i_1\cdots i_p} = 0$$

for all sequences $i_1 \cdots i_p$. This means nothing but $\bar{\Omega}_p u$ being zero itself and hence the restriction of d to the image of $\bar{\Omega}_p$, $d|_{\bar{\Omega}_p(A_p)} : \bar{\Omega}_p(A_p) \to W_p \subset \Lambda^2(\bar{\bar{A}}_{p-1})$, is injective. In particular we have

(2) $\dim \bar{A}_p = \dim W_p \geqslant \dim \bar{\Omega}_p(A_p)$.

On the other hand, we have

(3) $\dim \bar{\Omega}_p(A_p) = \mathrm{rank}\, \bar{\Omega}_p = \mathrm{rank}\, \Omega_p$.

Combining (1), (2) and (3), we complete the proof.

The main result of this section is

THEOREM 4.4. *Let X be a polyhedron whose cohomology ring with rational coefficients is isomorphic to $H^*(S^1 \vee \cdots \vee S^1; \mathbb{Q})$. Then $\mathcal{M}: \mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ is isomorphic to the canonical series of the* 1*-minimal model $\mathcal{M}_X$ for X.*

Proof. We prove this by induction on the length of a series. Suppose that $\mathbb{Q} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_{p-1}$ is isomorphic to the $p-1$st stage of the canonical series of $\mathcal{M}_X$. Then we have a d.g.a. mapping $\rho_{p-1}: \mathcal{M}_{p-1} \to \varepsilon(X)$ such that $\rho_{p-1}(x_{i_1\cdots i_q}) = \omega_{i_1\cdots i_q}$ for $q \leqslant p-1$, and $H^2(\mathcal{M}_{p-2}) \to H^2(\mathcal{M}_{p-1})$ is a zero map by the property of a canonical series, Lemma 3.1, (2). Also since $\dim H^1(\mathcal{M}_1) = n$ and $H^2(\mathcal{M}_{p-1})$ is generated by decomposable elements, Sullivan's theorem implicitly says that $\dim H^2(\mathcal{M}_{p-1})$ cannot exceed $\dim \mathcal{L}_p(F_n) \otimes \mathbb{Q} = W(n, p)$, that is

(1) $W(n, p) \geqslant \dim H^2(\mathcal{M}_{p-1})$.

To see that $\mathcal{M}_{p-1} \subset \mathcal{M}_p$ is isomorphic to the pth stage of the canonical series of $\mathcal{M}_X$, we need to construct a d.g.a. mapping $\rho_p: \mathcal{M}_p \to \varepsilon(X)$ and to show that the restriction of $H^2(\mathcal{M}_p) \to H^2(\varepsilon(X))$ to the image of $H^2(\mathcal{M}_{p-1})$ is injective. First of all, since $\rho_{p-1}(\sum_{k=1}^{p-1} x_{i_1\cdots i_k}\wedge x_{i_{k+1}\cdots i_p})=\sum_{k=1}^{p-1}\omega_{i_1\cdots i_k}\wedge\omega_{i_{k+1}\cdots i_p}$ is a closed 2-form of $\varepsilon(X)$ and $H^2(\varepsilon(X))=0$, there exists a 1-form $\omega_{i_1\cdots i_p}$ of $\varepsilon(X)$ so that

$$d\omega_{i_1\cdots i_p}=\sum_{k=1}^{p-1}\omega_{i_1\cdots i_k}\wedge\omega_{i_{k+1}\cdots i_p}.$$

We define $\rho_p:\mathcal{M}_p\to\varepsilon(X)$ as an extension of ρ_{p-1} by mapping $x_{i_1\cdots i_p}$ to $\omega_{i_1\cdots i_p}$. Since $d|_{\bar{A}_p}:\bar{A}_p\to W_p$ is an isomorphism and W_p can be identified with a subspace of $H^2(\mathcal{M}_{p-1})$, by Lemma 4.3 we have

$$\dim H^2(\mathcal{M}_{p-1})\geqslant\dim W_p=\dim\bar{A}_p\geqslant W(n,p).$$

These inequalities become equalities by (1) and W_p can be identified with $H^2(\mathcal{M}_{p-1})$ itself. Since W_p was the image of A_p by d, $H^2(\mathcal{M}_{p-1})\to H^2(\mathcal{M}_p)$ turns out a zero mapping, and we are done by induction.

Remark. Since $\dim\bar{A}_p=W(n,p)$, $\mathbb{Q}$-nil$(\pi_1(X))$ turns out to be isomorphic to $\mathbb{Q}$-nil(F_n) by Lemma 3.2. This consequence also follows from the result of [16].

§ 5. The 1-minimal model for $(S^1\vee\cdots\vee S^1)\times S^1$

In this section, we consider a family of minimal models for polyhedra which are cohomologically equivalent to the product of a bouquet of n circles with a circle.

We define the vector space B_1 over $\mathbb{Q}$ by adding one more generator, x_{n+1}, to A_1 and let B_p be equal to A_p for $p\geqslant 2$. The specific basis, $x_{i_1\cdots i_p}$'s, of A_p determines a homomorphism: $A_p\to A_p^*=\mathrm{Hom}(A_p,\mathbb{Q})$ and let I_p^* be the image of I_p by this mapping. Consider the subset $\Delta_p=\{x\in A_p;\ f(x)=0$ for all $f\in I_p^*\}$. Choose n elements from Δ_p for each $p\geqslant 3$ to form a set θ which will be called a system of twisting coefficients. Let us denote by $\theta_j^{i_1\cdots i_p}$ the coefficient of the $x_{i_1\cdots i_p}$-component of the jth element of degree p in θ. The system of twisting coefficients with $\theta_j^{i_1\cdots i_p}=0$ for all $1\leqslant j\leqslant n$, $1\leqslant i_1\cdots i_p\leqslant n$ and $p\geqslant 3$, will be denoted by 0. We now define the differential d_θ by

$$d_\theta x_{i_1\cdots i_p}=\sum_{k=1}^{p-1}x_{i_1\cdots i_k}\wedge x_{i_{k+1}\cdots i_p}-\sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\theta_j^{i_k\cdots i_{k+m}}x_{i_1\cdots i_{k-1}ji_{k+m+1}\cdots i_p}\wedge x_{n+1}$$

on a basis of $B=\bigoplus_{p\geqslant 1}B_p$ first of all and extend it to all of the exterior algebra $\Lambda(B)$ by linearity and the Leibniz rule. Notice that d_0 is the same as d in § 4 for $p\geqslant 2$.

LEMMA 5.1. $d_\theta^2=0$

Proof. It suffices to show this for a generator.

$$d_\theta^2 x_{i_1\cdots i_p} = d_\theta(d_0 x_{i_1\cdots i_p}) - \sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m} \theta_j^{i_k\cdots i_{k+m}} d_\theta(x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\wedge x_{n+1}).$$

Since $d_0^2=0$ by Lemma 4.1, the first term of the right side becomes

$$\begin{aligned} &d_\theta(d_0 x_{i_1\cdots i_p})\\ &=\sum_{s=1}^{p-1}((d_\theta-d_0)x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_p} - x_{i_1\cdots i_s}\wedge(d_\theta-d_0)x_{i_{s+1}\cdots i_p})\\ &=\sum_{s=1}^{p-1}\sum_{j=1}^{n}\left(\sum_{m=2}^{s-1}\sum_{k=1}^{s-m}\theta_j^{i_k\cdots i_{k+m}} x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p}\right.\\ &\quad\left.+\sum_{m=2}^{p-s-1}\sum_{k=s+1}^{p-m}\theta_j^{i_k\cdots i_{k+m}} x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\right)\wedge x_{n+1}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} &d_\theta(x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\wedge x_{n+1})\\ &=d_0 x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\wedge x_{n+1}\\ &=\left(\sum_{s=1}^{k-1} x_{i_1\cdots i_s}\wedge s_{i_{s+1}\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\right.\\ &\quad\left.+\sum_{s=k+m}^{p-1} x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p}\right)\wedge x_{n+1}, \end{aligned}$$

the second term of the first identity becomes

$$\begin{aligned} &\sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\theta_j^{i_k\cdots i_{k+m}} d_\theta(x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\wedge x_{n+1})\\ &=\sum_{j=1}^{n}\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\theta_j^{i_k\cdots i_{k+m}}\left(\sum_{s=1}^{k-1} x_{i_1\cdots i_s}\wedge x_{i_{s+1}\cdots i_{k-1} j i_{k+m+1}\cdots i_p}\right.\\ &\quad\left.+\sum_{s=k+m}^{p-1} x_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_s}\wedge x_{i_{s+1}\cdots i_p}\right)\wedge x_{n+1} \end{aligned}$$

Thus since

$$\sum_{s=1}^{p-1}\sum_{m=2}^{s-1}\sum_{k=1}^{s-m}=\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\sum_{s=k+m}^{p-1}$$

and

$$\sum_{s=1}^{p-1}\sum_{m=2}^{p-s-1}\sum_{k=s+1}^{p-m}=\sum_{m=2}^{p-1}\sum_{k=1}^{p-m}\sum_{s=1}^{k-1},$$

both are cancelled each other and we are done.

Let J_p be the subspace of B_p $(p\geqslant 2)$ inductively defined by $\{u\in B_p;\, d_\theta u=0$ in $\Lambda^2(B_1\oplus B_2/J_2\oplus\cdots\oplus B_{p-1}/J_{p-1})\}$. We simply denote it without θ because J_p actually does not depend on θ as we will see in Lemma 5.3. Again denote B_p/J_p by $\bar{B}_p$ and $B_1\oplus\bar{B}_2\oplus\cdots\oplus\bar{B}_p$ by $\bar{\bar{B}}_p$. Then $\mathcal{M}_p^\theta=\Lambda(\bar{\bar{B}}_p)$ with the induced differential (we use the same symbol d_θ) produces a series of Hirsch extensions of d.g.a.'s:

$$\mathbb{Q}\subset\mathcal{M}_1^\theta\subset\mathcal{M}_2^\theta\subset\cdots$$

of degree 1. Let us denote $\bigcup_{p\geqslant 1}\mathcal{M}_p^\theta$ by $\mathcal{M}^\theta$. Then

LEMMA 5.2. *The inclusion induces an isomorphism:* $H^1(\mathcal{M}_{p-1}^\theta)\to H^1(\mathcal{M}_p^\theta)$ *for all* $p\geqslant 2$.

LEMMA 5.3. J_p *is equal to* I_p. *In other words,* $d_0u=0$ *in* $\Lambda^2(\bar{\bar{B}}_{p-1})$ *for some* $u\in B_p$ *iff* $d_\theta u=0$ *in* $\Lambda^2(\bar{\bar{B}}_{p-1})$. *In particular,* $\dim\bar{B}_p=\dim\bar{A}_p=W(n,p)$ *for* $p\geqslant 2$.

Both lemmas are obvious when $p=2$. We prove these by mixed induction on p. Let us assume that both are true for $p-1$.

Proof of Lemma 5.3. Suppose that u is an element of B_p so that $d_0u=0$ in $\Lambda^2(\bar{\bar{B}}_{p-1})$. By the definition of d_θ, we can decompose $d_\theta u$ as

$$d_\theta u=d_0u+\nabla\wedge x_{n+1}$$

where ∇ is an element of $\Lambda^1(\bar{\bar{B}}_{p-1})$. Since $d_\theta^2=0$ and $d_0u=0$, we have

$$0=d_\theta^2u=d_\theta(\nabla\wedge x_{n+1})=d_\theta\nabla\wedge x_{n+1}=d_0\nabla\wedge x_{n+1},$$

which implies that $d_0\nabla=0$. Because ∇ was in $\Lambda^1(\bar{\bar{B}}_{p-1})$, we get $d_\theta\nabla=0$ by induction hypothesis. Since we also assumed that Lemma 5.2 is true for $p-1$, any closed 1-form of $\mathcal{M}_{p-1}^\theta$ is contained in $\mathcal{M}_1^\theta$, in particular so is ∇. Therefore if we let $\theta_j^p=\sum_{i_1\cdots i_p}\theta_j^{i_1\cdots i_p}x_{i_1\cdots i_p}$, then $\nabla=\sum_{j=1}^n u^*(\theta_j^p)x_j$. However since $u^*\in I_p^*$ and $\theta_j^p\in\Delta_p$, $u^*(\theta_j^p)$ must be zero for all $1\leqslant j\leqslant n$, which means that $\nabla=0$ itself. The converse is obvious and we are done

Proof of Lemma 5.2. Let $U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1})$ be the image of B_p by d_θ. W_p of the last section can be naturally identified with a subspace of $\bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j$, and we have the commutative diagram:

$$\begin{tikzcd} B_p \arrow[r, "d_\theta"] \arrow[dr, "d_0"'] & U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1}) \arrow[d] \\ & W_p \subset \bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j \end{tikzcd}$$

where the vertical line is the projection to the direct summand. Then since J_p is the kernel of $d_\theta|_{B_p}: B_p \to \Lambda^2(\bar{\bar{B}}_{p-1})$ and J_p is equal to I_p by Lemma 5.3,

$$\begin{tikzcd} \bar{B}_p \arrow[r, "d_\theta"] \arrow[dr, "d_0"'] & U_p^\theta \subset \Lambda^2(\bar{\bar{B}}_{p-1}) \arrow[d] \\ & W_p \subset \bigoplus_{i+j=p} \bar{B}_i \wedge \bar{B}_j \end{tikzcd}$$

becomes the commutative diagram of isomorphisms. In particular, U_p^θ and $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$ have no common points except zero. And since $\mathcal{M}_p^\theta = \mathcal{M}_{p-1}^\theta \otimes \Lambda(\bar{B}_p)$ and the image of $\Lambda^1(\bar{\bar{B}}_{p-1})$ by d_θ is contained in $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$, the Hirsch extension $\mathcal{M}_{p-1}^\theta \subset \mathcal{M}_p^\theta$ produces no new closed 1-forms.

LEMMA 5.4. *The image of* $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_2^\theta)$ *is injectively mapped to* $H^2(\mathcal{M}^\theta)$.

Proof. U_p^θ and $\bigoplus_{i+j<p} \bar{B}_i \wedge \bar{B}_j$ have no common points except zero, and hence the new exact 2-forms of $\mathcal{M}_p^\theta$ have no common points with $\bigoplus_{i+j=2} \bar{B}_i \wedge \bar{B}_j$ except zero for $p \geqslant 3$. Since the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_2^\theta)$ is actually generated by $x_i \wedge x_{n+1}$'s by the definition of d_θ, these do not become exact in $\mathcal{M}_p^\theta$ for any $p \geqslant 3$ and hence in $\mathcal{M}^\theta = \bigcup_{p \geqslant 1} \mathcal{M}_p^\theta$

The main theorem of this section is

THEOREM 5.5. *Let X be a polyhedron whose cohomology ring with rational coefficients is isomorphic to* $H^*((\overbrace{S^1 \vee \cdots \vee S^1}^{n}) \times S^1; \mathbb{Q})$. *Then there exists a system of twisting coefficients* θ *so that* $\mathcal{M}^\theta: \mathbb{Q} \subset \mathcal{M}_1^\theta \subset \mathcal{M}_2^\theta \subset \cdots$ *is isomorphic to the canonical series of the* 1*-minimal model* $\mathcal{M}_X$ *for X.*

Proof. By the assumption, there are 1-forms $\omega_1, \ldots, \omega_{n+1}$ of $\varepsilon(X)$ which generate $H^1(\varepsilon(X))$ such that

(i) $[\omega_i \wedge \omega_j] = 0$ for all $i, j \leq n$ and

(ii) $[\omega_i \wedge \omega_{n+1}]$'s form a basis of $H^2(\varepsilon(X))$.

The dual basis $g_1, \ldots, g_{n+1}$ of $\mathscr{L}_1(\pi_1(X))$ with respect to $\omega_1, \ldots, \omega_{n+1}$ satisfies the conditions of Lemma 2.5. We then prove this theorem by induction on the length of a series.

Suppose that $\mathbb{Q} \subset \mathcal{M}_1^\theta \subset \cdots \subset \mathcal{M}_{p-1}^\theta$ is isomorphic to the $p-1$st stage of the canonical series of $\mathcal{M}_X$ for some θ. Notice that since $\mathcal{M}_{p-1}^\theta = \Lambda(\bar{\bar{B}}_{p-1})$, we only need a system of twisting coefficients up to degree $p-1$. Then we have a d.g.a. mapping $\rho_{p-1}: \mathcal{M}_{p-1}^\theta \to \varepsilon(X)$ so that $\rho_{p-1}(x_{i_1 \cdots i_q}) = \omega_{i_1 \cdots i_q}$ for $q \leq p-1$, and the image of $H^2(\mathcal{M}_{p-2}^\theta) \to H^2(\mathcal{M}_{p-1}^\theta)$ is equal to the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_{p-1}^\theta)$ by the inclusions because of Lemma 3.1, (2), Lemma 5.4 and the structure of $H^2(\varepsilon(X))$. Also by Lemma 2.5 and Sullivan's theorem, we have

(1) $W(n, p) \geq \dim H^2(\mathcal{M}_{p-1}^\theta) - n$

where n means the dimension of $H^2(\varepsilon(X))$.

To see that $\mathcal{M}_{p-1}^\theta \subset \mathcal{M}_p^\theta$ for some θ is isomorphic to the pth stage of the canonical series of $\mathcal{M}_X$, we need to find appropriate n elements of Δ_p for θ, to construct a d.g.a. mapping $\rho_p: \mathcal{M}_p^\theta \to \varepsilon(X)$ and to show that the restriction of $H^2(\mathcal{M}_p^\theta) \to H^2(\varepsilon(X))$ to the image of $H^2(\mathcal{M}_{p-1}^\theta)$ is injective. First of all, since

$$\begin{aligned}
&\rho_{p-1}\Bigg(\sum_{k=1}^{p-1} x_{i_1 \cdots i_k} \wedge x_{i_{k+1} \cdots i_p} \\
&\quad - \sum_{j=1}^{n} \left(\sum_{m=2}^{p-1} \sum_{k=1}^{p-m} - \begin{pmatrix} k = 1 \text{ and} \\ m = p-1 \end{pmatrix}\right) \theta_j^{i_k \cdots i_{k+m}} x_{i_1 \cdots i_{k-1} j i_{k+m+1} \cdots i_p} \wedge x_{n+1}\Bigg) \\
&= \sum_{k=1}^{p-1} \omega_{i_1 \cdots i_k} \wedge \omega_{i_{k+1} \cdots i_p} \\
&\quad - \sum_{j=1}^{n} \left(\sum_{m=2}^{p-1} \sum_{k=1}^{p-m} - \begin{pmatrix} k = 1 \text{ and} \\ m = p-1 \end{pmatrix}\right) \theta_j^{i_k \cdots i_{k+m}} \omega_{i_1 \cdots i_{k-1} j i_{k+m+1} \cdots i_p} \wedge \omega_{n+1}
\end{aligned}$$

is a closed 2-form of $\varepsilon(X)$ for each $i_1 \cdots i_p$, and $H^2(\varepsilon(X))$ is generated by $\omega_i \wedge \omega_{n+1}$'s, it is cohomologous to

$$\sum_{j=1}^{n} \theta_j^{i_1 \cdots i_p} \omega_j \wedge \omega_{n+1}$$

for some $\{\theta_j^{i_1 \cdots i_p}\}_{j=1}^n$. For each j, $\sum_{i_1 \cdots i_p} \theta_j^{i_1 \cdots i_p} x_{i_1 \cdots i_p} \in A_p$ must be contained in Δ_p because ρ_{p-1} is a d.g.a. mapping. Adding $\theta_j^{i_1 \cdots i_p}$'s to θ, we get a system of twisting coefficients up to degree p. Then

$$\sum_{k=1}^{p-1} \omega_{i_1\cdots i_k} \wedge \omega_{i_{k+1}\cdots i_p}$$
$$-\sum_{j=1}^{n} \sum_{m=2}^{p-1} \sum_{k=1}^{p-m} \theta_j^{i_1\cdots i_p} \omega_{i_1\cdots i_{k-1} j i_{k+m+1}\cdots i_p} \wedge \omega_{n+1}$$

becomes an exact form and there exists a 1-form $\omega_{i_1\cdots i_p}$ of $\varepsilon(X)$ such that $d\omega_{i_1\cdots i_p}$ is equal to it, where d is the differential of $\varepsilon(X)$. Mapping $x_{i_1\cdots i_p}$ to $\omega_{i_1\cdots i_p}$, we define $\rho_p: \mathcal{M}_p^\theta \to \varepsilon(X)$ as an extension of ρ_{p-1}.

We finally show that the image of $H^2(\mathcal{M}_{p-1}^\theta) \to H^2(\mathcal{M}_p^\theta)$ is equal to the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_p^\theta)$ because if so, the proof is completed by Lemma 5.4. Since U_p^θ can be identified with a subspace of $H^2(\mathcal{M}_{p-1}^\theta)$ and has no common points with the image of $H^2(\mathcal{M}_1^\theta)$ except zero, we have by Lemma 5.3 that

$$\dim H^2(\mathcal{M}_{p-1}^\theta) - n \geq \dim U_p^\theta = \dim \bar{B}_p = W(n, p).$$

Thus by (1), the inequality becomes an equality and $H^2(\mathcal{M}_{p-1}^\theta)$ can be identified with the direct sum of the image of $H^2(\mathcal{M}_1^\theta)$ and U_p^θ. Since U_p^θ is the image of B_p by d_θ, the image of $H^2(\mathcal{M}_{p-1}^\theta) \to H^2(\mathcal{M}_p^\theta)$ turns out the image of $H^2(\mathcal{M}_1^\theta) \to H^2(\mathcal{M}_p^\theta)$, and we are done.

Here are corollaries of Theorem 5.5, Lemma 3.4 and Lemma 3.5.

COROLLARY 5.6. *Let X be a polyhedron such that $H^*(X; \mathbb{Q})$ is isomorphic to $H^*((S^1 \vee \cdots \vee S^1) \times S^1; \mathbb{Q})$ as a ring. Then $\mathcal{L}(\pi_1(X)) \otimes \mathbb{Q}$ is isomorphic to $\mathcal{L}(F_n \times \mathbb{Z}) \otimes \mathbb{Q}$.*

COROLLARY 5.7. *Let X be a polyhedron as in Corollary 5.6. If $\mathcal{L}_p(\pi_1(X))$ is free abelian for $p = 1$ and 2, then $\mathcal{L}(\pi_1(X))$ is isomorphic to $\mathcal{L}(F_n \times \mathbb{Z})$.*

Proof. Since $\mathcal{L}_1(\pi_1(X))$ is free abelian, we can choose a set of generators of $\mathcal{L}_1(\pi_1(X))$ as in Lemma 2.5. Furthermore since $\mathcal{L}_2(\pi_1(X))$ is also free abelian, it satisfies the conditions in Lemma 2.4. Thus this is an corollary of Theorem 5.5 and Lemma 3.5.

Remark. The condition of this corollary seems equivalent to saying that X is an integral cohomology $(S^1 \vee \cdots \vee S^1) \times S^1$ while I have no proof for this.

§ 6. Applications

To state corollaries of Theorem 4.4, following Kraines [6], let us define the Massey product on the first cohomology group. Given elements $\gamma_1, \ldots, \gamma_p \in H^1(\varepsilon(X))$, suppose that a collection of 1-forms $S = \{\omega_{ij} \in \varepsilon(X);\ 1 \leqslant i \leqslant j \leqslant p,\ j - i < p - 1\}$ satisfies the conditions

(1) ω_{ii} is a closed form representing γ_i for $1 \leqslant i \leqslant p$, and

(2) $d\omega_{ij} = \sum_{k=i}^{j-1} \omega_{ik} \wedge \omega_{k+1,j}$ if $i < j$.

Then the $\mathbb{Q}$-polynomial 2-form $\sum_{k=1}^{p-1} \omega_{1k} \wedge \omega_{k+1,p}$ turns out to be closed. We call S a defining system. The Massey product $\langle \gamma_1, \ldots, \gamma_p \rangle$ is defined as a subset of $H^2(\varepsilon(X))$ consisting of all elements produced by such systems. When $p = 2$, it is nothing but the wedge (cup) product. The Massey product $\langle \gamma_1, \ldots, \gamma_p \rangle$ will be understood as a cohomology class if it contains a single element. It is known that if any $(p-1)$-tuple Massey product on $H^1(\varepsilon(X))$ vanishes, that is, contains only the zero element, then every p-tuple Massey product contains a single element. See [9], Proposition 2.4 for the proof. We now have equivalent conditions for the vanishing of every p-tuple Massey products.

LEMMA 6.1. *Every p-tuple Massey product vanishes iff every q-tuple Massey product for any $1 < q \leqslant p$ vanishes.*

Proof. If every p-tuple Massey product vanishes, then for each $1 < q \leqslant p$, every q-tuple Massey product must contain the zero element. Thus any binary Massey product vanishes because it has no indeterminacy. Assume by induction that every $(q-1)$-tuple Massey product on $H^1(\varepsilon(X))$ vanishes, then every q-tuple Massey product contains a single element which is zero and we are done.

Here are corollaries of Theorem 4.4.

COROLLARY 6.2. *Let X be a polyhedron of* $\dim H^1(\varepsilon(X)) = n$. *Then, every p-tuple Massey product on $H^1(\varepsilon(X))$ vanishes iff $\mathbb{Q}$-nil$(\pi_1(X))$ is isomorphic to* $\mathbb{Q}$-nil(F_n) *up to the* pth *stage.*

Proof. To construct the 1-minimal model for X, we can use the vanishing of Massey products instead of the vanishing of $H^2(\varepsilon(X))$. Actually, the closed 2-forms in $\mathcal{M}_q$ were generated by $\sum_{k=1}^{q-1} x_{i_1\cdots i_k} \wedge x_{i_{k+1}\cdots i_q}$'s for $q \leqslant p$ which are mapped to $\sum_{k=1}^{q-1} \omega_{i_1\cdots i_k} \wedge \omega_{i_{k+1}\cdots i_q}$ of $\varepsilon(X)$ by ρ_q. This is nothing but the Massey product $\langle \omega_{i_1}, \ldots, \omega_{i_q} \rangle$.

Conversely, suppose that, for some $q \leqslant p$, some q-tuple Massey product does not vanish while every r-tuple Massey product does vanish for all $1 < r < q$. Then

$\rho_q^*: H^2(\mathcal{M}_q) \to H^2(\varepsilon(X))$ is not a zero map and rank $\mathcal{L}_q(\pi_1(X)) = \dim \operatorname{Ker} \rho_q^*$ is strictly less than $\dim H^2(\mathcal{M}_q) = W(n, q)$. Thus $\mathbb{Q}$-nil$(\pi_1(X))$ cannot be isomorphic to $\mathbb{Q}$-nil(F_n) at the qth stage.

By virtue of Lemma 3.3, we have

COROLLARY 6.3. *Let X be a polyhedron such that $\mathcal{L}_1(\pi_1(X))$ is a free abelian group of rank n, then every p-tuple Massey product on $H^1(\varepsilon(X))$ vanishes iff* Nil$(\pi_1(X))$ *is isomorphic to* Nil(F_n) *up to the* pth *stage.*

Remark. If we start with the Massey product on $H^1(\pi_1(X))$, this has been known by Dwyer [4], Corollary 4.5. I would like to thank the referee for pointing out this reference. For the link complement, there are much more detailed studies by Milnor [10] and Porter [15].

Remark. In [14] and [5], some higher intersectional properties of compact 4-manifolds have been detected by the nilpotent completion and the Massey product respectively. This corollary shows that these results are equivalent.

Let $L = K_1 \cup \cdots \cup K_n$ be a link of n components in S^3. Then $H^1(S^3 - L; \mathbb{Q})$ is generated by the Alexander duals ξ_i to the component K_i for $i = 1, \ldots, n$, and $H^2(S^3 - L; \mathbb{Q})$ is generated by the Lefshetz duals γ_{ij} to the path which connects K_i with K_j. These are subject to the relations in H^*:

$$\xi_i \wedge \xi_j = lk(K_i, K_j)\gamma_{ij}$$

and

$$\gamma_{ij} + \gamma_{jk} = \gamma_{ik}.$$

The next corollary has been conjectured by Murasugi.

COROLLARY 6.4. *Let G be a link group, $\pi_1(S^3 - L)$. If $lk(K_i, K_j) = 1$ for all $i \neq j$, then $\mathcal{L}(G)$ is isomorphic to $\mathcal{L}(F_{n-1} \times \mathbb{Z})$. In particular, rank $\mathcal{L}_p(G) = W(n-1, p)$ for all $p \geqslant 2$.*

Proof. Let $\omega_i = \xi_i - \xi_n$ for $i = 1, \ldots, n-1$ in this case. Then $\omega_i \wedge \omega_j = 0$ for all $i, j \leqslant n-1$ and $\omega_i \wedge \xi_n$'s form a basis of $H^2(S^3 - L: \mathbb{Q})$, and hence $S^3 - L$ is clearly a rational cohomology $(S^1 \vee \cdots \vee S^1) \times S^1$. Also $\mathcal{L}_1(\pi_1(X))$ and $\mathcal{L}_2(\pi_1(X))$ are free abelian because of Alexander duality and Chen's computations [3], Corollary 2, respectively. Thus we can apply Corollary 5.7 to this case.

COROLLARY 6.5. *Let L be a link of* 3 *component such that linking numbers of any two components are zero. Then* $\mathscr{L}(G) \otimes \mathbb{Q}$ *is isomorphic to* $\mathscr{L}(F_2 \times \mathbb{Z}) \otimes \mathbb{Q}$.

Proof. Let $\omega_1 = lk(K_2, K_3)\xi_1 - lk(K_1, K_2)\xi_3$ and $\omega_2 = lk(K_1, K_3)\xi_2 - lk(K_1, K_2)\xi_3$. Then $\omega_1 \wedge \omega_2 = 0$ and $\omega_1 \wedge \xi_3$, $\omega_2 \wedge \xi_3$ form a basis of $H^2(S^3 - L; \mathbb{Q})$ and hence $S^3 - L$ is a rational cohomology $(S^1 \vee S^1) \times S^1$. Applying Corollary 5.6, we are done.

$\mathscr{L}(G)$ is nilpotent if $\mathscr{L}_p(G) = 0$ for some p. This is equivalent to G/G_ω being nilpotent, where $G_\omega = \bigcap_{p \geq 1} G_p$.

COROLLARY 6.6. *Let G be the link group of a link L. Then*

(1) $\mathscr{L}(G)$ *is nilpotent iff either L is a knot or L is of* 2 *components whose mutual linking number is equal to* ± 1.

(2) $\mathscr{L}(G) \otimes \mathbb{Q}$ *is nilpotent iff either L is a knot or L is of* 2 *components whose mutual linking number is not zero.*

Proof. When L is a knot, $\mathscr{L}(G)$ is nilpotent of index 1 since $S^3 - L$ is a homology circle. If L has two components, then "if" part is obvious for both cases, (1), (2), because $\mathscr{L}_2(G)$ is isomorphic to a cyclic group of order $= |lk(K_1, K_2)|$. To see "only if" part, recall Murasugi's explicit computation [12] of the Chen groups. That is, roughly speaking, the Chen group $\mathrm{Ch}_p(G) = G_{p-1}[G_1, G_1]/G_p[G_1, G_1]$ of a 2 component link group G is infinite for all $p \geq 1$ if $lk(K_1, K_2) = 0$, and is nontrivially finite for all $p \geq 2$ if $lk(k_1, K_2) \neq 0, \pm 1$. Since there is an epimorphism of $\mathscr{L}_p(G)$ to $\mathrm{Ch}_p(G)$ for each p, $\mathscr{L}(G)$ cannot be nilpotent except when $lk(K_1, K_2) = \pm 1$. Also $\mathscr{L}(G) \otimes \mathbb{Q}$ cannot be nilpotent except when $lk(K_1, K_2) \neq 0$.

Let us think of the case where L has more than 3 components. When L contains two components whose mutual linking number is zero, then by forgetting the other components, we get a homomorphism of G onto the group of a link of 2 components whose mutual linking number is zero. When the linking numbers of any 2 components of L are not zero, then by forgetting some components, we get a homomorphism of G onto the group of a link of 3 components as in Corollary 6.5. Thus $\mathscr{L}(G) \otimes \mathbb{Q}$ cannot be nilpotent in either case. Of course neither does $\mathscr{L}(G)$ and we are done.

Remark. This remark was pointed out by Murasugi. It can be known by [1] and [12] that a link group itself is nilpotent iff it is abelian. Such a link must be either a trivial knot or a Hopf link by Dehn's lemma and Neuwirth's theorem [13].

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Local homology of groups of volume-preserving diffeomorphisms, II

DUSA MCDUFF*

§1. Introduction

This is the second in a series of three papers on the local homology of groups of volume-preserving diffeomorphisms: see [9], [10]. It may be read independently of the other two papers since it uses none of their results or methods of proof. Here is a statement of the main theorem. (A slightly sharper version is stated in §2.) We will explain later how it is related to the results of [9], [10].

We consider a compact, oriented, smooth manifold W_1 with boundary ∂W_1. Let $W_0 \subset W_1$ be the complement of an open collar neighbourhood of ∂W_1. If ω is a volume form on Int W_1, we write $\mathrm{Diff}_\omega\,(W_i, \mathrm{rel}\,\partial)$ for the discrete group of all ω-preserving diffeomorphisms of W_i which are the identity near ∂W_i. Clearly $\mathrm{Diff}_\omega\,(W_0, \mathrm{rel}\,\partial) \subset \mathrm{Diff}_\omega\,(W_1, \mathrm{rel}\,\partial)$.

THEOREM 1. *The inclusion* $\mathrm{Diff}_\omega\,(W_0, \mathrm{rel}\,\partial) \subset \mathrm{Diff}_\omega\,(W_1, \mathrm{rel}\,\partial)$ *induces an isomorphism on (untwisted) integer homology.*

This theorem holds for any volume form on Int W_1. In particular, taking $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $\mathbf{R}^n$, we see that the inclusion of the group of ω-preserving diffeomorphisms of $\mathbf{R}^n$ with support in the open unit disc into the group of compactly supported ω-preserving diffeomorphisms of $\mathbf{R}^n$ is a homology isomorphism.

Observe also that if we were considering the group $\mathrm{Diff}\,(W_i, \mathrm{rel}\,\partial)$ of all, not necessarily volume-preserving, diffeomorphisms with support in Int W_i, then the above result follows easily from the fact that $G_i = \mathrm{Diff}\,(W_i, \mathrm{rel}\,\partial)$ is the union of subgroups G_{ij}, where

$$G_{01} \subset G_{02} \subset \cdots \subset G_0 = G_{11} \subset G_{12} \subset \cdots \subset G_1,$$

and where, for any i, j, there is $g \in G_1$ which commutes with G_{0i} and conjugates

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G_{1j} to G_0. However these conjugating maps cannot preserve volume, and so one cannot argue in this way in the volume-preserving case.

The main application of Theorem 1 is to the study of the "local homology" of groups of volume-preserving diffeomorphisms. Recall from [7] that if $\mathscr{G}$ is a topological group whose underlying discrete group is G, then the homotopy fiber $\bar{B}\mathscr{G}$ of the natural map $BG \to B\mathscr{G}$ depends only on the algebraic and topological structure of a neighbourhood of the identity in $\mathscr{G}$. Therefore the homology of the space $\bar{B}\mathscr{G}$ is called the "local homology of $\mathscr{G}$ at the identity." When $\mathscr{G}$ is a Lie group, it follows from van Est's theorem that the "differentiable" part of its local cohomology is just the cohomology of the Lie algebra of $\mathscr{G}$. See [4]. If $\mathscr{Diff}_\omega\,(W_i, \mathrm{rel}\,\partial)$ denotes the group $\mathrm{Diff}_\omega\,(W_i, \mathrm{rel}\,\partial)$ in its usual C^∞-topology, it is not hard to see that the inclusion of $\mathscr{Diff}_\omega\,(W_0, \mathrm{rel}\,\partial)$ into $\mathscr{Diff}_\omega\,(W_1, \mathrm{rel}\,\partial)$ is a homotopy equivalence. For example, using [6] one can easily construct a family h_t of contractions of W_1 which preserve ω up to a constant and are such that $h_1(W_1) = W_0$. Then one can homotop $\mathscr{Diff}_\omega\,(W_1, \mathrm{rel}\,\partial)$ into $\mathscr{Diff}_\omega\,(W_0, \mathrm{rel}\,\partial)$ by conjugating by h_t. Thus Theorem 1 is equivalent to the statement:

THEOREM 1′. *The inclusion*

$$\bar{B}\,\mathscr{Diff}_\omega\,(W_0, \mathrm{rel}\,\partial) \hookrightarrow \bar{B}\,\mathscr{Diff}_\omega\,(W_1, \mathrm{rel}\,\partial)$$

induces an isomorphism on (untwisted) integer homology.

This is a very special case of a general theorem [10] which asserts that the local homology of groups such as $\mathscr{Diff}_\omega\,(W, \mathrm{rel}\,\partial)$ is a homotopy invariant of the pair consisting of W together with its tangent bundle. In fact, the local homology of $\mathscr{Diff}_\omega\,(W, \mathrm{rel}, \partial)$ is isomorphic to the homology of the space of sections of a bundle over W which is associated to the tangent bundle. In the ordinary, non-volume-preserving case, this theorem is due to Mather in dimension 1 and to Thurston in dimension >1. See [8]. In the volume-preserving case, Theorem 1′ is the crucial link which allows one to deduce the theorem for compact manifolds of finite volume from that for non-compact manifolds of infinite volume which is established in [9].

The proof of Theorem 1′ is surprisingly delicate. It is based on ideas of Thurston which he used in [11] to show that when $n = \dim W \geq 3$, the inclusion

$$\bar{B}\,\mathscr{Diff}^{\Phi}_{\omega 0}\,(V, \mathrm{rel}\,\partial) \hookrightarrow \bar{B}\,\mathscr{Diff}^{\Phi}_{\omega 0}\,(W, \mathrm{rel}\,\partial)$$

induces an isomorphism on H_1, where V is *any* compact n-dimensional submanifold of W, and where $\mathscr{Diff}^{\Phi}_{\omega 0}$ is the subgroup of $\mathscr{Diff}_\omega$ consisting of elements which are isotopic to the identity and have zero flux. This, in turn, is the key step in showing that $\mathscr{D}iff^{\Phi}_{\omega 0}(W, \mathrm{rel}\,\partial)$ is a simple group when $n \geq 3$. The case $n = 2$ coincides with the symplectic case and was considered by Banyaga in [1].

The proof of Theorem 1′ has two steps. First, one shows that the space $\bar{B}\,\mathscr{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ deformation retracts onto a subspace which is made up from diffeomorphisms of "small" support in W_1. An elegant proof of this deformation lemma in the non-volume-preserving case is given by Mather in [8] §15, following ideas of Thurston. However that proof does not work either in the volume-preserving or the C^0 case. The present proof is much more complicated, but it does work in both these cases as well as in the symplectic case. See Remark 4.15 below. In fact, it is just a generalization to higher dimensions of Thurston and Banyaga's proof of a similar result for the 2-skeleton. Second, one shows that any cycle on this subspace made from diffeomorphisms of small support may have its support conjugated into W_0. The techniques used here, notably the construction of the maps h_κ in Lemma 3.6, do not appear to generalize immediately to the symplectic case.

§2. Basic definitions

First let us recall some facts about the flux homomorphism. We assume throughout that W_1 is a connected n-dimensional manifold with non-empty boundary. Then the volume form ω is exact and the flux is a continuous homomorphism Φ from the identity component $\mathscr{D}iff_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ of $\mathscr{D}iff_{\omega}(W_1, \mathrm{rel}\,\partial)$ to $H^{n-1}_c(W_1; \mathbf{R}) \cong H^{n-1}(W_1, \partial W_1; \mathbf{R})$. It may be defined as follows. Given an $(n-1)$-cycle z in $(W_1, \partial W_1)$ then

$$\Phi(g)(z) = \int_c \omega,$$

where c is an n-chain with boundary $g_*(z) - z$. (This is independent of the choice of c because ω is exact.) We will write $\mathscr{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ for the kernel of Φ. Thurston shows in [11] that $\mathrm{Diff}^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial)$ is a perfect group when $n \geq 3$. In fact, he proves the slightly stronger result that $H_1(\bar{B}\,\mathscr{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial); \mathbf{Z}) = 0$. When $n = 2$, this is no longer true. There is a continuous surjective homomorphism

$$\rho : \mathscr{D}iff^{\Phi}_{\omega 0}(W_1, \mathrm{rel}\,\partial) \to \mathbf{R},$$

defined by Banyaga in [1] II.4.3, whose kernel we will denote by

$\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_1, \text{rel}\,\partial)$. Banyaga shows in [1] that $H_1(\bar{B}\,\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_1, \text{rel}\,\partial); \mathbf{Z}) = 0$. Let

$$\mathcal{G}_i = \begin{cases} \mathcal{D}iff^{\Phi}_{\omega 0}(W_i, \text{rel}\,\partial) & \text{if} \quad n \geqslant 3, \text{ and} \\ \mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_i, \text{rel}\,\partial) & \text{if} \quad n = 2. \end{cases}$$

We will prove Theorem 1′ in the following sharpened form.

THEOREM 2.1. *The inclusion $\bar{B}\mathcal{G}_0 \hookrightarrow \bar{B}\mathcal{G}_1$ induces an isomorphism on (untwisted) integer homology.*

Clearly, it suffices to consider the case when vol W_1 is finite. Therefore we will assume from now on that this is so. Also, it will be convenient to reformulate Theorem 2.1 slightly. Choose a point $x_1 \in \partial W_1$, and put $W_2 = W_1 -$(open disc *nbhd* of x_1). So W_2 has corners: see Fig. 1. Let $\mathcal{G}_2$ be $\mathcal{D}iff^{\Phi}_{\omega 0}(W_2, \text{rel}\,\partial)$ if $n \geqslant 3$ and $\mathcal{D}iff^{\Phi\rho}_{\omega 0}(W_2, \text{rel}\,\partial)$ when $n = 2$. If vol $W_0 =$ vol W_2, the discrete groups G_0 and G_2 are direct limits of subgroups which are conjugate in G_1. It follows that the inclusion $BG_0 \hookrightarrow BG_1$ induces an isomorphism on integer homology if and only if the inclusion $BG_2 \hookrightarrow BG_1$ does. Since the inclusions $\mathcal{G}_0 \hookrightarrow \mathcal{G}_1$ and $\mathcal{G}_2 \hookrightarrow \mathcal{G}_1$ are homotopy equivalences, a similar statement is true on the level of $\bar{B}\mathcal{G}$. Therefore it will suffice to show that $H_*(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2; \mathbf{Z}) = 0$ for any W_2. Clearly, this is an immediate consequence of the following lemma.

LEMMA 2.2. *If* $N > 2d^2 + 2$, *then* $H_d(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2; \mathbf{Z}) = 0$ *whenever* vol $(W_1 - W_2) < 1/N$ vol W_1.

Before beginning the proof we must make some definitions.

Let Sing $\mathcal{G}$ denote the singular complex of the topological group $\mathcal{G}$. The discrete group G acts freely on Sing $\mathcal{G}$ by multiplication on the right, and hence acts freely on the realization $|\text{Sing}\,\mathcal{G}|$ of Sing $\mathcal{G}$. The quotient space $|\text{Sing}\,\mathcal{G}|/G$ fits

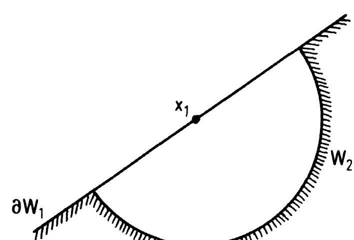

Fig. 1

into a fibration sequence

$$\mathcal{G} \simeq |\mathrm{Sing}\,\mathcal{G}| \to |\mathrm{Sing}\,\mathcal{G}|/G \to BG$$

and therefore is weakly equivalent to $\bar{B}\mathcal{G}$. In this paper, following [8], we will use $|\mathrm{Sing}\,\mathcal{G}|/G$ as our model for $\bar{B}\mathcal{G}$. Notice that $|\mathrm{Sing}\,\mathcal{G}|/G$ is the realization of the simplicial set $S = \mathrm{sing}\,\mathcal{G}/G$. If Δ^p denotes the standard p-simplex with vertices $(v_0, \ldots, v_p)$, a p-simplex σ of S is just a based continuous map

$$\theta_\sigma : (\Delta^p, v_0) \to (\mathcal{G}, \mathrm{id})$$

where id is the identity element of $\mathcal{G}$. Further, if Δ^q is a face of Δ^p with first vertex v_i, the corresponding face of σ is given by the map

$$\theta_\alpha \cdot \theta_\sigma(v_i)^{-1} \mid \Delta^q.$$

In other words, one must renormalize θ_σ as well as restricting its domain.

It turns out to be useful to represent elements of $H_*(\bar{B}\mathcal{G}; \mathbf{Z})$ by maps of *cubical* complexes into $\bar{B}\mathcal{G}$. Thus let K be a finite cell complex which is obtained from a disjoint union of oriented d-dimensional cubes by making certain linear identifications of the faces. We will suppose that the vertices of K are ordered and that for each cube $\kappa \subset K$ with first vertex v_κ we are given a map $f_\kappa : (\kappa, v_\kappa) \to (\mathcal{G}, \mathrm{id})$. If these maps are compatible, in other words, if

$$f_\lambda(a) = f_\kappa(a) f_\kappa(v_\lambda)^{-1}, \qquad a \in \lambda,$$

whenever $\lambda \subset \kappa$, then they fit together in a unique way to give a map $f : K \to \bar{B}\mathcal{G}$. To see this, let T be the triangulation of K obtained by starring each cube at its barycenter, and order the vertices of T lexicographically. Then each p-simplex σ in T is taken by f to the simplex in $\bar{B}\mathcal{G}$ which corresponds to the singular simplex

$$(\Delta^p, v_0) \xrightarrow{\iota} (\sigma, v_\sigma) \xrightarrow{f_\kappa \cdot f_\kappa(v_\sigma)^{-1}} (\mathcal{G}, \mathrm{id}),$$

where ι is the natural identification and where κ is some cube containing σ. The compatibility conditions ensure that this is independent of the choice of κ. Thus the f_κ define a d-chain (K, f). Its boundary $\partial(K, f)$ is obtained by restricting f to the $(d-1)$-cubes of K, where these are taken with the appropriate cancellations and multiplicities.

Notice that we do not collapse degeneracies here. Since the degenerate cubes are factored out when one defines homology by means of the cubical complex, it is necessary to check that every element of $H_*(\bar{B}\mathscr{G})$ may be represented by a cubical cycle (K, f) as above. However this follows because the standard simplex Δ^n, with barycentric coordinates $\lambda_0, \ldots, \lambda_n$, has a canonical subdivision into projectively embedded n-cubes $C_0, \ldots, C_n$. In fact, let C_p be the set of all points in Δ^n for which $\lambda_p = \max\{\lambda_0, \ldots, \lambda_n\}$. Then C_p is homeomorphic to the standard n-cube, with linear coordinates $0 \leqslant \lambda_i/\lambda_p \leqslant 1$ for $i \neq p$. Therefore one may obtain a suitable cubical representative of a homology class by subdividing a simplicial cycle.

An alternative way to describe the chain (K, f) is to define maps $f_C: C \to \mathscr{G}$, where C runs over a family of subcomplexes of K which cover K. These maps must satisfy the compatibility conditions

$$f_{C'}(a)f_{C'}(a_0)^{-1} = f_C(a)f_C(a_0)^{-1}, \qquad a \in C \cap C',$$

where a_0 is any fixed element of $C \cap C'$. For example, if K^* is a subdivision of K into little cubes, the f_κ, $\kappa \subset K$, define a chain (K^*, f^*). If we identify K and K^* as topological spaces, the maps f and f^* are not equal. However they are clearly homotopic.

If K' is a subcomplex of K, we will write (K', f) for the chain obtained by restricting f to K'.

EXAMPLE 2.3. Let K be the unit square $\kappa = \{(a, b): 0 \leqslant a, b \leqslant 1\}$ with vertices ordered as $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$, and define $f_\kappa(a, b) = h(a)g(b)$ where $h(0) = g(0) = \mathrm{id}$. Then (K, f) is a 2-chain in $\bar{B}\mathscr{G}$. Its boundary is the union of the two 1-chains $b \mapsto g(b)$ and $b \mapsto h(1)g(b)h(1)^{-1}$, since the chains corresponding to $b = 0, 1$ cancel. Thus (K, f) is a 2-cycle if $h(1)$ commutes with the $g(b)$.

Now consider the case $\mathscr{G} = \mathscr{G}_1$, and let (K, f) be a d-chain as above. The *support* $\operatorname{supp} f_\kappa$ *of a cube* κ in K is defined to be the closure in W_1 of the set $\{x \in W_1 : f_\kappa(a)(x) \neq x \text{ for some } a \in \kappa\}$. Clearly

$$\operatorname{supp} f_\lambda \subseteq \operatorname{supp} f_\kappa \quad \text{whenever} \quad \lambda \subseteq \kappa.$$

We define $\operatorname{supp}(K, f)$ to be the union of $\operatorname{supp} f_\kappa$ over all $\kappa \in K$. Observe that (K, f) is a relative cycle in $(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ if and only if $\operatorname{supp} \partial(K, f) \subset \operatorname{Int} W_2$. Thus, in Example 2.3, (K, f) is a relative 2-cycle if $g(b)$ and $h(1)g(b)h(1)^{-1}$ have supports in $\operatorname{Int} W_2$ for all b.

We aim to show that any relative d-cycle (K, f) in $(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ is null-homologous. To keep control on the boundary, it is convenient to consider cycles for which there is no need to make cancellations when passing to their boundaries. Therefore, we say that a complex K is *reduced* if it may be obtained from a union of disjoint d-cubes by identifying pairs of oppositely oriented $(d-1)$-dimensional faces. Its boundary ∂K is then the subcomplex of K which is spanned by the $(d-1)$-cubes which lie in only one d-cube. Further, we define a *reduced relative d-cycle* to be a d-chain (K, f) where K is reduced and where $\operatorname{supp} f_\kappa \subset \operatorname{Int} W_2$ for all $\kappa \subset \partial K$. Thus $\operatorname{supp}(\partial K, f) \subset \operatorname{Int} W_2$, so that we can consider the boundary of (K, f) to be $(\partial K, f)$. For example, suppose that in (2.3) above the edges $a = 0, 1$ have support in $\operatorname{Int} W_2$ while $b = 0, 1$ do not. Then (K, f) is a relative cycle, but it is not reduced since one must cancel the edges $b = 0, 1$ to obtain $\partial(K, f)$. However it may be reduced by identifying the edge $b = 0$ with the edge $b = 1$ and then subdividing κ into two embedded cubes by the line $b = \text{const}$. Note also that every homology class in $H_*(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ may be represented by a reduced relative d-cycle (K, f). This holds because every class is represented by a relative simplicial cycle, which may be reduced by changing the identifications of its $(d-1)$-dimensional faces and then subdividing. One then takes K to be a cubical subdivision of this simplicial cycle.

Now let $\mathscr{V} = \{V_i : i \in A\}$ be any open cover of W_1. The chain (K, f) will be said to be *supported by* $\mathscr{V}$ if there is a function α which assigns to every cube $\kappa \subset K$ a set $\alpha(\kappa) \subseteq A$ in such a way that

(2.4) (i) $|\alpha(\kappa)| \leq \dim \kappa$,
(ii) $\operatorname{supp} f_\kappa \subseteq \bigcup \{V_i : i \in \kappa(\kappa)\}$, and
(iii) $\alpha(\lambda) \subseteq \alpha(\kappa)$ if $\lambda \subseteq \kappa$.

Further, let $\mathscr{V}'$ be a subfamily $\{V_i : i \in A'\}$ of $\mathscr{V}$ and put $W' = \bigcup \{V_i : i \in A'\}$. If K' is a subcomplex of K, then we will say that the triple (K, K', f) is *supported by* $(\mathscr{V}, \mathscr{V}')$ if there is a function α which in addition to the above three conditions satisfies

(iv) $\alpha(\kappa) \subseteq A'$ for all $\kappa \subset K'$.

This condition clearly implies that $\operatorname{supp}(K', f) \subset W'$. A reduced relative d-cycle (K, f) will be said to be supported by $(\mathscr{V}, \mathscr{V}')$ if the triple $(K, \partial K, f)$ is so supported.

In §4 we will prove:

LEMMA 2.5 (Deformation Lemma). *Let (K, f) by any chain in $\bar{B}\mathscr{G}_1$ and let K' be a subcomplex of K such that* $\operatorname{supp}(K', f) \subset W'$. *Then there is a chain*

$(K \times I, F)$ such that:

(i) $(K \times 0, F) = (K, f)$,
(ii) $\operatorname{supp}(K' \times I, F) \subseteq W'$, *and*
(iii) *the triple* $(K \times 1, K' \times 1, F)$ *has a subdivision which is supported by* $(\mathcal{V}, \mathcal{V}')$,

COROLLARY 2.6. *Suppose that (K, f) is a reduced relative d-cycle such that* $\operatorname{supp}(\partial K, f) \subset W' \subset \operatorname{Int} W_2$. *Then (K, f) is homologous in $(\bar{B}\mathcal{G}_1, \bar{B}\mathcal{G}_2)$ to a reduced relative cycle which is supported by $(\mathcal{V}, \mathcal{V}')$.*

Notice that if a relative cycle (K, f) is supported by $(\mathcal{V}, \mathcal{V}')$ then the support of each cube in the cycle is small. However the support of the cycle (K, f) might still be almost the whole of $\operatorname{Int} W_1$. In the next section we describe a d-fold conjugation process which at each step takes a little more of the support of (K, f) into W_2.

§3. The conjugation lemma

Throughout this section we assume that (K, f) is a reduced relative d-cycle. Our first task is to construct a suitable cover $\mathcal{V}$.

(3.1) *The cover $\mathcal{V}$*

Let $N > 2d^2 + 2$. We will assume that $\operatorname{vol}(W_1 - W_2) < 1/N \operatorname{vol} W_1$ as in Lemma 2.2. The cover $\mathcal{V}$ will consist of sets $V_1, \ldots, V_N$ as in Fig. 2. Thus we require:

(i) $\bar{V}_i \cap \bar{V}_j = \emptyset$ if $|i-j| > 1$;
(ii) the sets $\bar{V}_2, \ldots, \bar{V}_{N-1}$ and $\bar{V}_1 \cap \bar{V}_2, \ldots, \bar{V}_{N-1} \cap \bar{V}_N$ are diffeomorphic to $D^{n-1} \times I$, where D^{n-1} is the closed $(n-1)$-disc;
(iii) each set $V_{(1,i)} = V_1 \cup \cdots \cup V_i$, $i < N$, is diffeomorphic to an open disc neighbourhood of x_1;
(iv) if $\mathring{V}_i = V_i - (\bar{V}_{i-1} \cup \bar{V}_{i+1})$, then

$$\operatorname{vol} \mathring{V}_1 < \operatorname{vol} \mathring{V}_i \quad \text{for} \quad 2 \leq i < N.$$

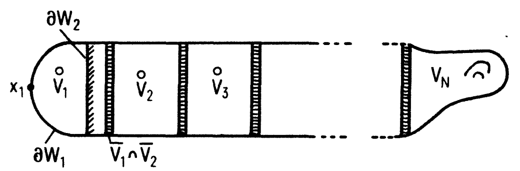

Fig. 2

We will assume the sets $V_i \cap V_{i+1}$ and V_N have very small volume so that the $\mathring{V}_i$, $1 \leq i < N$, fill out almost all of W_1.

If $\alpha \subset \{1, \ldots, N\}$ we write V_α for $\bigcup \{V_i : i \in \alpha\}$. Similarly $\mathring{V}_\alpha = \bigcup \{\mathring{V}_i : i \in \alpha\}$. Recall also the notation $V_{(1,i)} = V_1 \cup \cdots \cup V_i$ used in (ii) above.

Since supp $(\partial K, f)$ is a compact subset of Int W_2, we may choose V_1 and V_2 so that

(v) $\operatorname{supp}(\partial K, f) \subseteq W_1 - \bar{V}_1 \subseteq W_1 - \mathring{V}_1 \subseteq \operatorname{Int} W_2$.

This is compatible with (iv) above because $\operatorname{vol}(W_1 - W_2) < 1/N \operatorname{vol} W_1$. If $\mathcal{V}' = \{V_i : 2 \leq i \leq N\}$, the cycle (K, f) will then satisfy the conditions of Corollary 2.6 with respect to $(\mathcal{V}, \mathcal{V}')$, and so will be homologous to a cycle which is supported by $(\mathcal{V}, \mathcal{V}')$. Therefore, it will suffice to prove:

LEMMA 3.2 (Conjugation Lemma). *Suppose that (K, f) is a reduced relative d-cycle in $(\bar{B}\mathscr{G}_1, \bar{B}\mathscr{G}_2)$ which is supported by the cover $(\mathcal{V}, \mathcal{V}')$ of* (3.1) *above. Then (K, f) is null-homologous.*

Proof when $d = 1$. Because $\bar{B}\mathscr{G}_1$ has only one vertex, every 1-chain (K, f) is an (absolute) 1-cycle. In particular, since (K, f) is supported by $\mathcal{V}$, it is a sum of 1-cycles each supported by some V_i. Therefore we just have to show that any 1-cycle with support in V_1 is homologous to a cycle with support in Int W_2. Since $N > 2$, $\operatorname{vol} V_1 < \operatorname{vol}(V_1 \cap V_2) \cup \mathring{V}_2 < \operatorname{vol} W_2$. Therefore, given any compact subset S of $V_1 \cap \operatorname{Int} W_1$, there is a path h_t, $0 \leq t \leq 1$, in $\mathscr{G}_1$ with $h_0 = \mathrm{id}$ and such that $h_1(S) \subset \operatorname{Int} W_2$. (See Remark (3.3) below.) Thus the proof may be completed by constructing a 2-chain as in Example 2.3. □

Remark 3.3. In general, the only obstructions to constructing a volume-preserving isotopy which moves sets around in a prescribed way are the obvious ones involving volume. See [6]. One can ensure that h_t has zero flux by making its support lie in a contractible subset of W_1. When $n = 2$, one can also ensure that h_t lies in the kernel of ρ by replacing it by $h_t k_t$, where $\rho(k_t) + \rho(h_t) = 0$ and where k_t has support in a tiny disc which is disjoint from supp h_t. Then we will have $h_t \in \mathscr{G}_1$ in all cases.

The case $d = 1$ is so simple that one does not need the special properties of the cover $\mathcal{V}$. These will be useful later on, but first we must homotop (K, f) to a cycle (K_1, F) which is easier to manipulate.

(3.4) *The cycle* (K_1, F)

Let K^* be the first barycentric subdivision of K. It is an ordered cubical complex with one q-cube $D(\lambda, \kappa)$ for each pair of cubes λ, κ in K with $\lambda \subset \kappa$,

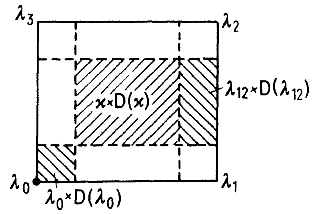

Fig. 3

where $q = \dim \kappa - \dim \lambda$. One can easily check that it is reduced. Let $D(\lambda)$ be the subcomplex $\bigcup \{D(\lambda, \kappa) : \kappa \supseteq \lambda\}$ of K^*. Now consider the subcomplex

$$K_1 = \bigcup_{\lambda \subset \kappa} \lambda \times D(\lambda, \kappa) = \bigcup_{\lambda} \lambda \times D(\lambda)$$

of $K \times K^*$. This is an ordered d-dimensional cubical complex whose d-cubes have the form $\lambda \times D(\lambda, \kappa)$ where $\dim \kappa = d$. See Fig. 3.

We will write $\mu(\lambda)$ for the subcomplex $\lambda \times D(\lambda)$ of K_1. If $\lambda \subset \kappa$ then $\mu(\lambda) \cap \mu(\kappa) = \lambda \times D(\kappa)$. It is not hard to check that K_1 is homeomorphic to K and so may be considered as a subdivision of K. Thus it is reduced and has boundary $\partial(K_1) = (\partial K)_1$.

There is a natural map $\pi : K_1 \to K$ which projects each $\mu(\lambda) = \lambda \times D(\lambda)$ onto the first factor λ. The maps $F_{\mu(\lambda)} = f_\lambda \circ \pi$ are clearly compatible. They fit together to form a reduced relative d-cycle (K_1, F) which is homologous to (K, f).

(3.5) *The conjugating map h*

We wish to define a map $h : K_1 \times I \to \mathcal{G}_1$ which will conjugate the support of (K_1, F) into Int W_2. Let Z be a set of the form $W_1 -$(open collar *nbhd* of ∂W_1) which contains supp (K, f), and let $T \subset V_{(1,d+1)} = V_1 \cup \cdots \cup V_{d+1}$ be a thin tube which intersects the sets $\mathring{V}_1, \ldots, \mathring{V}_{d+1}$ in turn and which lies outside Z. See Fig. 4.

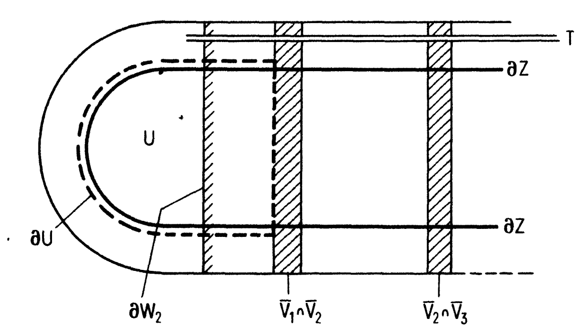

Fig. 4

Further, Let U be a neighbourhood of $Z \cap \mathring{V}_1$ in $\mathring{V}_1$ whose closure does not meet ∂W_1 or T. Then vol $U <$ vol $\mathring{V}_i$, $2 \leq i < N$, by (3.1)(iv). Therefore, one can find a path m_t, $0 \leq t \leq 1$, in $\mathscr{G}_1$ with support in $\mathring{V}_1 \cup T \cup \mathring{V}_2$ such that $m_0 = \mathrm{id}$ and $m_1(U) \subset \mathrm{Int}\, W_2$. See Remark (3.3). Observe that if $\operatorname{supp} g \subset Z - \mathring{V}_2$ then $\operatorname{supp}(m_1 g m_1^{-1}) = m_1(\operatorname{supp} g) \subset \mathrm{Int}\, W_2$.

Now recall from (2.4) that because (K, f) is supported by $\mathscr{V}$ there is a function α : (cubes of K) $\rightarrow$ (subsets of $\{1, \ldots, N\}$) such that

$$|\alpha(\kappa)| \leq \dim \kappa; \qquad \alpha(\lambda) \subseteq \alpha(\kappa) \quad \text{if} \quad \lambda \subseteq \kappa;$$

and

$$\operatorname{supp} f_\kappa \subset V_{\alpha(\kappa)} = \bigcup \{V_i : i \in \alpha(k)\}.$$

Choose a number $\beta(\kappa)$ in $\{1, \ldots, d+1\} - \alpha(\kappa)$ for each d-cube κ. In general, set $\beta(\lambda) = \bigcup \{\beta(\kappa) : \lambda \subset \kappa, \dim \kappa = d\}$. Then $\operatorname{supp} f_\lambda$ is disjoint from $\mathring{V}_{\beta(\lambda)}$ for all λ.

We are now ready to define the conjugating map h.

LEMMA 3.6. *There is a map* $h : K_1 \times I \rightarrow \mathscr{G}_1$ *such that*

(i) $h(\partial K_1 \times I \cup K_1 \times 0) = \mathrm{id}$;

(ii) *for each cube* κ *in* K *the restriction of* h *to* $\kappa \times D(\kappa) \times I$ *is a composite*

$$\kappa \times D(\kappa) \times I \xrightarrow{\text{proj.}} D(\kappa) \times I \xrightarrow{h_\kappa} \mathscr{G}_1;$$

(iii) *for each* κ *and* $b \in D(\kappa)$

$$\operatorname{supp} h_\kappa(b, t) \subset \mathring{V}_1 \cup T \cup \mathring{V}_{\beta(\kappa)},$$

where $\beta(\kappa)$ *is as above;*

(iv) *for each* $\kappa \subset K$, $\kappa \not\subset \partial K$ *and* $b \in D(\kappa)$

$$h_\kappa(b, 1)(U) \subseteq \mathrm{Int}\, W_2 \cap V_{(1,d+1)}.$$

Proof. We define the h_κ first for cubes of dimension d, then for those of dimension $d-1$ and so on. If $\dim \kappa = d$, then $|\beta(\kappa)| = 1$ and $D(\kappa)$ is a single point, b_κ say. Therefore, we may choose the path $h_\kappa(b_\kappa, t)$ to be a suitable conjugate of the path m_t defined in (3.5) above. In general, if $\dim \kappa = q$, and h_λ has been defined for all λ of dimension $>q$, then h_κ is determined on $\partial D(\kappa) \times I$. Since

$\beta(\kappa) \subset \beta(\lambda)$ whenever $\lambda \subset \kappa$, the support of $h_\kappa \mid \partial D(\kappa) \times I$ satisfies (ii) and (iv). It is not hard to see that one can extend h_κ to the whole of $D(\kappa) \times I$. If $\kappa \subset \partial K$, one must take care to satisfy (i) also. Further details will be left to the reader. □

(3.7) *The basic conjugation process*

We define a $(d+1)$-chain $(K_1 \times I, H)$ as follows:

$$\text{if } \mu = \kappa \times D(\kappa) \times I \subset K_1 \times I,$$

then

$$H_\mu(a, b, t) = h_\kappa(b, t) f_\kappa(a).$$

Observe that $H_\mu(a_0, b_0, 0) = \text{id}$ by (3.6)(i) if (a_0, b_0) is the first vertex of $\kappa \times D(\kappa)$. Hence H_μ is properly normalized. Since h is globally defined on $K_1 \times I$ and the f_κ are compatible, the H_μ are also compatible, and so fit together to form the chain $(K_1 \times I, H)$. The boundary of $(K_1 \times I, H)$ has three parts:

$$(K_1 \times 0, H), \qquad (K_1 \times 1, H) \quad \text{and} \quad (\partial K_1 \times I, H).$$

By Lemma 3.6(i) we have $(K_1 \times 0, H) = (K_1, F)$. Also $\operatorname{supp}(\partial K_1 \times I, H) \subset \operatorname{Int} W_2$. Hence (K, f) is homologous to $(K_1 \times 1, H)$.

Let us write $\bar{H}$ for the restriction of H to $K_1 \times 1 = K_1$. Then on the subcomplex $\mu(\kappa) = \kappa \times D(\kappa)$ of K_1 the map $\bar{H}$, when normalized at the vertex (a_0, b_0), takes the form:

$$\begin{aligned}\bar{H}_{\mu(\kappa)}(a, b) &= h_\kappa(b, 1) f_\kappa(a) h_\kappa(b_0, 1)^{-1} \\ &= l_\kappa(b) g_\kappa(a),\end{aligned}$$

where $l_\kappa(b) = h_\kappa(b, 1) h_\kappa(b_0, 1)^{-1}$ and $g_\kappa(a) = h_\kappa(b_0, 1) f_\kappa(a) h_\kappa(b_0, 1)^{-1}$. Observe that

$$\operatorname{supp} l_\kappa(b) \subset V_{(1,d+1)}$$

by (3.6)(iii). Also, because $\operatorname{supp} f_\kappa \subset Z \cap V_{\alpha(\kappa)}$, and because $T \cup \mathring{V}_{\beta(\kappa)}$ is disjoint from $Z \cap V_{\alpha(\kappa)}$, conditions (iii) and (iv) of (3.6) imply that $\operatorname{supp} g_\kappa \subset \operatorname{Int} W_2$ when $\kappa \not\subset \partial K$. This holds also when $\kappa \subset \partial K$ because $h_\kappa(b, t)$ commutes with $f_\kappa(a)$ by (3.1)(v). One may also take $(a_0, b_0) \in \partial K_1$ so that $h_\kappa(b_0, 1) = \text{id}$ by (3.6)(ii). Note further that

$$\operatorname{supp} l_\kappa(b) g_\kappa(a) l_\kappa(b)^{-1} \subset \operatorname{Int} W_2 \cap (V_{(1,d+1)} \cup V_{\alpha(\kappa)})$$

for all $b \in D(\kappa)$. In particular, if $\dim \kappa = d$, then $\operatorname{supp} \bar{H}_{\mu(\kappa)} \subset \operatorname{Int} W_2$, so that the cubes $\mu(\kappa)$, $\dim \kappa = d$, contribute nothing to $(K_1, \bar{H})$.

Proof of Lemma 3.2

We prove this by induction, using the following inductive hypothesis:

IH(k): There is a function

$$\alpha : (\text{cubes in } K) \to (\text{subsets of } \{1, 2kd+2, \ldots, N\})$$

such that

(i) $|\alpha(\kappa)| \leq d-k$,
(ii) $\alpha(\mu) \subset \alpha(\kappa)$ if $\mu \subset \kappa$; and
(iii) $\operatorname{supp} f_\kappa \subset V_{\alpha(\kappa)} \cup (\operatorname{Int} W_2 \cap V_{(1,2kd+1)})$.

Any reduced relative d-cycle which is supported by $(\mathcal{V}, \mathcal{V}')$ satisfies *IH*(0). Also, if (K, f) satisfies *IH*(d) then each $\alpha(\kappa)$ must be empty. Therefore (iii) implies that $\operatorname{supp}(K, f) \subseteq \operatorname{Int} W_2$. Hence it remains to prove:

LEMMA 3.8. *Any reduced cycle for which IH*(k) *holds for some* $k<d$ *is homologous to a reduced cycle for which IH*($k+1$) *holds.*

We begin with the following result.

LEMMA 3.9. *Suppose that* (K, f) *satisfies IH*(k) *for some* $k<d$. *Then* (K, f) *is homologous to a reduced cycle* $(\bar{K}, \bar{f})$ *which has a function* α *satisfying the conditions of IH*(k) *as well as:*

(iv) $1 \in \alpha(\kappa)$ *if and only if* $\kappa \not\subset \partial \bar{K}$.

Proof. Because (K, f) is a relative cycle, $\operatorname{supp} f_\kappa \subset \operatorname{Int} W_2$ for all $\kappa \subset \partial K$. therefore if we define

$$\alpha_1(\kappa) = \alpha(\kappa) \quad \text{for} \quad \kappa \not\subset \partial K,$$

and

$$\alpha_1(\kappa) = \alpha(\kappa) - \{1\} \quad \text{for} \quad \kappa \subset \partial K,$$

the function α_1 satisfies *IH*(k). Hence we may suppose that $1 \notin \alpha(\kappa)$ for all $\kappa \subset \partial K$.

Now consider the cycle (K_1, F) constructed in (3.4). Define α_1 for K_1 by setting

$$\alpha_1(\tau) = \bigcap_{\tau \subset \mu(\kappa)} \alpha(\kappa), \quad \text{for all cubes } \tau \text{ in } K_1,$$

where $\mu(\kappa) = \kappa \times D(\kappa) \subset K_1$. It is not hard to check that this satisfies $IH(k)$. Let $C = \{\kappa \in K : 1 \notin \alpha(\kappa)\}$ and put

$$\bar{K}_1 = K_1 - \mathrm{Int}\left(\bigcup_{\kappa \in C} \mu(\kappa)\right).$$

Since $\operatorname{supp} F_{\mu(\kappa)} = \operatorname{supp} f_\kappa \subset \mathrm{Int}\, W_2$ for all $\kappa \in C$, $(\bar{K}_1, F)$ is a (reduced) relative cycle homologous to (K, f). We claim that the function α_1 when restricted to $\bar{K}_1$ satisfies (iv) above. For, by construction, $\partial K \subset C$. It follows that $\tau \subset \partial \bar{K}_1$ if and only if the set $\{\kappa : \tau \subset \mu(\kappa)\}$ intersects C (but is not entirely contained in C.). Also, if $\tau \not\subset \partial \bar{K}_1$ then the set $\{\kappa : \tau \subset \mu(\kappa)\}$ is disjoint from C. Condition (iv) now follows easily. $\square$

Proof of Lemma 3.8

We will suppose as we may that the function α on (K, f) satisfies condition (3.9)(iv). Choose a function

$$\beta : (\text{cubes in } K) \to (\text{subsets of } \{2kd+2, \ldots, 2kd+d+1\})$$

so that

$$\beta(\kappa) \cap \alpha(\kappa) = \varnothing, \qquad \beta(\kappa \cap \lambda) = \beta(\kappa) \cup \beta(\lambda),$$

and

$$|\beta(\kappa)| = 1 \quad \text{if} \quad |\alpha(\kappa)| = d - k.$$

We now want to define a map $h : K_1 \times I \to \mathscr{G}_1$ as in Lemma 3.6 with respect to this β. We will choose Z and U as before, and will take T to be a tube in $V_{(1,2kd+d+1)}$ which is disjoint from Z. Since (K, f) is a relative cycle there is a compact subset A of $(\mathrm{Int}\, W_2) \cap V_1$ such that $\operatorname{supp}(\partial K, f) \subset A \cup V_{(2,N)}$. Then we require h to satisfy conditions (i) and (ii) of Lemma 3.6 as well as:

(iii)′ for each $\kappa \subset K$

$$\operatorname{supp} h_\kappa \subset (\mathring{V}_1 - A) \cup T \cup \mathring{V}_{\beta(\kappa)} \subset V_{(1,2kd+d+1)},$$

(iv)′ for each $\kappa \subset K$, $\kappa \not\subset \partial K$ and $b \in D(\kappa)$

$$h_\kappa(b, 1)(U) \subset \operatorname{Int} W_2 \cap V_{(1,2kd+d+1)}.$$

Now observe that if $|\alpha(\kappa)| = d-k$ then $\alpha(\kappa) = \alpha(\lambda)$ for all $\lambda \supset \kappa$. Also, if S is a subset $\{1, 2kd+2, \ldots, N\}$ with $|S| = d-k$, the subcomplex

$$K_1(S) = \bigcup \{\mu(\kappa) : \alpha(\kappa) = S\}$$

does not intersect $K_1(S')$ unless $S = S'$. Therefore we may define β and h so that:

(v) on each set $K_1(S) \times I$, h depends only on $t \in I$,
and then extend h to the rest of $K_1 \times I$.

Consider the cycle $(K_1, \bar{H})$ constructed from h as in (3.7). For each $\kappa \in K$ we have

$$\bar{H}_{\mu(\kappa)}(a, b) = l_\kappa(b) g_\kappa(a)$$

where $l_\kappa(b) = h_\kappa(b, 1) h_\kappa(b_0, 1)^{-1}$ and where

$$\operatorname{supp} g_\kappa(a) \subset \operatorname{Int} W_2 \cap (V_{(1,2kd+d+1)} \cup V_{\alpha(\kappa)}). \tag{*}$$

(When $\kappa \subset \partial K$, this holds by (iii)′ above.) In particular, $l_\kappa(b) = \mathrm{id}$ for all κ such that $\mu(\kappa) \subset K_1(S)$, by (v), so that the chain $(K_1(S), \bar{H})$ has support in Int W_2. Therefore, if

$$K_2 = K_1 - \bigcup_S \operatorname{Int} K_1(S),$$

then $(K_2, \bar{H})$ is a cycle homologous to $(K_1, \bar{H})$ and hence to (K, f).

This cycle nearly satisfies $IH(k+1)$. To see this, let

$$\mu(\partial K) = \bigcup \{\mu(\kappa) : \kappa \subset \partial K\}$$

and define α_1 on K_1 by

$$\begin{aligned} \alpha_1(\tau) &= \{1\} \cup \Big(\{2(k+1)d+2, \ldots, N\} \cap \Big(\bigcap_{\tau \subset \mu(\kappa)} \alpha(\kappa)\Big)\Big) \quad \text{if} \\ &\qquad \tau \subset \mu(\partial K) \text{ and } \tau \not\subset \partial K_1 \\ &= \{1, 2(k+1)d+2, \ldots, N\} \cap \Big(\bigcap_{\tau \subset \mu(\kappa)} \alpha(\kappa)\Big) \quad \text{otherwise.} \end{aligned}$$

It follows from condition (3.9)(iv) that $1\in\alpha_1(\tau)$ for all $\tau\not\subset\partial K_1$. Now, (*) together with conditions (iii)′ and (iv)′ above imply that

$$\operatorname{supp}\bar{H}_{\mu(\kappa)}\subset V_{(1,2(k+1)d+1)}\cup V_{\alpha(\kappa)}.$$

Therefore, if $\tau\not\subset\partial K_1$,

$$\operatorname{supp}\bar{H}_{\mu(\kappa)}\subset V_{\alpha_1(\tau)}\cup(\operatorname{Int}W_2\cap V_{(1,2(k+1)d+1)}).$$

On the other hand, if $\tau\subset\partial K_1$, then the function h is constant on τ by Lemma 3.6(i). Hence $l_\tau(b)=\mathrm{id}$, which implies that

$$\operatorname{supp}\bar{H}_\tau\subset\operatorname{Int}W_2\cap(V_{(1,2(k+1)d+1)}\cup V_{\alpha(\kappa)})$$

for all κ with $\tau\subset\mu(\kappa)$. Thus the function α_1 satisfies condition (iii) of $IH(k+1)$ for all $\tau\subset K_1$. It clearly also satisfies (ii). As for (i), observe that if $\tau\not\subset\mu(\partial K)$ then $|\alpha_1(\tau)|=d-k$ only if the sets $\alpha(\kappa)$, $\tau\subset\mu(\kappa)$, are all the same and all have $d-k$ elements. But this implies that $\tau\subset\operatorname{Int}K_1(S)$ for some S, so that τ is not in K_2. Observe also that because $1\in\alpha(\kappa)$ for all $\kappa\not\subset\partial K$, we must have $|\alpha(\kappa)|\leq d-k-1$ for $\kappa\subset\partial K$. Thus $\mu(\partial K)\subset K_2$ and $|\alpha_1(\tau)|\leq d-k$ for $\tau\subset\mu(\partial K)$. Therefore all we have to do now is cut out from K_2 the cubes in $\mu(\partial K)$ with $|\alpha_1(\tau)|=d-k$.

To do this, let R be a subset of $\{2kd+2,\dots,N\}$ with $d-k-1$ elements and define

$$L(R)=\bigcup\{\mu(\kappa):\kappa\subset\partial K,\alpha(\kappa)=R\}\subset\mu(\partial K),$$
$$L'(R)=\partial K_1\cap L(R).$$

Since $|\alpha(\kappa)|\leq d-k-1$ on ∂K, the subcomplexes $L(R)$ are disjoint for distinct R. Let $K_3=K_2-\bigcup_R\operatorname{Int}L(R)$. Then the restriction of α_1 to K_3 satisfies the conditions of $IH(k+1)$ since we have removed all the cubes for which (i) fails. However, $(K_3,\bar{H})$ is no longer a relative cycle. Our aim now is to define chains $(M(R),G)$ for each R so that $(L(R),\bar{H})+(M(R),G)$ is a relative boundary and so that $(K_3,\bar{H})+\sum_R(M(R),G)$ is a relative cycle which satisfies $IH(k+1)$.

Note that $L(R)=L'(R)\times[0,1]$. If we identify $L'(R)\times 0$ with $L'(R)\subset\partial K_1$, then it follows from (3.9)(iv) that $L'(R)\times 1\subset K_1(S)$, where $S=\{1\}\cup R$. We chose h so that h is constant on each $K_1(S)$: see (v) above. Clearly, we may also assume that on each set $L'(R)\times[0,1]$ the map h depends only on $s\in[0,1]$. Then, for each $\tau\subset L'(R)$, we will have

$$\bar{H}_{\tau\times[0,1]}(a,s)=l_\tau(s)F_\tau(a),\quad\text{for}\quad(a,s)\in\tau\times[0,1],$$

where $F_\tau = f_\kappa \circ \pi$ for some $\kappa \subset \partial K$ as in (3.4). Note also that the $l_\tau(s)$ commute with the $F_\tau(a)$ because of condition (iii)′ in the definition of h.

Now consider $\partial(L(R), \bar{H})$. The pieces $(L'(R)\times\{i\}, \bar{H})$, $i=0, 1$, have support in Int W_2. Therefore $\partial(L(R), \bar{H})$ is a sum of chains of the form $(\tau\times[0, 1], \bar{H})$, where τ is a $(d-2)$-cube in ∂K_1 with $\alpha(\tau)\subsetneqq R$. We will write $\partial L'(R)$ for this set of $(d-2)$-cubes. Since $|R|<d$, we may choose an integer $j\notin R$ such that $2kd+d+1 < j \leqslant 2(k+1)d+1$. Let m_t, $0\leqslant t\leqslant 1$, be a path in $\mathcal{G}_1$ with support in $(\mathring{V}_1 - A)\cup T'\cup \mathring{V}_j$ such that $m_1(U)\subset W_2$. Here T' is a tube in $V_{(1,2(k+1)d+1)}$ which does not meet supp F_τ for $\tau\subset L'(R)$. Therefore m_t commutes with the F_τ. Further, because supp $l_\tau \subset V_{(1,2kd+d+1)}$ by (iii)′, we may assume that $m_1 l_\tau(s) m_1^{-1}$ has support in Int W_2 for all s. Now define

$$G_\tau(t, a, s) = m_t l_\tau(s) F_\tau(a) \quad \text{for} \quad (t, a, s)\in I\times\tau\times[0, 1].$$

Because l_τ commutes with F_τ, the faces $(I\times\tau\times\{i\}, G_\tau)$, $i=0, 1$, of the chain $(I\times\tau\times[0, 1], G_\tau)$ cancel. Further $(\{1\}\times\tau\times[0, 1], G_\tau)$ has support in Int W_2, and $(\{0\}\times\tau\times[0, 1], G_\tau) = (\tau\times[0, 1], \bar{H})$. Therefore, if we put

$$(M(R), G) = \sum_{\tau\subset\partial L'(R)} (I\times\tau\times[0, 1], G_\tau),$$

then $(L(R), \bar{H}) + (M(R), G)$ is a relative boundary. Thus the cycle $(K_3, \bar{H}) + \sum_R (M(R), G)$ is homologous to $(K_1, \bar{H})$ and hence to (K, f). This cycle is not reduced since in the calculation of its boundary one must cancel the face $(I\times\tau\times\{1\}, G_\tau)$ with $(I\times\tau\times\{0\}, G_\tau)$ and must cancel $(\{0\}\times\tau\times[0, 1], G_\tau)$ with $(\tau\times[0, 1], \bar{H})$ in K_3. However, it is homologous to a reduced relative cycle $(K_4, \bar{G})$, where K_4 is formed from K_3 and the $M(R)$ by making the identifications which correspond to the above cancellations and then subdividing, and where $\bar{G}$ is induced by G and $\bar{H}$ in the obvious way.

We claim that $(K_4, \bar{G})$ satisfies $IH(k+1)$. To see this, define the function α_2 on K_4 by

$$\begin{aligned}\alpha_2(\lambda) &= \alpha_1(\lambda) && \text{if} \quad \lambda\subset K_3\\ &= \alpha_1(\tau)\cup\{1\} && \text{if} \quad \lambda\subset I\times\tau\times[0, 1]\subset M(R), \lambda\not\subset K_3.\end{aligned}$$

Since $|\alpha_1(\tau)| < d-k-1$ for $\tau\subset\partial L'(R)$, the function α_2 satisfies condition (i) of $IH(k+1)$. It is easy to check that the other conditions hold. This completes the proof of Lemma 3.8, and hence of Lemma 3.2. □

§4. The deformation lemma

This section is concerned with the proof of Lemma 2.5. In [11] Thurston gave a very brief outline of a proof in the case of a 1-chain on a manifold of dimension $\geqslant 3$. His method was later fully worked out in the symplectic case by Banyaga, both for 1-chains and 2-chains. The argument for $d > 2$ is essentially the same: one just has to be very systematic, so that one can keep track of what is going on. We will begin by making some definitions and will describe the strategy of the proof in (4.3). Throughout we consider a triple (K, K', f) such that $\operatorname{supp}(K', f) \subset X \subset W'$, for some compact submanifold X of W'.

(4.1) *Coverings associated to a triangulation*

Put a Riemannian metric on W_1 and choose $\varepsilon > 0$ so that ε-balls are geodesically convex and so that the ε-neighbourhood X_ε of X is contained in W'. Then choose a smooth convex triangulation $T = \{\Delta_i^k : i \in I_k, 0 \leqslant k \leqslant n\}$ of W_1 which restricts to a triangulation T' of X and is such that the ε-neighbourhood of any simplex in T, resp. T', is contained in a set of $\mathcal{V}$, resp. $\mathcal{V}'$. As in [1] ChIII.2, we associate to such a triangulation an open cover $\mathcal{U} = \{U_i^k : i \in I_k, 0 \leqslant k \leqslant n\}$ of W_1 with the following properties:

(a) each U_i^k is an ν-neighbourhood of a deformation retract of Δ_i^k for some $\eta < \varepsilon$,
(b) $\bar{U}_i^k \cap \bar{U}_j^l = \varnothing$ if either $k = l$ and $i \neq j$ or $k < l$ and Δ_i^k is not a face of Δ_j^l.
(c) For each k, the sets $U_i^j : 0 \leqslant j \leqslant k$, $i \in I_j$ cover the k-skeleton of T.

One should construct the U_i^k in order of increasing k. See Fig. 5. Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Also $\mathcal{U}' = \{U_i^k : \Delta_i^k \in T'\}$ refines $\mathcal{V}'$.

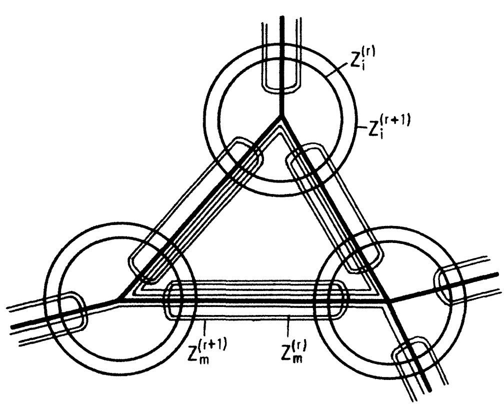

Fig. 5.

It will be convenient to renumber the sets U_i^k. Let $M_k = |I_0| + \cdots + |I_k|$ for each $0 \leq k \leq n$ and put $M = M_n$. We may assume that $I_k = \{1, \ldots, M_k - M_{k-1}\}$. Then U_i^k will be called U_m where $m = M_{k-1} + i$. In particular $U_m = U_m^0$ for $m \leq M_0$. (Our M_k slightly differ from Banyaga's N_k but the renumbering is the same as his.) We now choose a nested sequence of open covers $\mathscr{Z}^{(r)} = \{Z_m^{(r)} : 1 \leq m \leq M\}$, $-Q \leq r \leq Q + 2d + 1$, each of which is a slightly smaller version of $\mathscr{U}$, where $Q = 2Md$. Thus, for all r, s, m, l we have

$$Z_m^{(r)} \subset \bar{Z}_m^{(r)} \subset Z_m^{(r+1)} \subset \bar{Z}_m^{(Q+2d+1)} \subset U_m,$$
$$\bar{Z}_m^{(r)} \cap \bar{Z}_l^{(s)} \cong \bar{U}_m \cap \bar{U}_l.$$

A typical pair of such covers is shown in Fig. 5. We will often write $Z^{(r)}$ for a union of sets from $\mathscr{Z}^{(r)}$ and $Z^{(s)}$ for the corresponding union of sets from $\mathscr{Z}^{(s)}$. Further, we will write J' for the subset of $\{1, \ldots, M\}$ which corresponds to elements of T'. Thus $j \in J'$ if and only if $U_j = U_i^k$ where $\Delta_i^k \in T'$. We write $\mathscr{Z}'^{(r)}$ for $\{Z_j^{(r)} : j \in J'\}$. It is a refinement of $\mathscr{U}'$ and of $\mathscr{V}'$.

(4.2) *Neighbourhoods of the identity in $\mathscr{G}_1$*

Let $\mathscr{N}$ be any neighbourhood of the identity in $\mathscr{G}_1$. Then a k-simplex $\sigma \subset \bar{B}\mathscr{G}_1$ will be called *$\mathscr{N}$-small* if

$$\theta_\sigma(v)\theta_\sigma(w)^{-1} \in \mathscr{N}$$

for all v, w in the standard simplex Δ^k. Similarly, the chain (K, f) will be called $\mathscr{N}$-small if

$$f_\kappa(a) f_\kappa(b)^{-1} \in \mathscr{N}$$

for all a, b in κ and all cubes $\kappa \subset K$. Clearly, one may subdivide K to get a chain (K^*, f^*) which is $\mathscr{N}$-small and is homotopic to (K, f). Therefore, we may assume that our original triple (K, K', f) is $\mathscr{N}$-small for any given $\mathscr{N}$.

Let $\mathscr{M}$ be the set of all elements $g \in \mathscr{G}_1$ such that both g and g^{-1} take $Z_m^{(r)}$ into $Z_m^{(r+1)}$ for all m, r. Let $\mathscr{N}_i$, $0 \leq i \leq 2d+1$, be an increasing sequence of contractible C^1-neighbourhoods of the identity in $\mathscr{G}_1$ such that for each i we have:

(a) $\mathscr{N}_i = \mathscr{N}_i^{-1} \subset \mathscr{M}$; and
(b) for every union $Z^{(Q+i)}$ of sets from $\mathscr{Z}^{(Q+i)}$, every compact subset of the space $\{g \in \mathscr{N}_i\mathscr{N}_0 : \operatorname{supp} g \subset Z^{(Q+i)}\}$ contracts inside $\{g \in \mathscr{N}_{i+1} : \operatorname{supp} g \subset Z^{(Q+i+1)}\}$.

Finally $\mathcal{N}$ will be a very small neighbourhood of the identity which is contained in $\mathcal{N}_0$. Other conditions on $\mathcal{N}$ will be given later. We will assume from now on that (K, K', f) is $\mathcal{N}$-small.

(4.3) *The main construction*

For each p we identify the standard p-cube C^p with

$$\{x \in \mathbf{R}^p : 0 \leq x_1, \ldots, x_p \leq M\}$$

by a linear transformation. Then the hyperplanes $x_j \in \mathbf{Z}$ divide C^p into a collection of little cubes whose set of vertices is the integer lattice Λ in C^p. Since the hyperplanes $x_j \in \mathbf{Z}$ are preserved by the face inclusions $C^q \to C^p$, the cubical complex K has a corresponding subdivision K^*. Thus each p-cube κ in K is divided into M^p little cubes c in K^* whose vertices lie on the integer lattice Λ_κ. We will identify Λ_κ with Λ. In particular, the first vertex v_κ of κ is identified with $(0, \ldots, 0) \in \Lambda$.

Most of the effort involved in the proof of the deformation lemma is taken up in establishing the following result. It will be proved in (4.7)–(4.14) below.

LEMMA 4.4. *There is a family of maps $\psi_\kappa : \Lambda_\kappa \to \mathcal{N}_0$, $\kappa \subset K$, with the following properties.*

(a) (compatibility) *If $\iota \in \Lambda_\kappa$ and $\kappa \subset \mu$ then*

$$\psi_\kappa(\iota) = \psi_\mu(\iota)\psi_\mu(v_\kappa)^{-1}.$$

(b) (agreement with f_κ) *For each κ and all vertices v of κ*

$$\psi_\kappa(v) = f_\kappa(v).$$

(c) *For each p-cube κ we have*

$$\operatorname{supp}(\psi_\kappa(j_1, \ldots, j_l, \ldots, j_p) \circ \psi_\kappa(j_1, \ldots, j_l - 1, \ldots, j_p)^{-1}) \subset Z_{j_l}^{(Q)},$$

(d) *If $\kappa \in K'$, then*

$$\psi_\kappa(j_1, \ldots, j_l, \ldots, j_p) = \psi_\kappa(j_1, \ldots, j_l - 1, \ldots, l_p) \quad \textit{unless} \quad j \in J'.$$

Condition (c) implies that for every little cube c in κ there are p integers

$j_1, \ldots, j_p$ such that

(e) $\operatorname{supp} \psi_\kappa(\iota)\psi_\kappa(\iota')^{-1} \subseteq Z_{j_1}^{(Q)} \cup \cdots \cup Z_{j_p}^{(Q)}$,

where ι, ι' are any vertices of c. By (a), the diffeomorphism $\psi_\kappa(\iota)\psi_\kappa(\iota')^{-1}$ is independent of the choice of cube κ containing c. Therefore if ι_c is the first vertex of c we may put

$$\psi_c(\iota) = \psi_\kappa(\iota)\psi_\kappa(\iota_c)^{-1}$$

where $c \subset \kappa$. Clearly, this defines a compatible family of maps on the vertices of the subdivision K^* of K. Using (d) and (e), one can easily find a function α from the little cubes in K^* to the subsets of $\{1, \ldots, M\}$ which has the properties: $|\alpha(c)| \leq \dim c$; $\alpha(c) \subset \alpha(c')$ if $c \subset c'$; $\alpha(c) \in J'$ if $c \subset K'^*$;

$$\operatorname{supp} \psi_c(\iota) \subset Z_{\alpha(c)}^{(Q)} = \bigcup_{j \in \alpha(c)} Z_j^{(Q)}.$$

Therefore the triple (K^*, K'^*, ψ) is supported by $(\mathfrak{X}^{(Q)}, \mathfrak{X}'^{(Q)})$ and hence also by $(\mathcal{V}, \mathcal{V}')$. It remains to extend the ψ_c to the whole of K^* and to show that the resulting triple $(K^*, K'^*, \bar{\psi})$ is homotopic to (K, K', f). This will be the case because the ψ_c are close to the f_κ by (b).

(4.5) *Extending the ψ_c to $\bar{\psi}_c$*

We extend the ψ_c to a compatible family of maps $\bar{\psi}_c : c \to \mathcal{N}_p$, where $p = \dim c$, in such a way that

$$\operatorname{supp} \bar{\psi}_c \subset Z_{\alpha(c)}^{(Q+p)} \quad \text{for all} \quad c.$$

This may be done inductively over the skeleta of K^*. When $\dim c = 1$, the values of ψ_c at the two points of ∂c lie in $\mathcal{N}_0$ and one can extend ψ_c to the rest of c by (4.2)(b). Now suppose inductively that $\bar{\psi}_c$ has been defined for all c of dimension $<p$. If c has dimension p, then the compatibility conditions imply that $\bar{\psi}_c$ is already defined on ∂c. In fact, if c' is a face of c with first vertex $\iota_{c'}$, we must have

$$\bar{\psi}_c(a) = \bar{\psi}_{c'}(a)\psi_c(\iota_{c'}), \quad \text{for all} \quad a \in c'.$$

Hence $\bar{\psi}_c(a) \in \mathcal{N}_{p-1}\mathcal{N}_0$ for all $a \in \partial c$ by the inductive hypothesis. Also $\operatorname{supp} \bar{\psi}_c \mid \partial c \subset Z_{\alpha(c)}^{(Q+p-1)}$. Therefore one can extend $\bar{\psi}_c$ to the whole of c by condition (4.2)(b).

Thus we have a triple $(K^*, K'^*, \bar{\psi})$ which is supported by $(\mathscr{Z}^{(Q+d)}, \mathscr{Z}'^{(Q+d)})$ and hence also by $(\mathscr{V}, \mathscr{V}')$.

(4.6) *Construction of the homotopy* $(K \times I, F)$

Clearly there is a cycle $(K, \bar{\bar{\psi}})$ which gives $(K^*, \bar{\psi})$ upon the subdivision of K. In fact

$$\bar{\bar{\psi}}_\kappa(a) = \bar{\psi}_c(a)\psi_\kappa(\iota_c) \quad \text{for} \quad a \in c \subset \kappa.$$

Hence $\bar{\bar{\psi}}_\kappa(a) \in \mathcal{N}_{d+1}$ for all $a \in \kappa$ by (4.2)(b). Further, if $Z'^{(r)} = \bigcup \{Z_j^{(r)} : j \in J'\} \subset W'$, then for each $\kappa \in K'$ we have

$$\operatorname{supp} \bar{\bar{\psi}}_\kappa \subset \bigcup_{c \subset \kappa} Z'^{(Q+d+1)}_{\alpha(c)} \subset Z'^{(Q+d+1)} \subset W'$$

By repeating the argument of (4.5) one can easily define maps $F_\kappa : \kappa \times I \to \mathcal{N}_{d+p+1}$, where $p = \dim \kappa$, so that the following conditions are satisfied:

(i) $F_\kappa(a, 0) = f_\kappa(a)$ and $F_\kappa(a, 1) = \bar{\bar{\psi}}_\kappa(a)$ for $a \in \kappa$;
(ii) $F_\kappa(v, t) = f_\kappa(v)$ for each vertex v of κ;
(iii) $F_\kappa(a, t) = F_\lambda(a, t) f_\kappa(v_\kappa)$ if $a \in \lambda \subset \kappa$;
(iv) for each p-cube κ in K'

$$\operatorname{supp} F_\kappa \subset Z'^{(Q+d+p+1)} \subset W'.$$

Note that conditions (i) and (ii) are consistent by (4.4)(b). Also (i) and (iv) are consistent because $\operatorname{supp} f_\kappa \subset X \subset Z'^{(Q)}$ for all $\kappa \in K'$. By (iii) these F_κ are compatible and so define a chain $(K \times I, F)$ which clearly has all the properties required by Lemma 2.5. Observe in particular that the restriction of F to $K \times 1$ is just $(K, \bar{\bar{\psi}})$. This holds because, by (ii), no renormalization is needed: compare (3.7).

This completes the proof of the deformation lemma, modulo the proof of Lemma 4.4.

(4.7) *Proof of Lemma* 4.4

We prove the lemma first for $d = 1$ and 2. The general case will be proved by induction in (4.14). When $d = 1$, the compatibility conditions are irrelevant and it suffices to define, for each 1-cube κ in K, elements $\psi_\kappa(i)$, $0 \leq i \leq M$, of $\mathcal{N}_0$ such that $\psi_\kappa(0) = \mathrm{id}$, $\psi_\kappa(M) = f_\kappa(M)$ and

$$\operatorname{supp}(\psi_\kappa(j)\psi_\kappa(j-1)^{-1}) \subset Z_j^{(Q)}.$$

Our construction will be based on a very careful choice of these $\psi_\kappa(j)$.

If Y is a (compact) submanifold of Int W_1 and if $g \in \mathcal{G}_1$ we will say that **supp** $g \subset Y$ if g has support in Int Y and if the flux of g with respect to Y is zero. The second condition means that the element $\Phi_Y(g)$ of $H_c^{n-1}(\text{Int } Y; \mathbf{R})$ is zero. Clearly **supp** $g \subset W_i$ for all $g \in \mathcal{G}_i$, $i = 1, 2$.

The following lemma is essentially due to Thurston [11]. A complete proof is given by Ismagilov in [5] 2.4, and in Banyaga [1] III.3.2 for the case $n = 2$. We will give a proof here in a form convenient for our purposes, using Ismagilov's method.

LEMMA 4.8 (Fragmentation Lemma). *Let s be any integer, $1 \leq s \leq Q$, and let $\mathcal{M}_1$ be any neighbourhood of the identity in $\mathcal{G}_1$. Then there is a neighbourhood of the identity $\mathcal{M}_0 \subset \mathcal{M}_1$ such that every $g \in \mathcal{M}_0$ may be decomposed into a sequence $g(0) = \text{id}$, $g(1), \ldots, g(M) = g$ which satisfies the following conditions for each j:*

(i) $g(j) \in \mathcal{M}_1$;
(ii) $\text{supp}\,(g(j)g(j-1)^{-1}) \subset Z_j^{(s)}$;
(iii) $\mathbf{supp}\,(gg(j)^{-1}) \subset W_1 - \bigcup_{i \leq j} Z_i^{(-s)}$.

Proof. We construct the $g(j)$ by induction on j. Since the construction involves M steps, it will be clear that there is a C^1-neighbourhood $\mathcal{M}_0$ such that the $g(j)$ may all be chosen in $\mathcal{M}_1$. Moreover, we will assume that $\mathcal{M}_0$ is so small that any diffeomorphism which we encounter, for example $gg(j-1)^{-1}$ below, is in the neighbourhood $\mathcal{N}_0$ defined in (4.2).

If $g(j-1)$ is already defined, then $g(j)$ must have the form $s(j)^{-1}g(j-1)$ where

$$s(j) = \begin{cases} g(j-1)g^{-1} & \text{on } \bigcup\limits_{i \leq j} Z_i^{(-s)} \\ \text{id} & \text{outside } Z_j^{(s)}. \end{cases}$$

Thus, in order to define $g(j)$ we must first extend $s(j)$ over the whole of W_1. Second, we must check that the extension can be chosen so that $gg(j)^{-1}$ has zero flux in $W_1 - \bigcup_{i \leq j} Z_i^{(-s)}$. We will see that these two questions are related.

Now, $s(j)$ is defined on the complement of

$$Q_j^{(s)} = Z_j^{(s)} - \bigcup_{i \leq j} Z_j^{(-s)}$$

and is injective there because of our assumption that $gg(j-1)^{-1}$ is in $\mathcal{N}_0$. If j corresponds to a p-simplex in T, that is, if $M_{p-1} < j \leq M_p$, then one can easily check that $Q_j^{(s)} \cong S^{n-p-1} \times D^{p+1}$. Thus $Q_j^{(s)}$ is connected when $p < n-1$, and so $s(j)$ has an extension in $\mathcal{G}_1$. (See Remark (3.3).)

Let us now consider the second conditon. When $j \leq M_{n-3}$, the set $\bigcup_{i \leq j} Z_i^{(-s)}$ retracts onto part of the $(n-3)$-skeleton of T and so $H_c^{n-1}(\text{Int } W_1 - \bigcup_{i \leq j} Z_i^{(-s)}) = H_c^{n-1}(\text{Int } W_1)$. Therefore, for these values of j we just need $\Phi_{W_1}(gg(j)^{-1}) = 0$, which is true since g and $g(j)$ belong to $\mathscr{G}_1$. However, if $M_{n-3} < j \leq M_{n-2}$, the condition is significant. Let us suppose that $j = M_{n-3} + m$. Then $Z_j^{(s)}$ is a thickening of the $(n-2)$-simplex Δ_m^{n-2}, and $\bigcup_{i \leq j} Z_i^{(-s)}$ is a thickening of

$$T_j = ((n-3)\text{-skeleton of } T) \cup \{\Delta_l^{n-2} : l \leq m\}.$$

Let us denote the flux homomorphism relative to $W_1 - T_j$ by Φ_j. Then Φ_j is defined on those $g \in \mathscr{G}_1$ with support in $\text{Int } W_1 - T_j$, and it takes values in $H_c^{n-1}(\text{Int } W_1 - T_j)$. The inductive hypothesis implies that $\Phi_{j-1}(gg(j-1)^{-1}) = 0$, and we want to choose an extension $s(j)$ so that $\Phi_j(gg(j-1)^{-1}s(j)) = 0$. Consider the diagram

$$\begin{tikzcd}
H^{n-2}(\Delta_m^{n-2}, \partial\Delta_m^{n-2}) \arrow[r, "\delta"] \arrow[dr] & H_c^{n-1}(\text{Int } W_1 - T_j) \arrow[r, "j^*"] & H_c^{n-1}(\text{Int } W_1 - T_{j-1}) \\
& H_c^{n-1}(Q_j^{(s)}). \arrow[u, "i^*"'] &
\end{tikzcd} \tag{*}$$

Note that the top row is exact, and that δ is either injective or zero. Let $s'(j) \in \mathscr{G}_1$ be any extension of $s(j)$. Since it has support in $Z_j^{(s)} = Z_j^{(s)} - T_{j-1}$, the element $\Phi_{j-1}(s'(j))$ is defined. Moreover $\Phi_{j-1}(s'(j)) = 0$ because $Z_j^{(s)}$ is contractible. Hence

$$j^*\Phi_j(gg(j-1)^{-1}s'(j)) = \Phi_{j-1}(gg(j-1)^{-1}) + \Phi_{j-1}(s'(j)) = 0.$$

Therefore, when $\delta = 0$, any extension $s'(j)$ will do. If $\delta \neq 0$ the possible choices for the extension of $s(j)$ have the form $s'(j)t(j)$ where $\text{supp } t(j) \subset Q_j^{(s)} \subset \text{Int } W_1 - T_j$. Let Φ_j' be the flux homomorphism relative to $Q_j^{(s)}$. Then $\Phi_j = i^*\Phi_j'$ in the diagram above. It is not hard to see that Φ_j' is surjective. Therefore, because $\text{Im } \delta = \text{Im } i^*$, one can choose $t(j)$ so that

$$\Phi_j(gg(j-1)^{-1}s'(j)t(j)) = \Phi_j(gg(j-1)^{-1}s'(j)) + \Phi_j(t(j)) = 0.$$

Thus a suitable extension of $s(j)$ can be found when $j \leq M_{n-2}$.

Now consider j in the range $M_{n-2} < j \leq M_{n-1}$. Notice that $\text{supp } (gg(M_{n-2})^{-1}) \subset W_1 - T^{(n-2)}$, where $T^{(n-2)}$ is the $(n-2)$-skeleton of T. Using the definition of the flux homomorphism given in §2 above, one can check that for each $j = M_{n-2} + m$ the two components of $Z_j^{(s)} - gg(M_{m-2})^{-1}(\Delta_m^{n-1})$ have the same volume as the corresponding components of $Z_j^{(s)} - \Delta_m^{n-1} \cong Q_j^{(s)}$. (This holds because $gg(M_{n-2})^{-1}$

satisfies condition (iii).) Thus $s(j)$ can be extended for each such j. But for these j condition (iii) is automatically satisfied. Hence the $g(j)$ can be defined for these j. The $g(j)$ for $j>M_{n-1}$ are now uniquely determined, since $Q_j^{(s)}=\emptyset$ in this case. □

We will say that the elements $g(j)$, $0\leq j\leq M$, of Lemma 4.8 form a *canonical decomposition of* g *with respect to* $Z^{(s)}$. We will need the following sharpened version of this lemma.

LEMMA 4.9. *Given any neighbourhood $\mathcal{M}_1$ there is a neighbourhood $\mathcal{M}_0\subset\mathcal{M}_1$ such that if $g\in\mathcal{M}_0$ and if*

$$\mathbf{supp}\, g\subset Q_k^{(s)}=Z_k^{(s)}-\bigcup_{i\leq k} Z_i^{(-s)}$$

for some k and s then g has a canonical decomposition with respect to $\mathscr{X}^{(s+1)}$ such that

(i) $g(j)=\mathrm{id}$ *for $j\leq M_p$, where $M_{p-1}<k\leq M_p$; and*
(ii) *each $g(j)\in\mathcal{M}_1$ and has support in $Z_k^{(s+1)}$.*

Proof. This is a straightforward generalization of Lemma 4.8 and is proved in the same way. Condition (i) is possible because $Q_k^{(s)}$ is covered by the sets $Z_j^{(s+1)}$, $j>M_p$. Since there are only a finite number of s and k, the neighbourhood $\mathcal{M}_0$ clearly exists. □

Remark 4.10. If $\mathbf{supp}\, g\subset Z'^{(s)}\subset W'$, one can apply Lemmas 4.8 and 4.9 using the cover $\mathscr{X}'$ of W' instead of $\mathscr{X}$. Hence one can assume in addition that $g(j)=g(j-1)$ for $j\notin J'$.

(4.11) *Proof of Lemma* 4.4 *for* $d=2$

When $\dim\kappa=1$, the integer points in Λ_κ are j, $0\leq j\leq M$, and we define the $\psi_\kappa(j)$ to be a canonical decomposition of $f_\kappa(M)$ with respect to $\mathscr{X}^{(1)}$. If $\kappa\in K'$ we may by (4.10) assume that $\psi_\kappa(j)=\psi_\kappa(j-1)$ if $j\notin J'$. Thus (4.4)(d) is satisfied.

Let us now consider a 2-cube κ. Its integer points are (j,k), $0\leq j,k\leq M$. The compatibility conditions (4.4)(a) determine ψ_κ on $\partial\kappa$. We will also suppose that ψ_κ is defined along the diagonal (j,j), $0\leq j\leq M$, to be a canonical decomposition of $\psi_\kappa(M,M)=f_\kappa(M,M)$ with respect to $\mathscr{X}^{(1)}$. We will then show how to define the $\psi_\kappa(j,k)$ for $j\geq k$. The case $j\leq k$ may be obtained by symmetry.

Let $g=f_\kappa(M,M)$. then by (4.8)(iii) both $\mathbf{supp}\,(g\psi_\kappa(k,k)^{-1})$ and

supp $(g\psi_\kappa(M,k)^{-1})$ are contained in $W_1 - \bigcup_{i\le k} Z_i^{(-1)}$. Hence

(i) $\quad \mathbf{supp}\,(\psi_\kappa(M,k)\psi_\kappa(k,k)^{-1}) \subset W_1 - \bigcup_{i\le k} Z_i^{(-1)}$

Let us suppose that the $\psi_\kappa(j,k)$ have been defined for all (j,k) where $j \ge k$ and $k < l$ in such a way that:

(ii) $\quad \operatorname{supp}(\psi_\kappa(j,k)\psi_\kappa(j-1,k)^{-1}) \subset Z_j^{(2k+1)}$,
$\quad\quad \operatorname{supp}(\psi_\kappa(j,k)\psi_\kappa(j,k-1)^{-1}) \subset Z_k^{(2k+1)}$,

(iii) $\quad \mathbf{supp}\,(\psi_\kappa(M,k)\psi_\kappa(j,k)^{-1}) \subset W_1 - \bigcup_{i\le j} Z_i^{(-2k-1)}$.

(These conditions are satisfied when $l = 1$.) Put

$$h_{0l} = \psi_\kappa(l,l)\psi_\kappa(l,l-1)^{-1}, \qquad h_{1l} = \psi_\kappa(M,l)\psi_k(M,l-1)^{-1}.$$

Then we claim that

(iv) $\quad \operatorname{supp}(h_{1l}h_{0l}^{-1}) \subset Q_l^{(2l)}$.

To see this, note first that $\operatorname{supp} h_{1l} \subset Z_l^{(1)}$ by (4.8)(ii). Also,

$$\begin{aligned}\operatorname{supp} h_{0l} &\subset \operatorname{supp}(\psi_\kappa(l,l)\psi_\kappa(l-1,l-1)^{-1}) \cup \operatorname{supp}(\psi_\kappa(l-1,l-1)\psi_\kappa(l,l-1)^{-1})\\ &\subset Z_l^{(1)} \cup Z_l^{(2l-1)} = Z_l^{(2l-1)} \text{ by (ii) above.}\end{aligned}$$

Further

$$\begin{aligned}h_{1l}h_{0l}^{-1} &= \psi_\kappa(M,l)\psi_\kappa(M,l-1)^{-1}\psi_\kappa(l,l-1)\psi_\kappa(l,l)^{-1}\\ &= (\psi_\kappa(M,l)\psi_\kappa(M,l-1)^{-1}\psi_\kappa(l,l-1)\psi_\kappa(M,l)^{-1})(\psi_\kappa(M,l)\psi_\kappa(l,l)^{-1})\\ &= \text{id on } \psi_\kappa(M,l)Z_i^{(-2l+1)} \cap Z_i^{(-1)} \quad \text{for} \quad i \le l\end{aligned}$$

by (iii) above and (4.8)(iii). But $\psi_\kappa(M,l)Z_i^{(-2l+1)} \supset Z_i^{(-2l)}$ since $\psi_\kappa(M,l) \in \mathcal{N}_0 \subset \mathcal{M}$. Therefore (iv) holds.

When $M_{p-1} < l \le M_p$ for $p \ne n-2$, then $H_c^{n-1}(Q_l^{(2l)}) = 0$. In this case (iv) is equivalent to

(iv)′ $\quad \mathbf{supp}\,(h_{1l}h_{0l}^{-1}) \subset Q_l^{(2l)}$.

Hence Lemma 4.9 imples that $h_{1l}h_{0l}^{-1}$ has a canonical decomposition $h(j)$, $j \succ M_p$, with respect to $\mathscr{X}^{(2l+1)}$ and such that $\operatorname{supp} h(j) \subset Z_l^{(2l+1)}$ for all j. Now set

(v) $\quad \psi_\kappa(j, l) = h(j) h_{0l} \psi_\kappa(j, l-1) \quad$ for $\quad l \leq j \leq M.$

It is not hard to check that (ii) and (iii) above hold. For example

$$\psi_\kappa(M, l)\psi_\kappa(j, l)^{-1} = (h_{1l} h_{0l}^{-1} h(j)^{-1}) h(j) h_{0l} (\psi_\kappa(M, l-1)\psi_\kappa(j, l-1)^{-1}) h_{0l}^{-1} h(j)^{-1}$$

is the identity on $Z_i^{(-2l-1)} \cap h(j) h_{0l}(Z_i^{(-2l+1)}) = Z_i^{(-2l-1)}$ for all $i \leq j$.

It remains to consider the rows $\psi_\kappa(\cdot, l)$ where $M_{n-3} < l \leq M_{n-2}$. We have to ensure that (iv)′ holds. Consider diagram (*) in Lemma 4.8. If $\delta \neq 0$, then i^* is injective and so it suffices to prove that $\Phi_l(h_{1l} h_{0l}^{-1}) = 0$. If $g = \psi_\kappa(M, M)$, we know by (4.8)(iii) that

$$0 = \Phi_l(g\psi_\kappa(M, l)^{-1}) = \Phi_l(g\psi_\kappa(M, l-1)^{-1} h_{1l}^{-1})$$

and

$$0 = \Phi_l(g\psi_\kappa(l, l-1)^{-1} h_{0l}^{-1}).$$

But $\Phi_l(\psi_\kappa(M, l-1)\psi_\kappa(l, l-1)^{-1}) = 0$ by (iii) above. Hence $\Phi_l(h_{1l} h_{0l}^{-1}) = 0$ as required.

For those l for which $\delta = 0$, one argues rather differently. Notice that in the above construction the elements $\psi_\kappa(j, k)$ in the triangle $M_{p-1} < k \leq j \leq M_p$ depend only on the diagonal elements $\psi_\kappa(k, k)$ and the elements $\psi_\kappa(j, M_{p-1})$ in the $M_{p-1}{}^{\text{th}}$ row. (This is true because we chose the $h(j)$ in (v) so that $h(j) = \text{id}$ for $j \leq M_p$.) Therefore, once the elements $\psi_\kappa(j, M_{n-3})$, $M_{n-3} < j \leq M_{n-2}$ are chosen, the elements $h_{1k} h_{0k}^{-1}$, $M_{n-3} < k \leq M_{n-2}$, are determined. Two different choices of $\psi_\kappa(l, M_{n-3})$ differ by an element $t(l)$ with support in $Q_l^{(2l)}$. Further, if $\delta = 0$ for l, then $\Phi_l'(t(l))$ can be arbitrary. It is not hard to check that if one changes $\psi_\kappa(l, M_{n-3})$ by $t(l)$ then h_{0l} changes by a conjugate of $t(l)$ and so $\Phi_l'(h_{1l} h_{0l}^{-1})$ changes by $\Phi_l'(t(l))$. Hence one can choose the row $\psi_\kappa(\cdot, M_{n-3})$ so that (iv)′ is satisfied for all l with $\delta = 0$. This argument does not make sense when $n = 2$, but fortunately the map δ is never 0 in this case.

This completes the construction of the $\psi_\kappa(j, k)$, $j \geq k$. The $\psi_\kappa(j, k)$, $j \leq k$, are defined symmetrically. It remains to check that the conditions of Lemma 4.4 are satisfied. Now, conditions (a) and (b) are clear from the construction, (c) follows from (ii) above, and (d) follows by Remark 4.10. Finally, notice that in the construction of a particular $\psi_\kappa(j, k)$ we apply Lemma 4.8 three times to define ψ_κ on the edges of the 2-simplex which contains (j, k) and then apply Lemma 4.9

exactly $k-1$ times. It follows easily that one can choose the initial neighbourhood $\mathcal{N}$ which contains the $f_\kappa(v)$ to be so small that all the $\psi_\kappa(j,k)$ lie in $\mathcal{N}_0$. This completes the proof of Lemma 4.4 when $d=2$. $\square$

In the above proof we constructed the $\psi_\kappa(j,k)$ for $0\leqslant k\leqslant j\leqslant M$ from three given elements: $\psi_\kappa(0,0)$, $\psi_\kappa(M,0)$ and $\psi_\kappa(M,M)$. This may be thought of as a "two-dimensional" version of Lemma 4.8. In dimension p we want to define elements $\psi_\kappa(\iota)$, for all ι in the integer lattice of a p-simplex, given the values of ψ_κ at the vertices of that simplex. These $\psi_\kappa(\iota)$ should have certain properties which are formulated in the following definition.

DEFINITION 4.12. Let σ_p be the p-simplex $\{x:0\leqslant x_p\leqslant\cdots\leqslant x_1\leqslant M\}$ with set of vertices V_p and integer lattice Λ_p, and let $\mathcal{P}$ be a neighbourhood of the identity in $\mathcal{G}_1$. Suppose elements $\psi_\sigma(v)$, $v\in V_p$, are given where $\psi_\sigma(0,\ldots,0)=\mathrm{id}$. Then a *canonical decomposition of the $\psi_\sigma(v)$, $v\in V_p$ in $\mathcal{P}$ and with respect to $\mathcal{Z}^{(s)}$* is a collection

$$\psi_\sigma(j_1,\ldots,j_p):(j_1,\ldots,j_p)\in\Lambda_p,$$

of elements of $\mathcal{P}$ satisfying the conditions:

(i) $\operatorname{supp}(\psi_\sigma(j_1,\ldots,j_l,\ldots,j_p)\psi_\sigma(j_1,\ldots,j_l-1,\ldots,j_p)^{-1})\subseteq Z_{j_l}^{(s)}$
(ii) for each l, $1\leqslant l\leqslant p$,

$$\mathbf{supp}\,(\psi_\sigma(M,\ldots,M,M,j_{l+1},\ldots,j_p)\psi_\sigma(M,\ldots,M,j_l,j_{l+1},\ldots,j_p)^{-1})\subset W_1-\bigcup_{i\leqslant j_l}Z_i^{(-s)}.$$

This decomposition will be said to be *subordinate to $Q_k^{(s)}$* if, in addition,

(iii) $\psi_\sigma(j_1,\ldots,j_p)=\mathrm{id}$ for $j_p\leqslant\cdots\leqslant j_1\leqslant N_q$ where $N_{q-1}<k\leqslant N_q$
(iv) $\operatorname{supp}\psi_\sigma(j_1,\ldots,j_p)\subset Z_k^{(s)}$ for all $(j_1,\ldots,j_p)\in\Lambda_p$.

LEMMA 4.13. (a) *For any neighbourhood of the identity $\mathcal{M}_1$ there is a neighbourhood $\mathcal{M}_0$ such that, if $\psi_\sigma(\iota)$, $\iota\in\Lambda_p\cap\partial\sigma_p$, are any elements of $\mathcal{M}_0$ which satisfy* (4.12)(i), (ii) *for some s' wherever this makes sense, then one can define $\psi_\kappa(\iota)$ for the other $\iota\in\Lambda_p$ so that the $\psi_\kappa(\iota)$ form a canonical decomposition of the $\psi_\sigma(v)$ in $\mathcal{M}_1$ with respect to $\mathcal{Z}^{(r)}$, with $r=s'+2M$.*

(b) *If $\psi_\sigma(v)$, $v\in V_p$, are any elements of $\mathcal{M}_0$ such that*

$$\mathbf{supp}\,\psi_\sigma(v)\subset Q_k^{(s)}\quad\textit{for all}\quad v\in V_p,$$

then there is a canonical decomposition of the $\psi_\sigma(v)$ in $\mathcal{M}_1$ which is subordinate to $Q_k^{(s+1)}$.

Proof. Let us suppose that (a) and (b) have been proved for all $p'<p$, where $p\geq 3$. Further, we will suppose that $\psi_\sigma(j_1,\ldots,j_p)$ has been defined for all $(j_1,\ldots,j_p)$ with $j_p<l$ in such a way that (4.12)(i), (ii) is satisfied with $s=s'+2m$ when $j_p=m$. Consider the level $j_p=l$. The $(p-1)$-simplex $\sigma_p\cap(j_p=l)$ has vertices $v_i(l)$, $0\leq i\leq p-1$, where

$$v_i(j)=(M,\ldots,M,l,\ldots,l,j),\quad \text{with } i \text{ factors of } M.$$

Put $h_{il}=v_i(l)v_i(l-1)^{-1}$. For each i, the four elements $v_i(l)$, $v_i(l-1)$, $v_{p-1}(l)$ and $v_{p-1}(l-1)$ are contained in a 2-dimensional face of σ_p. (Here we need $p\geq 3$.) Therefore, our assumption that the $\psi_\sigma(\iota)$ satisfy (4.12)(i), (ii) on $\partial\sigma_p$, together with the calculation of (4.11), shows that

$$\mathbf{supp}\; h_{il}h_{p-1l}^{-1}\subset Q_l^{(r)}\quad \text{for}\quad r=s'+2l-1.$$

Hence

$$\mathbf{supp}\; h_{il}h_{0l}^{-1}\subset Q_l^{(r)}\quad \text{for}\quad 0\leq i\leq p-1.$$

Since (b) holds when $p'=p-1$, one can therefore find a canonical decomposition $h(j_1,\ldots,j_{p-1})$ of the $h_{il}h_{0l}^{-1}$ which is subordinate to $Q_l^{(r)}$, for $r=s'+2l$. One now checks as in (4.11) that the elements

$$\psi_\sigma(j_1,\ldots,j_{p-1},l)=h(j_1,\ldots,j_{p-1})h_{0l}\psi_\sigma(j_1,\ldots,j_{p-1},l-1)$$

satisfy the inductive hypothesis.

The proof of (b) is similar. One should insert an appropriate number of auxiliary covers in between $\mathcal{X}^{(s)}$ and $\mathcal{X}^{(s+1)}$, and then should choose the $\psi_\rho(\iota)$, where ρ is a q-dimensional face of σ, in order of increasing q. □

(4.14) *Proof of Lemma* 4.4 (*general case*)

One constructs the $\psi_\kappa(\iota)$ inductively over the skeleta of K using Lemma 4.13(a). The argument is just like that used when $d=2$, and its details will be left to the reader. □

(4.15) *Remark*

In the proof of Lemma 2.5 we have used two properties of the group $\mathcal{G}_1$: first, that it is locally contractible, so that the neighbourhoods of (4.2) exist, and second, that it has an appropriate isotopy extension theorem, so that the fragmentation Lemma 4.8 holds. Fathi shows in [3]§4 that the group of all homeomorphisms of a compact manifold which preserve a good measure has these properties. Hence Lemma 2.5 holds for this group. Indeed all the results of this paper are valid for this group.

Lemma 2.5 also holds for the group of all homeomorphisms of a compact manifold by [2]. Using this, one can presumably extend the proof given by Mather in [8] of the Mather–Thurston theorem to the C^0-case. See [8] §6.

Finally, note that Lemma 2.5 holds in the symplectic case. For the group of all symplectic diffeomorphisms of a symplectic manifold is locally contractible by [12], and Banyaga proves the equivalent of Lemma 4.8 in [1] III.3.2. In fact, let us say in this case that **supp** $g \subset Y$ if supp $g \subset \mathrm{Int}\, Y$ and if $S(g) = 0$ in $H^1_c(\mathrm{Int}\, Y)$, where S is the homomorphism defined by Banyaga in [1] II.1. Then Lemma 4.8 holds as stated, and the proof is the same, except that the obstruction to extending $s(j)$ in the required fashion now occurs for $j \leqslant M_0$ instead of for $M_{n-3} < j \leqslant M_{n-2}$. Therefore, just as when $n = 2$ in the volume preserving case, the map δ in the diagram corresponding to (*) is never zero. This means that condition (iv)′ in (4.11) is always satisfied, which slightly simplifies that proof.

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Corrigendum to: "Aspherical four-manifolds and the centres of two-knot groups"

JONATHAN A. HILLMAN

M. N. Dyer has pointed out that the proof of the key Lemma on "hopfian" rings in §2 of [1] is incorrect. As I have been unable to find a correct argument, the results on pages 465–469 are moot. (Corollaries 2 and 3 on page 470 are true as it is easy to see that the lemma holds for any commutative ring, while the results in §5 use only Kaplansky's original theorem, and not the lemma.) I hope that some ring-theorist may be able to prove the lemma.

In the first line of page 469, the map from $H^2(C^*)$ to $\mathrm{Hom}_\Gamma\,(H_2(C_*), \Gamma)$ given by the universal coefficient spectral sequence is, a priori, only a monomorphism. However the theorem is still true (without any essential change in the argument), modulo the lemma.

A. Suciu has pointed out that the map Φ in line 1 of page 471 should be replaced by its square Φ^2, to ensure that the mapping torus M be orientable.

I am grateful to Dyer and Suciu for their observations.

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Class numbers and periodic smooth maps

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§1. Introduction

For m a positive integer, let ζ_m be a primitive mth root of unity, and let $Z(\zeta_m)$ denote the ring of integers of the cyclotomic field $Q(\zeta_m)$ over the rationals Q. It is well-known that $Z(\zeta_m)$ is just the collection of polynomials in ζ_m with integral coefficients. Two ideals $\mathscr{a}$ and $\mathscr{b}$ of $Z(\zeta_m)$ are said to be equivalent if there exists $\mathscr{c} \in Q(\zeta_m)$ with $\mathscr{a} = \mathscr{b} \cdot \mathscr{c}$. The equivalence classes of ideals form the class group C_m, a finite group with order $h(m)$, the class number of m. Complex conjugation induces an involution on C_m; let C_m^- denote its (-1)-eigenspace. Then $h^-(m)$, the order of C_m^- is a factor of $h(m)$. This paper studies the significance of the parity of $h^-(m)$ for the behavior near fixed points of periodic diffeomorphisms of the sphere. From this definition, it is almost immediate that $h(m)$ and $h^-(m)$ actually have the same parity. For m a prime this relationship between $h(m)$ and $h^-(m)$, defined differently, was first noticed by Kummer [K].

Let Σ be a smooth manifold, $f:\Sigma \to \Sigma$ a diffeomorphism. Let x be a fixed point; i.e., $f(x) = x$. Then the derivative

$$(df)_x : \Sigma_x \to \Sigma_x$$

will be a linear isomorphism of the tangent space of Σ at x with itself. Then a question of P. A. Smith asks whether, for Σ a (homotopy) sphere, f periodic (i.e., $f^m = id_\Sigma$ for some m), and x and y isolated fixed points of f, $(df)_x$ and $(df)_y$ would be *linearly similar.* Linear similarity means by definition that there is a linear isomorphism $L:\Sigma_x \to \Sigma_y$ with $L(df)_x L^{-1} = (df)_y$. Results of Atiyah–Bott, Milnor (see [AB]), Bredon, and Sanchez established this in many cases. However, in [CS2], this was shown to be false even when Σ is a differentiable sphere. Petrie had previously given examples of actions of highly non-cyclic groups with inequivalent fixed point representations. His examples fail to satisfy the conclusions of classical "Smith theory"; they all have subgroups whose fixed point sets are

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disconnected but not isolated. The actions of [CS2] satisfy the conclusions of Smith theory in the strongest possible way, viz the fixed point set of f^j, any j, is also a differentiable sphere. Some more recent examples will appear in the Ph.D. thesis of Alan Siegel.

Given a periodic diffeomorphism f of Σ, a root of unity ξ, and a fixed point x of f, let $m_f(x, \xi)$ denote the multiplicity of ξ as an eigenvalue of $(df)_x$. The linear similarity of df_x, df_y is just the assertion that $m_f(x, \xi) = m_f(y, \xi)$, for all roots of unity ξ. By the *mod two Smith Conjecture* one means the following weaker statement: For each root of unity ξ,

$$m_f(x, \xi) \equiv m_f(y, \xi) \qquad (\text{mod } 2).$$

Unlike the original question, the mod two version is true for smooth maps of period a power of *any* prime, by the above quoted results for odd primes and by results of Bredon for the prime two.

Recall that a periodic map g is free if, for all j, either g^j is the identity or g^j fixes no points. For smooth maps of homotopy spheres of period q or $2q$, with q odd, that are free outside of a 1-dimensional set, it follows from the arguments of Atiyah–Bott, Milnor and Sanchez that the mod two Smith conjecture is actually true.

THEOREM 1.1. *For a positive odd integer q, the following statements are equivalent:*

1. *The mod two Smith conjecture holds for periodic diffeomorphisms of homotopy spheres, of period $4q$, that are free outside of a 1-dimensional set*; *and*
2. *q has at most two prime divisors and $h^-(q) \equiv 1$* (mod 2).

Thus, the mod two Smith Conjecture holds for periodic smooth maps of period $4q$, q odd, free outside a one-dimensional set, that have period less than 116. The first counter-example is a map of period 116 on the standard sphere of dimension 17. It can actually be shown that the mod two Smith Conjecture holds, for *all* periods less than 112, for periodic diffeomorphisms that are free outside a 1-dimensional set.

When $h^{-1}(q)$ is even or q has more than two prime divisors, it can also be shown that the minimal dimension of a counter-example to the mod two Smith Conjecture is at most $2\phi(q)+1$, and that in this counter-example, the eigenvalues of $(df)_x$ and $(df)_y$ other than -1 will (necessarily) be disjoint and of multiplicity one. (Here $\phi(q)$ is the Euler ϕ-function.) Moreover, there is a counter-example to the mod two Smith Conjecture, on the standard sphere, in every odd dimensional above the minimal one. An interesting problem relating topology and number theory is to find a formula for the minimal dimension of a counterexample to the mod two Smith Conjecture, when $h^-(q)$ is odd or q has more than two prime factors.

The class number actually plays a role even when q has more than two prime divisors. In fact, it will actually be shown below that 1. of Theorem 1.1 is equivalent to the assertion that the index of the Stickelberger ideal S^- in $Z[G(q)]^-$ is odd, $G(q)$ the Galois group of $Q(\zeta_q)$ over Q. Theorem 1.1 then follows from the results of Iwasawa that the index of this ideal is $h^-(q)$ for q divisible by at most two primes and of Sinnott that this index is $2^{g-2}h^-(q)$ for g the number of primes dividing q, $g>1$. (See [SI].) Thus, the failure of the mod two Smith Conjecture for more than three prime divisors is due to the extraneous powers of two in Sinnott's theorem. It is possible to relate with more refinement the type of behavior that can occur at the fixed points with the structure of $(Z[G(q)]^-/S^-)$. Even when $g>2$, it will still be possible to distinguish phenomena related to the class number $h^-(q)$ and to formulate, as above, a purely topological necessary and sufficient condition that $h^-(q)$ be odd. These matters will be the subject of a future paper.

The results of [CS2] [CS3] also give much, and in some cases complete, information on the possible pairs of linear maps $(df)_x$, $(df)_y$ that can arise as derivatives at fixed points of the same periodic smooth f map of a homotopy sphere Σ. A periodic smooth map f of Σ is said to be of Smith type if the fixed points set of f^j, all j, is either discrete or connected. The results of [CS2] [CS3] provide evidence for the revised conjecture that if f is of Smith type and x and y are fixed points, then $(df)_x$ and $(df)_y$ are *topologically similar*, i.e., there is a homeomorphism $\phi:\Sigma_x\to\Sigma_y$ with $\phi(df)_x\phi^{-1}=(df)_y$.

[*Historical Note:* It should be pointed out that the conjecture that for rotations topological and linear similarity would be equivalent was first proposed at the 1935 International Topology Conference in Moscow by deRham. He reduced it to the periodic case and made a number of other fundamental contributions to this problem as well. In [KR], this conjecture was extended to include all linear endomorphisms with all eigenvalues of modules one, and their extended conjecture also reduced to the periodic case. They and other authors (see [CS1] for details) produced further evidence. In [CS1], however, this conjecture was settled in the negative.]

The revised Smith Conjecture is established in [CS2] for periodic smooth maps that are free outside a 1-dimensional set. For period $4q$, q odd, the converse is also established (compare Theorem 4.1 below). That is, given A and B, linear isomorphisms of a m-dimensional vector space, that are periodic of period $4q$ and free outside of 1-dimensional sets, there exists a periodic smooth map f of S^{m+1}, with isolated fixed points x and y, with $(df)_x$ linearly similar to A and $(df)_y$ to B, if and only if A and B are typologically similar. Thus, to Theorem 1.1 one can add the following statement as equivalent to 1 or 2:

3. *If A and B are periodic topologically similar linear maps of real vector spaces, with A free outside a 1-dimensional set, then every root of unity appears as eigenvalues of A and of B with multiplicities congruent mod two.*

For $2 \mid q$, both the algebraic and geometric situations are more involved and will be taken up in later papers.

The first two sections of this paper will be purely algebraic and number-theoretic. Using results of Iwasawa–Sinnott, we relate the notion of tempered numbers to class-numbers of cyclotomic fields. The notion of tempered numbers is a measure of how multiples of integers distribute when reduced mod a given integer. In the final section, geometric results of [CS1] and [CS2] are compared with the algebra to obtain the theorem.

§2. Tempered numbers

Let n be a positive integer and a any integer. Let $R_n(x)$ denote the least non-negative number congruent to x mod n. Then, for x any *integer*, the function[(1)] whose value is 1 if $0<R(ax)\leq n/2$ and is zero otherwise depends only upon the congruence class of x mod n and so defines a map

$$f_a^{(n)}: Z/nZ \to \{0, 1\} = Z/2Z$$

which also depnds only upon the congruence class of a mod n.

Now let $n=4q$, with q odd, $q>1$, and consider only the functions $f_a^{(4q)}$ with the least common divisor $(a, 4q)=1$, a defined mod n (i.e., $a \in (Z/nZ)^*$), and $a \equiv 1 \pmod 4$. These functions satisfy the obvious linear relations

$$(\ddagger)_n : f_a + f_{2q-a} = f_1 + f_{2q-1}.$$

As in [CS1], we say that the number $n=4q$ is *tempered* if all linear relations among the functions $\{f_a^{(n)} \mid a \in (Z)^*,\ a \equiv 1 \pmod 4\}$ are consequences of the relations $(\ddagger)_n$.

In this section we wish to reduce the notion that $4q$ is tempered to a statement about modules over $Z_2[G(q)]$. The functions $f_a^{(q)}$, $(a, q)=1$, satisfy the obvious linear relations

$$(*)_q : f_a^{(q)} + f_{-a}^{(q)} = f_1^{(q)} + f_{-1}^{(q)}.$$

* It would be equivalent to consider the function that is zero for $R(ax) \leq n/2$ and 1 otherwise.

We will say that the *odd* number q is *tempered* if all linear relations among the functions $\{f_a^{(q)} \mid a \in (Z/qZ)^*\}$ are consequences of the relations $(*)_q$.

THEOREM 2.1. *Let q be odd. Then $4q$ is tempered if and only if q is tempered.*

Proof. The proof is based on a series of simple identities. For example, suppose that $y = 4k+1$. Then

$$(q-1)y = 4kq + (q-y)$$

and

$$(2q-2)y = 8kq + 2(q-y).$$

Hence, if $0 \leq y < 4q-1$, the sum

$$f_1^{(4q)}((q-1)y) + f_1^{(4q)}((2q-2)y)$$

is zero for $0 \leq y \leq 2q$ and 1 for $y > 2q$. Thus, if we set $\delta : Z/nZ \to \{0, 1\}$ to be function that is 1 except for $\delta(0) = 0$, we obtain that for $y \equiv 1 \pmod 4$

$$f_1^{(4q)}(y) = \delta + f_1^{(4q)}((q-1)y) + f_1^{(4q)}((2q-2)y).$$

Since $f_a^{(4q)}(x) = f_1^{(4q)}(ax)$, it follows that

(2.1.1) $\quad f_a^{(4q)}(x) = \delta + f_a^{(4q)}((q-1)x) + f_a^{(4q)}((2q-2)x)), \qquad x \equiv 1 \pmod 4)$

Similarly one can obtain the formula

(2.1.2) $\quad f_a^{(4q)}(x) = \delta_{2q} + f_a^{(4q)}((1-q)x) + f_a^{(4q)}((2-2q)x)), \qquad x \equiv 1 \pmod 4)$

Here

$$\delta_i(x) = \begin{cases} 0 & x \neq i \\ 1 & x = i \end{cases}.$$

This can actually be obtained from 2.1.1 by noting that $f_a(x) = f_{-a}(x)$ and $f_a + f_{-a} = \delta + \delta_{2q}$.

For $y = 4k+2, \quad 0 \leq y \leq 4q-1$,

$$(q-1)y = 4kq + (2q-y).$$

Hence,

$$f_1(y)=\delta_0+\delta_{2q}+f_1((q-1)y).$$

Thus, we obtain, as above

(2.1.3) For $x\equiv 2 \pmod 4$,

$$f_a^{(4q)}(x)=\delta_0+\delta_{2q}+f_a^{(4q)}((2q-2)x).$$

Finally, since $f_a(x)+f_a(-x)=\delta+\delta_{2q}$, from (2.1.1) and (2.1.2) one also obtains:

(2.1.4) $f_a^{(4q)}(x)=\delta_{2q}+f_a^{(4q)}((q-1)x)+f_a^{(4q)}((2q-2)x), \qquad x\equiv 3 \pmod 4;$

(2.1.5) $f_a^{(4q)}(x)=\delta+f_a^{(4q)}((1-q)x)+f_a^{(4q)}((2-2q)x), \qquad x\equiv 3 \pmod 4.$

Now let $H_q\subset(Z/4qZ)^*$ be the set of elements congruent to 1 mod 4. Then the above identities will be used to obtain;

(2.1.6) *Let* $\lambda_a\in\{0, 1\}$ *for each* $a\in H_q$. *Then*

$$\sum_{a\in H_q}\lambda_a f_a=0 \qquad (\text{in } Z/2Z)$$

if and only if

$$\sum_{a\in H_q}\lambda_a f_a(x)=o \qquad \forall x\equiv 0 \pmod 4.$$

One implication follows by restriction. Conversely, suppose

$$\sum_{a\in H_q}\lambda_a f_a(x)=0 \qquad \forall x\equiv 0 \pmod 4.$$

Then since

$$f_a+f_{-a}=\delta+\delta_{2q}$$

and since $f_a(-x)=f_{-a}(x)$, it follows that

$$\sum_{a\in H_q}\lambda_a(\delta(x)+\delta_{2q}(x))=0 \qquad \forall x\equiv 0 \pmod 4.$$

Let $x=4$, then $\delta(x)=1$, $\delta_{2q}(x)=0$. Hence

$$\sum_{a\in H_q} \lambda_a \equiv 0 \pmod 2.$$

Suppose now that $q\equiv 1 \pmod 4$. Then for $x\equiv 1 \pmod 4$,

$$\sum_{a\in H_q} \lambda_a f_a(x) = \sum_{a\in H_q} \lambda_a(\delta + f_a^{(4q)}((q-1)x) + f_a^{(4q)}((2q-2)x)).$$

Since $q-1\equiv 2q-2\equiv 0 \pmod 4$ and since $\sum_a \lambda_a \equiv 0 \pmod 2$, the right side vanishes. Hence

$$\sum_{a\in H_q} \lambda_a f_a(x) = 0 \qquad \text{for } x\equiv 1 \pmod 4.$$

The same argument also proves that

$$\sum_{a\in H_q} \lambda_a f_a(x) = 0,$$

for $x\equiv 2 \pmod 4$, using (2.1.2), and $x\equiv 3 \pmod 4$, using (2.1.4). For $q\equiv -1 \pmod 4$, we use (2.1.2) and (2.1.5) instead. Thus

$$\sum_{a\in H_q} \lambda_a f_a(x) = 0 \qquad \forall x \in Z/4qZ,$$

so (2.1.6) is proven.

Let $\pi : H_q \to (Z/qZ)^*$ be reduction mod q. Then π is an isomorphism, and the following identity is immediately verified:

$$\text{(2.1.7)} \quad f_a^{(4q)}(4x) = f_{\pi(a)}^{(q)}(x), \qquad x = 0, 1, \ldots, q-1.$$

Combining this with (2.1.6) yields

(2.1.8) $\sum_{a\in H_q} \lambda_a f_a^{(4q)} = 0$ *if and only if*

$$\sum_{b\in(Z/qZ)^*} \omega_b f_b^{(q)} = 0, \quad \textit{where} \quad \omega_b = \lambda_{\pi} - 1_{(b)}.$$

Since $2q\equiv 2 \pmod 4$, it follows that $\pi(2q-1) = -\pi(a)$, $a\in H_q$, or, equivalently, that $\pi^{-1}(-b) = 2q - \pi^{-1}(b)$. With (2.1.8) and this observation, Theorem 2.1 is proven.

From now on, we identify $(Z/qZ)^*$ with the Galois group $G(q)$ of $Q(\zeta_q)$ over Q, ζ_q a primitive qth root of unity. Under this identification, a number a mod q, $(a, q)=1$, corresponds to the unique element σ_a of $G(q)$ with $\sigma_a(\zeta_q)=\zeta_q^a$.

Let Λ_q be the vector space over $Z/2Z=\{0, 1\}$ of functions from Z/qZ to $Z/2Z$. Then Λ_q is a module over the ring $R_q = Z_2[G(q)]$, via the action of $G(q)$ defined by

$$(\sigma_a f)(x)=f(ax).$$

Let $A_q \subset R_q$ be the *augmentation ideal*; i.e., the kernel of the map to $Z/2Z$ that sends $\sum_a \lambda_a\sigma_a$ to $\sum_a \lambda_a$. If M is any module over a ring R and $m \in M$, let

$$(m)=\{\lambda m \mid \lambda \in R\} \text{ and let } \operatorname{Ann}(m)=\{\lambda \in R \mid \lambda m=0\}.$$

PROPOSITION 2.2. *The odd number q is tempered if and only if*

$$\operatorname{Ann}(f_1^{(q)})=A_q(\sigma_1+\sigma_{-1}).$$

Proof. Clearly $\sigma_a f_1^{(q)}=f_a^{(q)}$, $(a, 4q)=1$. Hence the relation $(*)_q$ can be rewritten as

$$(\sigma_1+\sigma_a)(\sigma_1+\sigma_{-1})f_1^{(q)}=0.$$

The ideal A_q is easily seen to be generated over $Z/2Z$ by the elements $\sigma_1+\sigma_a$, $(a, q)=1$. Thus the ideal $A_q(\sigma_1+\sigma_{-1})$ consists precisely of linear combinations over $Z/2Z$ of the elements of $(\sigma_1+\sigma_a)(\sigma_1+\sigma_{-1})$ for $(a, q)=1$. The Proposition follows easily.

In view of 1.1, the equation of 2.2 is also necessary and sufficient for $4q$ to be tempered. In a future paper the general problem of when is $2^\beta q$ tempered, q odd, will be solved.

§3. Stickelberger ideals

Throughout this section let q be a fixed odd integer, $q>1$, and let $G=G(q)$, the Galois group of $Q(\zeta_q)$ over Q as above. Recall from 2.2 that q is tempered iff $\operatorname{Ann}(f_1^{(q)})=(\sigma_1+\sigma_{-1})A_q$.

Let $S\subset Z[G]$ be the Stickelberger ideal, as defined in [SI]. For any real number x, let $\langle x\rangle$ denote the least non-negative residue of x mod Z. For $c\in Z$, let

$$\theta(c)=\sum_{a\in(Z/qZ)^*}\left\langle\frac{-ca}{q}\right\rangle\sigma_a^{-1},$$

where $\sigma_a \in G$ is the element with $\sigma_a(\zeta_q) = \zeta_q^a$. Then the subgroup (i.e., Z-module) S' of $Q[G]$ generated by the elements $\theta(c)$ is actually a $Z[G]$-module, as $\sigma_b\theta(c) = \theta(bc)$. Then, by definition, $S = S' \cap Z[G]$.

For any $Z[G]$-module, M, let $M^- = \{x \in M \mid \sigma_{-1}x = -x\}$.

LEMMA 3.1* ([L] Chap. 2, §1, Lemma 1). $Z[G]^- = (\sigma_1 - \sigma_{-1})$.

In view of this lemma, S^- is contained in the $Z[G]$ ideal $(\sigma_1 - \sigma_{-1})$ generated by $\sigma_1 - \sigma_{-1}$. Let $\gamma : Z[G] \to Z_2[G] = R_q$ be mod 2 reduction. Then $\gamma S^- \subset \gamma(\sigma_1 - \sigma_{-1}) = (\sigma_1 + \sigma_{-1})$. Note that $\operatorname{Ann}(\sigma_1 + \sigma_{-1}) \supset (\sigma_1 + \sigma_{-1})$; actually it will be seen that there are equal.

THEOREM 3.2. *The R_q-modules* $\operatorname{Ann}(f_1^{(q)})/A_q(\sigma_1 + \sigma_{-1})$ *and* $\operatorname{Ann}(\gamma S^-)/(\sigma_1 + \sigma_{-1})$ *are isomorphic.*

THEOREM 3.3. *The odd number q is tempered if and only if $\gamma S^- = (\sigma_1 + \sigma_{-1})$.*

COROLLARY 3.4. *The odd number q is tempered if and only if q has at most two prime factors and $h^-(q)$ is odd.*

Corollary 3.4 follows from Theorem 3.3 and the main result of [SI].

LEMMA 3.4* $\operatorname{Ann}(\sigma_1 + \sigma_{-1}) = (\sigma_1 + \sigma_{-1})$. (Hence $R_q/(\sigma_1 + \sigma_{-1})$ and $(\sigma_1 + \sigma_{-1})$ are isomorphic.)

Proof. $(\sigma_1 + \sigma_{-1})^2 = \sigma_1 + \sigma_1 = 0$, so one inclusion is obvious. Suppose

$$(\sigma_1 + \sigma_{-1})\Big(\sum_{g \in G} z(g)g\Big) = 0.$$

Then for all g,

$$z(g) + z(\sigma_{-1}g) = 0,$$

so that, for $\Gamma \subset G$ any set of representatives of $G/\{\sigma_1, \sigma_{-1}\}$,

$$\sum_{g \in G} z(g)g = \sum_{g \in \Gamma} z(g)g + \sum_{g \in \Gamma} z(g)\sigma_{-1}g = (\sigma_1 + \sigma_{-1}) \sum_{g \in \Gamma} z(g)g.$$

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* Results valid for all q.

LEMMA 3.5* $\mathrm{Ann}(\gamma S^-) = (\sigma_1 + \sigma_{-1})$ *if and only if* $\gamma S^- = (\sigma_1 + \sigma_{-1})$.

Proof. The if part follows from 3.4. To see the converse, note that multiplication induces a monomorphism.

$$\alpha : R_q/\mathrm{Ann}\ \gamma S^- \to \mathrm{Hom}_{R_q}(\gamma S^-, R_q),$$

$\alpha(x)(y) = xy$. Given $\beta \in \mathrm{Hom}_{R_q}(\gamma S^-, R_q)$, set

$$\beta(x) = \sum_{g \in G} \beta_g(x) g.$$

Then the assignment of β_{σ_1} to β is easily seen to induce an isomorphism of $\mathrm{Hom}_{R_q}(\gamma S^-, R_q)$ and $\mathrm{Hom}_{Z/2Z}(\gamma S^-; Z/2Z)$. Thus $\dim_{Z/2Z}(R_q/\mathrm{Ann}\ \gamma S^-) \leq \dim_{Z/2Z} \gamma S^-$. So if $\mathrm{Ann}\ \gamma S^- = (\sigma_1 + \sigma_{-1})$, then by Lemma 3.4

$$R_q/\mathrm{Ann}\ \gamma S^- = R_q/(\sigma_1 + \sigma_{-1}) \cong (\sigma_1 + \sigma_{-1}),$$

and so $\dim_{Z/2Z}(\sigma_1 + \sigma_{-1}) \leq \dim_{Z/2Z} \gamma S^-$. Hence, as $\gamma S^- \subset (\sigma_1 + \sigma_{-1})$, they must be equal.

Clearly Theorem 3.3 follows from Lemma 3.5 and Theorem 3.2. The remainder of this section will therefore be devoted to proving 3.2. To do this we first define a map

$$\psi : \Lambda_q \to Z_2[G] = R_q$$

as follows. Let $\delta_d \in \Lambda_q$, $0 \leq d < q$, be given by $\delta_d(x) = 1$ iff $x = d$. Then, if $d = sr_d$, $s \mid q$ and $(r_d, q) = 1$, let $\Gamma_d = \{a \mid 1 \leq a < q, (a, q) = 1, a \equiv r_d \pmod{q/s}\}$, and define

$$\psi(\delta_d) = \sum_{a \in \Gamma_d} \sigma_a^{-1} \left(= \sigma_{r_d}^{-1} \sum_{a \in \Gamma_s} \sigma_a^{-1} \right)$$

Then, for any $f \in \Lambda_q$, write $f = \sum_{d=0}^{q-1} \beta(d)\delta_d$ and let $\psi(f) = \sum_{d=0}^{q-1} \beta(d)\psi(\delta_d)$. Clearly ψ is a map of $Z/2Z$-vector spaces.

Let $(\sum_{g \in G} \alpha_g g)^- = \sum_g \alpha_g g^{-1}$, as usual.

LEMMA 3.6* *For* $\lambda \in R_q$, $\psi(\lambda f) = \bar{\lambda}\psi(f)$, (*i.e.*, ψ *is an* R_q*-module map, where* R_q *has the module structure* $\lambda \cdot \omega = \bar{\lambda}\omega$.)

Proof. If $(b, q) = 1$, $\Gamma_{db} = \sigma_b \cdot \Gamma_d$ (where r_b acts on $\{0, \ldots, q-1\}$ by $\sigma_b(x) =$

$R_q(xb))$. Hence

$$\psi(\sigma_b\delta_b)=\sum_{a\in\Gamma_d}\sigma_{ba}^{-1}=\sigma_b^{-1}\Big(\sum_{a\in\Gamma_d}\sigma_a^{-1}\Big),$$

which clearly implies the result.

For $s\mid m$, let $M_s\subset\Lambda_q$ be generated over $Z/2Z$ by the functions δ_d with $d=sr$, $(r,q)=1$. Clearly $\sigma_a M_s=M_s$, $(a,q)=1$ and so M_s is an R_q-submodule. For example, M_0 is the submodule generated by δ_0

LEMMA 3.7* $\Lambda_q=\bigoplus_{s\mid m} M_s$.

The proof is quite simple and so is omitted. If follows from this lemma that if we let h_s be the component of $f_1^{(q)}$ in M_s, then $\operatorname{Ann}(f_1^{(q)})=\bigcap_{s\mid m}\operatorname{Ann}(h_s)$.

LEMMA 3.8* $\psi\mid M_s:M_s\to Z_2[G]$ is a monomorphism.

Proof. For $\delta_d\in M_s$, $0\le d<q$, write $d=sr_d$, $(q,r_d)=1$. Suppose $\psi(\sum_{\Delta_s}\beta(d)\delta_d)=0$. where Δ_s consists of those d with $\delta_d\in M_s$, $0\le d<q$. Then

$$\sum_{\Delta_s}\beta(d)\Big(\sigma_{r_d}^{-1}\sum_{a\in\Gamma_s}\sigma_a^{-1}\Big)=0;$$

i.e.,

$$\Big(\sum_{\Delta_s}\beta(d)\sigma_{r_d}^{-1}\Big)\Big(\sum_{a\in\Gamma_s}\sigma_a^{-1}\Big)=0;$$

i.e.,

$$(3.8.1)\quad \sum_{\Delta_s}\sum_{\Gamma_s}\beta(d)\sigma_{r_da}^{-1}=0.$$

Suppose $\sigma_{r_da}=\sigma_{r_c}$ where $d,c\in\Delta_s$, $a\in\Gamma_s$. Then $r_da=r_c\pmod q$. Since $a\in\Gamma_s$, $a\equiv 1\pmod{q/s}$. Hence $r_da\equiv r_d \bmod (q/s)$; hence

$$da\equiv d\pmod q$$

as $d=sr_d$. Thus $d\equiv c\pmod q$, so $d=c$, as $0\le d$, $c<q$, and then $r_d=r_c$, and $a\equiv 1\pmod q$. Hence the left side of (3.8.1) has the form

$$\sum_{\Delta_s}\beta(d)\sigma_{r_d}^{-1}+\varepsilon,$$

where ε is a linear combination of elements of the set $\{\sigma_a^{-1} \mid a \not\equiv r_d \pmod q \text{ for all } d \in \Delta_s\}$. Clearly this implies that $\beta(d)=0$ for all $d \in \Delta_s$, i.e., $\sum_{\Delta_s} \beta(d)\delta_d = 0$. Thus $\psi \mid M_s$ is $1-1$.

In view of 3.7 and 3.8

$$\operatorname{Ann}(f_1^{(q)}) = \bigcap_{s \mid q} \operatorname{Ann}(\psi(h_s)).$$

If we let $\mathfrak{a}$ be the ideal generated by the $\psi(h_s)$, $s \mid q$, then the right side is just $\operatorname{Ann} \mathfrak{a} = \{\lambda \in R_q \mid \lambda x = 0 \text{ for all } x \in \mathfrak{a}\}$. So

$$\operatorname{Ann}(f_1^{(q)}) = \operatorname{Ann}(\mathfrak{a}).$$

Let $\Delta_s = \{d \mid 0 \le d < g-1,\ s \mid d, \text{ and } (d/s, q) = 1\}$, as above. Let $\Delta_s' = \{d \in \Delta_s \mid 0 < d \le [q/2]\}$. Then

$$h_s = \sum_{d \in \Delta_s'} \delta_d.$$

Hence, writing $d = sr_d$ as above for $d \in \Delta_s$,

$$\psi(h_s) = \sum_{d \in \Delta_s'} \sum_{a \in \Gamma_s} \sigma_{r_d}^{-1} \sigma_a^{-1}$$

So clearly

$$\psi(h_s) = \sum_{\substack{b=1 \\ (b,q)=1}}^{[q/2s]} \sum_{a \in \Gamma_s} \sigma_{ab}^{-1}. \qquad (\text{Thus, } \psi(h_s) = 0.)$$

Let $\mathcal{B} \subset \mathfrak{a}$ be the ideal consisting of linear combinations

$$\sum_{s \mid q} \lambda_s \psi(h_s), \quad \text{where} \quad \sum_{s \mid q} \lambda_s \in A_q, \quad \text{and} \quad \lambda_q = 0.$$

PROPOSITION 3.9. $\mathcal{B} = \gamma(S^-)$.

Proof. Recall

$$\theta(s) = \sum_{a \in (Z/qZ)^*} \left\langle \frac{-as}{q} \right\rangle \sigma_a^{-1}.$$

Hence

$$q\theta(s) = \sum_a R_q(-as)\sigma_a^{-1} \in S \subset Z[G].$$

Clearly $R_q(-cs)$ depends only on the class of $c \bmod q/s$, and for $1 \leq c < q/s$, $R_q(-cs) \equiv 1 \pmod 2$ iff c is even. Hence

$$\gamma(q\theta(s)) = \sum_{\substack{b=1 \\ (b,q)=1}}^{q/s} \sigma_b^{-1}\Big(\sum_{a \in \Gamma_s} \sigma_a^{-1}\Big).$$

Thus one easily sees that

$$\psi(q\theta(s)) = \sigma_2^{-1}\psi(h_s).$$

Hence, for $d \in \Delta_s$, $s \mid q$,

$$\gamma(q\theta(d)) = \sigma_{2r_d}^{-1}\psi(h_s).$$

According to [SI], S^- can be described as the intersection of $Z[G]$ with the subgroup of $Q[G]$ generated by the elements

$$e^-\theta(c) = \theta(c) - (1/2)s(G),$$

$s(G)$ the sum of the group elements. Let

$$\rho : Z[G] \to Z$$

be the augmentation. Given for each $c \geq 0$, $\omega_c \in Z[G]$, all but a finite number being 0,

$$\sum_c \omega_c\theta(c) = \sum_c \omega_c e^-\theta(c) + \tfrac{1}{2}\rho\Big(\sum_c \omega_c\Big)s(G).$$

Hence, since $q\theta(c) \in Z[G]\,\forall c$,

$$\sum_c \omega_c q\theta(c) \in S^- \quad \text{if} \quad \rho\Big(\sum_c \omega_c\Big) = 0.$$

Given $\lambda_s \in Z_2[G]$ with

$$\sum_{s \mid q} \lambda_s \in A_q \quad \text{and} \quad \lambda_q = 0,$$

as in the definition of $\mathcal{B}$, it is easy to see that there exist $\omega_s \in Z[G]$ with $\gamma(\omega_s) = \lambda_s$

and $\rho(\sum_{s|q} \omega_s) = 0$. Hence

$$\sum_{s|q} \lambda_s \psi(h_s) = \sum_{s|q} \lambda_s \gamma(q\sigma_2\theta(s)) = \sum_{s|q} \lambda_s \gamma(q\theta(2s))$$
$$= \gamma \sum_{s|q} \omega_s q\theta(2s)) \in \gamma S^-.$$

Hence $\mathcal{B} \subset \gamma S^-$.

Now suppose that $\xi \in S^-$. Then we can write

$$\xi = \sum_{c>0} \lambda_c(\theta(c) - \tfrac{1}{2}s(G)),$$

where λ_c, $c > 0$, $c \in Z$, are integers and all but a finite number are zero. Since q is odd and $q\theta(c) \in Z[G]$, it follows easily that $\omega = \frac{1}{2}(\sum_c \lambda_c)$ is an integer. Again, since q is odd,

$$\gamma(\xi) = \gamma(q\xi) = \sum_{c>0} \lambda_c \gamma(q\theta(c)) + \omega s(G) = \sum_{c>0} \lambda_c \sigma_{2r_c}^{-1} \psi(h_{s_c}) + \omega s(G),$$

where we now write $c \equiv s_c r_c \pmod q$ with $s_c \mid q$, $(r_c, q) = 1$, $0 \leq s$, $r_c < q$. Since $\sum_{c>0} \lambda_c \equiv 0(2)$, $\sum_{c>0} \lambda_c \sigma_{2r_c}^{-1} \in A_q$ and so

$$\sum_{c>0} \lambda_c \sigma_{2r_c}^{-1} \psi(h_{s_c}) \in \mathcal{B}.$$

Since $\Gamma_1 = \{\sigma_1\}$ and $\Delta_1 = \{a \mid (a, q) = 1\}$,

$$\psi(h_1) = \sum_{\substack{b=1 \\ (b,q)=1}}^{[q/2]} \sigma_b^{-1}.$$

Hence one sees easily that

(3.9.1) $\quad (\sigma_1 + \sigma_{-1})\psi(h_1) = s(G),$

and so $\omega s(G) \in \mathcal{B}$ also. Thus

$$\gamma S^- \subset \mathcal{B},$$

proving the proposition.

Since $(\sigma_1+\sigma_{-1}) = \operatorname{Ann}(\sigma_1+\sigma_{-1}) \subset \operatorname{Ann} \gamma S^-$, it follows from 3.9 that $(\sigma_1+\sigma_{-1}) \subset \operatorname{Ann} \mathcal{B}$ and that

$$\operatorname{Ann} \mathcal{B}/(\sigma_1+\sigma_{-1}) = \operatorname{Ann} \gamma S^-/(\sigma_1+\sigma_{-1}).$$

Since $\mathfrak{a} = \operatorname{Ann}(f_1^{(q)})$, the following lemma will complete the proof of 3.2

LEMMA 3.10. *The modules* $\operatorname{Ann} \mathcal{B}/(\sigma_1+\sigma_{-1})$ *and* $\operatorname{Ann} \mathfrak{a}/A_q(\sigma_1+\sigma_{-1})$ *are isomorphic.*

Proof. Since $\mathfrak{a} \subset \mathcal{B}$, $\operatorname{Ann} \mathfrak{a} \subset \operatorname{Ann} \mathcal{B}$, and this inclusion induces a mapping

$$\tau : \frac{\operatorname{Ann} \mathfrak{a}}{A_q \cdot (\sigma_1+\sigma_{-1})} \to \frac{\operatorname{Ann} \mathcal{B}}{(\sigma_1+\sigma_{-1})}$$

Lemma 3.10 will be proven by showing this map to be an isomorphism.

Suppose that $\lambda \in \operatorname{Ann} \mathfrak{a}$ and that λ represents an element in the kernel of τ. Then $\lambda = \omega \cdot (\sigma_1+\sigma_{-1})$ for some ω. Hence, since $(\sigma_1+\sigma_{-1})\psi(h_1) = s(G)$ (see (3.9.1) above), $o = \lambda\psi(h_1) = \omega s(G)$. Since $\sigma_a s(G) = s(G)$ for all $\sigma_a \in G(q)$,

$$\left(\sum_{g \in G} \alpha_g g\right) s(G) = \left(\sum_{g \in G} \alpha_g\right) s(G).$$

Hence $\omega s(G) = 0$ if and only if $\omega \in A_q$. Therefore, $\lambda \in A_q \cdot (\sigma_1+\sigma_{-1})$, which shows that τ is one-to-one.

Suppose that $\lambda \in \operatorname{Ann} \mathcal{B}$. Then, for $s \mid q$, $s \neq q$, $\lambda(\psi(h_s)+\psi(h_1)) = 0$, i.e.,

$$\text{(3.10.1)} \quad \lambda\psi(h_1) = \lambda\psi(h_s).$$

Now, if $\omega \in A_q$, $\lambda\omega\psi(h_1) = 0$. Hence, $\lambda\psi(h_1) \in \operatorname{Ann} A_q$. It is easy to see that $\operatorname{Ann} A_q = (s(G))$; hence

$$\lambda\psi(h_1) = \varepsilon s(G), \qquad \varepsilon = 0 \text{ or } 1,$$

as $(s(G)) = \{0, s(G)\}$. But then

$$(\lambda + \varepsilon(\sigma_1+\sigma_{-1}))\psi(h_1) = 0,$$

using 3.9.1. By (3.10.1), applied to $\lambda + \varepsilon(\sigma_1+\sigma_{-1})$,

$$\lambda + \varepsilon(\sigma_1+\sigma_{-1}) \in \operatorname{Ann} \alpha.$$

Thus τ is also surjective.

§4. Class numbers

In this section we prove the equivalence of statements 1. and 2. of Theorem 1.1. Recall that a linear periodic automorphism ϕ of a real vector space V is said to be pseudo-free if it is *free* outside of a 1-dimensional, invariant set L; i.e., $\phi(L)=L$ and ϕ^j is either the identity or fixes no points of $V-L$, for all j.

THEOREM 4.1. *Let ϕ_i be automorphisms of vector spaces V_i, $i=1,2$, with ϕ_1 of period m and pseudo-free. Then* (i) *implies* (ii) *and, for $m\neq 0 \pmod 8$ or* $\dim V_1<10$, (i) *is equivalent to* (ii) *where:*

(i) *There is a periodic smooth map $f:\Sigma\to\Sigma$ of a homotopy sphere Σ, free outside of a 1-dimensional set, with isolated fixed points, x_1, x_2 and with linear isomorphisms ψ_i: $\Sigma_{x_i}\to V_i$, $i=1,2$, so that $\psi_i^{-1}(df)_{x_i}\psi_i=\phi_i$ (i.e., $(df)_{x_i}$ and ϕ_i are linearly similar*); *and*

(ii) *ϕ_1 and ϕ_2 are topologically similar, i.e., there is a homeomorphism $\psi: V_1\to V_2$ with $\psi^{-1}\phi_1\psi=\phi_2$.*

(Note that ϕ_1 and ϕ_2 topologically similar and ϕ_1 pseudo-free implies ϕ_2 psuedo-free also.)

This is just part of Theorem 2 of [CS2]. Now consider pseudo-free automorphisms ϕ_i, of period $4q$, q odd, on vector spaces V_i. Let $p_i(t)=\det(\phi_i-tI)$ be the characteristic polynomial of ϕ_i.

THEOREM 4.2. *The following are equivalent:*

(i) *ϕ_1 and ϕ_2 are topologically similar*; *and*

(ii) *There are factorizations (over the reals) $p_1(t)=k(t)h(t)$ and $p_2(t)=k(t)h(-t)$, where $k(-1)=0$ if $h(t)\neq 1$, so that $h(t)=\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j)$, with $\xi_j=\exp((2\pi b_j)/4q)$, $(b_j,4q)=1$, $b_j\equiv 1 \pmod 4$, with l even, and with*

$$(4.2.1)\qquad (l/2)\Bigg(\sum_{\substack{c=1\\ c\neq q}}^{2q-1}\delta_{2c}^{(4q)}\Bigg)+\sum_{j=1}^{l}f_{b_j}^{(4q)}=0.$$

Theorem 4.2 is just a minor reformulation to suit the present notation of some of the results of [CS1]. Here $\delta_d^{(4q)}$ is just the function from $Z/4qZ$ to $\{0,1\}$ that vanishes on x iff $x\neq d$, $0\leq x<4q$. (See Theorems I, II, and compare 7.14 along with paragraph preceding 7.12, all in [CS1].)

It is easy to see that

$$\sum_{c=1,c\neq q}^{2q-1}\delta_{2c}=f_1^{(4q)}+f_{2q-1}^{(4q)}.$$

Hence (4.2.1) can be rewritten as

$$(4.3)\quad (l/2)(f_1^{(4q)}+f_{2q-1}^{(4q)})+\sum_{j=1}^{l} f_{b_j}^{(4q)}=0.$$

Suppose now that q has at most two prime factors and that $h^-(q)\equiv 1 \pmod 2$. Then, by Corollary 3.4 and Theorem 2.2, $4q$ is tempered. Let f be a periodic smooth map of the homotopy sphere Σ, with isolated fixed points x_1 and x_2, and suppose that f is free outside of a 1-dimensional set. Let $V_i=\Sigma_{x_i}$ and $(df)_{x_i}=\phi_i$, $i=1,2$. Then by 4.1 and 4.2, the characteristic polynomials of ϕ_i must factor as indicated in 4.2(ii). Hence, with b_j as in 4.2, (4.2) also holds. From the definition that $4q$ be *tempered*, it then follows easily that, after reordering of indices,

$$(b_i,\ldots,b_l)=(c_1,\ldots,c_s,2q-c_1,\ldots,2q-c_s,d_1,d_1,d_2,d_2,\ldots,d_t,d_t).$$

However, if $\xi=\exp(2\pi ib/4q)$, then $-\bar{\xi}=\exp(2\pi i(2q-b)/4q)$. It follows that multiplicity of a root of unity as a root of $h(t)=\prod_1^l(t-\xi_j)(t-\bar{\xi}_j)$ is congruent to its multiplicity as a root of $h(-t)$. From this and the fact that ϕ_1 and ϕ_2 have $k(t)h(t)$ and $k(t)h(-t)$ as characteristic polynomials, respectively, it is immediate the $\phi_1=(df)_{x_1}$ and $\phi_2=(df)_{x_2}$ satisfy the conclusion of the mod two Smith conjecture. Thus, in Theorem 1.1, it is proven that 2. implies 1.

To prove the converse, suppose that $h^-(q)\equiv 0 \pmod 2$ or that q has more than two prime factors. Then $4q$ is *not* tempered. Hence, there is a linear relation

$$\sum_{j=1}^{l} f_{b_j}=0,$$

with $1\le b_j<4q$, $b_j\equiv 1 \pmod 4$, $(b_j,4q)=1$, $j=1,2,\ldots,l$, that is *not* a consequence of the linear relations $(\mp)_q$ (See §2.). Without loss of generality, it may, of course, be assumed that $b_i\ne b_j$ if $i\ne j$.

Since $f_{b_j}(2q)=1$, it is immediate by evaluation at $2q$ that l is even. Let

$$\xi_j=\exp((2\pi ib_j)/4q),\qquad \tau=\exp((2\pi i)/4q),\qquad j=1,\ldots,l.$$

If $l/2$ is even, let

$$h(t)=\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j),$$

and if $l/2$ is odd, let

$$h(t)=(t-\tau)(t-\bar{\tau})(t+\tau)(t+\bar{\tau})\prod_{j=1}^{l}(t-\xi_j)(t-\bar{\xi}_j).$$

Let ϕ_1 and ϕ_2 be periodic automorphism of real vectors space V_i with characteristic polynomials $(t+1)h(t)$ and $(t+1)h(-t)$, respectively; these are elementary to construct. Then (4.3) and hence (4.2.1) is satisfied, so that, by 4.1 and 4.2, there is a periodic smooth map $f:\Sigma^{2l+1}\to\Sigma^{2l+1}$ (and actually even of S^{2l+1}) or Σ^{2l+5} (S^{2l+5}), free outside a 1-dimensional set, with fixed points x_1, x_2 and with $(df)_{x_i}$ linearly similar to ϕ_i, $i=1,2$.

We claim that f is a counter-example to the mod 2 Smith Conjecture. Since the b_j are pairwise distinct, ϕ_1 and ϕ_2 have all their eigenvalues of multiplicity one. Hence, in this case the mod 2 Smith Conjecture would actually imply the linear simplicity of ϕ_1 and ϕ_2; in particular, it would follows that $h(t)=h(-t)$. From this it follows easily that we must have

$$\{b_1,\ldots,b_l\}=\{c_1,\ldots,c_{l/2},2q-c_1,\ldots,2q-c_{l/2}\}.$$

Hence our original equation has the form

$$\sum_{j=1}^{l/2}(f_{c_j}+f_{2q-c_j})=0.$$

Since $(f_a+f_{2q-a})(0)=f_a(0)+f_a(2q)=1$, it follows by evaluation at zero that $l/2$ is also even; thus, our original equation $\sum_1^l f_{b_j}=0$ is a linear combination of relations of the form $(\mp)_q$, namely it is the sum of the relations

$$f_{c_j}+f_{2q-c_j}=f_1+f_{2q-1},\qquad 1\le j\le l/2,$$

a contradiction.

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Plongements d'espaces homogènes

D. LUNA et TH. VUST

Introduction

Soient G un groupe algébrique affine connexe, H un sous-groupe algébrique de G non nécessairement connexe (le corps de base sera algébriquement clos, de caractéristique nulle et même – lorsque cela nous arrange – non dénombrable). Un plongement de l'espace homogène G/H est une variété algébrique intègre dans laquelle G opère algébriquement et qui contient G/H comme orbite ouverte.

Ce travail propose un cadre pour l'étude des plongements. Aux §§1 et 2, nous précisons la définition des plongements en adoptant un point de veu "rationnel". Au §3, nous introduisons la notion de plongement élémentaire: il s'agit des plongements lisses composés de deux orbites, l'orbite ouverte G/H et une orbite fermée de codimension 1. Le charme et la maniabilité de cette notion viennent de ce qu'on peut la déguiser sous des apparences assez différentes: un plongement élémentaire est aussi une certaine valuation du corps des fonctions rationnelles sur G/H (voir le §3), puis également une classe d'équivalence de "germes de courbes formels divergents" dans G/H (voir le §4). Nous abordons l'étude des plongements élémentaires pour eux-mêmes au §5, où nous indiquons aussi leur lien avec l'immeuble sphérique. Mais les plongements élémentaires sont surtout intéressants pour le rôle qu'ils promettent de jouer dans l'étude des plongements généraux: par exemple, les critères valuatifs de séparation et de propreté s'expriment très naturellement, pour les plongements, en termes de plongements élémentaires (voir le §6).

Aux paragraphes suivants nous abordons une étude plus poussée des plongements, en supposant le groupe G réductif et la variété du plongement normale. Appelons "complication" de G/H la codimension minimale des orbites d'un sous-groupe de Borel de G dans G/H. L'analyse des plongements que nous faisons aux §§7 et 8 est surtout significative lorsque la "complication" de G/H est ≤ 1: dans ce cas nos résultats conduisent à une classification de tous les plongements normaux – même non nécessairement quasi-projectifs. Au dernier paragraphe, comme illustration de ce qui précède, nous examinons en détail le cas $G = SL(2)$ et $H = \{e\}$.

Nous devons notre point de départ bien évidemment à la théorie des plongements toriques ([5], [6]), mais aussi à l'article [10] de V. L. Popov, dans lequel est donnée la classification des espaces presque-homogènes *affines* normaux sous $SL(2)$. Nous remercions tous ceux qui, par l'intérêt qu'ils y ont pris et par leurs suggestions, ont contribué à la réalisation de ce travail – en particulier C. DeConcini, H. Kraft, M. Lejeune-Jalabert, C. Procesi, G. Rousseau et tout particulièrement F. Pauer ([20], [21]) qui nous a beaucoup aidés.

Nous dédions ces pages à notre ami Jacques Vey.

1. Préliminaires

Dans toute la suite, nous désignerons par G un groupe algébrique affine connexe, et par H un sous-groupe algébrique de G non nécessairement connexe, le corps de base k étant algébriquement clos et de caractéristique nulle.

Un plongement de l'espace homogène G/H est la donnée

1) d'une variété algébrique intègre X dans laquelle G opère algébriquement;
2) d'un plongement ouvert de l'espace homogène G/H dans X, plongement qui commute à l'opération de G.

Insistons sur le fait que, par définition, X contient un point privilégié (l'image du point H/H de G/H), dont l'orbite est ouverte et dont le groupe d'isotropie est H; c'est par ce détail que la notion de plongement diffère de celle d'espace presque-homogène (employée par exemple dans [10] et [12]). Dorénavant, on considère le plongement $G/H \hookrightarrow X$ comme une inclusion, et donc G/H comme un sous-ensemble de X.

Lorsque Z est une variété algébrique intègre, notons $k(Z)$ le corps des fonctions rationnelles, et $k[Z]$ l'algèbre des fonctions régulières. Pour tout plongement X, l'inclusion $G/H \hookrightarrow X$ identifie $k(X)$ et $k(G/H)$, et les algèbres locales $\mathcal{O}_{X,x}$, $x \in X$ se trouvent donc contenues dans $k(G/H)$. Comme on voit, lorsqu'on s'intéresse aux plongements, on est tout naturellement conduit à adopter un point de vue "rationnel" en géométrie algébrique (point de vue actuellement quelque peu délaissé, qui donne un rôle prépondérant au corps des fonctions rationnelles).

Pour la commodité du lecteur et pour fixer nos notations, nous commencerons par résumer brièvement ce point de vue (voir aussi [4]).

1.1 Soit K un corps, extension de type fini de k.

On appelle localités de K les sous-k-algèbres locales de K qui ont K comme corps des fractions. On dit qu'une localité est géométrique, si elle peut être obtenue comme localisé d'une sous-k-algèbre de type fini de K. On notera $\mathfrak{L}(K)$ l'ensemble des localités géométriques de K. On appelle sous-algèbres affines de K les sous-k-algèbres de type fini de K qui ont K comme corps des fractions. Les sous-ensembles de $\mathfrak{L}(K)$ qu'on obtient en localisant une sous-algèbre affine en ses différents idéaux premiers, forment la base d'une topologie de $\mathfrak{L}(K)$, la topologie de Zariski.

Désignons par $\mathfrak{X}(K)$ l'ensemble des localités géométriques de K dont le corps résiduel est isomorphe à k. On considère $\mathfrak{X}(K)$ muni de la topologie induite par celle de $\mathfrak{L}(K)$. Si A est une sous-algèbre affine de K, on désigne par X_A le sous-ensemble de $\mathfrak{X}(K)$ qu'on obtient en localisant A en ses divers idéaux maximaux; les X_A forment une base de la topologie de $\mathfrak{X}(K)$. Les "points" de $\mathfrak{X}(K)$ sont en fait des "germes de variétés algébriques intègres ayant K comme corps de fonctions rationnelles." On notera les éléments de $\mathfrak{X}(K)$ par $x, x', \ldots$ lorsqu'on les considérera comme points de l'ensemble $\mathfrak{X}(K)$, et par pur artifice, on écrira $\mathcal{O}_x$ lorsqu'on pense plutôt à la sous-algèbre locale qui est "associée" à x (et on notera $\mathfrak{m}_x$ l'idéal maximal de $\mathcal{O}_x$). On identifie de manière naturelle $\mathfrak{X}(K)\times\mathfrak{X}(K)$ à un sous-ensemble de $\mathfrak{X}(L)$, où L désigne le corps des fractions de $K\otimes_k K$.

Dans la présente perspective, une variété algébrique intègre ayant K comme corps de fonctions rationnelles n'est alors rien d'autre qu'un sous-ensemble X de $\mathfrak{X}(K)$ qui est ouvert, noethérien, et séparé (séparé signifie ici: la diagonale de $X\times X\subset\mathfrak{X}(L)$ est fermée dans $X\times X$). Si $x\in X\subset\mathfrak{X}(K)$, on écrira aussi $\mathcal{O}_{X,x}$ pour $\mathcal{O}_x$ (faisant ainsi le lien avec les notations habituelles).

Soit K' un sous-corps de K contenant k, et soient $X\subset\mathfrak{X}(K)$, $X'\subset\mathfrak{X}(K')$ deux ouverts. Un morphisme (dominant) $\varphi: X\to X'$ est une application telle que $\mathcal{O}_{X',\varphi(x)}\subset\mathcal{O}_{X,x}$, quel que soit $x\in X$.

1.2. L'opération naturelle de G dans G/H se reflète en une opération de G dans $k(G/H)$, G opérant par automorphismes de corps; on en déduit une opération (ensembliste) de G dans $\mathfrak{X}(k(G/H))$, et une opération de l'algèbre de Lie $\mathfrak{G}$ de G dans $k(G/H)$, $\mathfrak{G}$ opérant par dérivations. Désignons par $\mathfrak{X}(G/H)$ l'ensemble des $x\in\mathfrak{X}(k(G/H))$ tels que $\mathcal{O}_x$ soit stable par $\mathfrak{G}$.

PROPOSITION. *$\mathfrak{X}(G/H)$ est ouvert dans $\mathfrak{X}(k(G/H))$.*

Preuve. Soit $x\in\mathfrak{X}(G/H)$. Choisissons une sous-algèbre affine A de $k(G/H)$ telle que $x\in X_A$, puis un système de générateurs $f_1,\ldots,f_n$ de A, et enfin une base $X_1,\ldots,X_m$ de $\mathfrak{G}$. Puisque par hypothèse $X_if_j\in\mathcal{O}_x$, on peut trouver $g_{ij}\in A$ et $g\in A-m_x$ tels que $X_if_j=g_{ij}/g$. On vérifie sans peine que $A[g^{-1}]$ est stable par $\mathfrak{G}$, donc que $X_{A[g^{-1}]}\subset\mathfrak{X}(G/H)$ et que $x\in X_{A[g^{-1}]}$. La proposition en résulte.

1.3. On notera e l'élément neutre de G. L'opération de G dans G/H se reflète en une injection $\mu : k(G/H) \to \mathcal{O}_{G\times G/H, e\times G/H} \subset k(G\times G/H)$. Au numéro suivant, nous aurons besoin de connaître μ en termes de l'opération de $\mathfrak{B}$ dans $k(G/H)$.

Désignons par $S(\mathfrak{B}^*) = \bigoplus_{n\geq 0} S_n(\mathfrak{B}^*)$ l'algèbre symétrique sur le dual de $\mathfrak{B}$, et par $S_+(\mathfrak{B}^*) = \bigoplus_{n>0} S_n(\mathfrak{B}^*)$ son idéal maximal gradué. Pour toute k-algèbre A posons $A[\mathfrak{B}] = S(\mathfrak{B}^*)\otimes A$; c'est l'algèbre des fonctions polynômes sur $\mathfrak{B}$ à valeurs dans A. Désignons par $A[[\mathfrak{B}]]$ le complété de $A[\mathfrak{B}]$ pour l'idéal $S_+(\mathfrak{B}^*)\otimes A$; $A[[\mathfrak{B}]]$ s'identifie au produit des $S_n(\mathfrak{B}^*)\otimes A$, $n\in\mathbb{N}$.

Supposons maintenant que $\mathfrak{B}$ opère dans A par dérivations. Soit $f\in A$; pour tout $n\in\mathbb{N}$, f donne par $X\in\mathfrak{B}\mapsto(1/n!)X^n f\in A$ un élément de $S_n(\mathfrak{B}^*)\otimes A$; des propriétés bien connues de l'exponentielle résulte aussitôt que ceci définit un homomorphisme d'algèbres $\hat{\mu} : A \to A[[\mathfrak{B}]]$. Il est clair que $\hat{\mu}$ est fonctoriel en A.

Si $\mathcal{O}$ est un anneau local, nous désignerons par $\hat{\mathcal{O}}$ son complété pour l'idéal maximal. L'homomorphisme $\hat{\mu} : \mathcal{O}_{G,e} \to \mathcal{O}_{G,e}[[\mathfrak{B}]]$ et l'augmentation $\mathcal{O}_{G,e} \to k$ induisent une injection $\mathcal{O}_{G,e} \to k[[\mathfrak{B}]]$ qui permet d'identifier $\hat{\mathcal{O}}_{G,e}$ avec $k[[\mathfrak{B}]]$ (penser aux développements de Taylor). De l'inclusion $\mathcal{O}_{G,e} \hookrightarrow k[[\mathfrak{B}]]$ résulte par tensorisation une injection $\mathcal{O}_{G,e}\otimes k(G/H) \hookrightarrow k[[\mathfrak{B}]]\otimes k(G/H) \hookrightarrow k(G/H)[[\mathfrak{B}]]$, qui permet d'identifier $\hat{\mathcal{O}}_{G\times G/H, e\times G/H}$ avec $k(G/H)[[\mathfrak{B}]]$. Désignons par i l'inclusion de $\mathcal{O}_{G\times G/H, e\times G/H}$ dans $k(G/H)[[\mathfrak{B}]]$ qu'on en déduit.

LEMME. *On a* $i\circ\mu = \hat{\mu}$.

Preuve. Il suffit clairement de démontrer le lemme dans le cas où $H = \{e\}$, et il suffit de vérifier alors que $i\circ\mu = \hat{\mu}$ sur $k[G]$, l'algèbre des fonctions régulières sur G.

Par fonctorialité de $\hat{\mu}$, on a un diagramme commutatif

$$\begin{tikzcd} k[G] \arrow[r, "\mu"] \arrow[d, "\hat{\mu}"'] & k[G]\otimes k[G] \arrow[d, "\hat{\mu}"] \\ k[G][[\mathfrak{B}]] \arrow[r, "\mu[[\mathfrak{B}]]"] & (k[G]\otimes k[G])[[\mathfrak{B}]] \end{tikzcd}$$

où le $\hat{\mu}$ à droite est relatif à l'opération de $\mathfrak{B}$ dans $k[G]\otimes k[G]$ par "translations à gauche dans le premier facteur". Le diagramme suivant est clairement aussi commutatif

$$\begin{tikzcd} k[G]\otimes k[G] \arrow[rr, "\hat{\mu}\otimes k[G]"] \arrow[dr, "\hat{\mu}"'] & & k[G][[\mathfrak{B}]]\otimes k[G] \arrow[dl, "\text{évident}"] \\ & (k[G]\otimes k[G])[[\mathfrak{B}]] & \end{tikzcd}$$

d'où il résulte que i sur $k[G]\otimes k[G]\subset \mathcal{O}_{G\times G,e\times G}$ est donné par

$$k[G]\otimes k[G]\xrightarrow{\hat{\mu}}(k[G]\otimes k[G])[[\mathfrak{B}]]\xrightarrow{(e\otimes k[G])[[\mathfrak{B}]]}k[G][[\mathfrak{B}]],$$

où on considère e comme homomorphisme d'algèbres $k[G]\to k$. Comme $(e\otimes k[G])\circ\mu$ est l'identité de $k[G]$, il s'ensuit bien que $i\circ\mu$ et $\hat{\mu}$ coïncident sur $k[G]$.

1.4. L'opération ensembliste naturelle de G dans $\mathfrak{X}(k(G/H))$ laisse clairement stable $\mathfrak{X}(G/H)$.

PROPOSITION. *Soit X un ouvert de $\mathfrak{X}(k(G/H))$, stable par G. Pour que l'application $G\times X\to X$ donnée par l'opération de G dans X soit un morphisme, il faut et il suffit que X soit contenu dans $\mathfrak{X}(G/H)$.*

Preuve. L'application $G\times X\to X$ sera clairement un morphisme si et seulement si, pour tout $x\in X$, $\mu: k(G/H)\to k(G\times G/H)$ envoie $\mathcal{O}_{X,x}$ dans $\mathcal{O}_{G\times X,e\times x}$. Désignons par $\tilde{\mathcal{O}}$ le complété de $\mathcal{O}_{G\times X,e\times x}$ pour l'idéal des fonctions nulles sur $e\times X$. Les inclusions

$$\mathcal{O}_{G\times X,e\times x}\hookrightarrow\mathcal{O}_{G\times G/H,e\times G/H}\overset{i}{\hookrightarrow}k(G/H)[[\mathfrak{B}]]\simeq\hat{\mathcal{O}}_{G\times G/H,e\times G/H}$$

permettent d'identifier $\tilde{\mathcal{O}}$ avec $\mathcal{O}_{X,x}[[\mathcal{O}]]$. Puisque $\tilde{\mathcal{O}}\cap k(G\times G/H)=\mathcal{O}_{G\times X,e\times x}$ (voir par exemple [3], chap. III p. 73), et d'après 1.3, μ envoie $\mathcal{O}_{X,x}$ dans $\mathcal{O}_{G\times X,e\times x}$ si et seulement si $\hat{\mu}$ envoie $\mathcal{O}_{X,x}$ dans $\mathcal{O}_{X,x}[[\mathfrak{B}]]$. D'après la définition de $\hat{\mu}$, cette dernière condition est remplie si et seulement si $\mathfrak{B}$ laisse stable $\mathcal{O}_{X,x}$.

1.5. D'après ce qui précède, on peut reformuler la définition des plongements de la manière suivante.

DEFINITION. Un plongement de G/H est la donnée d'un sous-ensemble X de $\mathfrak{X}(G/H)$, qui est ouvert dans $\mathfrak{X}(G/H)$, noethérien, séparé, et stable par G.

Tourner ainsi la définition des plongements, permet d'énoncer les critères simples de noethérienté et de séparation que voici.

PROPOSITION. *Soit X un ouvert de $\mathfrak{X}(G/H)$. S'il existe un ouvert noethérien (resp. séparé) X' de $\mathfrak{X}(G/H)$ tel que $X \subset G \cdot X'$, alors X est noethérien (resp. séparé).*

Preuve. Il suffit de démontrer la proposition lorsque X est stable par G. Désignons par $\mu : G \times \mathfrak{X}(G/H) \to \mathfrak{X}(G/H)$ l'opération de G dans $\mathfrak{X}(G/H)$.

Supposons d'abord X' noethérien. Posons $X'' = (G \times X') \cap \mu^{-1}(X')$; X'' est un ouvert de $G \times X'$ contenant $e \times X'$. Faisons opérer G dans $G \times X'$ par translations à gauche dans le premier facteur. Puisque $G \times X'$ est noethérien, il existe $s_1, \ldots, s_m \in G$ tels que $G \times X' = \cup_{i=1}^m s_i X''$. Par suite,

$$X \subset G \cdot X' = \mu(G \times X') = \mu\left(\bigcup_{i=1}^m s_i X''\right) = \bigcup_{i=1}^m s_i \mu(X'') = \bigcup_{i=1}^m s_i X',$$

d'où il résulte bien que X est noethérien.

Montrons enfin que X non séparé entraîne X' non séparé. Si X est non séparé, on a $\bar{\Delta}_X \neq \Delta_X$ où Δ_X désigne la diagonale de $X \times X$. Le groupe G opère diagonalement dans $X \times X$ en laissant stable $\bar{\Delta}_X - \Delta_X$. Soit T une orbite de G dans $\bar{\Delta}_X - \Delta_X$. Puisqu'on suppose $X \subset G \cdot X'$, les deux ouverts $T \cap (X' \times X)$ et $T \cap (X \times X')$ de T ne sont pas vides. Par conséquent, puisque T est irréducible, on a $\varnothing \neq T \cap (X' \times X') \subset \bar{\Delta}_X \cap (X' \times X') = \bar{\Delta}_{X'}$. Il s'ensuit que $\bar{\Delta}_{X'} \neq \Delta_{X'}$, ce qui signifie bien que X' est non séparé.

1.6. La reformulation de la définition des plongements et le critère de noethérienté et de séparation de 1.5, permettent de "construire" des plongements: il suffit de choisir une sous-algèbre affine A de $k(G/H)$, stable par $\mathfrak{B}$, et de poser $X = G \cdot X_A$.

Illustrons ceci à l'aide d'un exemple simple. Posons $G = k^*$ et $H = \{e\}$; alors $k(G/H)$ s'identifie à $k(t)$. L'algèbre de Lie de k^* qui est de dimension 1, opère par $D = t(d/dt)$ dans $k(t)$. Posons $f = t/(1+t)^2$ et $g = t/(1+t)^3$. On vérifie sans peine qu'on a $Df = 2g - f$, $Dg = g - 3f^2$ et $f/g = 1 + t$. Il s'ensuit que la sous-algèbre A de $k(t)$ engendrée par f et g est une sous-algèbre affine stable par l'algèbre de Lie. On obtient donc un plongement de k^* par $X = k^* \cdot X_A$. Il est facile de voir que l'identité de k^* se prolonge en un morphisme $\varphi : \mathbb{P}_1 \to X$ tel que $\varphi(0) = \varphi(\infty)$, En regardant de plus près, on voit que X est obtenu à partir de $\mathbb{P}_1$ en identifiant 0 et ∞ en un point double ordinaire.

Cet exemple possède la particularité suivante: X est une courbe projective, mais X n'est pas un plongement projectif, à savoir X ne peut pas être plongé dans un espace projectif dans lequel k^* opère, par un morphisme qui est compatible avec les opérations de k^*. En effet, supposons qu'il existe un tel morphisme. La droite qui correspond alors au point fixe de k^* dans X, admet un hyperplan complémentaire stable par k^*. L'ensemble des points de X qui correspondent à des droites non contenues dans l'hyperplan, forme alors un ouvert affine de X, stable par k^* et contenant le point fixe de X. Mais un tel ouvert est forcément X tout entier, donc n'est pas affine, d'où une contradiction. •

2. Germes de plongements

Soit X un plongement de G/H, et soit Y un fermé de X, irréductible et stable par G. Le comportement de l'opération de G dans X au voisinage de Y est déterminé en grande partie par l'algèbre locale $O_{X,Y}$, qui se trouve contenue dans $k(G/H)=k(X)$. Dans ce §, nous caractérisons ces sous-algèbres locales de $k(G/H)$, et nous précisons leur signification géométrique. Les démonstrations de ce § sont formelles et sans imprévu.

2.1. Rappelons (voir 1.1), qu'une localité de $k(G/H)$ est une sous-k-algèbre locale de $k(G/H)$ dont le corps des fractions est $k(G/H)$. Une localité est appelée géométrique, si elle est le localisé d'une sous-k-algèbre de type fini de $k(G/H)$. Notons $\mathfrak{L}(k(G/H))$ l'ensemble des localités géométriques de $k(G/H)$; $\mathfrak{L}(k(G/H))$ est muni de la topologie de Zariski. On notera les éléments de $\mathfrak{L}(k(G/H))$ par $l, l', \ldots$ lorsqu'on les considérera comme points d'un ensemble, et par pur artifice, on écrira $\mathcal{O}_l$ lorsqu'on pense plutôt à la sous-algèbre locale de $k(G/H)$ qui est "associée" à l (et on notera m_l l'idéal maximal de $\mathcal{O}_l$, et $k_l=\mathcal{O}_l/m_l$ le corps résiduel).

On désigne par $\mathfrak{L}(G/H)$ l'ensemble des $l\in\mathfrak{L}(k(G/H))$ tels que $\mathcal{O}_l$ est stable par G et par $\mathfrak{B}$.

Remarquons qu'il existe des sous-algèbres de $k(G/H)$, stables par G mais non stables par $\mathfrak{B}$. Par exemple, considérons $G=k$ et $H=\{0\}$. Dans ce cas, $k(G/H)$ s'identifie à $k(t)$, et un générateur de $\mathfrak{B}=k$ opère par d/dt dans $k(t)$. La sous-algèbre de $k(t)$ engendrée par $1/t+1/(t+1)$ et ses translatés, n'est pas stable par dérivation. Toutefois, nous ne savons pas s'il existe des localités géométriques stables par G et non stables par $\mathfrak{B}$ (voir aussi 3.2).

Soit $l \in \mathfrak{L}(G/H)$. Soit X un plongement de G/H, et soit Y un fermé de X, irréductible et stable par G. On dit que le couple X, Y est une réalisation géométrique de $\mathfrak{L}$, si $\mathcal{O}_l = \mathcal{O}_{X,Y}$.

PROPOSITION. *Pour tout $l \in \mathfrak{L}(G/H)$, il existe des réalisations géométriques.*

Preuve. Puisque $\mathcal{O}_l$ est géométrique, on peut trouver une sous-algèbre affine A de $k(G/H)$, contenue dans $\mathcal{O}_l$ et telle que $\mathcal{O}_l$ soit égal au localisé de A en $A \cap \mathfrak{m}_l$. Puisque $\mathcal{O}_l$ est stable par $\mathfrak{B}$, en raisonnant comme dans 1.2, quitte à agrandir A, on peut supposer de plus A stable par $\mathfrak{B}$. Posons $X = G \cdot X_A$; c'est un plongement de G/H d'après 1.5. Désignons par Y_A le fermé de X_A qui correspond à l'idéal $A \cap \mathfrak{m}_l$ de A. Puisque $\mathcal{O}_l$ est stable par G, on voit que $(s \cdot Y_A) \cap X_A \subset Y_A$, quel que soit $s \in G$. Un nombre fini de translatés de X_A suffisent pour recouvrir X. On en déduit que $Y = G \cdot Y_A$ est un fermé de X, irréductible et stable par G, et qu'on a $Y \cap X_A = Y_A$. Enfin, $\mathcal{O}_{X,Y} = \mathcal{O}_{X_A,Y_A} = \mathcal{O}_l$, donc X, Y est une réalisation géométrique de l.

2.2. On désigne par $\mathfrak{L}_1(G/H)$ l'ensemble des $l \in \mathfrak{L}(G/H)$ tels que ${}^G k_l = k$.

PROPOSITION. *Il y a une bijection naturelle entre l'ensemble $\mathfrak{L}_1(G/H)$ et l'ensemble des orbites de G dans $\mathfrak{X}(G/H)$.*

Preuve. Soit T une orbite de G dans $\mathfrak{X}(G/H)$. Choisissons un plongement X de G/H (c'est-à-dire un ouvert de $\mathfrak{X}(G/H)$, noethérien, séparé et stable par G) qui contient T. Il est clair que $\mathcal{O}_{X,T}$ ne dépend que de T et non du plongement choisi; par conséquent, posons $\mathcal{O}_T = \mathcal{O}_{X,T}$ (et notons $\mathfrak{m}_T$ l'idéal maximal de $\mathcal{O}_T$, et $k_T = \mathcal{O}_T/\mathfrak{m}_T$ le corps résiduel). Il est clair que $\mathcal{O}_T$ est géométrique, et qu'il est stable par G et par $\mathfrak{B}$. Puisque $k_T \simeq k(T)$, on a aussi ${}^G k_T = k$. Par suite, à toute orbite T de G dans $\mathfrak{X}(G/H)$, on peut associer un $l \in \mathfrak{L}_1(G/H)$ bien déterminé, vérifiant $\mathcal{O}_l = \mathcal{O}_T$.

Soit $l \in \mathfrak{L}_1(G/H)$. Choisissons une réalisation géométrique X, Y de l. Puisque $k(Y) \simeq k_l$ et qu'on suppose ${}^G k_l = k$, G a une orbite ouverte T dans Y (voir par exemple [11]). Par construction, on a $\mathcal{O}_{X,T} = \mathcal{O}_{X,Y} = \mathcal{O}_l$. Montrons que l'orbite T ne dépend que de l et non de la réalisation géométrique choisie. Supposons que X', Y', soit une autre réalisation géométrique de l et désignons par T' l'orbite ouverte que G possède alors dans Y'. Choisissons des sous-algèbres affines A et A' de $k(G/H)$ telles que $X_A \subset X$, $T \cap X_A \neq \varnothing$ et $X_{A'} \subset X'$, $T' \cap X_{A'} \neq \varnothing$. Il n'est pas difficile de voir qu'il existe $f \in A - \mathfrak{m}_l$ et $f' \in A' - \mathfrak{m}_l$ tels que $A_f = A'_{f'}$. Il s'ensuit que $T \cap T' \neq \varnothing$, ce qui implique $T = T'$. La proposition est démontrée.

Soit $l \in \mathfrak{L}_1(G/H)$. On désignera par T_l l'orbite de G dans $\mathfrak{X}(G/H)$ qui lui correspond d'après la proposition précédente. Il est clair qu'une réalisation géométrique de l n'est rien d'autre qu'un ouvert de $\mathfrak{X}(G/H)$, noethérien, séparé, stable par G et contenant T_l.

2.3. Soit $l \in \mathfrak{L}(G/H)$. Tout localisé de $\mathcal{O}_l$ en un idéal premier stable par G est encore une localité géométrique de $k(G/H)$, stable par G et par $\mathfrak{V}$. Désignons par $\mathfrak{L}_f(G/H)$ l'ensemble des $l \in \mathfrak{L}(G/H)$ tels que tout localisé $\mathcal{O}$ de $\mathcal{O}_l$ en un idéal premier stable par G vérifie ${}^G(\mathcal{O}/\mathfrak{m}) = k$. Il est clair que $\mathfrak{L}_f(G/H) \subset \mathfrak{L}_1(G/H)$.

PROPOSITION. *Soit $l \in \mathfrak{L}(G/H)$. Les conditions suivantes sont équivalentes.*

(1) *On a $l \in \mathfrak{L}_f(G/H)$.*

(2) *Il existe une réalisation géométrique X, Y de l dont le nombre d'orbites est fini.*

Preuve. Soit X, Y une réalisation géométrique de l. Désignons par $\mathcal{F}$ l'ensemble des fermés irréductibles de X stables par G et contenant Y. Les idéaux premiers stables par G de $\mathcal{O}_l$ sont en bijection avec les éléments de $\mathcal{F}$. Si $Z \in \mathcal{F}$, le localisé de $\mathcal{O}_l$ par rapport à l'idéal qui correspond à Z, s'identifie *à* $\mathcal{O}_{X,Z}$. La condition (1) signifie alors: pour tout $Z \in \mathcal{F}$, on a ${}^Gk(Z) = k$, autrement dit G a une orbite ouverte dans Z. Il s'ensuit aussitôt que (2)$\Rightarrow$(1).

Réciproquement, supposons (1) vrai. Désignons par X_n l'ouvert de X constitué des orbites de G dans X dont la dimension est $\geq n$, et par X'_n la réunion des orbites dans X_n qui contiennent T_l dans leur adhérence (T_l est bien défini, car $\mathfrak{L}_f(G/H) \subset \mathfrak{L}_1(G/H)$). Montrons, par récurrence descendante, que X'_n est ouvert et que le nombre des orbites de G dans X'_n est fini. On a $X'_{\dim G/H} = G/H$. Supposons l'assertion démontrée pour $n+1$. Les composantes irréductibles du fermé $X_n - X'_{n+1}$ sont de deux espèces: ou bien elles ne contiennent pas T_l dans leur adhérence dans X, ou bien grâce à (1), il s'agit d'orbites de dimension n de X'_n. On voit qu'on obtient X'_n à partir de X'_{n+1} en ôtant de X_n ces composantes de la première espèce, et que ce faisant on n'ajoute qu'un nombre fini d'orbites à X'_{n+1}. La récurrence aboutit donc à $X'_{\dim T_l}$, qui est bien une réalisation géométrique de l dont le nombre d'orbites est fini. La preuve de la proposition est terminée.

Si $l \in \mathfrak{L}_f(G/H)$, on voit que l'intersection de toutes les réalisations géométriques de l est encore une réalisation géométrique de l. On l'appellera la réalisation géométrique minimale, et on la notera X_l. Il est clair que X_l est la réunion des orbites de $\mathfrak{X}(G/H)$ qui contiennent T_l dans leur adhérence.

2.4. – Soit H' un sous-groupe algébrique de G contenant H. Soit X' un plongement de G/H', T' une orbite de G dans X'.

PROPOSITION. *Soit $l \in \mathfrak{L}_1(G/H)$. Les conditions suivantes sont équivalentes.*
(1) *$\mathcal{O}_l$ domine $\mathcal{O}_{X',T'}$.*
(2) *Il existe une réalisation géométrique X de l possédant la propriété suivante: le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $\varphi : X \to X'$ tel que $\varphi(T_l) = T'$.*

Preuve. Il est clair que (2)⇒(1).

Supposons (1) vrai. Choisissons une sous-algèbre affine A' de $k(G/H')$ vérifiant: $X_{A'} \subset X'$ et $X_{A'} \cap T'$ est un fermé non vide de $X_{A'}$; en particulier, il s'ensuit que $A' \subset \mathcal{O}_{X',T'}$ et que le fermé $X_{A'} \cap T$ de $X_{A'}$ est associé à l'idéal $A' \cap \mathfrak{m}_{X',T'} = A' \cap \mathfrak{m}_l$ de A'. Choisissons ensuite une sous-algèbre affine A de $k(G/H)$ vérifiant: A est stable par $\mathfrak{B}$, on a $A' \subset A \subset \mathcal{O}_l$, et $\mathcal{O}_l$ est le localisé de A en l'idéal $A \cap \mathfrak{m}_l$. Posons $X = G \cdot X_A$. Il est clair que le morphisme $\varphi_A : X_A \to X_{A'}$ donné par l'inclusion $A' \subset A$, se prolonge en un morphisme $\varphi : X \to X'$, qui induit le morphisme naturel $G/H \to G/H'$ et qui envoie T_l sur T.

COROLLAIRE. *On suppose $l \in \mathfrak{L}_f(G/H)$. Les conditions suivantes sont équivalentes.*
(1) *$\mathcal{O}_l$ domine $\mathcal{O}_{X',T'}$.*
(2) *Le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $\varphi : X_l \to X'$ tel que $\varphi(T_l) = T'$.*

2.5. Soit H' un sous-groupe algébrique de G contenu dans H, et soit $l' \in \mathfrak{L}_1(G/H')$.

LEMME. *Soit A une sous-algèbre de $k(G/H)$ qui possède les propriétés suivantes: A est contenue dans $\mathcal{O}_{l'}$, A est de type fini, A est stable par $\mathfrak{B}$ et le corps des fractions de A est $k(G/H)$. On obtient un élément $l \in \mathfrak{L}_1(G/H)$ en prenant pour $\mathcal{O}_l$ le localisé de A en l'idéal premier $A \cap \mathfrak{m}_{l'}$.*

Preuve. Désignons par $\mathcal{O}$ le localisé de A en l'idéal premier $A \cap \mathfrak{m}_{l'}$. Il est clair que $\mathcal{O}$ est géométrique et qu'il est stable par $\mathfrak{B}$. Un peu moins clair est que $\mathcal{O}$ est aussi stable par G. Pour le prouver, considérons l'ouvert X_A de $\mathfrak{X}(G/H)$ associé à A, et choisissons un $x \in X_A$ tel que $f(x) = 0$ quel que soit $f \in A \cap \mathfrak{m}_{l'}$. Désignons par U l'ouvert des $s \in G$ tels que $s^{-1}x \in X_A$. Si $s \in U$, pour tout $f \in A$, puisque $f \in \mathcal{O}_{X_A, s^{-1}x}$, on a $s \cdot f \in \mathcal{O}_{X_A,x}$; par suite $sf = g/h$, avec $g, h \in A$ et $h(x) \neq 0$, d'où $sf \in \mathcal{O}$. Donc $sA \subset \mathcal{O}$, quel que soit $s \in U$, d'où l'on déduit sans peine que $\mathcal{O}$ est stable par G.

On obtient donc un élément $l \in \mathfrak{L}(G/H)$ par $\mathcal{O}_l = \mathcal{O}$. Puisque $\mathcal{O}_{l'}$ domine $\mathcal{O}_l$ et que $l' \in \mathfrak{L}_1(G/H)$, il est clair que $l \in \mathfrak{L}_1(G/H)$.

3. Plongements élémentaires et valuations invariantes

Les plongements élémentaires de G/H que l'on introduira dans ce §, sont en relation étroite avec certaines valuations sur $k(G/H)$. C'est pourquoi on commence par rappeler quelques généralités sur les valuations (pour plus de détails, voir [14] et [3], chap. VI).

3.1. Soit K un corps, extension de k. On pose $K^* = K - \{0\}$. Une valuation discrète de K sera pour nous une application $v : K^* \to \mathbb{Q}$ (qu'on prolonge sur K par $v(0) = +\infty$) vérifiant

1) $v(K^*) \simeq \mathbb{Z}$;
2) $v(fg) = v(f) + v(g)$ et $v(f+g) \geq \inf(v(f), v(g))$, quels que soient $f, g \in K$;
3) $v(f) = 0$ si $f \in k^*$.

On dit que la valuation est normalisée si $v(K^*) = \mathbb{Z}$. A toute valuation discrète v, on associe par $\mathcal{O}_v = \{f \in K, v(f) \geq 0\}$ une sous-k-algèbre locale de K, d'idéal maximal $\mathfrak{m}_v = \{f \in K, v(f) \geq 0\}$. On pose $k_v = \mathcal{O}_v/\mathfrak{m}_v$, qu'on appelle le corps résiduel de v. Si v est une valuation discrète de K, alors

1) le corps des fractions de $\mathcal{O}_v$ est K;
2) $\mathcal{O}_v$ est noethérienne, intégralement close et de dimension de Krull égale à 1.

Une sous-k-algèbre locale de K vérifiant 1) et 2) s'appelle une sous-algèbre de valuation discrète de K. L'application $v \to \mathcal{O}_v$ établit une bijection entre l'ensemble des valuations discrètes normalisées et l'ensemble des sous-algèbres de valuation discrète.

Supposons maintenant que K soit de type fini sur k, de degré de transcendance n. Soit v une valuation discrète de K. Nous dirons que v est géométrique, si $\mathcal{O}_v$ est géométrique (c'est-à-dire, si on peut obtenir $\mathcal{O}_v$ comme localisé d'une sous-k-algèbre de type fini de K). Si v est géométrique, son corps résiduel k_v est de type fini sur k et de degré de transcendance $n-1$ sur k. Inversement, s'il existe dans k_v $n-1$ éléments algébriquement indépendants sur k, on peut montrer que v est géométrique. Mais attention, pour tout nombre i compris entre 0 et $n-1$, on peut trouver des exemples de valuations discrètes dont le corps résiduel est de degré de transcendance i sur k; il peut même arriver que le corps résiduel ne soit pas de type fini sur k (voir [14]).

Le résultat suivant nous sera très utile. Soient K' une extension de type fini de K, v une valuation discrète de K, v' une valuation discrète de K', extension de v; alors le degré de transcendance de K' sur K est supérieur ou égal au degré de transcendance de $k_{v'}$ sur k_v (voir [3], chap. VI, §10 n° 3).

COROLLAIRE. *Si v' est géométrique, v l'est aussi.*

Preuve. Notons $d(\ ,\ )$ le degré de transcendance. On a $d(K', K) = d(K', k) - d(K, k)$ et $d(k_{v'}, k_v) = d(k_{v'}, k) - d(k_v, k)$. De $d(K', K) \geq d(k_{v'}, k_v)$ résulte alors que

$$d(k_v, k) \geq d(K, k) - d(K', k) + d(k_{v'}, k) = d(K, k) - 1.$$

D'après ce que nous avons rappelé, il s'ensuit bien que v est géométrique.

3.2. Soit v une valuation discrète de $k(G)$. Nous dirons que v est invariante par translations à gauche, si $v(s \cdot f) = v(f)$, quels que soient $s \in G$ et $f \in k(G)$.

LEMME (voir aussi [15]). *Pour toute valuation discrète v de $k(G)$ il existe une (unique) valuation discrète $\hat{v}$ de $k(G)$ qui possède la propriété suivante: pour tout $f \in k(G)$ il existe un ouvert non vide U de G tel que $\hat{v}(f) = v(s \cdot f)$ quel que soit $s \in U$. La valuation $\hat{v}$ est invariante par translations à gauche. De plus, si v est géométrique, $\hat{v}$ l'est également.*

Preuve. Posons $A = k(G) \otimes \mathcal{O}_v \subset k(G \times G)$ et $\mathfrak{p} = k(G) \otimes \mathfrak{m}_v$. Désignons par $\mathcal{O}$ le localisé de A en l'idéal premier $\mathfrak{p}$. Il est clair que $\mathcal{O}$ est une sous-algèbre de valuation discrète de $k(G \times G)$ à laquelle correspond donc une valuation discrète w de $k(G \times G)$. Désignons par $\mu : k(G) \hookrightarrow k(G \times G)$ l'homomorphisme injectif de corps qui correspond à la multiplication $G \times G \to G$. Posons $\hat{v} = w \circ \mu$.

Soit $g \in k[G]$. Les $s \cdot g$ $(s \in G)$ restent dans un espace vectoriel de dimension finie de $k[G]$. Désignons par U_g l'ouvert de G où la fonction $v(s \cdot g)$ atteint son minimum. Si $\mu(g) = \sum g_i \otimes g_i'$ et si $s \in G$, on a $s \cdot g = \sum g_i(s^{-1}) g_i'$. De là et de la définition de w on déduit que $\hat{v}(g) = \inf_{s \in G} v(s \cdot g) = v(s \cdot g)$ quel que soit $s \in U_g$. Si $f \in k(G)$, on écrit $f = gh^{-1}$, avec $g, h \in k[G]$; alors, pour tout $s \in U = U_g \cap U_h$, on a $\hat{v}(f) = \hat{v}(g) - \hat{v}(h) = v(s \cdot g) - v(s \cdot h) = v(s \cdot f)$.

Il est clair que $\hat{v}$ est invariante par translations à gauche. Enfin, si v est géométrique, w l'est manifestement aussi. De $\hat{v} = w \circ \mu$ et de la fin de 3.1 résulte alors que $\hat{v}$ est également géométrique.

COROLLAIRE 1. *Soit v une valuation discrète de $k(G/H)$, invariante par G. Il existe des valuations $\tilde{v}$ de $k(G)$, invariantes par translations à gauche, dont la* *restriction à $k(G/H)$ est égale à v. Si v est géométrique, on peut choisir $\tilde{v}$ géométrique.*

Preuve. Il existe des valuations discrètes w de $k(G)$, non invariantes par translations à gauche, dont la restriction à $k(G/H)$ est égale à v, et si v est géométrique, on peut choisir w géométrique (voir par exemple [3], chap. VI). Alors $\tilde{v} = \hat{w}$ répond aux exigences du corollaire 1.

COROLLAIRE 2. *Soit v une valuation discrète de $k(G/H)$, invariante par G. Soient $f, g \in k(G)$ et $s \in G$ tels que fg et $(s \cdot f)g$ appartiennent à $k(G/H)$. Alors*

$$v(fg) = v((s \cdot f)g).$$

Preuve. Soit $\tilde{v}$ une valuation de $k(G)$, invariante par translations à gauche, "au-dessus" de v comme dans le corollaire 1. Alors $v((s \cdot f)g) = \tilde{v}(s \cdot f) + \hat{v}(g) = \tilde{v}(f) + \tilde{v}(g) = v(fg)$.

COROLLAIRE 3. *Pour toute valuation discrète G-invariante v de $k(G/H)$, $\mathcal{O}_v$ est stable par $\mathfrak{B}$.*

Preuve. Grâce au corollaire 1, il suffit de considérer le cas où $H = \{e\}$. Puisque l'opération de G dans $k[G]$ est rationnelle, la $\mathbb{Z}$-filtration que v induit dans $k[G]$, étant stable par G, est aussi stable par $\mathfrak{B}$. Par conséquent, si $f \in k[G]$ et si $X \in \mathfrak{B}$, on a $v(Xf) \geq v(f)$, autrement dit $v(Xf/f) \geq 0$. Soit maintenant $f \in k(G)$. Ecrivons $f = g/h$, où $g, h \in k[G]$. On a $Xf/f = Xg/g - Xh/h$. Par suite, si $v(f) \geq 0$, il suit que

$$v(Xf) \geq v(Xf/f) \geq \min(v(Xg/g), v(Xh/h)) \geq 0,$$

ce qui signifie bien que $\mathcal{O}_v$ est stable par $\mathfrak{B}$.

3.3. On appellera plongement élémentaire de G/H tout plongement X vérifiant:

1) X est lisse;
2) X est composé de deux orbites, l'orbite ouverte G/H et une orbite fermée de codimension 1 dans X.

Lorsque G/H est affine, tout plongement normal composé de deux orbites est élémentaire: en effet, le complémentaire de tout ouvert affine dans une variété algébrique étant toujours pur de codimension 1, on voit que l'orbite fermée est de codimension 1; il s'ensuit que X est lisse, puisque l'ensemble singulier de X, qui est de codimension ≥ 2 à cause de la normalité et qui est aussi stable par G, est vide.

On notera $\mathcal{V}(G/H)$ l'ensemble des valuations discrètes normalisées de $k(G/H)$, géométriques et invariantes par G. On désignera par $\mathcal{V}_1(G/H)$ l'ensemble des $v \in \mathcal{V}(G/H)$ tels que ${}^G k_v = k$. On notera par $\mathcal{V}_2(G/H)$ le complémentaire de $\mathcal{V}_1(G/H)$ dans $\mathcal{V}(G/H)$.

PROPOSITION. *Il y a une bijection naturelle entre l'ensemble $\mathcal{V}_1(G/H)$ et l'ensemble des plongements élémentaires de G/H.*

Preuve. Le corollaire 3 de 3.2 permet d'identifier $\mathcal{V}_1(G/H)$ à un sous-ensemble de $\mathfrak{L}_1(G/H)$. Puisque les seuls idéaux premiers de $\mathcal{O}_v$ sont m_v et 0, on a même $\mathcal{V}_1(G/H) \hookrightarrow \mathfrak{L}_f(G/H)$. Soit $v \in \mathcal{V}_1(G/H)$. Le plongement minimal X_v associé à v (voir 2.3) est manifestement composé de deux orbites, G/H et T_v. Puisque T_v est de codimension 1 dans X_v et que $\mathcal{O}_v$ est intégralement clos, X_v est normal donc lisse. Par conséquent, X_v est un plongement élémentaire. Il est clair qu'on obtient tout plongement élémentaire de cette façon, d'où la proposition.

3.4. Soit H' un sous-groupe algébrique de G contenant H. Soit X un plongement élémentaire de G/H, d'orbite fermée T.

PROPOSITION. *De deux choses l'une: ou bien le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $X \to G/H'$; ou bien il existe un unique plongement élémentaire X' de G/H', d'orbite fermée T', tel que le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $\varphi: X \to X'$ vérifiant $\varphi(T) = T'$.*

Preuve. Cela résulte aussitôt de 2.4, 3.1 et 3.3.

3.5. PROPOSITION. *Pour tout $l \in \mathfrak{L}(G/H)$, il existe $v \in \mathcal{V}(G/H)$ tel que $\mathcal{O}_v$ domine $\mathcal{O}_l$.*

Preuve. Soit $G/H \hookrightarrow X$, Y une réalisation géométrique de l. Désignons par $\tilde{X}$ l'éclaté normalisé de X le long de Y, et notons $\pi: \tilde{X} \to X$ le morphisme naturel. Le groupe G opère dans $\tilde{X}$, et puisque $\pi: \pi^{-1}(G/H) \xrightarrow{\sim} G/H$ est un isomorphisme, $\tilde{X}$ est l'espace d'un plongement de G/H. Choisissons une composante irréductible $\tilde{Y}$ de $\pi^{-1}(Y)$; $\tilde{Y}$ est stable par G et de codimension 1 dans $\tilde{X}$. Par conséquent, il existe une valuation v dans $\mathcal{V}(G/H)$ telle que $\mathcal{O}_v = \mathcal{O}_{\tilde{X},\tilde{Y}}$. Par construction, $\mathcal{O}_v$ domine $\mathcal{O}_{X,Y} = \mathcal{O}_l$.

4. Plongements élémentaires et germes de courbes

On désigne par $k[[t]]$ l'algèbre des séries formelles en une indéterminée t, et par $k((t))$ le corps des fractions de $k[[t]]$. On note $(G/H)_{k((t))}$ (resp. $(G/H)_{k[[t]]}$)

l'ensemble des points de G/H à valeurs dans $k((t))$ (resp. dans $k[[t]]$), et on pose $(G/H)^*_{k((t))}=(G/H)_{k((t))}-(G/H)_{k[[t]]}$: c'est "l'ensemble des germes de courbe formels divergents dans G/H". Dans ce §, on fera correspondre à tout élément de $(G/H)^*_{k((t))}$ un plongement élémentaire, puis on étudiera cette correspondance.

4.1. Pour la commodité du lecteur, rappelons quelques généralités au sujet des points à valeur dans $k((t))$.

Soit X une variété algébrique (intègre) sur k. Un élément λ de $X_{k((t))}$ est la donnée d'une localité $\mathcal{O}, \mathfrak{M}$ de X et d'un homomorphisme de k-algèbres $\lambda:\mathcal{O}\to k((t))$ qui induit une injection $\mathcal{O}/\mathfrak{M}\to k((t))$. On appelle $\mathcal{O}$ le domaine de définition de λ. Le cas où X est affine est particulièrement simple: on a alors forcément $k[X]\subset\mathcal{O}$, et λ est déterminé par sa restriction à $k[X]$.

Un $\lambda\in X_{k((t))}$ est dit convergent, s'il existe $x\in X$ tel que $\mathcal{O}_x\subset\mathcal{O}$ et $\lambda(\mathcal{O}_x)\subset k[[t]]$; on appelle alors x la limite de λ et on écrit $x=\lim_{t\to 0}\lambda(t)$ (l'unicité de la limite résulte de la séparation de X). On note $X_{k[[t]]}$ l'ensemble des points convergents de $X_{k((t))}$. Si X est affine, pour que λ soit convergent, il faut et il suffit que $\lambda(k[X])\subset k[[t]]$, et la limite est alors donnée par $k[X]\xrightarrow{\lambda} k[[t]]\to k[[t]]/tk[[t]]=k$.

Nous utiliserons la structure naturelle de groupe sur $G_{k((t))}$, induite par la structure de groupe algébrique sur G; $G_{k[[t]]}$ est un sous-groupe de $G_{k((t))}$, et G s'identifie à un sous-groupe de $G_{k[[t]]}$. Nous utiliserons aussi l'opération naturelle de $G_{k((t))}$ dans $(G/H)_{k((t))}$.

4.2. Quel que soit le corps K, nous désignerons par $v_t:K((t))^*\to\mathbb{Z}$ la valuation discrète naturelle sur $K((t))$ (l'ordre en t des séries formelles). Dans ce qui va suivre, on considère $k[G]\otimes k((t))$, ainsi que son corps des fractions, plongé de manière naturelle dans $k(G)((t))$.

Soit $\lambda\in(G/H)_{k((t))}$. L'opération de G dans G/H donne un morphisme dominant

$$G\times\operatorname{Spec}k((t))\xrightarrow{1\times\lambda}G\times G/H\to G/H$$

d'où une injection de corps $i_\lambda:k(G/H)\to k(G)((t))$.

Posons $\mathcal{O}_\lambda=(i_\lambda)^{-1}(k(G)[[t]])$; c'est une algèbre locale dont nous noterons m_λ l'idéal maximal et k_λ le corps résiduel. Posons $v_\lambda=v_t\cdot i_\lambda:k(G/H)^*\to\mathbb{Z}$. Lorsque $\mathcal{O}_\lambda\neq k(G/H)$, v_λ est une valuation discrète (non nécessairement normalisée) de $k(G/H)$, dont $\mathcal{O}_\lambda$ est la sous-algèbre de valuation discrète (voir 3.1); désignons par n_λ l'entier positif tel que $(1/n_\lambda)v_\lambda$ soit normalisée.

Nous verrons plus loin que $(1/n_\lambda)v_\lambda \in \mathcal{V}_1(G/H)$. Le point délicat est la géométricité de $\mathcal{O}_\lambda$, que nous démontrerons en 4.6. Vérifions déjà que $\mathcal{O}_\lambda$ est stable par G et que ${}^G k_\lambda = k$: le premier résulte de ce que $k(G)[[t]]$ est stable sous l'opération de G par translations à gauche dans $k(G)$ et de ce que i_λ commute aux opérations de G; le second de ce que k_λ s'identifie à un sous-corps de $k(G)$.

4.3. Le morphisme canonique $G \to G/H$ induit une application $G_{k((t))} \to (G/H)_{k((t))}$. Si $\lambda \in G_{k((t))}$, notons $\bar{\lambda}$ son image dans $(G/H)_{k((t))}$. Du diagramme clairement commutatif

$$\begin{tikzcd} k(G) \arrow[dr, "i_\lambda"] & \\ & k(G)((t)) \\ k(G/H) \arrow[uu, hook] \arrow[ur, "i_{\bar{\lambda}}"'] & \end{tikzcd}$$

résulte que $v_{\bar{\lambda}}$ est la restriction de v_λ.

L'application $G_{k((t))} \to (G/H)_{k((t))}$ n'est pas en général surjective. Néanmoins, on obtient par un argument classique un résultat qui est presque aussi bon que la surjectivité.

LEMME. *Pour tout $\mu \in (G/H)_{k((t))}$, il existe $n \in \mathbb{N}^*$ et $\lambda \in G_{k((\sqrt[n]{t}))}$, tels que $\bar{\lambda} = \mu$ (où l'on considère μ comme point de G/H à valeurs dans $k((\sqrt[n]{t})) \supset k((t))$).*

Preuve. D'après le théorème de normalisation de Noether, on peut trouver des sous-algèbres affines A de $k(G/H)$ et B de $k(G)$ vérifiant

1) $A \subset B$;
2) $X_A \subset G/H$ *et* $X_B \subset G$;
3) il existe des éléments $g_1, \ldots, g_m$ de B algébriquement indépendants sur A, tels que B soit fini sur $A[g_1, \ldots, g_m]$.

Quitte à translater A et B par un $s \in G$ convenable, on peut supposer de plus que $\mu \in (X_A)_{k((t))}$. Il est alors clair que μ peut se relever en un point de G à valeurs dans une extension finie de $k((t))$, lesquelles sont isomorphes aux $k((\sqrt[n]{t}))$, $n \in \mathbb{N}^*$, ce qui démontre le lemme.

Ce lemme (et la remarque qui le précède) vont nous permettre dans l'étude des éléments de $(G/H)_{k((t))}$ de supposer qu'ils proviennent d'éléments de $G_{k((t))}$.

4.4. Le groupe $G_{k((t))}$ opère "par translations à droite" dans $k[G] \otimes k((t))$, en respectant sa structure de $k((t))$-algèbre. Précisons l'opération: soit $\lambda \in G_{k((t))}$, que nous considérons comme un homomorphisme de k-algèbres $\lambda : k[G] \to k((t))$;

alors R_λ, la "translation à droite par λ", est donnée sur $k[G]$ par

$$k[G] \xrightarrow{\text{comult.}} k[G] \otimes k[G] \xrightarrow{1\otimes\lambda} k[G] \otimes k((t)),$$

homomorphisme de k-algèbres qui se prolonge par $k((t))$-linéarité en un isomorphisme de $k[G]\otimes k((t))$ sur lui-même.

Si $\lambda \in G_{k[[t]]}$, on voit que R_λ laisse stable $k[G]\otimes k[[t]]$. Il s'ensuit que l'opération de $G_{k[[t]]}$ dans $k[G]\otimes k((t))$ se prolonge en une opération dans $k(G)((t))$ vérifiant $v_t \circ R_\lambda = v_t$, pour tout $\lambda \in G_{k[[t]]}$. Par contre, l'opération de $G_{k((t))}$ ne se prolonge pas à $k(G)((t))$; tout au plus peut-on la prolonger au corps des fractions de $k[G]\otimes k((t))$.

Remarquons que, dans le cas où $H=\{e\}$, l'injection i_λ définie dans 4.2 coïncide avec R_λ, modulo les inclusions $k(G) \hookrightarrow$ corps des fractions de $k[G]\otimes k((t)) \hookrightarrow k(G)((t))$; cette remarque nous servira dans la démonstration suivante.

LEMME. *Si* $\lambda \in (G/H)_{k((t))}$ *et si* $\mu \in G_{k[[t]]}$, *alors* $v_\lambda = v_{\mu\lambda}$.

Preuve. D'après 4.3, on peut supposer que $\lambda \in G_{k((t))}$. Quel que soit $f \in k(G)$, on a alors $v_{\mu\lambda}(f) = (v_t \circ i_{\mu\lambda})(f) = v_t(R_{\mu\lambda}f) = v_t(R_\mu R_\lambda f) = v_t(R_\lambda f) = (v_t \circ i_\lambda)(f) = v_\lambda(f)$.

4.5. On appellera germe de courbe (éventuellement divergent) dans G/H la donnée

1) d'une courbe lisse C,
2) d'un point c de C,
3) d'un morphisme $\gamma: C-\{c\} \to G/H$.

Tout isomorphisme $\hat{\mathcal{O}}_{C,c} \simeq k[[t]]$ donne une inclusion $k(C) \hookrightarrow k((t))$, qui permet d'associer au germe de courbe un point de $(G/H)_{k((t))}$, qu'on appellera un germe formel associé au germe de courbe. Soit $\lambda \in (G/H)_{k((t))}$, et désignons par $\mathcal{O}$ le domaine de définition de λ; pour que λ puisse s'obtenir comme germe formel associé à un germe de courbe, il faut et il suffit clairement que $\lambda(\mathcal{O})$ soit un sous-corps de $k((t))$ de degré de transcendance ≤ 1 sur k.

LEMME. *Pout tout* $\lambda \in G_{k((t))}$ *il existe* $\mu \in G_{k[[t]]}$ *tel que* $\mu\lambda$ *soit le germe formel associé à un germe de courbe dans* G.

Preuve. Plongeons G comme un sous-groupe fermé dans un $SL(n)$. Soient $p_1, \ldots, p_m \in k[x_{ij}, 1 \leq i, j \leq n]$ des équations polynômes qui définissent G dans l'ensemble des matrices $n \times n$. Si $\rho(t) = (\rho_{ij}(t))_{1\leq i,j\leq n}$, avec $\rho_{ij}(t) \in k((t))$, on a

$\rho(t) \in G_{k((t))}$ si et seulement si

$$(*) \quad p_r(\rho_{ij}(t),\quad 1 \le i,j \le n) = 0, \quad \text{pour} \quad r = 1, \ldots, m.$$

D'après un théorème d'Artin ([1]), pour tout $N \in \mathbb{N}$, il existe $\bar{\lambda}(t) = (\bar{\lambda}_{ij}(t))_{1 \le i,j \le n}$ vérifiant

1) les $\bar{\lambda}_{ij}(t)$ sont algébriques sur $k(t)$;
2) les $\bar{\lambda}_{ij}(t)$ vérifient les équations (*);
3) on a $\lambda_{ij}(t) - \bar{\lambda}_{ij}(t) \in t^N k[[t]]$.

Pour N assez grand, on a $\bar{\lambda}(t)\lambda(t)^{-1} = \mu(t) \in G_{k[[t]]}$, ce qui s'écrit aussi $\mu(t)\lambda(t) = \bar{\lambda}(t)$. Les coefficients de $\bar{\lambda}(t)$ étant tous algébriques sur $k(t)$, tous les éléments de $\bar{\lambda}(k[G])$ sont algébriques sur $k(t)$. Le sous-corps de $k((t))$ engendré par $\bar{\lambda}(k[G])$ est donc bien de degré de transcendance ≤ 1 sur k, ce qu'il fallait démontrer.

4.6. A tout $\lambda \in (G/H)_{k((t))}$ tel que $\mathcal{O}_\lambda \ne k(G/H)$, nous avons associé en 4.2 une valuation discrète normalisée $(1/n_\lambda)v_\lambda$ de $k(G/H)$.

PROPOSITION. *On a* $(1/n_\lambda)v_\lambda \in \mathcal{V}_1(G/H)$.

Preuve. Nous savons déjà que v_λ est invariante par G, et que ${}^G k_\lambda = k$ (voir 4.2). Reste à prouver que $\mathcal{O}_\lambda$ est géométrique.

D'après 4.3 et 3.1, il suffit de considérer le cas $H = \{e\}$. D'après 4.4 et 4.5, nous pouvons de plus supposer que λ est le germe formel d'un germe de courbe dans G.

Soit donc $C, c \in C, \gamma : C - \{c\} \to G$ un tel germe de courbe. Le morphisme dominant

$$G \times (C - \{c\}) \xrightarrow{1 \times \gamma} G \times G \xrightarrow{\text{mult.}} G$$

se reflète en une injection $i_\lambda : k(G) \to k(G \times C)$. L'isomorphisme $\hat{\mathcal{O}}_{C,c} \simeq k[[t]]$ induit une identification de $\hat{\mathcal{O}}_{G \times C, G \times \{c\}}$ avec $k(G)[[t]]$, d'où une injection de corps $j : k(G \times C) \hookrightarrow k(G)((t))$. Par construction, on a $i_\lambda = j \circ i_\gamma$. Puisque $k(G \times C) \cap \hat{\mathcal{O}}_{G \times C, G \times \{c\}} = \mathcal{O}_{G \times C, G \times \{c\}}$, il s'ensuit que $\mathcal{O}_\lambda = (i_\gamma)^{-1}(\mathcal{O}_{G \times C, G \times \{c\}})$, et $\mathcal{O}_\lambda$ est géométrique d'après 3.1.

4.7. Soit $\lambda \in (G/H)_{k((t))}$. Si $f \in k(G/H)$, posons $i_\lambda(f) = \sum_{n \gg -\infty} f_{\lambda,n} t^n$, où i_λ est l'injection de $k(G/H)$ dans $k(G)((t))$ introduite dans 4.2.

LEMME. *Pout tout $f \in k(G/H)$, il existe un ouvert non vide U de G vérifiant: pour tout $s \in U$, on a*

1) $f_{\lambda,n} \in \mathcal{O}_{G,s}$ *pour tout* n;
2) $s^{-1} \cdot f$ *est dans le domaine de définition de* λ;
3) $\lambda(s^{-1} \cdot f) = \sum_{n \gg -\infty} f_{\lambda,n}(s) t^n$.

Preuve. D'après 4.3, on peut supposer que $\lambda \in G_{k((t))}$. Dans ce cas λ est donné par un homomorphisme d'algèbres $\lambda : k[G] \to k((t))$, et l'application i_λ est donnée sur $k[G]$ par

$$k[G] \xrightarrow{\text{comult.}} k[G] \otimes k[G] \xrightarrow{1 \otimes \lambda} k[G] \otimes k((t)) \hookrightarrow k[G]((t));$$

il s'ensuit que, pour $f \in k[G]$, les trois propriétés sont vraies avec $U = G$. Dans le cas général, écrivons $f = g/h$, où $g, h \in k[G]$. Soit $i_\lambda(h) = \sum_{n \geq n_0} h_{\lambda,n} t^n$, avec $h_{\lambda,n_0} \neq 0$. On vérifie sans peine que l'ouvert $U = \{s \in G, h_{\lambda,n_0}(s) \neq 0\}$ convient pour f.

4.8. Soit $\lambda \in (G/H)_{k((t))}$ tel que $\mathcal{O}_\lambda \neq k(G/H)$. D'après 4.6 et 3.3, $\mathcal{O}_\lambda$ correspond à un plongement élémentaire de G/H. Nous le désignerons par X_λ, et par T_λ l'orbite fermée de G dans X_λ.

PROPOSITION. *Dans X_λ, $\lim_{t \to 0} \lambda(t)$ existe et appartient à T_λ.*

Preuve. Choisissons une sous-algèbre affine A de $k(G/H)$ vérifiant: $X_A \subset X_\lambda$ et $X_A \cap T_\lambda \neq \emptyset$; en particulier, on a $A \subset \mathcal{O}_{X_\lambda, T_\lambda} = \mathcal{O}_\lambda$. D'après 4.7, quitte à translater A par un $s \in G$ convenable, on peut supposer de plus que, pour tout $f \in A$, on a

1) $f_{\lambda,n} \in \mathcal{O}_{G,e}$ pour tout n;
2) f appartient au domaine de définition de λ;
3) $\lambda(f) = \sum_{n \gg -\infty} f_{\lambda,n}(e) t^n$.

Puisque $A \subset \mathcal{O}_\lambda$, on en déduit que $\lambda(A) \subset k[[t]]$, c'est-à-dire que $\lim_{t \to 0} \lambda(t)$ existe dans X_A. De plus, on voit que λ envoie l'idéal $A \cap \mathfrak{m}_\lambda$, qui correspond au fermé $X_A \cap T_\lambda$, dans $tk[[t]]$, ce qui signifie bien que $\lim_{t \to 0} \lambda(t) \in T_\lambda$.

Rappelons que $(G/H)^*_{k((t))} = (G/H)_{k((t))} - (G/H)_{k[[t]]}$.

COROLLAIRE. *Soit $\lambda \in (G/H)_{k((t))}$. Pour que $\mathcal{O}_\lambda \neq k(G/H)$, il faut et il suffit que $\lambda \in (G/H)^*_{k((t))}$.*

Preuve. Si $\mathcal{O}_\lambda \neq k(G/H)$, la proposition précédente montre que $\lambda \in (G/H)^*_{k((t))}$. Si $\mathcal{O}_\lambda = k(G/H)$, un argument très voisin de celui employé dans la démonstration précédente montre que $\lambda \in (G/H)_{k[[t]]}$.

4.9. Soient X un plongement de G/H, T une orbite de G dans X. Soit $\lambda \in (G/H)^*_{k((t))}$.

PROPOSITION. *Les conditions suivantes sont équivalentes*:
(1) $\lim_{t\to 0} \lambda(t)$ *existe dans* X *et appartient à* T;
(2) *l'identité de* G/H *se prolonge en un morphisme* $\varphi : X_\lambda \to X$ *tel que* $\varphi(T_\lambda) = T$;
(3) $\mathcal{O}_\lambda$ domine $\mathcal{O}_{X,T}$.

Preuve. L'équivalence (2)$\Leftrightarrow$(3) résulte de 2.4, l'implication (2)$\Rightarrow$(1) de 4.8.

Reste à montrer (1)$\Rightarrow$(3). Posons $x = \lim_{t\to 0} \lambda(t) \in T$. Choisissons une sous-algèbre affine A de $k(G/H)$ telle que $x \in X_A \subset X$. Choisissons ensuite un ouvert non vide U de G tel que $sx \in X_A$ quel que soit $s \in U$, et tel que U vérifie les trois conditions du lemme 4.7 pour tout $f \in A$. Si $f \in A$ et si $s \in U$, on a donc $\lambda(s^{-1}f) = \sum_{n \gg -\infty} f_{\lambda,n}(s)t^n$. Puisque $\lim_{t\to 0} s\lambda(t) = sx$ existe dans X_A, il s'ensuit que $f_{\lambda,n}(s) = 0$ pour $n < 0$ et $s \in U$, c'est-à-dire $f_{\lambda,n} = 0$ pour $n < 0$, autrement dit $v_\lambda(f) \geq 0$. De plus $v_\lambda(f) = 0$ équivaut à $f_{\lambda,0} \neq 0$, et cette dernière condition signifie qu'il existe $s \in U$ tel que $0 \neq f_{\lambda,0}(s) = \lim_{t\to 0} f(s)\lambda(t) = f(sx)$. Autrement dit, on a $A \subset \mathcal{O}_\lambda$, et $A \cap \mathfrak{m}_\lambda$ est l'idéal des $f \in A$ qui s'annulent sur $X_A \cap T$, ce qui entraîne bien que $\mathcal{O}_\lambda$ domine $\mathcal{O}_{X_A, X_A \cap T} = \mathcal{O}_{X,T}$.

4.10. Le groupe $\mathrm{Aut}_k\, k[[t]]$ opère de manière naturelle dans $G_{k[[t]]}$, par automorphismes de groupes. On peut donc former le produit semi-direct $\Gamma = G_{k[[t]]} \times \mathrm{Aut}_k\, k[[t]]$. Le groupe Γ opère dans $(G/H)^*_{k((t))}$ par

$$[(\mu, \alpha) \cdot \lambda](t) = \mu(t)\lambda(\alpha(t)), \quad (\mu, \alpha) \in \Gamma, \quad \lambda \in (G/H)^*_{k((t))}.$$

Considérons l'application $(G/H)^*_{k((t))} \to \mathcal{V}_1(G/H) \times \mathbb{N}^*$ qui envoie λ sur $[(1/n_\lambda)v_\lambda, n_\lambda]$.

PROPOSITION. *L'application précédente passe au quotient en une bijection*

$$\Gamma \backslash (G/H)^*_{k((t))} \xrightarrow{\sim} \mathcal{V}_1(G/H) \times \mathbb{N}^*.$$

Preuve. D'après 4.4, l'application est constante sur les orbites de $G_{k[[t]]}$; il est clair qu'elle l'est aussi sur les orbites de $\mathrm{Aut}_k\, k[[t]]$; elle passe donc bien au quotient par Γ.

Considérons un plongement élémentaire X d'orbite fermée T. Choisissons une courbe lisse C, un point $c \in C$ et un morphisme $\gamma : C \to X$ vérifiant: $\gamma(c) = x \in T$

et γ est transverse à T. Cette dernière condition implique que le morphisme $G\times C\to X$, qu'on déduit de l'opération de G dans X, est lisse. Soit $\lambda\in(G/H)^*_{k((t))}$ un germe formel associé à γ comme dans 4.5. En raisonnant comme en 4.6, on voit que $X_\lambda=X$; de la lissité de $G\times X\to X$ résulte en plus que $n_\lambda=1$. En considérant les $\lambda(t^n), n\in\mathbb{N}^*$, on voit que l'application $\Gamma\backslash(G/H)^*_{k((t))}\to \mathcal{V}_1(G/H)\times\mathbb{N}^*$ est surjective.

Soit $\lambda'\in(G/H)^*_{k((t))}$ tel que $X_{\lambda'}=X$. D'après 4.8, quitte à multiplier λ' par un $s\in G$ convenable, on peut supposer que $\lim_{t\to 0}\lambda'(t)=x$. Grâce à une propriété de relèvement bien connue des morphismes lisses, il existe $\mu\in G_{k[[t]]}$ (vérifiant $\lim_{t\to 0}\mu(t)=e$) et $\beta\in tk[[t]]$ tels que $\lambda'(t)=\mu(t)\lambda(\beta(t))$. Il suffit alors de prendre une racine n-ième de β, où n est l'ordre de β, pour voir que $\lambda'(t)$ est sur l'orbite de Γ passant par $\lambda(t^n)$. On a visiblement $n=n_{\lambda'}$, d'où il suit que l'application $\Gamma\backslash(G/H)^*_{k((t))}\to\mathcal{V}_1(G/H)\times\mathbb{N}^*$ est aussi injective.

Remarque. Résumons pour la suite une partie des résultats précédents, en les reformulant légèrement. A normalisation près, toute valuation de $\mathcal{V}_1(G/H)$ s'obtient comme restriction à $k(G/H)$ d'un v_λ, $\lambda\in G_{k((t))}$. De plus, si $(\mu,\nu,\alpha)\in G_{k[[t]]}\times H_{k((t))}\times(tk[[t]]-t^2k[[t]])$, et si $\lambda'(t)=\mu(t)\lambda(\alpha(t))\nu(t)$, alors v_λ et $v_{\lambda'}$ ont même restriction à $k(G/H)$.

4.11. Soit $v\in\mathcal{V}(G/H)$. Si V est un sous-espace vectoriel de $k(G/H)$ et si $j\in\mathbb{Q}$, on pose

$$F^j_vV=\{f\in V, v(f)\geq j\};$$

les F^j_vV sont des sous-espaces vectoriels de V, et $i<j$ implique $F^i_vV\supset F^j_vV$.

On dira qu'une suite v_n $(n\in\mathbb{N})$ dans $\mathcal{V}(G/H)$ converge géométriquement vers v dans $\mathcal{V}(G/H)$ si

1) il existe une suite r_n $(n\in\mathbb{N})$ de nombres rationnels positifs telle que $r_nv_n(f)$ converge vers $v(f)$, quel que soit $f\in k(G/H)$;
2) pour tout sous-espace vectoriel de dimension finie V de $k(G/H)$, il existe un entier $n(V)$ tel que, pour tout $n\geq n(V)$, chacun des F^i_vV $(i\in\mathbb{Z})$ est égal à l'un des $F^j_{v_n}V$ $(j\in\mathbb{Z})$.

Lorsqu'on connait déjà les valuations de $\mathcal{V}_1(G/H)$, pour déterminer celles de $\mathcal{V}_2(G/H)$, on peut parfois se servir de la proposition suivante (voir par exemple [16]).

PROPOSITION. *Tout élément de $\mathcal{V}(G/H)$ est limite géométrique d'une suite d'éléments de $\mathcal{V}_1(G/H)$.*

Soit $G/H \hookrightarrow X$ un plongement de G/H, avec la variété X affine. Soit Y un fermé de X, stable par G. Choisissons un point x dans Y. Soit $\lambda \in (G/H)_{k((t))}$ tel que $\lim_{t\to 0} \lambda(t) = x$.

LEMME 1. *Pour tout $f \in \mathcal{O}_{X,x}$ et pour tout voisinage assez petit U de e dans G, λ est défini en $s^{-1} \cdot f$ pour tout $s \in U$, et $v_\lambda(f) = \inf_{s\in U} v_t(\lambda(s^{-1} \cdot f))$.*

Preuve. Pour tout $f \in k(G/H)$, posons comme dans 4.7 $i_\lambda(f) = \sum_{n\gg-\infty} f_{\lambda,n} t^n$; par définition de v_λ on a $v_\lambda(f) = v_t(i_\lambda(f))$. Si $g \in k[G/H]$, on a $g_{\lambda,n} \in k[G]$ quel que soit $n \in \mathbb{Z}$, λ est définie sur $s^{-1} \cdot g$ quel que soit $s \in G$, et $\lambda(s^{-1} \cdot g) = \sum_{n\gg-\infty} g_{\lambda,n}(s)t^n$; par suite, quel que soit l'ouvert non vide U de G, on a $v_\lambda(g) = \inf_{s\in U} v_t(s^{-1} \cdot g)$. Si $h \in k[X] \subset k[G/H]$, $\lim_{t\to 0} \lambda(t) = x \in X$ implique $k[X] \subset \mathcal{O}_{v_\lambda}$, donc $h_{\lambda,n} = 0$ pour $n < 0$; si de plus pour un $s \in G$, $h(s \cdot x) \neq 0$, alors $\lambda(s^{-1} \cdot h) = \sum_{n\geq 0} h_{\lambda,n}(s)t^n \neq 0$, car $h_{\lambda,0}(s) = h(s \cdot x) \neq 0$ et $v_t(s^{-1} \cdot h) = v_\lambda(h) = 0$.

Soit maintenant $f \in \mathcal{O}_{X,x}$. On peut écrire $f = g/h$, avec $g,h \in k[X]$ et $h(x) \neq 0$. Soit U un voisinage de e dans G vérifiant $h(s \cdot x) \neq 0$ quel que soit $s \in U$. Alors, on a clairement

$$v_\lambda(f) = v_\lambda(g) = \inf_{s\in U} v_t(s^{-1} \cdot g) = \inf_{s\in U} v_t(s^{-1} \cdot f), \quad \text{c.q.f.d.}$$

Gardons les mêmes hypothèses que pour le lemme 1. Supposons de plus X normale, Y de codimension 1 dans X, et x lisse dans X et dans Y. Désignons par v la valuation de $\mathcal{V}(G/H)$ telle que $\mathcal{O}_v = \mathcal{O}_{X,Y}$.

LEMME 2. *Pour tout sous-espace vectoriel de dimension finie V de $\mathcal{O}_{X,x}$, il existe un entier p qui vérifie: pour tout $q > p$, on peut trouver $\lambda \in (G/H)^*_{k((t))}$ tel que*

1) $\lim_{t\to 0} \lambda(t) = x$;
2) $F^j_v V = F^{qj}_{v_\lambda} V$, *quel que soit* $j \in \mathbb{Z}$;
3) $\left| v(f) - \dfrac{1}{q} v_\lambda(f) \right| \leq p/q$, *quel que soit* $f \in V - \{0\}$.

Preuve. Choisissons $f_1, \ldots, f_r \in m_x \cap k[X] \subset k[G/H]$ des coordonnées locales en x, de manière à ce que $f_1 = 0$ définit Y au voisinage de x. On utilisera les identifications $\mathcal{O}_{X,x} \subset \hat{\mathcal{O}}_{X,x} = k[[f_1, \ldots, f_r]]$. Tout $f \in k[[f_1, \ldots, f_r]]$ peut s'écrire de manière unique $f = \sum_{i=0}^{\infty} c_i(f) f_1^i$, où $c_i(f) \in k[[f_2, \ldots, f_r]]$; on définit ainsi des applications linéaires $c_i : k[[f_1, \ldots, f_r]] \to k[[f_2, \ldots, f_r]]$ $(i \in \mathbb{N})$. Choisissons $\alpha_2, \ldots, \alpha_r \in tk[[t]]$ algébriquement indépendant sur k, et désignons par $\lambda' : k[[f_2, \ldots, f_r]] \to k[[t]]$ l'unique homomorphisme d'algèbres tel que $\lambda'(f_i) = \alpha_i$ $(i = 2, \ldots, r)$. Par construction, λ' est injectif en restriction à $k[f_2, \ldots, f_r]$. Il en résulte que λ' reste injectif en restriction à la sous-algèbre des éléments de

$k[[f_2, \ldots, f_r]]$ qui sont algébriques sur $k[f_2, \ldots, f_r]$. Tout $f \in \mathcal{O}_{X,x}$ est algébrique sur $k[f_1, \ldots, f_r]$; il s'ensuit facilement, par récurrence sur i, que $c_i(f)$ est algébrique sur $k[f_2, \ldots, f_r]$ quel que soit $i \in \mathbb{N}$. Par suite, on peut choisir N assez grand pour que $(c_0(f), \ldots, c_N(f)) \neq (0, \ldots, 0)$ quel que soit $f \in V - \{0\}$. Posons

$$p = \max\{v_t(\lambda'(c_i(f))), f \in V - \{0\}, c_i(f) \neq 0, i \in [0, N]\}.$$

Choisissons $q > p$ et désignons par $\lambda : k[[f_1, \ldots, f_r]] \to k[[t]]$ l'unique homomorphisme d'algèbres tel que $\lambda(f_1) = t^q$ et $\lambda(f_i) = \alpha_i$ $(i = 2, \ldots, r)$. Via les inclusions $k(G/H) \supset \mathcal{O}_{X,x} \subset k[[f_1, \ldots, f_r]]$, on peut considérer λ comme élément de $(G/H)_{k((t))}$. Reste à vérifier que p, q et λ possèdent les propriétés 1), 2) et 3) du lemme 2.

Par construction, il est clair que $\lim_{t \to 0} \lambda(t) = x$, d'où 1).

Soit $f \in V - \{0\}$. Supposons que $v(f) = j$. On peut alors écrire $f = f_1^j g$, où $g \in \mathcal{O}_{X,x} - f_1 \mathcal{O}_{X,x}$ (l'algèbre $\mathcal{O}_{X,x}$ est factorielle). Autrement écrit, cela devient

$$f = c_j(f) f_1^j + h f_1^{j+1} = f_1^j (c_j(f) + h f_1),$$

où $c_j(f) \in k[[f_2, \ldots, f_r]] - \{0\}$ et $h \in k[[f_1, \ldots, f_r]]$. On a clairement $j \leq N$. Par suite $v_t(\lambda(c_j(f))) = v_t(\lambda'(c_j(f))) \leq p$. Puisque $v_t(\lambda(hf_1) \geq v_t(\lambda(f_1)) = q$ et que par hypothèse $q > p$, on a $v_t(\lambda(g)) = v_t(\lambda(c_j(f))) \leq p$. D'après le lemme 1, il en résulte que $v_\lambda(g) \leq p$.

D'un autre côté, désignons par U l'ouvert des $s \in G$ tels que $s^{-1} \cdot f_1 = 0$ soit encore une équation de Y au voisinage de x. Si $s \in U$, on peut écrire $s^{-1} \cdot f_1 = f_1 u$, où $u \in \mathcal{O}_{X,x}$ vérifie $u(x) \neq 0$. D'où, pour tout $s \in U$, $v_t(\lambda(s^{-1} \cdot f_1)) = v_t(\lambda(f_1)) = q$, puisque $v_t(\lambda(u)) = 0$. Il en résulte, puisque $f_1 \in k[X] \subset k[G/H]$, que $v_\lambda(f_1) = \inf_{s \in U} v_t(\lambda(s^{-1} \cdot f)) = v_t(\lambda(f_1)) = q$.

En résumé, on a montré que pour tout $f \in V - \{0\}$, $v(f) = j$ entraîne

$$qj \leq v_\lambda(f) \leq qj + p < q(j+1).$$

D'où aussitôt 3) et l'inclusion $F_v^j V \subset F_{v_\lambda}^{qj} V$, quel que soit $j \in \mathbb{Z}$. Si $f \notin F_v^j V$, on a $v(f) = j' < j$, donc $v_\lambda(f) < q(j'+1) \leq qj$, d'où $f \notin F_{v_\lambda}^{qj}$, ce qui entraîne l'égalité de 2).

Preuve de la proposition. Grâce à 3.2, il suffit de considérer le cas $H = \{e\}$.

Montrons qu'on peut en plus supposer que $v \in \mathcal{V}(G/\{e\})$ prenne au moins une valeur strictement positive sur $k[G]$. Considérons $G = G \times \{e\} \subset G \times k^*$ et posons $k[G \times k^*] = k[G][t, t^{-1}]$. Il existe une unique valuation $\bar{v} \in \mathcal{V}((G \times k^*)/\{e\})$ dont la restriction à $k(G)$ est v et qui vérifie $\bar{v}(t) = 1$. Il est clair qu'il suffit de démontrer la proposition pour $\bar{v}$.

Supposons donc que $v \in \mathscr{V}(G/\{e\})$ et qu'il existe $f \in k[G]$ tel que $v(f) > 0$. Dans ce cas, on peut trouver un plongement $G = G/\{e\} \hookrightarrow X$, avec la variété X affine et normale, et un fermé Y dans X, stable par G et de codimension 1, tels que $\mathcal{O}_v = \mathcal{O}_{X,Y}$. Autrement dit, nous sommes dans les conditions du lemme 2.

Soit V_n ($n \in \mathbb{N}$) une suite croissante de sous-espaces vectoriels de dimension finie de $k[X]$, telle que $\bigcup_{n \in \mathbb{N}} V_n = k[X]$. Pour tout $n \in \mathbb{N}$, soit p_n l'entier que le lemme 2 associe à V_n; choisissons $q_n > p_n$ tel que la suite $p_n/q_n \to 0$; soit enfin λ_n l'élément de $G^*_{k((t))}$ que le lemme 2 associe à V_n et q_n. Désignons par v_n la valuation de $\mathscr{V}_1(G/\{e\})$ obtenue en normalisant v_{λ_n}.

Alors v_n tend géométriquement vers v. En effet, soit r_n ($n \in \mathbb{N}$) la suite des nombres rationnels positifs telle que $r_n v_n = (1/q_n) v_{\lambda_n}$. D'après la propriété 3) du lemme 2, si $f \in k[X]$, dès que $f \in V_n$, on a $|r_n v_n(f) - v(f)| \leq p_n/q_n$ qui tend vers 0. Si $f \in k(G/H)$, écrivons $f = g/h$ avec $g, h \in k[X]$; alors $r_n v_n(f) = r_n v_n(g) - r_n v_n(h)$ qui tend vers $v(g) - v(h) = v(f)$. D'autre part, si V est un sous-espace vectoriel de dimension finie de $k(G)$, il existe $g \in k[G]$ et $n = n(V)$ tels que $gV \subset V_n$. Alors, pour tout $i \in \mathbb{Z}$, si $j = (i + v(g))/r_n - v_n(g)$, de la propriété 2) du lemme 2 on déduit que $F^i_v V = F^j_{v_n} V$, ce qui termine la preuve de la proposition.

5. Compléments sur les plongements élémentaires

Les plongements élémentaires sont les plongements les plus simples possibles. A ce titre, ils méritent d'être étudiés, ce que nous commencerons à faire, après deux préliminaires, dans les trois derniers numéros de ce paragraphe.

5.1. Soit G' un sous-groupe algébrique de G contenant H.

Sous l'opération de G' par translations à droite, G est l'espace total d'un fibré principal de base G/G'. Pour toute G'-variété X', on peut donc former le fibré associé à ce fibré principal, de fibre type X'. On le notera $G *_{G'} X'$. L'opération de G dans lui-même par translations à gauche, passe au quotient en une opération de G dans $G *_{G'} X'$.

Il est clair que, pour que X' soit un plongement (resp. un plongement élémentaire) de G'/H, il faut et il suffit que $G *_{G'} X'$ soit un plongement (resp. un plongement élémentaire) de G/H.

LEMME. *Soit X un plongement élémentaire de G/H d'orbite fermée T. Pour qu'il existe un plongement élémentaire X' de G'/H tel que $X \simeq G *_{G'} X'$, il faut et il suffit que l'adhérence de G'/H dans X rencontre T.*

Preuve. Il est clair que la condition est nécessaire. Montrons qu'elle est aussi suffisante. Désignons par X' l'adhérence de G/H dans X. C'est un plongement de G'/H, mais a priori X' n'est pas nécessairement lisse. Considérons le G-morphisme naturel $\varphi : G *_{G'} X' \to X$. Modulo l'identification $G/H = G *_{G'} G'/H$, φ induit l'identité de G/H, et en particulier φ est birationnel. Pour tout $x \in T$, on a $\dim G_x = \dim H + 1$. Par suite, $\dim G'_x \leq \dim H + 1$, d'où

$$\dim G'/G'_x = \dim G' - \dim G'_x \geq \dim G' - \dim H - 1 = \dim G'/H - 1.$$

Puisque $\dim (T \cap X') < \dim G'/H$, il s'ensuit que G' a au plus un nombre fini d'orbites dans $X' \cap T$, et que les fibres de φ sont finies. D'après le théorème principal de Zariski, φ est un isomorphisme. Il s'ensuit que X' est un plongement élémentaire de G'/H, ce qui démontre le lemme.

5.2. La proposition du numéro suivant s'appuiera également sur le lemme que voici (qui concerne les groupes réductifs de transformations).

LEMME. *Soient K un groupe algébrique réductif connexe, X une variété algébrique affine lisse dans laquelle K opère algébriquement, Y une sous-variété lisse de X distincte de X et stable par K, et enfin x un point de Y fixé par K. On suppose que K n'a pas de point fixe dans $X - Y$. Il existe alors une orbite de K dans $X - Y$ dont l'adhérence dans X contient x.*

Ce lemme résulte par exemple sans peine des résultats de [7].

5.3. Dans la suite de ce paragraphe, on se bornera à considérer le cas $H = \{e\}$.

Soit X un plongement élémentaire de $G = G/\{e\}$ (on considère G comme espace homogène, G opérant dans lui-même par translations à gauche), d'orbite fermée T. Si $x \in T$, le groupe d'isotropie G_x est de dimension 1. Deux cas peuvent se produire: ou bien $(G_x)^0$, la composante connexe de G_x, est isomorphe au groupe multiplicatif k^*; ou bien $(G_x)^0$ est isomorphe au groupe additif k. Dans ce numéro et le suivant nous allons considérer le premier cas, et dans le numéro 5.5 nous pensons plutôt au second.

Voici une manière simple de construire des plongements élémentaires de G: choisissons un sous-groupe G' de G isomorphe à k^*, et considérons le plongement élémentaire $k^* \hookrightarrow k$, où k^* opère linéairement dans k; il induit par $G *_{G' \simeq k^*} k$ un plongement élémentaire de G. Ce qu'on a obtenu ainsi comme G-variété n'est rien d'autre qu'un G-fibré en droites sur G/G', dont la section nulle forme une orbite fermée, et dont le complémentaire de la section nulle est isomorphe à G.

PROPOSITION. *Soit X un plongement élémentaire de G d'orbite fermée T, et soit $x \in T$. On suppose $(G_x)^0 \simeq k^*$. Alors $G_x \simeq k^*$, et il existe un conjugué G' de G_x dans G tel que $X \simeq G *_{G' \simeq k^*} k$.*

Preuve. D'après un théorème de Sumihiro (voir [6]), $(G_x)^0$ étant un tore et X lisse donc normal, on peut trouver un voisinage de x dans X, ouvert affine et stable par $(G_x)^0$. D'après 5.2, il existe une orbite de $(G_x)^0$ dans $X-T$ dont l'adhérence dans X contient x. On en déduit l'existence d'un conjugué G' de $(G_x)^0$ dans G, dont l'adhérence dans X rencontre T. D'après 5.1, on peut trouver un plongement élémentaire X' de G' tel que $X \simeq G *_{G'} X'$. Puisque les seuls plongements élémentaires de k^* sont k et $\mathbb{P}_1 - \{0\}$, quitte à choisir convenablement l'isomorphisme $G' \simeq k^*$, on a $X \simeq G *_{G' \simeq k^*} k$. Il s'ensuit que G_x est un conjugué de G'; en particulier, G_x est connexe.

5.4. On désigne par $X_*(G)$ l'ensemble des morphismes de groupes algébriques $\lambda : k^* \to G$ non triviaux (c'est-à-dire tels que $\lambda(k^*) \neq \{e\}$). L'inclusion $k[t, t^{-1}] \subset k((t))$ permet de plonger $X_*(G)$ dans $G^*_{k((t))}$.

Pour tout $\lambda \in G^*_{k((t))}$, on note $G(\lambda)$ l'ensemble des $s \in G$ tels que $\lambda(t)s\lambda(t)^{-1} \in G_{k[[t]]}$; on vérifie sans peine que $G(\lambda)$ est un sous-groupe de G. Si $\lambda', \lambda \in X_*(G)$, on pose $\lambda' \sim \lambda$, s'il existe $n', n \in \mathbb{N}^*$ et $s \in G(\lambda)$ tels que $\lambda'(t^{n'}) = s\lambda(t^n)s^{-1}$.

Si $\lambda \in X_*(G) \subset G^*_{k((t))}$, rappelons qu'on désigne par X_λ le plongement élémentaire associé à λ (voir 4.8). Posons $G' = \lambda(k^*) \subset G$; c'est un sous-groupe algébrique de G isomorphe à k^*. Il est clair d'après 5.3 que $X_\lambda \simeq G *_{G' \simeq k^*} k$.

PROPOSITION. *Soient $\lambda', \lambda \in X_*(G)$. Pour que $X_{\lambda'} \simeq X_\lambda$, il faut et il suffit que $\lambda' \sim \lambda$.*

Preuve. Pour démontrer la proposition, on peut clairement se ramener au cas où les deux morphismes $\lambda', \lambda : k^* \to G$ sont injectifs; il suffit alors de montrer que $X_{\lambda'} \simeq X_\lambda$ si et seulement s'il existe $s \in G(\lambda)$ tel que $\lambda'(t) = s\lambda(t)s^{-1}$.

Supposons qu'il existe $s \in G(\lambda)$ tel que $\lambda'(t) = s\lambda(t)s^{-1}$. Alors, si l'on pose $\mu(t) = s\lambda(t)s^{-1}\lambda(t)^{-1}$, on a $\mu(t) \in G_{k[[t]]}$ et $\lambda'(t) = \mu(t)\lambda(t)$, d'où $X_{\lambda'} \simeq X_\lambda$ d'après 4.10.

Inversement, supposons $X_{\lambda'} \simeq X_\lambda$. Puisque $\lambda(k^*)$ et $\lambda'(k^*)$ s'interprètent comme groupes d'isotropie des orbites fermées dans X_λ et $X_{\lambda'}$, on voit qu'il existe $s \in G$ tel que $\lambda'(t) = s^{-1}\lambda(t)s$. Par ailleurs, en raisonnant comme dans 4.10, on peut trouver $(\alpha, \mu) \in \mathrm{Aut}_k\, k[[t]] \times G_{k[[t]]}$ tel que $\lambda'(t) = \mu(t)\lambda(\alpha(t))$. Puisque λ est un morphisme de groupes, on a $\lambda(\alpha(t))\lambda(t)^{-1} = \lambda(\alpha(t)/t) \in G_{k[[t]]}$. D'où $\lambda(t)s\lambda(t)^{-1} = s\lambda'(t)\lambda(t)^{-1} = s\mu(t)\lambda(\alpha(t))\lambda(t)^{-1} \in G_{k[[t]]}$. Il s'ensuit que $s \in G(\lambda)$, autrement dit que $\lambda' \sim \lambda$.

Remarque. Lorsque G est réductif, on peut montrer que les $G(\lambda)$, $\lambda \in X_*(G)$ sont des sous-groupes paraboliques de G, et l'ensemble quotient $X_*(G)/\sim$ n'est alors rien d'autre que la version de Mumford de l'immeuble sphérique de G (voir [8]), qu'on peut donc considérer comme plongé dans $\mathcal{V}_1(G/\{e\})$.

5.5. *Soit* X un plongement élémentaire de G d'orbite fermée T, et soit $x \in T$.

PROPOSITION. *Le groupe* $G_x/(G_x)^0$ *est cyclique.*

Preuve. Comme dans 4.10, soit λ un germe formel associé à une courbe dans X transverse à T, tel qu'on ait $\lim_{t\to 0} \lambda(t) = x$. Si $s \in G$, on a alors $s \in G_x$ si et seulement si $\lim_{t\to 0} s\lambda(t) = x$, c'est-à-dire si et seulement s'il existe $(\mu, \alpha) \in G_{k[[t]]} \times \mathrm{Aut}_k k[[t]]$ vérifiant $\lim_{t\to 0} \mu(t) = e$ et $s\lambda(t) = \mu(t)\lambda(\alpha(t))$.

Afin de pouvoir reformuler cette caractérisation de G_x de façon plus commode, introduisons quelques notations. Posons $\mathcal{A} = \mathrm{Aut}_k\, k[[t]]$, et désignons par $\mathcal{A}_n$ l'ensemble des $\alpha \in \mathcal{A}$ qui induisent l'identité dans $k[[t]]/t^{n+1}k[[t]]$. Les $\mathcal{A}_n$ sont des sous-groupes distingués dans $\mathcal{A} = \mathcal{A}_0$, et l'on a $\mathcal{A}/\mathcal{A}_1 \simeq k^*$ et $\mathcal{A}_n/\mathcal{A}_{n+1} \simeq k$ si $n \geq 1$. Designons par $\mathcal{A}(\lambda)$ l'ensemble des $\alpha \in \mathcal{A}$ tels que $\lim_{t\to 0} \lambda(\alpha(t))\lambda(t)^{-1}$ existe dans G. Définissons $h : \mathcal{A}(\lambda) \to G$ par $h(\alpha) = \lim_{t\to 0} \lambda(\alpha(t))\lambda(t)^{-1}$. On vérifie sans peine que $\mathcal{A}(\lambda)$ est un sous-groupe de $\mathcal{A}$, et que h est un homomorphisme de groupes. La caractérisation précédente de G_x peut se reformuler de la façon suivante: on a $h(\mathcal{A}(\lambda)) = G_x$.

De plus, on vérifie sans peine que, dès que n est assez grand:

1) $\mathcal{A}_n \subset \mathcal{A}(\lambda)$;
2) $\mathcal{A}(\lambda)/\mathcal{A}_n$ est un sous-groupe algébrique du groupe algébrique $\mathcal{A}/\mathcal{A}_n$;
3) h passe au quotient en un morphisme de groupes algébriques $\mathcal{A}(\lambda)/\mathcal{A}_n \to G$. Puisque $(\mathcal{A}(\lambda) \cap \mathcal{A}_1)/\mathcal{A}_n$, étant unipotent, est connexe, et que $\mathcal{A}(\lambda)/(\mathcal{A}(\lambda) \cap \mathcal{A}_1)$ s'injecte dans $\mathcal{A}/\mathcal{A}_1 \simeq k^*$, on voit que $(\mathcal{A}(\lambda)/\mathcal{A}_n)/(\mathcal{A}(\lambda)/\mathcal{A}_n)^0$ est cyclique. Par suite $G_x/(G_x)^0$ est cyclique, comme quotient d'un groupe cyclique.

6. Reformulation de la définition des plongements

Dans ce §, nous assemblons les résultats des §§1 à 4.

6.1. Rappelons que l'ensemble $\mathfrak{L}_1(G/H)$ est en bijection naturelle avec l'ensemble des orbites de G dans $\mathfrak{X}(G/H)$ (voir 2.2); si $l \in \mathfrak{L}_1(G/H)$; nous avons désigné par T_l l'orbite qui lui correspond. Si X est un sous-ensemble stable par G de $\mathfrak{X}(G/H)$, nous désignerons par $L(X)$ l'ensemble des $l \in \mathfrak{L}_1(G/H)$ tels que $T_l \subset X$.

PROPOSITION. *Soit X un sous-ensemble stable par G de $\mathfrak{X}(G/H)$.*

1) *Pour que X soit ouvert dans $\mathfrak{X}(G/H)$, il faut et il suffit que $L(X)$ soit ouvert dans $\mathfrak{L}_1(G/H)$.*
2) *Pour que X soit noethérien, il faut et il suffit que $L(X)$ le soit.*

Preuve. Soit A une sous-algèbre affine stable par $\mathfrak{B}$ de $k(G/H)$. Rappelons que nous avons noté X_A l'ensemble des $x \in \mathfrak{X}(G/H)$ tels que $\mathcal{O}_x$ soit un localisé de A; désignons de manière analogue par L_A l'ensemble des $l \in \mathfrak{L}_1(G/H)$ tels que $\mathcal{O}_l$ soit un localisé de A. Il est clair que $X_A \subset X$ si et seulement si $L_A \subset L(X)$. La première partie de la proposition en résulte aussitôt.

Si X est noethérien, on a $X \subset \bigcup_{i=1}^m X_{A_i}$, pour des sous-algèbres affines stables par $\mathfrak{B}$ de $k(G/H)$ convenables, d'où $L(X) \subset \bigcup_{i=1}^m L_{A_i}$, ce qui montre bien que $L(X)$ est noethérien. Inversement, si $L(X)$ est noethérien, on a $L(X) \subset \bigcup_{i=1}^m L_{A_i}$, où $A_1, \ldots, A_m$ sont certaines sous-algèbres affines stables par $\mathfrak{B}$ de $k(G/H)$, d'où $X \subset G(\bigcup_{i=1}^m X_{A_i})$, ce qui d'après 1.5 implique bien que X est noethérien.

6.2. Soit $l \in \mathfrak{L}_1(G/H)$. Nous désignerons par $\mathcal{F}_l$ l'ensemble des $v \in \mathcal{V}_1(G/H)$ tels que $\mathcal{O}_v$ domine $\mathcal{O}_l$. Nous appellerons $\mathcal{F}_l$ la facette de l. Il résulte de 4.9 que $\mathcal{F}_l \neq \emptyset$. La facette $\mathcal{F}_l$ constitue un renseignement important sur l; dans certains cas, les éléments de $\mathfrak{L}_1(G/H)$ sont même déterminés par leur facette (voir §9).

Si L est un sous-ensemble de $\mathfrak{L}_1(G/H)$, nous dirons que L est séparé si les facettes $\mathcal{F}_l$, $l \in L$ sont disjointes.

PROPOSITION. *Soit X un ouvert stable par G de $\mathfrak{X}(G/H)$. Pour que X soit séparé, il faut et il suffit que $L(X)$ le soit.*

Preuve. Il est bien connu qu'aucune localité ne peut dominer deux localités distinctes d'une variété séparée; par suite, si X est séparé, les facettes $\mathcal{F}_l$, $l \in L(X)$ sont disjointes.

Réciproquement, supposons X non séparé. On a alors $\bar{\Delta}_X \neq \Delta_X$, où Δ_X désigne la diagonale de $X \times X$. L'opération diagonale de G dans $X \times X$ laisse stable Δ_X, $\bar{\Delta}_x$ et $\bar{\Delta}_X - \Delta_X$. Il n'est pas difficile de voir que $\bar{\Delta}_X$ s'identifie à un ouvert stable par G de $\mathfrak{X}(G/H)$, et que les deux projections $X \xleftarrow{\pi_1} X \times X \xrightarrow{\pi_2} X$ induisent deux morphismes $X \xleftarrow{\pi_1} \bar{\Delta}_X \xrightarrow{\pi_2} X$ qui commutent à l'opération de G et qui induisent l'identité de G/H. Soit T une orbite de G dans $\bar{\Delta}_X - \Delta_X$. Puisque toute orbite de G dans $\mathfrak{X}(G/H)$ possède un voisinage séparé (1.5), on a $\pi_1(T) \neq \pi_2(T)$. Désignons par l_1 et l_2 les éléments de $L(X)$ qui correspondent à $\pi_1(T)$ et $\pi_2(T)$. Par construction, $\mathcal{O}_{\bar{\Delta}_X, T}$ domine $\mathcal{O}_{l_1}$ et $\mathcal{O}_{l_2}$. D'après 4.9, il existe $v \in \mathcal{V}_1(G/H)$ tel que $\mathcal{O}_v$ domine $\mathcal{O}_{\bar{\Delta}_X, T}$. Il s'ensuit que les facettes $\mathcal{F}_{l_1}$ et $\mathcal{F}_{l_2}$ ne sont pas disjointes.

6.3. Nous pouvons maintenant reformuler la définition des plongements.

DEFINITION. Un plongement de G/H est la donnée d'un sous-ensemble ouvert, noethérien et séparé de $\mathfrak{L}_1(G/H)$.

On voit comment cette reformulation se rattache à la définition des plongements donnée au §1: posons $X(L) = \bigcup_{l \in L} T_l \subset \mathfrak{X}(G/H)$; lorsque L parcourt les différents sous-ensembles ouverts noethériens et séparés de $\mathfrak{L}_1(G/H)$, grâce à 6.1 et 6.2, on obtient par $X(L)$ les différents plongements de G/H.

De 2.3, on déduit que $\mathfrak{L}_f(G/H)$ est ouvert dans $\mathfrak{L}_1(G/H)$; de plus, pour qu'un sous-ensemble L de $\mathfrak{L}_f(G/H)$ soit ouvert, il faut et il suffit manifestement qu'il soit saturé par localisation dans $\mathfrak{L}_f(G/H)$ (c'est-à-dire, tout l de $\mathfrak{L}_f(G/H)$, pour lequel il existe $l' \in L$ tel que $\mathcal{O}_l$ soit un localisé de $\mathcal{O}_{l'}$, appartient à L). Il s'ensuit, pour les plongements de G/H dont le nombre d'orbites est fini, une caractérisation particulièrement simple: ce sont les sous-ensembles finis de $\mathfrak{L}_f(G/H)$, saturés par localisation dans $\mathfrak{L}_f(G/H)$, et séparés.

Soit H' un sous-groupe algébrique de G contenant H. Soient X un plongement de G/H, X' un plongement de G/H'. De 2.3, on tire aussitôt que les assertions suivantes sont équivalentes.

1) Le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $X \to X'$.

2) Pour tout $l \in L(X)$ il existe un (unique) $l' \in L(X')$ tel que $\mathcal{O}_l$ domine $\mathcal{O}_{l'}$.

6.4. Soit H' un sous-groupe algébrique de G contenant H. Soient X un plongement de G/H, X' un plongement de G/H'. Si $v \in \mathcal{V}_1(G/H)$, rappelons que l'on désigne par X_v le plongement élémentaire correspondant.

On dira que v domine X si l'identité de G/H se prolonge en un morphisme $X_v \to X$; on dira que v domine X' si le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $X_v \to X'$. Si le morphisme naturel $G/H \to G/H'$ se prolonge en un morphisme $X \to X'$, il est clair que tout $v \in \mathcal{V}_1(G/H)$ qui domine X, domine aussi X'.

PROPOSITION. *Supposons que le morphisme naturel $G/H' \to G/H$ se prolonge en un morphisme $\varphi : X \to X'$. Pour que φ soit propre, il faut et il suffit que tout $v \in \mathcal{V}_1(G/H)$ qui domine X', domine déjà X.*

Preuve. Soient X, Y deux variétés algébriques intègres, U un ouvert non vide de X, et $\varphi : X \to Y$ un morphisme. Si $\lambda \in X_{k((t))}$, on note $\varphi \circ \lambda$ son image par φ dans $Y_{k((t))}$. On a le critère suivant de propreté: pour que φ soit propre, il faut et il suffit que, si $\lambda \in U_{k((t))}$, toutes les fois que $\lim_{t \to 0} (\varphi \circ \lambda)(t)$ existe dans Y, alors $\lim_{t \to 0} \lambda(t)$ existe déjà dans X. La proposition résulte aussitôt de là et de 4.9.

COROLLAIRE. *Soit X un plongement de G/H. Pour que X soit une variété complète, il faut et il suffit que tout $v \in \mathcal{V}_1(G/H)$ domine X.*

7. Valuations invariantes sous un groupe réductif

Dans la suite, nous supposerons le groupe G réductif et l'algèbre $k[G]$ factorielle. Au début de ce §, nous expliquons de quelle manière nous utiliserons ces hypothèses. Puis en 7.4, nous commençons par en tirer des conséquences pour les valuations invariantes. Comme première application, nous obtiendrons en 7.5 que, pour certains espaces homogènes, le nombre d'orbites dans tout plongement est fini. Enfin en 7.6, nous esquisserons une voie à suivre pour déterminer les valuations invariantes.

7.1. Supposons que l'algèbre $k[G]$ soit factorielle.

Pour l'étude des plongements, cette hypothèse n'est pas très restrictive: il est bien connu que pour tout groupe algébrique affine G, il existe un revêtement fini de groupes algébriques $p : \tilde{G} \to G$ tel que $k[\tilde{G}]$ est factorielle; et si $\tilde{H} = p^{-1}(H)$, on peut clairement identifier plongements de G/H et plongements de $\tilde{G}/\tilde{H}$.

Désignons par $k[G]^*$ l'ensemble des éléments inversibles de $k[G]$. Il est bien connu que tout élément de $k[G]^*$ est, à facteur scalaire près, un caractère (c'est-à-dire un morphisme de groupes algébriques $G \to k^*$).

Désignons par $\mathcal{D}(G)$ l'ensemble des fermés irréductibles de G de codimension 1. Pour tout $D \in \mathcal{D}$, choisissons un $f_D \in k[G]$ qui engendre l'idéal des $f \in k[G]$ nuls sur D. Tout $f \in k(G)$ s'écrit alors de manière unique

$$f = g \prod_{D \in \mathcal{D}(G)} f_D^{v_D(f)}$$

où $g \in k[G]^*$ et où les $v_D(f), D \in \mathcal{D}(G)$ sont des entiers presque tous nuls. Pour tout $D \in \mathcal{D}(G)$, la fonction $v_D : k(G)^* \to \mathbb{Z}$ est une valuation discrète de $k(G)$.

Soit H un sous-groupe algébrique de G (non nécessairement connexe). Désignons par $\pi : G \to G/H$ la projection naturelle et par $\mathcal{D}(G/H)$ l'ensemble des fermés irréductibles de G/H de codimension 1. Pour tout $D \in \mathcal{D}(G/H)$, H opère (par translations à droite) de façon transitive dans l'ensemble des composantes irréductibles $D_1, \ldots, D_r$ de $\pi^{-1}(D)$. Posons $f_D = f_{D_1} \cdot \cdots \cdot f_{D_r}$. Il est clair que les f_D $(D \in \mathcal{D}(G/H))$ sont des vecteurs propres de H. De plus, tout vecteur propre $f \in k(G)$ de H (et en particulier tout $f \in k(G/H)$) s'écrit de manière unique

$$f = g \prod_{D \in \mathcal{D}(G/H)} f_D^{v_D(f)},$$

où $g \in k[G]^*$ et où les $v_D(f)$, $D \in \mathscr{D}(G/H)$ sont des entiers presque tous nuls. Pour tout $D \in \mathscr{D}(G/H)$, la fonction $v_D : k(G/H)^* \to \mathbb{Z}$ est une valuation discrète de $k(G/H)$.

Soit $f \in k(G/H)$. Ecrivons $f = gh^{-1}$, avec $g, h \in k[G]$ sans diviseur commun. De ce qui précède résulte aussitôt que g et h sont des vecteurs propres de H, de même caractère.

Autre fait que nous utiliserons: pour tout $f \in k[G]$, il existe $s \in G$ tel que f et $s \cdot f$ sont sans diviseur commun. En effet, pour $s \in G$ en "position générale", le fermé de G où s'annulent f et $s \cdot f$ est de codimension ≥ 2.

7.2. Supposons que le groupe G soit réductif. Rappelons quelques faits de base sur les groupes réductifs et leurs représentations rationnelles (en caractéristique nulle).

Toute représentation rationnelle d'un groupe réductif est complètement réductible. Pour connaître un G-module rationnel N, il suffit donc de connaître tous les sous-G-modules irréductibles de N.

Désignons par $\hat{G}$ l'ensemble des (classes d'isomorphismes de) G-modules rationnels irréductibles. Pour décrire $\hat{G}$ on procède traditionnellement de la manière suivante. On fixe un sous-groupe unipotent maximal U de G. On pose $B = N_G(U)$, le normalisateur de U dans G; c'est un sous-groupe résoluble maximal de G, qui est connexe et dont le radical unipotent est U. On écrit T pour le quotient B/U qui est un tore, et on note $X(T)$ le groupe des caractères de T (et de B). Pour tout $M \in \hat{G}$, l'espace vectoriel ${}^U M$ est de dimension 1. L'opération naturelle de T dans ${}^U M$ fournit donc un caractère χ_M de T. L'application $\hat{G} \to X(T)$ qui envoie M sur χ_M est injective. On note P l'image de cette application, et on appelle P l'ensemble des poids dominants.

On se sert de P pour indexer $\hat{G}$ et pour manier les G-modules rationnels. De manière plus précise, soit N un G-module rationnel, et soit $M \in \hat{G}$ de poids dominant $\pi \in P$; alors un sous-G-module de N est irréductible et isomorphe à M si et seulement s'il est engendré par un vecteur propre de B dans N, de caractère π. Autrement dit, pour connaître l'opération de G dans N, il suffit de connaître les vecteurs propres de B dans N et leurs caractères.

7.3. Supposons maintenant à la fois G réductif et l'algèbre $k[G]$ factorielle.

Fixons un sous-groupe algébrique H et un sous-groupe unipotent maximal U de G. Dans ce qui suit, les groupes G, U et $B = N_G(U)$ opèreront toujours par "translations à gauche" et le groupe H toujours par "translations à droite".

On désignera par $\mathscr{P}$ l'ensemble des $f \in k(G)$ qui sont à la fois vecteur propre de B et vecteur propre de H; $\mathcal{P}$ est un sous-groupe multiplicatif de $k(G)^*$. On notera ${}^B\mathcal{D}(G/H)$ l'ensemble des $D\in\mathcal{D}(G/H)$ qui sont stables par B. Si $D\in\mathcal{D}(G/H)$, pour que $f_D\in\mathcal{P}$, il faut et il suffit que $D\in{}^B\mathcal{D}(G/H)$. De plus, il est clair que tout $f\in\mathcal{P}$ s'écrit de manière unique

$$f=g\prod_{D\in{}^B\mathcal{D}(G/H)} f_D^{v_D(f)},$$

où $g\in k[G]^*$.

Soit A un sous-espace vectoriel de $k(G/H)$. Nous dirons que A est quasi-G-stable, s'il existe une famille $M(h)$, $h\in\mathcal{P}$ de sous-G-modules de $k[G]$ telle que $A=\bigcup_{h\in\mathcal{P}} hM(h)$.

Nous dirons d'un sous-espace vectoriel N de A qu'il est "bon", s'il existe un ensemble $\mathcal{W}$ de valuations (discrètes, normalisées) G-invariantes de $k(G/H)$, et une famille d'entiers n_w $(w\in\mathcal{W})$, tels que N est l'ensemble des $f\in A$ qui vérifient $w(f)\geq n_w$, quel que soit $w\in\mathcal{W}$.

LEMME. *Soit A un sous-espace vectoriel quasi-G-stable de $k(G/H)$, et soient N et N' deux "bons" sous-espaces vectoriels de A. Pour que $N-N'\neq\emptyset$, il faut et il suffit que $\mathcal{P}\cap(N-N')\neq\emptyset$.*

Preuve. Si $h\in\mathcal{P}$, désignons par $N(h)$ (resp. $N'(h)$) l'ensemble des $g\in M(h)$ tels que $gh\in N$ (resp. N'). D'après le corollaire 2 de 3.2, on a $v((s\cdot g)h)=v(gh)$, quels que soient $s\in G$ et $g\in M(h)$ et quelle que soit la valuation G-invariante v de $k(G/H)$. Puisque N et N' sont "bons," il s'ensuit que $N(h)$ et $N'(h)$ sont des sous-G-modules de $M(h)$. Si $N-N'\neq\emptyset$, il existe $h\in\mathcal{P}$ tel que $N(h)-N'(h)\neq\emptyset$. Grâce aux propriétés des représentations rationnelles des groupes réductifs rappelées en 7.2, B possède un vecteur propre f dans $N(h)-N'(h)$. Puisque $h\in\mathcal{P}$ et que $hM(h)\subset A\subset k(G/H)$, tous les éléments de $M(h)$ sont des vecteurs propres de H. Par conséquent $f\in\mathcal{P}$ et $hf\in\mathcal{P}\cap(N-N')\neq\emptyset$, c.q.f.d.

Remarque. Nous avons vu en passant qu'un "bon" sous-espace vectoriel d'un espace vectoriel quasi-G-stable est encore quasi-G-stable.

7.4. Rappelons que, dans toute la suite, G sera un groupe réductif dont l'algèbre $k[G]$ est factorielle, H un sous-groupe algébrique de G, et U un sous-groupe unipotent maximal de G, qui resteront fixés. De plus, sauf mention expresse du contraire, les groupes G, U, $B=N_G(U)$ opèreront par "translations à gauche", et le groupe H par "translations à droite".

PROPOSITION. *Deux valuations discrètes G-invariantes de $k(G/H)$, qui coïncident en restriction à $\mathcal{P}\cap k(G/H)$, sont égales.*

Preuve. Désignons par A l'ensemble des $f \in k(G/H)$ qui peuvent s'écrire $f = gh$, avec $g \in k[G]$ et $h \in \mathcal{P}$; A est une sous-algèbre de $k(G/H)$.

Soit $f \in k(G/H)$. Ecrivons $f = g_1 g_2^{-1}$, où $g_1, g_2 \in k[G]$ sont sans diviseur commun. D'après 7.1, g_1 et g_2 sont des vecteurs propres de H, de même caractère. Dans le G-module engendré par g_1 choisissons un vecteur propre h de B (voir 7.2). Il est clair que h est aussi un vecteur propre de H de même caractère que g_1 et g_2. Par suite $h \in \mathcal{P}$ et $g_1 h^{-1}$, $g_2 h^{-1} \in A$. Il s'ensuit que f est dans le corps des fractions de A. Autrement dit, nous avons montré que A possède $k(G/H)$ comme corps des fractions.

Pour tout $h \in \mathcal{P}$, désignons par $M(h)$ l'ensemble des $g \in k[G]$ qui sont vecteur propre de H, de caractère inverse à celui de h. Il est clair que les $M(h)$, $h \in \mathcal{P}$ sont des sous-G-modules de $k[G]$, et que $A = \bigcup_{h \in \mathcal{P}} hM(h)$. Autrement dit, A est quasi-G-stable (voir 7.3).

Soient maintenant v_1 et v_2 deux valuations discrètes G-invariantes de $k(G/H)$, qui coïncident en restriction à $\mathcal{P} \cap k(G/H) = \mathcal{P} \cap A$. Puisque A est quasi-G-stable, de 7.3 résulte aussitôt que v_1 et v_2 coïncident sur A. Puisque le corps des fractions de A est $k(G/H)$, il s'ensuit que $v_1 = v_2$, c.q.f.d.

7.5. Si le degré de transcendance de ${}^B k(G/H)$ sur k est ≤ 1 (autrement dit, si B possède une orbite de codimension ≤ 1 dans G/H), la structure de ${}^B\mathcal{D}(G/H)$ est particulièrement simple.

1) Si $\deg \operatorname{tr}_k {}^B k(G/H) = 0$, c'est-à-dire si B a une orbite ouverte dans G/H, ${}^B\mathcal{D}(G/H)$ est l'ensemble fini des composantes irréductibles du complémentaire de cette orbite ouverte (l'orbite étant affine, son complémentaire est pur de codimension 1).
2) Si $\deg \operatorname{tr}_k {}^B k(G/H) = 1$, soit U l'ouvert de G/H formé des orbites de B de codimension 1 dans G/H; alors, si $D \in {}^B\mathcal{D}(G/H)$, ou bien D est l'adhérence dans G/H d'une orbite de B dans U, ou bien D est une composante du complémentaire de U dans G/H.

PROPOSITION (voir aussi [19]). *On suppose que B a une orbite ouverte dans G/H. Alors le nombre des orbites de G dans tout plongement de G/H est fini.*

Preuve. Si B a une orbite ouverte dans G/H, nous venons de voir que l'ensemble ${}^B\mathcal{D}(G/H)$ est fini. Il est clair que cela implique que les groupes $\mathcal{P}/k^*$ et $(\mathcal{P} \cap k(G/H))/k^*$ sont de type fini. Soient $f_1, \ldots, f_r$ des éléments de $\mathcal{P} \cap k(G/H)$ qui avec k^* engendrent le groupe $\mathcal{P} \cap k(G/H)$. D'après 8.1, toute valuation $v \in \mathcal{V}(G/H)$ est déterminée par $(v(f_1), \ldots, v(f_r)) \in \mathbb{Z}^r$. Il s'ensuit que l'ensemble $\mathcal{V}(G/H)$ est dénombrable.

Raisonnons maintenant à l'envers. Supposons qu'il existe un plongement X de

G/H dont le nombre des orbites est infini. Ce nombre est alors forcément non dénombrable (nous supposons le corps de base non dénombrable!). Soit $L \subset \mathfrak{L}_1(G/H)$ l'ensemble des orbites de G dans X. On sait que les facettes $\mathcal{F}_l$, $l \in L$ sont disjointes (cela exprime le fait que X est séparé, voir 6.2) et non vides (voir 4.9). De L non dénombrable suit donc $\mathcal{V}_1(G/H)$ non dénombrable, ce qui démontre la proposition.

Remarques. 1) Lorsque B a une orbite ouverte dans G/H, il résulte aussitôt de la proposition précédente que $\mathcal{V}(G/H) = \mathcal{V}_1(G/H)$, et plus généralement que $\mathfrak{L}(G/H) = \mathfrak{L}_1(G/H) = \mathfrak{L}_f(G/H)$.

2) De nombreux auteurs ont déjà étudié (et classé) les sous-groupes H de G tels que B possède une orbite ouverte dans G/H (en demandant parfois que H possède une orbite ouverte dans G/B, ce qui revient au même), voir par exemple [17]. Voici quelques cas de tels H: les sous-groupes de G qui contiennent un sous-groupe unipotent maximal; les sous-groupes de G fixés par un automorphisme involutif.

7.6. Terminons ce § par quelques indications pratiques sur la détermination de $\mathcal{V}(G/H)$.

Pour tout $\pi \in \mathcal{P}$, désignons par M_π un G-module rationnel irréductible de poids dominant π. Notons M'_π le dual de M_π. Il est bien connu que $k[G]$, en tant que $G \times G$-module (G opérant par "translations à gauche et à droite"), est isomorphe à $\bigoplus_{\pi \in P} M'_\pi \otimes M_\pi$. Il s'ensuit que ${}^U k[G]$, l'algèbre des invariants de U opérant par "translations à gauche", est isomorphe, en tant que G-module (G opérant par "translations à droite"), à $\bigoplus_{\pi \in P} M_\pi$. L'opération de $T = B/U$ dans ${}^U k[G]$ se retrouve dans la graduation de $\bigoplus_{\pi \in P} M_\pi$ par les poids $P \subset X(T)$.

Pour tout $\pi \in P$, choisissons un isomorphisme de M_π sur l'unique sous-G-module de ${}^U k[G]$ qui lui est isomorphe, et convenons dans la suite d'identifier M_π à son image (deux telles identifications ne diffèrent que par une homothétie). L'ensemble $\mathcal{P} \cap k[G]$ s'identifie alors à la réunion des vecteurs propres de H dans les M_π, $\pi \in P$.

Pour simplifier notre discussion, supposons que G/H soit quasi-affine. Dans ce cas, on peut trouver des f_σ ($\sigma \in \Sigma$) dans $\mathcal{P} \cap k[G/H]$ qui engendrent le groupe $\mathcal{P} \cap k(G/H)$. D'après 7.4, toute valuation v de $\mathcal{V}(G/H)$ est déterminée par les $v(f_\sigma)$ ($\sigma \in \Sigma$). Reste alors à trouver les familles d'entiers n_σ ($\sigma \in \Sigma$) pour lesquelles il existe $v \in \mathcal{V}(G/H)$ tel que $v(f_\sigma) = n_\sigma$.

Lorsque $\lambda \in G_{k((t))}$, on peut calculer $v_\lambda(f_\sigma)$ de la manière suivante. Supposons que $f_\sigma \in M_\pi = M_\pi \otimes k \subset M_\pi \otimes k((t))$. Le groupe $G_{k((t))}$ opère de manière naturelle dans $M_\pi \otimes k((t))$ et $v_\lambda(f_\sigma)$ n'est rien d'autre que l'ordre en t de la série formelle

$\lambda \cdot f_\sigma \in M_\pi \otimes k((t))$. De plus, le groupe

$$\Gamma = (G_{k[[t]]} \times H_{k((t))}) \times \mathrm{Aut}_k\, k[[t]]$$

opère de manière naturelle dans $G_{k((t))}$, et v_λ et $v_{\lambda'}$ coïncident en restriction à $k(G/H)$ lorsque $\lambda, \lambda' \in G_{k((t))}$ sont sur une même orbite de Γ (voir 4.10).

D'où une voie à suivre pour déterminer $\mathcal{V}_1(G/H)$: on cherche d'abord sur chaque orbite de Γ dans $G_{k((t))}$ un λ d'une forme simple, puis on calcule les $v_\lambda(f_\sigma)$ $(\sigma \in \Sigma)$. Enfin, pour trouver les valuations de $\mathcal{V}_2(G/H)$, on peut se servir de 4.11.

Ces indications semblent assez raisonnables lorsque le degré de transcendance de ${}^Bk(G/H)$ sur k est ≤ 1 (pour un exemple, voir [16]).

8. Germes de plongements normaux sous un groupe réductif

Dans ce §, nous étudierons les germes de plongements (lorsque la variété est normale et le groupe est réductif) dans le même esprit qu'au § précédent les valuations invariantes. Pour cela, nous nous appuierons beaucoup sur la théorie des anneaux de Krull (pour un bon exposé de cette théorie, voir par exemple [3], chap. 7).

8.1. Désignons par $\mathfrak{L}^n(G/H)$ l'ensemble des $l \in \mathfrak{L}(G/H)$ tels que $\mathcal{O}_l$ soit une algèbre intégralement close dans $k(G/H)$. Si $l \in \mathfrak{L}^n(G/H)$, $\mathcal{O}_l$ est un anneau noethérien intégralement clos, donc un anneau de Krull, dont les valuations essentielles sont manifestement de deux sortes:

1) un nombre fini de valuations appartenant à $\mathcal{V}(G/H)$ (si X est un plongement de G/H et si Y est un fermé stable par G de X, tels que $\mathcal{O}_{X,Y} = \mathcal{O}_l$, il s'agit des valuations correspondant aux composantes irréductibles de $X - G/H$, qui sont de codimension 1 dans X et qui contiennent Y); on notera $\mathcal{V}_l$ l'ensemble de ces valuations;
2) un certain nombre de valuations du type v_D, où $D \in \mathcal{D}(G/H)$ (si X, Y sont comme plus haut, il s'agit des D dont l'adhérence dans X contient Y); on notera $\mathcal{D}_l$ l'ensemble de ces éléments de $\mathcal{D}(G/H)$.

Puisque $\mathcal{O}_l$ est un anneau de Krull, $\mathcal{O}_l$ est déterminé par $\mathcal{V}_l$ et $\mathcal{D}_l$; plus précisément

$$\mathcal{O}_l = \bigcap_{v \in \mathcal{V}_l} \mathcal{O}_v \cap \bigcap_{D \in \mathcal{D}_l} \mathcal{O}_{v_D}.$$

Voici en gros le programme de ce §: nous montrerons que tout $l \in \mathfrak{L}^n(G/H)$ est déjà déterminé par $\mathcal{V}_l$ et ${}^B\mathcal{D}_l = \mathcal{D}_l \cap {}^B\mathcal{D}(G/H)$ (8.3), puis nous caractériserons les couples $\mathcal{V}_l$, ${}^B\mathcal{D}_l$ (8.8).

8.2. Si $N \subset k[G]$ et si $g \in k(G)$, on désignera par gN le sous-ensemble de $k(G)$ des gf ($f \in N$), et par $k[gN]$ la sous-algèbre de $k(G)$ engendrée par gN.

LEMME. *Soit $l \in \mathfrak{L}^n(G/H)$. Il existe un sous-$G$-module de dimension finite M de $k[G]$ et un $h \in \mathcal{P} \cap M$ vérifiant*:

1) $h^{-1}M \subset \mathcal{O}_l$;

2) *$\mathcal{O}_l$ est le localisé de $k[h^{-1}M]$ en l'idéal premier $k[h^{-1}M] \cap \mathfrak{m}_l$.*

De plus, h vérifie alors forcément $v_D(h) = 0$ quel que soit $D \in \mathcal{D}_l$.

Preuve. La localité l étant géométrique, on peut trouver une sous-algèbre de type fini A de $\mathcal{O}_l$ telle que $\mathcal{O}_l$ soit le localisé de A en l'idéal premier $A \cap \mathfrak{m}_l$. Choisissons un système de générateurs de A sous la forme $f_1 g^{-1}, \ldots, f_r g^{-1}$, où $f_1, \ldots, f_r$ et g sont des éléments de $k[G]$ sans diviseur commun. D'après 7.1, $f_1, \ldots, f_r$ et g sont alors des vecteurs propres de H, de même caractère. Si N désigne le sous-espace vectoriel de $k[G]$ engendré par $f_1, \ldots, f_r$ et g, on a $A = k[g^{-1}N]$. Puisque N et g sont dans diviseur commun, et que $g^{-1}N \subset A \subset \mathcal{O}_l$, $v_D(g) = 0$ quel que soit $D \in \mathcal{D}_l$.

Désignons par M le sous-G-module de $k[G]$ engendré par N. Puisque les opérations par translations à gauche et à droite commutent, les éléments de M sont encore des vecteurs propres de H, de même caractère que g. Par suite $g^{-1}M \subset k(G/H)$. D'après le corollaire 2 de 3.2, il s'ensuit que $v(g^{-1}(s \cdot f)) = v(g^{-1}f) \geq 0$, quels que soient $s \in G, f \in N$ et $v \in \mathcal{V}_l$. D'autre part, $v_D(g^{-1}f) \geq v_D(g^{-1}) = 0$, quels que soient $f \in M$ et $D \in \mathcal{D}_l$. Par suite, toutes les valuations essentielles de $\mathcal{O}_l$ restent positives sur $g^{-1}M$, d'où il suit que $g^{-1}M \subset \mathcal{O}_l$.

Choisissons un $v \in \mathcal{V}(G/H)$ tel que $\mathcal{O}_v$ domine $\mathcal{O}_l$ (voir 3.5). Le sous-espace vectoriel M' des $f \in M$ tels que $g^{-1}f \in \mathfrak{m}_l$ peut alors aussi se définir par l'inégalité $v(g^{-1}f) > 0$. Par suite, du corollaire 2 de 3.2 résulte que M' est stable par G. Par construction, $g \in M - M' \neq \emptyset$. Grâce aux propriétés des représentations rationnelles des groupes réductifs rappelées en 7.2, B possède un vecteur propre h dans $M - M'$. Puisque les éléments de M sont des vecteurs propres de H, $h \in \mathcal{P}$. Par construction $g^{-1}h \in \mathcal{O}_l^*$. Par suite, $h^{-1}M = (g^{-1}h)^{-1} \cdot g^{-1}M \subset \mathcal{O}_l$, d'où 1). De $g^{-1}M = (gh^{-1})^{-1} \cdot h^{-1}M$ résulte que $g^{-1}M$ (et donc aussi A) est contenu dans le localisé de $k[h^{-1}M]$ en l'idéal premier $k[h^{-1}M] \cap \mathfrak{m}_l$, d'où aussitôt 2).

Enfin, M est un sous-G-module de $k[G]$, $h \in M$ et $h^{-1}M \subset \mathcal{O}_l$; puisqu'on peut choisir $s \in G$ tel que h et $s \cdot h$ sont sans diviseur commun, de $(s \cdot h)h^{-1} \in \mathcal{O}_l$ suit aussitôt $v_D(h) = 0$ quel que soit $D \in \mathcal{D}_l$.

8.3. Soit $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$ et soit $\mathscr{W} \subset \mathscr{V}(G/H)$.

PROPOSITION. *Il existe au plus un $l \in \mathfrak{L}^n(G/H)$ tel que ${}^B\mathscr{D}_l = \mathscr{D}$ et $\mathscr{V}_l = \mathscr{W}$.*

Preuve. Désignons par $\mathscr{P}(\mathscr{D})$ l'ensemble des $h \in \mathscr{P}$ tels que $v_D(h) = 0$ quel que soit $D \in \mathscr{D}$. Désignons par $A(\mathscr{D})$ la sous-algèbre des $f \in k(G/H)$ qui peuvent s'écrire $f = gh$, où $g \in k[G]$ et $h \in \mathscr{P}(\mathscr{D})$. Enfin, désignons par $A = A(\mathscr{D}, \mathscr{W})$ la sous-algèbre des $f \in A(\mathscr{D})$ tels que $w(f) \geq 0$ quel que soit $w \in \mathscr{W}$.

Soit $l \in \mathfrak{L}^n(G/H)$ tel que ${}^B\mathscr{D}_l = \mathscr{D}$ et $\mathscr{V}_l = \mathscr{W}$. Puisque toute valuation essentielle de $\mathcal{O}_l$ reste positive sur A, $\mathcal{O}_l$ contient A. Si M et h sont comme dans le lemme de 8.2, il est clair que $h^{-1}M \subset A$. Il s'ensuit que $\mathcal{O}_l$ est le localisé de A en l'idéal premier $A \cap \mathfrak{m}_l$. Enfin, $\mathscr{P} \cap A \cap \mathfrak{m}_l$ peut aussi se décrire comme l'ensemble des $f \in \mathscr{P}$ qui vérifient $v_D(f) \geq 0$ ($D \in \mathscr{D} = {}^B\mathscr{D}_l$) et $w(f) \geq 0$ ($w \in \mathscr{W} = \mathscr{V}_l$), l'une au moins des inégalités étant stricte.

Soit maintenant l' un "autre" élément de $\mathfrak{L}^n(G/H)$ tel que ${}^B\mathscr{D}_{l'} = \mathscr{D}$ et $\mathscr{V}_{l'} = \mathscr{W}$. Il est clair que $A(\mathscr{D})$ est quasi-G-stable, au sens de 7.3. D'autre part, $A \cap \mathfrak{M}_l$ et $A \cap \mathfrak{M}_{l'}$, sont des "bons" sous-espaces de $A(\mathscr{D})$: en effet, si $v \in \mathscr{V}(G/H)$ est tel que $\mathcal{O}_v$ domine $\mathcal{O}_l$ (voir 3.5), $A \cap \mathfrak{M}_l$ peut aussi se décrire comme l'ensemble des $f \in A(\mathscr{D})$ tels que $w(f) \geq 0$ ($w \in \mathscr{W}$) et $v(f) > 0$, et pareil pour $A \cap \mathfrak{M}_{l'}$. D'après 7.3, de $\mathscr{P} \cap A \cap \mathfrak{M}_l = \mathscr{P} \cap A \cap \mathfrak{M}_{l'}$ résulte alors $A \cap \mathfrak{M}_l = A \cap \mathfrak{M}_{l'}$, d'où $l = l'$, c.q.f.d.

8.4. On notera $X(H)$ le groupe des caractères de H. Si $f \in k(G)$ est un vecteur propre de H, on notera χ_f son caractère, $\chi_f \in X(H)$. Soit $E \subset k(G)$. Si $\chi \in X(H)$, on notera E_χ l'ensemble des vecteurs propres de H dans E de caractère χ. Enfin, on désignera par $X_E(H)$ l'ensemble des $\chi \in X(H)$ tels que $E_\chi \neq \varnothing$.

En général, on a $X_{k[G]}(H) \neq X(H)$; $X_{k[G]}(H)$ est seulement un sous-monoïde de $X(H)$ qui engendre $X(H)$ en tant que groupe; pour que $X_{k[G]}(H) = X(H)$, il faut et il suffit que l'espace homogène G/H soit une variété quasi-affine (voir [8]).

Posons $\mathscr{P}^+ \doteq \mathscr{P} \cap k[G]$. Pour tout $\chi \in X(H)$, $k[G]_\chi$ est un sous-G-module de $k[G]$. Par suite, grâce aux propriétés des représentations rationnelles des groupes réductifs rappelées en 7.2, si $k[G]_\chi \neq 0$, B possède un vecteur propre dans $k[G]_\chi$. Il s'ensuit que $X_{k[G]}(H) = X_{\mathscr{P}^+}(H)$.

Soit $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$. Rappelons que $\mathscr{P}(\mathscr{D})$ est le sous-groupe des $f \in \mathscr{P}$ tels que $v_D(f) = 0$ quel que soit $D \in \mathscr{D}$, et que $A(\mathscr{D})$ est la sous-algèbre des $f \in k(G/H)$ qui peuvent s'écrire $f = gh$, avec $g \in k[G]$ et $h \in \mathscr{P}(\mathscr{D})$.

LEMME. *Pour que le corps des fractions de $A(\mathscr{D})$ soit égal à $k(G/H)$, il faut et il suffit que $\mathscr{D}$ vérifie la condition*

(D) *$X(H)$ est engendré, en tant que monoïde, par $X_{\mathscr{P}^+}(H)$ et $X_{\mathscr{P}(\mathscr{D})}(H)$.*

Preuve. Supposons que $\mathscr{D}$ vérifie (D). Soit $f \in k(G/H)$. Ecrivons $f = f_1 f_2^{-1}$, où

$f_1, f_2 \in k[G]$ sont sans diviseur commun. Nous savons que f_1 et f_2 sont alors des vecteurs propres de H, de même caractère. D'après (D), il existe un vecteur propre g de H dans $k[G]$ et un $h \in \mathcal{P}(\mathcal{D})$, tels que $\chi_{f_1}^{-1} = \chi_{f_2}^{-1} = \chi_g \chi_h$. Puisque $f_1 gh$ et $f_2 gh$ appartiennent alors à $A(\mathcal{D})$, f est dans le corps des fractions de $A(\mathcal{D})$. Par conséquent, le corps des fractions de $A(\mathcal{D})$ est bien $k(G/H)$.

Inversement, supposons le corps des fractions de $A(\mathcal{D})$ égal à $k(G/H)$. Puisque $X_{\mathcal{P}^+}(H)$ engendre le groupe $X(H)$, pour montrer que $\mathcal{D}$ vérifie (D), il suffit de voir que, pour tout vecteur propre g de H dans $k[G]$, χ_g est dans le monoïde engendré par $X_{\mathcal{P}^+}(H)$ et $X_{\mathcal{P}(\mathcal{D})}(H)$. Choisissons $s \in G$ tel que g et $s \cdot g$ sont sans diviseur commun. On a $(s \cdot g)g^{-1} \in k(G/H)$. Par hypothèse, il existe $f_1, f_2 \in k[G]$ et $h \in \mathcal{P}(\mathcal{D})$ tels que $f_1 h, f_2 h \in k(G/H)$ et $(s \cdot g)g^{-1} = f_1 h (f_2 h)^{-1} = f_1 f_2^{-1}$. Les éléments g et $s \cdot g$ étant sans diviseur commun, il existe $f \in k[G]$ tel que $gf = f_2$. Il est clair que f_2 et f sont des vecteurs propres de H. De $g^{-1} = f f_2^{-1} = fh(f_2 h)^{-1}$, on tire alors $\chi_g^{-1} = \chi_g \chi_h$, c.q.f.d.

8.5. Soit $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ et soit $\mathcal{W} \subset \mathcal{V}(G/H)$. Rappelons que nous avons désigné par $A(\mathcal{D}, \mathcal{W})$ la sous-algèbre des $f \in A(\mathcal{D})$ tels que $w(f) \geq 0$ quel que soit $w \in \mathcal{W}$.

Pour tout $D \in \mathcal{D}(G/H)$, choisissons $s \in G$ tel que f_D et $s \cdot f_D$ sont dans diviseur commun (pour la définition de f_D, voir 7.1). Posons $g_D = (s \cdot f_D) f_D^{-1}$. On vérifie sans peine les assertions suivantes: $g_D \in k(G/H)$, $v_D(g_D) < 0$, $v_{D'}(g_D) \geq 0$ quel que soit $D' \in \mathcal{D}(G/H) - \{D\}$, et $v(g_D) = 0$ quel que soit $v \in \mathcal{V}(G/H)$ (cette dernière assertion résulte du corollaire 2 de 3.2).

LEMME 1. *Soit $D \in {}^B\mathcal{D}(G/H)$. Pour que $D \in \mathcal{D}$, il faut et il suffit que v_D reste positif sur $A(\mathcal{D}, \mathcal{W})$.*

Preuve. Si $D \in \mathcal{D}$, il est clair que v_D reste positif sur $A(\mathcal{D}, \mathcal{W})$. Si $D \in {}^B\mathcal{D}(G/H) - \mathcal{D}$, alors $g_D \in A(\mathcal{D}, \mathcal{W})$ et $v_D(g_D) < 0$.

Dans la suite de ce numéro, on supposera que $\mathcal{W}$ est un sous-ensemble fini de $\mathcal{V}(G/H)$.

LEMME 2. *Pour que le corps des fractions de $A(\mathcal{D}, \mathcal{W})$ soit égal à $k(G/H)$, il faut et il suffit que $\mathcal{D}, \mathcal{W}$ vérifient les conditions* (D) et

(W) *Il existe $f \in \mathcal{P} \cap A(\mathcal{D})$ tel que $w(f) > 0$ quel que soit $w \in \mathcal{W}$.*

Preuve. Posons $A = A(\mathcal{D}, \mathcal{W})$.

Supposons que le corps des fractions de A est $k(G/H)$. Puisque $A \subset A(\mathcal{D}) \subset k(G/H)$, le corps des fractions de $A(\mathcal{D})$ est alors aussi $k(G/H)$. D'après 8.4, il s'ensuit que $\mathcal{D}$ vérifie (D). D'autre part, pour tout $w \in \mathcal{W}$, $A \cap \mathfrak{m}_w \neq \{0\}$: en effet, sinon, puisque le corps des fractions de A est $k(G/H)$, w s'annulerait sur $k(G/H)^*$, ce qui n'est pas possible. Il s'ensuit que $A \cap \bigcap_{w \in \mathcal{W}} \mathfrak{m}_w \neq \{0\}$. Il est clair que $A(\mathcal{D})$ est quasi-G-stable (au sens de 7.3), et que $A \cap \bigcap_{w \in \mathcal{W}} \mathfrak{m}_w$ est un "bon" sous-espace de $A(\mathcal{D})$. D'après 7.3, il s'ensuit que $\mathcal{P} \cap A \cap \bigcap_{w \in \mathcal{W}} \mathfrak{m}_w \neq \emptyset$, ce qui signifie que $\mathcal{D}, \mathcal{W}$ vérifient (W).

Inversement, supposons que $\mathcal{D}, \mathcal{W}$ vérifient (D) et (W). Soit $g \in A(\mathcal{D})$. Si $f \in \mathcal{P} \cap A(\mathcal{D})$ possède les propriétés de (W), alors $f \in A$ et $gf^N \in A$, dès que $N \in \mathbb{N}$ est assez grand. Il s'ensuit que A et $A(\mathcal{D})$ ont même corps de fractions. D'après 8.4, puisque $\mathcal{D}$ vérifie (D), ce corps est $k(G/H)$.

Posons $\tilde{\mathcal{D}} = \mathcal{D} \cup (\mathcal{D}(G/H) - {}^B\mathcal{D}(G/H)$.

LEMME 3. *On suppose que $\mathcal{D}, \mathcal{W}$ vérifient (D) et (W). L'algèbre $A(\mathcal{D}, \mathcal{W})$ est un anneau de Krull dont les valuations essentielles sont les v_D $(D \in \tilde{\mathcal{D}})$ et certaines des valuations de $\mathcal{W}$.*

Preuve. Posons $A = A(\mathcal{D}, \mathcal{W})$. D'après le lemme 2 le corps des fractions de A est $k(G/H)$. Désignons par A' le localisé de $k[G]$ en la partie multiplicative $k[G] \cap \mathcal{P}(\mathcal{D})$; A' est un anneau factoriel. Du fait que $A = A' \cap \bigcap_{w \in \mathcal{W}} \mathcal{O}_w$, il résulte que A est un anneau de Krull (toute intersection finie d'anneaux de Krull est encore un anneau de Krull). Puisque $A = \bigcap_{D \in \tilde{\mathcal{D}}} \mathcal{O}_{v_D} \cap \bigcap_{w \in \mathcal{W}} \mathcal{O}_w$, il est clair que les valuations essentielles de A se trouvent parmi les v_D $(D \in \tilde{\mathcal{D}})$ et les w $(w \in \mathcal{W})$. Si $D \in \tilde{\mathcal{D}}$, on a $v_D(g_D) < 0$, donc $g_D \notin A$, mais $v_{D'}(g_D) \geq 0$ quel que soit $D' \in \tilde{\mathcal{D}} - \{D\}$, et $w(g_D) = 0$ quel que soit $w \in \mathcal{W}$; par suite, les valuations v_D $(D \in \tilde{\mathcal{D}})$ sont toutes essentielles pour A.

Considérons encore les deux conditions suivantes portant sur $\mathcal{D}, \mathcal{W}$.

$(W')_1$ Pour tout $w \in \mathcal{W}$, il existe $g_w \in \mathcal{P} \cap A(\mathcal{D})$ tel que $w(g_w) < 0$.

$(W')_{\geq 2}$ Si card $\mathcal{W} \geq 2$, pour tout $w \in \mathcal{W}$, il existe $f_w \in \mathcal{P} \cap A(\mathcal{D})$ tel que

$$w(f_{w'}) \begin{cases} >0 & \text{si} \quad w \in \mathcal{W} - \{w'\} \\ =0 & \text{si} \quad w = w'. \end{cases}$$

Remarques. 1) Si l'espace homogène G/H est affine, la condition $(W')_1$ est toujours remplie: en effet, toute valuation de $\mathcal{V}(G/H)$ prend alors des valeurs strictement négatives sur $k[G/H]$, donc aussi sur $\mathcal{P} \cap k[G/H]$.

2) Lorsque card $\mathcal{W} \geq 2$, $(W')_{\geq 2}$ implique (W) (il suffit de prendre pour f le produit de deux des f_w, $w \in \mathcal{W}$).

Pour abréger, nous désignerons par (W') la réunion des conditions $(W')_1$ et $(W')_{\geq 2}$.

LEMME 4. *On suppose que $\mathcal{D}, \mathcal{W}$ vérifient (D) et (W). Pour que toutes les valuations de $\mathcal{W}$ soient essentielles pour $A(\mathcal{D}, \mathcal{W})$, il faut et il suffit que $\mathcal{D}, \mathcal{W}$ vérifient (W').*

Preuve. Posons $A = A(\mathscr{D}, \mathscr{W})$. Par hypothèse, le corps des fractions de A est $k(G/H)$.

Si toutes les valuations de $\mathscr{W}$ sont essentielles pour A, on a

$$A(\mathscr{D}) - A(\mathscr{D}, \{w\}) \neq \varnothing$$

et

$$\Big(A(\mathscr{D}) \cap \bigcap_{w \in \mathscr{W} - \{w'\}} \mathfrak{m}_w\Big) - \mathfrak{m}_{w'} \neq \varnothing,$$

quels que soient $w, w' \in \mathscr{W}$. D'après 7.3, ces ensembles contiennent des éléments de $\mathscr{P}$, d'où aussitôt $(W')_1$ et $(W')_{\geq 2}$.

Soit $w \in \mathscr{W}$. De $(W')_1$ suit que w est une valuation essentielle pour $A(\mathscr{D}) \cap \mathcal{O}_w = A(\mathscr{D}, \{w\})$. Cela termine la preuve si card $\mathscr{W} = 1$. En tout cas, $\mathcal{O}_w$ est alors le localisé de $A(\mathscr{D}) \cap \mathcal{O}_w$ en l'idéal premier $A(\mathscr{D}) \cap \mathfrak{M}_w$. Soit $g \in A(\mathscr{D}) \cap \mathcal{O}_w$. Si card $\mathscr{W} \geq 2$, d'après $(W')_{\geq 2}$, il existe $f_w \in A(\mathscr{D}) \cap \mathcal{O}_w$, inversible dans $\mathcal{O}_w$, et tel que $gf_w^N \in A$ dès que $N \in \mathbb{N}$ est assez grand. Il s'ensuit que $\mathcal{O}_w$ est encore le localisé de A en l'idéal premier $A \cap \mathfrak{m}_w$, ce qui montre bien que w est une valuation essentielle pour A.

8.6. Soit $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$ et soit $\mathscr{W} \subset \mathscr{V}(G/H)$. On dira d'un sous-ensemble qu'il est cofini si son complémentaire est fini.

LEMME 1. *Si $A(\mathscr{D}, \mathscr{W})$ est une algèbre de type fini, alors $\mathscr{D}$ est cofini dans ${}^B\mathscr{D}(G/H)$.*

Preuve. Soit $f_1, \ldots, f_r$ un système de générateurs de l'algèbre $A(\mathscr{D}, \mathscr{W})$. D'après le lemme 1 de 8.5, $\mathscr{D}$ peut se caractériser comme l'ensemble des $D \in {}^B\mathscr{D}(G/H)$ tels que v_D reste positif sur $A(\mathscr{D}, \mathscr{W})$. Mais v_D reste positif sur $A(\mathscr{D}, \mathscr{W})$ si et seulement si $v_D(f_1) \geq 0, \ldots, v_D(f_r) \geq 0$, condition qui détermine un sous-ensemble confini de ${}^B\mathscr{D}(G/H)$.

LEMME 2. *Pour que l'algèbre $A(\mathscr{D}, \mathscr{W})$ soit de type fini, il faut et il suffit que $\mathscr{D}, \mathscr{W}$ vérifient la condition*

(F) *$\mathscr{P} \cap A(\mathscr{D}, \mathscr{W})$ engendre une sous-algèbre de type fini de $k(G)$.*

Preuve. L'opération de B dans $k(G)$ laisse stable $A(\mathscr{D}, \mathscr{W})$. La sous-algèbre de $k(G)$ engendrée par $\mathscr{P} \cap A(\mathscr{D}, \mathscr{W})$ n'est rien d'autre que ${}^UA(\mathscr{D}, \mathscr{W})$. On doit donc montrer que $A(\mathscr{D}, \mathscr{W})$ est de type fini si et seulement si ${}^UA(\mathscr{D}, \mathscr{W})$ l'est. Soit t une indéterminée. Pour tout $h \in \mathscr{P}(\mathscr{D})$, notons $\rho_h : k[G][t] \to k(G)$ l'homomorphisme d'algèbres qui prolonge l'inclusion $k[G] \subset k(G)$ et qui envoie t sur h. Faisons opérer G dans $k[G][t] = k[G] \otimes k[t]$ par "translations à gauche" dans $k[G]$ et trivialement dans $k[t]$; ρ_h commute seulement à l'opération de U. Nous savons que $A(\mathscr{D}, \mathscr{W})$ est quasi-G-stable (plus précisément, si $h \in \mathscr{P}(\mathscr{D})$ et si $M(h)$ est l'ensemble des $g \in k[G]$ tels que $hg \in A(\mathscr{D}, \mathscr{W})$, alors les $M(h)$, $h \in \mathscr{P}(\mathscr{D})$ sont des sous-G-modules de $k[G]$, et $A(\mathscr{D}, \mathscr{W}) = \bigcup_{h \in \mathscr{P}(\mathscr{D})} hM(h)$). Si A est une sous-algèbre de $k[G][t]$, stable par G, grâce aux propriétés des représentations rationnelles des groupes réductifs, il s'ensuit que $\rho_h(A) = A(\mathscr{D}, \mathscr{W})$ si et seulement si $\rho_h({}^UA) = {}^UA(\mathscr{D}, \mathscr{W})$. Pour conclure, il suffit alors de savoir que A est de type fini si et seulement si ${}^UA$ l'est. Puisque G opère rationnellement dans A, ce résultat est bien connu.

8.7. Soit $l \in \mathfrak{L}^n(G/H)$ et soit $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$. Nous dirons que $\mathscr{D}$ est adapté à l, si

1) $A(\mathscr{D}, \mathscr{V}_l)$ est une algèbre de type fini;
2) $A(\mathscr{D}, \mathscr{V}_l) \subset \mathcal{O}_l$;
3) $\mathcal{O}_l$ est le localisé de $A(\mathscr{D}, \mathscr{V}_l)$ en l'idéal premier $A(\mathscr{D}, \mathscr{V}_l) \cap \mathfrak{m}_l$.

D'après 8.5 et 8.6, ces conditions entraînent que $\mathscr{D}_l \subset \mathscr{D}$, et que $\mathscr{D}, \mathscr{V}_l$ vérifient (D), (W), (W'), (F) (d'où en particulier que $\mathscr{D}$ est cofini dans ${}^B\mathscr{D}(G/H)$).

Si $\mathscr{D}$ est adapté à l, l'algèbre $A(\mathscr{D}, \mathscr{V}_l)$ est une sous-algèbre affine de $k(G/H)$, qui est clairement stable par l'algèbre de Lie de G; il lui correspond donc un ouvert $X_{A(\mathscr{D}, \mathscr{V}_l)}$ de $\mathfrak{X}(G/H)$ (voir §1).

PROPOSITION. *Soit $l \in \mathfrak{L}^n(G/H)$. Il existe des $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$ adaptés à l. De plus, pour toute réalisation géométrique X', Y' de l, on peut trouver $\mathscr{D} \subset {}^B\mathscr{D}(G/H)$, adapté à l et tel que $X_{A(\mathscr{D}, \mathscr{V}_l)} \subset X'$.*

Preuve. Soit X', Y' une réalisation géométrique de l. Puisque $\mathcal{O}_l$ est intégralement clos, l'ouvert des points normaux de X' rencontre Y'. Par conséquent, quitte à rétrécir X', on peut supposer la variété X' normale.

Soient M et h comme dans 8.2, et posons $A = k[h^{-1}M]$; c'est une sous-algèbre affine de $k(G/H)$, stable par l'algèbre de Lie de G. Notons X_A l'ouvert de $\mathfrak{X}(G/H)$ qui correspond à A, et posons $X = G \cdot X_A$; c'est l'espace d'un plongement de G/H. Désignons par Y le fermé de X tel que $\mathcal{O}_l = \mathcal{O}_{X,Y}$; par $Y_1, \ldots, Y_\alpha, Y_{\alpha+1}, \ldots, Y_\beta$ les composantes irréductibles de $X - G/H$ qui sont de codimension 1 dans X, où $Y_1, \ldots, Y_\alpha$ sont celles qui contiennent Y et qui correspondent aux éléments de $\mathscr{V}_l$; enfin désignons par $Y'_1, \ldots, Y'_\gamma$ les composantes irréductibles de $X - X'$.

L'inclusion $M \subset k[G]$ induit des applications naturelles $S^n(M) \to k[G]$ (où $S^n(M)$ est la puissance symétrique n-ième de M). Notons $\bar{S}^n(M)$ l'image de

$S^n(M)$ dans $k[G]$. Il est clair que tout $f \in A$ peut s'écrire $f = h^{-n}g$, avec $n \in \mathbb{N}^*$ et $g \in \bar{S}^n(M)$. Il s'ensuit que A est quasi-G-stable.

Pour tout fermé Z de X, désignons par $\mathfrak{a}(Z)$ l'idéal des éléments de A qui s'annulent sur $Z \cap X_A$. Si Z est irréductible et stable par G, on peut choisir $v \in \mathcal{V}(G/H)$ tel que $\mathcal{O}_v$ domine $\mathcal{O}_{X,Z}$; $\mathfrak{a}(Z)$ peut alors aussi se décrire comme l'ensemble des $f \in A$ tels que $v(f) > 0$.

Puisque Y n'est pas contenu dans $Y_{\alpha+1}, \ldots, Y_\beta, Y'_1, \ldots, Y'_\gamma$, on a

$$(\mathfrak{a}(Y_{\alpha+1}) \cap \cdots \cap \mathfrak{a}(Y_\beta) \cap \mathfrak{a}(Y'_1) \cap \cdots \cap \mathfrak{a}(Y'_\gamma)) - \mathfrak{a}(Y) \neq \varnothing.$$

D'après 7.3, dans cet ensemble existe au moins un élément de $\mathcal{P}$, disons h'. Posons $A' = A[(h')^{-1}]$; A' est une algèbre de type fini. La variété $X_{A'}$ s'identifie à l'ouvert de X_A où la fonction h' ne n'annule pas. Puisque h' s'annule sur $X - X'$, et que nous supposons X' normale, A' est intégralement clos, donc un anneau de Krull. Puisque h' s'annule sur $Y_{\alpha+1}, \ldots, Y_\beta$ mais ne s'annule pas sur Y, on voit que les valuations essentielles de A' sont celles de $\mathcal{V}_l$ et certaines des v_D, $D \in \mathcal{D}(G/H)$.

Désignons par $\mathcal{D}$ l'ensemble des $D \in {}^B\mathcal{D}(G/H)$ tels que $v_D(h) = v_D(h') = 0$. Si $D \in \mathcal{D}(G/H)$, il est clair que $A' \subset \mathcal{O}_{v_D}$ si et seulement si $D \in \tilde{\mathcal{D}}$. Par suite, on a $A' = A(\mathcal{D}, \mathcal{V}_l)$. Il s'ensuit que $A(\mathcal{D}, \mathcal{V}_l)$ est une algèbre de type fini. Puisque h' ne s'annule par sur Y, on a $A(\mathcal{D}, \mathcal{V}_l) = A' \subset \mathcal{O}_l$. Puisque $A \subset A' = A(\mathcal{D}, \mathcal{V}_l)$, on voit que $\mathcal{D}$ est adapté à l. Enfin, puisque h' s'annule sur $X - X'$, on a $X_{A(\mathcal{D}, \mathcal{V}_l)} = X_{A'} \subset X'$.

8.8. Soit $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ et soit $\mathcal{W}$ un sous-ensemble fini de $\mathcal{V}(G/H)$. Si $l \in \mathfrak{L}_1^n(G/H)$, on désignera par $\mathcal{D}(\mathcal{W}, l)$ l'ensemble des $D \in \mathcal{D}$ tels que v_D s'annule sur $\mathcal{P} \cap A(\mathcal{D}, \mathcal{W}) \cap \mathcal{O}_l^*$, et par $\mathcal{W}(\mathcal{D}, l)$ l'ensemble des $w \in \mathcal{W}$ qui s'annulent sur $\mathcal{P} \cap A(\mathcal{D}, \mathcal{W}) \cap \mathcal{O}_l^*$.

Soit $v \in \mathcal{V}_1(G/H)$. Voici deux conditions portant sur $\mathcal{D}, \mathcal{W}, v$.

(V) Pour tout $f \in \mathcal{P} \cap A(\mathcal{D}, \mathcal{W})$, on a $v(f) \geq 0$.

(V′) Tout $w \in W$ s'annule sur $\mathcal{P} \cap A(\mathcal{D}, \mathcal{W}) \cap \mathcal{O}_v^*$ (autrement dit, $\mathcal{W} = \mathcal{W}(\mathcal{D}, v)$).

PROPOSITION. *On suppose que $\mathcal{D}, \mathcal{W}$ vérifient* (D), (W), (W′), (F).

1) *Soit $v \in \mathcal{V}_1(G/H)$ tel que $\mathcal{D}, \mathcal{W}, v$ vérifient* (V). *Il existe alors $l \in \mathfrak{L}_1^n(G/H)$ tel que $\mathcal{O}_l$ est le localisé de $A(\mathcal{D}, \mathcal{W})$ en l'idéal premier $A(\mathcal{D}, \mathcal{W}) \cap \mathfrak{m}_v$. On a $v \in \mathcal{F}_l$, ${}^B\mathcal{D}_l = \mathcal{D}(\mathcal{W}, v)$ et $\mathcal{V}_l = \mathcal{W}(\mathcal{D}, v)$.*
2) *Inversement, soit $l \in \mathfrak{L}_1^n(G/H)$ qui vérifie ${}^B\mathcal{D}_l = \mathcal{D}(\mathcal{W}, l)$ et $\mathcal{V}_l = \mathcal{W}(\mathcal{D}, l)$, et soit $v \in \mathcal{F}_l$. Alors $\mathcal{D}, \mathcal{W}, v$ vérifient* (V) *et $\mathcal{O}_l$ est le localisé de $A(\mathcal{D}, \mathcal{W})$ en l'idéal premier $A(\mathcal{D}, \mathcal{W}) \cap \mathfrak{m}_v$.*

Preuve. Posons $A = A(\mathscr{D}, \mathscr{W})$. D'après 8.5 et 8.6, A est une sous-algèbre affine de $k(G/H)$, et A est un anneau de Krull, dont les valuations essentielles sont les v_D $(D \in \tilde{\mathscr{D}})$ et les w $(w \in \mathscr{W})$.

Soit $v \in \mathscr{V}_1(G/H)$ tel que $\mathscr{D}, \mathscr{W}, v$ vérifient (V). L'espace vectoriel A étant quasi-G-stable, la condition (V) entraîne $\mathcal{O}_v \supset A$ (voir 7.3). L'algèbre A étant stable par l'algèbre de Lie de G, il s'ensuit que le localisé de A en l'idéal premier $A \cap \mathfrak{M}_v$ détermine un $l \in \mathfrak{L}_1(G/H)$ (voir 2.5); A étant intégralement clos, $l \in \mathfrak{L}_1^n(G/H)$. Par construction, $v \in \mathscr{F}_l$.

Soit $D \in \mathscr{D}$. Pour que $D \in {}^B\mathscr{D}_l$, c'est-à-dire pour que la valuation v_D reste essentielle pour le localisé de A en l'idéal premier $A \cap \mathfrak{m}_v$, il faut et il suffit que

$$(*) \quad A \cap \mathfrak{m}_{v_D} \subset \mathfrak{m}_v.$$

Par définition de $\mathscr{D}(\mathscr{W}, v)$, pour que $D \in \mathscr{D}(\mathscr{W}, v)$, il faut et il suffit que

$$(**) \quad \mathscr{P} \cap A \cap \mathfrak{m}_{v_D} \subset \mathfrak{m}_v.$$

Il n'est pas difficile à voir que $A \cap \mathfrak{m}_{v_D}$ est quasi-G-stable: en effet, tout $f \in A \cap \mathfrak{m}_{v_D}$ peut s'écrire $f = hg$, où $h \in \mathscr{P}$ vérifie $v_D(h) > 0$ et $v_{D'}(h) = 0$ $(D' \in \mathscr{D} - \{D\})$, et où g appartient au sous-G-module des éléments de $k[G]$ qui vérifient $hg \in k(G/H)$ et $w(hg) \geq 0$ $(w \in \mathscr{W})$.

D'après 7.3, il s'ensuit que $(*) \Leftrightarrow (**)$.

De même, si $w \in \mathscr{W}$, pour que $w \in \mathscr{V}_l$, il faut et il suffit que

$$(*') \quad A \cap \mathfrak{m}_w \subset \mathfrak{m}_v.$$

Pour que $w \in \mathscr{W}(\mathscr{D}, v)$, il faut et il suffit que

$$(**') \quad \mathscr{P} \cap A \cap \mathfrak{m}_w \subset \mathfrak{m}_v.$$

L'espace vectoriel A étant quasi-G-stable, $(*') \Leftrightarrow (**')$.

Soit $l \in \mathfrak{L}_1^n(G/H)$ tel que ${}^B\mathscr{D}_l = \mathscr{D}(\mathscr{W}, l)$ et $\mathscr{V}_l = \mathscr{W}(\mathscr{D}, l)$, et soit $v \in \mathscr{F}_l$. De ${}^B\mathscr{D}_l \subset \mathscr{D}$ et $\mathscr{V}_l \subset \mathscr{W}$ résulte que $A \subset \mathcal{O}_l \subset \mathcal{O}_v$, d'où il suit que $\mathscr{D}$, $\mathscr{W}$, v vérifient (V). D'après la première partie de la proposition, il existe $l' \in \mathfrak{L}_1^n(G/H)$ tel que $\mathcal{O}_l$ soit le localisé de A en $A \cap \mathfrak{M}_v$. De ${}^B\mathscr{D}_l = \mathscr{D}(\mathscr{W}, \mathscr{V}) = \mathscr{D}(\mathscr{W}, l) = {}^B\mathscr{D}_l$ et de $\mathscr{V}_{l'} = \mathscr{W}(\mathscr{D}, v) = \mathscr{W}(\mathscr{D}, l) = \mathscr{V}_l$ résulte alors $l = l'$ (voir 8.3), ce qui termine la preuve de la proposition.

COROLLAIRE 1. *Soient* $l \in \mathfrak{L}_1^n(G/H)$, $v \in \mathcal{F}_l$ *et* $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$. *Pour que* $\mathcal{D}$ *soit adapté à* l, *il faut et il suffit que* $\mathcal{D}, \mathcal{V}_l, v$ *vérifient* $(D), (W), (W'), (F), (V), (V')$ *et que* ${}^B\mathcal{D}_l = \mathcal{D}(\mathcal{V}_l, v)$.

COROLLAIRE 2. *Soit* $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ *et soit* $\mathcal{W}$ *un sous-ensemble fini de* $\mathcal{V}(G/H)$. *Pour qu'il existe* $l \in \mathfrak{L}_1^n(G/H)$ *tel que* ${}^B\mathcal{D}_l = \mathcal{D}$ *et* $\mathcal{V}_l = \mathcal{W}$, *il faut et il suffit qu'on puisse trouver* $\mathcal{D}' \subset {}^B\mathcal{D}(G/H)$ *et* $v \in \mathcal{V}_1(G/H)$ *possédant les propriétés suivantes*: $\mathcal{D}', \mathcal{W}, v$ *vérifient* $(D), (W), (W'), (F), (V), (V')$ *et* $\mathcal{D} = \mathcal{D}'(\mathcal{W}, v)$.

COROLLAIRE 3. *Soit* $l \in \mathfrak{L}_1^n(G/H)$, *soit* $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ *adapté à* l, *et soit* $v \in \mathcal{V}_1(G/H)$ *tel que* $\mathcal{D}, \mathcal{V}_l, v$ *vérifient* (V). *Pour que* $v \in \mathcal{F}_l$, *il faut et il suffit que* $\mathcal{D}, \mathcal{V}_l, v$ *vérifient* (V') *et que* ${}^B\mathcal{D}_l = \mathcal{D}(\mathcal{V}_l, v)$.

8.9. Soit $l \in \mathfrak{L}_1^n(G/H)$ et soit $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$. Désignons par $L(\mathcal{D}, l)$ l'ensemble des $l' \in \mathfrak{L}_1^n(G/H)$ possédant la propriété suivante: il existe $v' \in \mathcal{V}_1(G/H)$ tel que $\mathcal{D}, \mathcal{V}_l, v'$ vérifient (V) et tel que ${}^B\mathcal{D}_{l'} = \mathcal{D}(\mathcal{V}_l, v')$ et $\mathcal{V}_{l'} = \mathcal{V}_l(\mathcal{D}, v')$.

PROPOSITION. *La famille des* $L(\mathcal{D}, l)$, $\mathcal{D}$ *adapté à* l, *forme une base de voisinages de* l *dans* $\mathfrak{L}_1(G/H)$ (*pour la topologie de Zariski*).

Preuve. Soit $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ adapté à l; $A(\mathcal{D}, \mathcal{V}_l)$ est alors une algèbre affine de $k(G/H)$, stable par l'algèbre de Lie de G. D'après 8.8, $L(\mathcal{D}, l)$ peut aussi se décrire comme l'ensemble des $l' \in \mathfrak{L}_1(G/H)$ tel que $\mathcal{O}_{l'}$ soit un localisé de $A(\mathcal{D}, \mathcal{V}_l)$; par définition de la topologie de Zariski, $L(\mathcal{D}, l)$ est donc ouvert dans $\mathfrak{L}_1(G/H)$. D'autre part, si L est un voisinage quelconque de l dans $\mathfrak{L}_1(G/H)$, de 8.7 résulte qu'on peut trouver $\mathcal{D}$ adapté à l tel que $l \in L(\mathcal{D}, l) \subset L$.

8.10. Supposons maintenant que B ait une orbite ouverte U dans G/H. Désignons par $Y_1, \ldots, Y_{r'}$ les composantes irréductibles de $G/H - U$; U étant affine, $Y_1, \ldots, Y_{r'}$ sont de codimension 1 dans G/H, et constituent visiblement les différents éléments de ${}^B\mathcal{D}(G/H)$. Désignons par r'' le rang de $X(G)$, le groupe des caractères de G, et posons $r = r' + r''$. Il est clair que le groupe $\mathcal{P}/k^*$ est isomorphe à $\mathbb{Z}^r$.

Soit $\mathcal{D} \subset {}^B\mathcal{D}(G/H)$ et soit $\mathcal{W}$ un sous-ensemble fini de $\mathcal{V}(G/H) = \mathcal{V}_1(G/H)$. Rappelons un résultat classique: quelles que soient les applications $\mathbb{Q}$-linéaires $\varphi_1, \ldots, \varphi_\alpha, \varphi_{\alpha+1}, \ldots, \varphi_\beta$ de $\mathbb{Q}^r$ dans $\mathbb{Q}$, le monoïde formé des $f \in \mathbb{Z}^r$ tels que $\varphi_1(f) \geq 0, \ldots, \varphi_\alpha(f) \geq 0$ et $\varphi_{\alpha+1}(f) = \cdots = \varphi_\beta(f) = 0$, est de type fini. Il en résulte que $(\mathcal{P} \cap A(\mathcal{D}, \mathcal{W}))/k^*$ est un monoïde de type fini; en particulier, $\mathcal{P} \cap A(\mathcal{D}, \mathcal{W})$ engendre une sous-algèbre de type fini de $k(G)$. On voit que la condition (F) est toujours remplie.

Soit $l \in \mathfrak{L}^n(G/H) = \mathfrak{L}_1^n(G/H) = \mathfrak{L}_f^n(G/H)$. De la finitude de ${}^B\mathcal{D}(G/H)$ et de ce qui précède, résulte que ${}^B\mathcal{D}_l$ est déjà adapté à l. La résultat du corollaire 2 de 8.8 se simplifie alors comme suit: soit $\mathcal{D}\subset {}^B\mathcal{D}(G/H)$ et soit $\mathcal{W}$ un sous-ensemble fini de $\mathcal{V}(G/H)$; pour qu'il existe $l\in\mathfrak{L}_1^n(G/H)$ tel que ${}^B\mathcal{D}_l=\mathcal{D}$ et $\mathcal{V}_l=\mathcal{W}$, il faut et il suffit qu'il existe $v\in\mathcal{V}_1(G/H)$ tel que $\mathcal{D}, \mathcal{W}, v$ vérifient $(D), (W), (W'), (V), (V')$ et que $\mathcal{D}=\mathcal{D}(\mathcal{W}, v)$.

Nous allons reformuler ce résultat, afin de faire ressortir davantage l'analogie avec le cas particulier des plongements toriques (voir [5], [6]). Désignons par V le $\mathbb{Q}$-espace vectoriel de dimension finie obtenu en tensorisant par $\mathbb{Q}$ le $\mathbb{Z}$-module libre $(\mathcal{P}\cap k(G/H))/k^*$. Les éléments de $\mathcal{V}(G/H)$ et les v_D $(D\in{}^B\mathcal{D}(G/H))$ se laissent interpréter comme des éléments de V', l'espace vectoriel dual de V. On observera que l'application $D\mapsto v_D$ de ${}^B\mathcal{D}(G/H)$ dans V' n'est pas injective en général.

Soit C un cône convexe de $\mathbb{Q}^r$. Rappelons qu'on dit que C est saillant, si C ne contient aucune droite de $\mathbb{Q}^r$. On notera C^0 l'intérieur relatif de C (c'est-à-dire l'intérieur de C dans le sous-espace vectoriel engendré par C). Si C' est un autre cône convexe de $\mathbb{Q}^r$, on dit que C' est une facette de C, s'il existe une forme linéaire $\varphi:\mathbb{Q}^r\to\mathbb{Q}$, positive sur C, et telle que $C'=C\cap\varphi^{-1}(0)$.

Si $\mathcal{D}\subset{}^B\mathcal{D}(G/H)$ et si $\mathcal{W}\subset\mathcal{V}(G/H)$, désignons par $C(\mathcal{D},\mathcal{W})$ le cône convexe de V' engendré par $\mathcal{W}$ et les v_D $(D\in\mathcal{D})$. Si $l\in\mathfrak{L}^n(G/H)$, posons $C(l)=C({}^B\mathcal{D}_l,\mathcal{V}_l)$. Considérons les quatre conditions suivantes:

(a) Le cône $C(\mathcal{D},\mathcal{W})$ est saillant.

(b) Les droites $\mathbb{Q}w$ $(w\in\mathcal{W})$ sont des droites extrémales de $C(\mathcal{D},\mathcal{W})$ et ne coïncident pas avec l'une des $\mathbb{Q}v_D(D\in\mathcal{D})$.

(c) $C(\mathcal{D},\mathcal{W})^0\cap\mathcal{V}(G/H)\neq\emptyset$.

(d) $=(D)$.

PROPOSITION. 1) *Soit* $\mathcal{D}\subset{}^B(G/H)$ *et soit* $\mathcal{W}$ *un sous-ensemble fini de* $\mathcal{V}(G/H)$. *Pour qu'il existe* $l\in\mathfrak{L}_1^n(G/H)$ *tel que* ${}^B\mathcal{D}_l=\mathcal{D}$ *et* $\mathcal{V}_l=\mathcal{W}$, *il faut et il suffit que* $\mathcal{D},\mathcal{W}$ *vérifient les conditions* (a), (b), (c), (d).

2) *Pour tout* $l\in\mathfrak{L}_1^n(G/H)$, *on a* $\mathcal{F}_l=\mathcal{V}_1(G/H)\cap C(l)^0$.

3) *Si* $l, l'\in\mathfrak{L}_1^n(G/H)$, *pour que* $\mathcal{O}_{l'}$ *soit un localisé de* $\mathcal{O}_l$, *il faut et il suffit que* $C(l')$ *soit une facette de* $C(l)$ *et que* $\mathcal{D}_{l'}$ *soit l'ensemble des* $D\in\mathcal{D}_l$ *vérifiant* $v_D\in C(l')$.

Preuve. Il est clair que (a) = (W), que (b) = (W'), et que (c) remplace avantageusement $(V), (V')$ et $\mathcal{D}=\mathcal{D}(\mathcal{W}, v)$, d'où 1). Les assertions 2) et 3) résultent aussitôt de 8.8 et 8.9.

9. Classification des plongements normaux de $SL(2)$

Dans ce §, nous appliquerons les résultats des §§ précédents au cas où $G=SL(2)$ et $H=\{e\}$, pour obtenir la classification des plongements normaux de $SL(2)/\{e\}=SL(2)$.

9.1. Comme d'habitude, $SL(2)$ est le groupe des matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ de déterminant $ad-bc=1$. Pour avoir des notations plus intrinsèques, désignons par $M=k^2$ l'unique $SL(2)$-module rationnel irréductible de dimension 2.

Désignons par U le sous-groupe unipotent maximal de $SL(2)$ formé des matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, et posons $B=N_{SL(2)}(U)$. Sauf mention expresse du contraire, $SL(2)$, U et B opèreront toujours par "translations à gauche".

Le $SL(2)$-module rationnel ${}^Uk[SL(2)]$ ($SL(2)$ opérant par "translations à droite") contient un unique sous-module isomorphe à M. Vu la structure particulièrement simple de $SL(2)$, tout morphisme injectif de $SL(2)$-modules $M \to {}^Uk[SL(2)]$ s'étend en un isomorphisme de $SL(2)$-algèbres $S(M) \xrightarrow{\sim} {}^Uk[SL(2)]$, où $S(M)$ désigne l'algèbre symétrique de M. Nous fixerons dans la suite un tel isomorphisme et nous le traiterons comme une égalité.

Il y a une bijection naturelle entre $\mathbb{P}_1$, l'ensemble des droites de M, et ${}^B\mathscr{D}(SL(2))$, l'ensemble des fermés irréductibles de codimension 1 de $SL(2)$, stables par B. Si D est une droite de M, nous écrirons v_D pour la valuation qui lui correspond, lorsqu'on interprète D comme élément de ${}^B\mathscr{D}(SL(2))$; si $f\in M\subset {}^Uk[SL(2)]\subset k(SL(2))$, on a

$$v_D(f)=\begin{cases} 0 & \text{si } f\in M-D \\ 1 & \text{si } f\in D-\{0\}. \end{cases}$$

En suivant les notations du § précédent, désignons par $\mathscr{P}$ l'ensemble des $f\in k(SL(2))$ qui sont des vecteurs propres de B (l'autre condition tombe, puisque $H=\{e\}$!). Il est clair que $\mathscr{P}\cap k[SL(2)]$ s'identifie à l'ensemble des éléments homogènes (non nuls) de $S(M)$ (homogènes au sens de la graduation naturelle $S(M)=\oplus_{n\in\mathbb{N}} S_n(M)$.

PROPOSITION. 1) *Toute valuation de $\mathscr{V}(SL(2)/\{e\})$ est déterminée par sa restriction à $M-\{0\}$.*

2) *Pour qu'une fonction $w: M-\{0\}\to\mathbb{Z}$ soit la restriction d'une valuation dans $\mathscr{V}_1(SL(2)/\{E\})$, il faut et il suffit qu'il existe une droite D de M et des entiers p, q premiers entre eux vérifiant $p<0$ et $p<q\le -p$, tels que $w(M-D)=p$ et $w(D-\{0\})=q$.*

3) *Il n'existe qu'une seule valuation dans $\mathscr{V}_2(SL(2)/\{e\})$, et sa restriction à $M-\{0\}$ est identiquement égale à -1.*

Preuve. D'après 7.4, toute valuation de $\mathscr{V}(SL(2)/\{e\})$ est déterminée par sa restriction au groupe multiplicatif $\mathscr{P}$. D'après ce qui précède, il est clair que ce groupe est engendré par $M-\{0\}$, ce qui montre la partie 1) de la proposition.

Soit $v \in \mathcal{V}(SL(2)/\{e\})$. Si v restait positif sur $M-\{0\}$, v resterait aussi positif sur $\mathcal{P} \cap k[SL(2)]$, donc aussi sur $k[SL(2)]$. Mais cela n'est pas possible, car $k[SL(2)] \cap \mathfrak{m}_v$ serait alors un idéal propre non nul de $k[SL(2)]$, qui serait stable par translations à gauche. Par suite, v prend au moins une valeur strictement négative sur $M-\{0\}$.

Désignons par $M(2)$ l'espace vectoriel des matrices 2×2. Faisons opérer $SL(2)$ dans $M(2)\oplus k$, par multiplication à gauche et à droite dans $M(2)$ et trivialement dans k. Désignons par X la sous-variété lisse de $\mathbb{P}_4 = \mathbb{P}(M(2)\oplus k)$ définie par l'équation $ad-bc-t^2=0$. L'inclusion $SL(2) \hookrightarrow M(2)$ passe au quotient en un plongement $SL(2)\hookrightarrow X$, dans lequel opère $SL(2)\times SL(2)$. On vérifie sans peine que $Y = X - SL(2)$ est une seule orbite sous $SL(2)\times SL(2)$, isomorphe à $\mathbb{P}_1\times\mathbb{P}_1$; $SL(2)\hookrightarrow X$, considéré comme plongement sous $SL(2)\times SL(2)$, est donc un plongement élémentaire. Désignons par v la valuation de $k(SL(2))$ qui lui correspond. Puisque les orbites de $SL(2)$ dans Y (pour l'opération par multiplication à gauche) sont toutes de dimension 1, on a $v \in \mathcal{V}_2(SL(2)/\{e\})$. De l'invariance par "translations à droite" de v, il résulte que v reste constant sur $M-\{0\}$. Puisque v prend une valeur strictement négative sur $M-\{0\}$, et que v est normalisé, il s'ensuit que v est égal à -1 sur $M-\{0\}$.

Soit D une droite de M, et soient p, q deux entiers vérifiant $p<0$ et $p<q\leq -p$. Choisissons un $s\in SL(2)$ qui envoie le second vecteur de la base canonique de $k^2=M$ dans D, et considérons

$$\lambda(t) = \begin{pmatrix} t^p & t^q \\ 0 & t^{-p} \end{pmatrix} \cdot s^{-1} \in SL(2)_{k((t))}.$$

Il est clair que $v_\lambda(M-D)=p$ et que $v_\lambda(D-\{0\})=q$. Pour que v_λ soit normalisé, il faut et il suffit que p et q soient premiers entre eux.

Pour terminer la preuve de la proposition, il suffit de montrer que pour tout $v\in\mathcal{V}(SL(2)/\{e\})$, on a $q\leq -p$, où p (resp. q) désigne le minimum (resp. le maximum) de v dans $M-\{0\}$. On peut supposer que $q>0$. On voit alors facilement que l'ensemble des $f\in k[SL(2)]$ vérifiant $v(f)\geq 0$ forme une sous-algèbre de type fini A de $k[SL(2)]$ (voir aussi 9.2), et que A possède comme corps des fractions $k(SL(2))$. Puisque A est stable par $SL(2)$, on peut considérer A comme algèbre des fonctions régulières d'une $SL(2)$-variété affine X, et l'inclusion $A\subset k[SL(2)]$ se reflète en un plongement de $SL(2)/\{e\}$ dans X.

D'après le critère de Hilbert-Mumford (voir [8]) il existe un sous-groupe à 1-paramètre multiplicatif λ de $SL(2)$ tel que $\lim_{t\to 0}\lambda(t)$ existe dans X, autrement dit (voir §4) tel que $\mathcal{O}_{v_\lambda}\supset A$. Choisissons h et g linéairement indépendants dans M tels que $v(h)=q$, $v(g)=p$. Vu la manière dont on obtient v_λ sur M et l'hypothèse $h\in A\subset\mathcal{O}_{v_\lambda}$, il est clair que $v_\lambda(h)>0$ et $v_\lambda(g)=-v_\lambda(h)$. On a $h^{-p}g^q\in A$, car

$v(h^{-p}g^q) = -pq + pq = 0$. Par suite $v_\lambda(h^{-p}g^q) = -(p+q)v_\lambda(h) \geq 0$, d'où il suit bien que $q \leq -p$.

Profitons de cette description explicite des valuations de $\mathcal{V}(SL(2)/\{e\})$ pour changer légèrement nos conventions et pour introduire des notations commodes dans la suite.

Désormais, nous considérons les éléments de $\mathcal{V}(SL(2)/\{e\})$ renormalisés par la condition suivante: si $v \in \mathcal{V}(SL(2)/\{e\})$, on suppose que le minimum de v dans M est égal à -1 (pour pouvoir le faire, il faut bien sûr admettre des valuations à valeur dans $\mathbb{Q}$!). De manière plus précise, si D est une droite de M et si $r \in \mathbb{Q}$ vérifie $-1 < r \leq 1$, on désignera par $v(D, r)$ l'unique élément de $\mathcal{V}_1(SL(2)/\{e\})$ qui vaut -1 sur $M - D$ et r sur $D - \{0\}$ (avec les notations de la proposition, on a $r = -q/p$).

Nous désignerons enfin par $v(\ ,-1)$ l'unique élément de $\mathcal{V}_2(SL(2)/\{e\})$ qui vaut -1 sur $M - \{0\}$; parfois, on écrira aussi $v(D, -1)$ pour $v(\ ,-1)$, où D est une droite quelconque de M.

Si $E \subset [-1, 1]$, nous désignerons par $v(D, E)$ l'ensemble des $v(D, r), r \in E$.

9.2. Soit $\mathcal{D} \subset {}^B\mathcal{D}(SL(2)) = \mathbb{P}_1$ et soit $\mathcal{W} \subset \mathcal{V}(SL(2)/\{e\})$.

PROPOSITION. *Si $\mathcal{D}$ est cofini dans $\mathbb{P}_1$ et si $\mathcal{W}$ est fini, alors $A(\mathcal{D}, \mathcal{W})$ est une algèbre de type fini.*

Preuve. Posons $A = A(\mathcal{D}, \mathcal{W})$. D'après 8.6, il suffit de montrer que ${}^UA$ est une algèbre de type fini. Choisissons $h \in \mathcal{P} \cap k[SL(2)]$ vérifiant: $v_D(h) = 0$ quel que soit $D \in \mathcal{D}$, et $v_{D'}(h) > 0$ quel que soit $D' \in \mathbb{P}_1 - \mathcal{D}$. Désignons par $D_1, \ldots, D_\beta$ les différentes droites de M sur lesquelles au moins un des $w \in \mathcal{W}$ n'est pas égal à -1. Choisissons $h_j \in D_j - \{0\}$ $(j = 1, \ldots, \beta)$, et choisissons deux éléments linéairement indépendants g_1, g_2 dans $M - \bigcup_{j=1}^{\beta} D_j$. Désignons par S le sous-monoïde des $(n_1, \ldots, n_\beta, m_1, m_2, n) \in \mathbb{N}^{\beta+3}$ tels que $h_1^{n_1} \cdot \cdots \cdot h_\beta^{n_\beta} \cdot g_1^{m_1} \cdot g_2^{m_2} \cdot h^{-n} \in {}^UA$ (autrement dit, tels que $n_1 w(h_1) + \cdots + n_\beta w(h_\beta) - m_1 - m_2 - nw(h) \geq 0$ quel que soit $w \in \mathcal{W}$). Il est classique que de tels monoïdes sont de type fini; désignons par S' un système fini de générateurs de S. On voit facilement que tout élément de ${}^UA$ est somme d'éléments de la forme $h_1^{n_1} \cdot \cdots \cdot h_\beta^{n_\beta} \cdot f_1 \cdot \cdots \cdot f_m \cdot h^{-n}$, où $f_1, \ldots, f_m \in M - \bigcup_{j=1}^{\beta} D_j$ et où $(n_1, \ldots, n_\beta, m, 0, n) \in S$. En écrivant $f_1, \ldots, f_m$ comme combinaison linéaire de g_1 et g_2, on voit que tout élément de ${}^UA$ est combinaison linéaire d'éléments de la forme $h_1^{n_1} \cdot \cdots \cdot h_\beta^{n_\beta} \cdot g_1^{m_1} \cdot g_2^{m_2} \cdot h^{-n}$, où $(n_1, \ldots, n_\beta, m_1, m_2, n) \in S$. Il s'ensuit que les $h_1^{n_1} \cdot \cdots \cdot h_\beta^{n_\beta} \cdot g_1^{m_1} \cdot g_2^{m_2} \cdot h^{-n}$, $(n_1, \ldots, n_\beta, m_1, m_2, n) \in S'$ forment un système fini de générateurs de ${}^UA$.

9.3. Soient $\mathcal{D}$ un sous-ensemble cofini de ${}^B\mathcal{D}(SL(2)) = \mathbb{P}_1$, $\mathcal{W}$ un sous-ensemble fini de $\mathcal{W}(SL(2)/\{e\})$ et $v \in \mathcal{V}_1(SL(2)/\{e\})$. Posons $\mathcal{W} = \{w_1, \ldots, w_\alpha\}$ et

$w_j = v(D_j, r_j)$ $(j=1, \ldots, \alpha)$. Nous aurons à considérer les six types suivants de tels $\mathcal{D}$, $\mathcal{W}$, v.

Type A_α $(\alpha \geq 1)$

$\mathcal{D} = \mathbb{P}_1 - \{D_1, \ldots, D_\alpha\}$, où $D_1, \ldots, D_\alpha$ sont des droites différentes de M;

$$-1 < r_j \leq 1 \; (j=1, \ldots, \alpha) \text{ et } \sum_{j=1}^{\alpha} \frac{1}{1+r_j} > 1$$

(compte tenu des premières inégalités, cette dernière condition est vide si $\alpha \geq 3$; si $\alpha = 2$, elle signifie que r_1 ou r_2 est <1; si $\alpha = 1$, elle signifie que $r_1 < 0$);

$$v \in \bigcup_{D \in \mathcal{D}} v(D,]-1, 1]) \cup \bigcup_{j=1}^{\alpha} v(D_j,]-1, r_j[).$$

Type AB $(\alpha = 2)$

$D_1 \notin \mathcal{D}$ et $\mathcal{D} \neq \mathbb{P}_1 - \{D_1\}$; $\quad D_1 = D_2$ et $-1 \leq r_2 < r_1 \leq 1$; $\quad v \in v(D_1,]r_2, r_1[)$.

Type B_+ $(\alpha = 1)$

$D_1 \in \mathcal{D} \neq \mathbb{P}_1$; $\quad -1 \leq r_1 < 1$; $\quad v \in v(D_1,]r_1, 1])$.

Type B_- $(\alpha = 1)$

$\mathcal{D} = \mathbb{P}_1 - \{D_1\}$; $\quad 0 < r_1 < 1$; $\quad v \in v(D_1,]r_1, 1])$.

Type B_0 $(\alpha = 1)$

$\mathcal{D} = \mathbb{P}_1$; $\quad 0 < r_1 < 1$; $\quad v \in v(D_1,]r_1, 1])$.

Type C

C'est le cas banal $\mathcal{W} = \{v\}$.

PROPOSITION. *Pour que* $\mathcal{D}$, $\mathcal{W}$, v *vérifient* (W), $(W')_{\geq 2}$, (V), (V'), *il faut et il suffit qu'ils appartiennent à l'un des types* A_α, AB, B_+, B_-, B_0, C.

Voici d'abord un lemme qui servira deux fois dans la démonstration de la proposition.

LEMME. *Soient* $D_1, \ldots, D_\beta$ *des droites différentes de* M *et* $r_1, \ldots, r_\beta$ *des* *nombres rationnels vérifiant* $-1 < r_j \leq 1$ $(j = 1, \ldots, \beta)$ *et*

$$t = \left(\sum_{j=1}^{\beta} \frac{1}{1+r_j}\right) - 1 > 0.$$

Posons $\mathcal{D} = \mathbb{P} - \{D_1, \ldots, D_\beta\}$ *et* $w_j = v(D_j, r_j)$ $(j = 1, \ldots, \beta)$. *Alors, pour tout*

$$v \in \bigcup_{D \in \mathcal{D}} v(D, [-1, 1]) \cup \bigcup_{j=1}^{\beta} v(D_j, [-1, r_j[),$$

il existe des nombres rationnels $\lambda_j > 0$ $(j = 1, \ldots, \beta)$ *et un homomorphisme de groupes* $z : \mathcal{P} \to \mathbb{Q}$, *positif sur* $\mathcal{P} \cap A(\mathcal{D})$, *tels que* $v = \lambda_1 w_1 + \cdots + \lambda_\beta w_\beta + z$.

Preuve. Posons $v = v(D, r)$. Si D coïncide avec l'un des D_i $(i = 1, \ldots, \beta)$, posons

$$\lambda_j = \frac{1}{t} \frac{r_i - r}{(r_i + 1)(r_j + 1)} + \delta_{ij} \frac{r+1}{r_i + 1}$$

(où δ_{ij} est le symbole de Kronecker), et définissons z sur $M - \{0\}$ par

$$z(f) = \begin{cases} 0 & \text{si } f \in \bigcup_{j=1}^{\beta} D_j \\ \dfrac{1}{t} \dfrac{r_i - r}{r_i + 1} & \text{sinon;} \end{cases}$$

si $D \in \mathcal{D}$, posons $\lambda_j = 1/t(r_j + 1)$, et définissons z sur $M - \{0\}$ par

$$z(f) = \begin{cases} 0 & \text{si } f \in \bigcup_{j=1}^{\beta} D_j \\ r + 1 + \dfrac{1}{t} & \text{si } f \in D \\ \dfrac{1}{t} & \text{dans les autres cas.} \end{cases}$$

Il reste alors à vérifier, dans les deux cas, l'égalité du lemme sur $M - \{0\}$, ce qui n'est pas difficile.

Preuve de la proposition.

a) Démontrons que les $\mathcal{D}, \mathcal{W}, v$ des types $A_\alpha, AB, B_+, B_-, B_0, C$ vérifient les conditions $(W), (W')_{\geq 2}, (V), (V')$.

Le cas A_α

Choisissons $h_j \in D_j - \{0\}$ $(j = 1, \ldots, \alpha)$ et $g \in M - \cup_{j=1}^{\alpha} D_j$. Soit p un entier strictement positif tel que

$$\frac{p}{1+r_j} \in \mathbb{Z} \qquad (j = 1, \ldots, \alpha).$$

Posons

$$n_{ij} = -p\frac{1+\delta_{ij}}{1+r_j} \in \mathbb{Z}, \qquad m_i = p\left(t + \frac{1}{1+r_i}\right) \in \mathbb{N}$$

$$\text{et} \quad f_i = h_1^{n_{i1}} \cdot \cdots \cdot h_\alpha^{n_{i\alpha}} \cdot g^{m_i} \in \mathcal{P} \cap A(\mathcal{D}) \quad (i, j = 1, \ldots, \alpha).$$

On a

$$\begin{aligned} w_j(f_i) &= -\sum_{k=1}^{\alpha} n_{ik} + n_{ij}(1+r_j) - m_i \\ &= p\left(\sum_{k=1}^{\alpha} \frac{1+\delta_{ik}}{1+r_k} + \frac{1+\delta_{ij}}{1+r_j}(1+r_j) - t - \frac{1}{1+r_i}\right) \\ &= p\left(t - 1 + \frac{1}{1+r_i} + 1 + \delta_{ij} - t - \frac{1}{1+r_i}\right) = p\delta_{ij}. \end{aligned}$$

D'où aussitôt (W). Si $\alpha \geq 2$, alors $g_j = \prod_{i \neq j} f_i \in \mathcal{P} \cap A(\mathcal{D})$, et

$$w_i(g_j) \begin{cases} >0 & \text{si} \quad i \neq j \\ =0 & \text{si} \quad i = j, \end{cases}$$

d'où $(W')_{\geq 2}$.

Du lemme précédent résulte l'existence de nombres rationnels $\lambda_j > 0$ $(j = 1, \ldots, \alpha)$ et d'une application $z : \mathcal{P} \to \mathbb{Q}$, positive sur $\mathcal{P} \cap A(\mathcal{D})$, tels que $v = \lambda_1 w_1 + \cdots + \lambda_\alpha w_\alpha + z$. Les propriétés (V) et (V') en résultent aussitôt.

Le cas AB

Choisissons $D \in \mathbb{P}_1 - (\mathcal{D} \cup \{D_1\})$, $h \in D_1 - \{0\}$ et $g \in D - \{0\}$. Soit p un entier strictement positif tel que $pr_1, pr_2 \in \mathbb{Z}$. Posons $f_1 = h^p g^{pr_2}$ et $f_2 = h^{-p} g^{-pr_2}$. On a

$f_1, f_2 \in \mathcal{P} \cap A(\mathcal{D})$ et on vérifie facilement que $w_i(f_j) = \delta_{ij} p(r_1 - r_2)$ $(i, j = 1, 2)$. Puisque par hypothèse $r_2 < r_1$, les propriétés (W) et $(W')_{\geq 2}$ en résultent.

Si $v = v(D_1, r)$, et si

$$\lambda_1 = \frac{r - r_2}{r_1 - r_2} \quad \text{et} \quad \lambda_2 = \frac{r_1 - r}{r_1 - r_2}$$

on vérifie sans peine que $v = \lambda_1 w_1 + \lambda_2 w_2$. Puisque par hypothèse $r_2 < r < r_1$, on a $\lambda_1 > 0$ et $\lambda_2 > 0$, d'où aussitôt les propriétés (V) et (V').

Le cas B_+

Choisissons $D \in \mathbb{P}_1 - \mathcal{D}$ et $g \in D - \{0\}$. On a $g^{-1} \in \mathcal{P} \cap A(\mathcal{D})$ et $w_1(g^{-1}) = 1 > 0$, d'où la propriété (W).

Si $v = v(D_1, r)$, on vérifie sans peine que $v = w_1 + z$, où $z : \mathcal{P} \to \mathbb{Q}$ est l'homomorphisme de groupes déterminé par

$$z(f) = \begin{cases} r - r_1 & \text{si} \quad f \in D_1 - \{0\} \\ 0 & \text{si} \quad f \in M - D_1. \end{cases}$$

Puisque par hypothèse $r_1 < r$, les propriétés (V) et (V') en résultent aussitôt.

Les cas B_- et B_0

Si $h \in D_1 - \{0\}$, on a $w_1(h) = r_1 > 0$, d'où la propriété (W).

Si $v = v(D_1, r)$, on vérifie sans peine que $v = (r/r_1)w_1 + z$, où $z : \mathcal{P} \to \mathbb{Q}$ est l'homomorphisme de groupes déterminé par

$$z(f) = \begin{cases} \dfrac{r - r_1}{r_1} & \text{si} \quad f \in M - D_1 \\ 0 & \text{si} \quad f \in D_1 - \{0\}. \end{cases}$$

Par hypothèse, on a $0 < r_1 < r$, d'où aussitôt les propriétés (V) et (V').

Le cas C est banal

b) Démontrons que les $\mathcal{D}, \mathcal{W}, v$ qui vérifient les propriétés $(W), (W')_{\geq 2}, (V), (V')$ sont forcément de l'un des types $A_\alpha, AB, B_+, B_-, B_0, C$.

Posons $\mathcal{W} = \{w_1, \ldots, w_\alpha\}$ et $w_j = v(D_j, r_j)$ $(j = 1, \ldots, \alpha)$. On peut supposer que $D_1, \ldots, D_\beta$ $(\beta \leq \alpha)$ sont les différentes droites de M sur lesquelles au moins un des w_i n'est pas égal à -1. On peut supposer également, pour tout $i = 1, \ldots, \beta$, que r_i est maximal parmi les r_j tels que $D_j = D_i$. Choisissons $h_i \in D_i - \{0\}$ $(i = 1, \ldots, \beta)$. Enfin posons $v = v(D_0, r_0)$.

Supposons d'abord $\beta \geq 2$.

Montrons que $\mathbb{P}_1 - \{D_1, \ldots, D_\beta\} \subset \mathcal{D}$. Raisonnons par l'absurde: supposons qu'il existe $g \in M - \bigcup_{i=1}^{\beta} D_i$ tel que $g \in \mathcal{P}(\mathcal{D})$. Puisque $\beta \geq 2$, on peut trouver $i \in [1, \beta]$ tel que $D_i \neq D_0$. On a $h_i g^{-1} \in \mathcal{P} \cap A(\mathcal{D})$ et

$$w_j(h_i g^{-1}) = \begin{cases} 1 + r_j & \text{si} \quad w_j \in v(D_i,]-1, 1]) \\ 0 & \text{sinon.} \end{cases}$$

Par suite, $h_i g^{-1} \in \mathcal{P} \cap A(\mathcal{D}, \mathcal{W})$. Si $g \in D_0$, on aurait $v(h_i g^{-1}) = -1 - r_0 < 0$, ce qui contredit (V). Si $g \notin D_0$, on aurait $v(h_i g^{-1}) = 0$ et $v_i(h_i g^{-1}) = 1 + r_i > 0$, ce qui contredit (V').

Supposons $\beta = 2$ et $r_1 = r_2 = 1$. Puisque $\mathbb{P}_1 - \{D_1, D_2\} \subset \mathcal{D}$, tout $f \in \mathcal{P} \cap A(\mathcal{D})$ peut s'écrire $f = h_1 h_2 g$, où $n_1, n_2 \in \mathbb{Z}$ et où $g \in \mathcal{P} \cap k[SL(2)]$ vérifie $v_{D_1}(g) = v_{D_2}(g) = 0$. Pour tous ces f, on a $w_1(f) + w_2(f) = 2w_1(g) \leq 0$, ce qui contredit (W). Par suite, r_1 ou r_2 est <1.

D'après $(W')_{\geq 2}$, il existe $f_i \in \mathcal{P} \cap A(\mathcal{D})$ $(i = 1, \ldots, \beta)$ tels que

$$w_j(f_i) \begin{cases} >0 & \text{si} \quad i \neq j \\ =0 & \text{si} \quad i = j. \end{cases}$$

Puisque $\mathcal{D} \supset \mathbb{P}_1 - \{D_1, \ldots, D_\beta\}$, on peut écrire

$$f_i = h_1^{n_{i1}} \cdot \cdots \cdot h_\beta^{n_{i\beta}} \cdot g_i$$

où $n_{ij} \in \mathbb{Z}$ et où $g_i \in \mathcal{P} \cap k[SL(2)]$ vérifie $v_{D_j}(g_i) = 0$ $(i, j = 1, \ldots, \beta)$. Posons $m_i = w_1(g_i) = \cdots = w_\beta(g_i) \leq 0$ et $N_i = -\sum_{j=1}^{\beta} n_{ij} + m_i$. On a $w_j(f_i) = (1 + r_j)n_{ij} + N_i$. Par suite,

$$0 \geq m_i = N_i + \sum_{j=1}^{\beta} n_{ij} > N_i\left(1 - \sum_{j=1}^{\beta} \frac{1}{1 + r_j}\right) = -tN_i.$$

Puisque nous savons déjà que $t > 0$, il s'ensuit que $N_i > 0$. Par conséquent,

$$n_{ii} = \frac{-N_i}{1 + r_i} < 0 \qquad (i = 1, \ldots, \beta),$$

ce qui montre que $\mathcal{D} = \mathbb{P}_1 - \{D_1, \ldots, D_\beta\}$.

Si $\beta<\alpha$, d'après le lemme, pour tout $j=\beta+1,\ldots,\alpha$, il existerait des $\lambda_{ij}>0$ $(i=1,\ldots,\beta)$ et un homomorphisme de groupes $z_j:\mathcal{P}\to\mathbb{Q}$, positif sur $\mathcal{P}\cap A(\mathcal{D})$, vérifiant $w_j=\lambda_{1j}w_1+\cdots+\lambda_{\beta j}w_\beta+z_j$. Manifestement cela contredirait $(W')_{\geq 2}$. Donc $\alpha=\beta$.

Choisissons un entier strictement positif p tel que

$$n_j=\frac{-p}{1+r_j}\in\mathbb{Z}\qquad (j=1,\ldots,\beta).$$

Choisissons $g\in M-\bigcup_{j=1}^{\beta}D_j$, et posons $f=h_1^{n_1}\cdot\cdots\cdot h_\beta^{n_\beta}\cdot g^{pt}$. On vérifie sans peine que $w_j(f)=0$ $(j=1,\ldots,\beta)$. Si $D_0=D_i$, de (V) résulte alors que

$$0\leq v(f)=\frac{p}{1+r_i}(r_i-r_0).$$

Puisque $\beta\geq 2$, on a $v\notin\mathcal{W}$, d'où $r_0<r_i$.

En résumé, si $\beta\geq 2$, nous avons montré que $\alpha=\beta$ et que $\mathcal{D},\mathcal{W},v$ est du type A_α.

Supposons maintenant $\beta=1$ et $\alpha\geq 2$.

Quitte à renuméroter $w_1,\ldots,w_\alpha$, on peut supposer que $w_j=v(D_1,r_j)$, où $-1\leq r_\alpha<r_{\alpha-1}<\cdots<r_2<r_1\leqslant 1$. Pour tout $i=2,\ldots,\alpha-1$, l'égalité

$$w_i=\frac{r_i-r_\alpha}{r_1-r_\alpha}w_1+\frac{r_1-r_i}{r_1-r_\alpha}w_\alpha$$

n'est alors pas compatible avec $(W')_{\geq 2}$. Il s'ensuit que $\alpha=2$.

Montrons que $D_1\notin\mathcal{D}$ et que $\mathcal{D}\neq\mathbb{P}_1-\{D_1\}$. D'après $(W')_{\geq 2}$, il existe $f_1,f_2\in\mathcal{P}\cap A(\mathcal{D})$ tels que

$$w_i(f_j)\begin{cases}>0 & \text{si}\quad i\neq j\\ =0 & \text{si}\quad i=j.\end{cases}$$

Posons $f_i=h_1^{n_i}g_i$ $(i=1,2)$, où $n_1,n_2\in\mathbb{Z}$ et où $g_1,g_2\in\mathcal{P}$ vérifient $v_{D_1}(g_1)=v_{D_1}(g_2)=0$. De $w_1(f_1)<w_2(f_1)$, on déduit, puisque $w_1(g_1)=w_2(g_1)$, que $n_1r_1<n_1r_2$; d'ou, puisque $r_2<r_1$, que $n_1<0$. Il s'ensuit que $D_1\notin\mathcal{D}$. De $w_2(f_2)<w_1(f_2)$, on déduit de la même façon que $n_2>0$. Si $r_1>0$, de $0=w_1(f_1)=n_1r_1+w_1(g_1)$ résulte $w_1(g_1)>0$, d'où $g_1\notin k[SL(2)]$. Si $r_1\leq 0$, de $r_2<r_1$ et de $0=w_2(f_2)=n_2r_2+w_2(g_2)$ résulte $w_2(g_2)>0$, d'où $g_2\notin k[SL(2)]$. Dans les deux cas, il s'ensuit que $\mathcal{D}\neq\mathbb{P}_1-\{D_1\}$.

Montrons que $v \in v(D_1,]r_2, r_1[)$. Choisissons un entier strictement positif p tel que $pr_1, pr_2 \in \mathbb{Z}$. Choisissons $g \in M - D_1$ tel que $g \in \mathcal{P}(\mathcal{D})$. Lorsque $D_0 = D_1$, de $w_1(h_1^p g^{pr_2}) = p(r_1 - r_2) > 0$ et $w_2(h_1^p g^{pr_2}) = 0$ résulte, grâce à (V), que $v(h_1^p g^{pr_2}) = p(r_0 - r_2) \geq 0$, c'est-à-dire $r_2 \leq r_0$; de $w_1(h^{-p} g^{-pr_1}) = 0$ et $w_2(h_1^{-p} g^{-pr_1}) = p(r_1 - r_2) > 0$ résulte, grâce à (V) que $v(h_1^p g^{pr_2}) = p(r_0 - r_2) \geq 0$, c'est-à-dire $r_0 \leq r_1$. Montrons que $D_0 \neq D_1$ n'est pas possible: en effet, si $g \in D_0$, on a $v(h_1^p g^{pr_2}) = p(r_0 r_2 - 1) < 0$ ce qui contredit (V); si $g \notin D_0$, on a

$$v(h_1^p g^{pr_2}) = -p(1 + r_2) \begin{cases} < 0 & \text{si} \quad -1 < r_2 \\ = 0 & \text{si} \quad -1 = r_2 \end{cases}$$

ce qui contredit (V) ou (V').

En résumé, si $\beta = 1$ et $\alpha \geq 2$, nous avons montré que $\alpha = 2$ et que $\mathcal{D}, \mathcal{W}, v$ est du type AB.

Il reste à examiner le cas $\alpha = 1$.

Supposons d'abord que $w_1 = v(D_1, r_1)$ vérifie $-1 < r_1$. Choisissons un entier strictement positif tel que $pr_1 \in \mathbb{Z}$.

Supposons qu'il existe $D \in \mathbb{P}_1 - (\mathcal{D} \cup \{D_1\})$. Choisissons $g \in D - \{0\}$. On a $h_1^p g^{pr_1} \in \mathcal{P} \cap A(\mathcal{D})$ et $w_1(h_1^p g^{pr_1}) = 0$. Il en résulte que $D_0 \neq D_1$ n'est pas possible: en effet, si $g \in D_0$, on aurait $v(h_1^p g^{pr_1}) = p(r_1 r_0 - 1) < 0$, et si $g \notin D_0$, on aurait $v(h_1^p g^{pr_1}) = -p(1 + r_1) < 0$, deux inégalités qui contredisent (V). Si $D_0 = D_1$, de (V) résulte que $v(h_1^p g^{pr_1}) = p(r_0 - r_1) \geq 0$, c'est-à-dire $r_0 \geq r_1$; si $r_0 > r_1$, on a $D_1 \in \mathcal{D}$ (en effet, sinon $h_1^{-p} g^{-pr_1} \in \mathcal{P} \cap A(\mathcal{D})$, $w_1(h_1^{-p} g^{-pr_1}) = 0$ et $v(h_1^{-p} g^{-pr_1}) = p(r_1 - r_0) < 0$, ce qui contredit (V)); autrement dit, nous sommes en type C ou B_+.

Supposons que $\mathcal{D} = \mathbb{P}_1 - \{D_1\}$. Choisissons $g \in M - D_1$. Si $r_1 < 0$, et si $D_0 = D_1$, on a $h_1^{-p} g^{-pr_1} \in \mathcal{P} \cap A(\mathcal{D})$ et $w_1(h_1^{-p} g^{-pr_1}) = 0$, donc d'après (V) on a $v(h_1^{-p} g^{-pr_1}) = p(r_1 - r_0) \geq 0$, c'est-à-dire $r_0 \leq r_1$ et nous sommes en type C ou A_1. Le cas $r_1 = 0$ n'est pas possible: en effet, on aurait alors $w_1(h_1^n g) \leq 0$ quel que soit $n \in \mathbb{Z}$ et $g \in \mathcal{P} \cap k[SL(2)]$, ce qui contredit (W). Si $r_1 > 0$, on a $h_1^p g^{pr_1} \in \mathcal{P} \cap A(\mathcal{D})$ et $w_1(h_1^p g^{pr_1}) = 0$. Comme plus haut, de (V) résulte alors que $D_0 = D_1$ et $r_0 \geq r_1$, et nous sommes en type C ou B_-.

Supposons $\mathcal{D} = \mathbb{P}_1$. Si $r_1 \leq 0$, w_1 resterait négatif ou nul sur $\mathcal{P} \cap k[SL(2)]$, ce qui contredit (W), donc $r_1 > 0$. Choisissons $g \in M - D_1$. De $w_1(h_1^p g^{pr_1}) = 0$ et de (V) on déduit alors comme plus haut que $D_0 = D_1$ et $r_0 \geq r_1$. Nous sommes donc en type C ou B_0.

Enfin, considérons le cas $w_1 = v(\ , -1)$. Choisissons $h_0 \in D_0 - \{0\}$ et $g \in M - D_0$. Puisque $v(\ , -1)$ est négatif ou nul sur $\mathcal{P} \cap k[SL(2)]$, $\mathcal{D} \neq \mathbb{P}_1$. Montrons que $D_0 \in \mathcal{D}$: en effet, sinon $gh_0^{-1} \in \mathcal{P} \cap A(\mathcal{D})$, $w_1(gh_0^{-1}) = 0$ et $v(gh_0^{-1}) = -(1 + r_0) < 0$, ce qui contredit (V). Par conséquent, nous sommes en type B_+.

9.4. Soient $\mathscr{D}, \mathscr{W}, v$ comme dans le numéro précédent. Supposons que $\mathscr{D}, \mathscr{W}, v$ vérifient $(W), (W')_{\geq 2}, (V), (V')$. D'après 9.2, $\mathscr{D}, \mathscr{W}$ vérifient aussi (F). D'autre part, $SL(2)/\{e\}$ étant un espace homogène affine, $\mathscr{D}, \mathscr{W}$ vérifient aussi (D) et $(W')_1$. Par conséquent, d'après 8.8, on peut associer à $\mathscr{D}, \mathscr{W}, v$ un $l \in \mathfrak{L}_1^n(SL(2)/\{e\})$ tel que $\mathscr{V}_l = \mathscr{W}$ et ${}^B\mathscr{D}_l = \mathscr{D}(\mathscr{W}, v)$.

PROPOSITION. *Si $\mathscr{D}, \mathscr{W}, v$ sont du type A_α (resp. AB, B_+, B_-, B_0), alors ${}^B\mathscr{D}_l = \mathbb{P}_1 - \{D_1, \ldots, D_\alpha\}$ (resp. $\varnothing, \{D_1\}, \mathbb{P}_1 - \{D_1\}, \mathbb{P}_1$).*

Preuve. Ces assertions se déduisent machinalement à partir de la définition de $\mathscr{D}(\mathscr{W}, v)$ et de 9.3.

Compte tenu de 8.8 et 9.3, la proposition précédente constitue une classification des éléments de $\mathfrak{L}_1^n(SL(2)/\{e\})$. Nous utiliserons dans la suite les notations suivantes.

Si $D_1, \ldots, D_\alpha$ sont des éléments différents de $\mathbb{P}_1$, et si $r_1, \ldots, r_\alpha$ vérifient

$$-1 < r_j \leq 1 \ (j = 1, \ldots, \alpha) \quad \text{et} \quad \sum_{j=1}^{\alpha} \frac{1}{1+r_j} > 1, \qquad l(D_1, r_1; \ldots; D_\alpha, r_\alpha)$$

sera l'élément $l \in \mathfrak{L}_1^n(SL(2)/\{e\})$ tel que ${}^B\mathscr{D}_l = \mathbb{P}_1 - \{D_1, \ldots, D_\alpha\}$ et $\mathscr{V}_l = \{v(D_1, r_1), \ldots, v(D_\alpha, r_\alpha)\}$.

Si $D \in \mathbb{P}_1$ et si $-1 \leq r_2 < r_1 \leq 1$, $l(D, r_2, r_1)$ sera l'élément $l \in \mathfrak{L}_1^n(SL(2)/\{e\})$ tel que ${}^B\mathscr{D}_l = \varnothing$ et $\mathscr{V}_l = \{v(D, r_1), v(D, r_2)\}$.

Si $D \in \mathbb{P}_1$ et si $-1 \leq r < 1$, $l_+(D, r)$ sera l'élément $l \in \mathfrak{L}_1^n(SL(2)/\{e\})$ tel que ${}^B\mathscr{D}_l = \{D\}$ et $\mathscr{V}_l = \{v(D, r)\}$.

Si $D \in \mathbb{P}_1$ et si $0 < r < 1$, $l_-(D, r)$ (resp. $l_0(D, r)$) sera l'élément $l \in \mathfrak{L}_1^n(SL(2)/\{e\})$ tel que ${}^B\mathscr{D}_l = \mathbb{P}_1 - \{D\}$ (resp. $\mathbb{P}_1$) et $\mathscr{V}_l = \{v(D, r)\}$.

D'après ce qui précède, l'ensemble $\mathfrak{L}_1^n(SL(2)/\{e\})$ est composé des $l(D_1, r_1; \ldots; D_\alpha, r_\alpha)$, $l(D, r_2, r_1)$, $l_+(D, r)$, $l_-(D, r)$, $l_0(D, r)$ et de $\mathscr{V}_1(SL(2)/\{e\})$.

9.5. La proposition suivante décrit les facettes des éléments de $\mathfrak{L}_1^n(SL(2)/\{e\})$.

PROPOSITION 1. *Si $l = l(D_1, r_1; \ldots; D_\alpha, r_\alpha)$, alors*

$$\mathscr{F}_l = \bigcup_{D \in \mathbb{P}_1 - \{D_1, \ldots, D_\alpha\}} v(D,]-1, 1]) \cup \bigcup_{j=1}^{\alpha} v(D_j,]-1, r_j[).$$

Si $l = l(D, r_2, r_1)$, alors $\mathscr{F}_l = v(D,]r_2, r_1[)$.

Si $l = l_+(D, r)$ (ou $l_-(D, r)$, ou $l_0(D, r)$), alors $\mathscr{F}_l = v(D,]r, 1])$.

Preuve. Ces assertions résultent directement du corollaire 3 de 8.8 et de 9.3.

Désignons par $\mathfrak{L}'$ l'ensemble des $l_+(D,-1), D \in \mathbb{P}_1$. La proposition suivante précise la topologie de Zariski de $\mathfrak{L}_1^n(SL(2)/\{e\})$.

PROPOSITION 2. a) *Les* $l(D_1, r_1; \ldots; D_\alpha, r_\alpha)$, $l_-(D, r)$, $l_0(D, r)$ *appartiennent à* $\mathfrak{L}_f^n(SL(2)/\{e\})$. *Pour que* $l(D, r_2, r_1)$ (*resp.* $l_+(D, r)$) *appartienne à* $\mathfrak{L}_f^n(SL(2)/\{e\})$, *il faut et il suffit que* $-1 < r_2$ (*resp.* $-1 < r$).

b) *Si* $l \in \mathfrak{L}_f^n(SL(2)/\{e\})$, *l'ouvert de* $\mathfrak{L}_f^n(SL(2)/\{e\})$ *qui correspond à la réalisation géométrique minimale de* l *est* $\{l\} \cup \mathcal{V}_l$.

c) *Si* $l = l(D, -1, r_1)$ (*resp.* $l_+(D, -1)$), *on obtient un système fondamental de voisinages de* l *dans* $\mathfrak{L}_1(SL(2)/\{e\})$ *par* $\{l\} \cup \{v(D, r_1)\} \cup L$ (*resp.* $\{l\} \cup L$), *où* L *parcourt les sous-ensembles cofinis de* $\mathfrak{L}'$.

Preuve. Ces assertions résultent directement de 8.9.

Remarque. Ce sont les réalisations géométriques minimales des $l_0(D, r)$ qui sont les seuls plongements normaux *affines* de $SL(2)$, étudiés par V. L. Popov dans [10].

9.6. D'après le §6, la donnée d'un plongement X de $SL(2)/\{e\}$ équivaut à la donnée d'un sous-ensemble ouvert, noethérien et séparé L de $\mathfrak{L}_1(SL(2)/\{e\})$; il est clair que X sera une variété normale si et seulement si $L \subset \mathfrak{L}_1^n(SL(2)/\{e\})$.

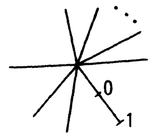
PROPOSITION. *Soit* L *un sous-ensemble de* $\mathfrak{L}_1^n(SL(2)/\{e\})$. *Pour que* L *soit ouvert, noethérien et séparé, il faut et il suffit que* L *vérifie les conditions suivantes.*

1) *Pour tout* $l \in L, \mathcal{V}_l \cap \mathcal{V}_1(SL(2)/\{e\}) \subset L$.
2) *S'il existe* $l \in L$ *tel que* $v(\ ,-1) \in \mathcal{V}_l$, *alors* L *contient un sous-ensemble cofini de* $\mathfrak{L}'$.
3) *L'ensemble* $L - \mathfrak{L}'$ *est fini.*
4) *Les* $\mathcal{F}_l$, $l \in L$ *sont disjoints.*

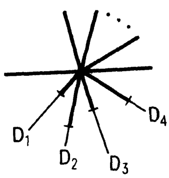
Preuve. D'après la proposition 2 de 9.5, 1) et 2) signifient que L est ouvert. Par définition, 4) signifie que L est séparé. On vérifie alors sans peine, pour que L soit noethérien, qu'il faut et qu'il suffit que L vérifie 3).

Remarque. La condition 4) restreint considérablement le choix des L: par exemple, elle implique que L contient au plus un des $l(D_1, r_1; \ldots; D_\alpha, r_\alpha)$; ou encore que, pour tout $D \in \mathbb{P}_1, L$ contient au plus un parmi les $l_+(D, r)$, $l_-(D, r)$, $l_0(D, r)$; etc...

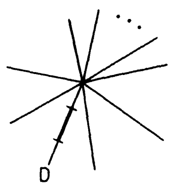
9.7. Terminons ce travail par une présentation graphique de notre classification des plongements normaux de $SL(2)/\{e\}$. Le "support" de la classification est l'ensemble $\mathcal{V}(SL(2)/\{e\})$, qu'on peut dessiner comme suit



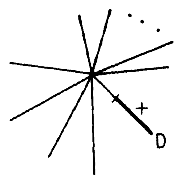
(ce dessin représente autant d'intervalles rationnels $[-1, 1]$ qu'il y a de points dans $\mathbb{P}_1$, recollés par leur extrémité gauche -1). Puisque les localités dans $\mathfrak{L}_1^n(SL(2)/\{e\})$ sont presque déterminées par leur facette, on peut les représenter par les dessins suivants:



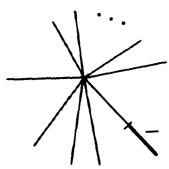
$l(D_1, r_1; \ldots; D_4, r_4)$



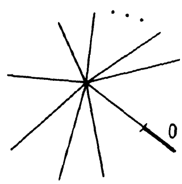
$l(D, r_2, r_1)$



$l_+(D, r)$

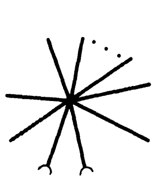


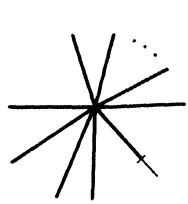
$l_-(D, r)$



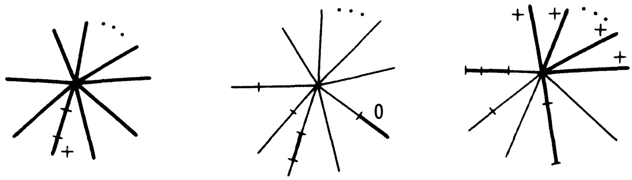
$l_0(D, r)$

N'oublions pas pour les localités $l(D_1, r_1; \ldots; D_\alpha r_\alpha)$ la conditions $\sum_{j=1}^{\alpha} 1/(1+r_j) > 1$: elle signifie que les dessins suivants ne sont pas permis





Les plongements normaux de $SL(2)/\{e\}$ sont alors classés par ce que nous pourrions appeler, si nous voulions suivre la terminologie de Demazure ([5]), des éventails coloriés: il s'agit d'ensembles d'éléments de $\mathfrak{L}_1^n(SL(2)/\{e\})$ satisfaisant aux quatre conditions de 9.6. Ne voulant par répéter ces conditions ici, donnons seulement trois exemples



Le dessin de gauche correspond à un plongement complet, les deux autres à des plongements non complets; le nombre d'orbites du plongement correspondant au dessin de droite est infini, les deux autres plongements contiennent respectivement six et huit orbites.

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On the Kähler form of the moduli space of once punctured tori

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A Riemann surface R of negative Euler characteristic has a unique hyperbolic metric. Provided R has finite area in this metric the Teichmüller space $T(R)$ of R will be a complex manifold. The complex structure of $T(R)$ is characterized by describing the holomorphic cotangent space at R. A natural identification exists of the holomorphic cotangent space and $Q(R)$, the space of holomorphic quadratic differentials on R. Consequently a Hermitian structure on $Q(R)$ naturally gives rise to one on $T(R)$. An example is the Petersson inner product. Given φ, $\psi \in Q(R)$ define

$$\langle \varphi, \psi \rangle = \int_R \varphi \bar{\psi} \lambda^{-2}$$

where λ^2 is the hyperbolic area element of R. The corresponding Hermitian structure on $T(R)$ is that of the Weil–Petersson metric. The metric is invariant with respect to the Teichmüller modular group and hence can be used to study the geometry of the moduli space of R. Ahlfors and Weil established that the metric is Kähler. We are concerned with the Kähler form ω of the metric.

A relationship exists between the geometry of ω and that of the vector fields derived from a construction of Fenchel–Nielsen. A Fenchel–Nielsen vector field $t(\alpha)$ on Teichmüller space is associated to each closed geodesic α of R. In the manuscript [10] the quantity $\omega(t(\alpha), t(\beta))$ is evaluated as the sum of the cosines of the intersection angles of α and β. It is also shown that the vector fields $t(\alpha)$ are Hamiltonian for ω; ω is invariant under the flow of $t(\alpha)$. The form ω and vector fields $t(\alpha)$ are the elements of a symplectic geometry for $T(R)$. The geometry is natural in the sense that ω is invariant with respect to the Teichmüller modular group. The quotient of $T(R)$ by the modular group is the classical moduli space of R. The Kähler form ω projects to the moduli space.

The simplest example of the above discussion is provided in the case of the once punctured torus. In the first section we describe natural global coordinates

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for the Teichmüller space $\mathcal{T}$ of the once punctured torus. These coordinates have no apparent relationship to the complex structure of $\mathcal{T}$. Teichmüller space in these coordinates is a simplex and the modular group $\mathcal{M}$ acts as a group of rational maps. An analysis of the action of $\mathcal{M}$ is given. The discussion is concluded with a description of a fundamental domain Δ. In the second section the Kähler form ω is calculated in the global coordinates and is found to be rational. Using the descriptions of ω and Δ the integral $\int_\Delta \omega$ is computed. It reduces to that of the dilogarithm. The final result is $\pi^2/6$, the area of the moduli space $\mathcal{T}/\mathcal{M}$.

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The Teichmüller space

We begin with an exposition of the Teichmüller theory of the once punctured torus. Our goal is to describe coordinates for the Teichmüller space, and to describe the action of the modular group. The material was in part previously considered by Keen, [3].

A once punctured torus is uniformized by a Fuchsian group Γ, $\Gamma \subset PSL(2;\mathbb{R})$. We shall use the following normalized form for the presentation of Γ. Hyperbolic transformations $A, B \in PSL(2;\mathbb{R})$ freely generate Γ with $ABA^{-1}B^{-1}$ parabolic; the repelling (resp. attracting) fixed point of A is 0 (resp. ∞) and the attracting fixed point of B is 1. In fact the group Γ can be lifted into $SL(2;\mathbb{R})$ such that $\operatorname{tr} A$, $\operatorname{tr} B$, $\operatorname{tr} AB$ become positive, where tr denotes the trace of a matrix. We shall consider Γ both as a subgroup of $SL(2;\mathbb{R})$ and of $PSL(2;\mathbb{R})$ without making the proper distinction. The quantities $x = \operatorname{tr} A$, $y = \operatorname{tr} B$, and $z = \operatorname{tr} AB$ uniquely characterize the above description of Γ. The transformation $ABA^{-1}B^{-1}$ is parabolic. An elementary argument shows that the commutator $ABA^{-1}B^{-1}$ has negative trace and consequently $\operatorname{tr} ABA^{-1}B^{-1} = -2$. The equation $\operatorname{tr} ABA^{-1}B^{-1} = -2$ is equivalent to the identity $x^2 + y^2 + z^2 = xyz$. This is the unique relation satisfied by the triple (x, y, z).

THEOREM (Fricke Klein [2], Keen [3]). *The Teichmüller space $\mathcal{T}$ of the once punctured torus is the sublocus of $x^2 + y^2 + z^2 = xyz$ satisfying $x, y, z > 2$.*

It will be necessary to consider two other coordinate systems for $\mathcal{T}$. We begin by introducing the invariants a, b and c where $a = x/yz$, $b = y/xz$ and $c = z/xy$. Teichmüller space is now the sublocus of $a + b + c = 1$ satisfying $a, b, c > 0$, a simplex. The third coordinate system will be introduced in the next section.

Distinct triples (x, y, z) and $(\tilde{x}, \tilde{y}, \tilde{z})$ may describe conjugate Fuchsian groups and thus isometric punctured tori. We wish to better understand this phenomenon. It is the direct consequence of the non-uniqueness of a choice of generators in the presentation for Γ. The automorphism group of Γ will be used to study the different choices of generators for Γ. Let G be the free group with generators A and B. The automorphism group $\mathrm{Aut}(G)$ of G has generators, σ, P and U where

$$\begin{matrix} \sigma(A)=A^{-1} & P(A)=B & & U(A)=AB \\ \sigma(B)=B & P(B)=A & \text{and} & U(B)=B \end{matrix},$$

[4]. A representation of $\mathrm{Aut}(G)$ in $GL(2;\mathbb{Z})$ is obtained by letting $\mathrm{Aut}(G)$ act on $G/[G, G]\approx\mathbb{Z}\oplus\mathbb{Z}$. Choosing the cosets of A and B as generators for $G/[G, G]$ we have under the representation

$$\sigma\to\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} \quad P\to\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \text{and} \quad U\to\begin{Bmatrix}1 & 1\\ 0 & 1\end{Bmatrix}.$$

We shall be concerned with $\mathrm{Aut}^+(G)$ the preimage of $SL(2;\mathbb{Z})\subset GL(2;\mathbb{Z})$. Denote by $\mathrm{Inn}(G)$ the inner automorphism group of G. The essential properties of the representation are given in the following theorem of Nielsen, [4].

THEOREM. *Let G be the free group on two generators. Then*

$$\mathrm{Out}^+(G)=\mathrm{Aut}^+(G)/\mathrm{Inn}(G)\approx SL(2;\mathbb{Z}).$$

In fact, by the representation

$$\sigma P\to\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad U\to\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

the classical generators of $SL(2;\mathbb{Z})$. By definition the Teichmüller modular group $\mathcal{M}$ for $\mathcal{T}$ is $\mathrm{Out}^+(G)$. The modular group $\mathcal{M}$ does not act effectively on $\mathcal{T}$; the kernel of the action is $\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$. Accordingly $PSL(2;\mathbb{Z})$ does act effectively on $\mathcal{T}$. It is this action that we wish to understand, in particular to describe a fundamental domain.

For clarification we note that the upper half plane H also provides a coordinate system for $\mathcal{T}$ where the action of $\mathcal{M}\approx SL(2;\mathbb{Z})$ is by fractional linear transformations. We shall describe the map from the (x, y, z) coordinates to the H coordinates. To the triple (x, y, z) the point $\tau\in H$ is determined as follows. First let Γ be the Fuchsian group associated to (x, y, z). Corresponding to the commutator sub-group $[\Gamma,\Gamma]\subset\Gamma$ is the abelian covering $H/[\Gamma,\Gamma]$ of H/Γ. The covering surface $H/[\Gamma,\Gamma]$ is planar and conformally equivalent to $\mathbb{C}-L$ for a Euclidean lattice L. A conformal map of $H/[\Gamma,\Gamma]$ to $\mathbb{C}-L$, equivariant with respect to $\Gamma/[\Gamma,\Gamma]$ and L, exists such that the cosets of A and B in $\Gamma/[\Gamma,\Gamma]$ are conjugated into the generators of L. The lattice L is normalized such that its generators are τ and 1 with $\tau\in H$. The quantity τ is the image of the triple (x, y, z).

We now wish to focus our attention on the principal congruence subgroup $\Gamma(2)$ of level 2 in $SL(2;\mathbb{Z})$, where

$$\Gamma(2)=\left\{C\in SL(2;\mathbb{Z})\mid C\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\mathrm{mod}\ 2\right\}.$$

Denote by $P\Gamma(2)$ the image of $\Gamma(2)$ in $PSL(2;\mathbb{Z})$. By an elementary argument the indices satisfy

$$[SL(2;\mathbb{Z}):\Gamma(2)]=[PSL(2;\mathbb{Z}):P\Gamma(2)]=6.$$

We wish to characterize the preimage of $P\Gamma(2)$ in $\mathrm{Aut}\,(G)$. Let $\mathcal{M}_2$ be the subgroup of $\mathrm{Aut}\,(G)$ generated by $\rho_1=\sigma U^2$, $\rho_2=\sigma$ and $\rho_3=P\sigma U^2P$. Under the representation of $\mathrm{Aut}\,(G)$ in $GL(2;\mathbb{Z})$

$$\rho_1\to\begin{pmatrix}-1&-2\\0&1\end{pmatrix}\qquad\rho_2\to\begin{pmatrix}-1&0\\0&1\end{pmatrix}\quad\text{and}\quad\rho_3\to\begin{pmatrix}1&0\\-2&-1\end{pmatrix}.$$

Denote by $\mathcal{M}_2^+$ the intersection $\mathcal{M}_2\cap\mathrm{Aut}^+\,(G)$. Now the representations of ρ_1, ρ_2 and ρ_3 each have negative determinant; consequently $\mathcal{M}_2^+\subset\mathcal{M}_2$ is the subgroup of words in ρ_1, ρ_2 and ρ_3 of even length. Under the representation of $\mathrm{Aut}\,(G)$ we claim that $\mathcal{M}_2^+\subset\Gamma(2)$, and that $\mathcal{M}_2^+$ surjects onto $P\Gamma(2)$. Indeed the representations of ρ_1, ρ_2 and ρ_3 are each congruent to $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ mod 2, the inclusion $\mathcal{M}_2^+\subset\Gamma(2)$ is immediate. The images of $\rho_2\rho_1$ and $\rho_2\rho_3$ are respectively $\begin{pmatrix}1&2\\0&1\end{pmatrix}$ and $\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$ the lifts of the generators of $P\Gamma(2)$. We shall establish below that $\mathcal{M}_2^+$ acts effectively and thus conclude that the natural map $\mathcal{M}_2^+\to P\Gamma(2)$ is a bijection.

First we shall consider the action of $\mathcal{M}_2$ in the (x, y, z) coordinates. We begin with the action of ρ_1,ρ_2 and ρ_3 on the generators A,B of Γ

$$\begin{array}{lll}\rho_1(A)=B^{-2}A^{-1} & \rho_2(A)=A^{-1} & \rho_3(A)=A\\ \rho_1(B)=B & \rho_2(B)=B & \rho_3(B)=A^{-2}B^{-1}\end{array}$$

There is natural induced action on the traces of the generators of Γ. Recalling that $x=\operatorname{tr} A$, $y=\operatorname{tr} B$, $z=\operatorname{tr} AB$ we have

$$\begin{array}{lll}
\rho_1(x)=yz-x & \rho_2(x)=x & \rho_3(x)=x \\
\rho_1(y)=y & \rho_2(y)=y & \rho_3(y)=xz-y \\
\rho_1(z)=z & \rho_2(z)=xy-z & \rho_3(z)=z
\end{array}$$

The following identity has been used: given C, $D\in SL(2;\mathbb{R})$ then $\operatorname{tr} C \operatorname{tr} D=\operatorname{tr} CD+\operatorname{tr} C^{-1}D$. The above triples satisfy $x^2+y^2+z^2=xyz$, x, y, $z>2$. For example consider the triple $(\rho_1(x), \rho_1(y), \rho_1(z))$. By the quadratic formula $2x=yz\pm(y^2z^2-4(y^2+z^2))^{1/2}<2yz$ hence $yz-x$ is positive. From the definition of Aut (G) there exists a unique normalized Fuchsian group $\tilde{\Gamma}$ with generators $\tilde{A}$, $\tilde{B}$ satisfying $|\operatorname{tr} \tilde{A}|=yz-x$, $|\operatorname{tr} \tilde{B}|=y$, $|\operatorname{tr} \tilde{A}\tilde{B}|=z$ and $\operatorname{tr} \tilde{A}$, $\operatorname{tr} \tilde{B}$, $\operatorname{tr} \tilde{A}\tilde{B}>2$. Necessarily we have that $\operatorname{tr} \tilde{A}=yz-x$, $\operatorname{tr} \tilde{B}=y$ and $\operatorname{tr} \tilde{A}\tilde{B}=z$. The description of the $\mathcal{M}_2$ action is complete. We also give the action of ρ_1, ρ_2 and ρ_3 in the (a, b, c) coordinates

$$\begin{array}{lll}
\rho_1(a)=1-a & \rho_2(a)=\dfrac{ca}{1-c} & \rho_3(a)=\dfrac{ba}{1-b} \\[2ex]
\rho_1(b)=\dfrac{ab}{1-a} & \rho_2(b)=\dfrac{cb}{1-c} & \rho_3(b)=1-b \\[2ex]
\rho_1(c)=\dfrac{ac}{1-a} & \rho_2(c)=1-c & \rho_3(c)=\dfrac{bc}{1-b}
\end{array}$$

We shall now describe a fundamental domain for the action of $\mathcal{M}_2$ on $\mathcal{T}$. Consider the domain $\Delta\subset\mathcal{T}$, $\Delta=\{(a, b, c)\in\mathcal{T}\mid a, b, c\leq\frac{1}{2}\}$; Δ will be a fundamental domain for $\mathcal{M}_2$. Indeed this is an immediate consequence of the observation that the function $E(x, y, z)=x+y+z$ achieves a unique minimum on the orbit $\mathcal{M}_2(p)$, $p\in\mathcal{T}$ at the unique point of the orbit in Δ. We shall make the argument in several stages. E is the sum of positive traces from a fixed group Γ; the set of traces of elements of Γ is discrete. The minimum of E on the orbit $\mathcal{M}_2(p)$ exists. For the remaining arguments we require the following elementary formulas

$$\frac{E\circ\rho_1-E}{2yz}=\frac{1}{2}-\frac{x}{yz} \qquad \frac{E\circ\rho_2-E}{2xy}=\frac{1}{2}-\frac{z}{xy} \qquad \frac{E\circ\rho_3-E}{2xz}=\frac{1}{2}-\frac{y}{xz} \tag{1}$$

Now if $q\in\mathcal{M}_2(p)$ represents a minimum then necessarily $E(\rho_j(q))\geq E(q)$, $1\leq j\leq 3$. In particular the formulas (1) show that the coordinates a,b,c of q are each bounded by $\frac{1}{2}$, hence $q\in\Delta$.

First we shall establish a convexity property of E along the orbit $\mathcal{M}_2(p)$. Let $w_n \cdots w_1$ be a reduced word in ρ_1, ρ_2 and ρ_3 (note that $\rho_j = \rho_j^{-1}$, $1 \le j \le 3$). Then we claim the finite sequence $E(s)$, $E(w_1(s)), \ldots, E(w_n \cdots w_1(s))$, $s \in \mathcal{T}$ is strictly convex. Consider the alternative: $n > 2$ and $r = w_j \cdots w_1(s)$, $j < n$ exist with $E(r) \ge E(w_{j-1} \cdots w_1(s)) = E(w_j(r))$ and $E(r) \ge E(w_{j+1}(r))$. Letting the coordinates of r be $(\tilde{a}, \tilde{b}, \tilde{c})$ then the inequalities $E(r) \ge E(w_j(r))$ and $E(r) \ge E(w_{j+1}(r))$ combined with (1) establish that two of the inequalities $\tilde{a} \ge \frac{1}{2}$, $\tilde{b} \ge \frac{1}{2}$ and $\tilde{c} \ge \frac{1}{2}$ necessarily hold, contradicting $(\tilde{a}, \tilde{b}, \tilde{c}) \in \mathcal{T}$. The convexity property is established. Consider now $q, r \in \Delta$ points of the orbit $\mathcal{M}_2(p)$. Using (1) we observe that $E(\rho_j(q)) \ge E(q)$, $E(\rho_j(r)) \ge E(r)$, $1 \le j \le 3$. Now these inequalities and the convexity of E imply that either $q = r$ or $\rho_k(q) = r$ for some k, $1 \le k \le 3$. In the latter case $E(r) = E(\rho_k(q)) \ge E(q)$ and $E(q) = E(\rho_k(r)) \ge E(r)$; the definitions of E and ρ_k show that actually $q = r$. In conclusion the minimum of E on an orbit $\mathcal{M}_2(p)$ occurs at its unique intersection with Δ.

We observe in closing that $\mathcal{M}_2$ acts effectively. Otherwise a nontrivial reduced word $w_n \cdots w_1$, $n > 2$ exists with $E(w_n \cdots w_1(q)) = E(q)$, $q \in \Delta$ contradicting the convexity of E. The subgroup $\mathcal{M}_2^+ \subset \mathcal{M}_2$ necessarily acts effectively; the natural map $\mathcal{M}_2^+ \to P\Gamma(2)$ is a bijection. As a consequence we have that the index $[PSL(2;\mathbb{Z})\colon \mathcal{M}_2^+]$ is 6.

The Kähler form

Our goal is to derive the expression for the Weil–Petersson Kähler form ω in the (a, b, c) coordinates and then intergrate ω over the fundamental domain Δ. We begin with the formula evaluating ω on the Fenchel Nielsen vector fields in terms of the hyperbolic geometry of closed geodesics on a Riemann surface. Then we proceed by a change of variables to obtain the desired formula. Finally the integral of ω over Δ is considered.

We consider only Riemann surfaces R with a hyperbolic, constant curvature -1, metric of finite area. A deformation of the metric R is defined by the following construction. Our point of view throughout is that a fixed topological surface underlies the conformal structure of R. Let γ be a simple closed geodesic. Cut R along γ, rotate one *side* of the cut relative to the other, and then glue the sides together in their new position. Perform this deformation continuously such that the distance between two points, one on each *side* of the cut, is measured by the time elapsed. A tangent vector of this deformation is an infinitesimal twist $t(\gamma)$. Our approach centers on the Fenchel–Nielsen tangents $t(\gamma)$. An important invariant of the hyperbolic metric on R is $l(\gamma)$, the length of the unique geodesic in the free homotopy class of γ.

In the manuscripts [8, 9, 10] the geometry of the quantities ω, $t(\alpha)$ and $l(\beta)$ is investigated. The fundamental formula is $\omega(t(\alpha),\ ) = -dl(\alpha)$. An immediate consequence is that the vector fields $t(\alpha)$ are Hamiltonian for the symplectic form ω. Indeed a symplectic geometry is associated to the quantities ω, $t(\alpha)$ and $l(\alpha)$. We shall only require the following two formulas. Denote by $\alpha \# \beta$ the intersection locus of the geodesics α and β then

$$\omega(t(\alpha), t(\beta)) = t(\alpha)l(\beta)$$

and

$$\omega(t(\alpha), t(\beta)) = \sum_{p\in\alpha\#\beta} \cos\theta_p$$

where θ_p is the intersection angle at p measured from α to β [9, 10].

Let $T(R)$ be the Teichmüller space of R and s its complex dimension. It is classical that free homotopy classes $\gamma_1, \ldots, \gamma_{2s}$ can be chosen such that the lengths $l(\gamma_j)$, $1 \le j \le 2s$ provide local real coordinates for $T(R)$ near R. We begin with an expression for ω in these coordinates.

LEMMA.

$$\omega = \sum_{j<k} W_{kj}\, dl_j \wedge dl_k$$

where the matrix (W_{jk}) *is the inverse of the matrix* $(t(\gamma_j)l(\gamma_k))$.

Proof. We abbreviate t_j for $t(\gamma_j)$ and l_j for $l(\gamma_j)$ and use repeated indices to indicate summation. By the chain rule $t_jl_m(\partial/\partial l_m) = t_j$, $1 \le j \le 2s$ and hence $\partial/\partial l_j = W_{jm}t_m$. Now we calculate

$$\begin{aligned}\omega &= \sum_{j<k} \omega\left(\frac{\partial}{\partial l_j}, \frac{\partial}{\partial l_k}\right) dl_j \wedge dl_k \\ &= \sum_{j<k} \omega(W_{jm}t_m, W_{kn}t_n)\, dl_j \wedge dl_k \\ &= \sum_{j<k} W_{jm}W_{kn}\omega(t_m, t_n)\, dl_j \wedge dl_k.\end{aligned}$$

Using the equation $\omega(t_m, t_n) = t_m l_n$ we proceed

$$\omega = \sum_{j<k} W_{jm} W_{kn} t_m l_n \, dl_j \wedge dl_k$$
$$= \sum_{j<k} W_{kj} \, dl_j \wedge dl_k,$$

and the calculation is complete.

The volume form dV of the Weil–Petersson metric by definition is $(1/s!)\omega^s = (1/s!)\omega \wedge \cdots \wedge \omega$. $\mathcal{T}$ has complex dimension 1; ω is the volume form of $\mathcal{T}$. Note that the Kähler metric on $\mathcal{T} \approx H$ is not complete and thus is not a multiple of the hyperbolic metric, [7]. We now argue that in the (x, y, z) coordinates

$$\omega = 4\frac{dx \wedge dy}{xy - 2z}$$

and then in the (a, b, c) coordinates

$$\omega = \frac{da \wedge db}{abc}.$$

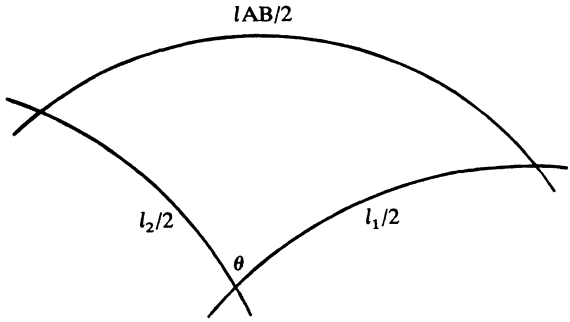
By definition if Γ represents a point of $\mathcal{T}$ with generators A, B then

$$x = \operatorname{tr} A = 2 \cosh l_1/2$$
$$y = \operatorname{tr} B = 2 \cosh l_2/2$$
$$z = \operatorname{tr} AB = 2 \cosh l_{AB}/2$$

where l_* is the length of the appropriate geodesic and x, y are local coordinates for $\mathcal{T}$ provided $xy - 2z \neq 0$. Applying the above lemma we have that

$$\omega = (t_1 l_2)^{-1} \, dl_1 \wedge dl_2$$

where $t_1 l_2 = \cos \theta$, and θ is measured *from* A to B. The following diagram indicates the geometry of the geodesics corresponding to A, B and AB



where the lengths are for the sides of the triangle, [7]. By the law of cosines

$$\cos\theta = \frac{\cosh l_1/2 \cosh l_2/2 - \cosh l_{AB}/2}{\sinh l_1/2 \sinh l_2/2}.$$

and thus

$$\omega = \frac{\sinh l_1/2 \sinh l_2/2}{\cosh l_1/2 \cosh l_2/2 - \cosh l_{AB}/2}\, dl_1 \wedge dl_2.$$

Now $dx = \sinh l_1/2\, dl_1$, $dy = \sinh l_2/2\, dl_2$ and on substituting we obtain

$$\omega = \frac{4\, dx \wedge dy}{xy - 2z}$$

the first expression.

The second expression is obtained by the rational change of variables. Beginning with the formulas $x^{-2} = bc$, $y^{-2} = ac$ and $z^{-2} = ab$ we have $-2\, dx/x = db/b + dc/c$ and $-2\, dy/y = da/a + dc/c$ thus

$$\frac{4\, dx \wedge dy}{xy} = \frac{db \wedge da}{ba} + \frac{db \wedge dc}{bc} + \frac{dc \wedge da}{ca}$$

or using $a + b + c = 1$

$$\frac{4\, dx \wedge dy}{xy} = \frac{a + b - c}{abc}\, da \wedge db.$$

Finally we obtain

$$\omega = \frac{a+b-c}{abc(1-2c)}\, da \wedge db = \frac{da \wedge db}{abc},$$

which is valid throughout $\mathcal{T}$ since ω is real analytic.

We are ready to consider the ω integral

$$\int_{\Delta} \omega = \iint_{\Delta} \frac{da\, db}{ab(1-a-b)}.$$

Using the description of Δ we have

$$\int_{\Delta} \omega = \int_0^{1/2} \int_{1/2-b}^{1/2} \frac{da\, db}{ab(1-a-b)} = \int_0^{1/2} \int_{1/2-b}^{1/2} \left(\frac{1}{a} + \frac{1}{1-b-a}\right) \frac{da\, db}{b(1-b)}$$
$$= \int_0^{1/2} \frac{2}{b(1-b)} \log \frac{1}{1-2b}\, db$$

and substituting $v = 1-2b$ yields the common dilogarithm integral

$$= -4 \int_0^1 \frac{1}{1-v^2} \log v\, dv = \frac{\pi^2}{2}$$

[1].

We complete the calculation by considering the index of $\mathcal{M}_2^+ \subset PSL(2;\mathbb{Z})$. The form ω is $\mathcal{M}_2$ invariant and Δ is a fundamental domain for $\mathcal{M}_2$. Using the previously computed indices

$$\int_{\mathcal{T}/\mathcal{M}_2^+} \omega = [\mathcal{M}_2 : \mathcal{M}_2^+] \int_{\mathcal{T}/\mathcal{M}_2} \omega = \pi^2$$
$$\int_{\mathcal{T}/\mathcal{M}} \omega = \frac{1}{[PSL(2;\mathbb{Z}) : \mathcal{M}_2^+]} \int_{\mathcal{T}/\mathcal{M}_2^+} \omega = \frac{\pi^2}{6}$$

the area of moduli space for the once punctured torus.

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Quasiaspherical knots with infinitely many ends

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A smooth n-knot K in S^{n+2} is called *quasiaspherical* [3] if $H_{n+1}(U)=0$ where U is the universal cover of the exterior of K. Let G be a finitely generated group such that $G/G' \approx Z$ and let H be a subgroup of G which is not contained in G'. We say that (G, H) is *unsplitable* if G does not have a free product with amalgamation decomposition $A \ast_F B$ with F finite and H contained in A.[1]

THEOREM 1. *K is quasiaspherical if and only if $(\pi_1(S^{n+2}-K), H)$ is unsplitable, where H is the subgroup generated by a meridian.*

The "only if" part of this theorem was proved by Swarup [7]. A sketch of the "if" part was given in [2]; for the sake of completeness we give the details in § 1.

A knot K has *infinitely many ends* if for each integer m there is a compact set in U whose complement has more than m components with non compact closure.

The property of having infinitely many ends depends only on $\pi_1(S^{n+2}-K)$.

THEOREM 2. [5]. *K has infinitely many ends if and only if either*

(i) *$\pi_1(S^{n+2}-K) = A \ast_F B$ where F is finite; or*

(ii) *$\pi_1(S^{n+2}-K) = A \underset{F}{\leftarrow\!\supset} \phi$ where F is finite and properly contained in A and $\phi: F \to A$ is a monomorphism.*[2]

Therefore, a knot which is not quasiaspherical has infinitely many ends. There are examples of n-knots which are not quasiaspherical, for $n \geq 2$ [2] [4].

Ratcliffe conjectures ([4, p. 323], [3, Problem 3]) that n-knots with infinitely many ends are not quasiaspherical. We give counter-examples to this conjecture for $n \geq 2$. Thus, by the results of Lomonaco [3; Theorem 10.1], even in the class

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[1] Whenever we write $A \ast_C B$ it is understood that C is a proper subgroup of A and B.

[2] The HNN group $A \underset{F}{\leftarrow\!\supset} \phi$ is $(A \ast \|t: -\|)/N$, where N is the normal closure of $\{tft^{-1}\phi(f)^{-1} : f \in F\}$. Here $\|t: -\|$ is an infinite cyclic group generated by t.

of infinitely many ended knots there are knots for which the homotopy type of the complement is determined by its algebraic 2-type.

First we obtain sufficient conditions for a pair $(A \underset{F}{\hookleftarrow} \phi, H)$ to be unsplitable; then we realize geometrically examples of such pairs. An affirmative answer to the question we ask in § 1 would characterize unsplitable pairs $(A \underset{F}{\hookleftarrow} \phi, H)$. We settle it when A has at most one end and H is generated by the stable letter. In § 2 we construct a 2-knot whose group is $(Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\hookleftarrow} \psi$ where $Z_m \cup \psi(Z_m)$ generates the semidirect product $Z_m \ltimes Z_{2^m-1}$, a meridian being represented by the stable letter. Using § 1 one shows that this is a quasiaspherical knot with infinitely many ends.

We thank Professor Milnor for his comments on the paper.

§ 1. Algebraic part

Let G be a finitely generated group and let H be a subgroup of G. Viewing ZG as a left G-module by left multiplication, we consider the restriction homomorphism $r: H^1(G; ZG) \to H^1(H; ZG)$. Swarup [7, Th. 4] proved:

PROPOSITION 1. *If $r: H^1(G; ZG) \to H^1(H; ZG)$ is not injective then $G = A \underset{F}{*} B$ or $G = A \underset{F}{\hookleftarrow} \phi$ where F is finite and $H \subset A$.*

The converse of this theorem is valid [10, Theorem 5.2]:

PROPOSITION 2. *If $G = A \underset{F}{*} B$ or $G = A \underset{F}{\hookleftarrow} \phi$ with F finite and if $H \subset A$ then $r: H^1(G; ZG) \to H^1(H; ZG)$ is not injective.*

COROLLARY 1. *Let G be a finitely generated group such that $G/G' \approx Z$ and let H be a subgroup of G such that $H \not\subset G'$. Then (G, H) is unsplitable if and only if the restriction $r: H^1(G; ZG) \to H^1(H; ZG)$ is injective.*

Proof. G cannot be of the form $A \underset{F}{\hookleftarrow} \phi$ with $H \subset A$ because $A \subset G'$. The result then follows from Propositions 1 and 2.

Now if U is the universal cover of the exterior of a knot K then using the exact sequence of $(U, \partial U)$, Poincaré duality and the isomorphisms $H^1_c(U) \approx H^1(G; ZG)$ $H^1_c(\partial U) \approx H^1(H; ZG)$ it follows that $H_{n+1}(U)$ is isomorphic to the kernel of r.

From these observations and Corollary 1, Theorem 1 follows.

If $G = A \underset{F}{*} B$, where F is finite, we say that A is a *factor* of G.

In the remainder of this section we let $G = A \underset{F}{\hookleftarrow} \phi$ where F is finite and

$G/G' \approx Z$, let $m = a_0 t^{\varepsilon_1} a_1 \cdots t^{\varepsilon_n} a_n$ $a_i \in A$ $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \varepsilon_i = 1$ and let H be the (infinite cyclic) subgroup of G generated by m.

PROPOSITION 3. *Let C be the subgroup of A generated by $F \cup \phi(F) \cup \{a_0, \ldots, a_n\}$. If C is a finite proper subgroup of A or if C is contained in a factor of A then (G, H) is not unsplitable.*

Proof. Suppose C is a finite proper subgroup of A. Then the homomorphism from $G = A \underset{F}{\hookleftarrow} \phi$ to $(C \underset{F}{\hookleftarrow} \phi) \underset{C}{*} A$ whose restriction to A is the natural inclusion and which sends the stable letter of $A \underset{F}{\hookleftarrow} \phi$ to the stable letter of $C \underset{F}{\hookleftarrow} \phi$ is easily seen to be an isomorphism. Since $C \underset{F}{\hookleftarrow} \phi$ contains the image of H it follows that (G, H) is not unsplitable.

Similarly one shows that if C is contained in a factor P of $A = P \underset{E}{*} Q$ then there is an isomorphism from G onto $(P \underset{F}{\hookleftarrow} \phi) \underset{E}{*} Q$ where E is finite and H is mapped into $P \underset{F}{\hookleftarrow} \phi$.

Question. Is the converse of Proposition 3 valid?

A partial answer is the following:

THEOREM 3. *Let $G = A \underset{F}{\hookleftarrow} \phi$ where F is finite and $G/G' \approx Z$; let H be the subgroup generated by the stable letter t and let C be the subgroup of A generated by $F \cup \phi(F)$. Assume*

(i) *A has at most one end, and*

(ii) *C is not a finite proper subgroup of A. Then (G, H) is unsplitable.*

Proof. Associated to a *HNN*-group there is a natural exact sequence of cohomology groups [1, Th. 3.1]. The homomorphism of the *HNN* group $H = 1 \underset{1}{\hookleftarrow}$ to the *HNN* group $G = A \underset{F}{\hookleftarrow} \phi$ sending the stable letter t of H to the stable letter t of G induces a commutative diagram with exact rows

$$\begin{tikzcd}
 & ZG \arrow[r, "(1-t)\cdot"] \arrow[d, equal] & ZG \arrow[d, equal] & & \\
0 \arrow[r] & H^0(1; ZG) \arrow[r, "(1-t)\cdot"] & H^0(1; ZG) \arrow[r] & H^1(H; ZG) \arrow[r] & 0 \\
0 \arrow[r] & H^0(A; ZG) \arrow[r, "(1-t)\cdot"] \arrow[u, "i"] \arrow[d, equal] & H^0(F; ZG) \arrow[r] \arrow[u, "j"] \arrow[d, equal] & H^1(G; ZG) \arrow[r] \arrow[u, "r"] & H^1(A; ZG) = 0 \\
 & (ZG)^A \arrow[r, "(1-t)\cdot"] & (ZG)^F & &
\end{tikzcd}$$

Here i can be identified with the inclusion of $(ZG)^A$ in ZG and j, with the inclusion of $(ZG)^F$ in ZG. Notice that $H^1(A; ZG) \approx H^1(A; ZA) \otimes_{ZA} ZG = 0$ because A has at most one end [8, page 145].

LEMMA. *Let $w \in ZG$. If $(1-t) \cdot w \in (ZG)^F$ then $w \in (ZG)^C$.*

Proof. Write $w = \sum_{g \in G} n_g \cdot g$. Then $(1-t)w = \sum_{g \in G} m_g \cdot g$ where $m_g = n_g - n_{t^{-1}g}$. Since $(1-t) \cdot w \in (ZG)^F$ we have $m_g = m_{fg}$ that is

$$n_g - n_{t^{-1}g} = n_{fg} - n_{t^{-1}fg}, \qquad g \in G, \qquad f \in G. \tag{$*$}$$

We only need to show (i) $n_g = n_{fg}$ and (ii) $n_g = n_{\phi(f)g}$ for $f \in F$, $g \in G$.

For a sufficiently large k we have $n_{t^{-k}g} = n_{t^{-k}fg} = 0$. From $(*)$ it follows that $n_{t^{-i}g} = n_{t^{-i}fg}$ for $k \geq i \geq 0$. This proves (i).

To prove (ii) notice that $n_{tg} - n_g = n_{tg} - n_{t^{-1}tg} = n_{f(tg)} - n_{t^{-1}f(tg)} = n_{ftg} - n_{\phi(f)g}$. By (i) $n_{tg} = n_{ftg}$. Hence $n_g = n_{\phi(f)g}$. This proves the lemma.

An element $x \in \ker r$ is the image of an element $y \in (ZG)^F$. Then $j(y) = y$ is of the form $(1-t) \cdot w$ where $w \in ZG$. By the lemma $w \in (ZG)^C$. If C is infinite then $w = 0$ so that $x = 0$; if $C = A$ then y is in the image of $(ZG)^A$ and therefore $x = 0$. Hence, r is injective and, by Corollary 1, (G, H) is unsplitable. This completes the proof of the theorem.

§2. Geometric realization

Let L be a smooth n-link in S^{n+2}, $n > 1$, with components $L_1, \ldots, L_r$. L has a unique framing. Denote by N^{n+2} the manifold obtained by surgery on L. Then L is replaced by $M = m_1 \cup \cdots \cup m_r$ where each m_i is a 1-sphere. M has a natural framing so that if we perform surgery on M using this framing we recover S^{n+2}.

If G is a group, a *cyclic word* of G is a subset of G which is the union $[g]$ of the conjugacy classes of g and g^{-1}, for some $g \in G$. The cyclic word of $\pi_1 N^{n+2}$ determined by m_i will also be denoted by m_i and will be called a *meridian*. It corresponds to a meridian of $\pi_1(S^{n+2} - L)$ under the isomorphism $\pi_1(S^{n+2} - L) = \pi_1(N^{n+2} - M) \approx \pi_1(N^{n+2})$. We remark that a finite system of cyclic words $c_1, \ldots, c_r$ of $\pi_1 N$ determines disjoint 1-spheres (which we also denote by $c_1, \ldots, c_r$), well defined up to isotopy, which represent them.

Let (G, m, c) be a triple where G is a group, m is a system of r cyclic words $m_1, \ldots, m_r$ of G, and c is also a system of r cyclic words $c_1, \ldots, c_r$ of G.

If, for some i, we replace c_i by $c_i' = [g_i g_j]$ where $g_i \in c_i$ $g_j \in c_j$ $i \neq j$ we obtain a new system c' of cyclic words of G. We say that (G, m, c') is obtained from (G, m, c) by a *band move*.

If in the triple (G, m, c) some cyclic word m_i of m coincides with a cyclic word c_j of c consider the projection $G \to \hat{G}$ where $\hat{G} = G/\langle m_i \rangle$.[4] Let $\hat{m}$ be the system $\hat{m}_1, \ldots, \hat{m}_{i-1}, \hat{m}_{i+1}, \ldots, \hat{m}_r$ and let $\hat{c}$ be the system $\hat{c}_1, \ldots, \hat{c}_{j-1}, \hat{c}_{j+1}, \ldots, \hat{c}_r$. Then we say that $(\hat{G}, \hat{m}, \hat{c})$ is obtained from (G, m, c) by a *collapse.*

PROPOSITION 4. *Let $c = \{c_1, \ldots, c_r\}$ be a system of cyclic words of $\pi_1 N^{n+2}$; let $m = \{m_1, \ldots, m_r\}$ be the system of meridians of $\pi_1 N^{n+2}$. Assume the triple $(1, \varnothing, \varnothing)$ can be obtained from the triple (G, m, c) by a finite sequence of band moves and collapses. Then, if we perform surgery on $c_1 \cdots c_r$ using suitable framings, we obtain S^{n+2}.*

Proof. Consider the $(n+2)$-manifold $\chi(L_1, L_2, \ldots, L_r;\ c_1, \ldots, c_r)$ obtained from S^{n+2} by surgery on $L_1, L_2, \ldots, L_r$ and then by surgery on $c_1, \ldots, c_r$; the framing of $L_1, \ldots, L_r$ is unique; the framings of $c_1, \ldots, c_r$ are specified later.

A band move on $c_1, \ldots, c_r$ can be realized by a "band move" among the 1-dimensional surgeries. By this we understand the effect on the boundary of a cobordism when we perform handle slidings; these handle slidings do not change the cobordism. Thus if $c' = \{c'_1, \ldots, c'_r\}$ is obtained from $c = \{c_1, \ldots, c_r\}$ by band moves then $\chi(L_1, \ldots, L_r;\ c_1, \ldots, c_r) = \chi(L_1, \ldots, L_r;\ c'_1, \ldots, c'_r)$.

If now some cyclic word of c', say c'_r, equals some cyclic word of m, say m_r, then if we endow m_r with the natural framing $\chi(L_1, \ldots, L_r;\ c'_1, \ldots, c'_{r-1}, m_r) = \chi(L_1, \ldots, L_{r-1};\ c'_1, \ldots, c'_{r-1})$ because the surgeries on L_r and m_r cancel. We want the framings of $c_1, \ldots, c_r$ be such that the framing of c'_r coincides with the framing of m_r. Then we have

$$\chi(L_1, \ldots, L_r; c_1, \ldots, c_r) \approx \chi(L_1, \ldots, L_{r-1}; c'_1, \ldots, c'_{r-1})$$

Proceeding this way we eventually obtain

$$\chi(L_1, \ldots, L_r, c_1, \ldots, c_r) = \chi(\varnothing; \varnothing) = S^{n+2}.$$

This proves the proposition because we can find the framings of $c_1, \ldots, c_r$ working all the process backwards.

Suppose $c_1, \ldots, c_r$ are cyclic words of $\pi_1 N^{n+2}$ such that by a finite sequence of band moves and collapses, it is possible to obtain the triple $(1, \varnothing, \varnothing)$ from $(\pi_1 N; m_1, \ldots, m_r;\ c_1, \ldots, c_r)$. Perform surgery on $c_1 \cup \cdots \cup c_r$ using suitable framings to obtain S^{n+2}. Then $c_1 \cup \cdots \cup c_r$ is replaced by a disjoint union of n-spheres $S_1, \ldots, S_r$ in S^{n+2}.

The following proposition is clear.

---

[4] $\langle\ \rangle$ denotes normal closure.

PROPOSITION 5. *Let* $1 \le k \le r$. *Then* $\bigcup_{i=1}^{k} S_i$ *is a link in* S^{n+2} *with group* $\pi_1 N / \bigcup_{i>k} \langle c_i \rangle$. *The meridian corresponding to* S_i, $i \le k$, *is represented by* c_i.

Remark. This construction of links generalizes the construction introduced in [2, § 1].

Now, we will construct quasiaspherical knots with infinitely many ends. Let $L = L_1 \cup L_2$ be a smooth 2-link in S^4 such that $\pi_1 N^4 \approx \|a, t, x\colon a^m = 1, t^{-1}at = a^{-1}\|$ where m is odd and t, x are the meridians. For example L can be taken to be a split link one of whose components is a 2-twist spun torus knot and the other one is trivial. Now let c_1, c_2 be the cyclic words of $\pi_1 N^4$ represented by xt^{-1} and $a^{-1}xax^{-2}$ respectively. It is easy to find a sequence of band moves changing $\{c_1, c_2\}$ into $\{x, t\}$. According to Proposition 5 there is a knot K_m in S^4 whose group is $\|a, t, x\colon a^m = 1,\ t^{-1}\ at = a^{-1},\ a^{-1}xax^{-2} = 1\| \approx (Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\leftrightarrows} \phi$ where $Z_m \ltimes Z_{2^m-1}$ is the semidirect product $\|a, t\colon a^m = x^{2^m-1} = 1,\ a^{-1}xa = x^2\|$; the domain of ϕ is the subgroup generated by a; and $\phi(a) = a^{-1}$. Moreover xt^{-1} represents a meridian of K_m.

THEOREM 4. *The* 2-*knot* K_m *is quasiaspherical and has infinitely many ends.*

Proof. By Theorem 2 ii) K_m has infinitely many ends. To see that it is quasiaspherical notice that $\pi_1(S^4 - K_m) \approx \|a, x, t\colon a^m = a^{-1}xax^{-2} = 1,\ t^{-1}at = a^{-1}\| \xrightarrow{\overset{f}{\approx}} \|a, x, s\colon a^m = a^{-1}xax^{-2} = 1\ \ s^{-1}as = x^{-1}a^{-1}\| \approx (Z_m \ltimes Z_{2^m-1}) \underset{Z_m}{\leftrightarrows} \psi$ where $f(a) = a$, $f(x) = x$, $f(t) = sx$; the domain of ψ is the subgroup generated by a and $\psi(a) = x^{-1}a^{-1}$. Since $Z_m \cup \psi(Z_m)$ generates $Z_m \ltimes Z_{2^m-1}$ and the stable letter s is a meridian, it follows from Theorems 3 and 1 that K_m is quasiaspherical. This proves the theorem.

Since the spinning construction preserves meridian, we have:

COROLLARY 2. *For* $n \ge 2$ *there are quasiaspherical* n-*knots with infinitely many ends.*

Remark. The knot K_m has the same group as the corresponding knot in [2, pag. 95]. However, the latter is not quasiaspherical (see [4] or Proposition 3).

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Complete minimal hypersurfaces in hyperbolic n-manifolds

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This paper is concerned with the existence and basic properties of minimal hypersurfaces in hyperbolic n-manifolds. A powerful and general method fo constructing minimal hypersurfaces in complete Riemannian manifolds N^n is given by geometric measure theory. For example, it is known that there exists an area-minimizing hypersurface, with small singular set, in any codimension one homology class of N. More recently, Schoen–Yau [SY] and Sachs–Uhlenbeck [SU] have constructed smooth branched minimal immersions of surfaces $f:\Sigma_g \to N$, area-minimizing in a conjugacy class of homomorphisms $\pi_1(\Sigma_g) \to \pi_1(N)$, provided $f_\#$ is injective on π_1. In case N is a 3-manifold, these surfaces are smooth immersions and in fact embeddings in case f is homotopic to an embedding (see [FHS]).

Restricting ourselves to hyperbolic manifolds (or more generally manifolds of negative curvature), we prove existence theorems for minimal hypersurfaces related to the above results, but distinct in several ways. The method, briefly stated, is as follows. Let N^n be a complete manifold of strictly negative sectional curvature $c_2 \leq K_N \leq c_1 < 0$ and let $\tilde{N}^n$ be its universal cover. Using geometric measure theory, we produce complete area-minimizing hypersurfaces in $\tilde{N}^n$, with prescribed behaviour at infinity; if Γ is a discrete group of isometries of $\tilde{N}^n$ whose action at infinity is sufficiently tame, we prove the existence of Γ-invariant area-minimizing hypersurfaces in $\tilde{N}^n$. Thus when Γ acts freely, one obtains complete immersed minimal hypersurfaces in N^n, provided $\Gamma \subset \pi_1(N^n)$.

In dimensions greater than three, these existence results are new; however, the generality of the result is unclear, since the action at infinity of discrete subgroups of isometries is not well understood in these dimensions.

In dimension three, these results partially overlap with those of [SY] and [SU]; in many respects, their results are much stronger. However, the lifts of least area incompressible surfaces to the universal cover are not in general area-minimizing, so that there is reason to believe the two methods may produce different surfaces in the quotient 3-manifolds. We show in sections §4 and §5 that this is in fact so and is related to the non-uniqueness of minimal surfaces in a given homotopy class. Prèvious examples of such non-uniqueness are due to Thurston and discussed in [SU]; see also the interesting work of Uhlenbeck [U] for related discussion.

From a somewhat different point of view, the results for Γ-invariant minimal surfaces complement the construction of Lawson [L] on complete minimal surfaces in S^3 and Nagano–Smyth [NS] on surfaces in $\mathbb{R}^3$ invariant under discrete groups of isometries. The construction of surfaces in H^3 and H^n is simpler and more complete than in the other space forms, due to the structure of H^n at infinity.

We now present our results and organization of the paper in more detail. The first section is of a preliminary nature, providing the necessary background in geometric measure theory and hyperbolic geometry. In §2, we prove a general existence theorem for complete area-minimizing hypersurfaces in H^n with prescribed behavior at infinity; for example, one may choose the boundary at infinity in H^3 to be an arbitrary Jordan curve (perhaps non-rectifiable). The constructions used in this theorem occur repeatedly throughout the paper. We also remark that a similar result holds for manifolds of negative curvature $c_2 \leq K_N \leq c_1 < 0$, although we do not give a proof here.

In §3, we discuss the action of discrete groups Γ of isometries on H^n ("Kleinian groups") and prove the existence of Γ-invariant area-minimizing hypersurfaces provided the limit set Λ_Γ is sufficiently tame; this class includes in particular the case of quasi-Fuchsian groups in all dimensions. This leads to a new method of constructing closed minimal hypersurfaces in manifolds of negative curvature in dimensions greater than three.

The last two sections are concerned with dimension 3, where a great deal more can be said. We first prove that for any torsion free quasi-Fuchsian group Γ acting on H^3, there is a complete smoothly embedded Γ-invariant minimal disc; when $\Gamma \subset \pi_1(M^3)$ for M^3 a hyperbolic 3-manifold, one obtains in this fashion stable incompressible minimal surfaces in M^3 in the given homotopy class. This duplicates a special case of general results of [SY] and [SU] in the case Γ has no cusps or torsion. (Our method encompasses this case also.) The method of proof relies on the work of Almgren–Simon [AS] on embedded solutions to the Plateau problem; based on this work, one may prove the existence of curves γ on $S^2(\infty)$ in H^3 such that any complete absolutely area-minimizing surface Σ asymptotic to γ has genus greater than a fixed g_0.

In §5, these results are used to prove certain non-uniqueness and non-finiteness results. First, we note that there are naturally occurring quasi-circles γ (limit sets of quasi-Fuchsian groups Γ) for which any Γ-invariant area minimizing surface asymptotic to γ at infinity has infinite genus. As corollaries of this, it is shown that such curves must bound an infinite number of complete smoothly embedded minimal surfaces at infinity. Second, such groups Γ have at least two distinct Γ-invariant minimal discs; thus one finds non-uniqueness of incompressible minimal surfaces in a given homotopy class, for a large class of quasi-Fuchsian manifolds of a given genus. Further, such manifolds provide examples where the least area incompressible surfaces of [SY] are not homologically area-minimizing. Finally, we establish a general finiteness result for compact area-minimizing surfaces in hyperbolic 3-manifolds, based on the method of Tomi [To]. These last results answer some questions of Uhlenbeck in [U].

This paper may be viewed as a sequel to [An], which we refer to occasionally. A portion of the results in this paper are based on part of the author's Ph.D. Thesis at U.C. Berkeley. I wish to thank my advisor, H. Blaine Lawson, for his unending guidance and encouragement. Also, I wish to thank Bill Dunbar for helpful conversations on 3-manifolds and orbifolds.

§1. Preliminaries

We discuss briefly in this section basic concepts from geometric measure theory and hyperbolic geometry used throughout the paper.

A natural class of objects in which the Plateau problem admits a solution with desired smoothness properties is the class of integral p-currents; these may be thought of as suitable generalizations of smooth oriented p-manifolds. Recall that given an oriented smooth Riemannian manifold N^n, the space of p-currents on N is defined to be the space of continuous linear functionals $(\Omega^p)^*$ on the space of p-forms of N, endowed with the weak topology. Clearly, there is a natural embedding of the set of smooth oriented p-manifolds S^p of finite volume in $(\Omega^p)^*$, given by

$$[S](\alpha)=\int_S \alpha \qquad \alpha\in\Omega^p(N).$$

More generally, a rectifiable p-current is a convergent sum of such currents $\mathcal{S}=\sum_{j=1}^{\infty} j[S_j]$, where $\{S_j\}_1^{\infty}$ is a collection of mutually disjoint oriented p-rectifiable sets and

$$\mathbf{M}(\mathcal{S})\equiv\sum_{j=1}^{\infty} j\mathcal{H}^p(S_j)<\infty;$$

here $\mathcal{H}^p$ is Hausdorff p-measure for the metric on N. There is a natural mass norm on the space $\mathcal{R}_p(N)$ of rectifiable p-currents, given as

$$\mathbf{M}(\mathcal{S})=\sup\{\mathcal{S}(w):M(w)\leq 1\},$$

where $M(w)=\sup_{x\in N}|w_x|$, $|w_x|=\sup\{w_x(\xi):\xi$ a unit simple p-vector$\}$. The support of $\mathcal{S}=\Sigma j[S_j]$ is defined as $\operatorname{supp}\mathcal{S}=\overline{\bigcup_{j=1}^{\infty} S_j}$; finally, the boundary operator on $(\Omega^p)^*$ is given by

$$(\partial\mathcal{S})(w)=\mathcal{S}(dw).$$

One now defines the space of *integral p-currents* $\mathcal{I}_p(N)$ on N to be the set of currents $\mathcal{S}$ such that $\mathcal{S}$ and $\partial\mathcal{S}$ are rectifiable. One of the deep theorems of geometric measure theory is the

COMPACTNESS THEOREM ([FF]). *Let $K\subset N^n$ be a compact set and $C\in\mathbb{R}^+$. Then the set*

$$\{\mathcal{S}\in\mathcal{I}_p(N):\operatorname{supp}\mathcal{S}\subset K,\ \mathbf{M}(\mathcal{S})+\mathbf{M}(\partial\mathcal{S})\le C\}$$

is compact in the weak topology.

It follows easily from the definition that the mass norm is lower semicontinuous in the weak topology; this, together with the compactness theorem, allows one to solve the Plateau problem in the category of integral currents. Thus, if B^{p-1} is a $(p-1)$ manifold (or integral $(p-1)$-current) such that $B^{p-1}=\partial\mathcal{S}$, for some $\mathcal{S}\in\mathcal{I}_p(N)$, then there is an $\mathcal{S}_0\in\mathcal{I}_p(N)$ satisfying $\partial\mathcal{S}_0=B$ and

$$\mathbf{M}(\mathcal{S}_0)\le\mathbf{M}(\mathcal{S}),\quad \forall\mathcal{S}\text{ s.t. }\ \partial\mathcal{S}=B.$$

One says that $\mathcal{S}_0$ is absolutely area minimizing for the boundary B. We will often work with currents of non-compact support. One defines the group $\mathcal{I}_p^{\mathrm{loc}}$ of locally integral p-currents as the currents $\mathcal{S}$ such that for all $x\in N$, there is a $\tau\in\mathcal{I}_p(N)$ of compact support such that $x\notin\operatorname{supp}(\mathcal{S}-\tau)$. We then say $\mathcal{S}\in\mathcal{I}_p^{\mathrm{loc}}(N)$ is absolutely area-minimizing if, for all compact sets $K\subset N$, one has

$$\mathbf{M}(\mathcal{S}\llcorner K)\le\mathbf{M}(\tau),$$

for any $\tau\in\mathcal{I}_p(N)$ with $\partial(\mathcal{S}\llcorner K)=\partial\tau$.

Next, we briefly discuss the regularity properties of area-minimizing currents. A point $a\in\operatorname{supp}(\mathcal{S})\setminus\operatorname{supp}(\partial\mathcal{S})$ is regular if there is a neighborhood W of p such that $W\cap\operatorname{supp}(S)$ is a connected p-dimensional C^2-submanifold of N^n. If a is regular, then the manifold $B=W\cap\operatorname{supp}\mathcal{S}$ is oriented by $\vec{\mathcal{S}}|_B$ and $\mathcal{S}$ is given by integration over B, up to multiplicity. A fundamental theorem in the subject is the

REGULARITY THEOREM (c.f. [F]). *Let $\mathcal{S}$ be an absolutely area-minimizing integral $(n-1)$-current in $U \subset N^n$. Then the interior singular set Z of $\mathcal{S}$ has codimension ≥ 8, i.e. $\mathcal{H}^q(Z)=0$, for all $q>n-8$.*

In particular, if $n \leq 7$, then any area-minimizing $(n-1)$-current $\mathcal{S}$ is the standard orientation current over a smoothly embedded hypersurface.

For further information and details regarding geometric measure theory, we refer to the basic references [Al], [F].

Throughout much of this paper, the ambient space N^n will be hyperbolic space H^n of constant curvature -1, or a quotient of H^n by a discrete group of isometries. Usually we identify H^n with the unit ball $B^n(1)$ of Euclidean space via the Poincaré model. In this model, the unit sphere represents the *sphere at infinity* $S^{n-1}(\infty)$ of H^n and provides a natural conformal compactification of H^n; every point $p \in S^{n-1}(\infty)$ represents an asymptote class of geodesics in H^n. Analogously, we define the *asymptotic boundary* $\mathcal{A}$ of a locally integral p-current Σ in H^n by

$$\mathcal{A} = \overline{\operatorname{supp} \Sigma} \cap S^{n-1}(\infty),$$

where $\overline{}$ denotes closure in the Euclidean topology.

Recall that in the Poincaré model, geodesics are arcs of circles intersecting the sphere at infinity orthogonally; similarly, totally geodesic k-planes are domains on Euclidean k-spheres having orthogonal intersection with $S^{n-1}(\infty)$. One defines the *convex hull* $\mathscr{C}(S)$ of a set S in $\overline{H^n}$ as the intersection of all half-spaces containing S; a half-space is a component of $\overline{H^n}-P$, where P is a totally geodesic hyperplane.

Finally, we use standard notation and results from Riemannian geometry; geodesic balls of radius r are denoted by B_r^p or $B^p(r)$, where p is dimension.

§2. The boundary-value problem at infinity

In this section, we will prove the existence of complete area-minimizing hypersurfaces in H^n asymptotic to a rather general class of boundaries in $S^{n-1}(\infty)$; such boundaries arise naturally as limit sets of discrete groups acting on H^n.

Given compact sets A, B in a metric space (X, d), recall that the Hausdorff distance between A and B is given by

$$\rho(A, B) = \max(\rho_A(B), \rho_B(A)),$$

where $\rho_A(B) = \sup\{d(x, B) : x \in A\}$.

We now state the main existence theorem of this section; both the theorem and its proof will be used often in the sequel.

THEOREM 2.1. *Let $S \subset S^{n-1}(\infty)$ be a closed set such that $S^{n-1}(\infty)\setminus S$ has exactly 2 connected components. Suppose there are $(n-2)$-dimensional smooth, closed, connected manifolds $M_j \subset S^{n-1}(\infty)$ such that*

$$\lim_{j\to\infty} \rho(M_j, S) = 0.$$

Then there exists an absolutely area-minimizing integral $(n-1)$-current Σ asymptotic to S at infinity.

Proof. The outline of the proof resembles that of Theorem 4 of [An], where an analogous theorem was proved for the case of S a k-manifold in $S^{n-1}(\infty)$. We choose $O \in H^n$ as an origin and view $M_j \subset S^{n-1}(j)$ via geodesic projection from O.

Let Σ_j be an integral $(n-1)$-current representing a solution to the Plateau problem with boundary M_j; thus we have $\partial\Sigma_j = M_j$ and

$$\mathbf{M}(\Sigma_j) \le \mathbf{M}(\mathscr{S}),$$

for $\mathscr{S}$ any integral $(n-1)$-current with $\partial\mathscr{S} = M_j$. The proof is based on establishing the estimates

$$c_r \le \mathbf{M}(\Sigma_j \llcorner B_r) \le C_r \tag{2.2}$$

on the mass of Σ_j inside the geodesic r-ball B_r centered at O.

[A] *Existence of C_r*

We begin with

LEMMA 2.3. *Let Σ be an area-minimizing $(n-1)$ current in $B^n(s)$ with $\partial\Sigma = M$ a connected manifold in $S^{n-1}(s)$. Then* supp Σ *is connected and disconnects $B^n(s)$ into two components $\Omega^{\pm}$.*

Proof. We recall that supp Σ is an analytic submanifold outside a closed subset Z of Hausdorff dimension at most $n-8$. The work of Hardt–Simon [HS] on boundary regularity shows that $Z \cap \operatorname{supp} \partial\Sigma = \varnothing$. Thus the boundary of each component of supp Σ is M, and so it follows that supp Σ is connected. Since Z is of high codimension, it follows that $\pi_1(B^n(s) - Z) = 0$; see, e.g., [HP: Theorem 4.1b].

Suppose $B^n(s)\setminus\operatorname{supp}\Sigma$ were connected; choose a regular point $x\in\operatorname{supp}\Sigma$ and L a transverse curve so that $L\cap\operatorname{supp}\Sigma=x$. We may join the endpoints ∂L in $B^n(s)\setminus\operatorname{supp}\Sigma$ and obtain an embedding $f\colon S^1\to B^n(s)$ such that $f(S^1)\cap\operatorname{supp}\Sigma=x$. It follows that f extends to a map $f\colon D^2\to B^n(s)$; assume w.l.o.g. that f is transverse to $\operatorname{supp}(\Sigma-Z)$. Thus $f^{-1}(\operatorname{supp}(\Sigma-Z))$ is a 1-manifold with single boundary component x, a contradiction.

To see there are at most two components of $B^n(s)\setminus\operatorname{supp}\Sigma$, let x, L be as above and for any $y\in B^n(s)\setminus\operatorname{supp}\Sigma$, let τ_y be a shortest geodesic from y to $\operatorname{supp}\Sigma$. If p_y is the endpoint of τ_y, then p is regular and one may join p and x by a path γ in the regular set of Σ. By sliding γ in the direction normal to $\operatorname{supp}\Sigma$, one may join y to one endpoint of ∂L by a path in $B^n(s)\setminus\operatorname{supp}\Sigma$. ■

We apply Lemma 2.3 to the current Σ_j in $B^n(j)$ and see that $\operatorname{supp}\Sigma_j$ separates $B^n(j)$ into 2 components. The current Σ_j is of multiplicity 1, so that Σ_j represents a boundary of least area in $B^n(j)$; in other words, letting $B^n(j)\setminus\operatorname{supp}\Sigma_j=\Omega_j^+\cup\Omega_j^-$, we have $\Sigma_j=\partial\Omega_j^+$ and

$$\operatorname{vol}(\partial\Omega_j^+\cap K)\le\operatorname{vol}(\partial K\cap\Omega_j^+),$$

for any compact $K\subset B^n(j)$. Choosing $K=B^n(r)$, $r<j$, it follows that

$$\mathbf{M}(\Sigma_j \llcorner B_r)\le\tfrac{1}{2}\operatorname{vol}S(r), \tag{2.4}$$

for all j. This gives the upper bound $C_r=\frac{1}{2}\operatorname{vol}S(r)$.

[B] *Existence of* c_r

Recall that given a set $T\subset\overline{H^n}$ one may define the convex hull $\mathscr{C}(T)$ of T as the smallest geodesically convex set containing T. It is not difficult to prove that if Σ is a stationary p-current in H^n, then

$$\operatorname{supp}\Sigma\subset\mathscr{C}(\operatorname{supp}\partial\Sigma); \tag{2.5}$$

see e.g. [An], [AS]. We note also the useful fact that for $T\subset S^{n-1}(\infty)$

$$\mathscr{C}(T)\cap S^{n-1}(\infty)=\bar{T}. \tag{2.6}$$

Now choose points x, y in different components of $S^{n-1}(\infty)\setminus S$ and let γ be the unique geodesic asymptotic to x and y. For j sufficiently large, it is clear that the intersection $\gamma\cap S^{n-1}(j)$ consists of two points x_j, y_j with $x_j\to x$ and $y_j\to y$ as $j\to\infty$ and x_j, y_j lie in distinct components of $S^{n-1}(j)\setminus M_j$. Since, by Lemma 2.3

again, supp Σ_j separates $B^n(j)$ into two components, it follows that

$$\text{supp}\,\Sigma_j \cap \gamma \neq \varnothing,$$

for all j sufficiently large. Since supp $\Sigma \subset \mathscr{C}(M_j)$ and $\mathscr{C}(M_j)$ converges to $\mathscr{C}(S)$ as $j \to \infty$, we see that the sequence

$$\{\text{supp}\,\Sigma_j \cap \gamma\} \subset K,$$

for some compact set $K \subset \mathring{H}^n$. In particular, it follows that there is a $p \in \gamma$ and $R > 0$ such that

$$\text{dist}\,(p, \text{supp}\,\Sigma_j) < R, \quad \text{for all } j.$$

Thus, supp Σ_j intersects a fixed ball of radius R in H^n, for each j. The existence of the lower bound c_r now follows from standard monotonicity estimates on the mass of stationary currents in geodesic balls, see e.g., [An], [L_2].

The proof of Theorem 2.1 is now straightforward. The estimate (2.2) together with the compactness theorem for integral currents show that the sequence $\{\Sigma_j \llcorner B_i\}_{j=1}^{\infty}$ has a weakly convergent subsequence for each fixed i. Choosing such for each i and taking the diagonal subsequence, we find there is a subsequence $\{\Sigma_{j'}\}$ of $\{\Sigma_j\}$ and an integral $(n-1)$-current Σ such that

$$\Sigma_{j'} \to \Sigma$$

on any compact set, in the weak topology. The current Σ is absolutely area minimizing, being a limit of area-minimizing currents, and is easily seen to have asymptotic boundary S, using (2.5) and (2.6) again. ■

Remark 1. We note that these currents Σ are smoothly embedded complete submanifolds in case $n \leq 7$ and have singular set Z of Hausdorff codimension at least 8 in higher dimensions. As examples of boundaries S to which the theorem applies, we mention the following.

EXAMPLE 1. In dimension 3, we may choose S to be an arbitrary Jordan curve (not necessarily rectifiable) on $S^2(\infty)$. This follows from the fact that any Jordan curve may be approximated, in the Hausdorff distance, by inscribed polygons.

EXAMPLE 2. In higher dimensions, let S be the image of the equator $S^{n-2} \subset S^{n-1}$ under a homeomorphism h of S^{n-1}. Then S satisfies the hypothesis of the theorem. In fact, for any $\varepsilon > 0$, let $T = \{x : d(x, S^{n-2}) < \varepsilon\}$ be the ε-tubular neighborhood of the equator S^{n-2}. Define

$$f : T \to \mathbb{R} \quad \text{by} \quad f(x) = \frac{1}{\varepsilon^2 - d(x, S^{n-2})^2}.$$

Then $f_h = f \circ h^{-1} : h(T) \to \mathbb{R}$ is a proper exhaustion function of $h(T)$. We may choose a uniform approximation to f_h by a C^∞ function $\tilde{f}_h$ and, for any regular value q, define

$$M_q = \tilde{f}_h^{-1}(q).$$

Thus, $\rho(S, M_q) < \varepsilon$, as desired.

Remark 2. It is unknown whether a result analogous to Theorem 2.1 holds in higher codimension; the estimation (2.4) is no longer valid.

Remark 3. We note that a result analogous to Theorem 2.1 holds in complete manifolds of curvature $c_2 \le K_N \le c_1 < 0$; the proof will appear elsewhere.

§3. Kleinian groups and invariant solutions

In this section, we will study the existence of area-minimizing hypersurfaces invariant under a discrete group of isometries acting on H^n.

Let Γ be a discrete subgroup of $O^+(n, 1)$, the group of orientation-preserving isometries of H^n. The limit set Λ_Γ of Γ is the set of accumulation points of an orbit Γ_x, $x \in H^n$ on $S^{n-1}(\infty)$; this turns out to be independent of the choice of $x \in H^n$. Λ_Γ is a closed set, minimal under the conformal action of Γ on $S^{n-1}(\infty)$; we have

$$S^{n-1}(\infty) = \Omega_\Gamma \cup \Lambda_\Gamma,$$

where Ω_Γ is the 'domain of discontinuity' of Γ; Γ acts properly discontinuously on Ω_Γ. Ω_Γ may be empty, or have one, two or infinitely many components. We will call Γ *quasi-Fuchsian* if Ω_Γ has exactly two components. In case Γ acts freely (Γ is torsion free), we see that Γ is quasi-Fuchsian if and only if the quotient manifold $\mathscr{C}^n$

$$\mathscr{C}^n = \frac{\mathscr{C}(\Lambda_\Gamma)}{\Gamma} \subset M^n = \frac{H^n \cup \Omega_\Gamma}{\Gamma}$$

is a 'convex' hyperbolic manifold with two boundary components strictly contained in $\mathring{M}^n$; we note that

$$\pi_1(M^n) \cong \pi_1(\partial M).$$

[A manifold N is convex if any path in N is homotopic to a geodesic in N, relative to the endpoints.] In H^3, Maskit [M] has shown that if Γ is finitely generated and torsion free, then Γ is quasi-Fuchsian if and only if Γ is a quasi-conformal deformation of a Fuchsian group, i.e. a discrete subgroup of Isom (H^2); in this case, Λ_Γ is the image of a circle S^1 under a quasi-conformal homeomorphism of S^2.

Remark. In dimension 3, if Γ is a surface group, i.e. $\Gamma \cong \pi_1(\Sigma)$ where Γ is a (punctured) surface, Ω_Γ has either 0, 1 or 2 components; it is conjectured that the 'degenerate' groups with Ω_Γ having 0 or 1 component are suitable limits of quasi-Fuchsian groups. Thus quasi-Fuchsian groups play a central role in dimension 3.

The main result of this section is the following.

THEOREM 3.1. *Let Γ be a quasi-Fuchsian group acting on H^n. Then there exist complete Γ-invariant absolutely area-minimizing $(n-1)$-currents Σ_Γ in H^n.*

Proof. Let $\mathscr{C}(\Lambda_\Gamma)$ be the convex hull of Λ_Γ and let M_j be a sequence of smooth manifolds in the interior of $\mathscr{C}(\Lambda_\Gamma)$ eventually lying outside any compact set in H^n. We may apply Theorem 2.1, since $S^{n-1}(\infty)\setminus\Lambda_\Gamma$ has exactly two components; let Σ be a complete area-minimizing hypersurface in H^n asymptotic to Λ_Γ. We may assume that supp Σ is connected, since we may replace it by a component of supp Σ. Then, by Lemma 2.3, $\overline{H^n\setminus \operatorname{supp}\Sigma}$ has two components $\Omega^\pm$ such that $\Omega^\pm\cap S^{n-1}(\infty)$ are the two components of $S^{n-1}(\infty)\setminus\Lambda_\Gamma$; we note these latter are Γ-invariant. Consider the currents $g\Sigma$ defined by

$$(g\Sigma)(\omega)=\Sigma(g^*\omega), \quad \text{for} \quad g\in\Gamma.$$

Each $g\Sigma$ is a minimizing integral $(n-1)$-current; in fact $g\Sigma$ is a boundary of least area;

$$\partial(g\Omega^+)=g\Sigma,$$

where $g\Omega^\pm$ are the components of $H^n\setminus\operatorname{supp}(g\Sigma)$. Consider

$$\Omega_1=\bigcap_{g\in\Gamma} g\Omega^+.$$

It is clear that Ω_1 is Γ-invariant and so it follows that $\partial\Omega_1$ is also Γ-invariant. If $\partial\Omega_1$ is a boundary of least area, we are done. If not, then we proceed to solve the Plateau problem in Ω_1 as follows. Let B_i be a sequence of smooth connected $(n-2)$ manifolds in $\Omega_1 \cap \mathscr{C}(\Lambda_\Gamma)$, eventually lying outside any compact set $K \subset H^n$. Let $\mathscr{S}_i$ be a solution to the Plateau problem with boundary B_i. We now claim that $\mathscr{S}_i \subset \Omega_1$, for all i. To see this, one has $B_i \subset \Omega_1$, so that in particular $B_i \subset g\Omega^+$, for any $g \in \Gamma$. Since $g\Omega^+$ has a boundary of least area, it follows that $\mathscr{S}_i \subset g\Omega^+$, for any g; this gives the claim. Thus there is a sequence of boundaries of least area $\{\mathscr{S}_i\}$ in Ω_1, with $\{\partial\mathscr{S}_i\}$ converging to Λ_Γ in the sense of Hausdorff distance. Apply the proof of Theorem 2.1 to $\{\mathscr{S}_i\}$; it follows there is a convergent subsequence, call it $\{\mathscr{S}_i\}$ again, such that

$$\mathscr{S}_i \to \mathscr{S}^1 \text{ weakly,}$$

with $\operatorname{supp} \mathscr{S}^1 \subset \Omega_1$. Now $\mathscr{S}^1$ is a boundary of least area with support 'above' all $g\Sigma$, $g \in \Gamma$. In other words, one may define an ordering $<$ on the set of complete minimal currents asymptotic to Λ_Γ by

$$\Sigma_1 < \Sigma_2 \Leftrightarrow \Omega_1^+ \supset \Omega_2^+,$$

where $\Omega_i^+ \cap S^{n-1}(\infty)$ is the $+$ component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$. We thus have

$$g\Sigma < \mathscr{S}^1, \quad \text{for all} \quad g \in \Gamma.$$

Now repeat on $\mathscr{S}^1$ the process above. If $\mathscr{S}^1$ is not Γ-invariant, let

$$\Omega_2 = \bigcap_{g \in \Gamma} g(\Omega_1)^+$$

where $(\Omega_1)^+$ gives the positive component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$. Continuing in this fashion, we produce a sequence of boundaries of least area $\mathscr{S}^i$ such that

$$\Sigma = \mathscr{S}^0 < \mathscr{S}^1 < \cdots < \mathscr{S}^k < \cdots.$$

and also

$$g\mathscr{S}^i < \mathscr{S}^{i+1},$$

for all $g \in \Gamma$, and for all i. Each $\mathscr{S}^i$ is a complete area-minimizing $(n-1)$ current asymptotic to Λ_Γ satisfying

$$\operatorname{supp} \mathscr{S}^i \subset \mathscr{C}(\Lambda_\Gamma).$$

One may again apply the proof of Theorem 2.1 to obtain a convergent subsequence $\{\mathcal{S}^{k'}\} \subset \{\mathcal{S}^k\}$ with

$$\mathcal{S}^{k'} \to \Sigma_\Gamma \quad \text{as} \quad k \to \infty, \quad \text{weakly.}$$

It is now clear that Σ_Γ is a complete area-minimizing integral $(n-1)$-current asymptotic to Λ_Γ. To see that Σ_Γ is Γ-invariant, note that $\Sigma_\Gamma = \lim_{k\to\infty} \mathcal{S}^k$ so that $g\Sigma_\Gamma = \lim_{k\to\infty} g\mathcal{S}^k$; by construction, $g\mathcal{S}^k < \mathcal{S}^{k+1}$ so that

$$g\Sigma_\Gamma \leq \Sigma_\Gamma,$$

for any $g \in \Gamma$. Replacing g by g^{-1}, it follows that $g\Sigma_\Gamma = \Sigma_\Gamma$, for all $g \in \Gamma$. ∎

We now discuss some applications to closed minimal hypersurfaces in hyperbolic manifolds. The theory is most complete for surfaces in 3-manifolds, so we begin with this.

Let Γ be an arbitrary quasi-Fuchsian group (not necessarily finitely generated). The orbit space

$$M^3 = H^3/\Gamma$$

is a 3-manifold with boundary equal to Ω_Γ/Γ; note that $M^3 \approx (\Omega_\Gamma^+/\Gamma) \times I$, where $\Omega_\Gamma^\pm$ are the components of Ω_Γ. Conversely, recall the simultaneous uniformization theorem of Bers [B] which states that, given any pair of homeomorphic Riemann surfaces Σ_1, Σ_2 (possibly having punctures and branch points), there is a quasi-Fuchsian group Γ such that $\Omega_\Gamma/\Gamma = \Sigma_1 \cup \Sigma_2$; Γ is unique up to conjugation in $\mathrm{PSL}(2, \mathbb{C})$. In case that Γ acts freely, M^3, and its boundary Ω_Γ/Γ, inherit complete hyperbolic metrics. On the other hand, there are, for example, groups Γ with $M^3/\Gamma \approx S^2 \times I$ topologically; clearly Γ does not act freely, since $S^2 \times I$ does not admit any complete hyperbolic metric.

The following is a simple consequence of Theorem 3.1.

COROLLARY 3.2. *Let $M^3 = H^3/\Gamma$ be a quasi-Fuchsian 3-manifold. Then M^3 contains a branched minimal embedding of a Riemann surface S satisfying*

$$\pi_1(S) \to \pi_1(M^3) \to 0.$$

In case Γ acts freely, S is a smoothly embedded complete stable minimal surface *with*

$$0 \to \pi_1(\tilde{S}) \to \pi_1(S) \to \Gamma \to 0,$$

where $\tilde{S}$ is the Γ-covering of S in H^3.

Proof. The first statement follows from Theorem 3.1 by passing to the orbit space H^3/Γ; in this context, minimality means vanishing of the mean curvature away from the branch points of S. The second statement follows similarly; it a consequence of [F-CS, Theorem 1] that the embedded surface S is stable when Γ acts freely.

Remark 1. It is not necessarily true that $\pi_1(\tilde{S}) = 0$; in §4 and §5, we will prove the existence of smoothly embedded minimal surfaces S in certain $M^3 = H^3/\Gamma$ with $\pi_1(\tilde{S}) \neq 0$; in particular, these surfaces are not incompressible. On the other hand, in §4 (see Theorem 4.4), we will also show the existence of embedded minimal surfaces S in M^3 with $\pi_1(S) = \pi_1(M^3) = \Gamma$, for every torsion-free quasi-Fuchsian group Γ.

Remark 2. In case Γ acts freely and represents a compact surface, $\Gamma \cong \pi_1(\Sigma_g)$, Schoen–Yau [SY] and Sachs–Uhlenbeck [SU] have obtained very strong results on the existence of incompressible minimal surfaces in Riemannian manifolds. It is clear however that in general, the surfaces produced above are inequivalent; in particular, the lifts of incompressible minimal surfaces in compact 3 manifolds to H^3 are not necessarily area-minimizing. Further, our constructions apply to surfaces having cusps and branch points, as well as infinitely generated π_1.

For higher dimensions, one obtains the following.

COROLLARY 3.3. *Let N^n be a compact convex hyperbolic n-manifold with exactly two boundary components. Then there is a closed minimal hypersurface (integral $(n-1)$-current) Σ satisfying*

$$0 \to \pi_1(\operatorname{supp} \Sigma_\Gamma) \to \pi_1(\operatorname{supp} \Sigma) \to \pi_1(N^n) \to 0$$

where Σ_Γ is the Γ-lift of Σ to H^n. In case $n \leq 7$, Σ is a smoothly embedded stable submanifold.

Proof. Theorem 3.1 gives the existence of complete area-minimizing integral $(n-1)$-currents Σ_Γ invariant under the action of $\Gamma = \pi_1(N^n)$ on H^n. Passing to the orbit space gives the desired current Σ; stability follows as in Corollary 3.2.

Remark 3. Corollary 3.3 proves the existence of closed stable minimal hypersurfaces Σ in compact hyperbolic n-manifolds N^n which are covered by a hyperbolic manifold $\tilde{N}^n$ having two ends and compact convex hull; furthermore, we have

$$\pi_1(\text{supp}\,\Sigma) \to \pi_1(\tilde{N}) \subset \pi_1(N).$$

A similar result holds for N^n of pinched negative curvature. However, the class of manifolds satisfying the above conditions is not well understood.

§4. Minimal surfaces in hyperbolic 3-manifolds

In this section, we will work exclusively with hyperbolic 3-manifolds. Almgren–Simon in [AS] have proved the existence of embedded minimal discs in Riemannian 3-manifolds provided the boundary is constrained to lie on a convex set. More precisely, given a C^2 Jordan curve $\gamma \subset \partial C$, for C a convex set, consider the space $\mathcal{M}_0$ of smooth embeddings

$$f: D^2 \to M^3 \quad \text{such that} \quad f|_{S^1} = \gamma.$$

They show that the area functional achieves a minimum on $\mathcal{M}_0$ giving the existence of an embedded minimal disc $f_0(D^2)$ in M^3 with boundary γ. The work of Meeks–Yau [MY] actually shows that $f_0(D^2)$ realizes the minimum area over all branched immersions $D^2 \to M^3$; however, we shall not be using their techniques here.

We begin by using the result and method of proof of Almgren-Simon to construct complete embedded minimal discs in H^3.

THEOREM 4.1. *Let γ be a Jordan curve on $S^2(\infty)$. Then there exists a complete embedded minimal surface D in H^3 of the topological type of the disc, asymptotic to γ. Further, D minimizes area in the category of embedded discs.*

Proof. Let $\gamma_i \subset S^2(i)$ be a sequence of C^2-Jordan curves in H^3 whose limit is γ, in the sense of Hausdorff distance (see §1, Example 1). Then the work of [AS] gives existence of smoothly embedded minimal discs D_i with $\partial D_i = \gamma_i$. We apply the proof of Theorem 2.1 to $\{D_i\}$ (in place of $\{\Sigma_j\}$ there). The estimate

$$\mathbf{M}(D_i \llcorner B_r) \le \tfrac{1}{2} \operatorname{vol} S(r), \tag{4.2}$$

will follow easily from the following lemma.

LEMMA 4.2. *Let D^2 be a minimally embedded disc with $\partial D^2 \subset S^2(r)$. Then $D^2 \cap B^3(s)$ is a disjoint union of discs, for almost all $s \leq r$.*

Proof. Let $j: D^2 \to H^3$ be the inclusion and let s be a regular value of $d \circ j: D^2 \to \mathbb{R}$, where d is the distance function from 0. Then $j^{-1}(S^2(s))$ is a disjoint collection of circles $\{S_\alpha\}$ in $\mathring{D}^2$. Consider $j^{-1}(B(r)\setminus \mathring{B}(s)) \subset D^2$: this is a compact set K in D^2 with boundary equal to $\partial D^2 \cup \bigcup S_\alpha$. It follows easily from the convex hull property (2.5) that K is connected; thus none of the curves S_α are nested and so the complement $j^{-1}(B(s))$ is a union of discs. ■

Returning to the proof of Theorem 4.1, we now have by Lemma 4.2 that $D_i \llcorner B_r$ is a finite collection of discs. The area-minimizing property of D_i among embedded discs then gives (4.2) immediately. We may now copy the proof of Theorem 2.1 for $\{D_i\}$ and produce a stationary integral 2-current $\tilde{D}$ such that a subsequence converges

$$D_{i'} \to \tilde{D} \quad \text{weakly on compact sets.}$$

One sees that $\tilde{D}$ is a complete stationary integral 2-current asymptotic to γ and area minimizing among comparision discs in the following sense: if $\gamma \subset \operatorname{supp} \tilde{D}$ is a smooth Jordan curve wtih $\partial T = \gamma$, where T is a stationary 2-current and $\operatorname{supp} T \subset \operatorname{supp} \tilde{D}$, then

$$\mathbf{M}(T) \leq \operatorname{vol}(V),$$

where V is any embedded disc in H^3, $\partial V = \gamma$.

Our aim is to prove that $\tilde{D}$ is in fact a smoothly embedded disc. Thus, consider $x \in \operatorname{supp} \tilde{D}$. The slices $\partial(\tilde{D} \llcorner B_x(\varepsilon))$ are closed rectifiable 1-currents, for almost all $\varepsilon > 0$. Similarly, by means of Sard's theorem, the restriction $D_i \cap B_x(\varepsilon)$ is a union of smoothly embedded submanifolds with smooth Jordan curves as boundary, for almost all $\varepsilon > 0$. By Lemma 4.2, each component of $D_i \cap B_x(\varepsilon)$ is in fact a smooth embedded disc.

We claim there is a $\delta > 0$, with perhaps $\delta \ll \varepsilon$, such that at most four components of $D_i \cap B_x(\varepsilon)$ intersect $B_x(\delta)$, for all i. To see this, let C_i^j, $j = 1, 2, \ldots, K_i$ denote the components of $D_i \cap B_x(\varepsilon)$ intersecting $B_x(\delta)$; by a simple area comparison, we have

$$\sum_{j=1}^{K_i} \mathbf{M}(C_i^j) < \mathbf{M}(\partial(B_x(\varepsilon))).$$

Further, by the local monotonicity of stationary currents (see [An], [L_2]), it follows that

$$\mathbf{M}(C_i^j) \geq 1 \cdot \operatorname{vol}(B^2(\varepsilon-\delta)), \quad \text{for each } i, j.$$

Thus we find

$$K_i \cdot \operatorname{vol} B^2(\varepsilon-\delta) < \mathbf{M}(\partial(B_x(\varepsilon))) \approx 4\pi\varepsilon^2,$$

for ε sufficiently small. Since $\operatorname{vol} B^2(\varepsilon-\delta) \approx \pi(\varepsilon-\delta)^2$, we see that

$$K_i \leq 4, \quad \text{for any } i.$$

Thus the limiting current $\tilde{D} \llcorner B_x(\delta)$ is the limit of regular currents $D_i \llcorner B_x(\delta)$ having at most four components, each a smoothly embedded disc. By relabelling and passing to a subsequence, we may assume the sequence of components $\{C_i^j\}_{i=1}^{\infty}$ converges weakly to a current W^j

$$\tilde{D} \llcorner B_x(\delta) = \sum_j W^j.$$

The regularity of the current $\tilde{D}$ follows from the methods of Almgren–Simon. In fact, let $\tilde{T}_x$ be the (varifold) tangent cone to $\tilde{D}$ at x: it is known that $\tilde{T}_x$ either has support contained in a plane or is locally a union of half-discs with common diameter L (see [AS, Corollary 2]). Let T_x^j denote the varifold tangent cones to W^j at x; we then have

$$\sum_j T_x^j = \tilde{T}_x.$$

For fixed j, we choose a sequence $r_k \to \infty$ so that the expansions $N_k \equiv \mu_{r_k}(C_k^j)$ converge to the varifold tangent

$$\mu_{r_k}(C_k^j) \to T_x^j \text{ weakly}, \quad \text{as} \quad k \to \infty;$$

here μ_{r_k} denotes geodesic dilation of the ambient space H^3 centered at x_k, $x_k \to x$ as $k \to \infty$. Now the interior regularity results, Theorems 2 and 3 of [AS], applied to the sequence $\{N_k\}$, show that T_x^j is a plane (with multiplicity 1), for each j. Since the components C_k^j for fixed k are disjoint, it follows that the tangent planes T_x^j are identical. We apply the basic regularity theorem of Allard [Al, §8] to

$(\tilde{D}, \tilde{T}_x)$ and find that $\tilde{D}$ is a regular varifold in a neighborhood of x:

$$\tilde{D} \llcorner B(x, \delta') = \rho \cdot [S],$$

for some integer $\rho \leq 4$, where S is an analytic, embedded minimal surface in $\overline{B(x, \delta')}$. Clearly, ρ is independent of x and we now see that $\tilde{D}$ is a regularly embedded minimal surface in H^3, asymptotic to γ.

The Allard regularity result also shows that the convergence $D_i \to D$ is smooth. Since for almost all r, $D_i \cap B(r)$ is a disjoint union of discs, it follows that $\tilde{D} \cap B(r)$ is as well; we thus find that $\tilde{D}$ is a complete embedded disc. ■

COROLLARY 4.3. *Let Γ be a quasi-Fuchsian group acting on H^3. Then there is a complete smoothly embedded Γ-invariant minimal disc $\tilde{D}$ in H^3. As above, $\tilde{D}$ minimizes area among embedded discs.*

Proof. Let Λ_Γ be the limit set of Γ on $S^2(\infty)$; since Γ is quasi-Fuchsian, Λ_Γ is a Jordan curve. By Theorem 4.1, there exists a complete embedded minimal disc D asymptotic to Λ_Γ. We now use the proof of Theorem 3.1 to construct a Γ-invariant minimal disc. If D is not Γ-invariant define gD as in Theorem 3.1 by

$$(gD)(\omega) = D(g^*\omega), \qquad g \in \Gamma.$$

Then each gD is a smoothly embedded minimal disc; let

$$\Omega_1 = \bigcap_{g \in \Gamma} g\Omega^+,$$

where $g\Omega^+$ is the component of $H^n \setminus \operatorname{supp}(gD)$ containing the positive component of $S^{n-1}(\infty) \setminus \Lambda_\Gamma$ in its closure. We see that Ω_1 and $\partial\Omega_1$ are Γ-invariant currents; if $\partial\Omega_1$ is a smoothly embedded disc, we are done; if not, choose extreme C^2 Jordan curves γ_i in $\Omega_1 \cap \mathscr{C}(\Lambda_\Gamma)$ eventually lying outside any compact set in H^3. Let $\mathscr{S}_i$ be an Almgren-Simon solution with boundary γ_i: thus $\mathscr{S}_i$ is a smoothly embedded minimal disc with boundary γ_i, area-minimizing among embedded discs with the same boundary. We see as before that $\mathscr{S}_i \subset g\Omega^+$, for all $g \in \Gamma$, so that $\mathscr{S}_i \subset \Omega_1$, for all i. Now repeat the process carried out in Theorem 3.1, using the regularity results of Theorem 4.1. In fact, we see that $\{\mathscr{S}_i\}$ subconverges to a stationary integral 2-current $\mathscr{S}^1$; by the proof of Theorem 4.1, $\mathscr{S}^1$ is a smooth embedded disc, asymptotic to Λ_Γ. One thus obtains a sequence $\{\mathscr{S}^i\}$ by repetition of the above argument. It follows that $\{\mathscr{S}^i\}$ will subconverge to a Γ-invariant stationary integral 2-current $\tilde{D}$; the fact that $\tilde{D}$ is a complete smoothly embedded minimizing disc follows from Theorem 4.1. ■

Remark 1. In connection with Remark 1 of §3, Corollary 4.3 produces complete embedded incompressible minimal surfaces Σ in quasi-Fuchsian 3-manifolds $M^3 \cong H^3/\Gamma$, $\Gamma \cong \pi_1(\Sigma)$. For example, there are complete minimal embeddings of a k-fold punctured S^2 in certain quasi-Fuchsian 3-manifolds, for any $k > 3$. As far as the author knows, these give the first non-trivial examples of non-compact complete minimal surfaces of finite volume.

The complete minimal discs constructed in Theorem 4.1 and Corollary 4.3 need not be absolutely area-minimizing. In case they are not, one may construct surfaces in H^3 of higher genus. To begin, we recall the results of Almgren-Simon [AS] in the compact case. Let γ be an extreme C^2-Jordan curve in H^3. Let $\mathcal{M}_g(\gamma)$ be the space of connected, oriented embedded C^2-surfaces $M \subset H^3$ with boundary γ, with genus $M = g$. Let

$$\begin{aligned}\alpha_g(\gamma) &= \inf\{\text{area}\,(M) : M \in \mathcal{M}_g(\gamma)\}\\ &= \inf\{\text{area}\,(M) : M \in \mathcal{M}_h(\gamma) : h \leq g\}.\end{aligned}$$

Then it is proved in [AS] that if $\alpha_g(\gamma) < \alpha_{g-1}(\gamma)$, there is a surface $M \in \mathcal{M}_g(\gamma)$ with area $(M) = \alpha_g(\gamma)$.

For complete surfaces in H^3, we then prove:

THEOREM 4.4. *Let Σ_g be a complete embedded minimal surface of genus $\leq g$ in H^3 asymptotic to γ and area-minimizing among embedded surfaces of genus $\leq g$. If Σ_g is not absolutely area-minimizing, then there exists a complete embedded minimal surface $\Sigma_{g'}$ in H^3, of genus $\leq g'$, for some finite $g' > g$, asymptotic to γ. Further, $\Sigma_{g'}$ is area-minimizing among comparison surfaces of genus $h \leq g'$.*

Proof. As in the proof of Theorem 4.1, let γ_i be a sequence of extreme C^2-Jordan curves on Σ_g, with $\gamma_i \to \gamma$ as $i \to \infty$. By hypothesis, there is an i_0 and $g' > g$ such that

$$\alpha_{g'}(\gamma_{i_0}) < \alpha_g(\gamma_{i_0}).$$

Since we may assume that Σ_g is an annulus outside of γ_{i_0}, it is clear that $\alpha_{g'}(\gamma_i) < \alpha_g(\gamma_i)$, for all $i \geq i_0$. By [AS, Theorem 8], there exists smoothly embedded surfaces S_i, all of genus g', satisfying $\partial S_i = \gamma_i$ and area $(S_i) = \alpha_{g'}(\gamma_i)$. Consider the sequence of integral 2-currents $\{S_i\}$. The proof of Theorem 2.1 applies and gives, after passage to a subsequence, a weak limit

$$S_i \to \Sigma_{g'},$$

where $\Sigma_{g'}$ is a complete stationary integral 2-current asymptotic to γ. The regularity arguments of Theorem 4.1 apply here and prove that $\Sigma_{g'}$ is a smoothly embedded submanifold. Since $S_i \to \Sigma_{g'}$ in the C^2-topology and S_i has genus g', it follows genus $\Sigma_{g'} \leq g'$. Finally, the area-minimizing properties of $\Sigma_{g'}$ follow from those of $\{S_i\}$.

Remark 2. Of course, the surfaces Σ_g and $\Sigma_{g'}$ constructed above are geometrically distinct, since they have distinct area-minimizing properties.

Remark 3. The proof above does not show that genus $\Sigma_{g'} = g'$, or even genus $\Sigma_g \geq$ genus Σ_g, although it is likely that one can find surfaces with these properties.
In order to show that such a 'hierarchy' of complete minimal surfaces actually occurs, we use the following Proposition.

PROPOSITION 4.5. *There exist Jordan curves* γ *on* $S^2(\infty)$ *such that any absolutely area-minimizing surface* Σ *asymptotic to* γ *has genus* $g \geq g_0$, *for any prescribed* $g_0 \geq 0$.

Proof. The proof is a simple modification of work in [AS]; the case $g_0 = 1$ is given below. Let γ_0 be the curve consisting of two concentric circles of radii r_1, r_2 centered at the origin in $\mathbb{R}^2$, viewed as infinity in the upper half space model of H^3. It is not difficult to see that for $\frac{1}{2}r_2 \leq r_1 \leq r_2$, any area-minimizing surface Σ_0 asymptotic to γ does not intersect the line $l_1 = \{x = y = 0\}$. To justify this, we note that any area-minimizing surface asymptotic to γ_0 is invariant under rotation about l_1; if Σ_0 intersects l_1, it follows Σ_0 is the union of two totally geodesic hyperplanes asymptotic to γ_0. Now simple area-comparision with an annulus spanning γ_0 shows that Σ_0 cannot be area-minimizing, given the bounds on r_1, r_2 above.

Let γ_ε be the oriented Jordan curve obtained by joining the circles of γ_0 by line segments of Euclidean separation ε, and let Σ_ε be an area-minimizing surface asymptotic to γ_ε (see Figure 1). As $\varepsilon \to 0$, Σ_ε converges to Σ_0 in the weak topology on varifolds.

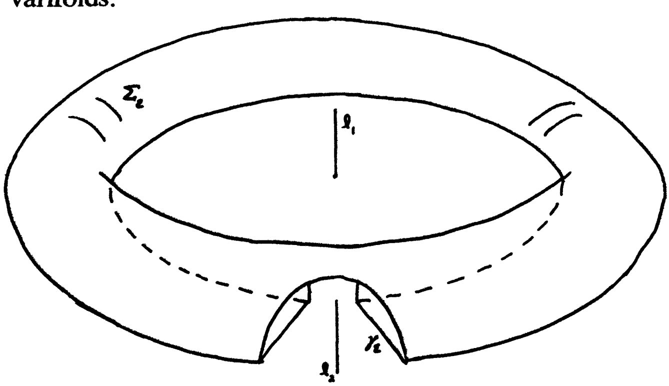

Figure 1.

Let l_2 be the ray consisting of the negative x-axis; we assume $\gamma_\varepsilon \cap l_2 = \emptyset$. Choose a ball B such that B is tangent to the plane $\{z=0\}$ at a point on l_2 but $B \cap \Sigma_0 = \emptyset$. It follows from the area-minimizing property and the convergence $\Sigma_\varepsilon \to \Sigma_0$ that for all ε sufficiently small, $B \cap \Sigma_\varepsilon = \emptyset$. Thus there is a loop σ in $H^3 - \Sigma_\varepsilon$ such that σ does not bound in $H^3 - \Sigma_\varepsilon$. It follows that Σ_ε is not a disc for ε sufficiently small. ∎

Remark 4. We note that the curves γ satisfying the above Proposition are stable under small perturbations; thus, if $\gamma \subset S^2(\infty)$ has only absolutely area-minimizing solutions of genus $\geq g_0$, then any Jordan curve γ' sufficiently close to γ in the Euclidean flat topology (or Hausdorff distance) also has only least area solutions of genus $\geq g_0$. One proves this by contradiction: if $\{\gamma_i\} \subset S^2(\infty)$ converge to γ in the flat topology, then after passing to a subsequence, any least area solutions Σ_i asymptotic to γ_i will converge smoothly to a least area solution Σ asymptotic to γ; thus for i sufficiently large, genus $\Sigma_i \geq$ genus $\Sigma \geq g_0$.

§5. Non-uniqueness, finiteness, and non-finiteness

In this section, we will continue the study of minimal surfaces in hyperbolic 3-manifolds, using the results of §4 in particular. We begin by using Proposition 4.5 to show that complete area-minimizing surfaces of infinite genus arise naturally in H^3.

THEOREM 5.1. *There exist torsion-free quasi-Fuchsian groups Γ_g such that any complete absolutely area-minimizing Γ_g-invariant surface in H^3 has infinite genus.*

Proof. Let γ be a curve as in Proposition 4.5, given explicitly as in Figure 2. Then there is a band B around γ, given as in Figure 2 also, with the following property: if Σ is any area-minimizing surface asymptotic to a Jordan curve $\gamma' \subset B$, then genus $\Sigma \geq 1$. This follows by using the arguments of Proposition 4.5.

Now inscribe successively, within the band B, N Euclidean circles C_i so that C_i intersects C_{i+1} at an angle of $\pi/2$ and $C_i \cap C_{i+k} = \emptyset$, for all $k \geq 2$. It is not difficult to see that this can be done for any $N \geq N_0 = 30$, for example.

Let Γ' be the Kleinian group acting on H^3 generated by reflections through hyperplanes asymptotic to C_i and let $\Gamma_0 \subset \Gamma'$ be the subgroup of orientation preserving mappings. It is well known that Λ_{Γ_0} is a Jordan curve lying inside the circles C_i (see [B]): in particular $\Lambda_{\Gamma_0} \subset B$ and Γ_0 is quasi-Fuchsian. We now claim

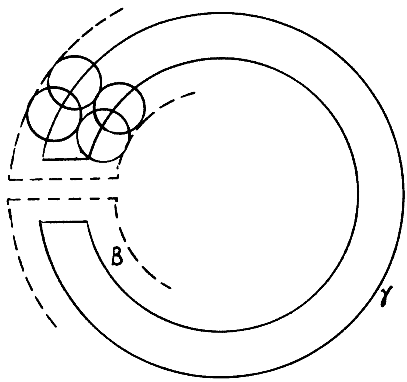

Figure 2.

that Γ_0 has a torsion-free surface subgroup Γ of index 2 such that

$$\Gamma = \pi_1(\Sigma_g),\ \Sigma_g \text{ a surface of genus g, where } g \doteq N.$$

To see this, we note that $M^3 = H^3/\Gamma_0$ is a 3-orbifold in the sense of Thurston [T]; topologically $M^3 \approx S^2 \times I$ where S^2 has $2N$ elliptic points (branch points) with group $\mathbb{Z}_2$ determined by the circle intersections at infinity in H^3. In fact, the action of Γ' on $S^2(\infty)$ has a disc with $2N$ corner angles of $\pi/2$ on the boundary as fundamental domain; passing to Γ_0, its fundamental domain is two copies of this disc glued together along the boundary to give the desired S^2. Now such orbifolds have a surface Σ_g of genus $g = N$ as 2-fold orbifold covers. In fact, embed Σ_g in $\mathbb{R}^3$ in such a way that the z-axis L passes through all the "holes" of Σ_g and $\Sigma_g \cap L$ consists of $2N+2$ points; assume that Σ_g is invariant under rotation by 180° in the z-axis. Under this $\mathbb{Z}_2$ action on Σ_g, the quotient space $\Sigma_g/\mathbb{Z}_2$ is easily seen to be an S^2 with $2N$ elliptic points with group $\mathbb{Z}_2$.

The quasi-Fuchsian group Γ has limit set $\Lambda_\Gamma = \Lambda_{\Gamma_0}$, since Γ is normal in Γ_0 ([T:8.1.3]). Applying Theorem 3.1, we may construct complete, Γ-invariant area-minimizing surfaces $\tilde{\Sigma}$ in H^3; it is clear that such surfaces have genus either 0 or ∞. Since $\tilde{\Sigma}$ is asymptotic to $\Lambda_\Gamma \subset B$, $\tilde{\Sigma}$ cannot have genus 0. ■

Remark 1. Define the Bers isomorphism

$$T(\Sigma_g) \times T(\Sigma_g) \xrightarrow{\cong} QF_g$$

by associating to any pair of points in the Teichmüller space of a surface of genus g the associated quasi-Fuchsian group. Then we have shown that for any $g \geq 30$, e.g., there are quasi-Fuchsian groups Γ_g having Γ_g-invariant area-minimizing surfaces of infinite genus. In the orbit space $M^3 = H^3/\Gamma_g$, these surfaces descend to compact embeded stable minimal surfaces $\Sigma_{\bar{g}}$ of genus $\bar{g} > g$; clearly, these surfaces are not incompressible. Further examination of the proof shows that for any g, there is a lower bound $N(g)$ on the number of quasi-Fuchisan groups of genus g having such surfaces: we have $N(g) \to \infty$ as $g \to \infty$. In the other direction, fixing the genus g, if one takes a sequence in $T(\Sigma_g) \times T(\Sigma_g)$ tending to "infinity" in both factors (but not diagonally), it seems likely that again the number of area-minimizing surfaces of infinite genus becomes unbounded; see the discussion in [U] and [T, §9].

Let $\mathscr{G}$ be the class of quasi-Fuchsian groups such that any Γ-invariant area-minimizing surface is of infinite genus, $\mathscr{G}_g$ the subset of $\Gamma \in \mathscr{G}$ such that $\pi_1(H^3/\Gamma) = \pi_1(\Sigma_g)$. Thus the above Remark shows that the cardinality of $\mathscr{G}_g$ is unbounded in g.

We may use these surfaces to construct infinitely many geometrically distinct complete minimal surfaces asymptotic to a given boundary.

THEOREM 5.2. *Let Λ_Γ be the limit circle of a quasi-Fuchsian group $\Gamma \in \mathscr{G}$. Then there exist infinitely many complete, smoothly embedded minimal surfaces asymptotic to Λ_Γ; furthermore, there is a finite bound on the maximal normal distance between these surfaces.*

Proof. By Theorem 4.1, we know there is a complete Γ-invariant embedded minimal disc Σ_0. By definition of Λ_Γ, Σ_0 is not absolutely area-minimizing; thus we may choose extreme Jordan curves γ_i on Σ_0 and embedded minimal surfaces Σ_i of fixed genus $g_1 > 0$ with $\partial \Sigma_i = \gamma_i$. By the techniques of Theorems 4.1 and 4.4, $\{\Sigma_i\}$ will subconverge to a smoothly embedded surface Σ_{g_1} of genus $\leq g_1$. If Σ_{g_1} happens to be area-minimizing, the translates $h \cdot (\Sigma_{g_1})$, for $h \in \Gamma$ give an infinite family of distinct (but isometric) minimal surfaces asymptotic to Λ_Γ. If Σ_{g_1} is not area-minimizing, we may repeat the process on Σ_{g_1}: in either case we obtain an infinite family of distinct surfaces.

To verify the second statement, note that all surfaces are contained in the convex hull $\mathscr{C}(\Lambda_\Gamma)$: note also that the diameter of $\mathscr{C}(\Lambda_\Gamma)$

$$d_\Gamma = \sup_{x \in \partial(\mathscr{C}(\Lambda_\Gamma))} \{\mathrm{dist}\,(x, \partial(\mathscr{C}(\Lambda_\Gamma))\} < \infty;$$

in particular, there is an upper bound to the distances of all minimal surfaces asymptotic to Λ_Γ. ■

Note. One expects that Σ_{g_1} constructed above is not area-minimizing; this would then give an infinite sequence of isometrically distinct surfaces.

Next we prove a non-uniqueness result for incompressible minimal surfaces in a given homotopy class in hyperbolic 3-manifolds.

THEOREM 5.3. *Let Γ be a quasi-Fuchsian group in $\mathscr{G}_g$, so $\pi_1(\Sigma_g)=\Gamma$. Then in the homotopy class of the inclusion*

$$\Sigma_g \xrightarrow{\approx} M^3 = H^3/\Gamma,$$

there are at least two geometrically distinct compact stable embedded minimal surfaces of genus g.

Proof. Let Σ_∞ be a Γ-invariant area-minimizing surface of infinite genus in H^3 and let $\Omega^\pm$ be the Γ-invariant components of $H^3 \backslash \Sigma_\infty$: we will construct Γ-invariant stably embedded minimal discs in $\Omega^\pm$. It suffices to work in Ω^+: let γ_i be a sequence of smooth extreme Jordan curves in $\Omega^+ \cap \mathscr{C}(\Lambda_\Gamma)$ converging to Λ_Γ as $i \to \infty$. It is well known one may solve the Plateau problem for minimal discs in Ω^+, see e.g. [MY]. By the work of [AS] or [MY], any solution S_i is an embedded minimal disc, area-minimizing among embedded minimal discs in Ω^+. Letting $i \to \infty$, the techniques of Theorem 4.1 show that $\{S_i\}$ subconverges to a complete embedded minimal disc D^+ in Ω^+ asymptotic to Λ_Γ. If D^+ is not Γ-invariant, we may use the methods of Corollary 4.4 to produce a Γ-invariant minimal disc, call it again D^+ in Ω^+. (In fact there are at least two such in Ω^+ if D^+ was not Γ-invariant to begin with.) The quotient surfaces D^+/Γ, D^-/Γ are then stable minimal surfaces embedded in M^3, inducing an isomorphism on π_1. ■

Remark 2. This result contrasts with the result that harmonic maps $f: M \to N$ are unique in their homotopy class, provided $K_N < 0$ and N is compact. Thurston has shown that there are infinitely many (isometric) minimal surfaces in $M^3 = H^3/\Gamma$, where Γ is a "doubly degenerate group", i.e. $\Gamma = \pi_1(\Sigma_g)$, where $\Sigma_g \to N^3 \to S^1$ is a smooth fibration over S^1, N^3 having a hyperbolic structure. In this case, Γ is a suitable 'limit' of quasi-Fuchsian groups: see [T§9].

The following theorem shows that least area incompressible surfaces constructed by Schoen–Yau [SY] are not necessarily area-minimizing in their homology class.

THEOREM 5.4. *Let $\Sigma_g \overset{i}{\hookrightarrow} M^3$ be a least area incompressible surface in M^3, where $M^3 = H^3/\Gamma$, $\pi_1(\Sigma_g) \approx \Gamma$. Then if $\Gamma \in \mathscr{G}_g$, there exists $\Gamma' \in \mathscr{G}_{g'}$, with $\Gamma' \lhd \Gamma$ of finite index such that the lift*

$$H^2/\Gamma' \approx \Sigma_{g'} \overset{i'}{\hookrightarrow} M' = H^3/\Gamma'$$

covering i is a least area incompressible embedding, but $[\Sigma_{g'}] \in H_2(M', \mathbb{Z})$ *is not of least area in its homology claass.*

Proof. Since $\Sigma_g \hookrightarrow M^3$ is incompressible, the lift $\tilde{\Sigma}_g \hookrightarrow H^3$ is a complete (embedded) disc, asymptotic to Λ_Γ. Since $\Gamma \in \mathscr{G}_g$, $\tilde{\Sigma}_g$ is not absolutely area minimizing. Let D be a domain in $\tilde{\Sigma}_g$ such that D is not area-minimizing w.r.t. ∂D. Now choose $\Gamma' \lhd \Gamma$ such that D is contained in a fundamental domain of Γ'; this is possible since Γ is residually finite [H]. Let $D' = \tilde{\Sigma}_g \cap \Gamma'$ and let S' be an area minimizing surface in H^3 with $\partial S' = \partial D'$; clearly D' and S' are homologous in H^3. It follows that $D'/\Gamma' = \Sigma_{g'}$ and S'/Γ' are homologous in $M' = H^3/\Gamma'$ and since area $(S') <$ area (D'), area $(S'/\Gamma) <$ area (Σ_g'). On the other hand, it is not difficult to see that $\Sigma_{g'}$ is of least area in its homotopy class; see [FHS] (Lemma 3.3) for the details.

Finally we prove a general finiteness result for stable minimal surfaces in compact Riemannian 3-manifolds; this will show in particular that "most" hyperbolic 3-manifolds admit only finitely many stable minimal surfaces of a given genus.

Define a surface S in N^3 to be R-locally area-minimizing if for any geodesic R-ball $B(x, R)$ in N^3, the surface $S \cap B(x, R)$ is area-minimizing with respect to its boundary.

THEOREM 5.5. *Let N^3 be a compact oriented 3-manifold with an analytic Riemannian metric. Then for any given $R > 0$, either*

(1) N^3 contains only finitely many compact stable, oriented, R-locally minimizing surfaces of uniformly bounded area, or

(2) N^3 fibres over S^1 with fibres smooth compact minimal surfaces.

We expect the added condition of R-locally minimizing may be dropped, but have not been able to do so.

Proof. The proof is based on the method of Tomi [To] on the finite solvability of the Plateau problem in $\mathbb{R}^3$. We suppose (1) does not hold; let $\{M_i\}$ be a sequence of R-locally area-minimizing surfaces in N^3 with area $(M_i) < K$. The compactness theorem for integral currents implies that $\{M_i\}$ converges, after passing to a subsequence, to an R-locally minimizing integral 2-current $\mathscr{M}$. Since each M_i is stable, the regularity theorem of Schoen–Simon [SS] implies that $\mathscr{M}$ is a smooth stable minimal surface; furthermore the fact that $\mathscr{M}$ is R-locally area-minimizing implies that $\mathscr{M}$ and M_i may be locally graphed over the tangent planes of $\mathscr{M}$, for i sufficiently large, see e.g. [P]. Thus, in sufficiently small geodesic balls, $\mathscr{M}$ and M_i are embedded discs D, D_i, and the convergence $D_i \to D$ is C^2 (in fact analytic).

These results show that each M_i may be graphed globally over $\mathcal{M}$ in the following sense: for any $f \in C^{2,\alpha}(\mathcal{M})$, define M_f to be the graph of f over $\mathcal{M}$, i.e.,

$$M_f = \{y : y = \exp_x f(x) \cdot E_0\},$$

where E_0 is the unit normal to $\mathcal{M}$ in N. Thus M_i defines a unique function $f_i \in C^{2,\alpha}$ such that $M_i = M_{f_i}$, where $f_i \to 0$ as $i \to \infty$ and $\mathcal{M} = M_0$. Define

$$H : C^{2,\alpha}(\mathcal{M}) \to C^{0,\alpha}(\mathcal{M}) \quad \text{by}$$
$$H(f) = \text{mean curvature function of } M_f.$$

Using the fact that $N, \mathcal{M}$ and the normal exponential map of $\mathcal{M}$ in N are analytic, it is a straightforward, but lengthy, computation to show that H is an analytic mapping in a neighborhood of $0 \in C^{2,\alpha}(\mathcal{M})$.

The arguments of Tomi [To] then show that $H^{-1}(0)$ is an analytic 1-manifold V in a neighborhood of $\mathcal{M}$. Using the compactness theorem again, we see V is a compact analytic 1-manifold (diffeomorphic to S^1) parametrizing diffeomorphic stable minimal surfaces M_t in N^3. It now follows that the natural projection

$$\pi : N^3 \to V$$
$$\pi(x) = t, \quad \text{where} \quad x \in M_t$$

gives the desired fibration. □

COROLLARY 5.6. *A quasi-Fuchsian 3-manifold $M = H^3/\Gamma$ has only finitely many stable, locally area-minimizing compact surfaces of a given genus.*

Proof. The convex hull property shows that all compact minimal surfaces in M^3 are contained in the convex part of M: since this latter does not fiber over S^1 isometrically, it follows from the proof of Theorem 5.5 that M contains only finitely many R-locally area-minimizing surfaces of bounded area. Now we have, for Σ_g a minimally immersed surface of genus g in a hyperbolic manifold that,

$$\text{vol}(\Sigma_g) = \int_{\Sigma_g} 1 \leq -\int_{\Sigma_g} K = -2\pi\chi(\Sigma_g),$$

where K is the Gaussian curvature of Σ_g, $\chi(\Sigma_g) = (2-2g)$ is the Euler characteristic. Thus a bound on genus gives a bound on area, proving the corollary. ■

Remark 3. As noted above in Remark 2, the Corollary is false if we drop the assumption that Γ is quasi-Fuchsian. On the other hand, it does hold for any compact hyperbolic 3-manifold which does not fibre over S^1 with fibres being minimal surfaces. We conjecture that no hyperbolic 3-manifold has this property: more generally, we conjecture that if M^3 is a closed hyperbolic 3-manifold, then there does not exist a local 1-parameter family of closed minimal surfaces in M^3.

A result of this type, together with Theorem 3.4 would provide a good basis in understanding the moduli spaces of minimal surfaces in negatively curved 3-manifolds.

Finally, one obtains a purely topological result from Theorem 5.5.

COROLLARY 5.7. *Let N^3 be a compact 3-manifold admitting a metric of curvature $K_N \leq c < 0$. Then for any given g, the set of homotopy classes $[\Sigma_g, N^3]_i$ of incompressible surfaces in N^3 is finite, up to conjugacy.*

Proof. It follows from [SY] that in any class of $[\Sigma_g, N^3]_i$, there is an immersed least area incompressible surface. By the estimate in Corollary 5.6, any such surface has a bound on its area. If there was infinitely many such homotopy classes, the proof of Theorem 5.5 implies the least area surfaces must subconverge to a limiting surface; thus all surfaces will eventually be homotopic. ■

Corollary 5.7 has been proved by Thurston [T: 8.8.6] by means of pleated surfaces.

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$D(\mathbb{Z}\pi)^+$ and the Artin cokernel

ROBERT OLIVER

Let p be an odd prime. If π is a p-group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order containing $\mathbb{Z}\pi$, then we set

$$D(\mathbb{Z}\pi) = \operatorname{Ker}[K_0(\mathbb{Z}\pi) \to K_0(\mathfrak{M})]$$

as usual. Since $D(\mathbb{Z}\pi)$ is a p-group [2], the involution $g \mapsto g^{-1}$ induces a natural splitting

$$D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+ \oplus D(\mathbb{Z}\pi)^-;$$

where $D(\mathbb{Z}\pi)^+$ and $D(\mathbb{Z}\pi)^-$ are the $+1$ and -1 eigenspaces, respectively. The involution can be extended to a linear action of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$; and this induces further eigenspace decompositions

$$D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi) \quad \text{and} \quad D(\mathbb{Z}\pi)^- = \sum_{i=0}^{(p-3)/2} {}^{2i+1}D(\mathbb{Z}\pi).$$

The groups $D(\mathbb{Z}\pi)^-$ and ${}^{2i+1}D(\mathbb{Z}\pi)$ have been studied extensively in [3] and [14]; and the order of $D(\mathbb{Z}\pi)^-$ for abelian π is computed in [3]. In this paper, attention is focused on ${}^0D(\mathbb{Z}\pi)$. This is the part of $D(\mathbb{Z}\pi)^+$ which is independent of number theoretic properties of p; and in fact, ${}^0D(\mathbb{Z}\pi) = D(\mathbb{Z}\pi)^+$ whenever p is regular.

For any finite group π, we define the "Artin cokernel" $A_{\mathbb{Q}}(\pi)$ to be the group

$$A_{\mathbb{Q}}(\pi) = \operatorname{Coker}[\operatorname{Ind}: \sum \{R_{\mathbb{Q}}(\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \to R_{\mathbb{Q}}(\pi)],$$

where $R_{\mathbb{Q}}(\pi)$ denotes the rational representation ring. The main result here is that ${}^0D(\mathbb{Z}\pi) \cong A_{\mathbb{Q}}(\pi)$ for any p-group π (p odd). Among other consequences, this gives new insight into Martin Taylor's result [13] that the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism has order equal to the Artin exponent of $\pi : T(\mathbb{Z}\pi)$ corresponds under this isomorphism to multiples of the identity in $R_{\mathbb{Q}}(\pi)$.

We end by deriving a formula for $|{}^0D(\mathbb{Z}\pi)|$ – in fact, a formula for the order of $A_{\mathbb{Q}}(\pi)$ for arbitrary finite π. Also, for the sake of completeness, the formula for $|D(\mathbb{Z}\pi)^-|$ in [3] is generalized to cover arbitrary p-groups; thus giving a complete calculation of $|D(\mathbb{Z}\pi)|$ when π is a p-group and p any odd regular prime.

For $n \geqq 1$, $\mathbb{Q}\zeta_n$ always denotes the field generated by the n-th roots of unity (and similarly for $\mathbb{Z}\zeta_n$, $\hat{\mathbb{Z}}_p\zeta_n$, etc.) Also, $\varphi(n)$ will always mean the Euler φ-function.

Throughout the paper, unless otherwise stated, p will be a fixed odd prime. We start with the following well known description of $\mathbb{Q}\pi$ when π is a p-group.

PROPOSITION 1. *Let π be a p-group. Then $\mathbb{Q}\pi$ is a product of matrix rings over fields $\mathbb{Q}\zeta_{p^s}$ for various $s \geqq 0$. Furthermore, for each $s \geqq 0$, the number of simple summands isomorphic to matrix rings over $\mathbb{Q}\zeta_{p^s}$ is equal to the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ such that*

$$|\sigma/[\sigma, N(\sigma)]| = |\sigma| \cdot |Z(\sigma)|/|N(\sigma)| = p^s.$$

Proof. That $\mathbb{Q}\pi$ is a product of matrix algebras over the $\mathbb{Q}\zeta_{p^s}$ is shown (though not explicitly stated) by Roquette in [11]. In Section 2 of [11] he shows that the division algebra for any irreducible representation M of π is isomorphic to the division algebra of a primitive faithful representation of some subquotient of π; and in Section 3 he shows that the only p-groups with primitive faithful representations are the cyclic groups.

Now, for all $s \geqq 0$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$; and let v_s be the number of conjugacy classes of cyclic subgroups $\sigma \subseteq \pi$ with $|\sigma/[\sigma, N(\sigma)]| = p^s$. Let $p^n = |\pi|$. We say that two elements $g, h \in \pi$ are $\mathbb{Q}\zeta_{p^m}$-conjugate (any $m \geqq 0$) if g is conjugate to h^a for some

$$a \in \mathrm{Gal}\,(\mathbb{Q}\zeta_{p^n}/\mathbb{Q}\zeta_{p^m}) \subseteq \mathrm{Gal}\,(\mathbb{Q}\zeta_{p^n}/\mathbb{Z}) = (\mathbb{Z}/p^n)^*.$$

In other words, $a \equiv 1 \pmod{p^m}$ if $m \geqq 1$, or $p \nmid a$ if $m = 0$.

For each cyclic $\sigma \subseteq \pi$, the number of conjugacy classes of generators of σ is just $\varphi(|\sigma/[\sigma, N(\sigma)]|)$. So for any $m \geqq 0$, the number of $\mathbb{Q}\xi_{p^m}$-conjugacy classes in π is

$$\sum_{s \geqq 0} v_s \cdot \min\{\varphi(p^m), \varphi(p^s)\}. \tag{1}$$

On the other hand

$$\mathrm{rk}\, K_0(\mathbb{Q}\xi_{p^m}[\pi]) = \sum_{s \geqq 0} w_s \cdot \min\{\varphi(p^m), \varphi(p^s)\}; \tag{2}$$

and by Theorem 21.5 in [1], the numbers in (1) and (2) are equal for all m. So $v_s = w_s$ for all $s \geqq 0$. $\square$

Thus, if π is a p-group and $L = Z(\mathbb{Q}\pi)$ is the center, then the reduced norm

$$\nu : K_1(\hat{\mathbb{Q}}_p\pi) \xrightarrow{\cong} (\hat{L}_p)^*$$

is just the product of the determinant maps for the simple summands of $\hat{\mathbb{Q}}_p\pi$. If $\mathfrak{M} \subseteq \mathbb{Q}\pi$ and $\mathcal{O} \subseteq L$ are maximal orders, then ν induces an isomorphism of $K_1(\hat{\mathfrak{M}}_p)$ with $(\hat{\mathcal{O}}_p)^*$: by [9, Theorem 21.6], $\hat{\mathfrak{M}}_p$ is a product of matrix algebras over the components of $\hat{\mathcal{O}}_p$. In particular, $K_1(\hat{\mathfrak{M}}_p)$ can be regarded as a subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$. We let $K_1'(\mathfrak{M})$ and $K_1'(\hat{\mathbb{Z}}_p\pi)$ denote the images of $K_1(\mathfrak{M})$ and $K_1(\hat{\mathbb{Z}}_p\pi)$ in $K_1(\hat{\mathbb{Q}}_p\pi)$; and let $K_1'(\mathfrak{M})\hat{}$ denote the p-adic closure of $K_1'(\mathfrak{M})$.

We will use the following description of $D(\mathbb{Z}\pi)$, based on the formulas in [4].

PROPOSITION 2. *Let π be a p-group, and let $\mathfrak{M} \subseteq \mathbb{Z}\pi$ be a maximal order in $\mathbb{Q}\pi$. Then there is an isomorphism*

$$D(\mathbb{Z}\pi) \cong \operatorname{Coker}\left[K_1'(\hat{\mathbb{Z}}_p\pi) \to K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{}\,\right]$$

which is natural in π.

Proof. Let $L \subseteq \mathbb{Q}\pi$ be its center, and let $\mathcal{O} \subseteq L$ be its maximal order. Since $\hat{\mathbb{Z}}_q[\pi] = \hat{\mathfrak{M}}_q$ for all primes $q \neq p$ [9, Theorem 41.1], Theorems 1 and 2 in [4] reduce to the formula

$$D(\mathbb{Z}\pi) \cong K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M}) \cdot K_1'(\hat{\mathbb{Z}}_p\pi).$$

Furthermore, $K_1(\hat{\mathfrak{M}}_p)/K_1'(\hat{\mathbb{Z}}_p\pi)$ is finite [12, Proposition 8.15]; and so $K_1'(\mathfrak{M})$ can be replaced by its p-adic closure. $\square$

Now let

$$\kappa : \mathbb{F}_p^* \to (\hat{\mathbb{Z}}_p)^*$$

denote the inclusion into the group of $(p-1)$-st roots of unity (with $\kappa(a) = a \pmod p$ for $a \in \mathbb{F}_p^*$). If π is an abelian p-group, then the group homomorphisms $g \to g^{\kappa(a)}$ for $a \in \mathbb{F}_p^*$ and $g \in \pi$ induce actions of $\mathbb{F}_p^*$ on $K_i(\mathbb{Z}\pi)$, $K_i(\hat{\mathbb{Z}}_p\pi)$, $D(\mathbb{Z}\pi)$, etc. The next problem is to find a natural way to do this when π is non-abelian.

Let π be an arbitrary p-group. Then by Proposition 1, the center $Z(\mathbb{Q}\pi)$ is a product of fields $\mathbb{Q}\zeta_{p^s}$ for various $s \geqq 0$. The natural embedding $\mathbb{F}_p^* \subseteq \text{Gal}(\mathbb{Q}\zeta_{p^s}/\mathbb{Q})$ (where $a \in \mathbb{F}_p^*$ sends ζ_{p^n} to $\zeta_{p^n}^{\kappa(a)}$) induces actions of $\mathbb{F}_p^*$ on the centers $Z(\mathbb{Q}\pi)$ and $Z(\hat{\mathbb{Q}}_p\pi)$; and hence on $K_1(\hat{\mathbb{Q}}_p\pi)$. If $\alpha : \pi \to \pi'$ is a homomorphism of p-groups, then

$$\alpha_* : K_1(\hat{\mathbb{Q}}_p\pi) \to K_1(\hat{\mathbb{Q}}_p\pi')$$

is a product of norm, includion, and diagonal maps between the groups of units of the field components of the centers; and is hence $\mathbb{F}_p^*$-linear.

PROPOSITION 3. *Let π be a p-group. Then*
(i) *$K_1'(\hat{\mathbb{Z}}_p\pi)$ is an $\mathbb{F}_p^*$-invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$.*
(ii) *For any $0 \leqq t \leqq p-2$, set*

$${}^tK_1'(\hat{\mathbb{Z}}_p\pi) = \{x \in K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)} : \tau_a(x) = \kappa(a)^t \cdot x \text{ for all } a \in \mathbb{F}_p^*\}$$

(here τ_a denotes the action of $a \in \mathbb{F}_p^*$). Then for $t \neq 1$, ${}^tK_1'(\hat{\mathbb{Z}}_p\pi)$ is generated by induction from cyclic subgroups.

Proof. By [8, Theorem 2], there is an exact sequence

$$0 \to \langle \lambda g : \lambda \in \hat{\mathbb{Z}}_p^*, g \in \pi \rangle \hookrightarrow K_1'(\hat{\mathbb{Z}}_p\pi) \xrightarrow{\Gamma} \overline{I(\hat{\mathbb{Z}}_p\pi)} \xrightarrow{\omega} \pi^{ab} \to 0,$$

natural in π, where

$$\overline{I(\hat{\mathbb{Z}}_p\pi)} = \text{Ker}\,[\varepsilon : \hat{\mathbb{Z}}_p\pi \to \hat{\mathbb{Z}}_p]/\langle gxg^{-1} - x : g \in \pi, x \in \hat{\mathbb{Z}}_p\pi \rangle$$

and

$$\varepsilon(\textstyle\sum \lambda_i g_i) = \sum \lambda_i; \qquad \omega(\sum \lambda_i g_i) = \prod g_i^{\lambda_i}.$$

For $a \in \mathbb{F}_p^*$, let $\hat{\tau}_a$ be the action of a on $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ given by: $\hat{\tau}_a(\sum \lambda_i g_i) = \sum \lambda_i g_i^{\kappa(a)}$. This clearly leaves $\text{Ker}(\omega) = \text{Im}(\Gamma)$ invariant. By the definition of Γ in [8], Γ is $\mathbb{F}_p^*$-linear when π is abelian.

Now set

$$X(\pi) = \text{Im}\,[\text{Ind:} \textstyle\sum \{K_1'(\hat{\mathbb{Z}}_p\sigma) : \sigma \subseteq \pi, \sigma \text{ cyclic}\} \to K_1'(\hat{\mathbb{Z}}_p\pi)]$$

and

$$Y(\pi) = \Gamma^{-1}({}^1\overline{I(\hat{\mathbb{Z}}_p\pi)}) \cap \text{Ker}\,[\varepsilon_* : K_1'(\hat{\mathbb{Z}}_p\pi) \to \hat{\mathbb{Z}}_p^*]. \tag{1}$$

Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, $\mathrm{Ker}\,(\Gamma)\subseteq X(\pi)$, and

$$^t\overline{I(\hat{\mathbb{Z}}_p\pi)} = {}^t\mathrm{Ker}\,(\omega) = {}^t\mathrm{Im}\,(\Gamma) \tag{2}$$

for $t\neq 1$ $(0\leqq t\leqq p-2)$; it follows that

$$K_1'(\hat{\mathbb{Z}}_p\pi) = X(\pi) + Y(\pi). \tag{3}$$

By naturality, $X(\pi)$ is an $\mathbb{F}_p^*$-invariant subgroup of $K_1(\hat{\mathbb{Q}}_p\pi)$, and $\Gamma\,|\,X(\pi)$ is $\mathbb{F}_p^*$-linear. It follows that for n large,

$$Y(\pi)^{p^n} \subseteq {}^1X(\pi). \tag{4}$$

Furthermore,

$$\mathrm{tors}\,(K_1(\hat{\mathbb{Q}}_p\pi))_{(p)} \subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi):$$

the torsion in $K_1(\hat{\mathbb{Q}}_p\pi)$ comes from roots of unity in the center. It follows by (4) that $Y(\pi)\subseteq {}^1K_1(\hat{\mathbb{Q}}_p\pi)$; and hence by (3) that $K_1'(\hat{\mathbb{Z}}_p\pi)$ is $\mathbb{F}_p^*$-invariant. Furthermore, by (1), this shows that Γ is $\mathbb{F}_p^*$-linear. Since $\overline{I(\hat{\mathbb{Z}}_p\pi)}$ is generated by cyclic induction, $^tK_1'(\hat{\mathbb{Z}}_p\pi)$ is generated by cyclic induction for $t\neq 1$ by (2). □

Propositions 2 and 3 now imply the existence of natural actions of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$:

PROPOSITION 4. *For any p-group π, there is a natural linear action of $\mathbb{F}_p^*$ on $D(\mathbb{Z}\pi)$ such that the isomorphism of Proposition 2 is $\mathbb{F}_p^*$-linear.* □

In particular, for any p-group π and $0\leqq t\leqq p-2$, set

$$^tD(\mathbb{Z}\pi) = \{x\in D(\mathbb{Z}\pi): \tau_a(x) = \kappa(a)^t\cdot x \text{ for all } a\in\mathbb{F}_p^*\}.$$

Since $D(\mathbb{Z}\pi)$ is a p-group [2], and $p\nmid|\mathbb{F}_p^*|$,

$$D(\mathbb{Z}\pi) = \sum_{t=0}^{p-2} {}^tD(\mathbb{Z}\pi) \quad\text{and}\quad D(\mathbb{Z}\pi)^+ = \sum_{i=0}^{(p-3)/2} {}^{2i}D(\mathbb{Z}\pi).$$

Here, $D(\mathbb{Z}\pi)^+$ is the group of elements invariant under the involution τ_{-1}; induced by complex conjugation on $Z(\mathbb{Q}\pi)$.

If p is regular, then results of Iwasawa (see, *fx*, [7, Theorem 7.5.2]) show that for any $s \geqq 0$ and any even $0 < t \leqq p-3$,

$${}^t[(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,] = {}^t(\hat{\mathbb{Z}}_p\zeta_{p^s})^*.$$

So by Proposition 1, if $\mathfrak{M} \supseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then ${}^tK_1'(\mathfrak{M})\hat{} = {}^tK_1(\hat{\mathfrak{M}}_p)$; and hence ${}^tD(\mathbb{Z}\pi) = 0$ for such t. In other words:

PROPOSITION 5. *If p is an odd regular prime and π is a p-group, then*

$$D(\mathbb{Z}\pi)^+ = {}^0D(\mathbb{Z}\pi). \quad \square$$

In order to study the groups ${}^0D(\mathbb{Z}\pi)$, we must first describe

$${}^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,]$$

for any $s \geqq 0$. The following result must be well known, but we have been unable to find a reference.

PROPOSITION 6. *For any $s \geqq 0$,*

$${}^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{}\,]_{(p)} \cong \hat{\mathbb{Z}}_p.$$

Proof. This is clear if $s = 0$. So fix $s > 0$, let $K \subseteq \mathbb{Q}\zeta_{p^s}$ be the fixed subfield of $\mathbb{F}_p^*$, and let $R \subseteq K$ be the ring of integers. Then $\Gamma = \mathrm{Gal}\,(K/\mathbb{Q})$ is cyclic of order p^{s-1}. Set $\zeta = \zeta_{p^s}$, and let $\gamma \in \Gamma$ be the generator: $\gamma(\zeta) = \zeta^{p+1}$.

Let

$$\mathfrak{p} = \left\langle z = \prod_{a=1}^{p-1} (1-\zeta^{\kappa(a)}) \right\rangle \subseteq R$$

be the prime ideal over p. Set

$$U' = \mathrm{Ker}\,[N : (\hat{R}_p)^* \to (\hat{\mathbb{Z}}_p)^*],$$

where N is the norm of $\hat{K}_p/\hat{\mathbb{Q}}_p$. Note that $U' \subseteq 1 + p\hat{R}_p$.

Fix $u \in U'$. By Hilbert's Theorem 90, there is $x \in \hat{K}_p^*$ such that $u = \gamma(x)/(x)$. Write $x = z^i v$, where $v \in (\hat{R}_p)^*$ and z is the element defined above. Then

$$\gamma(z^i)/z^i = \prod_{a=1}^{p-1} \left(\frac{1-\zeta^{(p+1)\kappa(a)}}{1-\zeta^{\kappa(a)}}\right)^i \in R^* \cap (1+\mathfrak{p}).$$

Furthermore, $N(v) \in \kappa(\mathbb{F}_p^*) \times (1+p^s\hat{\mathbb{Z}}_p)$ (the norm group has index p^{s-1} by local class field theory). So there exists $w \in (\hat{\mathbb{Z}}_p)^*$ such that $N(w) = w^{p^{s-1}} = N(v)$. Then

$$N(vw^{-1}) = 1 \quad \text{and} \quad \gamma(vw^{-1})/(vw^{-1}) = \gamma(v)/v = x \cdot (\gamma(z^i)/z^i)^{-1}.$$

In other words,

$$U' = \{\gamma(v)/v : v \in U'\} \cdot (R^* \cap (1+\mathfrak{p})). \tag{1}$$

But U' is a $\hat{\mathbb{Z}}_p[\Gamma]$-module, and the closure of $R^* \cap (1+\mathfrak{p})$ is a $\hat{\mathbb{Z}}_p[\Gamma]$-submodule. Since $\hat{\mathbb{Z}}[\Gamma]$ is a local ring with maximal ideal generated by p and $\gamma - 1$, no proper submodule of U' can generate

$$U'/\langle \gamma(v)/v,\, v^p : v \in U' \rangle.$$

So by (1), $R^* \cap (1+\mathfrak{p})$ is dense in $U' = \operatorname{Ker}(N)$; and

$${}^0[(\hat{\mathbb{Z}}_p\zeta_{p^s})^*/(\mathbb{Z}\zeta_{p^s}^*)\hat{\ }]_{(p)} = [(\hat{R}_p)^*/(R^*)\hat{\ }]_{(p)} \cong (\operatorname{Im}(N))_{(p)} \cong \hat{\mathbb{Z}}_p. \quad \square$$

By Proposition 6, if π is a p-group and $\mathfrak{M} \subseteq \mathbb{Q}\pi$ is a maximal order, then ${}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)}$ is a sum of one copy of $\hat{\mathbb{Z}}_p$ for each irreducible $\mathbb{Q}\pi$-module; and is thus (abstractly, at least) isomorphic to $\hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi)$. The key remaining step is to construct a natural isomorphism between these groups; once this is done the isomorphism between ${}^0D(\mathbb{Z}\pi)$ and the Artin cokernel will follow easily.

We temporarily allow p to be an arbitrary prime (possibly $p = 2$). If A is a $\hat{\mathbb{Q}}_p$-algebra, and V is an A-module with $\dim_{\hat{\mathbb{Q}}_p}(V) < \infty$; let $\det(u, V)$, for $u \in A$, denote the determinant over $\hat{\mathbb{Q}}_p$ of $u : V \to V$. Define

$$L : (\hat{\mathbb{Z}}_p)^* \to \hat{\mathbb{Z}}_p$$

by setting $L(u) = 1/p \log(u/\kappa(\bar{u}))$ for $u \in (\hat{\mathbb{Z}}_p)^*$ and $\bar{u} \in \mathbb{F}_p^*$ its reduction mod p (note that $u/\kappa(\bar{u}) \in 1 + p\hat{\mathbb{Z}}_p$).

Now assume A is a finite dimensional semisimple $\hat{\mathbb{Q}}_p$-algebra, and let $\mathfrak{A} \subseteq A$ be any order. Let $V_1, \ldots V_k$ be the distinct irreducible A-modules, and set

$$n_i = [\operatorname{End}_A(V_i) : \hat{\mathbb{Q}}_p].$$

Define a homomorphism

$$\delta = \delta_{\mathfrak{A}} : K_1(\mathfrak{A}) \to \hat{\mathbb{Q}}_p \otimes_{\mathbb{Z}} K_0(A)$$

by setting, for any matrix $u \in GL_r(\mathfrak{A})$,

$$\delta([u]) = \sum_{i=1}^{k} \frac{1}{n_i} L(\det(u, V_i')) \cdot [V_i].$$

PROPOSITION 7. *For any prime p, the maps $\delta_{\mathfrak{A}}$ are natural with respect to homomorphisms between orders in semisimple $\hat{\mathbb{Q}}_p$-algebras.*

Proof. We must show, for any homomorphism $\alpha : A \to B$, orders $\mathfrak{A} \subseteq A$ and $\mathfrak{B} \subseteq B$ such that $\alpha(\mathfrak{A}) \subseteq \mathfrak{B}$, and $u \in \mathfrak{A}^*$, that

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \delta_{\mathfrak{B}}(\alpha(u)) \in \hat{\mathbb{Q}}_p \otimes K_0(B).$$

Let $V_1, \ldots, V_s$ be the irreducible A-modules, and $W_1, \ldots, W_t$ the irreducible B-modules. Define a_{ij}, $b_{ij} \in \mathbb{Z}$ by setting

$$\alpha_*(V_i) = \sum_{j=1}^{t} a_{ij} W_j, \qquad \alpha^*(W_j) = \sum_{i=1}^{s} b_{ij} V_i$$

(where $\alpha_*(V_i) = B \otimes_A V_i$, and $\alpha^*(W_j)$ is W_j regarded as an A-module). We also set

$$m_i = [\mathrm{End}_A(V_i) : \hat{\mathbb{Q}}_p], \qquad n_j = [\mathrm{End}_B(W_j) : \hat{\mathbb{Q}}_p],$$

and write $L_i = L(\det(u, V_i))$ for short. Then

$$\alpha_*(\delta_{\mathfrak{A}}(u)) = \alpha_*\left(\sum_{i=1}^{s} (L_i/m_i)[V_i]\right) = \sum_{i,j} (a_{ij} L_i/m_i)[W_j]$$

and

$$\delta_{\mathfrak{B}}(\alpha(u)) = \sum_{j=1}^{t} n_j^{-1} L(\det(u, \alpha^*(W_j)))[W_j] = \sum_{i,j} (b_{ij} L_i/n_j)[W_j].$$

It remains to check that $(b_{ij}/n_j) = (a_{ij}/m_i)$ for all i, j. But

$$\dim \mathrm{Hom}_A(V_i, W_j) = m_i b_{ij}, \quad \dim \mathrm{Hom}_B(\alpha_* V_i, W_j) = n_j a_{ij};$$

and these two dimensions are equal by [1, Theorem 2.19]. $\square$

We now again restrict to the case where p is odd.

PROPOSITION 8. *Let π be a p-group, let $\mathfrak{M} \subseteq \mathbb{Q}\pi$ be any maximal order, and set $\delta_\pi = \delta_{\hat{\mathbb{Q}}_p\pi}$. Then*

$$\operatorname{Im}[\delta_\pi : K_1(\hat{\mathfrak{M}}_p) \to \hat{\mathbb{Q}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi)] = \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi);$$

and δ_π induces an isomorphism

$$\delta' = \delta'_\pi : {}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(\hat{\mathbb{Q}}_p\pi) \cong \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi).$$

Proof. Using Proposition 1, it will suffice to show that whenever $A \cong M_r(\mathbb{Q}\zeta_{p^s})$ and $\mathfrak{M} \subseteq A$ is a maximal order, then $\delta = \delta_{\hat{A}_p}$ induces an isomorphism

$$\delta' : {}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)} \xrightarrow{\cong} \hat{\mathbb{Z}}_p \otimes K_0(A).$$

By [9, Theorem 21.6], we may assume that $\mathfrak{M} = M_r(\mathbb{Z}\zeta_{p^s})$.

Let $V \cong (\hat{\mathbb{Q}}_p\zeta_{p^s})^r$ be the irreducible $\hat{A}_p$-representation. For any $u \in 1 + J(\hat{\mathfrak{M}}_p)$ (where $J(\hat{\mathfrak{M}}_p)$ is the Jacobson radical),

$$\begin{aligned}\delta(u) &= \frac{1}{\varphi(p^s)} L(\det(u, V)) \cdot [V] \\ &= \frac{1}{p\varphi(p^s)} \log\left(N_{\hat{\mathbb{Q}}_p\zeta_{p^s}/\hat{\mathbb{Q}}_p}(\det_{\hat{\mathbb{Q}}_p\zeta_{p^s}}(u)\right) \cdot [V].\end{aligned}$$

Furthermore, by local class field theory,

$$N \circ \det(1 + J(\hat{\mathfrak{M}}_p)) = 1 + p^s\hat{\mathbb{Z}}_p$$

(or $1 + p\hat{\mathbb{Z}}_p$ if $s = 0$). Since $\log(1 + p^s\hat{\mathbb{Z}}_p) = p^s\hat{\mathbb{Z}}_p$ for $s \geqq 1$, we have

$$\delta(K_1(\hat{\mathfrak{M}}_p)_{(p)} = \delta(1 + J(\hat{\mathfrak{M}}_p)) = \hat{\mathbb{Z}}_p \cdot [V] = \hat{\mathbb{Z}}_p \otimes K_0(A).$$

If u is a global unit, then $N(\det(u)) = \pm 1$, and so $\delta(u) = 0$. Furthermore, δ is $\mathbb{F}_p^*$-linear when $K_0(A)$ is given the trivial action; and so δ induces a surjection

$$\delta' : {}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)} \to \hat{\mathbb{Z}}_p \otimes K_0(A) \cong \hat{\mathbb{Z}}_p.$$

But the two groups are isomorphic by Proposition 6, and so δ' is an isomorphism. □

We can now prove the main result. Recall that the Artin cokernel $A_{\mathbb{Q}}(\pi)$ is defined by

$$A_{\mathbb{Q}}(\pi) = \operatorname{Coker}\left[\operatorname{Ind}: \sum \{R_{\mathbb{Q}}(\sigma) : \sigma \subseteq \pi,\ \sigma \text{ cyclic}\} \to R_{\mathbb{Q}}(\pi)\right].$$

THEOREM 9. *For any p-group π (p-odd), δ' induces an isomorphism*

$$\delta''_{\pi} : {}^0D(\mathbb{Z}\pi) \xrightarrow{\cong} A_{\mathbb{Q}}(\pi).$$

Proof. Let C be the set of cyclic subgroups of π. Propositions 2, 7, and 8 combine to give the following commutative diagram with exact rows:

$$\begin{tikzcd}
\sum_{\sigma\in C} {}^0K_1'(\hat{\mathbb{Z}}_p\sigma) \arrow[r, "\Sigma\eta_\sigma"] \arrow[d, "I_1"] & \sum_{\sigma\in C} \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma) \arrow[r, "\Sigma\theta_\sigma"] \arrow[d, "I_2"] & \sum_{\sigma\in C} {}^0D(\mathbb{Z}\sigma) \arrow[r] \arrow[d, "I_3"] & 0 \\
{}^0K_1'(\hat{\mathbb{Z}}_p\pi) \arrow[r, "\eta_\pi"] & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi) \arrow[r, "\theta_\pi"] & {}^0D(\mathbb{Z}\pi) \arrow[r] & 0
\end{tikzcd}$$

Here I_1, I_2, and I_3 are the induction maps; and θ_π and η_π are the composites ($\mathfrak{M} \subseteq \theta\pi$ a maximal order):

$$\theta_\pi : \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi) \xrightarrow[\cong]{(\delta')^{-1}} {}^0[K_1(\hat{\mathfrak{M}}_p)/K_1'(\mathfrak{M})\hat{\ }]_{(p)} \longrightarrow {}^0D(\mathbb{Z}\pi)$$

(the second map being the map of Proposition 2), and

$$\eta_\pi : {}^0K_1'(\hat{\mathbb{Z}}_p\pi) \to {}^0K_1(\hat{\mathfrak{M}}_p) \xrightarrow{\delta_\pi} \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\pi).$$

By Proposition 3(ii), I_1 is onto. Assume that ${}^0D(\mathbb{Z}\sigma) = 0$ for any cyclic p-group σ. It then follows by diagram chasing that

$${}^0D(\mathbb{Z}\pi) \cong \operatorname{Coker}(I_2) \cong A_{\mathbb{Q}}(\pi).$$

($A_{\mathbb{Q}}(\pi)$ is a p-group by the Artin induction theorem: see, for example, [1, Theorem 15.4]).

It remains to check that ${}^0D(\mathbb{Z}\sigma) = 0$ for cyclic σ: This is implicit in [6], [5], and [15]; but doesn't seem to be stated explicitly. If $|\sigma| \leqq p$, then $D(\mathbb{Z}\sigma) = 0$ by [10, Theorem 6.24].

So assume $|\sigma| = p^n$ for $n \geqq 2$. Let $\rho \subseteq \sigma$ be the order p subgroup, and assume inductively that ${}^0D(\mathbb{Z}[\sigma/\rho]) = 0$. There is a commutative diagram

$$\begin{array}{ccccc}
\hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\rho) & \xrightarrow{i_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma) & \xrightarrow{j_*} & \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho) \\
\downarrow{\scriptstyle \theta_\rho} & & \downarrow{\scriptstyle \theta_\sigma} & & \downarrow{\scriptstyle \theta_{\sigma/\rho}} \\
0 = {}^0D(\mathbb{Z}\rho) & \longrightarrow & {}^0D(\mathbb{Z}\sigma) & \longrightarrow & {}^0D(\mathbb{Z}[\sigma/\rho]) = 0,
\end{array}$$

where i_* and j_* are induced by inclusion and projection. Since $K_1(\hat{\mathbb{Z}}_p\sigma)$ maps onto $K_1(\hat{\mathbb{Z}}_p[\sigma'\rho])$,

$$j_*(\mathrm{Ker}\,(\theta_\sigma)) = \mathrm{Ker}\,(\theta_{\sigma/\rho}) = \hat{\mathbb{Z}}_p \otimes R_{\mathbb{Q}}(\sigma/\rho).$$

In other words, $\theta_\sigma \mid \mathrm{Ker}\,(j_*)$ is onto. Furthermore, $\mathrm{Ker}\,(j_*) \subseteq \mathrm{Im}\,(i_*)$: if $V \cong \mathbb{Q}\zeta_{p^n}$ and $W \cong \mathbb{Q}\zeta_p$ are the faithful irreducible $\mathbb{Q}\sigma$- and $\mathbb{Q}\rho$-representations, then $[V]$ generates $\mathrm{Ker}\,(j_*)$, and $V = \mathrm{Ind}_\rho^\sigma(W)$. We thus get that $\theta_\sigma \mid \mathrm{Im}\,(i_*)$ is onto, but $\theta_\sigma \circ i_* = 0$, and so ${}^0D(\mathbb{Z}\sigma) = 0$.

One easy consequence of Theorem 9 is an alternate proof, for odd p-groups, of Martin Taylor's theorem [13] involving the image $T(\mathbb{Z}\pi)$ of the Swan homomorphism. $T(\mathbb{Z}\pi)$ is the group of all elements

$$[\Sigma, n] - [\mathbb{Z}\pi] \in D(\mathbb{Z}\pi),$$

for $(n, |\pi|) = 1$, where $[\Sigma, n]$ is the projective module

$$[\Sigma, n] = n\mathbb{Z}\pi + \mathbb{Z} \cdot \Big(\sum_{g \in \pi} g\Big) \subseteq \mathbb{Z}\pi.$$

So if π is a p-group and $\mathfrak{M} \subseteq \mathbb{Z}\pi$ is a maximal order in $\mathbb{Q}\pi$, then $[\Sigma, n] - [\mathbb{Z}\pi]$ corresponds, under the identification in Proposition 2, to the element of $K_1(\hat{\mathfrak{M}}_p)$ which is $n \in (\hat{\mathbb{Z}}_p)^*$ at the identity component and 1 at all other components (in particular, $T(\mathbb{Z}\pi) \subseteq {}^0D(\mathbb{Z}\pi)$). The isomorphism of Theorem 9 thus sends $T(\mathbb{Z}\pi)$ to the group of multiples of the identity in

$$R_{\mathbb{Q}}(\pi)/\sum \{\mathrm{Ind}_\sigma^\pi(R_{\mathbb{Q}}(\sigma)) : \sigma \subseteq \pi \text{ cyclic}\}.$$

In other words:

THEOREM 10. (*M. Taylor* [13]) *For any p-group π, $T(\mathbb{Z}\pi)$ is cyclic of order equal to the Artin exponent of π.* □

The computation of $|{}^0D(\mathbb{Z}\pi)|$ can now be carried out, using the same idea as for the calculation in [3]: that of comparing discriminants. We first consider the Artin cokernel of an arbitrary finite group.

THEOREM 11. *Let π be any finite group, and write*

$$\mathbb{Q}\pi \cong \prod_{i=1}^{k} M_{r_i}(D_i),$$

where the D_i are division algebras. Let X be a set of conjugacy class representatives for cyclic subgroups $\sigma \subseteq \pi$. Then

$$|A_{\mathbb{Q}}(\pi)| = \left[\left(\prod_{\sigma\in X} \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma|\right) \Big/ \left(\prod_{i=1}^{k} [D_i:\mathbb{Q}]\right)\right]^{1/2}$$

Proof. For convenience, set

$$G = \sum_{\sigma\in X} \operatorname{Ind}_{\sigma}^{\pi}(R_{\mathbb{Q}}(\sigma)) \subseteq R_{\mathbb{Q}}(\pi).$$

Then

$$|R_{\mathbb{Q}}(\pi)/G| = [d(G)/d(R_{\mathbb{Q}}(\pi))]^{1/2}; \tag{1}$$

where $d(-)$ denotes discriminant with respect to the usual inner product

$$\langle [V], [W] \rangle = \frac{1}{|\pi|} \sum_{g\in\pi} \chi_V(g)\chi_W(g).$$

For each i, let V_i denote the irreducible representation of $M_{r_i}(D_i)$; then

$$\begin{aligned} \langle [V_i], [V_j] \rangle = \dim_{\mathbb{Q}} (\operatorname{Hom}_{\mathbb{Q}\pi}(V_i, V_j)) &= 0 && \text{if } i \neq j \\ &= [D_i:\mathbb{Q}] && \text{if } i = j. \end{aligned}$$

So

$$d(R_{\mathbb{Q}}(\pi)) = \prod_{i=1}^{k} [D_i:\mathbb{Q}]. \tag{2}$$

To compute $d(G)$, consider first the set

$$S = \{[\mathbb{Q}(\pi/\sigma)] : \sigma \in X\} \subseteq R_{\mathbb{Q}}(\pi);$$

where $\mathbb{Q}(\pi/\sigma)$ denotes the permutation representation with $\mathbb{Q}$-basis π/σ. These elements generate G: since

$$\mathbb{Q}(\pi/\sigma) = \mathrm{Ind}_\sigma^\pi(\mathbb{Q}) \in G,$$

and $R_\mathbb{Q}(\sigma)$ $(\sigma \in X)$ is generated by the elements

$$\{[\mathbb{Q}(\sigma/\tau)] = \mathrm{Ind}_\tau^\sigma([\mathbb{Q}]) : \tau \subseteq \sigma\}.$$

Also, $\mathrm{rk}\,(R_\mathbb{Q}(\pi)) = |X|$ (see [1, Theorem 21.5]); and so S is a basis for G.

It follows that

$$d(G) = \det(M) \tag{3}$$

where $M = (M_{\sigma\tau})_{\sigma,\tau\in X}$ is the matrix defined by

$$M_{\sigma\tau} = \langle[\mathbb{Q}(\pi/\sigma)], [\mathbb{Q}(\pi/\tau)]\rangle.$$

For $\sigma \in X$, let χ_σ denote the character of $\mathbb{Q}(\pi/\sigma)$. For any $x \in \pi$,

$$\chi_\sigma(x) = \frac{1}{|\sigma|}\cdot \#\{g \in \pi : xg\sigma = g\sigma\} = \frac{1}{|\sigma|}\cdot \#\{g \in \pi : x \in g\sigma g^{-1}\}.$$

Hence, for σ, $\tau \in X$,

$$\begin{aligned} M_{\sigma\tau} &= \frac{1}{|\pi|}\sum_{x\in\pi}\chi_\sigma(x)\chi_\tau(x) = \frac{1}{|\sigma|\cdot|\tau|}\cdot\frac{1}{|\pi|}\sum_{g,h\in\pi}|g\sigma g^{-1}\cap h\tau h^{-1}| \\ &= \frac{1}{|\sigma|\cdot|\tau|}\sum_{g\in\pi}|\sigma\cap g\tau g^{-1}|. \end{aligned} \tag{4}$$

To simplify what follows, define, for $n \geqq 1$ and $m \geqq 1$,

$$\varphi_m(n) = n - \sum_{\substack{d|n \\ d<m}} \varphi(d).$$

Note in particular that $\varphi_1(n) = n$, $\varphi_n(n) = \varphi(n)$, and $\varphi_m(n) = 0$ for $m > n$. Let $N = \max\{|\sigma| : \sigma \in X\}$; and define, for $1 \leqq m \leqq N$:

$$X_m = \{\sigma \in X : |\sigma| = m\} \qquad Y_m = \{\sigma \in X : |\sigma| \geqq m\} = \bigcup_{i\geqq m} X_i.$$

For all $0 \leqq m \leqq N$, define a matrix $M^{(m)} = (M^{(m)}_{\sigma\tau})_{\sigma,\tau \in Y_m}$, by setting

$$M^{(m)}_{\sigma\tau} = \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|).$$

In particular, $M^{(1)} = M$.

Fix $1 \leqq m \leqq N$. For $\sigma, \tau \in X_m$ (i.e., $|\sigma| = |\tau| = m$),

$$\begin{aligned} M^{(m)}_{\sigma\tau} &= m^{-2} \sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) \\ &= (\varphi(m)/m^2) \cdot \#\{g \in \pi : \sigma = g\tau g^{-1}\} = 0 \qquad \text{if} \quad \sigma \neq \tau \\ &= \frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma| \qquad \text{if} \quad \sigma = \tau \end{aligned}$$

In particular,

$$\det(M^{(N)}) = \prod_{\sigma \in X_N} \left[\frac{\varphi(|\sigma|)}{|\sigma|} |N(\sigma)/\sigma|\right]. \tag{5}$$

If $1 \leqq m < N$ and $\sigma, \tau \in Y_{m+1}$ (i.e., $|\sigma|, |\tau| > m$), consider the entries $M^{(m)}_{\sigma\rho}$ for $\rho \in X_m$ ($|\rho| = m$). By definition, $M^{(m)}_{\sigma\rho} = 0$ unless $|\sigma \cap g\rho g^{-1}| \geqq m$ for some g; i.e., unless $g\rho g^{-1} \subseteq \sigma$. If $m \nmid |\sigma|$, then these $M^{(m)}_{\sigma\rho}$ all vanish; and also $M^{(m)}_{\sigma\tau} = M^{(m+1)}_{\sigma\tau}$ ($\varphi_m(n) = \varphi_{m+1}(n)$ if $m \nmid n$). If $m \mid |\sigma|$, let $\rho \in X_m$ be the unique element conjugate to a subgroup of σ; then

$$\begin{aligned} M^{(m)}_{\sigma\tau} - (M^{(m)}_{\sigma\rho}/M^{(m)}_{\rho\rho}) \cdot (M^{(m)}_{\rho\tau}) &= \frac{1}{|\sigma| \cdot |\tau|} \left[\sum_{g \in \pi} \varphi_m(|\sigma \cap g\tau g^{-1}|) - \sum_{g \in \pi} \varphi_m(|\rho \cap g\tau g^{-1}|)\right] \\ &= \frac{1}{|\sigma| \cdot |\tau|} \sum_{g \in \pi} \varphi_{m+1}(|\sigma \cap g\tau g^{-1}|) = M^{(m)}_{\sigma\tau}. \end{aligned}$$

In other words, for all $\sigma, \tau \in Y_{m+1}$,

$$M^{(m+1)}_{\sigma\tau} = M^{(m)}_{\sigma\tau} - \sum_{\rho \in X_m} (M^{(m)}_{\sigma\rho}/M^{(m)}_{\rho\rho}) \cdot M^{(m+1)}_{\rho\tau};$$

and $M^{(m+1)}$ is obtained from $M^{(m)}$ by elementary operations which eliminate all entries $M^{(m)}_{\rho\tau}$ for $\rho \in X_m$, $\tau \in Y_{m+1}$. It follows that

$$\begin{aligned} \det(M^{(m)}) &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} M^{(m)}_{\sigma\sigma} \\ &= \det(M^{(m+1)}) \cdot \prod_{\sigma \in X_m} \left[\frac{\varphi(|\sigma|)}{|\sigma|} \cdot |N(\sigma)/\sigma|\right]. \end{aligned}$$

Combining this with (5) gives

$$d(G) = \det(M) = \det(M^{(1)}) = \prod_{\sigma\in X}\left[\frac{\varphi(|\sigma|)}{|\sigma|}\cdot|N(\sigma)/\sigma|\right].$$

Finally, combined with (1) and (2), this gives the desired formula for $|R_{\mathbb{Q}}(\pi)/G|$. □

When π is a p-group (recall that p is always odd), the above formula can be reformulated solely in terms of cyclic subgroups:

THEOREM 12. *Let π be a p-group, and let X be a set of conjugacy class representatives for cyclic subgroups $\sigma\subseteq\pi$. Then*

$$|{}^0D(\mathbb{Z}\pi)| = \left[\prod_{\sigma\in X}\frac{|N(\sigma)/\sigma|^2}{|Z(\sigma)|}\right]^{1/2}.$$

Proof. By Proposition 1, $\mathbb{Q}\pi\cong\prod_{i=1}^k M_{r_i}(D_i)$, when the D_i are fields, and

$$\prod_{i=1}^k [D_i:\mathbb{Q}] = \prod_{\sigma\in X}\varphi(|\sigma|\cdot|Z(\sigma)|/|N(\sigma)|)$$

$$= \prod_{\sigma\in X}[\varphi(|\sigma|)\cdot|Z(\sigma)|/|N(\sigma)|].$$

The result now follows by substitution into the formula of Theorem 11. □

Finally, for the sake of completeness, we extend Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ in [3] to arbitrary (not necessarily abelian) p-groups π. For any such π, $\overline{\hat{\mathbb{Q}}_p\pi}$ will denote the group ring modulo conjugation:

$$\overline{\hat{\mathbb{Q}}_p\pi} = \hat{\mathbb{Q}}_p\pi/\langle x-gxg^{-1}: x\in\hat{\mathbb{Q}}_p\pi,\ g\in\pi\rangle = \hat{\mathbb{Q}}_p\pi/\langle xy-yx: x, y\in\hat{\mathbb{Q}}_p\pi\rangle.$$

This can be regarded as the $\hat{\mathbb{Q}}_p$-vector space with basis the set of conjugacy classes in π. Let $\overline{\hat{\mathbb{Z}}_p\pi}\subseteq\overline{\hat{\mathbb{Q}}_p\pi}$ be the image of $\hat{\mathbb{Z}}_p\pi$; and let $\overline{\mathfrak{M}}\subseteq\overline{\hat{\mathbb{Q}}_p\pi}$ denote the image of any maximal order $\mathfrak{M}\subseteq\hat{\mathbb{Q}}_p\pi$.

If F is any field and $r\geqq 1$, it is easy to check that

$$\langle xy-yx: x, y\in M_r(F)\rangle = \operatorname{Ker}[\operatorname{tr}: M_r(F)\to F].$$

Thus, if $\hat{\mathbb{Q}}_p\pi\cong\prod M_{r_i}(F_i)$, then $\overline{\hat{\mathbb{Q}}_p\pi}\cong\prod F_i$, and the projection $\hat{\mathbb{Q}}_p\pi\to\overline{\hat{\mathbb{Q}}_p\pi}$ is the product of the trace maps. If $R_i\subseteq F_i$ is the ring of integers, then any maximal order $\mathfrak{M}_i \subseteq M_{r_i}(F_i)$ is conjugate to $M_{r_i}(R_i)$ [9, Theorem 21.6]; and so $\operatorname{tr}(\mathfrak{M}_i) = R_i$. In particular, $\bar{\mathfrak{M}} = \prod R_i$ under the above identification (and is thus independent of the choice of maximal order).

PROPOSITION 13. *Let π be a p-group, and let $\overline{\hat{\mathbb{Z}}_p\pi}$, $\bar{\mathfrak{M}} \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$ be as above. Then, for any odd $1 \leqq t \leqq p-2$,*

$$\begin{aligned} |{}^tD(\mathbb{Z}\pi)| &= |{}^t(\bar{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p\pi})| && \text{if} \quad t \neq 1 \\ &= |{}^1(\bar{\mathfrak{M}}/\overline{\hat{\mathbb{Z}}_p\pi})| \cdot \frac{|\pi^{ab}|}{|\operatorname{tors}_p(\mathbb{Q}\pi)^*|} && \text{if} \quad t = 1. \end{aligned}$$

Proof. Let $\mathfrak{M} \supseteq \mathbb{Z}\pi$ be a maximal order, and write

$$\hat{\mathbb{Q}}_p\pi = \prod_{i=1}^k A_i; \qquad A_i \cong M_{r_i}(F_i); \qquad \hat{\mathfrak{M}}_p \cong \prod_{i=1}^k \mathfrak{M}_i;$$

where F_i are fields and $\mathfrak{M}_i \subseteq A_i$ is a maximal order for all i. Given any $x \in \mathfrak{M}_i$ which is topologically nilpotent (i.e., $p \mid x^n$ for some n), the series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

converges in A_i. We claim that for such x,

$$\operatorname{tr}(\log(1+x)) = \log(\det(1+x)) \in F_i. \tag{1}$$

To see (1), choose n such that $p \mid x^{p^n}$. Then for any $m \geqq 0$, $(1+x)^{p^{m+n}} = 1 + p^{m+1}y$ for some $y \in \mathfrak{M}$, and

$$\begin{aligned} \log(\det(1+x)^{p^{n+m}}) &= \log(\det(1+p^{m+1}y)) \equiv p^{m+1} \cdot \operatorname{tr}(y) \\ &\equiv \operatorname{tr}(\log(1+p^{m+1}y)) = \operatorname{tr}(\log(1+x)^{p^{n+m}}) \pmod{p^{2m+2}}. \end{aligned}$$

So for all $m \geqq 0$,

$$\log(\det(1+x)) \equiv \operatorname{tr}(\log(1+x)) \pmod{p^{m-n+2}};$$

and (1) holds.

In particular, $\log(1+x)$ for $x \in J(\hat{\mathbb{Z}}_p\pi)$ or $x \in J(\hat{\mathfrak{M}}_p)$ induces homomorphisms

$$L_1 : K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)} \to \overline{\hat{\mathbb{Q}}_p\pi}; \qquad L_2 : K_1(\hat{\mathfrak{M}}_p)_{(p)} \to \overline{\hat{\mathbb{Q}}_p\pi}$$

such that $L_1 = L_2 \mid K_1'(\hat{\mathbb{Z}}_p\pi)_{(p)}$. (Here J means Jacobson radical; note that $J(\hat{\mathbb{Z}}_p\pi) \not\subseteq J(\hat{\mathfrak{M}}_p)$ in general.) Furthermore,

$$\operatorname{Ker}(L_2) = \operatorname{tors}_p(\hat{\mathfrak{M}}_p^*) = \operatorname{tors}_p(\mathbb{Q}\pi)^*; \qquad \operatorname{Ker}(L_1) \cong \pi^{ab};$$

and so by Proposition 2, for all odd t:

$$|{}^t D(\mathbb{Z}\pi)| = [{}^t\operatorname{Im}(L_2) : {}^t\operatorname{Im}(L_1)]. \tag{2}$$

For any $\hat{\mathbb{Z}}_p$-lattices $M_1, M_2 \subseteq \overline{\hat{\mathbb{Q}}_p\pi}$, we write for short

$$[M_1 : M_2] = [M_1 : M_1 \cap M_2]/[M_2 : M_1 \cap M_2].$$

By Theorem 2 in [8], for any $1 \leqq t \leqq p-2$,

$$\begin{aligned}\left(1 - \frac{1}{p}\Phi\right)({}^t\operatorname{Im}(L_1)) &= {}^t\overline{\hat{\mathbb{Z}}_p\pi} && \text{if} \quad t \neq 1 \\ &= \operatorname{Ker}[{}^1\overline{\hat{\mathbb{Z}}_p\pi} \to \pi^{ab}] && \text{if} \quad t = 1\end{aligned}$$

Here, $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$; and Φ is nilpotent (${}^t\overline{\hat{\mathbb{Q}}_p\pi}$ lies in the augmentation ideal, since $t \neq 0$). So

$$\det\left(1 - \frac{1}{p}\Phi\right) = 1,$$

and hence

$$\begin{aligned}[{}^t\overline{\hat{\mathbb{Z}}_p\pi} : {}^t\operatorname{Im}(\mathbb{L}_1)] &= 1 && \text{if} \quad t \neq 1 \\ &= |\pi^{ab}| && \text{if} \quad t = 1.\end{aligned} \tag{3}$$

Finally, note that for $s \geqq 0$,

$$\begin{aligned}[{}^t\hat{\mathbb{Z}}_p\zeta_{p^s} : \log({}^t(\hat{\mathbb{Z}}_p\zeta_{p^s})^*)] &= 1 && \text{if} \quad t \neq 1 \\ &= p^s && \text{if} \quad t = 1:\end{aligned} \tag{4}$$

this follows by noting that $\log(1 + p\hat{\mathbb{Z}}_p\zeta_{p^s}) = p\hat{\mathbb{Z}}_p\zeta_{p^s}$, and then counting orders of the quotients. Since by (1),

$$\operatorname{Im}(L_2) = \prod_{i=1}^{k} \log(R_i^*)_{(p)} \subseteq \prod_{i=1}^{k} F_i \cong \overline{\hat{\mathbb{Q}}_p\pi}$$

($R_i \subseteq F_i$ the ring of integers), (4) implies that

$$\begin{aligned} [{}^t\mathfrak{M} : {}^t\mathrm{Im}\,(L_2)] &= 1 && \text{if} \quad t \neq 1 \\ &= \prod |\mathrm{tors}_p\,(R_i^*)| = |\mathrm{tors}_p\,(\mathbb{Q}\pi)^*| && \text{if} \quad t = 1. \end{aligned} \tag{5}$$

So (2), (3), and (5) combine to prove the proposition. □

Generalizing Fröhlich's formula for $|D(\mathbb{Z}\pi)^-|$ is now straightforward:

THEOREM 14. *Let π be a p-group (p odd). Let $S \subseteq \pi$ be a set of conjugacy class representatives for all $1 \neq g \in \pi$. Set*

$$p^n = |\pi^{ab}| \quad \textit{and} \quad p^k = \prod_{g \in S} |Z(g)|.$$

For $s \geqq 1$, let w_s be the number of simple summands of $\mathbb{Q}\pi$ which are matrix algebras over $\mathbb{Q}\zeta_{p^s}$. Then $|D(\mathbb{Z}\pi)^-| = p^N$, where

$$N = \frac{1}{4}\left[k + 4n - \sum_{s \geqq 1} w_s(sp^s - (s+1)p^{s-1} + 4s + 1)\right].$$

Proof. Let $\hat{\mathbb{Z}}_p\pi \subseteq \mathfrak{M} \subseteq \hat{\mathbb{Q}}_p\pi$ be as above. By Proposition 13,

$$|D(\mathbb{Z}\pi)^-| = |(\mathfrak{M}/\hat{\mathbb{Z}}_p\pi)^-| \cdot p^n \cdot \left[\prod_{s \geqq 1} p^{sw_s}\right]^{-1}. \tag{1}$$

Write $\hat{\mathbb{Q}}_p\pi = \prod_{i=1}^k A_i$, where $A_i \cong M_{r_i}(F_i)$ and the F_i are fields. As before, the trace maps $\mathrm{tr}_i : A_i \to F_i$ induce an identification of $\hat{\mathbb{Q}}_p\pi$ with $\prod F_i$. Let $\mathrm{pr}_i : \hat{\mathbb{Q}}_p\pi \to A_i$ be the projection; and define an inner product on $\hat{\mathbb{Q}}_p\pi$ by setting

$$\langle x, y \rangle = \sum_{i=1}^{k} \mathrm{tr}_{F_i/\hat{\mathbb{Q}}_p}\,(\mathrm{tr}_i \circ \mathrm{pr}_i\,(x) \cdot \mathrm{tr}_i \circ \mathrm{pr}_i\,(y)) \quad (x, y \in \hat{\mathbb{Q}}_p\pi).$$

Since $\mathfrak{M} \subseteq \prod F_i$ is the product of the rings of integers, we have by definition discriminants

$$d(\mathfrak{M}) = \prod_i \Delta(F_i) \quad \text{and} \quad d(\mathfrak{M}^+) = 2^{rk(\mathfrak{M})-1} \cdot \prod_i \Delta(F_i \cap \mathbb{R}).$$

Here $\Delta(F_i)$, $\Delta(F_i \cap \mathbb{R})$ denote the discriminants over $\mathbb{Q}$; and the power of 2 arises due to using the trace over F_i instead of $F_i \cap \mathbb{R}$.

By [16, Proposition 7-5-7], for $s \geqq 1$,

$$\Delta(\mathbb{Q}\zeta_{p^s}) = p^{p^s(ps-s-1)}.$$

By the same proof, or by the composition formula applied to the fields $\mathbb{Q}\zeta_{p^s}/\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}$ [16, Corollary 3-7-20]:

$$\Delta(\mathbb{R} \cap \mathbb{Q}\zeta_{p^s}) = p^{\frac{1}{2}[sp^s-(s+1)p^{s-1}-1]}.$$

Hence, $d(\mathfrak{M}^-) = p^{N_0}$, where

$$N_0 = \tfrac{1}{2} \sum_{s \geqq 1} (sp^s - (s+1)p^{s-1} + 1). \tag{2}$$

Now fix $g, h \in \pi$. For any given $1 \leqq i \leqq k$,

$$\mathrm{tr}_{F_i/\hat{\mathbb{Q}}_p}(\mathrm{tr}_i \circ \mathrm{pr}_i\,(g) \cdot \mathrm{tr}_i \circ \mathrm{pr}_i\,(h)) = \sum_{j=1}^{t} \chi_j(g)\chi_j(h),$$

where $\chi_1, \ldots, \chi_t$ are the distinct irreducible (complex) characters contained in the summand A_i. Let π^* denote the set of all irreducible complex characters. Then

$$\begin{aligned} \langle g, h\rangle = \sum_{\chi \in \pi^*} \chi(g)\chi(h) &= 0 && \text{if} \quad g \text{ not conjugate to } h^{-1} \\ &= |Z(g)| && \text{if} \quad g \text{ is conjugate to } h^{-1} \end{aligned}$$

by the second orthogonality relation [1, Proposition 9.26]. Hence, eliminating factors prime to p,

$$d(\overline{\hat{\mathbb{Z}}_p\pi}) = \left[\prod_{g \in S} |Z(g)|\right]^{1/2} = p^{k/2}. \tag{3}$$

By (2) and (3), $|(\mathfrak{M}'\overline{\hat{\mathbb{Z}}_p\pi})^-| = p^{N_1}$, where

$$N_1 = \frac{1}{4}\left[k - \sum_{s \geqq 1} (sp^s - (s+1)p^{s-1} + 1)\right].$$

Together with (1), this proves the theorem. □

This can also be reformulated solely in terms of cyclic subgroups of π:

THEOREM 15. *Let π be any p-group, and let X_0 be a set of conjugacy class representatives for cyclic subgroups $1 \neq \sigma \subseteq \pi$. For each $\sigma \in X_0$, set*

$$a_\sigma = \mathrm{ord}_p\, |N(\sigma)/\sigma|; \qquad b_\sigma = \mathrm{ord}_p\, (|\sigma| \cdot |Z(\sigma)|/|N(\sigma)|).$$

Then

$$\mathrm{ord}_p\, |D(\mathbb{Z}\pi)^-| = \mathrm{ord}_p\, |\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1].$$

Proof. Let $w_s (s \geqq 1)$ and k be as in Theorem 14. Then each $\sigma \in X_0$ has $\varphi(p^{b_\sigma})$ conjugacy classes of generators, and so

$$k = \sum_{\sigma \in X_0} \varphi(p^{b_\sigma}) \cdot (a_\sigma + b_\sigma).$$

By Proposition 1, w_s is the number of $\sigma \in X_0$ such that $b_\sigma = s$. So Theorem 14 takes the form

$$\begin{aligned} \mathrm{ord}_p\, |D(\mathbb{Z}\pi)^-| &= \mathrm{ord}_p\, |\pi^{ab}| \\ &\quad + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma + b_\sigma)\varphi(p^{b_\sigma}) - b_\sigma p^{b_\sigma} + (b_\sigma + 1)p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \mathrm{ord}_p\, |\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [a_\sigma \varphi(p^{b_\sigma}) + p^{b_\sigma - 1} - 4b_\sigma - 1] \\ &= \mathrm{ord}_p\, |\pi^{ab}| + \tfrac{1}{4} \sum_{\sigma \in X_0} [(a_\sigma - 1)\varphi(p^{b_\sigma}) + p^{b_\sigma} - 4b_\sigma - 1]. \quad \square \end{aligned}$$

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Satake compactifications

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Introduction

Let G be a semi-simple algebraic group defined over $\mathbf{Q}$, and let X denote the corresponding symmetric space, which we assume to be non-compact. Let Γ be an arithmetic subgroup of G. The quotients $\Gamma\backslash X$ are interesting spaces, in general non-compact. For example, when G is the standard form of $\mathrm{Sp}(2n, \mathbf{R})$, the group of $(2n)\times(2n)$ symplectic matrices, X is the Siegel upper half-space of genus n, and $\Gamma\backslash X$ is, for suitable Γ, the moduli space of n-dimensional principally polarized Abelian varieties with corresponding level structure. In order to discuss the geometry of these spaces, people have introduced various methods for compactifying them.

Though there are some ideas in the work of Siegel, the modern starting point for the theory of compactifications is the work of Satake (see [8], [9]). To each locally faithful finite-dimensional representation τ of G, he constructed an embedding of X in some real projective space, and took the closure, X^*_τ (see our (2.1)). In fact, as he observed, the homeomorphism type of X^*_τ can be described explicitly, and depends on τ only through the orthogonality relations between its restricted highest weight and the simple $\mathbf{R}$-roots of G. The boundary of X^*_τ can be written as a union of so-called *boundary components*. In [9], Satake observed that by taking only the *rational* boundary components (defined suitably), one could use the space X^*_τ to construct a Hausdorff compactification of $\Gamma\backslash X$ in certain cases. This included the case of the Siegel upper half-spaces that he had worked out slightly earlier. The construction was extended somewhat by Borel in [3], where notions from the theory of algebraic groups were added to the discussion.

Assume now that X is Hermitian symmetric. In [2], Baily and Borel compactified all arithmetic quotients of such X. The procedure used the realization of X as a bounded symmetric domain to generate the boundary. In these cases, it is known that the closure of the bounded domain is homeomorphic to X^*_τ for suitably chosen τ (see our (3.11)). Thus, the Baily–Borel compactification of $\Gamma\backslash X$

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is a *generalized Satake compactification.*[1] Moreover, it is proved in [2] that the compactification becomes, via the projective embedding defined by automorphic forms of sufficiently high weight, a normal complex algebraic variety. The rather nasty nature of the singularities of these spaces has led algebraic geometers to seek smooth models. Mumford et al. have an elaborate procedure for constructing desingularizations of the varieties by means of toroidal embeddings [1].

The invention of intersection homology, a theory with Poincaré duality even for singular varieties, by Goresky and MacPherson has opened the possibility of avoiding the complicated construction, and non-canonical nature, of desingularizations. This is especially promising in light of the link we discovered between the L_2 cohomology of $\Gamma\backslash X$ with respect to natural metrics, and the intersection homology of the Baily–Borel compactification [10, (6.11)ff.]. In fact, we have conjectured that they are always isomorphic. Borel has verified this for all groups of $\mathbf{Q}$-rank one. We would, of course, like to prove it in higher rank, where the local topology is more complicated, as well.

The most natural place on which to study the L_2 cohomology is the manifold with corners $\Gamma\backslash\bar{X}$ defined by Borel and Serre in [4], or what comes to almost the same thing, on the maximal Satake compactification (where τ is generic; see [10, §4(a)]). The reason for this is that there are distinguished neighborhoods of compact subsets of the faces of the boundary of $\Gamma\backslash\bar{X}$; with respect to associated coordinates, the metric has explicit asymptotic formulas. Now, the verification of our conjecture is equivalent to certain vanishing assertions for the L_2 cohomology of neighborhoods of points on the Baily–Borel compactification. It seems to be a good idea, then, to express these neighborhoods in terms of the distinguished neighborhoods on $\Gamma\backslash\bar{X}$, for then we could try to patch together the local L_2 cohomology by the methods of [10]. This approach works for arithmetic quotients of the Siegel upper half-space of genus two.

Thus grew the idea of realizing the Baily–Borel compactification as the natural quotient of $\Gamma\backslash\bar{X}$. The possibility of doing this has been conceded for some time, but it had not been carried out, perhaps for lack of incentive. In this paper, we realize all generalized Satake compactifications as quotients of $\Gamma\backslash\bar{X}$. We require two assumptions on the representation τ (see (3.3) and (3.4)) in order to carry out the construction; these hypotheses are met in the cases covered in [2], [3] and [9], and as such, one can regard our discussion as a mild generalization of these works.

We will reconstruct the Satake compactifications in such a way that they really do look like quotients of the manifold with corners. Let ${}_{\mathbf{Q}}A$ denote the identity component of the real points of a maximal $\mathbf{Q}$-split torus of G. By making a choice

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[1] By this, what we mean is a compactification whose topology is induced from that of the closure in some X^*_τ of a Siegel set, by the procedure introduced in [9] (see our (3.9)).

of positive simple roots, one obtains an identification

$$_{\mathbf{Q}}A \simeq (0, \infty)^r,$$

where r is the $\mathbf{Q}$-rank of G. One puts

$$_{\mathbf{Q}}\bar{A} \simeq (0, \infty]^r.$$

Roughly speaking, the basic point in the construction of the manifold with corners $\bar{X}$ is the adjoining of $_{\mathbf{Q}}\bar{A}$ along images of $_{\mathbf{Q}}A$ in X. In effect, one allows the simple roots to go to infinity independently and in all possible ways. For the Satake compactifications, if a simple root goes to infinity, it is irrelevant whether certain other ones do or not. As such, we are led to introduce an $_{\mathbf{Q}}A$-equivariant quotient $_{\mathbf{Q}}A^*_\tau$ of $_{\mathbf{Q}}\bar{A}$. By adjoining $_{\mathbf{Q}}A^*_\tau$ along $_{\mathbf{Q}}A$, and then doing a little more, we define a *crumpled corner* and the manifold with crumpled corners in a way that mimics the construction in [4, §§5–7]. After taking an arithmetic quotient, one sees that one has reproduced the topology of $\Gamma\backslash X^*_\tau$. Having done this, one can write down rather easily a description of the fibers of the quotient mapping from $\Gamma\backslash\bar{X}$.[(2)] We remark that the smooth compactifications of [1], however, are not in general natural quotients of the manifold with corners.

In §1, we give an exposition of basic facts about restricted root systems, following the treatments in [5] and [8]. In §2, we present a discussion of the Satake compactifications X^*_τ. The one ingredient which could be called new is the introduction of A^* in (2.6). In §3, we carry out the construction of Satake compactifications via manifolds with crumpled corners, as described above.

I wish to express my gratitude to Armand Borel for encouraging this project, and for patiently and critically listening to the details of the construction.

1. Restricted root systems associated to real algebraic groups

(1.1) Let $\mathbf{F}$ be a subfield of the real numbers $\mathbf{R}$. We establish the following abuse of language as convention: "H is an algebraic group over $\mathbf{F}$" shall mean "H is the set of real points of an algebraic group defined over $\mathbf{F}$, and we regard it as a Lie group"; for the set of $\mathbf{F}$-rational points of H, we write $H_{\mathbf{F}}$.

Let G be a semi-simple algebraic group over $\mathbf{F}$, with Lie algebra $\mathfrak{g}$. Let $_{\mathbf{F}}A$ be

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[2] Our description of the quotient mapping has been used by Charney and Lee in their calculation of the cohomology of the (Baily–Borel–)Satake compactification of the Siegel modular varieties ($G = \mathrm{Sp}(2n, \mathbf{R})$, $\Gamma = \mathrm{Sp}(2n, \mathbf{Z})$) [6].

the identity component of a maximal $\mathbf{F}$-split torus ${}_{\mathbf{F}}T$ of G, with Lie algebra ${}_{\mathbf{F}}\mathfrak{a}$. Let K be a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$, which can be chosen so that under the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{1}$$

(orthogonal with respect to the Killing form B), we have ${}_{\mathbf{F}}\mathfrak{a} \subset \mathfrak{p}$. One can enlarge ${}_{\mathbf{F}}\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h}_{\mathbf{C}}$ of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$ such that $\mathfrak{h}_{\mathbf{C}} = (\mathfrak{h}_{\mathbf{C}} \cap \mathfrak{k}_{\mathbf{C}}) \oplus (\mathfrak{h}_{\mathbf{C}} \cap \mathfrak{p}_{\mathbf{C}})$. Then

$$\mathfrak{h} = i(\mathfrak{h}_{\mathbf{C}} \cap \mathfrak{k}) \oplus (\mathfrak{h}_{\mathbf{C}} \cap \mathfrak{p}) \tag{2}$$

is a real form of $\mathfrak{h}_{\mathbf{C}}$ on which the roots of $\mathfrak{g}_{\mathbf{C}}$ are real-valued, and to which the restriction of B is positive-definite.

(1.2) Let $\mathbf{E} \subseteq \mathbf{R}$ or $\mathbf{E} = \mathbf{C}$ denote a field containing $\mathbf{F}$. In the latter case, we write ${}_{\mathbf{C}}\mathfrak{a}$ for $\mathfrak{h}$. We assume that ${}_{\mathbf{E}}\mathfrak{a}$ has been chosen so that ${}_{\mathbf{F}}\mathfrak{a} \subseteq {}_{\mathbf{E}}\mathfrak{a}$. Let

$${}_{\mathbf{F}}\rho_{\mathbf{E}} : {}_{\mathbf{E}}\mathfrak{a}^* \to {}_{\mathbf{F}}\mathfrak{a}^*$$

denote the restriction mapping.

Let ${}_{\mathbf{F}}\Phi \subset {}_{\mathbf{F}}\mathfrak{a}^*$ denote the system of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to ${}_{\mathbf{F}}\mathfrak{a}$, etc. By selecting positive chambers in ${}_{\mathbf{F}}\mathfrak{a}$ and ${}_{\mathbf{E}}\mathfrak{a}$ consistently, one obtains systems of positive simple roots ${}_{\mathbf{F}}\Delta$ in ${}_{\mathbf{F}}\Phi$, ${}_{\mathbf{E}}\Delta$ in ${}_{\mathbf{E}}\Phi$, with ${}_{\mathbf{F}}\rho_{\mathbf{E}}$ inducing a mapping, denoted by the same symbol,

$${}_{\mathbf{F}}\rho_{\mathbf{E}} : {}_{\mathbf{E}}\Delta \to {}_{\mathbf{F}}\Delta \cup \{0\} \tag{1}$$

(see [5, (6.8)]). For $\beta \in {}_{\mathbf{F}}\Delta \cup \{0\}$, we put

$${}_{\mathbf{E}}\Delta^{\beta} = {}_{\mathbf{F}}\rho_{\mathbf{E}}^{-1}(\beta). \tag{2}$$

A subset of ${}_{\mathbf{E}}\Delta$ will be said to be $\mathbf{F}$-*rational* whenever it is of the form

$${}_{\mathbf{F}}\rho_{\mathbf{E}}^{-1}(Y \cup \{0\}) = \bigcup_{\beta \in Y \cup \{0\}} {}_{\mathbf{E}}\Delta^{\beta}. \tag{3}$$

for some $Y \subseteq {}_{\mathbf{F}}\Delta$.

(1.3) The Killing form B defines an inner product ${}_{\mathbf{E}}B$ on ${}_{\mathbf{E}}\mathfrak{a}$ for any $\mathbf{E}$. We may then identify ${}_{\mathbf{E}}\mathfrak{a}^*$ with ${}_{\mathbf{E}}\mathfrak{a}$, which in turn identifies ${}_{\mathbf{F}}\mathfrak{a}^*$ as the orthogonal complement of $(\ker {}_{\mathbf{F}}\rho_{\mathbf{E}})$ in ${}_{\mathbf{E}}\mathfrak{a}$, and ρ as orthogonal projection onto ${}_{\mathbf{F}}\mathfrak{a}^*$. With this inner product, ${}_{\mathbf{E}}\Phi$ satisfies the axioms of a root system (see [5, (2.1), (5.8)]). In particular, there is a Weyl group ${}_{\mathbf{E}}W$ acting orthogonally on $({}_{\mathbf{E}}\mathfrak{a}^*, {}_{\mathbf{E}}\Phi)$; for $\mathbf{F} \subset \mathbf{E}$, ${}_{\mathbf{F}}W$ can be seen as the set of restrictions of those elements in ${}_{\mathbf{E}}W$ that leave ${}_{\mathbf{F}}\mathfrak{a}^*$ invariant (see [5, (6.10)]).

(1.4) Let $\mathscr{G} = \mathrm{Aut}\,(\mathbf{C}/\mathbf{E})$. Then $\mathscr{G}$ acts on ${}_{\mathbf{C}}\Phi$ as follows. As $\sigma \in \mathscr{G}$ acts on both $\mathfrak{h}_{\mathbf{C}}$ and $\mathbf{C}$, one can set for $\alpha \in {}_{\mathbf{C}}\Phi$

$$\sigma(\alpha) = \sigma\alpha\sigma^{-1}. \tag{1}$$

Under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$, extended linearly to the complexifications, this becomes the standard action of $\mathscr{G}$ on $\mathfrak{h}_{\mathbf{C}}$.

While $\sigma(\alpha)$ need not be in ${}_{\mathbf{C}}\Delta$ when $\alpha \in {}_{\mathbf{C}}\Delta$, it is clear that $\sigma({}_{\mathbf{C}}\Delta)$ is a base of ${}_{\mathbf{C}}\Phi$. Thus, there exists a unique $w_\sigma \in {}_{\mathbf{C}}W$ such that $w_\sigma[\sigma({}_{\mathbf{C}}\Delta)] = {}_{\mathbf{C}}\Delta$; then

$$\sigma^+(\alpha) = w_\sigma\sigma(\alpha) \tag{2}$$

defines an action of $\mathscr{G}$ on Δ.

The above actions extend linearly to $\mathfrak{h}^*$. With respect to either, ${}_{\mathbf{C}}B$ is invariant.

For $\sigma \in \mathscr{G}$, $\alpha \in {}_{\mathbf{C}}\Delta$, it is clear that

$${}_{\mathbf{E}}\rho_{\mathbf{C}}(\sigma(\alpha)) = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha). \tag{3}$$

It follows that if ${}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha) \neq 0$, there exist a unique $\tilde{\alpha}_\sigma \in {}_{\mathbf{C}}\Delta$ such that

$$\sigma(\alpha) = \tilde{\alpha}_\sigma + \sum_{\delta \in {}_{\mathbf{C}}\Delta^0} n_\delta \delta, \tag{4}$$

where the n_δ's are non-negative integers. In fact, $\tilde{\alpha}_\sigma = \sigma^+(\alpha)$ (cf. [5, (6.7)]), and therefore

$${}_{\mathbf{E}}\rho_{\mathbf{C}}(\sigma^+(\alpha)) = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha). \tag{5}$$

(1.5) We recall some basic facts about the inner products. From here on, we will write

$$\langle \alpha, \beta \rangle = {}_{\mathbf{E}}B(\alpha, \beta) \tag{1}$$

for $\alpha, \beta \in {}_{\mathbf{E}}a^*$. It follows from the basic properties of roots that for $\alpha, \alpha' \in {}_{\mathbf{E}}\Delta$,

$$\langle \alpha, \alpha' \rangle \leq 0 \quad \text{if} \quad \alpha \neq \alpha'. \tag{2}$$

Let $\hat{\mathscr{G}} \subset \mathscr{G}$ be a set of representatives for the image of $\mathscr{G}$ in the permutation group of ${}_{\mathbf{C}}\Phi$. If $\beta \in {}_{\mathbf{E}}\Delta$, and $\beta = {}_{\mathbf{E}}\rho_{\mathbf{C}}(\alpha)$ for $\alpha \in \Delta$, the identification in (1.3) gives

$$\beta = (\#\hat{\mathscr{G}})^{-1} \sum_{\sigma \in \hat{\mathscr{G}}} \sigma(\alpha), \tag{3}$$

and thus one obtains for any $h \in \mathscr{h}^*$

$$(\#\hat{\mathscr{G}})\langle \rho(\alpha), \rho(h) \rangle = \sum_{\sigma \in \hat{\mathscr{G}}} \langle \sigma(\alpha), h \rangle. \tag{4}$$

(1.6) Let S be a subset of an inner product space. The *graph* of S consists of a vertex for each element of S; two vertices are connected by an edge if and only if the inner product of the corresponding elements is non-zero. One says that S is *connected* if its graph is connected. One can speak of the connected components of S, etc.

(1.7) For any $h \in \mathscr{h}^*$, we have a unique expression

$$h = \sum_{\alpha \in {}_{\mathbf{C}}\Delta} c_\alpha \alpha \qquad (c_\alpha \in \mathbf{R}). \tag{1}$$

We define

$$\operatorname{supp}(h) = \{\alpha \in {}_{\mathbf{C}}\Delta : c_\alpha \neq 0\}. \tag{2}$$

Viewing $\mathbf{E} \subset \mathbf{C}$ as fixed, we put ${}_{\mathbf{C}}\Delta = \Delta$, ${}_{\mathbf{E}}\rho_{\mathbf{C}} = \rho$.

PROPOSITION. *Suppose that* $\rho(\alpha) \neq 0$, $\alpha' \in \Delta - \bigcup_{\sigma \in \hat{\mathscr{G}}} \operatorname{supp}(\sigma(\alpha) - \sigma^+(\alpha))$, *and* $\rho(\alpha') \neq \rho(\alpha)$. *Then* $\langle \rho(\alpha), \rho(\alpha') \rangle = 0$ *if and only if* α' *is orthogonal to* $\sigma^+(\alpha)$ *and* $\operatorname{supp}(\sigma(\alpha) - \sigma^+(\alpha))$ *for every* $\sigma \in \mathscr{G}$.

Proof. We may, of course, replace $\mathscr{G}$ by $\hat{\mathscr{G}}$ in the above statement. By (1.5(3)), we have

$$(\#\hat{\mathscr{G}})\langle \rho(\alpha), \rho(\alpha') \rangle = \sum_{\sigma \in \hat{\mathscr{G}}} \langle \sigma(\alpha), \alpha' \rangle.$$

Using (1.4(4)) with the definition (2), we rewrite this as

$$(\#\hat{\mathscr{G}})\langle\rho(\alpha),\rho(\alpha')\rangle=\sum_{\sigma\in\hat{\mathscr{G}}}\langle\sigma^+(\alpha),\alpha'\rangle + \sum_{\sigma\in\hat{\mathscr{G}}}\Bigg(\sum_{\delta\in\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))} n^\sigma_\delta\langle\delta,\alpha'\rangle\Bigg). \tag{3}$$

With the hypotheses on α', we have by (1.5(2)) that all terms in the above sum are non-positive. Since the n^σ_δ's in (3) are all positive, it follows that $\langle\rho(\alpha),\rho(\alpha')\rangle=0$ if and only if all inner products in the right-hand side are zero.

COROLLARY. *$\bigcup_{\sigma\in\hat{\mathscr{G}}}\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))$ is a union of connected components of Δ^0.*

(1.8) PROPOSITION. *A component C of Δ^0 is orthogonal to all $\sigma^+(\alpha)$ if and only if C is not contained in $\bigcup_{\sigma\in\mathscr{G}}\operatorname{supp}(\sigma(\alpha)-\sigma^+(\alpha))$.*

Proof. One direction is contained in the statement of the proposition in (1.7). For the other, suppose that for all $\beta\in C$, $\sigma\in\hat{\mathscr{G}}$, we have $\langle\sigma^+(\alpha),\beta\rangle=0$. Then as in (1.7(3)), we have for any $\beta\in C$

$$\sum_{\sigma\in\hat{\mathscr{G}}}\sum_{\delta\in\Delta^0} n^\sigma_\delta\langle\delta,\beta\rangle=(\#\hat{\mathscr{G}})\langle\rho(\alpha),\rho(\beta)\rangle=0. \tag{1}$$

If $C=\{\beta_1,\ldots,\beta_m\}$, we obtain from (1) the system of equations:

$$\sum_{j=1}^m\langle\beta_j,\beta_k\rangle\Bigg(\sum_{\sigma\in\hat{\mathscr{G}}} n^\sigma_{\beta_j}\Bigg)=0 \qquad k=1,\ldots,m. \tag{2}$$

Now, the matrix $[\langle\beta_j,\beta_k\rangle]$ is invertible. Therefore, we must have that $\sum_{\sigma\in\hat{\mathscr{G}}} n^\sigma_{\beta_j}=0$ for all j. Since the $n^\sigma_{\beta_j}$'s are non-negative, they must all equal zero.

One combines (1.7) and (1.8) to obtain

COROLLARY. *For $\alpha,\alpha'\in\Delta-\Delta^0$, $\langle\rho(\alpha),\rho(\alpha')\rangle\neq0$ if and only if for some $\sigma\in\mathscr{G}$, $\Psi\subseteq\Delta^0$, the set $\Psi\cup\{\sigma^+(\alpha),\alpha'\}$ is connected.*

(1.9) PROPOSITION. (i) *If $\Psi\subseteq\Delta$ is connected, then $\rho(\Psi)-\{0\}$ is connected.*
(ii) *If $\Theta\subseteq_{\mathbf{E}}\Delta$ is connected, then $\tilde{\Theta}=\rho^{-1}(\Theta)\cup\{\delta\in\Delta^0:\delta$ is not orthogonal to $\rho^{-1}(\Theta)\}$ has at most $(\#\hat{\mathscr{G}})$ connected components, each of which projects onto Θ.*

Proof. From the corollary in (1.8), (i) follows immediately. Suppose, on the other hand, that $\Theta \subseteq {}_{\mathbf{E}}\Delta$ is connected, and let $\tilde{\Theta}$ be as in (ii). Since $\mathscr{G}$ acts orthogonally on Δ (1.4), we see that $\mathscr{G}$ permutes the connected components of $\tilde{\Theta}$. It follows from the corollary in (1.8) that each component contains at least one element from each $\mathscr{G}$-orbit in $\rho^{-1}(\Theta)$. Since $\tilde{\Theta}$ has at most as many components as does $\rho^{-1}(\Theta)$, we obtain the first assertion of (ii). By (1.4(5)), all components are seen to have the same image in ${}_{\mathbf{E}}\Delta$, namely Θ.

COROLLARY 1. *$\Theta \subseteq {}_{\mathbf{E}}\Delta$ is connected if and only if there is a connected subset $\Psi \subseteq \Delta$ with $\rho(\Psi) - \{0\} = \Theta$.*

COROLLARY 2. *Let $\mathbf{F} \subset \mathbf{E} \subseteq \mathbf{R}$. Then $Y \subseteq {}_{\mathbf{F}}\Delta$ is connected if and only if there is a connected subset $\Theta \subseteq {}_{\mathbf{E}}\Delta$ with ${}_{\mathbf{F}}\rho_{\mathbf{E}}(\Theta) - \{0\} = Y$.*

(*Proof.* Apply Corollary 1 twice, once with $\mathbf{E}$ replaced by $\mathbf{F}$, and recall that ${}_{\mathbf{F}}\rho_{\mathbf{C}} = {}_{\mathbf{F}}\rho_{\mathbf{E}} \circ {}_{\mathbf{E}}\rho_{\mathbf{C}}$.)

2. Satake's compactifications of G/K

(2.1) Let G be a semi-simple real algebraic group, K a maximal compact subgroup of G, and $X = G/K$ the associated symmetric space.

Let $\tau: G \to SL(V)$ be a finite-dimensional representation of G, with finite kernel. There exists an *admissible inner product* on V, which means that

$$\tau(g)\tau(\theta(g))^* = I. \tag{1}$$

where θ denotes the Cartan involution of G with respect to K, and * denotes adjoint with respect to the inner product. It follows that the mapping

$$\tau_0(g) = \tau(g)\tau(g)^* \tag{2}$$

descends to X, and has values in the space $S(V)$ of self-adjoint endomorphisms of V. By taking the quotient by the action of the scalars, one obtains a mapping, which we also denote τ_0

$$\tau_0: X \to \mathbf{P}S(V), \tag{3}$$

which is easily seen to be an embedding. It is G-equivariant with respect to the natural action on X and the projectivization of the action

$$g \cdot M = gMg^* \qquad M \in S(V). \tag{4}$$

The *Satake compactification* determined by τ is the closure of the image of τ_0:

$$X_\tau^* = \overline{\tau_0(X)}. \tag{5}$$

If τ is fixed, we write X^* instead of X_τ^*, the dependence on τ being understood.

(2.2) We prefer an intrinsic description of the topological structure of X^*. From now on, we will assume that τ is irreducible.

To facilitate the discussion to come in §3, we suppose that G is defined over $\mathbf{F} \subseteq \mathbf{R}$. Let $A = {}_{\mathbf{F}}A$ and $\mathfrak{a} = {}_{\mathbf{F}}\mathfrak{a}$ be as in (1.1). Then V has an orthogonal weight-space decomposition with respect to $\mathfrak{a}$:

$$V = \bigoplus V_\mu, \tag{1}$$

where V_μ is the subspace of V on which $\mathfrak{a}$ acts with weight μ.[3] With respect to a basis of weight vectors, τ, and therefore also τ_0, maps A to diagonal matrices. Thus, we see that points in the boundary of X^* which lie in the closure of $\tau_0(A)$ are determined by the behavior of weights, as we shall next describe.

(2.3) The simple roots ${}_{\mathbf{F}}\Delta$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ define characters on A, whose values are denoted a^α ($a \in A$, $\alpha \in {}_{\mathbf{F}}\Delta$). From these, one obtains a canonical isomorphism

$$\iota : A \to (\mathbf{R}^+)^{{}_{\mathbf{F}}\Delta}, \tag{1}$$

where $\mathbf{R}^+$ denotes the interval $(0, \infty)$. By adjoining $\{\infty\}$ to each factor of $\mathbf{R}^+$, one gets via ι a partial compactification $\bar{A}$ of A.[4]

One puts for $Y \subseteq {}_{\mathbf{F}}\Delta$

$$A_Y = \bigcap_{\beta \in Y} (\ker \beta) \tag{2}$$

(so $A = A_\varnothing$), and for $Y \subseteq \Theta \subseteq {}_{\mathbf{F}}\Delta$

$$A_{Y,\Theta} = A_Y \cap \bigcap_{\beta \notin \Theta} (\ker \beta), \tag{3}$$

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[3] These weights are, in fact, real.

[4] In [4], $\{0\}$ is adjoined to $\mathbf{R}^+$, but G is acting there on the right. Our adjoining $\{\infty\}$ is consistent with [8], and also with current convention, where G is also assumed to act on the left.

so that

$$A_Y = A_{Y,\Theta} \times A_\Theta \quad \text{if} \quad Y \subseteq \Theta. \tag{4}$$

We then have an A-orbit space decomposition

$$\bar{A} = \coprod_{Y \subseteq {}_{\mathbf{F}}\Delta} A'_Y, \tag{5}$$

where

$$A'_Y = \{a \in \bar{A} : a^\beta \neq \infty \text{ if and only if } \beta \in Y\}. \tag{6}$$

The closure $\bar{A}'_Y$ of A'_Y in $\bar{A}$ can be written as

$$\bar{A}'_Y = \coprod_{\Psi \subseteq Y} A'_\Psi. \tag{7}$$

For $Y \subseteq \Theta$, there is an obvious projection

$$p_{\Theta,Y} : A'_\Theta \to A'_Y, \tag{8}$$

determined by setting the characters in $\Theta - Y$ to infinity. This mapping extends continuously to yield

$$\bar{p}_{\Theta,Y} : \bar{A}'_\Theta \to \bar{A}'_Y, \tag{9}$$

with

$$p_{\Theta,Y}|_{A'_\Psi} = p_{\Psi,Y\cap\Psi} \quad \text{if} \quad \Psi \subseteq \Theta. \tag{10}$$

Finally, we remark that as the A-orbit of

$$1_Y = p_{{}_{\mathbf{F}}\Delta,Y}(1), \tag{11}$$

we identify

$$A'_Y = A/A_Y. \tag{12}$$

(2.4) Let $\mu_0 = {}_{\mathbf{F}}\rho_{\mathbf{C}}(\lambda_0)$, where $\lambda_0 \in \mathfrak{h}^*$ is the highest weight of τ (relative to Δ), with notation as in (1.1) and (1.2). All weights of τ with respect to ${}_{\mathbf{F}}\mathfrak{a}$ (cf. (2.2(1))) are of the form

$$\mu=\mu_0-\sum_{\beta\in{}_{\mathbf{F}}\Delta} m_\beta\beta, \tag{1}$$

where the m_β's are non-negative integers.

As in (1.7(2)), one can define for $\nu\in\mathfrak{a}^*$ the subset $\operatorname{supp}(\nu)$ of ${}_{\mathbf{F}}\Delta$.

PROPOSITION [5, (12.16)]. *Let $Y\subseteq{}_{\mathbf{F}}\Delta$. Then Y equals* $\operatorname{supp}(\mu_0-\mu)$ *for some weight μ of τ with respect to ${}_{\mathbf{F}}\mathfrak{a}$ if and only if $Y\cup\{\mu_0\}$ is connected.*

In view of the above, the following definition is warranted. One says that a subset Y of ${}_{\mathbf{F}}\Delta$ is *τ-connected* (or *τ-open* [8]) if $Y\cup\{\mu_0\}$ is connected.

COROLLARY. *Let $\mathbf{F}\subset\mathbf{E}$. Then $Y\subseteq{}_{\mathbf{F}}\Delta$ is τ-connected if and only if there is a τ-connected subset $\Theta\subseteq{}_{\mathbf{E}}\Delta$ with ${}_{\mathbf{F}}\rho_{\mathbf{E}}(\Theta)-\{0\}=Y$.*

Remark. An alternate treatment of the proposition, along the lines of [8, (2.3)], can be carried out, by the use of the following analogue of the corollary in our (1.8). If $\beta={}_{\mathbf{F}}\rho_{\mathbf{C}}(\alpha)\in{}_{\mathbf{F}}\Delta$, then $\langle\beta,\mu_0\rangle\neq 0$ if and only if for some $\sigma\in\operatorname{Aut}(\mathbf{C}/\mathbf{F})$ and $\Psi\subseteq\Delta^0$, the set $\Psi\cup\{\sigma^+(\alpha),\lambda_0\}$ is connected.

(2.5) One should always keep in mind that (if we divide out the common factor of $a^{2\mu_0}$) $\tau_0(a)|_{V_\mu}$ is equal to the scalar $a^{-2\sum m_\beta\beta}$. Thus, as $a\in A$ approaches a point in A'_Y, $\tau_0(a)$ degenerates to zero on the weight-space V_μ if and only if $\operatorname{supp}(\mu_0-\mu)\nsubseteq Y$.

Every subset Y of ${}_{\mathbf{F}}\Delta$ contains a largest τ-connected subset, which we call its *τ-connected component.* The assignment of τ-connected components defines a mapping

$$\kappa: 2^{({}_{\mathbf{F}}\Delta)}\to 2^{({}_{\mathbf{F}}\Delta)}. \tag{1}$$

Given $\Theta\subseteq{}_{\mathbf{F}}\Delta$, one puts

$$\omega(\Theta)=\Theta\cup\Theta', \tag{2}$$

where

$$\Theta'=\{\beta\in{}_{\mathbf{F}}\Delta:\beta \text{ is orthogonal to } \Theta\cup\{\mu_0\}\}. \tag{3}$$

The following is immediate:

LEMMA. (i) *Let Y be a τ-connected subset of ${}_{\mathbf{F}}\Delta$. Then $\kappa(\Theta)=Y$ if and only if* $Y\subseteq\Theta\subseteq\omega(Y)$.

(ii) *If μ is a weight of τ with respect to ${}_{\mathbf{F}}a$, then* $\operatorname{supp}(\mu_0-\mu)\subseteq\Theta$ *if and only if* $\operatorname{supp}(\mu_0-\mu)\subseteq\kappa(\Theta)$.

Thus, the behavior of $\tau_0(a)$ as a approaches A'_Θ is determined by the projection $p_{\Theta,\kappa(\Theta)}$ of the limit.

(2.6) The preceding observation suggests that we define the following A-equivariant quotient A^* of $\bar{A}$. As underlying set, we take

$$A^*=\coprod_{Y\ \tau\text{-connected}} A'_Y. \tag{1}$$

There is a surjective mapping $p:\bar{A}\to A^*$, defined by

$$p|_{A'_\Theta}=p_{\Theta,\kappa(\Theta)}. \tag{2}$$

We will show that A^*, equipped with the quotient topology induced by p, is a Hausdorff space.

If $a\in\bar{A}$ and $p(a)$ is in A'_Θ, we set

$$Y(a)=\Theta; \tag{3}$$

that is, $Y(a)$ is the τ-connected component of

$$\{\alpha\in{}_{\mathbf{F}}\Delta : a^\alpha<\infty\}.$$

The equivalence relation on $\bar{A}$ induced by p can then be described as:

a and b are identified if and only if

$$Y(a)=Y(b) \text{ and } a^\alpha=b^\alpha \text{ whenever } \alpha\in Y(a). \tag{4}$$

Consider a point $a_0\in A'_\Theta$. Then $a_0^\alpha<\infty$ if and only if α is in the τ-connected subset Θ of ${}_{\mathbf{F}}\Delta$. For each α, let J_α denote a neighborhood of a_0^α in $\mathbf{R}^+\cup\{\infty\}$. Then

$$\{a\in\bar{A} : a^\alpha\in J_\alpha \text{ if } \Theta\cup\{\alpha\} \text{ is } \tau\text{-connected}\} \tag{5}$$

is an open neighborhood of $p^{-1}(a_0)$ in $\bar{A}$. As it is a union of fibers of p, as follows from (4), it projects onto a neighborhood of a_0 in A^*.

Let $b_0 \in A'_\Psi \subset A^*$, with $b_0 \neq a_0$. If $\Theta = \Psi$, then it is clear from (5) that we can find disjoint neighborhoods of a_0 and b_0, since these elements are distinguished by α for some $\alpha \in \Theta$. Otherwise, if $\Psi - \Theta \neq \varnothing$ (as we may assume without loss of generality), there must exist $\alpha \in \Psi - \Theta$ with $\Theta \cup \{\alpha\}$ τ-connected. As $\Psi \cup \{\alpha\} = \Psi$ is τ-connected and α distinguishes a_0 and b_0, we can likewise find disjoint neighborhoods of the form (5). Thus, A^* is Hausdorff.

We remark that p is almost never an open mapping.

The closing remark of (2.5) implies:

PROPOSITION. *The embedding of A by τ_0 in $\mathbf{PS}(V)$ extends to a continuous mapping τ_0^* of A^*.*

In fact, we will later see that τ_0^* is an embedding.

(2.7) For any $t \in A$, we put

$$A(t) = \{a \in A : a^\beta \geq t^\beta \text{ for all } \beta \in {}_{\mathbf{F}}\Delta\}. \tag{1}$$

We let $\bar{A}(t)$ (resp. $A^*(t)$) denote the closure of $A(t)$ in $\bar{A}$ (resp. A^*).

PROPOSITION. *Take $\mathbf{F} = \mathbf{R}$. Then there is a continuous mapping*

$$\phi : G \times A^* \to X^*, \tag{2}$$

defined by $\phi(g, a) = g\tau_0^*(a)g^*$, whose restriction to $K \times A^*(1)$ is already surjective.

Proof. This follows from the proposition of (2.6) and the fact that $KA(1)K = G$.

(2.8) In order to describe the structure of the mapping ϕ (2.7(2)), it is necessary to discuss the role of the parabolic subgroups of G. We continue to assume that $\mathbf{F} = \mathbf{R}$.

To each subset Θ of ${}_{\mathbf{R}}\Delta$ is associated a "standard" parabolic subgroup Q_Θ of G, whose definition we recall. Let $\mathfrak{g}_\alpha$ denote the α root space of $\mathfrak{g}$, if $\alpha \in {}_{\mathbf{R}}\Phi$. Then

$$Q_\Theta = M_\Theta A_\Theta U_\Theta \tag{1}$$

is the connected algebraic subgroup of G with (correspondingly decomposed) Lie algebra

$$\mathfrak{q}_\Theta = \mathfrak{m}_\Theta \oplus \mathfrak{a}_\Theta \oplus \mathfrak{u}_\Theta, \tag{2}$$

where $\mathfrak{a}_\Theta$ is the Lie algebra of A_Θ (2.3(2)); $\mathfrak{m}_\Theta$ is the sum of the (semi-simple) Lie algebra ℓ_Θ generated by $\{\mathfrak{g}_\beta, \mathfrak{g}_{-\beta} : \beta \in \Theta\}$ and a sub-algebra of $\mathfrak{k}$, which coincides with the orthogonal complement of $\mathfrak{a}_\Theta$ in the centralizer of $\mathfrak{a}_\Theta$; and

$$\mathfrak{u}_\Theta = \bigoplus_{0<\alpha \notin \mathrm{Span}(\Theta)} \mathfrak{g}_\alpha$$

is the nilpotent radical of $\mathfrak{q}_\Theta$. Every parabolic subgroup of G is conjugate, by an element of G, to a unique Q_Θ (see [5, (4.6), (4.13c)]).

(2.9) Let $K_\Theta = K \cap M_\Theta$; put

$$X_\Theta = M_\Theta / K_\Theta, \tag{1}$$

a symmetric space of rank $(\#\Theta)$. If Θ is τ-connected, then X_Θ is embedded in X^* as follows.

Let

$$V_\Theta = \bigoplus_{\mathrm{supp}(\mu_0 - \mu) \subseteq \Theta} V_\mu. \tag{2}$$

It is evident that V_Θ is stable under $\tau(M_\Theta)$. In other words, τ determines a representation τ_Θ of M_Θ on V_Θ. It can be seen [8, (2.4)] that τ_Θ has finite kernel, and that all of its irreducible constituents are equivalent. One gets, as in (2.1(3)), an embedding

$$(\tau_\Theta)_0 : X_\Theta \to \mathbf{P}S(V_\Theta). \tag{3}$$

We identify $S(V_\Theta)$ as the linear subspace of $S(V)$ given by transformations which are zero on the weight spaces complementary to V_Θ, and thereby regard $(\tau_\Theta)_0$ as having its image in $\mathbf{P}S(V)$.

We observe that M_Θ contains the subgroup

$$\tilde{A}_{\blacksquare\Delta,\Theta} = M_\Theta \cap A, \tag{4}$$

whose Lie algebra is spanned by the elements of Θ, viewed as elements of $\mathfrak{a}$ via the identification of (1.3). Since $\tilde{A}_{\blacksquare\Delta,\Theta}$, as does $A_{\blacksquare\Delta,\Theta}$, projects isomorphically onto A/A_Θ, we can see from the descriptions of τ_0^* and $(\tau_\Theta)_0$ that

$$\tau_0^*(A'_\Theta) = (\tau_\Theta)_0(\tilde{A}_{_{\mathbf{R}}\Delta,\Theta}) \tag{5}$$

canonically. In particular, τ_0^* is an embedding of A^*; moreover, X_Θ is embedded in X^* as the M_Θ-orbit of $\tau_0^*(1_\Theta)$.

(2.10) The G-translates of the X_Θ's are called the *boundary components* of X^*. It is clear from (2.7) that X^* is the union of its boundary components (note that $X = X_{_{\mathbf{R}}\Delta}$). The best way to index them is by means of their normalizers. The normalizer of X_Θ in G is equal to the normalizer of V_Θ under τ, namely the parabolic subgroup $Q_{\omega(\Theta)}$; thus the normalizer of gX_Θ is the conjugate ${}^gQ_{\omega(\Theta)} := gQ_{\omega(\Theta)}g^{-1}$.

The preceding discussion makes it apparent that as a topological space, X^* depends on τ only to the extent that A^* does, namely on the collection of τ-connected subsets of ${}_{\mathbf{R}}\Delta$. Thus, there are only finitely many topologically distinct Satake compactifications of X.

The preceding can be reformulated as follows. Let $\xi_1, \ldots, \xi_r$ be the fundamental dominant weights of ${}_{\mathbf{R}}\Delta$; i.e., if ${}_{\mathbf{R}}\Delta = \{\alpha_1, \ldots, \alpha_r\}$, then $\{\xi_1, \ldots, \xi_r\}$ is the dual basis of ${}_{\mathbf{R}}\mathfrak{a}^*$. The restricted highest weight μ_0 of τ can be written

$$\mu_0 = \sum_{j=1}^{r} c_j \xi_j, \tag{1}$$

where the c_j's are non-negative integers. We put

$$\operatorname{supp}^*(\mu_0) = \{\xi_j : c_j \neq 0\}. \tag{2}$$

Equivalently,

$$\xi_j \in \operatorname{supp}^*(\mu_0) \quad \text{if and only if} \quad \langle \alpha_j, \mu_0 \rangle \neq 0. \tag{3}$$

Then we have asserted:

PROPOSITION. *Let μ_0, μ'_0 be the respective restricted highest weights of τ, τ'. If* $\operatorname{supp}^*(\mu_0) = \operatorname{supp}^*(\mu'_0)$, *then the identity mapping of X extends to a homeomorphism of X^*_τ and $X^*_{\tau'}$.*

(2.11) Suppose now that for τ, τ' we have only that $\operatorname{supp}^*(\mu'_0) \subset \operatorname{supp}^*(\mu_0)$. Let Θ be a τ'-connected subset of ${}_{\mathbf{R}}\Delta$. This is equivalent to asserting that every connected component of Θ is not orthogonal to $\operatorname{supp}(\mu_0')$, so Θ is also τ-connected. Thus, for any Θ,

$$\kappa_{\tau'}(\Theta) \subseteq \kappa_\tau(\Theta). \tag{1}$$

It follows that there is an A-equivariant continuous surjection

$$\psi_{\tau,\tau'}: A_\tau^* \to A_{\tau'}^*. \tag{2}$$

PROPOSITION. *The mapping*

$$1 \times \psi_{\tau,\tau'}: K \times A_\tau^* \to K \times A_{\tau'}^*$$

induces a continuous surjection

$$f_{\tau,\tau'}: X_\tau^* \to X_{\tau'}^*.$$

Proof. Since X_τ^* (resp. $X_{\tau'}^*$) is a quotient of the domain (resp. range) of $1 \times \psi_{\tau,\tau'}$, and these are compact spaces, it suffices to show that points identified under ϕ_τ (2.7) are mapped by $1 \times \psi_{\tau,\tau'}$ to points identified under $\phi_{\tau'}$. We should be explicit about these identifications.

(2.12) Throughout this paragraph, we assume that Ξ and Θ are subsets of ${}_{\mathbf{R}}\Delta$ which satisfy $\kappa_\tau(\Xi) = \Theta$. Let I_Θ denote the isometry group of X_Θ. As Q_Ξ normalizes X_Θ, there is a continuous homomorphism

$$h_\Xi : Q_\Xi \to I_\Theta, \tag{1}$$

such that for $\Xi' \subset \Xi$ one has

$$h_\Xi|_{Q_{\Xi'}} = h_{\Xi'}. \tag{2}$$

It is a tautology that $Q_{\omega_\tau(\Theta)}$ acts on X_Θ via $h_{\omega_\tau(\Theta)}$.

Put $K'_\Theta = h_\Theta(K_\Theta)$, the "distinguished" maximal compact subgroup of I_Θ. Since $K_\Xi \supseteq K_\Theta$ and $h_\Xi(K_\Xi)$ is compact, it follows that

$$h_\Xi(K_\Xi) = K'_\Theta. \tag{3}$$

LEMMA (cf. [8, (4.4)]). *Under ϕ_τ, one has the identification* $(k_1, a_1^*) \sim (k_2, a_2^*)$

if and only if the following three statements hold:

(i) a_1^* *and* a_2^* *are in the same* A*-orbit* (*call it* A'_Θ, *where* Θ *is* τ*-connected*) *in* A_τ^*,

(ii) $k_2^{-1}k_1 \in K_{\omega_\tau(\Theta)}$,

(iii) *with* a_1^*, a_2^* *regarded as elements of* $\tilde{A}_{\mathbf{R}\Delta,\Theta}$, $(k_2^{-1}k_1)\cdot a_1^* K_\Theta = a_2^* K_\Theta$.

Since $h_\Theta | \tilde{A}_{\mathbf{R}\Delta,\Theta}$ is one-to-one, we may assume without loss of generality that $I_\Theta = M_\Theta$. Then, condition (iii) of the above lemma can be rewritten as

$$\operatorname{Int}[h_{\omega_\tau(\Theta)}(k_2^{-1}k_1)]a_1^* = a_2^*, \tag{4}$$

where Int denotes the action of K_Θ on $\tilde{A}_{\mathbf{R}\Delta,\Theta}$ by inner automorphisms. Also, K_Θ centralizes A_Θ, so the preceding action of K_Θ is canonically isomorphic to that on A/A_Θ.

(2.13) We return to complete the proof of the proposition in (2.11). Suppose that Θ is τ-connected. Then

$$\Psi := \kappa_{\tau'}(\Theta) \subseteq \Theta. \tag{1}$$

As Ψ is the τ'-connected component of Θ, it follows that

$$\Theta - \Psi \subseteq \Psi'_{\tau'} \tag{2}$$

(Ψ' as in (2.5(3))); moreover, it is evident that

$$\Theta'_\tau \subseteq \Theta'_{\tau'} \subseteq \Psi'_{\tau'}. \tag{3}$$

Therefore, we conclude that

$$\omega_\tau(\Theta) \subseteq \omega_{\tau'}(\Psi), \tag{4}$$

so also

$$Q_{\omega_\tau(\Theta)} \subseteq Q_{\omega_{\tau'}(\Psi)}. \tag{5}$$

In particular,

$$K_{\omega_\tau(\Theta)} \subseteq K_{\omega_{\tau'}(\Psi)}. \tag{6}$$

That $f_{\tau,\tau'}$ exists now follows from (6), (2.12(2), (4)), and G-equivariance.

3. Quotients by arithmetic groups

Throughout this section, we assume that G is defined over $\mathbf{Q}$.

(3.1) We choose the maximal compact subgroup K and the maximal split tori of (1.1), for $\mathbf{F}=\mathbf{Q}, \mathbf{R}$, so that ${}_{\mathbf{Q}}A \subseteq {}_{\mathbf{R}}A$; of course we now retain the subscripts for distinction. Superseding the convention of (1.7), we put $\rho = {}_{\mathbf{Q}}\rho_{\mathbf{R}}$. *To simplify the notation, if $Y \subseteq {}_{\mathbf{Q}}\Delta$, we denote the* $\mathbf{Q}$*-rational subset $\rho^{-1}(Y \cup \{0\})$ of ${}_{\mathbf{R}}\Delta$ by $\tilde{Y}$; if $\Theta \subseteq {}_{\mathbf{R}}\Delta$, we put $\hat{\Theta} = \rho(\Theta) - \{0\} \subseteq {}_{\mathbf{Q}}\Delta$.* We observe that $(\tilde{Y})\hat{} = Y$, and that $\tilde{\hat{\Theta}}$ is the smallest $\mathbf{Q}$-rational subset of ${}_{\mathbf{R}}\Delta$ which contains Θ. Also,

$$ {}_{\mathbf{Q}}A_Y = {}_{\mathbf{R}}A_\Theta \cap {}_{\mathbf{Q}}A \quad \text{whenever} \quad \hat{\Theta} = Y. \tag{1}$$

Let $\Theta \subseteq {}_{\mathbf{R}}\Delta$. The standard parabolic subgroup Q_Θ is defined over $\mathbf{Q}$ if and only if Θ is $\mathbf{Q}$-rational [5, (6.3)]. In this case, $\Theta = \tilde{Y}$ for some $Y \subseteq {}_{\mathbf{Q}}\Delta$, and we put

$$ {}_{\mathbf{Q}}Q_Y = Q_\Theta. \tag{2}$$

One redecomposes Q_Θ "rationally":

$$ {}_{\mathbf{Q}}Q_Y = ({}_{\mathbf{Q}}M_Y)({}_{\mathbf{Q}}A_Y)({}_{\mathbf{Q}}U_Y), \tag{3}$$

where: ${}_{\mathbf{Q}}U_Y = U_\Theta$; ${}_{\mathbf{Q}}A_Y$ is as in (2.3(2)); the Lie algebra ${}_{\mathbf{Q}}\mathfrak{m}_Y$ of ${}_{\mathbf{Q}}M_Y$ is orthogonal to the Lie algebra ${}_{\mathbf{Q}}\mathfrak{a}_Y$ of ${}_{\mathbf{Q}}A_Y$, $({}_{\mathbf{Q}}M_Y)({}_{\mathbf{Q}}A_Y) = M_\Theta A_\Theta$ is the Levi subgroup of Q_Θ stable under the Cartan involution of G that fixes K, and if ${}_{\mathbf{R}}\mathfrak{a}$ (or equivalently, the corresponding torus ${}_{\mathbf{R}}T$ of G) is defined over $\mathbf{Q}$, then ${}_{\mathbf{Q}}M_Y$ is isogenous to the product of M_Θ and (the real points of) the maximal $\mathbf{Q}$-anisotropic sub-torus of ${}_{\mathbf{R}}T$.

In what follows, the representation τ, or more properly its equivalence class in the sense of (2.10), used in constructing X^* shall be considered fixed. We also write κ and ω (see (2.5(1), (2))) for subsets of ${}_{\mathbf{Q}}\Delta$, ${}_{\mathbf{R}}\Delta$ and ${}_{\mathbf{C}}\Delta$, since it is always clear which root system one is discussing.

(3.2) Let X_Θ ($\Theta \subseteq {}_{\mathbf{R}}\Delta$ τ-connected) be one of the "standard" boundary components (2.9) of X^*. As was observed in (2.10), the normalizer N_Θ in G of X_Θ is $Q_{\omega(\Theta)}$. One can likewise identify the centralizer Z_Θ of X_Θ as

$$ Z_\Theta = \{h \in N_\Theta : \tau_\Theta(h) \text{ is a multiple of } I\}, \tag{1}$$

where I denotes the identity transformation of V_Θ. Then

$$G_\Theta = N_\Theta / Z_\Theta \tag{2}$$

is an algebraic group over $\mathbf{R}$, with Lie algebra isomorphic to ℓ_Θ.

For a general boundary component gX_Θ, the preceding discussion can be repeated if only we conjugate all groups above by g.

DEFINITION (cf. [2, (3.5)]). *One says that a boundary component is rational if*

(i) *its normalizer N is defined over $\mathbf{Q}$, and*

(ii) *its centralizer Z contains a normal subgroup Z' of N that is defined over $\mathbf{Q}$, such that Z/Z' is compact.*

What (ii) above asserts is that N/Z is the quotient of an algebraic group over $\mathbf{Q}$ by a normal compact subgroup.

(3.3) For X_Θ, the condition (i) in the definition in (3.2) requires that $\omega(\Theta)$ be a $\mathbf{Q}$-rational subset of ${}_{\mathbf{R}}\Delta$. We will impose the following *quasi-rationality* hypothesis on τ:

ASSUMPTION 1. If Y is a τ-connected subset of ${}_{\mathbf{Q}}\Delta$, then the τ-connected component of $\tilde{Y}$ contains $\rho^{-1}(Y)$.

In case G is split over $\mathbf{R}$ (or in general, if we replace ${}_{\mathbf{R}}\Delta$ by ${}_{\mathbf{C}}\Delta$), the above condition is equivalent to the assertion that $\kappa(\tilde{Y})$ be invariant under the action (1.4(2)) of $\mathscr{G} = \mathrm{Aut}\,(\mathbf{C}/\mathbf{Q})$, since this action is transitive on the fibers of ρ other than ${}_{\mathbf{C}}\Delta^0$ [5, (6.4(2))]. If τ is *projectively rational* over $\mathbf{Q}$ in the sense of [5, (12.3a)], e.g. if τ is defined over $\mathbf{Q}$, then Assumption 1 holds, for its highest weight λ_0 is $\mathscr{G}$-invariant [5, (12.6)].

PROPOSITION. (i) *If $\Theta \subseteq {}_{\mathbf{R}}\Delta$ is τ-connected, and $\omega(\Theta)$ is $\mathbf{Q}$-rational, then* $\Theta = \kappa(\widetilde{\Theta^\wedge})$.

(ii) *Suppose that τ satisfies Assumption 1. If Θ is the τ-connected component of $\widetilde{\Theta^\wedge}$, then $\omega(\Theta)$ is $\mathbf{Q}$-rational, and $\widetilde{\omega(\Theta^\wedge)} = \omega(\Theta)$.*

Proof. We set $Y = \Theta^\wedge$, and let $\Psi = \kappa(\tilde{Y})$. By construction, $\Theta \subseteq \Psi$. If $\Theta \neq \Psi$, $\tilde{Y} \subseteq \omega(\Theta)$ is impossible, so we have (i). It is clear (cf. (2.13)) that we always have

$$\omega(\dot{\Psi}) \supseteq \widetilde{\omega(Y)}.$$

If equality failed to hold, there would exist $\beta \in \omega(\Psi)$ with

$$0 \neq \rho(\beta) \not\leq \omega(Y),$$

and then $Y \cup \{\rho(\beta)\}$ is τ-connected. Thus, by the corollary of (2.4), there is $\beta' \in {}_{\mathbf{R}}\Delta$ with $\rho(\beta') = \rho(\beta)$ such that $\Psi \cup \{\beta'\}$ is τ-connected. Under Assumption 1,

$$\Psi = \kappa(\tilde{Y}) \supseteq \rho^{-1}(Y),$$

so $\Psi \cup \{\beta\}$ is also τ-connected (see Remark of (2.4)), a contradiction. Thus

$$\omega(\Psi) = \widetilde{\omega(Y)}.$$

We therefore obtain:

COROLLARY. *When Assumption* 1 *holds, the boundary components of* X^* *which satisfy condition* (i) *in the definition in* (3.2) *are precisely those of the form* gX_Θ, *where* $g \in G_{\mathbf{Q}}$ *and* $\Theta = \kappa(\tilde{Y})$ *for some* τ*-connected* $Y \subseteq {}_{\mathbf{Q}}\Delta$.

(3.4) It is sometimes the case that for a given representation τ, condition (ii) in the definition in (3.2) is a consequence of (i). For our construction, *we will assume that* τ *is as such* (*Assumption* 2). In other words, if τ satisfies Assumption 2, then the corollary of (3.3) describes the set of rational boundary components. *Whenever we need to be explicit, we will for the sake of simplicity assume that a boundary component is standard; the general case is covered by acting by* $G_{\mathbf{Q}}$, *or equivalently, by making a different choice of* ${}_{\mathbf{Q}}A$.

We now show:

PROPOSITION (cf. [3, (4.3)]). *If* τ *is defined over* $\mathbf{Q}$, *then* Z_Θ (3.2(1)) *is defined over* $\mathbf{Q}$ *whenever* $\omega(\Theta)$ *is* $\mathbf{Q}$*-rational, and thus Assumption* 2 *is satisfied.*

Proof. It is enough to show that the subspaces V_Θ (2.9(2)), with Θ as in (i) of the proposition of (3.3), of V are defined over $\mathbf{Q}$. For that, it suffices to show that if λ is a weight of V with respect to ${}_{\mathbf{C}}\mathfrak{a}$, and $\operatorname{supp}(\lambda - \lambda_0)$ – which is τ-connected by (2.4) – is contained in

$$\Psi = \kappa({}_{\mathbf{R}}\rho_{\mathbf{C}}^{-1}(\Theta \cup \{0\})), \tag{1}$$

then so is $\operatorname{supp}(\sigma(\lambda) - \lambda_0)$ for all $\sigma \in \mathscr{G}$. As $\sigma^+(\lambda_0) = \lambda_0$, we write

$$\sigma(\lambda) - \lambda_0 = \sigma^+(\lambda - \lambda_0) + (\sigma(\lambda) - \sigma^+(\lambda)) \tag{2}$$

From this and (1.4(4)), we see that

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \sigma^+(\Psi)\cup {}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(0).$$

and hence, by the proposition in (2.4),

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \kappa[\sigma^+(\Psi)\cup {}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(0)]. \tag{3}$$

In view of the corollary in (2.4), if $\Theta=\kappa(\tilde{Y})$ we can rewrite (1) as

$$\Psi=\kappa({}_{\mathbf{Q}}\rho_{\mathbf{C}}^{-1}(Y\cup\{0\})).$$

Since λ_0 is invariant under $\mathscr{G}$, we see that Ψ, as the τ-connected component of a $\mathscr{G}$-stable set, is $\mathscr{G}$-stable. It follows that (3) gives

$$\operatorname{supp}(\sigma(\lambda)-\lambda_0)\subseteq \Psi,$$

as desired.

Remark. If ${}_{\mathbf{Q}}A={}_{\mathbf{R}}A$, then Assumption 2 is satisfied for any τ, for one knows that the Lie algebra of the isometry group of X_Θ is defined over $\mathbf{Q}$ (see (3.8)). Of course, Assumption 1 is also satisfied, for trivial reasons.

We will not address the issue of determining for which τ Assumption 2 holds (see [2, §3] for some discussion). The importance of this assumption is that it permits a nice description of the set of rational boundary components. The case studied in [2] is the only general instance we know in which Assumption 2 is satisfied, beyond those described in the above proposition; it would be nice to have a reasonable, more general representation-theoretic condition which guarantees it.

(3.5) Let ${}_{\mathbf{Q}}X^*$ denote the union of all rational boundary components of X^*. It is best to regard ${}_{\mathbf{Q}}X^*$ at this point as only a set, i.e. without a topology. We will eventually define a natural surjective mapping of the manifold with corners $\bar{X}$ constructed by Borel and Serre in [4] onto ${}_{\mathbf{Q}}X^*$.

We recall the definition of $\bar{X}$. The torus ${}_{\mathbf{Q}}A$, or more precisely a certain isomorphic image, operates on X via the so-called *geodesic action* [4, §3]. For a $\mathbf{Q}$-parabolic subgroup $P={}_{\mathbf{Q}}Q_Y$ of G, one defines the *corner* $X(P)$ [4, §5]:

$$X({}_{\mathbf{Q}}Q_Y)={}_{\mathbf{Q}}\bar{A}_Y\times^{{}_{\mathbf{Q}}A_Y}X, \tag{1}$$

where ${}_{\mathbf{Q}}\bar{A}_Y$ is the closure of ${}_{\mathbf{Q}}A_Y$ in ${}_{\mathbf{Q}}\bar{A}$. We should mention that $X(P)$ is intrinsic to P – if $P = {}_{\mathbf{Q}}Q_Y$, ${}_{\mathbf{Q}}A_Y$ is a lifting of the maximal $\mathbf{Q}$-split torus of the center of ${}_{\mathbf{Q}}Q_Y/{}_{\mathbf{Q}}U_Y$. Letting

$$e({}_{\mathbf{Q}}Q_Y) = {}_{\mathbf{Q}}A_Y \backslash X, \tag{2}$$

one has for any $\mathbf{Q}$-parabolic subgroup P

$$X(P) = \coprod_{\substack{Q\ \mathbf{Q}\text{-parabolic} \\ Q \supseteq P}} e(Q) \tag{3}$$

in a natural way [4, (5.1)]. Moreover, if $P' \subset P$, there is a natural embedding of $X(P)$ in $X(P')$ as an open subset. One then defines

$$\bar{X} = \bigcup X(P) = \coprod e(Q), \tag{4}$$

where the unions are taken over the set of all $\mathbf{Q}$-parabolic subgroups of G. The space $\bar{X}$ is Hausdorff, and hence is a manifold with corners [4, (7.8)]. The action of $G_{\mathbf{Q}}$ on X extends to an action on $\bar{X}$ as a group of diffeomorphisms [4, (7.6)].

(3.6) Inspired by (2.6), we will define for a $\mathbf{Q}$-parabolic subgroup P the *crumpled corner* $X^*(P)$ in three steps. First, let

$$X_1^*({}_{\mathbf{Q}}Q_Y) = {}_{\mathbf{Q}}A_Y^* \times^{{}_{\mathbf{Q}}A_Y} X, \tag{1}$$

where ${}_{\mathbf{Q}}A_Y^*$ denotes the closure of ${}_{\mathbf{Q}}A_Y$ in ${}_{\mathbf{Q}}A^*$. The surjection p of (2.6(2)) induces

$$p_1^*(P) : X(P) \to X_1^*(P). \tag{2}$$

From (2.6(1)), we see that

$$X^*({}_{\mathbf{Q}}Q_Y) = \coprod_{\substack{\Xi \supseteq \kappa(Y) \\ \Xi\ \tau\text{-connected}}} e({}_{\mathbf{Q}}Q_{\Xi}), \tag{3}$$

and

$$p_1^*(P)\,|_{e({}_{\mathbf{Q}}Q_{\Psi})} : e({}_{\mathbf{Q}}Q_{\Psi}) \to e({}_{\mathbf{Q}}Q_{\Xi}) \qquad \Xi = \kappa(\Psi) \tag{4}$$

is the quotient mapping $\nu_{\Psi,\Xi}$ by the geodesic action of ${}_{\mathbf{Q}}A_{\Xi}/{}_{\mathbf{Q}}A_{\Psi} \simeq {}_{\mathbf{Q}}A_{\Psi,\Xi}$ (see [4, (5.1(8))]).

It is clear that P operates as a group of homeomorphisms of $X_1^*(P)$. For $P = {}_{\mathbf{Q}}Q_Y$, let Y be a cross-section to the geodesic action of ${}_{\mathbf{Q}}A_Y$ (see [4, (5.4)]). Then, we have a homeomorphism

$$X_1^*({}_{\mathbf{Q}}Q_Y) = {}_{\mathbf{Q}}A_Y^* \times Y. \tag{5}$$

It follows immediately that $X_1^*(P)$ is a Hausdorff space.

If $P \subset P'$, then $X_1^*(P')$ is naturally embedded as an open subset of $X_1^*(P)$, in analogy with the corresponding assertion for corners in (3.5). We should be aware, however, that $X_1^*({}_{\mathbf{Q}}Q_Y)$ depends only on $\kappa(Y)$; if $\kappa(Y) = \kappa(Y')$, then $X_1^*({}_{\mathbf{Q}}Q_Y)$ and $X_1^*({}_{\mathbf{Q}}Q_{Y'})$ are, by (3) and (4), canonically homeomorphic.

The next step is to "derationalize" $X_1^*(P)$. There is a geodesic action of ${}_{\mathbf{R}}A$ on X (which restricts to that of ${}_{\mathbf{Q}}A$), and therefore we can define ${}_{\mathbf{R}}e(Q)$ for any $\mathbf{R}$-parabolic subgroup Q of G, such that whenever $\Xi \subseteq {}_{\mathbf{Q}}\Delta$ there is a projection

$$d_\Xi : e({}_{\mathbf{Q}}Q_\Xi) \to {}_{\mathbf{R}}e(Q_{\underline{\Xi}}), \tag{6}$$

given as the quotient by the geodesic action of ${}_{\mathbf{R}}A_{\underline{\Xi}}/{}_{\mathbf{Q}}A_\Xi$.

Let Ξ be a τ-connected subset of ${}_{\mathbf{Q}}\Delta$. Then we have

$$Q_{\kappa(\underline{\Xi})} \subseteq Q_{\underline{\Xi}} = {}_{\mathbf{Q}}Q_\Xi, \tag{7}$$

from which there is a projection (cf. (4))

$${}_{\mathbf{R}}\nu_{\underline{\Xi},\kappa(\underline{\Xi})} : {}_{\mathbf{R}}e(Q_{\underline{\Xi}}) \to {}_{\mathbf{R}}e(Q_{\kappa(\underline{\Xi})}). \tag{8}$$

We define the space

$$X_2^*({}_{\mathbf{Q}}Q_Y) = \coprod_{\substack{\Xi \supseteq Y \\ \Xi\ \tau\text{-connected}}} {}_{\mathbf{R}}e(Q_{\kappa(\underline{\Xi})}) \tag{9}$$

and mapping

$$p_2^*(P) : X_1^*(P) \to X_2^*(P), \tag{10}$$

where

$$p_2^*({}_{\mathbf{Q}}Q_Y)|_{e({}_{\mathbf{Q}}Q_\Xi)} = {}_{\mathbf{R}}\nu_{\underline{\Xi},\kappa(\underline{\Xi})} \circ d_\Xi \tag{11}$$

and $X_2^*(P)$ is given the quotient topology induced by $p_2^*(P)$. Thus, $X_2^*(P)$ is obtained from $X_1^*(P)$ by collapsing the orbits of further geodesic actions. We note that ${}_{\mathbf{R}}A_{\kappa(\Xi)}/{}_{\mathbf{Q}}A_{\Xi}$ is a subgroup of ${}_{\mathbf{R}}A_{\kappa(\Xi')}/{}_{\mathbf{Q}}A_{\Xi'}$, whenever $\Xi' \subset \Xi$.

Finally, for a parabolic subgroup Q, let U_Q denote its unipotent radical. The projection of Q onto Q/U_Q induces a principal U_Q-bundle (see [4, (7.2)])

$$r_Q : {}_{\mathbf{R}}e(Q) \to {}_{\mathbf{R}}\hat{e}(Q), \tag{12}$$

and moreover, one can identify

$${}_{\mathbf{R}}\hat{e}(Q_\Theta) = X_\Theta. \tag{13}$$

Let

$$X^*({}_{\mathbf{Q}}Q_Y) = \coprod_{\substack{\Xi \supseteq Y \\ \Xi\ \tau\text{-connected}}} {}_{\mathbf{R}}\hat{e}(Q_{\kappa(\Xi)}); \tag{14}$$

use (12) to define a surjection

$$p_3^*(P) : X_2^*(P) \to X^*(P), \tag{15}$$

and equip $X^*(P)$ with the quotient topology. Since also $U_{Q'} \subset U_Q$ whenever $Q \subset Q'$, we can see that $X^*(P)$ (and for a similar reason, $X_2^*(P)$) is a Hausdorff space, as a consequence of the following lemma (cf. [10, (4.2)]):

LEMMA. *Let S be a stratified Hausdorff space, with strata $\{S_j\}$, and let Y be a homogeneous space for the Lie group H. Let, for each j, H_j be a closed normal subgroup of H such that $H_i \supseteq H_j$ whenever S_i is in the closure of S_j. On the product $S \times Y$ define an equivalence relation by: $(s, y_1) \sim (s, y_2)$ if and only if $y_1 \in H_j y_2$, where $S_j \ni s$. Then the quotient space is Hausdorff.*

We remark that the quotient mapping, under the conditions of the above lemma, is seldom an open mapping.

(3.7) For $P \subset P'$, we have $X^*(P')$ embedded as an open subset of $X^*(P)$, as was the case with the $X_1^*(P)$'s. We form the identification space

$${}_{\mathbf{Q}}\tilde{X}^* = \bigcup X^*(P), \tag{1}$$

in which P runs over all $\mathbf{Q}$-parabolic subgroups of G. Observe that if $h \in {}_{\mathbf{Q}}Q_Y \dot{-} {}_{\mathbf{Q}}Q_{\kappa(Y)}$, then the distinct [5, (5.18)] crumpled corners $X^*({}_{\mathbf{Q}}Q_{\kappa(Y)})$ and

$X^*({}^h_{\mathbf{Q}}Q_{\kappa(Y)}) = hX^*({}_{\mathbf{Q}}Q_{\kappa(Y)})$ are identified homeomorphically. Thus, we can see that an efficient way to describe ${}_{\mathbf{Q}}\tilde{X}^*$ as a set is to allow P in (1) to range only over those parabolic subgroups of the form ${}^g_{\mathbf{Q}}Q_{\omega(Y)}$, where $g \in G_{\mathbf{Q}}$ and Y is τ-connected (cf. (2.10)); sometimes, it is also useful to allow repetitions and let P range over the $G_{\mathbf{Q}}$-conjugates of the ${}_{\mathbf{Q}}Q_Y$'s.

By construction, ${}_{\mathbf{Q}}\tilde{X}^*$ is a quotient space of $\bar{X}$ under the mapping

$$p^* : \bar{X} \to {}_{\mathbf{Q}}\tilde{X}^*, \tag{2}$$

where $p^* = p_3^* p_2^* p_1^*$. It is also clear from the construction that the action of $G_{\mathbf{Q}}$ on $\bar{X}$ respects the fibers of p^*. It follows that $G_{\mathbf{Q}}$ acts on ${}_{\mathbf{Q}}\tilde{X}^*$ as a group of homeomorphisms.

We remark that the construction of ${}_{\mathbf{Q}}\tilde{X}^*$ requires neither Assumption 1 nor Assumption 2.

(3.8) We will examine more carefully the structure of the quotient mapping p^*. If one pursues the consequences of the definitions of the geodesic action and the manifold with corners, one sees that, *a priori*, the equivalence relation on $\bar{X}$ that determines the identifications in (3.7(1)) could be unexpectedly large.[5] It will become apparent that things are in actuality fairly nice, because we have in (3.6(4)) that

$$\Xi = \kappa(\Psi); \tag{1}$$

in particular, Ξ is a union of connected components of Ψ.

Let $\Lambda = \omega(\Xi)$, with Ξ τ-connected. According to [4, (7.2)], $e({}_{\mathbf{Q}}Q_\Lambda)$ is a principal ${}_{\mathbf{Q}}U_\Lambda$ fibration over the symmetric space Y_Λ of ${}_{\mathbf{Q}}M_\Lambda$, and moreover, this fibration extends to one of $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$ over the manifold with corners $\bar{Y}_\Lambda$.

Because of (1), with Ψ taken to be Λ, a certain decomposition is possible. First, we can write a product decomposition of identity components

$$({}_{\mathbf{Q}}M_\Lambda)^0 = (H_\Lambda)^0({}_{\mathbf{Q}}L_\Lambda)^0, \tag{2}$$

where ${}_{\mathbf{Q}}L_\Lambda$ is a semi-simple algebraic group over $\mathbf{Q}$ whose Lie algebra ${}_{\mathbf{Q}}\ell_\Lambda$ is generated by the $\mathbf{Q}$-root spaces

$$\{{}_{\mathbf{Q}}\mathfrak{g}_\beta, {}_{\mathbf{Q}}\mathfrak{g}_\beta : \beta \in \Lambda\}; \tag{3}$$

---

[5] See Appendix.

H_Λ is defined, and is anisotropic, over $\mathbf{Q}$; and $H_\Lambda \cap {}_{\mathbf{Q}}L_\Lambda$ is finite (see [2, (2.2)]). From (2), we get an induced decomposition

$$Y_\Lambda \simeq W_{\varnothing,\Lambda} \times W_\Lambda, \tag{4}$$

where $W_{\varnothing,\Lambda}$ and W_Λ are the symmetric spaces of H_Λ and ${}_{\mathbf{Q}}L_\Lambda$ respectively. Next, (1) implies the almost-direct product decomposition

$$({}_{\mathbf{Q}}L_\Lambda)^0 = ({}_{\mathbf{Q}}L_\Xi)^0({}_{\mathbf{Q}}L_Y)^0, \tag{5}$$

with $Y = \Lambda - \Xi$. Therefore

$$W_\Lambda \simeq W_\Xi \times W_Y. \tag{6}$$

We put $\Theta = \kappa(\tilde{\Xi})$. We also let $c(\tilde{\Xi})$ denote the union of the connected components of $\tilde{\Xi}$ that meet $\rho^{-1}(\Xi)$; it is a subset of Θ since we have made Assumption 1 (3.3). We then put

$$\Delta^0(\Xi) = \Theta - c(\tilde{\Xi}). \tag{7}$$

We can restate (3), for Ξ instead of Λ, as: the Lie algebra of ${}_{\mathbf{Q}}L_\Xi$ is $\ell_{c(\tilde{\Xi})}$ (2.8); we then write

$${}_{\mathbf{Q}}L_\Xi = {}_{\mathbf{R}}L_{c(\tilde{\Xi})}. \tag{8}$$

By [2, (3.6,iii)], condition (ii) in the definition of "rational boundary component" for X_Θ is equivalent to the existence of a normal subgroup B of $N_\Theta = {}_{\mathbf{Q}}Q_\Lambda$, defined over $\mathbf{Q}$ and containing $({}_{\mathbf{R}}L_\Theta)^0({}_{\mathbf{Q}}U_\Lambda)$, such that $B/({}_{\mathbf{R}}L_\Theta)^0({}_{\mathbf{Q}}U_\Lambda)$ is compact. It is enough to find a normal $\mathbf{Q}$-subgroup B' of ${}_{\mathbf{Q}}M_\Lambda$ that contains $({}_{\mathbf{R}}L_\Theta)^0$, with $B'/({}_{\mathbf{R}}L_\Theta)^0$ compact. (It is useful to recall that X_Θ is the symmetric space of ${}_{\mathbf{R}}L_\Theta$, or of its identity component.) Now, there is an almost-direct product

$$({}_{\mathbf{R}}L_\Theta)^0 = ({}_{\mathbf{Q}}L_\Xi)^0({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0, \tag{9}$$

and moreover

$$({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0 \subset (H_\Lambda)^0. \tag{10}$$

Thus, we see that X_Θ is a rational boundary component if and only if there is a normal algebraic subgroup H'_Λ of H_Λ, defined over $\mathbf{Q}$, such that $({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0 \subset H'_\Lambda$

and $H'_\Lambda/({}_{\mathbf{R}}L_{\Delta^0(\Xi)})^0$ is compact. By our Assumption 2 (3.4), we have the existence of such H'_Λ.

Write $(H_\Lambda)^0$ as the almost-direct product of $\mathbf{Q}$-groups

$$(H_\Lambda)^0 = (H'_\Lambda)^0 \cdot (H''_\Lambda)^0, \tag{11}$$

and decompose accordingly

$$W_{\varnothing,\Lambda} \simeq W'_\Lambda \times W''_\Lambda. \tag{12}$$

We have

$$X_\Theta \simeq W_\Xi \times W'_\Lambda. \tag{13}$$

The following is now apparent:

PROPOSITION. *With notation as above:*
(i) $\overline{e({}_{\mathbf{Q}}Q_\Lambda)} \simeq (\bar{W}_\Xi \times W'_\Lambda) \times \bar{W}_Y \times W''_\Lambda \times {}_{\mathbf{Q}}U_\Lambda$,
(ii) *The subset of $\bar{X}$ that crumples onto ${}_{\mathbf{R}}\hat{e}(\mathbf{Q}_\Theta)$ under p^* is the subset*

$$\coprod_{\substack{\kappa(\Psi)=\Xi \\ h\in({}_{\mathbf{Q}}Q_\Lambda)_{\mathbf{Q}}/({}_{\mathbf{Q}}Q_\Psi)_{\mathbf{Q}}}} e({}^h_{\mathbf{Q}}Q_\Psi) \simeq X_\Theta \times \bar{W}_Y \times W''_\Lambda \times {}_{\mathbf{Q}}U_\Lambda, \tag{14}$$

of $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$, and the mapping p^* is given by projecting onto the first factor.

COROLLARY. *The fibers of p^* over ${}_{\mathbf{R}}\hat{e}(Q_\Theta)$ are naturally isomorphic to $\bar{W}_Y \times W''_\Lambda \times {}_{\mathbf{Q}}U_\Lambda$. They are closed in $\overline{e({}_{\mathbf{Q}}Q_\Lambda)}$.*

(3.9) We recall the set ${}_{\mathbf{Q}}X^*$ introduced in (3.5). One puts a topology on ${}_{\mathbf{Q}}X^*$ as follows (cf. [9, §2], [3, §4]).

Let $\mathcal{S}$ be a *generalized Siegel set* in X, which we should feel free to take as large as is necessary. By definition,

$$\mathcal{S} = (C\,{}_{\mathbf{Q}}A(t))x_0 \tag{1}$$

where C is a compact subset of ${}_{\mathbf{Q}}M_\varnothing\,{}_{\mathbf{Q}}U_\varnothing$, and x_0 is the basepoint (the coset K) of X. One can choose $\mathcal{S}$ so that

$$X = G_{\mathbf{Q}}\mathcal{S} \tag{2}$$

(see [11]; cf. [3, (2.4)]).

Let Γ be an *arithmetic* subgroup of G. This means that Γ is commensurable with the set of elements of G that are represented by matrices with integer entries under a general rational finite-dimensional representation of G. Then there exists a finite subset F_Γ of $G_{\mathbf{Q}}$ such that

$$\Omega = \Omega_\Gamma := F_\Gamma \mathscr{S} \tag{3}$$

is a *fundamental set* in X for Γ, namely

$$X = \Gamma\Omega, \textit{ and for all } g \in G, \{\gamma \in \Gamma : \gamma\Omega \cap g\Omega \neq \varnothing\} \textit{ is finite} \tag{4}$$

(see [11, §§10, 12]; compare [3, (2.4)], [2, (4.3)]).

Let Ω^* denote the closure of Ω in X^* (with respect to the "usual" topology of (2.1)); since the closure $\mathscr{S}^*$ of $\mathscr{S}$ is contained in ${}_{\mathbf{Q}}X^*$, we have

$$\Omega^* \subset {}_{\mathbf{Q}}X^*. \tag{5}$$

LEMMA (cf. [3, (4.3)]). (i) *Ω^* intersects only finitely many rational boundary components.*

(ii) ${}_{\mathbf{Q}}X^* = \Gamma \cdot \Omega^*$ *if Ω is sufficiently large.*

(iii) *There is a finite subset Γ_0 of Γ such that if $\gamma \in \Gamma$ and $\gamma\Omega^* \cap \Omega^* \neq \varnothing$, then there exists $\gamma_0 \in \Gamma_0$ with $\gamma_0 x = \gamma x$ for all $x \in \gamma\Omega^* \cap \Omega^*$.*

Proof. By construction, $\mathscr{S}^*$ meets only the standard boundary components. From the definition of Ω (3), we have (i). Given (2) and (4), Assumption 2 (3.4), and the corollary of (3.3), in order to prove (ii) it suffices to verify that $G_{\mathbf{Q}} \cdot \mathscr{S}^*$ contains all of the standard rational boundary components. For this, and also for (iii) – since $\Gamma_\Theta := \Gamma \cap N_\Theta$ is arithmetic and, with Assumption 2, its image in the automorphism group of X_Θ is arithmetically defined (cf. [2, (3.4)–(3.6)]) – it is enough to know that $\Omega^* \cap X_\Theta$ is a fundamental set in X_Θ, for rational boundary components X_Θ. When $\Xi \neq \varnothing$, this follows from the fact that $\mathscr{S}^* \cap X_\Theta$ is a Siegel set in X_Θ, and all Siegel sets in X_Θ are of this form (cf. [2, (4.5)] and [4, (6.2)]). To see this, note that for a Siegel set (1) in X, $\mathscr{S}^* \cap X_\Theta$ consists of the translates of the base point of X_Θ by the Siegel set $\tilde{\mathscr{S}} = C_{\mathbf{Q}}A(t)(K \cap N_\Theta)$ in N_Θ. We are taking, to replace the centralizer (see (3.8)),

$$Z'_\Theta = H''_\Lambda({}_{\mathbf{Q}}L_Y)({}_{\mathbf{Q}}A_\Lambda)({}_{\mathbf{Q}}U_\Lambda);$$

remembering the remaining factors from N_Θ, we see that the projection of $\tilde{\mathscr{S}}$ in

N_Θ/Z'_Θ is in fact a Siegel set, and the desired conclusion holds. In the remaining case where $\Xi = \emptyset$, where the quotient group is anisotropic, both the projection of $\tilde{\mathscr{S}}$ and $\Gamma_\Theta \backslash X_\Theta$ are compact; if $\mathscr{S}$ is large enough, we get a fundamental set.

With the above lemma proved, we can now assert:

PROPOSITION [9, §2]. *There exists a unique topology on* ${}_{\mathbf{Q}}X^*$ *for which*
(i) *the relative topology induced on* Ω^* *is the given* ("*usual*") *one,*
(ii) Γ *acts as a group of homeomorphisms of* ${}_{\mathbf{Q}}X^*$,
(iii) *if* $x, x' \in {}_{\mathbf{Q}}X^*$ *are in different* Γ*-orbits, then there exist neighborhoods* U, U' *of* x, x' *respectively, which satisfy* $\Gamma U \cap U' = \emptyset$;
(iv) *let* Γ_x *be the isotropy group in* Γ *of* $x \in {}_{\mathbf{Q}}X^*$. *Then* x *has a neighborhood base consisting of* Γ_x*-invariant open sets* U *for which* $\gamma U \cap U = \emptyset$ *if* $\gamma \notin \Gamma_x$.
It follows that ${}_{\mathbf{Q}}X^*$ *becomes a Hausdorff space, and that the quotient* $\Gamma \backslash {}_{\mathbf{Q}}X^*$ *is compact Hausdorff. Moreover, the topology on* ${}_{\mathbf{Q}}X^*$ *does not, in fact, depend on* Γ.

From the construction of the topology [9, p. 562], one can be more explicit about (i) and (iv). Let $x \in \Omega^*$. A fundamental system of neighborhoods of x in ${}_{\mathbf{Q}}X^*$ can be described as follows. Let $\{U_j\}$ be a neighborhood base for x in Ω^*. Then $\{\Gamma_x U_j\}$ forms a neighborhood base of x in ${}_{\mathbf{Q}}X^*$.

We can compare compactifications determined by different τ's:

COROLLARY. *If* τ *and* τ' *are related as in* (2.11), *then the identity mapping of* $\Gamma \backslash X$ *extends to a continuous surjection*

$$\Gamma \backslash {}_{\mathbf{Q}}X^*_\tau \to \Gamma \backslash {}_{\mathbf{Q}}X^*_{\tau'}.$$

(3.10) We have been aiming toward the following result:

THEOREM. *The identity mapping of* X *extends to a continuous bijection of* ${}_{\mathbf{Q}}\tilde{X}^*$ *onto* ${}_{\mathbf{Q}}X^*$.

Proof. From the construction of ${}_{\mathbf{Q}}\tilde{X}^*$, it is apparent that as a set, ${}_{\mathbf{Q}}\tilde{X}^*$ is the union of all rational boundary components of X^*. In other words, there is an obvious one-to-one mapping of ${}_{\mathbf{Q}}\tilde{X}^*$ onto ${}_{\mathbf{Q}}X^*$. Henceforth, we identify ${}_{\mathbf{Q}}\tilde{X}^*$ and ${}_{\mathbf{Q}}X^*$ as sets. In order to show that this mapping is continuous, we need only verify that the sets described at the end of (3.9) are open in the topology of ${}_{\mathbf{Q}}\tilde{X}^*$. Since $G_{\mathbf{Q}}$ acts by homeomorphisms, we can see that

$$\Gamma_x[(p^*)^{-1}(U_j) \cap \tilde{\mathscr{S}}]$$

is open in $\bar{X}$ by checking that the topology induced on $\mathscr{S}^*$ is the usual one. (Note that for $x \in {}_{\mathbf{R}}\hat{e}(\mathbf{Q}_\Theta)$, Γ_x contains, and is commensurable with, the centralizer of the boundary component.) Now, $\mathscr{S}^*$ is the one-to-one continuous image of the (compact) closure of $\mathscr{S}$ in the crumpled corner $X^*({}_{\mathbf{Q}}Q_\varnothing)$. The latter inherits the usual topology, and hence $\mathscr{S}^*$ does as well. By taking larger and larger $\mathscr{S}$, we see that $\Gamma_x(p^*)^{-1}(U_j) = (p^*)^{-1}(\Gamma_x U_j)$ is open in $\bar{X}$. This completes the proof.

COROLLARY. *For any arithmetic subgroup Γ of G, the Satake compactification $\Gamma\backslash_{\mathbf{Q}}X^*$ is a quotient of the compact manifold*[6] *with corners $\Gamma\backslash\bar{X}$. The fibers of the quotient mapping can be deduced from* (3.8(8)) *by passing to the quotient by the action of ${}_{\mathbf{Q}}Q_\Lambda \cap \Gamma$.*

Remark. One can see that in general the topology of ${}_{\mathbf{Q}}\tilde{X}^*$ is finer than that of ${}_{\mathbf{Q}}X^*$, since it contains neighborhoods of points x that lack uniformity under the action of Γ_x.

(3.11) Our motivation has been to realize the compactification of $\Gamma\backslash X$, when X is Hermitian, constructed by Baily and Borel in [2] as a quotient of the manifold with corners. In order to apply the results of the preceding section, we must know that the Baily–Borel compactification is, as a topological space, of the form $\Gamma\backslash_{\mathbf{Q}}X^*_\tau$ for some representation τ of G. In some instances, the answer is already in the earlier [9]. The issue clearly lies prior to the taking of the quotient by Γ.

We first assume that X is irreducible. Then the $\mathbf{R}$-root system of G is either of classification type C_r, or is the non-reduced system BC_r, for some r (see [2, (1.2)]). These systems contain one simple root α_r that is respectively longer or shorter than the other simple roots. The construction in [2, §4] uses maximal $\mathbf{Q}$-parabolic subgroups, the closure X^c of the realization of X as a bounded domain, and a construction like that in the proposition of (3.9). We appeal to:

PROPOSITION [7, §3], [8, (5.2)]. *Let τ be any representation of G whose restricted highest weight is a multiple of the fundamental dominant weight dual to α_r. Then the Satake compactification X^*_τ is homeomorphic to X^c.*

If X is reducible, one makes the two constructions on each irreducible factor and takes the product; i.e., X^c is homeomorphic to X^*_τ, where τ is the tensor product of the chosen representations for the factors. For such τ, Assumptions 1 and 2 of (3.3) and (3.4) hold (see [2, (2.9), (3.7)]).

---

[6] There are finite quotient singularities if Γ contains torsion elements.

To discuss ${}_{\mathbf{Q}}X^*_\tau$, we may assume without loss of generality that ${}_{\mathbf{Q}}\Delta$ is irreducible. The root system is then of type C_r or BC_r ([2, (2.9(a))]). Let Ξ be a τ-connected subset of ${}_{\mathbf{Q}}\Delta$. Then $\omega(\Xi)$ omits but one simple root. Therefore, ${}_{\mathbf{Q}}Q_{\omega(\Xi)}$ is maximal $\mathbf{Q}$-parabolic; conversely, every standard maximal $\mathbf{Q}$-parabolic subgroup is of this form. We see that the rational boundary components of X^c [2, (3.5)] and X^*_τ correspond. We obtain:

THEOREM. *The Baily–Borel compactification is a quotient of the manifold with corners $\Gamma\backslash\bar{X}$.*

We remark that although the Baily–Borel compactification is also the quotient of the smooth compactifications defined in [1, Ch. III, §5], one cannot realize the latter as quotients of $\Gamma\backslash\bar{X}$ (we allow no identifications in $\Gamma\backslash X$). An example of one which cannot be so realized is the Hirzebruch resolution (see [1, Ch. I, §5] for the definition) of a Hilbert modular surface.

Appendix: Comparison of geodesic actions

The geodesic action is denoted $a\circ x$ for $a\in{}_{\mathbf{Q}}A$ and $x\in X$ (or $x\in e({}_{\mathbf{Q}}Q_\Xi)$). We recall from [4, (3.2)] two basic properties of its definition. *For simplicity, we will state things for X, though there are parallel assertions for the general case.*

First, the geodesic action of ${}_{\mathbf{Q}}A_\Xi$ commutes with translations by ${}_{\mathbf{Q}}Q_\Xi$. Secondly, if x_0 is the basepoint of X associated to the choice of K, then the geodesic action of ${}_{\mathbf{Q}}A$ on x_0 coincides with the usual translation by ${}_{\mathbf{Q}}A$. One sees that if $a\in{}_{\mathbf{Q}}A_\Xi$ and $x=qx_0$ for $q\in{}_{\mathbf{Q}}Q_\Xi$, there is the formula

$$a\circ x = qax_0 = (qaq^{-1})(qx_0). \tag{1}$$

This makes sense for all $x\in X$, since the parabolic subgroups act transitively on X.

We compare the geodesic actions of ${}_{\mathbf{Q}}A_\Xi$ and ${}^g_{\mathbf{Q}}A_\Xi$, with $g\in{}_{\mathbf{Q}}Q_\Psi$ ($\Xi\subset\Psi$), on $e({}_{\mathbf{Q}}Q_\Psi)$. Write $x\in e({}_{\mathbf{Q}}Q_\Psi)$ as gq_gx_0 (more precisely, the projection of this element of X onto $e({}_{\mathbf{Q}}Q_\Psi)$), with $q_g\in{}_{\mathbf{Q}}Q_\Xi$. Then

$${}^g_{\mathbf{Q}}A_\Xi\circ x = {}^g_{\mathbf{Q}}A_\Xi\circ(gq_gg^{-1})gx_0 = (gq_gg^{-1}){}^g_{\mathbf{Q}}A_\Xi(gx_0) = gq_g\,{}_{\mathbf{Q}}A_\Xi x_0. \tag{2}$$

If we select $p\in{}_{\mathbf{Q}}Q_\Xi$ such that $x=px_0$, we can then rewrite (2) as

$${}^g_{\mathbf{Q}}A_\Xi\circ x = (gq_gp^{-1})\,{}_{\mathbf{Q}}A_\Xi\circ x. \tag{3}$$

We see that the two geodesic orbits are ${}_{\mathbf{Q}}Q_{\Psi}$-translates of each other. More specifically:

PROPOSITION. *Let* $x \in e({}_{\mathbf{Q}}Q_{\Psi})$, $\Xi \subset \Psi \subseteq {}_{\mathbf{Q}}\Delta$, $g \in {}_{\mathbf{Q}}Q_{\Psi}$. *Then the projections* $\bar{x}$ *and* $\bar{x}_g$ *of* x, *in* $e({}_{\mathbf{Q}}Q_{\Xi})$ *and* $e({}^{g}_{\mathbf{Q}}Q_{\Xi})$ *respectively, are related by the formula*

$$\bar{x}_g = g(q_g p^{-1} \bar{x}), \tag{4}$$

where q_g *and* p *are defined above.*

Of course, in the construction of ${}_{\mathbf{Q}}\tilde{X}^*$ (3.7(1)), the images of certain $\bar{x}$ and $\bar{x}_g$ are identified. We observe that, in general, the choices of q_g and p, and therefore also the point $\bar{x}_g$, depend not only on $\bar{x}$, but also on x. This is the source of the remark in the second sentence of (3.8).

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Non-Linear Parametric Optimization

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In this book various classes of non-linear parametric optimization problems (including integer problems) are studied theoretically and under the aspect of possible practical applications. The behaviour of the parameter-dependent constraint set, solution set (optimal set) and extreme values of the corresponding optimization problem are examined; this leads, in the main, to an analysis of concrete point-to set mappings. Topological and structural properties are demonstrated for special classes of optimization problems (convex quadratic problems, linear complementarity problems with positive semi-definite matrix, non-linear optimization problems with parameter-dependent objective function and mixed-integer problems). As in linear parametric optimization, the concept of local stability sets plays a central role. Special attention is paid to the study of the restriction of the optimal set mapping to a local stability set, thus the behaviour of the optimal set under perturbations of the parameters in a local stability set may be characterized.

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Cheeger's inequality with a boundary term

M. N. Huxley

1. Introduction

Cheeger's inequality refers to the eigenvalues of the Laplacian on an oriented manifold M. For ease of exposition we consider two dimensional manifolds in this note, with local coordinates x, y and length l and measure μ given by

$$dl^2 = g(x, y)(dx^2 + dy^2), \qquad d\mu = g(x, y)\, dx\, dy,$$

where $g(x, y)$ is an analytic function of x and y. We say $f(x, y)$ is a Dirichlet eigenfunction on M if

$$\iint_M f^2(x, y) g(x, y)\, dx\, dy \text{ converges,}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\lambda f(x, y) g(x, y)$$

(for some λ) on the interior of M, and if f is not identically zero, but $f(x, y) = 0$ on ∂M, the boundary of M.

CHEEGER'S THEOREM. *Let h be a constant such that*

$$l(\partial S) \geqslant h\mu(S)$$

for all subsets S of M with finite measure and piecewise smooth boundary ∂S. Then if λ is the eigenvalue of a Dirichlet eigenfunction,

$$2\sqrt{\lambda} > h.$$

For a compact manifold without boundary the constant h is zero, but [2] the infimum defining h may now be taken over subsets S whose measure is at most half that of M. In fact, the sets S considered are connected components of the inverse image of $[\delta, \infty)$ or of $(-\infty, -\delta]$ under f. We may replace h by the infimum $h(f)$ over such sets S; the constant h now furnishes a lower bound for $h(f)$.

If M is multiply connected, it is difficult to estimate the bound h, or even $h(f)$. Cutting the manifold introduces extra boundaries. We prove an appropriate extension of Cheeger's theorem:

Let N be a two-dimensional oriented manifold with metric given by

$$dl^2 = g(x, y)(dx^2 + dy^2), \qquad d\mu = g(x, y)\, dx\, dy,$$

and boundary $N = C \cup D$. Let f satisfy

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\lambda f(x, y) g(x, y)$$

on the interior of N, and the mixed boundary conditions

$$f = 0 \text{ on } C, \qquad \frac{\partial f}{\partial n} = 0 \text{ on } D.$$

Let the manifold M be obtained by cutting N along E, a finite union of simple curves; the boundary of M thus consists of C, D and two copies E_1 and E_2 of E, with periodic boundary conditions identifying E_1 and E_2. Then if f is non-constant, we have

$$(2\lambda^{1/2} - h(f)) \iint_M f^2\, d\mu > - \oint_{\partial M} f^2\, dl.$$

2. Proof

We follow Buser's account [2] of Cheeger's theorem. By periodicity

$$\left(\int_{E_1} + \int_{E_2}\right) f \operatorname{grad} f \cdot d\boldsymbol{s} = 0,$$

where $\operatorname{grad} f$ and ds are with respect to the coordinates x, y. We have $f = 0$ on C

and the normal derivative of f vanishes on D in both metrics. Hence

$$0=\oint_{\partial M} f \operatorname{grad} f \cdot ds=\iint_M \operatorname{div}(f \operatorname{grad} f)\, dx\, dy$$

$$=\iint_M |\operatorname{grad} f|^2\, dx\, dy+\iint_M f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right) dx\, dy$$

$$=\iint_M |\operatorname{grad} f|^2\, dx\, dy-\lambda \iint_M f^2 g\, dx\, dy.$$

It is convenient to suppose f normalised so that

$$\iint_M f^2 g\, dx\, dy=1.$$

Next we have

$$\iint_M |\operatorname{grad} f^2|\, g^{1/2}\, dx\, dy=2 \iint_M |f|\, |\operatorname{grad} f|\, g^{1/2}\, dx\, dy$$

$$<\left\{4 \iint_M f^2 g\, dx\, dy \iint_M |\operatorname{grad} f|^2\, dx\, dy\right\}^{1/2}=2\lambda^{1/2}$$

by the normalisation. Since f is non-constant we have grad f zero but f nonzero at the extrema of f, and so $f^2 g$ and $|\operatorname{grad} f|^2$ are not proportional. Thus the inequality is strict.

Next we take curvilinear coordinates t, u on M, with $t=f(x, y)$, u running along the curves $f(x, y)=$ constant. These coordinates are orthogonal, with area element $dt\, du=|\operatorname{grad} f|\, dx\, dy$. We deduce that

$$2\lambda^{1/2}>\iint_M 2\,|f|\, |\operatorname{grad} f|\, g^{1/2}\, dx\, dy=\iint_M 2\,|t|\, g^{1/2}\, du\, dt.$$

Next we let $M(t)$ be the set of points with $f(x, y) \geqslant t$, $M'(t)$ be the set of points with $f(x, y) \leqslant t$, and $L(t)$ be the curve $f(x, y) = t$ of length

$$l(t) = \int_{L(t)} g^{1/2}\, du$$

in the metric on M. The boundary of $M(t)$ consists of $L(t)$ and that part of ∂M that lies in $M(t)$, so that

$$\int_{L(t)} g^{1/2}\, du + \int_{M(t)\cap\partial M} g^{1/2}\, ds \geqslant h(f)\mu(M(t)),$$

where du and ds are Euclidean lengths. Hence

$$2\lambda^{1/2} + 2\int_{t>0} t \int_{M(t)\cap\partial M} g^{1/2}\, ds\, dt + 2\int_{t<0} |t| \int_{M'(t)\cap\partial M} g^{1/2}\, ds\, dt$$

$$\geqslant h(f)\int_{t>0} 2t \iint_{M(t)} g\, dx\, dy\, dt + h(f)\int_{t<0} 2|t| \iint_{M'(t)} g\, dx\, dy\, dt.$$

Interchanging the order of integration, we have

$$2\lambda^{1/2} + \int_{M(0)\cap\partial M} t^2 g^{1/2}\, ds + \int_{M'(0)\cap\partial M} t^2 g^{1/2}\, ds$$

$$\geqslant h(f)\left\{\iint_{M(0)} g\, t^2\, dx\, dy + \iint_{M'(0)} g\, t^2\, dx\, dy\right\},$$

so that

$$2\lambda^{1/2} + \oint_{\partial M} f^2\, dl \geqslant h(f),$$

where dl is the differential of distance in the metric.

3. Applications

Consider the upper half plane H as hyperbolic space of curvature -1, with $g(x, y) = 1/y^2$. For sets S of finite measure the isoperimetric inequality [1] states

$$l^2(\partial S) \geqslant \mu^2(S) + 4\pi\mu(S).$$

Let Γ be a group acting discontinuously on H, for which the quotient space $\Gamma\backslash H$ has finite measure. If Γ contains no rotations or transvections (limit rotations about points at infinity, the 'cusps'), then $\Gamma\backslash H$ is a compact Riemann surface, and every simply connected subset carries the hyperbolic metric. The compact case has been studied extensively, cf. Elstrodt's survey [4]. If Γ does contain transvections, there is a continuous spectrum $\lambda > \frac{1}{4}$ whose multiplicity is the number of inequivalent cusps. The generalised eigenfunctions of the continuous spectrum are given by the values of the Eisenstein series introduced by Maass [8] on the line Re $s = \frac{1}{2}$, with $\lambda = s(1-s)$. The continuation of the Eisenstein series to Re $s = \frac{1}{2}$ is difficult (except in special cases); see [5, 9, 10, 11]. The Eisenstein series has a pole at $s = 1$ with constant residue f_0, the trivial constant eigenfunction, and any other poles in $\frac{1}{2} \leqslant s \leqslant 1$ on the real axis correspond to square-integrable eigenfunctions, again by $\lambda = s(1-s)$. All other eigenfunctions are 'cusp forms', zero at all cusps of $\Gamma\backslash H$.

The modular group PSL(2, Z) and its congruence subgroups are of particular interest [8, 9]. Recently Kuznietsov [6] and Deshouillers and Iwaniec [3] have used the Kuznietsov Trace Formulae to study them. These formulae differ from that of Selberg by taking the group elements not in conjugacy classes, but in double cosets of the Borel subgroup of upper triangular matrices, and using the Fourier theory for the transvection group at the cusp ∞. Eigenfunctions with $\lambda < \frac{1}{4}$ complicate asymptotic formulae as in the Selberg theory [4]. The Linnik-Selberg conjecture on averages of Kloosterman sums [7] would imply $\lambda \geqslant \frac{1}{4}$, and Kuznietsov [6] has shown that an averaged form of the conjecture holds in the absence of such exceptional eigenvalues. For congruence subgroups of the modular group the Eisenstein series $E(z, s)$ is a linear combination of Epstein zeta-functions in the variable s, and $E(z, s)$ is easily seen to be regular for Re $s > \frac{1}{2}$ except for the pole at $s = 1$; cf. [8]. Accordingly exceptional eigenfunctions, if any, must be cusp forms.

Congruence subgroups of the modular group of level N are those subgroups containing $\Gamma(N)$, the principal subgroup of level N, which consists of matrices congruent mod N to the identity. The lengths of translations on $\Gamma(N)$ tend to infinity with N, and for $N \geqslant 2$ $\Gamma(N)$ contains no rotations. For $N \geqslant 3$ $\Gamma(N)$ has index

$$I(N) = \tfrac{1}{2}N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

with $I(N)/N$ distinct cusps, and genus

$$1 + \frac{(N-6)I(N)}{12N}.$$

Other interesting subgroups consist of the matrices which become upper triangular ($\Gamma_0(N)$), lower triangular ($\Gamma^0(N)$) or diagonal ($\Gamma_0^0(N)$) when reduced mod N. We note that $\Gamma_0(N)$ and $\Gamma^0(N)$ are conjugate in PSL $(2, Z)$, and that $\Gamma_0^0(N)$ is conjugate in $\mathrm{PSL}(2, R)$ to $\Gamma_0(N^2)$ and to $\Gamma^0(N^2)$. The eigenfunctions on conjugate groups differ only by a rigid motion in H.

For certain small values of N we can rule out exceptional eigenvalues by purely combinatorial arguments.

THEOREM. *Let the group Γ act discontinuously on the upper half plane H, and let $\Gamma\backslash H$ have finite area. Let f be a non-constant eigenfunction of the Laplacian that vanishes at all the cusps. If either*

(1) $\Gamma\backslash H$ has genus zero, and f vanishes at all the fixed points of rotations with at most three exceptions, or (2) $\Gamma\backslash H$ has genus one, and f vanishes at all the fixed points of rotations with at most one exception, then the corresponding eigenvalue satisfies $\lambda > \frac{1}{4}$.

Proof. Since f is non-constant, it has at least two nodal domains. When $\Gamma\backslash H$ has genus zero, at least two of them are topological discs, and one of these contains at most one fixed point in its interior. When $\Gamma\backslash H$ has genus one, either one nodal domain is a disc (which may contain a fixed point), or at least two nodal domains are topological annuli, and one of these contains no fixed point in its interior.

A disc on $\Gamma\backslash H$ containing one fixed point, that of a rotation group of order n, lifts to a disc on H that covers it n times. The cusps are also singularities of the hyperbolic metric on $\Gamma\backslash H$, but they lie on nodal lines–this may be verified directly from the Fourier series expansion of f–and so cannot lie inside a nodal domain. Hence Cheeger's theorem applies, and $\lambda > \frac{1}{4}$.

An annulus on $\Gamma\backslash H$ may lift to an annulus on H. In this case we may apply Cheeger's theorem at once. Otherwise we must make a cut, and lift to a disc D in H for which two connected arcs E and τE of the boundary are identified by some τ in Γ. If τ is a rotation of order n, then n copies of D fit together to form an annulus in H, and $\lambda > \frac{1}{4}$ again by Cheeger's theorem. If τ has infinite order, we unite n copies $D, \tau D, \ldots, \tau^{n-1}D$ and the arcs $\tau E, \ldots, \tau^{n-1}E$ to form a simply connected region whose boundary consists of E, $\tau^n E$ and two nodal lines of f. The cut E can be taken away from the cusps, so that

$$\int_E f^2\, dl < \infty,$$

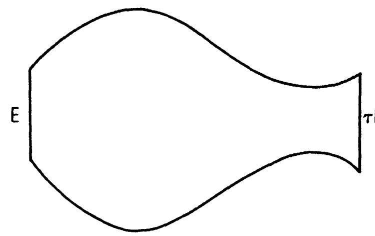

Figure 1

and using the value $h=1$ appropriate to hyperbolic space, we have

$$(2\lambda^{1/2}-1)n\iint_D f^2\,d\mu > -2\int_E f^2\,dl.$$

Hence $\lambda \geqslant \frac{1}{4}$, since n can be taken arbitrarily large. This inequality is in fact strict, as we have shown

$$\iint_D |f \operatorname{grad} f|\frac{dx\,dy}{y} \geqslant \tfrac{1}{4}\iint_D f^2\,d\mu,$$

and the left hand side is strictly less than

$$2\lambda^{1/2}\iint_D f^2\,d\mu.$$

COROLLARY. *We have* $\lambda > \frac{1}{4}$ *on* $\Gamma(N)\backslash H$ *for* $N=1,\ldots,5$ (*genus* 0), *and* $N=6$ (*genus* 1), *there being no rotations in the group for* $N \geqslant 2$. *We have* $\lambda > \frac{1}{4}$ *on* $\Gamma^0(N)\backslash H$ *for* $N=1,\ldots,12$, 14, 15, 16, 18, 20, 24, 25, 27, 32, *and* 36. (*The genus is* 0 *for* $N=1,\ldots,5,7,\ldots,10$, 12, 16, 18 *and* 25. *There are two fixed points for* $N=1,3,5,7$ *and* 10, *and one for* $N=2$. *The other values of N listed give genus* 1 *and no rotations*.)

We remark that if Γ has neither cusps nor rotations, then $\Gamma\backslash H$ has genus at least two. The rotations all lie outside a subgroup of finite index in Γ (index at most six for congruence subgroups of the modular group), but the corresponding surface has a larger genus. One might hope that any eigenfunction vanishing at all cusps would have $\lambda \geqslant \frac{1}{4}$; but using ideas of Buser we can construct a group Γ for which $\Gamma\backslash H$ has one cusp, four fixed points of rotations and genus zero, λ_1 is arbitrarily small, and the first non-constant eignefunction is skewsymmetric about an axis through the cusp, and so is zero there.

Our Theorem can be generalised by allowing some of the singular points of the metric at which f does not vanish to be cusps, not fixed points. This uses a different argument to deal with a cusp in the interior of a nodal domain.

For the congruence subgroups the fact that $\lambda > \frac{1}{4}$ for the modular group and for $\Gamma^0(2)$ is implicit in Maass [8], who was interested mainly in the value $\lambda = \frac{1}{4}$. It was proved explicitly by Roelcke [9]. For the modular group Roelcke showed $\lambda > 3\pi^2/2$, better than the bound $\lambda > 25/4$ obtained from estimating $h(f)$ in Cheeger's theorem. We shall discuss numerical estimates and particular examples more fully elsewhere.

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The Novikov conjecture and low-dimensional topology

SHMUEL WEINBERGER*

It is well known that many interesting manifolds can be obtained by cutting some standard manifold along a separating codimension-one submanifold and glueing the pieces together by a homeomorphism (or diffeomorphism) homotopic to the identity. (These manifolds will be simple homotopy equivalent.) For instance, in dimension at least five, all smooth homotopy spheres can be obtained from the standard sphere in this fashion. In [We 1] we studied the question of identifying those homotopy equivalences that can be so produced as well as related problems. In high dimensions, e.g. dimension at least five, a complete solution is often possible. Through dimension three, the result is trivial, no "new" homotopy equivalences are obtainable since for surfaces homeomorphisms homotopic to the identity are in fact isotopic to the identity. Thus, a gap is left in our knowledge in dimension four.

It is known that for a large class of three-manifolds, homotopy implies isotopy for homeomorphisms, and no example is known of this failing for any three-manifold. This suggests that the situation for four-manifolds should be no different from that for three manifolds and should therefore be very different from the higher dimensional theory.

Most of this paper is devoted to studying homotopy equivalences $h: S^1 \times L_1 \to S^1 \times L_2$ where L_1 and L_2 are classical lens spaces. In §1 we review enough of [We 1] to get the flavor of the high dimensional theory and see why it would be anomalous for h *not* to be obtainable by cutting and pasting. (In fact, $h \times 1_{S^1}: T^2 \times L_1 \to T^2 \times L_2$ can be obtained in such a manner.) In §2 we show by low dimensional techniques that if the codimension-one (three-) manifold cut along lies in the Poincaré category then h cannot result. In §4 we remove this restriction by an algebraic technique that also shows that many other homotopy equivalences are not cut-pastable. It is in this algebra (§3) that the Novikov conjecture enters as an ingredient in calculating the image of the L-theory of three-manifold groups in the L-theory of a certain class of groups.

This paper is an extension of part of the author's thesis. It is a pleasure to

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1. Partial review of [We 1]

Let M be a manifold and N a codimension one submanifold which divides M into two components, i.e. $M = M_+ \cup_N M_-$. The homotopy equivalence obtained by cutting and pasting with (N, h, H), where $h: N \to N$ is a homeomorphism and $H: N \times I \to N \times I$ is a homotopy from h to the identity, is

$$\bar{H}: M(N, h) \equiv M_+ \cup N \times I \cup_{h^{-1}} M_- \xrightarrow{1_{M_+} \cup H \cup 1_{M_-}} M_+ \cup N \times I \cup M_- = M.$$

Observe that if H is an isotopy, $\bar{H}$ is a homeomorphism. A homotopy equivalence $g: M' \to M$ is *CP* (*cut-pastable*) if there is a triple (N, h, H) as above and a homeomorphism $G: M' \to M(N, H)$ such that $g \sim \bar{H} \circ G$. In [We 1] the following is proven:

THEOREM A. *Let $h: M' \to M^n$ be a homotopy equivalence between closed n manifolds, $n \geq 5$, and suppose that*

(a) *$H^2(\pi_1 M; Z_2) = 0$ and $H_*(\pi_1 M; Z_{(2)}) = 0$ for $* \geq n-3$ or*

(b) *$Sq^2: H^2(M; Z_2) \to H^4(M; Z_2)$ is injective, or*

(c) *$\pi_1 M$ is cyclic.*

Let $\nu(h)$ denote the normal invariant of h. Then h is CP *iff*

(1) *h is a simple homotopy equivalence, and*

(2) *$\nu(h): M \to G/\mathrm{Top}$ lifts to $\sum \Omega(G/\mathrm{Top})$.*

Remarks. 1. Condition (a) holds if π is of odd order or a classical knot group; (b) holds for many manifolds e.g. projective spaces or $S^1 \times L_p^{2k+1}$ for p odd. There are many other choices of technical hypotheses available to replace (a) and (b), but some condition is necessary. That h is CP always implies that (1) and (2) hold, but for $M = \partial$ (regular neighborhood of the two-skeleton of T^7), the converse fails (see [We 1]).

For the next result recall that extensions $1 \to Z_2 \to E \to \pi \to 1$ are naturally in a one to one correspondence with $H^2(\pi; Z_2)$. Also recall that if $h: M' \to M$ is a homotopy equivalence then $k_2(h) \in H^2(M; Z_2)$ is $\nu(h)^*(k_2)$ where k_2 is the (unique) nonzero element of $H^2(G/\mathrm{Top}; Z_2)$.

THEOREM B. *Define*

$$E(h)=\begin{cases} E & \text{if } k_2(h)\in \operatorname{Im} H^2(\pi_1 M; Z_2)\to H^2(M; Z_2) \\ & \text{with preimage corresponding to } E \\ \pi_1 M & \text{otherwise} \end{cases}$$

Then the obstruction to the normal cobordism invariance within the class of h of a simple homotopy equivalence's being CP *lies in a quotient of* $\operatorname{cok}: L^s_{n+1}(E(h))\to L^s_{n+1}(\pi_1 M)$.

COROLLARY C. *Any simple homotopy equivalence* $h: M'\to M^n$, $n\geq 5$ *normally cobordant to the identity is* CP.

Concretely, let $h: L_1\to L_2$ be a homotopy equivalence between classical lens spaces. It is easy to see that $\nu(h)=0$. However, h is not a simple homotopy equivalence, but $h\times 1_{S^1}: L_1\times S^1\to L_2\times S^1$ is simple (by the product formula for torsion). Corollary C suggests that $h\times 1_{S^1}$ is CP. We will see in §4 that this is not true! However $h\times 1_{T^2}: L_1\times T^2\to L_1\times T^2$ is CP.

Interestingly, one can combine these results to show that diffeomorphisms of four-manifolds do not up to pseudoisotopy preserve many codimension-one submanifolds. For instance:

THEOREM 3. *There is a diffeomorphism of* $T^2\times S^2 \# k(S^2\times S^2)$ *which is not pseudoisotopic to one preserving* $\{1\}\times S^1\times S^2$, *for k large enough.*

In higher dimensions one can often show that diffeomorphisms up to pseudoisotopy preserve codimension-one submanifolds under appropriate π_1 conditions by analogy to the "Straightening Lemma" techniques of [We 1]. (See also [Ch].) We will not pursue these ideas further here.

2. Low-dimensional considerations

THEOREM 1. *If* L_1 *and* L_2 *are nondiffeomorphic three-dimensional lens spaces, then* $S^1\times L_1\neq S^1\times L_2(N, h)$ *for any* $N^3\subset S^1\times L_2$ *and* $h: N^3\to N^3$ *homotopic to the identity. In other words,* $h: S^1\times L_1\to S^1\times L_2$ *is not cut-pastable.*

In this section we prove this under the additional hypothesis that N is in the Poincaré category, i.e. that any $C^3\subset N$, a contractible compact three manifold, must be the three-disk. More precisely we prove:

PROPOSITION. *If* $M^4 \to S^1 \times L_2$ *is* CP *in the Poincaré category, then* $S^1 \times \tilde{M}^4 \approx S^1 \times R \times L_2$ *where* $\tilde{M}^4$ *is the cover of* M *corresponding to* $Z_p \subset \pi_1 M$.

Proof of Theorem 1 *from the Proposition.* If $S^1 \times L_1 \to S^1 \times L_2$ is CP, then by the proposition $S^1 \times R \times L_1 \approx S^1 \times R \times L_2$ so $S^1 \times L_1$ is h-cobordant to $S^1 \times L_2$. The classical proof that L_1 and L_2 are not h-cobordant was based on a multisignature calculation [AB] and therefore shows that $T^k \times L_1$ is not h-cobordant to $T^k \times L_2$ for any k. □

Proof of the Proposition. For clarity, we first give a proof when the codimension-one submanifold N of $S^1 \times L_2$ is assumed prime, and then describe the changes necessary to reduce the general case to one that can be handled by the same technique.

There are two cases to consider; N sufficiently large or otherwise. Notice that N is orientable. For prime sufficiently large 3-manifolds diffeomorphisms homotopic to the identity are isotopic to it (see [Wd 1] for the irreducible case and [La] for $S^1 \times S^2$), and therefore does not change the diffeomorphism type by cutting and pasting. If N is not sufficiently large then we can homotop N off $L_2 \times \{1\}$. If it were isotopic to an embedding not intersecting $L_2 \times \{1\}$, then all the cutting and pasting would take place in a regular neighborhood of $L_2 \times \{-1\}$ and the resulting manifold would therefore be (by 5-dimensional surgery) s-cobordant to $L_2 \times S^1$ and the conclusion would follow. Unfortunately, if it is not clear how to perform such an isotopy (or even if one exists) so instead look at the cover $L_2 \times R$ of $L_2 \times S^1$ and the induced cover of N, i.e. $\bigcup_{i=-\infty}^{\infty} T^i \tilde{N}$ where T generates the covering translates. The cover $\tilde{M}^4$ is the result of cutting and pasting a $L_2 \times R$ along $\bigcup T^i N$.

Claim. If K is obtained by CP of $L \times R$ along a collection $\mathscr{C}$ of compact submanifolds such that $\mathscr{C}_1 = \{C \in \mathscr{C} \mid C \cap L \times [-1, 1] \neq \emptyset\}$ is finite, then $S^1 \times K \approx S^1 \times L \times R$.

Proof of claim. First CP $S^1 \times K$ along $S^1 \times \bar{\mathscr{C}}_1$. The inclusion of $S^1 \times L \times 0$ in the resulting manifold is a homotopy equivalence, and each end has π_1^∞ mapping isomorphically to the fundamental group of the ambient manifold, so by Siebenmann's criterion [Si] the diffeomorphism type remains $S^1 \times L \times R$. Now embed a copy of $S^1 \times L$ disjoint from the elements of $\mathscr{C}_1$ and the same argument applies proving the claim. □

Now for the general case, write $N = N_1 \# N_2$ where N_1 is the connected sum of the sufficiently large connect summands of N and N_2 the remaining summands.

The argument in [La V S.4] shows that the glueing map h is isotopic to a diffeomorphism, still called h, which is the identity on N_1. (Here we are using the assumption that N lie in the Poincaré category.)

Note that $h\,|_{\partial N_2=S^2}$ is the identity so we choose a base point $*$ in ∂N_2. Consider $N_2 \# -N_2$ with the autodiffeomorphism $h \# 1$. This diffeomorphism induces the identity on $\pi_1(N_2 \# -N_2, *)$ and homology of all covers. Regarding $N_2 \# -N_2$ as being the boundary of a regular neighborhood of $N_2 \subset S^1 \times L_2$, it is easy to see $S^1 \times L_2(N, h) \approx S^1 \times L_2(N_2 \# -N_2, h \# 1)$. Now we argue as before to get the desired conclusion, since it is easy to justify the claim in the more general context where the pasting maps induce the identity on π_1 and homology of all covers.

Remark. In the next two sections we develop a proof of Theorem 1 without any restriction on the submanifold. The above proof has the virtue of being more geometric and not throwing away 2-torsion. For instance the above proof implies that if M^4 is obtained by CP in the Poincaré category from $S^1 \times S^3$, then M^4 is s-cobordant to $S^1 \times S^3$. On the other hand, the results of the next two sections yield no information on this.

3. Algebraic preliminaries

Let $Z = \langle x \mid \rangle$ and $p \in \pi$. Define $h_p : Z \to \pi$ by $h_p(x) = p$. $L_1(Z) = Z$ with canonical generator t. Let $I : \pi \to L_1(\pi)$ denote the function given by $I(p) = h_{p*}(t)$ where $h_{p*} : L_1(Z) \to L_1(\pi)$ is the induced map on L-theory.

PROPOSITION 1. *I is a homomorphism, and any other natural homomorphism $\pi \to L_1(\pi)$ is a multiple of I.*

Proof. Notice that I is defined to be natural. Therefore, by considering $h_{g,g'} : Z * Z \to \pi$ defined as above, it suffices to show $I(xy) = I(x) + I(y)$ for $Z * Z = \langle x, y \mid \ \rangle$. Cappell [C1] has shown that $L_1(Z * Z) \xrightarrow{p_{1*} \oplus p_{2*}} L_1(Z) \oplus L_1(Z) = Z \oplus Z$ is an isomorphism. Now

$$\begin{aligned} p_{1*} I(xy) &= I(p_1(xy)) = I(p_1(x)) = I(p_1(x)) + I(p_1(y)) \\ &= p_{1*}(I(x) + I(y)) \end{aligned}$$

Similarly for p_{2*}.

The last statement is obvious. □

Remarks. 1. An elementary proof of this proposition based just on first principles can be given. (I would like to thank Andrew Ranicki for pointing this out to me.) However, the above is shorter and more natural.

2. The above proof also yields a homomorphism $K:\pi\to L_3(\pi)$ but this will not concern us here.

The significance of I is two-fold. First, the invariant of homotopy equivalences that is relevant to us lies in $G(\pi)=\operatorname{cok} I\otimes Q$. Second, since L-groups are abelian, I actually factors through group homology and lifts the first of a sequence of homomorphisms $I_i: H_i(\pi; Q)\to L_i(\pi)\otimes Q$ defined first by Wall [Wa 1].

These homomorphisms are crucial for calculation of surgery obstructions for problems on closed manifolds. They are also intimately related to the Novikov higher signature conjecture (cf. [Wa 2]). For our purposes we shall regard the Novikov conjecture as the statement that

$$\bigoplus_{i\equiv n(4)} I_i: \bigoplus_{i\equiv n(4)} H_i(\pi; Q)\to L_n(\pi)\otimes Q$$

is injective. The Novikov conjecture has been verified in many cases, cf. [C2], [FH 1], [Lu]. In particular, $G(\pi)$ can be arbitrarily large. (For the group in [We 2], $G(\pi)$ has infinite rank.) Actually, there is no known example of a torsion free group where the homomorphism $\oplus I_i$ is not an isomorphism. We call this the strong Novikov conjecture. In fact [C2] and [FH 2] show that for the Waldhausen class [Wd 2] and Bieberbach groups respectively, this conjecture holds. Since fundamental groups of irreducible sufficiently large three manifolds lie in the Waldhausen class [C2] calculates their rational surgery groups.

DEFINITION. A group π is *very large* if for any infinite finitely generated subgroup H of π, there is a homomorphism $h_H: H\to Z$ of H onto the infinite cyclic group Z.

EXAMPLES. 1. Finite groups are very large.

2. Free groups, abelian groups and surface groups are all very large.

3. Given an exact sequence

$$1\to\pi\to\pi'\to G$$

with G a torsion free very large group, then π is very large iff π' is. In particular, Poly-Z groups are very large.

4. Fundamental groups of irreducible sufficiently large three manifolds need not be very large. (For instance, the first Betti number can vanish.)

THEOREM 2. *If π is a very large group and $K\approx\pi_1M^3$, $h: K\to\pi$ an arbitrary homomorphism, then $h_*: G(K)\to G(\pi)$ vanishes.* ($G(\pi)=\operatorname{cok} I\otimes Q$.)

Proof. By [C1] $G(A * B) \cong G(A) \oplus G(B)$ so without loss of generality M can be assumed irreducible. If Im h is finite then h_* factors through G (finite group) which vanishes as odd Wall groups of finite groups are finite [Wa 3]. Thus, assume Im H is infinite, so that the composite $K \xrightarrow{h} \operatorname{Im} h \xrightarrow{h_{\operatorname{Im} h}} Z$ is onto which implies that K is in the Waldhausen class. If $K \neq Z$, M is aspherical; in particular in all cases $H_{4k+1}(K; Q) = 0$ for $k > 0$, and $I: H_1(K; Q) \to L_1(K) \otimes Q$ is an isomorphism ([C2]) so that $G(K) = 0$. □

Remarks. 1. If the strong Novikov conjecture were true for all three-manifold groups, then the conclusion of Theorem 2 would hold for all groups.

2. For torsion free very large groups one can modify the above to calculate $\operatorname{Im} \bigoplus_{K = \pi_1 M^3} L_i(K) \otimes Q \to L_i(\pi) \otimes Q$.

4. An invariant of 4-dimensional homotopy equivalences

Let $h: M' \to M^4$ be a homotopy equivalence. In this section we define an h-cobordism invariant $\eta(h) \in G(\pi_1 M)$ which, for $\pi_1 M$ very large, is an obstruction to h being CP. Stably, the entire image of $L_1^s(\pi_1 M)$, in $G(\pi)$ is realizable as the η-invariant of some simple homotopy equivalence. It will be clear that for $h \# 1_{k(S^2 \times S^2)}: M' \# k(S^2 \times S^1) \to M \# k(S^2 \times S^2)$; $\eta(h \# 1_{k(S^2 \times S^2)}) = \eta(h)$ so that these are stable obstructions to CP, violating the usual philosophy that the stable geometric topology of dimension four is no different than the high dimensional theory. The proof of Theorem 1 will merely be the calculation that $\eta(1_{S^1} \times h: S^1 \times L_1 \to S^1 \times L_2) \neq 0$.

Recall the surgery exact sequence of [Wa 2]

$$\cdots \to \left[\sum M : G/\mathrm{Top}\right] \to L_{n+1}(\pi_1 M) \xrightarrow{\phi} h\,\mathrm{Top}\,(M^n) \xrightarrow{n} [M : G/\mathrm{Top}] \xrightarrow{\theta} L_n(\pi_1 M)$$

which is exact for $n \geq 5$. By [KS essay V Appendix C], there is an H-space structure on G/Top and an abelian group structure on $h\,\mathrm{Top}\,(M)$ such that the surgery sequence becomes an exact sequence of groups and homomorphism (rather than merely pointed sets). For four manifolds $\ker \theta: [M^4 : G/\mathrm{Top}] \to L_4(\pi_1 M)$ is easily seen to be a two-group. If $h: M' \to M^4$ is a homotopy equivalence it determines an element $[h] \in h\,\mathrm{Top}\,(M)$. Taking the product with S^1 we get an element $[1_{S^1} \times h] \in h\,\mathrm{Top}\,(S^1 \times M)$. Now $2^k n([h]) = 0$ so $0 = 2^k n([h \times 1_{S^1}]) = n(2^k [h \times 1_{S^1}])$. Thus $2^k [h \times 1_{S^1}] = \phi(\alpha)$ for some $\alpha \in L_6(Z \times \pi_i M)$. Let $S: L_6(Z \times \pi) \to L_5(\pi)$ denote the Shaneson homomorphism [Sh] and

$p: L_5(\pi) = L_1(\pi) \to G(\pi)$ the quotient map. We define

$$\eta(h) = \frac{1}{2^k} pS(\alpha) \in G(\pi).$$

PROPOSITION 2. *$\eta(h)$ is well defined and vanishes for* CP *homotopy equivalences, if $\pi_1 M$ is very large.*

Proof. First, the power of 2 used is clearly irrelevant since ϕ is a homomorphism. Suppose $\phi(\alpha) = \phi(\alpha')$; then $(\alpha - \alpha') \in \operatorname{Im}[\sum (S^1 \times M) : G/\mathrm{Top}]$. Thus it suffices to show that the composite

$$[\textstyle\sum (S^1 \times M) : G/\mathrm{Top}] \to L_6(Z \times \pi_1 M) \to L_5(\pi_1 M) \to G(\pi_1 M)$$

vanishes. Notice that the map

$$\begin{tikzcd}
{[\sum (S^1 \times M) : G/\mathrm{Top}]} \arrow[r] & {\left[\sum^2 M : G/\mathrm{Top}\right] \oplus [\sum M : G/\mathrm{Top}]} \arrow[d] \\
 & L_6(\pi_1 M) \oplus L_5(\pi_1 M)
\end{tikzcd}$$

preserves the factorization so that it suffices to show that $\operatorname{Im}[\sum M : G/\mathrm{Top}] \subset L_5(\pi_1 M)$ actually lies in Im I, at least rationally. Analogous to [Wa 2 §13 B] there is a factorization

$$\begin{tikzcd}
 & {[\sum M : G/\mathrm{Top}] \otimes Z[\frac{1}{2}]} \arrow[rr] \arrow[d] & & L_5(\pi_1 M) \otimes Z[\frac{1}{2}] \\
H_1(\pi; Z[\frac{1}{2}]) \arrow[r, "\approx"] & \tilde{\Omega}_4(K(\pi, 1) \times \Omega(G/\mathrm{Top})) \otimes Z[\frac{1}{2}] \arrow[urr, "\hat{\theta}"'] & &
\end{tikzcd}$$

The map $H_1(\pi; Z[\frac{1}{2}]) \to L_5(\pi_1 M) \otimes Z[\frac{1}{2}]$ is a homomorphism and hence multiple of I (Proposition 1) so even away from 2 we are done. This proves well definedness. □

Remark. All of the proofs above can be refined from Q to $Z[\frac{1}{2}]$ so the invariant η can be made to keep track of odd torsion. If one is willing (as we have been) to throw away the odd torsion then [Wa 1] or [TW] can be used to prove that $\operatorname{Im}[\sum M : G/\mathrm{Top}] \otimes Q \subset \operatorname{Im} I \otimes Q$.

Now, by definition η is natural with respect to inclusions. Therefore if $h: M' \to M^4$ is CP by (N, g, H) $\eta(h) = i_* \eta(H : N \times I \text{ rel } \partial \to N \times I \text{ rel } \partial)$ which vanishes for $\pi_1 M$ very large by Theorem 2. □

Proof of Theorem 1. First we calculate $G(Z\times Z_n)$.

$$L_1^s(Z\times Z_n)\approx L_0^h(Z_n), \qquad \operatorname{Im} I=\operatorname{Im} L_0(0)$$

so that $G(Z\times Z_n)\to \tilde{L}_0^h(Z_n)\otimes Q$ is an isomorphism, so that a codimension one multisignature (modulo multiples of the regular representation) detects $G(Z\times Z_n)$. Let $h:S^1\times L_1\to S^1\times L_2$, $\eta(h)$ is the (reduced) multisignature of a normal cobordism between L_1 and L_2 which is nonzero when $L_1 \not\approx L_2$, by [AB]. □

5. Concluding remarks

1. The following conjecture seems plausible.

CONJECTURE. *If* $h:M'\to M^4$ *is* CP, *h is homotopic to a homeomorphism.*

The results of §§3, 4 verify this modulo two-torsion and up to s-cobordism if $\pi_1 M$ is very large. If the strong Novikov conjecture were true, then the same would hold independently of $\pi_1 M$. Of course the whole conjecture would follow from "homotopy implies isotopy for homeomorphisms of three-manifolds".

An interesting question is whether the fake RP^4 of [CS 2] is CP. Neither of the techniques of this paper yield any interesting information.

2. It may perhaps be interesting to compare the approaches of §2 and that of §§3, 4. Philosophically, both are a reduction of the general situation to one where only irreducible sufficiently large 3-manifolds occur. Geometrically this is accomplished by [La] and algebraically by [C1]. (Interestingly, the heart of the geometry of §1 deals with showing that the nonsufficiently large summands contribute nothing. Algebraically, they trivially cause no difficulty.) The sufficiently large cases are dealt with either by [Wd 1] [La] or [C2]. The analogy between [Wd 1] and [C2] is well known (cf. [Wd 2]).

3. The application of the Novikov conjecture given here is a "global" one, in that to get a result about even a single homotopy equivalence requires the verification of the conjecture for infinitely many groups. For another geometric application of the strong Novikov conjecture, see [FH 3], but that application only requires knowledge of the conjecture for $\pi_1 M$, and is a "local" application.

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The Conway potential function for links

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Conway introduced the potential function of a link in 1970, [1]. This potential function, closely allied to the Alexander link polynomial, nevertheless has important properties which the Alexander polynomial does not have. However, despite this fact, no proof has appeared either for the properties, or even for the existence of Conway's potential function. That, then, is the purpose of this paper. Kauffman [3] showed how to define what may be called the reduced potential function of a link in terms of a Seifert matrix. This reduced potential function is an L-polynomial in one variable. However, the potential function is essentially a function of several variables, and I can see no way of generalising Kauffman's method to obtain the full potential function. Quite a different approach is therefore indicated.

The potential function is determined except for sign by the Alexander polynomial, since for a link with n components,

$$\begin{aligned} &(t_1 - t_1^{-1}) \cdot \nabla(t_1) = \Delta(t_1^2) \cdot t_1^{\mu_1} \quad \text{if} \quad n = 1 \\ &\nabla(t_1, \ldots, t_n) = \Delta(t_1^2, \ldots, t_n^2) \cdot t_1^{\mu_1} \cdots t_n^{\mu_n} \quad \text{if} \quad n > 1 \end{aligned} \tag{1.1}$$

where ∇ is the potential function, Δ is the Alexander polynomial properly chosen with correct sign and μ_i are integers which are uniquely determined by the requirement that ∇ should satisfy the symmetry condition (5.5). But the Alexander polynomial is not usually defined with a well determined sign. It is shown here, however, how by defining a simple correspondence between the rows and columns of an Alexander matrix obtained from a Wirtinger presentation, the Alexander polynomial can be defined with a well determined sign. Then, one may define a symmetric potential function using (1.1).

However, in order to derive properties of the potential function, and in particular the replacement relations which are of central importance, it is necessary to be able to determine in advance the values of the μ_i in (1.1) directly from the link projection. This is perhaps the most delicate step in the definition of the potential function. The values of the μ_i turn out to depend on the *curvature* of the projection of the i-th component of the link.

The method of proof of invariance of the potential function is somewhat old fashioned, by means of the three PL moves of Reidemeister [5]. This is perhaps justified by the fact that the potential function is not an algebraic invariant, and a proof of its invariance must contain some geometric element. It is often the case that a theorem is easily proven once one makes the correct definition. This is the case here, and for that reason, tedious detail is often omitted.

The contents of this paper overlap in part with some of the results of a recent monograph of Kauffman, [4], in which the Conway polynomial is treated from a different point of view. Kauffman also notes the connection with what is in fact the Whitney degree of the planar knot projection, called here the curvature, and by Kauffman, curliness.

Finally, the notion of defining a correspondence between rows and columns in an Alexander matrix was suggested to me by J. H. Conway in a brief conversation in Galway in 1973, and this paper has developed as an expansion of that idea. It was written down while I was a visitor at the J. W. Goethe University in Frankfurt am Main in the summer of 1982, and I should like to express my appreciation for the hospitality that was extended to me there.

§2. Definition of the potential function

We consider an oriented link, the components of which are numbered 1 to n ($n \geq 1$) in some way. It will be described how a potential function is assigned to the link. If the link has more than one component, the potential function will be an integral L-polynomial in the variables $t_1, \ldots, t_n$, that is an element of the polynomial ring $Z[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. If the link has one component, then the potential function is of the form $f(t_1)/(t_1 - t_1^{-1})$, where $f(t_1) \in Z[t_1, t_1^{-1}]$.

We start with a regular projection of the link. If some connected component of the link projection has no crossing points, define $\nabla(t_1) = (t_1 - t_1^{-1})^{-1}$ if L has one component (L is a trivial knot) and $\nabla(t_1, \ldots, t_n) = 0$ if L has more than one component (L is a split link). From now on we exclude this possibility. At a crossing point of the projection, two arcs meet, one passing under and one over. By cutting the undercrossing arc at the point where it crosses under, the link is cut into m arcs (where m is equal to the number of crossing points) called *generating arcs*. Thus at each crossing point, P, of the projection three generating arcs meet, one arc passing over at P, one arc terminating at P and one exiting from P (with regard to the link orientation). These last two arcs together make up the undercrossing arc at P. Now, number the crossing points $P_1, \ldots, P_m$ and the generating arcs $u_1, \ldots, u_m$ in such a way that u_i is the generating arc which exits from P_i. If generating arc u_i belongs to the j-th link component, then give u_i the *label* t_j.

To a path, a, in the plane of projection, which does not start or end at a point on the link projection, and which avoids the crossing points, we can associate an element, $\mathfrak{a}$, of the free group, $F(u_1, \ldots, u_m)$ generated by the u_i as follows. One moves along the path writing down the sequence of generating arcs crossed, more precisely writing u_i if u_i is crossed from right to left and u_i^{-1} if it is crossed from left to right. Using this, we read off a Wirtinger relator, R_i, at each crossing point, P_i, of the projection, as follows. R_i is the word in the u_i corresponding to a small loop which starts at a point to the right of both over- and undercrossing arcs at P_i and proceeds anticlockwise around P_i. Thus, for a positive crossing, P_i (the undercrossing arc crosses under the overcrossing arc from right to left), relator R_i is $u_k u_i u_k^{-1} u_j^{-1}$ and for a negative crossing (the undercrossing crosses under from left to right), R_i is $u_i u_k u_j^{-1} u_k^{-1}$, where in each case u_k is the overcrossing arc. Now let θ be the map from $ZF(u_1, \ldots, u_m)$ to $Z[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ which takes each u_i to its label, and define the $m \times m$ *Jacobian matrix*, M, by $M_{ij} = (\partial R_i/\partial u_j)^\theta$.

From a basic formula of the free differential calculus, we have

$$\sum_{j=1}^{m} (u_j^\theta - 1) \cdot (j\text{-th column of } M) = 0 \tag{2.1}$$

The link projection divides the plane of projection into regions. Let w_i be a path from a base point b in the unbounded region to a point close to P_i and to the right of both under- and overcrossing arcs, $\mathfrak{w}_i$ the corresponding word in $F(u_1, \ldots, u_m)$. If the w_i are chosen so that they do not intersect except at b, then for some permutation, σ, of degree m representing the anticlockwise order of the w_i about b we have

$$\mathfrak{w}_{\sigma(1)} R_{\sigma(1)} \mathfrak{w}_{\sigma(1)}^{-1} \cdot \mathfrak{w}_{\sigma(2)} R_{\sigma(2)} \mathfrak{w}_{\sigma(2)}^{-1} \cdots \mathfrak{w}_{\sigma(m)} R_{\sigma(m)} \mathfrak{w}_{\sigma(m)}^{-1} = \mathrm{id} \quad \text{in} \quad F(u_1, \ldots, u_m)$$

from which it follows that

$$\sum_{i=1}^{m} \mathfrak{w}_i^\theta \cdot (i\text{-th row of } M) = 0 \tag{2.2}$$

Now, if $M^{(ij)}$ denotes the matrix obtained from M by deleting the i-th row and j-th column, then from (2.1) and (2.2) we have that

$$(-1)^{i+j} \det (M^{(ij)})/\mathfrak{w}_i^\theta (u_j^\theta - 1) = (-1)^{k+l} \det (M^{(kl)})/\mathfrak{w}_k^\theta (u_l^\theta - 1)$$

So, defining

$$D(t_1, \ldots, t_n) = (-1)^{i+j} \det (M^{(ij)})/\mathfrak{w}_i^\theta (u_j^\theta - 1) \tag{2.3}$$

for any i and j, we see that D is independent of the choice of i and j. (If M is a 1×1 matrix, define $\det(M^{(11)})=1$.) It is also clear that D does not depend on the original numbering of the crossing points and generating arcs, since a renumbering corresponds to a simultaneous identical permutation of the rows and columns of M. Thus, D depends only on the link projection and numbering of the components of the link. By its very definition, if $n>1$, $D(t_1,\ldots,t_n)$ is the Alexander polynomial of the link, and if $n=1$, then $D(t_1)=(t_1-1)^{-1}\cdot\Delta(t_1)$. It will turn out that for different projections of the same link, the value of D differs only by a factor $t_1^{\beta_1}\cdots t_n^{\beta_n}$. Hence the value of D is determined as to sign, and so represents a signed form of the Alexander polynomial.

We now need to determine the factor $t_1^{\mu_1}\cdots t_n^{\mu_n}$ in (1.1) required to make the potential function symmetric. For each component of the link, trace out the Seifert circuits in the projection of that component, and let its *curvature* equal (number of anticlockwise circuits) − (number of clockwise circuits). Let κ_i be the curvature of the i-th component. Further, for each i, let ν_i equal the number of crossing points in the link projection for which the overcrossing arc has label t_i (belongs to the i-th component of the link). Now define

$$\nabla(t_1,\ldots,t_n)=D(t_1^2,\ldots,t_n^2)\cdot t_1^{\mu_1}\cdots t_n^{\mu_n}\quad\text{where}\quad \mu_i=\kappa_i-\nu_i. \tag{2.4}$$

This, then, is Conway's potential function.

§3. The potential function is a link invariant

We have shown in the previous section that the potential function defined there is uniquely determined by the link projection. We now show that it remains invariant under transition from one projection to another via the three basic Reidemeister moves, and hence it is a link invariant.

In the definition of the potential function the numbering of the generating arcs is immaterial and may be suited to our convenience. Similarly, since we may choose to delete any row and column from the Jacobian matrix we will assume that the row and column deleted are not among those specifically considered. This is always possible as long as the projection has at least one more generating arc besides those explicitly shown. Once we have shown that the introduction or removal of trivial loops (first basic move) does not change the potential function, this desirable situation may be achieved by the introduction of redundant trivial loops. For the same reason we may always assume that the generating arcs shown in diagrams are all different. The only exceptions to these rules, therefore, are in the verification of invariance for the removal of trivial loops from a component which has at most one other crossing point (which must also belong to a trivial loop). This must be treated as a rather trivial special case. Details are omitted.

In that part of the link projection which is altered by the Reidemeister move there are at most three link components involved. For convenience we give them labels r, s, t instead of t_{i_1}, t_{i_2}, t_{i_3}, write κ_r, κ_s, κ_t instead of κ_{i_j} and ν_r, ν_s, ν_t instead of ν_{i_j}. For each of the three Reidemeister moves one must consider various cases depending on the orientation of the link components, and in the case of removal of trivial loops, whether the loop is clockwise or anticlockwise. We consider explicitly only one representative case for each type of move. Quantities with primes (′) refer to the diagram on the left, unprimed quantities the diagram on the right in each case.

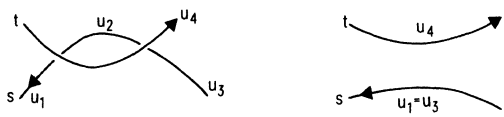
First basic move:

Here $R_1' = u_2 u_1 u_2^{-1} u_2^{-1}$ and so (with c_i standing for column i),

$$\det(M'^{(ij)}) = \left\| \begin{array}{cc|c} t & -t & 0 \\ \hline c_1 & c_2 & * \end{array} \right\| = t \cdot \|c_1 + c_2 \mid *\| = t \cdot \det(M^{(ij)}).$$

Since the factor $(-1)^{i+j}/\mathfrak{w}_i^\theta(u_j^\theta - 1)$ is unchanged we have $D' = t \cdot D$. However, $\nu_t' = \nu_t + 1$, $\kappa_t' = \kappa_t - 1$, so $\nabla' = \nabla$ from (2.4).

Second basic move:



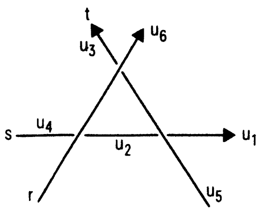
Now, $R_1' = u_1 u_4 u_2^{-1} u_4^{-1}$, $R_2' = u_4 u_2 u_4^{-1} u_3^{-1}$. Thus,

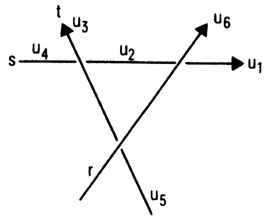
$$\det(M'^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & -t & 0 & s-1 & 0 \\ 0 & t & -1 & 1-s & 0 \\ \hline c_1 & 0 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & t & -1 & 1-s & 0 \\ \hline c_1 & 0 & c_3 & c_4 & * \end{array} \right\|$$

$$= \left\| \begin{array}{ccc|c} t & -1 & 1-s & 0 \\ \hline 0 & c_1 + c_3 & c_4 & * \end{array} \right\| = t \cdot \|c_1 + c_3 \quad c_4 \mid *\| = t \cdot \det(M^{(ij)})$$

Hence as before, $D' = t \cdot D$. However, $\nu'_t = \nu_t + 2$ and other values are unchanged. Thus, $\nabla' = \nabla$.

Third basic move:





Here

$$R'_1 = u_1u_5u_2^{-1}u_5^{-1}, \qquad R'_2 = u_2u_6u_4^{-1}u_6^{-1}, \qquad R'_3 = u_6u_3u_6^{-1}u_5^{-1},$$
$$R_1 = u_1u_6u_2^{-1}u_6^{-1}, \qquad R_2 = u_2u_3u_4^{-1}u_3^{-1}, \qquad R_3 = u_6u_3u_6^{-1}u_5^{-1}.$$

Thus,

$$\det(M'^{(ij)}) = \left\| \begin{array}{cccccc|c} 1 & -t & 0 & 0 & s-1 & 0 & 0 \\ 0 & 1 & 0 & -r & 0 & s-1 & 0 \\ 0 & 0 & r & 0 & -1 & 1-t & 0 \\ \hline \mathbf{c}_1 & 0 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 & * \end{array} \right\|$$
$$= \left\| \begin{array}{ccccc|c} 1 & 0 & -rt & s-1 & st-t & 0 \\ 0 & r & 0 & -1 & 1-t & 0 \\ \hline \mathbf{c}_1 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 & * \end{array} \right\|$$
$$= \left\| \begin{array}{ccccc|c} 1 & r(s-1) & -rt & 0 & s-1 & 0 \\ 0 & r & 0 & -1 & 1-t & 0 \\ \hline \mathbf{c}_1 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 & * \end{array} \right\| = A.$$

The first step is by adding t times the second row to the first then eliminating the second row and column. The second step is by adding $s-1$ times the (now) second row to the first. Similarly,

$$\det(M^{(ij)}) = \left\| \begin{array}{cccccc|c} 1 & -r & 0 & 0 & 0 & s-1 & 0 \\ 0 & 1 & s-1 & -t & 0 & 0 & 0 \\ 0 & 0 & r & 0 & -1 & 1-t & 0 \\ \hline \mathbf{c}_1 & 0 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 & \mathbf{c}_6 & * \end{array} \right\|$$

which is transformed to A by adding r times the second row to the first and eliminating the second row and column. Thus, $D' = D$, and since the values of the ν's and the κ's are unchanged, $\nabla' = \nabla$.

From this we conclude that the potential function is a link invariant.

§4. The reduced potential function

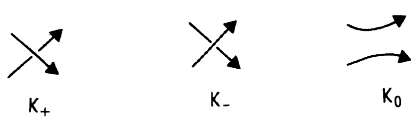
We may define a reduced potential function, $\bar{\nabla}$, for a link by

$$\bar{\nabla}(t) = (t - t^{-1}) \cdot \nabla(t, \ldots, t).$$

This is an integral L-polynomial. It has two basic properties.

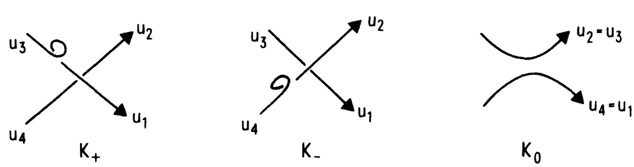
(4.1) *For the trivial knot,* $\bar{\nabla}(t) = 1$.

(4.2) (*Replacement relation.*) *For three links,* K_+, K_- *and* K_0 *which differ only in one place as shown,*



the potential functions satisfy $\bar{\nabla}_+(t) = \bar{\nabla}_-(t) + (t - t^{-1})\, \bar{\nabla}_0(t)$.

Proof of (4.2). For convenience we introduce an extra trivial loop in K_+ and K_-, which does not alter the potential function.



Let $\theta: ZF(u_1, \ldots, u_n) \to Z[t, t^{-1}]$ take all u_i to t, and denote $(\partial R_i/\partial u_j)^\theta$ by $\bar{M}$. If $\bar{D}(t) = (-1)^{i+j} \det(\bar{M}^{(ij)})/\mathfrak{w}_i^\theta$, then $\bar{\nabla}(t) = \bar{D}(t^2) \cdot t^{\kappa - \nu - 1}$ where now ν is the number of crossing points and κ is the sum of curvatures of all components. Then,

$$\det(\bar{M}_+^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 1-t & t & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 1 & t & -t & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

and

$$\det(\bar{M}^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & t-1 & -t & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\| = \left\| \begin{array}{cccc|c} 1 & t & -t & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

Hence,

$$\det(\bar{M}_+^{(ij)}) - \det(\bar{M}_-^{(ij)}) = \left\| \begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & t & -t & -1 & 0 \\ \hline c_1 & c_2 & c_3 & c_4 & * \end{array} \right\|$$

$$= \left\| \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & t & 0 & 0 & 0 \\ \hline c_1 & c_2 & c_3+c_2 & c_1+c_4 & * \end{array} \right\| = (t-1)\, \| c_2+c_3 \quad c_1+c_4 \mid * \|$$

$$= (t-1) \cdot \det(\bar{M}_0^{(ij)}).$$

This shows that $\bar{D}_+(t) - \bar{D}_-(t) = (t-1) \cdot \bar{D}_0(t)$. However, $\nu_+ = \nu_- = 2 + \nu_0$, and $\kappa_+ = \kappa_- = \kappa_0 + 1$, and so (4.2) follows.

We now show that the properties (4.1) and (4.2) characterise the reduced potential function. (This was also proven by Kauffman [3].) The following part of the proof deserves to be singled out.

(4.3) (*Induction principle*). *Let* $\mathfrak{C}$ *be a class of links satisfying* (i) *the trivial knot is in* $\mathfrak{C}$, (ii) *all split links are in* $\mathfrak{C}$, (iii) *if* K_0 *is in* $\mathfrak{C}$ *and one of* K_+ *and* K_- *is in* $\mathfrak{C}$, *then both* K_+ *and* K_- *are in* $\mathfrak{C}$. *Then* $\mathfrak{C}$ *contains all links.*

Proof. Consider a link, L, with m crossings. By interchanging overcrossing and undercrossing for some number, $h(L)$ of crossing points, L may be transformed either to a split link or a trivial knot. Consider one of these crossings. Suppose it is positive and denote L by L_+. Then L_0 has $m-1$ crossings, whereas L_- has m crossings but $h(L_-) < h(L_+)$. By induction on m and h one deduces that L_+ is in $\mathfrak{C}$.

Now we prove

(4.4) (*Uniqueness of the reduced potential function.*) $\bar{\nabla}(t)$ *is the unique link invariant, an* L*-polynomial defined for all links, which satisfies* (4.1) *and* (4.2).

We assume that $\bar{\nabla}'(t)$ also satisfies (4.1) and (4.2) and let $\mathfrak{C}$ be the class of links for which $\bar{\nabla}'_L(t) = \bar{\nabla}_L(t)$. It follows easily from (4.1) and (4.2) that $\bar{\nabla}'_L(t) = \bar{\nabla}_L(t) = 0$ for split links. (See for instance Kauffman [3].) By (4.3), then, $\mathfrak{C}$ contains all links.

From (4.4) we deduce the following important corollary.

(4.5) (*Symmetry of the reduced potential function.*) *For a link of n components,* $\bar{\nabla}(t) = (-1)^{n-1}\,\bar{\nabla}(t^{-1})$.

Proof. Let $\bar{\nabla}'(t) = (-1)^{n-1}\,\bar{\nabla}(t^{-1})$. It is easily verified that $\bar{\nabla}'(t)$ satisfies (4.1) and (4.2) since $\bar{\nabla}(t)$ does. The vital point is that K_+ and K_- have the same number of components, whereas the number of components of K_0 differs by one.

Similar in style is the following proposition.

(4.6) *If* $\sim L$ *is the mirror image of* L, *a link with* n *components, then* $\bar{\nabla}_L(t) = (-1)^{n-1}\bar{\nabla}_{\sim L}(t)$.

To prove this, observe that $(-1)^{n-1}\,\bar{\nabla}_{\sim L}(t)$ satisfies (4.1) and (4.2).

Let L be a link with n components and let G_n be the complete graph on n vertices. We give the edge joining the vertices i and j of G_n a *weight* equal to λ_{ij}, the linking number of the i-th and j-th components of L. We say that G_n is *weighted by* L. Define the weight of a subgraph of G_n to be the product of the weights of all its edges. We can now determine more exactly the form of $\bar{\nabla}(t)$.

(4.7) *For a link* L *of* n *components,* $\bar{\nabla}(t) = (t - t^{-1})^{n-1}H(t)$ *where* $H(t)$ *is an integral* L*-polynomial in even powers of* t *and* t^{-1}. *For* $n = 1$, $H(1) = 1$. *For* $n > 1$, $H(1)$ *is equal to the sum of the weights of all spanning trees in* G_n *weighted by* L.

Let $\mathfrak{C}$ be the class of links for which the proposition is true. It is trivially true for the trivial knot and for split links. We assume (4.7) holds for K_0 and K_+ (or K_-). If the two arcs crossing in K_+ are from the same link component, then that component splits into two components in K, and it follows that $H_+(t) = H_-(t) + (t - t^{-1})^2 H_0(t)$. If however the two arcs are from different components, then these two components are amalgamated to one component in K, and one has $H_+(t) = H_-(t) + H_0(t)$. Thus, all statements but the last are easily proven. The value of $H(1)$ may be deduced by induction continuing this line of argument, however the details are omitted as the result will not be used further.

Of course, $H(t)$ is nothing but a disguised and signed form of the Hosokawa polynomial (see [2]), just as $\nabla(t)$ is a disguised form of the Alexander polynomial. In fact, with this viewpoint, (4.7) contains the main results of [2]. Corresponding to Theorem 2 of Hosokawa, we may give a different description of $H(1)$ as follows: Let L be the matrix given by

$$L_{ij} = -\lambda_{ij} \quad \text{if} \quad i \neq j$$

$$L_{jj} = \sum_{\substack{i=1 \\ i \neq j}}^{n} \lambda_{ij}$$

Let $L^{(kl)}$ be the minor obtained by deleting the k-th row and l-th column of L. Then $H(1)=(-1)^{k+l}\cdot\det(L^{(kl)})$. It is a simple matter to prove this by induction using the recursion relations for H derived above.

It follows from (4.7) that for knots of one component, $\bar{\nabla}(1)=1$, so $\nabla(t)$ is determined uniquely by the Alexander polynomial. For $n\geq 2$, if $H(1)\neq 0$, in particular if all the linking numbers are positive, then the sign of $H(1)$ determines the correct sign for the potential function.

From the uniqueness of the reduced potential function it follows that our $\bar{\nabla}(t)$ is equal to Kauffman's $\Omega(t)$. In particular, $\bar{\nabla}(t)=\det(tV-t^{-1}V^*)$ where V is a Seifert matrix and V^* its transpose. An important property of $\bar{\nabla}(t)$ which is most easily proven using the Seifert matrix is

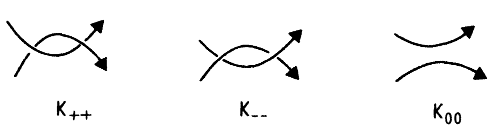
(4.8) (*Signature and nullity of links.*) $\bar{\nabla}_L(i)=0$ *if and only if* $\text{nullity}(L)>1$. *Otherwise,* $\bar{\nabla}_L(i)=R\cdot i^{\sigma}$. *Here* $i^2=-1$, R *is a positive real number and* σ *is the signature of the link.*

Proof. Suppose V is a $k\times k$ matrix. Then $\bar{\nabla}(i)=\det(iV+iV^*)=i^k\cdot\det(V+V^*)$. Now $V+V^*$ is congruent to a diagonal matrix, J, with p ones, q minus-ones and r zeros on the diagonal, $r=\text{nullity}(L)-1$. Further, $\det(V+V^*)=R\cdot\det(J)$ for a positive real R. Now $\det(J)=0$ if and only if $r\neq 0$. If $r=0$, then $\bar{\nabla}(i)=i^k\cdot(-1)^q=i^k\cdot(-1)^{-q}=i^{k-2q}=i^{p-q}$ (since $k=p+q$) $=i^{\sigma}$.

§5. Properties of the potential function

Similar to the replacement relation (4.2) we have for the (unreduced) potential function

(5.1) (*Replacement relation.*) $\nabla_{++}+\nabla_{--}=(t_{i_1}t_{i_2}+t_{i_1}^{-1}t_{i_2}^{-1})\,\nabla_{00}$ *for links containing the tangles*



the components having labels t_{i_1} *and* t_{i_2}.

Similarly,

(5.2) (*Replacement relation.*) $\nabla_{++}+\nabla_{--}=(t_{i_1}t_{i_2}^{-1}+t_{i_1}^{-1}t_{i_2})\,\nabla_{00}$ *in the case where one of the two arcs in* (5.1) *is oppositely oriented.*

The proof of these relations is similar to the proof of (4.2) and is omitted. A further property which is easily proven is

(5.3) *If L is a link with n components, then*

$$\nabla(1, t_2, \ldots, t_n) = (t_2^{\lambda_{12}} \cdots t_n^{\lambda_{1n}} - t_2^{-\lambda_{12}} \cdots t_n^{-\lambda_{1n}}) \cdot \nabla'(t_2, \ldots, t_n)$$

where $\nabla'(t_2, \ldots, t_n)$ is the potential function of the link obtained by eliminating the first component of L and λ_{ij} is the linking number between the i-th and j-th components of the link.

Indeed, if we number the generating arcs of L such that $u_1, \ldots, u_k$ are the consecutive generating arcs of the first component, we obtain, setting $t_1 = 1$ in the matrix M, a matrix of the form $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$. A is a $k \times k$ matrix which gives rise to the first half of the expression on the right of (5.3) and B gives rise to $\nabla'(t_2, \ldots, t_n)$. See Torres [6] for a proof of this result for the Alexander polynomial.

Applying (5.3) $n-1$ times we have the formula

(5.4) $\nabla_L(1, \ldots, 1, t_i, 1, \ldots, 1) = \nabla_i(t_i) \cdot \prod_{k=1, k\neq i}^{n} (t_i^{\lambda_{ik}} - t_i^{-\lambda_{ik}})$ *where $\nabla_i(t_i)$ is the potential function of the i-th component of L.*

We are now able to prove

(5.5) (*Symmetry of the potential function.*) $\nabla(t_1, \ldots, t_n) = (-1)^n \nabla(t_1^{-1}, \ldots, t_n^{-1})$.

Proof. We assume the well known symmetry of the Alexander polynomial [6] which implies that $\nabla(t_1, \ldots, t_n) = \varepsilon t_1^{\gamma_1} \cdots t_n^{\gamma_n} \cdot \nabla(t_1^{-1}, \ldots, t_n^{-1})$. From (5.4) $\nabla(1, \ldots, t_i, \ldots, 1) = \nabla_i(t_i) G_i(t_i)$ where $G_i(t_i) = (-1)^{n-1} G_i(t_i^{-1})$ and $\nabla_i(t_i) = -\nabla_i(t_i^{-1})$ from (4.6). Then $\nabla_i(t_i) G_i(t_i) = \nabla(1, \ldots, t_i, \ldots, 1) = \varepsilon t_i^{\gamma_i} \nabla(1, \ldots, t_i^{-1}, \ldots, 1) = \varepsilon t_i^{\gamma_i} \nabla_i(t_i^{-1}) G_i(t_i^{-1}) = \varepsilon t_i^{\gamma_i} \cdot (-1)^n \nabla_i(t_i) G_i(t_i)$. Now $\nabla_i(t_i) \neq 0$, and $G_i(t_i) \neq 0$ as long as all linking numbers are non-zero. In this case, therefore, $\gamma_i = 0$ and $\varepsilon = (-1)^n$, and (5.5) is proven for the case where all λ_{ij} are non-zero.

Now assume $\lambda_{i_0 j_0} = 0$. From (5.1) we have a formula $\nabla_{++++} + \nabla_{00} = (t_{i_0} t_{j_0} + t_{i_0}^{-1} t_{j_0}^{-1}) \nabla_{++}$. Identifying the link L as K_{00} we see that $\nabla_L = \nabla_{00}$ may be expressed in terms of the potential functions of K_{++++} (for which $\lambda_{i_0 j_0} = 2$) and K_{++} (for which $\lambda_{i_0 j_0} = 1$). If ∇_{++++} and ∇_{++} satisfy (5.5) then so does $\nabla_{00} = \nabla_L$. So, (5.5) follows by induction on the number of λ_{ij} equal to zero.

Next we consider the mirror image of L.

(5.6) *If $\sim L$ is the mirror image of L then* $\nabla_{\sim L}(t_1, \ldots, t_n) = (-1)^{n-1}\, \nabla_L(t_1, \ldots, t_n)$.

As is well known, the Alexander polynomials of L and $\sim L$ are equal, so $\nabla_{\sim L}(t_1, \ldots, t_n) = \varepsilon\, \nabla_L(t_1, \ldots, t_n)$. Using (5.4) we deduce that $\varepsilon = (-1)^{n-1}$ as long as all linking numbers are non-zero, since for a knot of one component, $\nabla_K(t) = \nabla_{\sim K}(t)$ by (4.6). This may be extended to all links using (5.1) just as in the previous proof.

Finally, we consider the effect of changing the orientation of one component of a link.

(5.7) *If L^* is obtained from L by reversing the orientation of the first component, then* $\nabla_{L^*}(t_1, t_2, \ldots, t_n) = -\nabla_L(t_1^{-1}, t_2, \ldots, t_n)$.

Once again from the properties of the Alexander polynomial we have $\nabla_{L^*}(t_1, \ldots, t_n) = \varepsilon\, \nabla_L(t_1^{-1}, t_2, \ldots, t_n)$. Using (5.4) we deduce that $\varepsilon = -1$ for links with all linking numbers non-zero and extend to all links using (5.1) and (5.2).

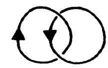
§6. Axiomatic determination of the potential function?

The proofs in the last section of properties of the potential function unfortunately rely on properties of the Alexander polynomial. Hence, they are more cumbersome than the proofs of properties of the reduced potential function which rely only on the two properties (4.1) and (4.2). For links with more than one component, however, a simple set of defining "axioms" for the potential function are not known, at least to me. As an exercise the reader may like to attempt to calculate the potential function of the Borromean rings using the derived properties of potential functions but without resorting to matrix calculations. (I cannot do it.)

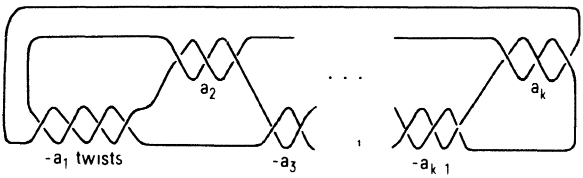
However, for many links, a simple set of properties suffice for the determination of the potential function. As an example, we show that the two properties

(6.1) *For a split link,* $\nabla = 0$.

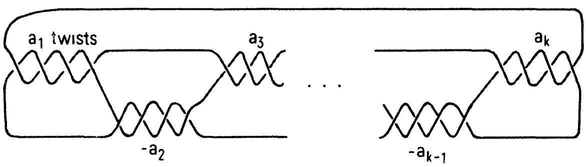
(6.2) *For a simple positive clasp,*  , $\nabla = 1$.

along with the replacement relations (5.1) and (5.2) are enough to determine the potential function of a 2-bridged link of two components. Note first that these conditions imply that $\nabla = -1$ for a negative clasp  .

Following Conway [1], one denotes a 2-bridged link by a sequence of integers $[a_1 \cdots a_k]$ which represents the link



if k is even, and



if k is odd. (We do not worry too much about link orientation in this explanation.)

The links $[a_1 \cdots a_k]$ and $[b_1 \cdots b_l]$ are the same if the continued fractions

$$a_k + \frac{1}{a_{k-1}} + \cdots + \frac{1}{a_1} \quad \text{and} \quad b_l + \frac{1}{b_{l-1}} + \cdots + \frac{1}{b_1}$$

are equal. Every 2-bridged link has a notation $[a_1 \cdots a_k]$ with all a_i positive, and since $[1 \quad a_1 \cdots a_k] = [a_1 + 1 \quad a_2 \cdots a_k]$ we may assume $a_1 > 1$. Now using (5.1) or (5.2) we see that

$$\nabla_{[a_1 \quad a_2 \cdots a_k]} = -\nabla_{[a_1 - 4 \quad a_2 \cdots a_k]} + A(t_1, t_2) \cdot \nabla_{[a_1 - 2 \quad a_2 \cdots a_k]} \tag{**}$$

where $A(t_1, t_2)$ is one of $(t_1t_2 + t_1^{-1}t_2^{-1})$ or $(t_1t_2^{-1} + t_1^{-1}t_2)$ depending on the orientation of the strings crossing in the part of the diagram represented by a_1. (The two strings must belong to different components if L is to have two components.) However, $[0 \quad a_2 \cdots a_k] = [a_3 \cdots a_k]$, $[-1 \quad a_2 \cdots a_k] = [a_2 - 1 \quad a_3 \cdots a_k]$ and $[-2 \quad a_2 \cdots a_k] = [2 \quad a_2 - 1 \quad a_3 \cdots a_k]$. (If $a_2 = 1$, this last one is equal to $[2 + a_3 \quad a_4 \cdots a_k]$.) Therefore, in all cases, the two links on the right hand side of (**) have smaller crossing number than the left hand side. Eventually, the calculation reduces to the potential functions of [0] (split link) and [2] (simple clasp) given by (6.1) and (6.2).

As a result of this calculation we see that

(6.4) *The potential function of a 2-component 2-bridged link is an integral polynomial in* $t_1t_2+t_1^{-1}t_2^{-1}$ *and* $t_1t_2^{-1}+t_1^{-1}t_2$.

In view of the success for 2-bridged links, one is disposed to hope that the potential function of any two-component link is uniquely determined by simple "axioms." It indeed seems possible that the replacement relations, (5.1), (5.2) and (4.1) along with values for the trivial knot, split links and the simple clasp may uniquely determine the potential function of a two component link.

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On the cohomology of groups of p-length 1

THOMAS DIETHELM

1. Introduction

Let G be a finite group, whose order is divisible by the prime p, and let k denote the field of p elements. We consider the cohomology $H^n(G, A)$, where A is a simple kG-module. It is well known that $H^n(G, A) \neq 0$ implies that A lies in the principal block of kG. We ask, if the converse is true, i.e. if to every simple kG-module A in the principal block there is an $n \in \mathbb{N}$ with $H^n(G, A) \neq 0$.

Swan proved that this is true for the trivial module k. Therefore the above question has a positive answer for p-nilpotent groups ($G = O_{p'p}G$). In this paper we show: (Theorem 5.3) if $G = O_{p'pp'p}G$, then there are infinitely many $n \in \mathbb{N}$ with $H^n(G, A) \neq 0$.

In §3 we first consider the case where G is of p-length 1. In order to show the nontriviality of $H^n(G, A)$ we analyze the action of the p'-group $Q = G/O_{p'p}G$ on the cohomology ring $H^*(O_{p'p}G/O_{p'}G, k)$ of the p-group $P = O_{p'p}G/O_{p'}G$. We prove the following result, which is of interest in its own right (Theorem 4.5):

If the p'-group Q acts faithfully on the p-group P, then every simple kQ-module A appears infinitely often in $H^*(P, k)$ as a direct summand.

The proof of this result is by induction on the length of a central series of P with elementary abelian factors. With the aid of this result we can prove Theorem 4.6:

Let G be a group of p-length 1, and let A be a simple kG-module lying in the principal block of kG. Then $H^n(G, A) \neq 0$ for infinitely many $n \in \mathbb{N}$.

In §5 we show how the result for groups G of p-length 1 can be used to treat the case where $G = O_{p'pp'p}G$. We do that by considering the extension

$$O_{p'pp'}G \rightarrowtail G \twoheadrightarrow G/O_{p'pp'}G.$$

Most of the results of this paper first appeared in the author's doctoral thesis (ETH, Zürich, Switzerland, 1981; adviser: U. Stammbach).

2. Techinal lemmas

As a preparation we state the following well known results:

LEMMA 2.1. *Let G be an extension of a p'-group N by a group H, $N \rightarrowtail G \twoheadrightarrow H$. If V is an indecomposable kG-module lying in the principal block, then:*

$$H^n(G, V) \cong H^n(H, V); \qquad n \geqslant 0.$$

Proof. Since N is a p'-group, V is centralised by N and the spectral sequence of the extension $N \rightarrowtail G \twoheadrightarrow H$ collapses.

LEMMA 2.2. *Let G be an extension of a group N by a p'-group H. If V is a kG-module, then $H^n(G, V) \cong H^n(N, V)^H$; $n \geqslant 0$.*

Proof. Since H is a p'-group, the spectral sequence of the extension $N \rightarrowtail G \twoheadrightarrow H$ collapses.

LEMMA 2.3. *Let G be an extension of N by a group H, and let A be a kG-module with $C_G(A) \supseteq N$. Then:*

$$H^n(N, A)^H \cong \operatorname{Hom}_{kH}(H_n(N, k), A); \qquad n \geqslant 0.$$

Proof. Since A is a trivial kN-module, the universal coefficient theorem holds

$$H^n(N, A) \cong \operatorname{Hom}_k(H_n(N, k), A).$$

The above isomorphism is natural and thus H acts diagonally on the right hand side. Hence

$$H^n(N, A)^H = \operatorname{Hom}_{kH}(H_n(N, k), A).$$

3. The cohomology of groups of p-length 1

Let G be a group of p-length 1 ($G = O_{p'pp'}G$), and let A be a simple kG-module lying in the principal block of kG. Then $O_{p'p}G \subseteq C_G(A)$. ([6] p. 164.)
From Lemma 2.1 we obtain

$$H^i(G/O_{p'}G, A) \cong H^i(G, A)$$

and from Lemmas 2.1, 2.2

$$H^i(G, A) \cong H^i(O_{p'p}G/O_{p'}G, A)^{G/O_{p'p}G}.$$

Let Q denote the p'-group $G/O_{p'p}G$, and let P denote the p-group $O_{p'p}G/O_{p'}G$. Then Lemma 2.3 yields

$$H^n(G, A) \cong \mathrm{Hom}_{kQ}(H_n(P, k), A).$$

This preparation allows the proof of the following result.

THEOREM 3.1. *Let G be a group of p-length* 1, *and let A be a simple kG-module lying in the principal block of kG. Then:*

$$H^n(G, A) \neq 0$$

if and only if A is a direct summand of $H_n(P, k)$.

Proof.

"$\Rightarrow$" If $H^n(G, A)$ is nontrivial, then $\mathrm{Hom}_{kQ}(H_n(P, k), A)$ is nontrivial, and the simple kQ-module A is a direct summand of $H_n(P, k)$.

"$\Leftarrow$" By Maschke's theorem $H_n(P, k)$ is semi-simple. If A is a direct summand of $H_n(P, k)$ the projection onto A is a nontrivial kQ-module homomorphism $f: H_n(P, k) \to A$. But the nontriviality of $\mathrm{Hom}_{kQ}(H_n(P, k), A)$ implies the nontriviality of $H^n(G, A)$.

Note 3.1. It follows from Theorem 3.1, that it is necessary to analyze the G/P-module structure of $H_*(P, k)$ induced by conjugation of G in P. Since the cohomology $H^*(P, k)$ is the dual of $H_*(P, k)$, this is equivalent to analyze the G/P-module structure of $H^*(P, k)$. The advantage of working in cohomology is, that we may use its algebra structure which is induced by the cup-product.

Note 3.2. Clearly the p'-group $Q = G/O_{p'p}G$ acts faithfully on the p-group $P = O_{p'p}G/O_{p'}G$.

4. The kQ-module structure of $H^*(P, k)$

By Note 3.2, the p'-group Q acts faithfully on the p-group P. This action induces an action of Q on the cohomology ring $H^*(P, k)$.

Our problem is to determine these kQ-modules which are direct summands of $H^*(P, k)$.

LEMMA 4.1. *Let the p'-group Q act faithfully on the elementary abelian p-group $E=C_p^{(1)}\times\cdots\times C_p^{(m)}$. Then every simple kQ-module A is infinitely often a direct summand of $H^*(E,k)$.*

Proof. It is well known, that the cohomology ring $H^*(E,k)$ contains the polynomial ring $k[x_1,x_2,\dots,x_m]$; $x_i\in H^2(C_p^{(i)},k)$ as a subring. The generators $x_1,x_2,\dots,x_m$ correspond to a basis of E, and Q acts faithfully on the subspace $\langle x_1,x_2,\dots,x_m\rangle$ of $H^2(E,k)$. By the theorem of Steinberg [7], every simple kQ-module A is infinitely often a direct summand of $k[x_1,x_2,\dots,x_m]$, and $k[x_1,x_2,\dots,x_m]$ is a direct summand of $H^*(E,k)$.

Note 4.1. The map $\phi_s:k[x_1,\dots,x_m]\to k[x_1^{p^s},\dots,x_m^{p^s}]$, $f(x_1,\dots,x_m)\mapsto f(x_1,\dots,x_m)^{p^s}=f(x_1^{p^s},\dots,x_m^{p^s})$; $s=0,1,2,\dots$ is a kQ-module isomorphism. Therefore $k[x_1,x_2,\dots,x_m]$ contains infinitely many copies of itself.

LEMMA 4.2. *Let $E=C_p^{(1)}\times C_p^{(2)}\times\cdots\times C_p^{(m)}$ be an elementary abelian central subgroup of the p-group P. Then for some $s\in\mathbb{N}$ the polynomial ring $k[x_1^{p^s},x_2^{p^s},\dots,x_m^{p^s}]$ lies in the image of the restriction map*

$$\mathrm{res}:H^*(P,k)\to H^*(E,k).$$

Proof. We consider the spectral sequence $E_2^{i,j}\cong H^i(P/E,H^j(E,k))\Rightarrow H^{i+j}(P,k)$ of the extension $E\rightarrowtail P\twoheadrightarrow P/E$. Since E is a central subgroup, we get $E_2^{0,j}=H^j(E,k)^{P/E}=H^j(E,k)$.

There is a cup-product [4]

$$E_r^{i,j}\otimes E_r^{i',j'}\xrightarrow{\cup}E_r^{i+i',j+j'}$$

with the following rules

(i) $a\cdot b=(-1)^{ii'+jj'}b\cdot a$;

(ii) $d_r(a\cdot b)=d_ra\cdot b+(-1)^{i+j}a\cdot d_rb$

$$a\in E_r^{i,j};\qquad b\in E_r^{i',j'}.$$

Suppose $0\neq x\in E_2^{0,2}$. Since $\operatorname{char}k=p$, one easily checks that $d_2(x^p)=pd_2x\cdot x^{p-1}=0$.

Now x^p is a nontrivial cocycle of $E_3^{0,2p}$ and $d_3(x^{p^2})=p\cdot d_3x^p\cdot x^{p(p-1)}=0$. Iteration of this process yields $0\neq x^{p^s}\in E_{s+2}^{0,2p^s}$. By a theorem of Evens [1] the spectral sequence of a finite group extension stops, i.e. there is a $t\in\mathbb{N}$ with $E_t=E_\infty$. Now $s=t-2$ yields $0\neq x^{p^s}\in E_\infty^{0,2p^s}$, but x^{p^s} then lies in the image of the restriction map

$$\mathrm{res}:H^{2p^s}(P,k)\to H^{2p^s}(E,k).$$

It follows that the polynomial ring $k[x_1^{p^s}, \ldots, x_m^{p^s}]$ lies in the image of the restriction map.

Note 4.2. It follows from the naturality of the *LHS*-spectral sequence, that, if the p'-group Q acts on the extension $E \rightarrowtail P \twoheadrightarrow P/E$, then the restriction map

$$\mathrm{res}: H^*(P, k) \to H^*(E, k)$$

is a kQ-module homomorphism.

LEMMA 4.3. *Let the p'-group Q act on the central extension $E \rightarrowtail P \twoheadrightarrow P/E$. Let N denote the centralisator $C_Q(E)$. Then every simple $k(Q/N)$-module A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. The group Q/N acts faithfully on E. By Lemma 4.1 and Note 4.1 every simple $k(Q/N)$-module A is infinitely often a direct summand of $k[x_1^{p^s}, \ldots, x_m^{p^s}]$. By Lemma 4.2 A is infinitely often a direct summand in the image of the restriction map

$$\mathrm{res}: H^*(P, k) \to H^*(E, k),$$

and by Note 4.2 A is infinitely often a direct summand of $H^*(P, k)$.

THEOREM 4.4. *Let the p'-group Q act on the central extension $E \rightarrowtail P \twoheadrightarrow P/E$. If the simple kQ-module A is a direct summand of $H^*(P/E, k)$, then A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. We consider the spectral sequence $E_2^{i,j} \cong H^i(P/E, H^j(E, k)) \Rightarrow H^{i+j}(P, k)$ associated with the extension $E \rightarrowtail P \twoheadrightarrow P/E$. Let $B_1, B_2, \ldots, B_m$ be the simple direct summands of $E_2^{0,*}$ and let $A_1, A_2, \ldots, A_n$ be the simple direct summands of $E_2^{*,0}$.

Since E is a central subgroup, we get

$$H^i(P/E, H^j(E, k)) \cong H^i(P/E, k) \otimes_k H^j(E, k) \cong E_2^{i,0} \otimes E_2^{0,j},$$

and $E_2^{i,j}$ is a direct sum of tensorproducts $A_a \otimes B_b$. If we let the p'-group Q act diagonally on $E_r^{i,0} \otimes E_r^{0,j}$, then the map $E_r^{i,0} \otimes E_r^{0,j} \xrightarrow{\cup} E_r^{i,j}$ is a kQ-module homomorphism.

First we prove that there is a simple kQ-module A_s depending on A such that A_s is a direct summand of $E_\infty^{i',0}$. Secondly we show that A is infinitely often a direct summand in the image of the map

$$E_\infty^{i',0} \otimes E_\infty^{0,*} \xrightarrow{\cup} E_\infty^{i',*}.$$

(1) Let i' be the smallest i such that A is a direct summand in some tensorproduct $A_s \otimes B_u$ with $A_s \subseteq E_2^{i',0}$ and $B_u \subseteq E_2^{0,*}$. We show that A_s is a direct summand in $E_\infty^{i',0}$:

If A_s lies in the image of the differential $d_r: E_r^{i'-r-1,r} \to E_r^{i',0}$, then A_s is a direct summand in some tensorproduct $A_t \otimes B_v$ with

$$A_t \subseteq E_2^{i'-r-1,0} \quad \text{and} \quad B_v \subseteq E_2^{0,r}.$$

The module A is then a direct summand in the tensorporoduct $(A_t \otimes B_v) \otimes B_u = A_t \otimes (B_v \otimes B_u)$.

But $B_v \otimes B_u = \bigoplus_w B_w$ and therefore A is a direct summand in $A_t \otimes B_w$. Since B_w is a direct summand of $E_2^{0,*}$ it follows that A is a direct summand of $E_2^{i'-r-1,*}$. This contradicts the minimality of i'. Hence A_s is a direct summand of $E_\infty^{i',0}$.

(2) By Lemma 4.2 and Lemma 4.3 B_u is infinitely often a direct summand in the image of the restriction map

$$\text{res}: H^*(P, k) \to H^*(E, k) \quad \text{i.e.}$$

B_u is infinitely often a direct summand of $E_\infty^{0,*}$. Hence A is infinitely often a direct summand in $E_\infty^{i',0} \otimes E_\infty^{0,*}$.

If A is contained in the kernel of the map $E_\infty^{i',0} \otimes E_\infty^{0,*} \xrightarrow{\cup} E_\infty^{i',*}$, then A lies in the image of some differential $d_r: E_r^{i'-r-1,*} \to E_r^{i',*}$.

This contradicts the minimality of i'. It thus follows that A is infinitely often a direct summand of $E_\infty^{i',*}$.

THEOREM 4.5. *If the p'-group Q acts faithfully on the p-group P, then every simple kQ-module A is infinitely often a direct summand of $H^*(P, k)$.*

Proof. Let us consider the lower central series of P

$$P = P^{(0)} \trianglerighteq P^{(1)} \trianglerighteq \cdots \trianglerighteq P^{(m)} = e.$$

We obviously can refine this series to a central series

$$P = \tilde{P}^{(0)} \trianglerighteq \tilde{P}^{(1)} \trianglerighteq \cdots \trianglerighteq \tilde{P}^{(n)} = e,$$

with elementary abelian factor groups $\tilde{P}^{(i)}/\tilde{P}^{(i+1)}$ and $\tilde{P}^{(0)}/\tilde{P}^{(1)} = P/\Phi(P)$.

If Q acts faithfully on P, Q acts faithfully on $P/\Phi(P)$, see for example [5] p. 102.

By Lemma 4.1 every simple kQ-module A is infinitely often a direct summand of $H^*(P/\tilde{P}^{(1)}, k)$. By Theorem 4.4 A is infinitely often a direct summand of $H^*(P/\tilde{P}^{(2)}, k)$. Iterating this step for the factor groups $P/\tilde{P}^{(i)}$ yields the result that A is infinitely often a direct summand of $H^*(P, k)$.

THEOREM 4.6. *Let G be a group of p-length* 1, *and let A be a simple kG-module lying in the principal block of kG. Then*

$$H^n(G, A) \neq 0 \quad \textit{for infinitely many} \quad n \in \mathbb{N}.$$

Proof. Let A^* denote the dual of A. By Theorem 4.5 A^* is infinitely often a direct summand of $H^*(P, k)$. Dualisation yields the fact that A is infinitely often a direct summand of $H_*(P, k)$. By Theorem 3.1 $H^n(G, A)$ is nontrivial for infinitely many $n \in \mathbb{N}$.

5. The case $G = O_{p'pp'p}G$

LEMMA 5.1. *Let $N \rightarrowtail G \twoheadrightarrow P$ be a group extension with $|P| = p^a$; $a \in \mathbb{N}$, and let A be a kG-module. Then*

$$H^n(N, A) \neq 0 \Rightarrow H^n(G, A) \neq 0.$$

Proof. We consider the long exact sequence [3] p. 224

$$\rightarrow H^n(G, A) \rightarrow H^n(N, A) \rightarrow \mathrm{Ext}^n_G(IP, A) \rightarrow$$

where IP denotes the augmentation ideal of the factor group P. Let IP^* denote the dual of IP^* then there is a natural isomorphism

$$\mathrm{Ext}^n_G(IP, A) \cong H^n(G, IP^* \underset{k}{\otimes} A).$$

Since P is a p-group, all composition factors of IP^* are isomorphic to the trivial module k. A composition series of IP^* induces a composition series of $IP^* \otimes_k A$, of which all composition factors are isomorphic to A.

If $H^n(G, IP^* \otimes_k A)$ is nontrivial, it follows by induction, that $H^n(G, A)$ is nontrivial.

From $H^n(N, A) \neq 0$ and from the above sequence we may conclude that $H^n(G, A)$ or $H^n(G, IP^* \otimes_k A)$ and hence again $H^n(G, A)$ is nontrivial.

LEMMA 5.2. *Let G be a p-solvable group with normal subgroup N, and let A be a simple kG-module lying in the principal block of kG. Then:*

(i) $A = \bigoplus_{i=1}^m B_i$ *as a kN-module and all* B_i *are simple kN-modules.*

(ii) *The simple kN-modules* B_i *lie in the principal block of kN.*

Proof. (i) is a consequence of Clifford's theorem.

(ii) For p-solvable groups the following holds [2] p. 279

$$C_G(A) \supseteq O_{p'p}G \Leftrightarrow A \quad \text{lies in the principal block of } kG$$

Since $O_{p'p}G$ is the maximal p-nilpotent normal subgroup of G, $O_{p'p}N$ is a subgroup of $O_{p'p}G$. Therefore we get $O_{p'p}N \subseteq O_{p'p}G \subseteq C_G(A)$, and thus all B_i are simple kN-modules lying in the principal block of kN.

THEOREM 5.3. *Let* $G = O_{p'pp'p}G$, *and let A be a simple kG-module lying in the principal block of kG. Then*

$$H^n(G, A) \neq 0 \quad \textit{for infinitely many} \quad n \in \mathbb{N}.$$

Proof. We consider the extension

$$O_{p'pp'}G \rightarrowtail G \twoheadrightarrow G/O_{p'pp'}G.$$

The factor groups $G/O_{p'pp'}G$ is a p-group, and the normal subgroup $O_{p'pp'}G$ has p-length 1.

By Lemma 5.2 A is a direct sum of simple $k(O_{p'pp'}G)$-modules B_i lying in the principal block of $k(O_{p'pp'}G)$. By Theorem 4.6 $H^n(O_{p'pp'}G, B_i)$ is nontrivial for infinitely many $n \in \mathbb{N}$, and Lemma 5.1 yields

$$H^n(G, A) \neq 0 \quad \text{for infinitely many} \quad n \in \mathbb{N}.$$

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Two types of birational models

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1. Introduction

Let V be a quasiprojective irreducible algebraic variety over an algebraically closed field k. Using the terminology of [8] we call a birational model $Y \xrightarrow{\varphi} V$ in which Y is a Cohen–Macaulay (CM) variety a *Macaulayfication* of V. We say that a birational model $Y \xrightarrow{\varphi} V$ *preserves a* given local *property* P if a point p of Y satisfies P whenever its image $\varphi(p)$ in V does. The aim of this paper is to describe two classes of blow-up, which for certain varieties V furnish a very simple way to get Macaulayfications that preserve normality and regularity. In view of the fact that V admits nonsingular models if k is of characteristic 0 [15] or of dimension $\leq$ 3 [1] [22] our construction should satisfy two requirements: It should work in any characteristic and it should be essentially simpler than the processes of desingularization. Our results will show that both requirements are satisfied.

Let W be the closed subset of the non-CM points of V. We only will deal with the case $\dim(W) \leq 1$, as it is done in [8] where a very effective method is given to construct Macaulayfications in this case. Unfortunately the Macaulayfications described in [8] do not preserve normality nor regularity. So using them means losing a lot of information on the basic variety V. Our main results are:

(1.1) THEOREM. *Assume that* $\dim(W)=0$. *Then there is a curve* $C \subseteq V$ *such that the blow-up* $Bl_V(C) \xrightarrow{\pi} V$ *of* V *at* C *is a Macaulayfication which preserves normality and regularity.*

(1.2) THEOREM. *Assume that* $\dim(W)=1$. *Then there is a surface* $S \subseteq V$ *and a two-codimensional closed subvariety* T *of the blow-up* $X := Bl_V(S) \xrightarrow{\varphi} V$ *of* V *at* S *such that the blow-up* $Y := Bl_X(T) \xrightarrow{\psi} X$ *of* X *at* T *is a Macaulayfication and such that* φ *and* ψ *preserve normality and* $\varphi \circ \psi$ *preserves regularity. So* $Y \xrightarrow{\varphi\psi} V$ *is a Macaulayfication which preserves normality and regularity.*

In [8] Macaulayfications also are constructed by one or two consecutive blow-up according to whether $\dim(W)$ equals 0 or 1. (According to [4] it is possible in the case $\dim(W)=1$ to replace the second blow-up by a finite birational covering. This at least gives the preservation of normality and regularity for the second step). But the blow-up are centered at ideals which in non-trivial cases never may occur as ideals of sections vanishing at a subvariety. This is contrary to our construction which makes essentially use of the reducedness of the ideals which define the blow-up. It turns out to be important for the preservation of normality that the occurring centers C, S and T are normally torsion-free (This implies more than the mere preservation of normality, namely that normality is preserved even "arithmetically": e.g. the morphism from the projecting cone of the blow-up to V preserves normality). The essential feature to guarantee the CM-propery of our blow-up is that the centers C, S and T define ideals which locally are the unmixed part of "standard-ideals" (e.g. ideals which are generated by "standard sequences"). Standard-sequences (S-sequences) already have been introduced and studied in [4]. Standard-ideals may be considered as a cohomological analogue to the permissible subvarieties which occur as the centers of blow-up in Hironakas' resolution of singularities [15]. So one of the main properties of blowing-up at a S-ideal is the preservation of the cohomology type of the exceptional fiber [4], [5], [6]. Under the mentioned analogy this corresponds to the 'preservation" of the local Hilbert-functions under a permissible blow-up [2], [14], [20]. The above property of the blow-up at a S-ideal makes it basically useful for Macaulayfication, as this latter is nothing else than an improvement of the local cohomological properties by means of blow-up.

(1.1) and its proof allow to draw the following consequences:

(1.3) COROLLARY. *Let V be a normal and assume that* $\dim(W)=0$. *Then there is a curve $C \subseteq V$ such that $Bl_V(C) \to V$ is a Macaulayfication which preserves regularity and such that $Bl_V(C)$ is arithmetically normal.*

(1.4) COROLLARY. *Let V be a normal and of dimension* 3. *Then there is a curve $C \subseteq V$ such that $Bl_V(C)$ is arithmetically normal and arithmetically* CM *and such that $Bl_V(C)$ preserves regularity.*

(1.5) *Remark.* The proof of (1.1) will show more. Namely, if $\dim(W)=0$ we may write $W=\{p_1, \ldots, p_t\}$. Then there are $\mathfrak{m}_{p_i}$-primary ideals $\mathfrak{q}_i \subseteq \mathcal{O}_{V,p_i}$ such that the following holds: The general curve C which is the restriction of a set-theoretic complete intersection in the projective closure of V and whose defining ideal at the point p_i is contained in $\mathfrak{q}_i$ has the property requested in (1.1), (1.3), (1.4). So, embed V into a projective closure V'. Then there are natural numbers $\nu_1, \ldots, \nu_t$ (which may be estimated by the lengths of the local cohomology of $\mathcal{O}_V$ at the points p_i) such that $C := C'|_V$ is of the requested type as soon as C' is a set-theoretic complete intersection which vanishes of order $\geqslant \nu_i$ at p_i and which is in a sufficiently general position.

As a consequence of (1.2) and its proof we get:

(1.6) COROLLARY. *Let V be normal and assume that* $\dim(W) = 1$. *Then there is a surface $S \subseteq V$ and a subvariety $T \subseteq Bl_V(S) =: X$ such that $Bl_V(C)$ and $Bl_X(T)$ are arithmetically normal, such that the morphism $Y := Bl_X(T) \to V$ preserves regularity and such that T is CM.*

(1.7) COROLLARY. *Let V be normal and of dimension ≤ 4. Then there are $S \subseteq V$ and $T \subseteq Bl_V(S)$ which are as in* (1.6).

(1.8) COROLLARY. *Let V be of dimension ≤ 4. Then there is a Macaulayfication $Y \to V$ of V which preserves regularity and such that Y is normal.*

(1.9) *Remark.* For the varieties S and $T \subseteq X = Bl_V(S)$ it holds a statement similar to (1.5): S may be realized by the restriction of a set-theoretic complete intersection vanishing of sufficiently high order at W and being in a sufficiently general position (defined in a projective closure of V). Once having chosen S, there are finitely many points $p_1, \ldots, p_t \in V$ such that T may be obtained as the restriction of a set-theoretic complete intersection hypersurface in a projective closure of the exceptional fiber $\bar{X}$ of $X \to V$, which moreover vanishes of sufficiently high order at the fibers of $p_1, \ldots, p_t$ and which is in a sufficiently general position.

The main difficulty of the proofs in fact consists in showing some corresponding local results. The local features needed to proof (1.1) are some results given in [6], where already a local version of (1.1) is given. These results also are needed to show (1.2). They are presented in the second section. Here we also list a number of rather technical results from [6] which will be used currently in the sequel.

(1.2) also needs some additional algebraic background, mainly the concept of "double-standard-sequences". We already used this concept (in a slightly different way) in [4]. The corresponding results are given in section 3.

Section 4 gives the conclusive globalization of the previously local results and so completes all the proofs. Here we mainly will use arguments of Bertini-type of the kind which are found in [9].

We use the following notations:

Concerning graded rings: If $A = R_0 \oplus R_1 \oplus \cdots$ is a graded ring and if $M = \bigoplus_{h \in \mathbf{Z}} M_h$ is a graded A-module, $M_{>n}$ is the A-submodule $\bigoplus_{h>n} M_h$ of M. If $N = \bigoplus_{h \in \mathbf{Z}} N_h$ is another graded A module we write $\varphi: M \xrightarrow[(d)]{} N$ to express that φ is a homogeneous A-homomorphism of degree d.

Concerning Rees-rings and associated graded rings: If I is an ideal of the ring R, $\mathfrak{R}(I)$ denotes the *Rees-ring* $\oplus_{n\geq 0} I^n$ of I, $\mathrm{Gr}(I)$ the corresponding *associated graded ring* $\mathfrak{R}(I)/I\mathfrak{R}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$. If $\mathcal{J}$ is an ideal in the structure sheaf of a scheme, $\mathfrak{R}(\mathcal{J})$ and $\mathrm{Gr}(\mathcal{J})$ are defined analogous.

Let $x \in I$. Then x^* denotes the one-form in $\mathfrak{R}(I)$ which is induced by $x: x^* = (0, x, 0, \ldots)$. $\bar{x}$ is the corresponding one-form in $\mathrm{Gr}(I)$. $\mathrm{Bl}_R(I)$ stands for the blow-up $\mathrm{Proj}(\mathfrak{R}(I))$ of R at I. If $\mathcal{J}$ is an ideal in the structure-sheaf of a scheme X, $\mathrm{Bl}_X(\mathcal{J})$ stands for the blow-up of X at $\mathcal{J}$. If $\mathcal{J}$ is the ideal of sections vanishing at a closed subset $Y \subseteq X$, we write $\mathrm{Bl}_X(Y)$ for $\mathrm{Bl}_X(\mathcal{J})$ and speak of the blow-up X at Y.

Concerning local cohomology (s. [11], [12]): Let R be a noetherian ring and let $J \subseteq R$ be an ideal. Then $\Gamma_J(M)$ stands for the *J-torsion* $\bigcup_{n\geq 0}(0:J^n)_M = \varinjlim_n \mathrm{Hom}_R(R/J^n, M)$ of the R-module M. H_J^i stands for the ith *local cohomology functor supported at J* which is the ith right derived functor $R^i\Gamma_J$ of the J-torsion and which also may be written as $\varinjlim_n \mathrm{Ext}_R^i(R/J^n, \cdot) \cdot D_J$ stands for the functor of *J-transform* $\varinjlim_n \mathrm{Hom}(J^n, \cdot) \cdot \overline{M^J}$ (or $\bar{M}$ if no confusion is possible) stands for the *J-reduction* $M/\Gamma_J(M)$ of the R-module M. We frequently shall use the following well known relations between these functors:

(1.10) $\quad H_J^0(\overline{M^J}) = 0, \qquad H_J^i(\overline{M^J}) = H_J^i(M) \quad$ for all $i > 0$.

(1.11) $\quad D_J(\overline{M^J}) = D_J(M), \qquad R^iD_J = H_J^{i+1} \quad$ for all $i > 0$.

Moreover we may write

(1.12) $\quad D_J(M) = \bigcup_{n\geq 0}(\overline{M^J}:J^n)_{S^{-1}\overline{M^J}}$, where S is any multiplicatively closed set of R which consists of non-zerodivisors with respect to $\overline{M^J}$ and meets $\sqrt{J}$ (such S exist if M is of finite type).

Moreover there is an exact sequence

(1.13) $\quad 0 \to \Gamma_J(M) \to M \to D_J(M) \to H_J^1(M) \to 0$

for each R-module M, which induces in particular

(1.14) $\quad H_J^i(D_J(M)) = 0, \;$ if $\; i \leq 1; \qquad H_J^i(D_J(M)) = H_J^i(M), \;$ if $\; i > 1$.

If X is a locally noetherian scheme and if $Z \subseteq X$ is stable under specialization, the corresponding local cohomology functors supported in Z are denoted by H_Z^i

(These functors are obtained by globalization of the previous one.) The corresponding transform is denoted by D_Z (D_Z is the Z-closure introduced in [10]; if Z is closed and if $\mathcal{F}$ is a quasicoherent sheaf over X, $D_Z(\mathcal{F})$ is the direct image of the restriction $\mathcal{F}|_{X-Z}$).

Concerning loci: Let X be a locally noetherian scheme. Then by $\mathrm{Reg}(X)$, $\mathrm{Sing}(X)$, $\mathrm{Nor}(X)$, $\mathrm{Fac}(X)$, $\mathrm{CM}(X)$ we respectively denote the set of its regular, singular, normal, factorial and CM-points. If $X = \mathrm{Spec}(R)$, where R is a noetherian ring, these loci are denoted respectively by $\mathrm{Reg}(R)$, $\mathrm{Sing}(R)$, $\mathrm{Nor}(R)$, $\mathrm{Fac}(R)$, $\mathrm{CM}(R)$.

As for the unexplained notations and terminology see [13] (algebraic geometry) and [17] (commutative algebra).

(1.15) ***Remark***: Our results hold in fact over any infinite field k. The adjustment needed to treat this case is the use of double standard sequences, which have this property universally, e.g. under finite extensions of the base field k. We used this concept in [4]. To keep our arguments less technical we decided only to present the case of an algebraically closed field k.

2. Standard-sequences

In this section we present some notions and results from [4], [5] and [6]. These will furnish the algebraic background of our proofs.

In the sequel let R be a noetherian ring, let $J \subseteq R$ be an ideal and let M be a finitely generated R-module. An element $x \in R$ is said to be *J-filter-regular* (resp. a J-f-regular element or a J-f-element) *with respect to M* if it belongs to the set $\mathrm{reg}(\overline{M^J})$ of regular elements (= non-zerodivisors) with respect to $\overline{M^J}$. A sequence $x_1, \ldots, x_r \in R$ is called J-f-regular (or a J-f-sequence) with respect to M if x_i is J-f-regular with respect to $M/(x_1, \ldots, x_{i-1})M$ for all $i \leq r$. These concepts have been introduced in [21] for the special case of a local ring with a maximal ideal J. $x_1, \ldots, x_{i-1}$ is a J-f-sequence with respect M iff it is a regular sequence with respect to $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R) - V(J)$. J-f-regularity with respect to M is equivalent to J-f-regularity with respect to $\overline{M^J}$ and to J-f-regularity with respect to $D_J(M)$. J-f-regularity with respect to M induces JR'-f-regularity with respect to $R' \otimes M$, where R' is a noetherian and flat R-algebra. The following easy statements will be used frequently (grade (J) denotes the common length of all R-regular sequences in J):

(2.1) LEMMA. (I) *Assume that* $\mathrm{Spec}(R) - V(J)$ *is* CM *and let* $x_1, \ldots, x_r \in \varepsilon\sqrt{J}$ *be such that* $ht(x_1, \ldots, x_r) = r$. *Then* $x_1, \ldots, x_r$ *is* J-f-*regular with respect to* R.

(ii) *Let* $x_1, \ldots, x_r \in \sqrt{J}$ *be* J-f-*regular with respect to* R. *Then, if* $r \leq ht(J)$, *it holds* $ht(x_1, \ldots, x_r) = r$ if $r \leq \text{grade}\,(J)$ $x_1, \ldots, x_r$ *even constitute a regular sequence with respect to* R.

Now, we define the *cohomological* J-*finiteness-dimension* of M:

(2.2) $e_J(M) := \inf\{i \mid H_J^i(M)$ is not finitely generated over $R\}$.

Using (1.10) we obtain $e_J(M) = e_J(\overline{M^J})$. If $M \neq \Gamma_J(M)$ it is well known that $e_J(M) \leq \text{ht}\,(J \cdot R/\text{ann}\, M)$. In particular we have $e_J(M) = \infty \Leftrightarrow M = \Gamma_J(M)$. If x is J-f-regular with respect to M, there is a short exact sequence $0 \to \overline{M^J} \xrightarrow{x} \overline{M^J} \to \overline{M^J}/x\overline{M^J} \to 0$, which shows that $e_J(M/xM) \geq e_J(M) - 1$.

In [4] we introduced the notion of *standard-sequences*. In [5] we systematically studied a certain class of *truncated standard-sequences* (but only for the case where J was the maximal ideal of a local ring).

(2.3) DEFINITION. A sequence $x_1, \ldots, x_r \in R$ is called a J-standard-sequence (J-S-sequence) with with respect to M if

(i) $r \leq e_J(M)$,

(ii) $x_1, \ldots, x_r$ is a J-f-sequence with respect to m,

(iii) $(x_1, \ldots, x_r)H_J^i(M/(x_1, \ldots, x_j)M) = 0$, for all i, j with $i + j < r$. $x_1, \ldots, x_r$ is called a truncated J-standard-sequence (J-S^+-sequence) with respect to M if $r < e_J(M)$ and if (iii) holds for all pairs i, j for which $i + j < r$. It is the same to say that there is a $y \in R$ such that $x_1, \ldots, x_r, y$ is a J-S-sequence with respect to M.

$x_1, \ldots, x_r$ is a J-S-sequence (resp. a J-S^+-sequence) with respect to M iff it is with respect to $\overline{M^J}$. If $e_J(M) > 1$ (which implies in particular that $D_J(M)$ is finitely generated), the same statement holds for the pair of modules M and $D_J(M)$. If $x_1, \ldots, x_r$ is a J-f-sequence with respect to M, $x_2, \ldots, x_r$ is a J-S (resp. a J-S^+)-sequence with respect to M/x_1M if $x_1, \ldots, x_r$ has the corresponding property with respect to M.

The following result has been shown in [4]:

(2.4) LEMMA. *Let* $x_1, \ldots, x_r \in R$ *be a* J-f-*sequence with respect to* M. *Then*:

$$\text{ann}\,(H_J^i(M/(x_1, \ldots, x_r)M \supseteq \prod_{j=0}^{r} [\text{ann}\,(H_J^{i+j}(M))]^{\binom{r}{j}}.$$

If $H_J^i(M)$ *is finitely generated, it holds* $\sqrt{\text{ann}\,(H_J^i(M))} \supseteq J$.

So (2.4) induces

(2.5) COROLLARY. *There is an ideal* $\mathfrak{a} \subseteq R$ *such that* $\sqrt{\mathfrak{a}} \supseteq J$ *and such that each* J-f-*sequence* $x_1, \ldots, x_r \in \mathfrak{a}$ *with respect to* M (*with* $r \leq e_J(M)$) *is a* J-S-*sequence with respect to* M. *Consequently if* $r < e_J(M)$, $x_1, \ldots, x_r$ *is a* J-S^+-*sequence under these assumptions.* (*It suffices to choose* $\mathfrak{a} = [\prod_{j<e_J(M)} \operatorname{ann}(H_J^j(M))]^{2^{e_J(M)-1}}$.)

If a sequence keeps having one of these properties under all its permutations, we express this in using the prefix *p*. So we shall speak of J-*pf*-sequences (= permutable J-f-sequences (= permutable J-standard-sequences) and J-pS^+-sequences (= permutable truncated J-standard-sequences).

(2.6) *Remark.* We later mainly shall use the concept of truncated standard-sequence. The situation in which they will come up is as follows: Let V and W be as in the introduction. Let $\mathcal{J} \subseteq \mathcal{O}_V$ be such that $V(\mathcal{J})$ is of codimension h in V and such that $W \subseteq V(\mathcal{J})$. Then $V - V(\mathcal{J})$ is CM. So if $p \in V - V(\mathcal{J})$ it holds $\operatorname{depth}(\mathcal{O}_{V,p}) + \operatorname{codim}(\overline{\{p\}}, \overline{\{p\}} \cap V(\mathcal{J})) = \operatorname{codim}(V, \overline{\{p\}}) + \operatorname{codim}(\overline{\{p\}}, \overline{\{p\}} \cap V(\mathcal{J})) \geq \operatorname{codim}(V, V(\mathcal{J})) = h$, where for a locally noetherian scheme X and a closed subscheme Y $\operatorname{codim}(X, Y)$ denotes the codimension of Y with respect to X. If $h > 0$ and if p is the generic point of V, equality holds in the above estimate. So by Grothendiecks finiteness theorem for local cohomology [11] the sheaves $H^i_{V(\mathcal{J})}(\mathcal{O}_V)$ are coherent for all $i < b$. Now, let $p \in V(\mathcal{J})$ and put $R = \mathcal{O}_{V,p}$ and $J = \mathcal{J}_p$. Then it follows that $e_J(R) = h$ if $h > 0$. Let $\mathfrak{b}$ be the following ideal of $\mathcal{O}_V$: $\mathfrak{b} = [\prod_{j<h} \operatorname{ann}(H^j_{V(\mathcal{J})}(\mathcal{O}_V))]^{2^{h-1}}$ ($h > 0$) and put $\mathfrak{a} = \mathfrak{b}_p$. Then $\mathfrak{a} \subseteq R$ is as in (2.5). Moreover $\operatorname{Spec}(R) - V(J)$ is CM. So (2.1) (i) implies that each partial system of parameters $x_1, \ldots, x_r \in \mathfrak{a}$ is a J-pS-sequence with respect to R. If $r < h$, $x_1, \ldots, x_r$ is even a J-pS^+-sequence with respect to R.

For the rest of this paragraph we fix the following notations: Let $0 < r < e_J(M)$ and let $x_1, \ldots, x_r \in \sqrt{J}$ be a J-pS^+-sequence with respect to M. Let $L = (x_1, \ldots, x_r)$ and put $\overline{M}^J = \bar{M}$. In [4] we have proved the following results:

(2.7) LEMMA. (i) *The canonical maps* $H^1_J(L^n M) \to H^1_J(L^{n-1} M)$ *vanish for all* $n > 0$.

(ii) $LM \cap \Gamma_J(M) = 0$.

(iii) $D_J(L^n M) = \bigcup_j (L^n \bar{M} : J^j)_{\bar{M}} \subseteq L^{n-1} \bar{M}$ ($n > 0$).

(iv) $\overline{M/L^n M}^J = \bar{M}/D_J(L^n M)$.

(2.8) LEMMA. $\overline{L^n M/L^{n+1} M}^J = \overline{M/LM}^{J\binom{n+r-1}{r-1}}$.

Now, let $M' = M/x_1 M$, $L' = (x_2, \ldots, x_r)$. Then $x_2, \ldots, x_r$ is a J-pS^+-sequence with respect to M'. According to [4] we have

(2.9) LEMMA. *If x_1^* denotes the maps induced by multiplication with x_1, we have the following canonical exact sequences*

$$\text{(i)}\quad 0 \to L^{n-2}\bar{M}/D_J(L^{n-1}M) \xrightarrow{x_1^*} L^{n-1}\bar{M}/D_J(L^nM) \to L'^{n-1}\overline{M'}/D_J(L'^nM') \to 0$$

for all $n > 1$. (Thereby this sequences split).

$$\text{(ii)}\quad 0 \to D_J(L^{n-1}M) \xrightarrow{x_1} D_J(L^nM) \to D_J(L'^nM') \to 0,$$

$$0 \to D_J(L^{n-1}M)/D_J(L^nM) \xrightarrow{x_1} D_J(L^nM)/D_J(L^{n+1}M) \to D_J(L'^nM')/D_J(L'^{n+1}M') \to 0, \textit{ for all } n > 0.$$

$$\text{(iii)}\quad 0 \to L^{n-1}\bar{M} \xrightarrow{x_1} L^nM \to L'^nM' \to 0,$$

$$0 \to L^{n-1}\bar{M}/L^nM \xrightarrow{x_1} L^nM/L^{n+1}M \to L'^nM'/L'^{n+1}M' \to 0, \textit{ for all } n > 0.$$

We denoted the above injections with x_1^* by the following reason:

$$\bigoplus_n L^n\bar{M}/D_J(L^{n+1}M) = \bigoplus_n \overline{L^nM/L^{n+1}M}^J, \qquad \bigoplus_n D_J(L^nM) = D_J\Big(\bigoplus_n L^nM\Big),$$

$$\bigoplus_n D_J(L^nM)/D_J(L^{n+1}M) = D_J\Big(\bigoplus_n L^nM\Big)\Big/D_J\Big(\bigoplus_n L^{n+1}M\Big),$$

$$\bigoplus_n L^m\bar{M} \quad\text{and}\quad \bigoplus_n L^n\bar{M}/L^{n+1}\bar{M}$$

all are in a canonical way graded modules over the Rees-algebra $\mathfrak{R}(L)$. So the operation of the one-form x_1^* on these modules is exactly the corresponding map x_1^* of (2.9). The purpose of the above sequences is of merely technical nature, as they repeatedly come up to perform different induction arguments. They come close to generalize the fact that the associated graded module with respect to an M-regular sequence $x_1, \ldots, x_r$ is a polynomial extension of $M/(x_1, \ldots, x_r)M$ [18].

The following result was shown in [6] for the special case where R is local and J its maximal ideal. But in fact the proof works in full generality.

(2.10) LEMMA. *For all $n > 0$ it holds $L^nD_J(LM) = D_J(L^{n+1}M)$.*

As a consequence we get:

(2.11) COROLLARY. *Let* $0<r<e_J(R)$. *Put* $\bar{R}=\overline{R^J}$. *Let* $x_1,\ldots,x_r\in\sqrt{J}$ *be a* J-*pS*$^+$-*sequence with respect to* R *and put* $L=(x_1,\ldots,x_r)$, $\tilde{L}=\bigcup_j(L\bar{R}:J^j)_{\bar{R}}$. *Then it holds:*

(i) $\tilde{L}^n=D_J(L^n)$, $(n>0)$,

(ii) $\mathrm{Ass}\,(\bar{R}/\tilde{L}^n)=\mathrm{Ass}\,(\bar{R}/\tilde{L})=\mathrm{Ass}\,(R/L)-V(J)$.

Proof. By (2.7) (iii) we have $\tilde{L}^n\subseteq D_J(L^n)$ and $\tilde{L}=D_J(L)$. (2.10) implies $D_J(L^n)=L^{n-1}D_J(L)\subseteq\tilde{L}^{n-1}\tilde{L}=\tilde{L}^n$. This proves (i).

To show (ii) we first observe that $L^{n-1}\bar{R}/\tilde{L}^n=\overline{L^{n-1}/L^n}$ is a free module over $\overline{R/L}=\bar{R}/\tilde{L}$ (use (2.7) (iii), (2.8) and (2.11) (i)). So we have $\mathrm{Ass}\,(L^{n-1}\bar{R}/\tilde{L}^n)=\mathrm{Ass}\,(\bar{R}/\tilde{L})$. By the short exact sequence $0\to L^{n-1}\bar{R}/\tilde{L}^n\to\tilde{L}^{n-1}/\tilde{L}^n\to\tilde{L}^{n-1}/L^{n-1}\bar{R}\to 0$ we see that $\mathrm{Ass}\,(\bar{R}/\tilde{L})=\mathrm{Ass}\,(L^{n-1}\bar{R}/\tilde{L}^n)\subseteq\mathrm{Ass}\,(\tilde{L}^{n-1}/\tilde{L}^n)\subseteq\mathrm{Ass}\,(\bar{R}/\tilde{L})\cup\mathrm{Ass}\,(\tilde{L}^{n-1}/L^{n-1}\bar{R})$. As $\tilde{L}^{n-1}=\bigcup_j(L^{n-1}\bar{R}:J^j)_{\bar{R}}$ we have $J^\nu\tilde{L}^{n-1}\subseteq L^{n-1}\bar{R}$ for some ν. So we have $\mathrm{Ass}\,(\tilde{L}^{n-1}/L^{n-1}\bar{R})\subseteq V(J)$. As $\tilde{L}^{n-1}/\tilde{L}^n=\overline{\tilde{L}^{n-1}/\tilde{L}^n}$ has no J-torsion it follows that $\mathrm{Ass}\,(\tilde{L}^{n-1}/\tilde{L}^n)\cap V(J)=\emptyset$. So we get $\mathrm{Ass}\,(\tilde{L}^{n-1}/\tilde{L}^n)=\mathrm{Ass}\,(\bar{R}/\tilde{L})$. Now, using the exact sequences $0\to\tilde{L}^{n-1}/\tilde{L}^n\to\bar{R}/\tilde{L}^n\to\bar{R}/\tilde{L}^{n-1}\to 0$ we conclude by induction on n that $\mathrm{Ass}\,(\bar{R}/\tilde{L}^n)=\mathrm{Ass}\,(\bar{R}/\tilde{L})$.

$\mathrm{Ass}\,(\bar{R}/\tilde{L})=\mathrm{Ass}\,(\overline{R/L^J})=\mathrm{Ass}\,(R/L)-V(J)$ is clear by the definition of the functor $\overline{\cdot}^J$.

(2.12) *Remark.* (2.11) (ii) will be of importance in our later applications as it states that L is *normally torsion-free* (An ideal ideal of R is said to be normally torsion-free, if $\mathrm{Gr}\,(I)$ is torsion-free over R/I or – equivalently – if $\mathrm{Ass}\,(R/I^n)\subseteq\mathrm{Ass}\,(R/I)$, $\forall n$.)

Finally we shall make use of the following result, which deals with the special case where R is local and J its maximal ideal.

(2.13) PROPOSITION. *Let* $(R,\mathfrak{m})$ *be a local ring of dimension* $d>1$ *and assume that* $e_{\mathfrak{m}}(R)=d$. *Let* $x_1,\ldots,x_{d-1}\in\mathfrak{m}$ *be a* $\mathfrak{m}$-*pS*$^+$-*sequence with respect to* R *and put* $L=(x_1,\ldots,x_{d-1})$, $\bar{R}=\overline{R^{\mathfrak{m}}}$, $\tilde{L}=\bigcup_j(L\bar{R}:\mathfrak{m}^j)_{\bar{R}}$. *Then:*

(i) $D_{\mathfrak{m}}(\mathfrak{R}(L))$ *is finitely generated as a* $\mathfrak{R}(L)$-*module*;

(ii) $\mathrm{Bl}_{\bar{R}}(\tilde{L})=\mathrm{Proj}\,(\mathfrak{R}(\tilde{L}))=\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathfrak{R}(L))$ *is* CM;

(iii) *If* R *is* CM, *we have* $D_{\mathfrak{m}}(R)=R$, $L=\tilde{L}$ *and* $\mathfrak{R}(\tilde{L})$ *is* CM *in this case.*

Proof. Let $\mathfrak{n}$ be the homogeneous maximal ideal $\mathfrak{m}\oplus L\oplus L^2\oplus\cdots$ of $\mathfrak{R}(L)$. Let D be the graded $\mathfrak{R}(L)$-module $D_{\mathfrak{m}}(\mathfrak{R}(L))=\bigoplus_n D_{\mathfrak{m}}(L^n)$. We know by [6] that D is finitely generated over $\mathfrak{R}(L)$ and that its local cohomology supported in $\mathfrak{n}$

satisfies

$$(*)\quad H^i_{\mathfrak{n}}(D)=\begin{cases}0, & \text{if}\quad i\leqslant \min d' := \min(3,d),\\ \bigoplus_{j=-i+3}^{-1}[H^{i-1}_{\mathfrak{m}}(R)]_j, & \text{for}\quad d'<i\leqslant d.\end{cases}$$

Thereby – for an R-module H – $[H]_j$ stands for the graded $\mathfrak{R}(L)$-module whose tth component is 0 or H, according to whether $t\neq j$ or $t=j$.

This shows that the $X:=\operatorname{Proj}(\mathfrak{R}(L))$-sheaf $\tilde{D}$ induced by D is coherent and CM (the latter is a consequence of the weak part of Grothendiecks finiteness theorem [11]). But – as D and $\mathfrak{R}(L)$ differ only in degree 0 – $\tilde{D}$ also is the structure-sheaf of $\mathrm{Bl}_R(\tilde{L})$. This proves the first statement. If R is CM we have $H^i_{\mathfrak{m}}(R)=0$ for $i\leqslant 1$. This induces $R=D_{\mathfrak{m}}(R)$ (s. (1.13)). Moreover we then have $\overline{R/L^{\mathfrak{m}}}=R/L$, thus $\tilde{L}=L$. This shows that $D=\mathfrak{R}(\tilde{L})$. By (*) we have $H^i_{\mathfrak{n}}(D_{\mathfrak{n}})=0$ for all $i<d+1=\dim D_{\mathfrak{n}}$. So $D_{\mathfrak{n}}$ is CM. So the same is true for $D=\mathfrak{R}(\tilde{L})$ by [16].

(2.14) *Complement. Let R, L and $\tilde{L}$ be as in* (2.13) *and assume moreover that $d\leqslant 3$ and that* depth $(R)\geqslant 2$ (*which latter is the case if R is normal*). *Then $\mathfrak{R}(\tilde{L})$ is* CM.

Proof. Our assumptions imply that $R=D_{\mathfrak{m}}(R)$. So, in the notations of the previous proof we have $\mathfrak{R}(\tilde{L})=D$. Now we conclude as above by (*).

(2.15) *Remark.* We shall use these results mainly in the following context: Let V and W be as in the introduction, assuming that $\dim(W)=0$. Put $d=\dim(V)$ and let $\mathcal{J}$ be the ideal of sections vanishing at W. By (2.6) there is an ideal $\mathfrak{b}\subseteq\mathcal{O}_V$ such that $V(\mathfrak{b})=W$ which has the following property: For each $p\in V$ and each partial system of parameters $x_1,\dots,x_{d-1}$ of $\mathcal{O}_{V,p}$ contained in $\mathfrak{b}_p$, $x_1,\dots,x_{d-1}$ is a $\mathfrak{m}_p$-pS^+-sequence with respect to $\mathcal{O}_{V,p}$. So, let $\mathcal{L}\subseteq\mathfrak{b}$ be a locally complete intersection of codimension $d-1$ with respect to V (This means that for any point $p\in V(\mathfrak{b})$ $\mathcal{L}_p$ is generated by a partial system of parameters $x_1,\dots,x_{d-1}$ with respect to $\mathcal{O}_{V,p}$). Let $\tilde{\mathcal{L}}=\bigcup_j(\mathcal{L}:\mathcal{J}^j)\mathcal{O}_V$. Then, using the previous notations and putting $R=\mathcal{O}_{V,p}$, $\mathfrak{m}=\mathfrak{m}_p$, $L=(x_1,\dots,x_{d-1})=\mathcal{L}_p$ and $\tilde{L}=\bigcup_j(L:\mathfrak{m}^j)_R$ we have $\tilde{L}=\tilde{\mathcal{L}}_p$. So by (2.13) $\mathcal{L}_p$ is a normally torsion-free ideal of $\mathcal{O}_{V,p}$ for all $p\in V(\mathcal{L})$ and $\mathrm{Bl}_V(\tilde{\mathcal{L}})$ is CM. If $d\leqslant 3$ and if V is normal (It suffices that V is S_2) we see by (2.14) that $Bl_V(\tilde{\mathcal{L}})$ is even arithmetically CM.

3. Double standard-sequences

Let J, $T\subseteq R$ be ideals of the noetherian ring R such that $J\subseteq T$ and such that there is an element $t\in T$ with $\sqrt{T}=\sqrt{(J,t)}$. In this section we consider double standard-sequences and truncated double standard-sequences with respect to J and T. These are J-S-sequences (resp. J-S^+-sequences) subject to another condition, involving local cohomology supported in T. In [4] we developed this concept in a slightly different way, without assuming the existence of an element t as above. Note that the existence of such an element induces $H_T^i(H_J^j(M))=0$ for all $i>1$, all $j\geqslant 0$ and all R-modules M (s. for example [7]).

(3.1) LEMMA. *Let* $x_1, \ldots, x_r$ R *be a* J-f-*sequence with respect to* M. *Then it holds* $\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_r)M^J})) \supseteq \prod_{j=0}^r [\operatorname{ann}(H_T^{i+j}(\overline{M^J}))]^{\binom{r}{j}}$

Proof. (Induction on r). If $r=0$ or $i=0$, all is clear. So let $r, i>0$. As x_r is J-f-regular with respect to $M/(x_1, \ldots, x_{r-1})M$ we have an exact sequence

$$0 \to \overline{M/(x_1, \ldots, x_{r-1})M^J} \xrightarrow{x_r} \overline{M/(x_1, \ldots, x_{r-1})M^J} \to \overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)} \to 0.$$

The corresponding $H_T^{\cdot}$-sequence furnishes the relation

$$\begin{aligned} &\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)})) \\ &\quad \supseteq \operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J})) \cdot \operatorname{ann}(H_T^{i+1}(\overline{M/(x_1, \ldots, x_r)M^J})). \end{aligned}$$

By the exact sequence

$$\begin{aligned} 0 &\to \Gamma_J(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}) \to \overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)} \\ &\to \overline{M/(x_1, \ldots, x_r)M^J} \to 0 \end{aligned}$$

we also obtain an exact sequence

$$H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)}) \to H_T^i(\overline{M/(x_1, \ldots, x_r)M^J}) \to H_T^{i+1}(\Gamma_J(\cdots))=0,$$

which shows that

$$\operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_r)M^J})) \supseteq \operatorname{ann}(H_T^i(\overline{M/(x_1, \ldots, x_{r-1})M^J/(x_r)})).$$

So, using induction, we get

$$\begin{aligned}
&\operatorname{ann}(H^i_T(\overline{M/(x_1,\dots,x_r)M^J})) \supseteq \operatorname{ann}(H^i_T(\overline{M/(x_1,\dots,x_{r-1})M^J}))\\
&\quad\times \operatorname{ann}(H^{i+1}_T(\overline{M/(x_1,\dots,x_{r-1})M^J})) \supseteq \prod_{0\leq j\leq r-1} [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r-1}{j}}\\
&\quad\times \prod_{0\leq 1\leq r-1} [\operatorname{ann}(H^{i+1+l}_T(\overline{M^J}))]^{\binom{r-1}{l}}\\
&= \operatorname{ann}(H^i_T(\overline{M^J}))\Big[\prod_{0<j<r} [\operatorname{ann}(H^{i+j}_T)(\overline{M^J}))]^{\binom{r-1}{j}} \cdot [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r-1}{j-1}}\Big]\\
&\quad\times \operatorname{ann}(H^{i+r}_T(\overline{M^J})) = \prod_{0\leq j\leq r} [\operatorname{ann}(H^{i+j}_T(\overline{M^J}))]^{\binom{r}{j}}.
\end{aligned}$$

For a finitely generated R-module M let us introduce the following notations

(3.2) (i) $\lambda_{J,T}(M) = \sup\{n \mid \sqrt{\operatorname{ann}(H^i_T(M))} \supseteq J, \text{ for all } i \leq n\}$,
(ii) $e_{J,T}(M) = \min\{\lambda_{J,T}(M), e_J(M)\}$.

We call $e_{J,T}(M)$ the J, T-*finiteness-dimension* of the R-module M.

(3.3) DEFINITION. Let M be a finitely generated R-module. A sequence $x_1, \dots, x_r \in R$ is called a J, T-*standard-sequence* ($= J, T$-S-sequence) *with respect to* M if

(i) $r \leq e_{J,T}(M)$
(ii) $x_1, \dots, x_r$ is a J-S-sequence with respect to M
(iii) $(x_1, \dots, x_r)H^i_T(\overline{M/(x_1, \dots, x_j)M^J}) = 0$ for all i, j with $i + j \leq r$.

$x_1, \dots, x_r$ is said to be a *truncated* J, T-*standard-sequence* ($= J, T$-S^+-sequence) with respect to M if $r < e_{J,T}(M)$, if $x_1, \dots, x_r$ is a J-S^+-sequence with respect to M and if (iii) holds for all pairs i, j for which $i + j \leq r + 1$. So $x_1, \dots, x_r$ is a $J, T - S^+$-sequence with respect to M iff there is an element $y \in R$ such that $x_1, \dots, x_r, y$ is a J, T-S^+-sequence with respect to M.

This is the concept of the previously announced (truncated) double-standard-sequences.

Using (2.5) and (3.1) we obtain:

(3.4) LEMMA. *Let M be a finitely generated R-module. Then there is an ideal $\mathfrak{a} \subseteq R$ such that $\sqrt{\mathfrak{a}} \supseteq J$ and such that each J-f-sequence $x_1, \dots, x_r \in \mathfrak{a}$ with respect to M of length $r \leq e_{J,T}(M)$ is a J, T-S-sequence with respect to M. If $r < e_{J,T}(M)$,*

$x_1, \ldots, x_r$ will be a J, T-S^+-sequence with respect to M under the above assumptions. (It suffices to choose $\mathfrak{a}$ as the ideal

$$\left[\prod_{j<e_{J,T}(M)} (\operatorname{ann}(H^j_J(M)) \cdot \operatorname{ann}(H^{j+1}_T(\overline{M^J})))\right]^{2^{e_{J,T}(M)-1}}).$$

If a sequence $x_1, \ldots, x_r$ is a J, T-S-sequence under all its permutations we speak again of a J, T-pS-sequence. Similarly we use the notation of J, T-pS^+-sequence in case of a permutable truncated J, T-standard-sequence.

(3.5) *Remark.* We shall apply the above concept in the following context: Let V and W be as in the introduction and put $\dim(V) = d$. Assume that $\dim(W) \leq 1$ and let $\mathscr{J} \subseteq \mathcal{O}$ be an ideal such that $V(\mathscr{J}) \supseteq W$. Let $Z \subseteq V$ be the set of all closed points of V. Z is stable under specialization. So the local cohomology functors H^i_Z are defined in the category of quasicoherent $\mathcal{O}_V$-sheaves. As $V - V(\mathscr{J})$ is CM [7] guarantees that $\sqrt{\operatorname{ann}}(H^{j+1}_Z(\mathcal{O}_V)) \supseteq \mathscr{J}$ and $\sqrt{\operatorname{ann}}(H^j_{V(\mathscr{J})}(\mathcal{O}_V)) \supseteq \mathscr{J}$ for all $j < d-1$ (The second statement also follows by (2.6)). Now, put

$$\mathfrak{b} = \left[\prod_{j<d-1} \operatorname{ann}(H^j_{V(\mathscr{J})}(\mathcal{O}_V)) \cdot \operatorname{ann}(H^{j+1}_Z(\mathcal{O}_V))\right]^{2^{d-2}}.$$

Let $p \in Z$, $R = \mathcal{O}_{V,p}$, $J = \mathscr{J}_p$, $\mathfrak{a} = \mathfrak{b}_p$, $\mathfrak{m} = \mathfrak{m}_{V,p}$. Then we have $H^j_{V(\mathscr{J})}(\mathcal{O}_V)_p = H^j_J(R)$, $H^j_Z(\mathcal{O}_V)_p = H^j_{\mathfrak{m}}(R)$, $e_{J,\mathfrak{m}}(R) = d-1$ and $\mathfrak{a} = [\prod_{j<d-1} \operatorname{ann}(H^j_J(R)) \cdot \operatorname{ann}(H^{j+1}_{\mathfrak{m}}(R))]^{2^{d-2}}$. By (2.1) (i) each partial system of parameters $x_1, \ldots, x_{d-2} \in \mathfrak{a}$ for the local ring R is a J-f-sequence with respect to R, thus a $J, \mathfrak{m}$-pS^+-sequence with respect to R (3.4). So, let $\mathscr{L} \subseteq \mathfrak{b}$ be a locally complete intersection-ideal of codimension $d-2$. This means in particular, that $\mathscr{L}_p$ is generated by a partial system of parameters $x_1, \ldots, x_{d-2}$ for the local ring $\mathcal{O}_{V,p}$ for each closed $p \in V(\mathscr{J})$. If $p \in V(\mathscr{J})$ this system is a $\mathscr{J}_p, \mathfrak{m}_p$-$pS^+$-sequence by the previous remark. If $p \notin V(\mathscr{J})$ the same is true as $x_1, \ldots, x_{d-2}$ is a regular sequence in the CM-ring $\mathcal{O}_{V,p}$. So $\mathscr{L}_p$ is generated by a $\mathscr{J}_p, \mathfrak{m}_p$-$pS^+$-sequence of length $d-2$ for all closed $p \in V(\mathscr{L})$. Let $\tilde{\mathscr{L}} = \bigcup_j (\mathscr{L} : \mathscr{J}^j)_{\mathcal{O}_V}$. Then, by (2.13) $\tilde{\mathscr{L}}_p$ is a normally torsion-free ideal of the local ring $\mathcal{O}_{V,p}$ for each closed $p \in V(\mathscr{L})$.

The following result is an extension of (2.7)(i) to double-standard-sequences. Similar to (2.7)(i), which is of fundamental significance for the treatment of standard-sequences, the result to come is a strong tool to treat double-standard-sequences.

(3.6) LEMMA. *Let $x_1, \ldots, x_r$ $(r < e_{J,T}(M))$ be a J, T-pS^+-sequence with respect to M (which is assumed to be finitely generated). Put $L = (x_1, \ldots, x_r)R$. Then the canonical maps*

$$H^2_T(D_J(L^nM)) \to H^2_T(L^{n-1}\overline{M^J}) \qquad (n > 0)$$

(induced by the injections $D_J(L^nM) \hookrightarrow L^{n-1}\overline{M^J}$ (2.7) (iii)) vanish.

Proof. The case $r = 0$ is trivial. First we treat the case $r = 1$. Clearly we may assume $\Gamma_J(M) = 0$. By the commutative diagram

$$\begin{tikzcd} L^nM \arrow[r, hook] & L^{n-1}M \\ M \arrow[u, "\cong"] \arrow[r, "x_1"'] & M \arrow[u, "\cong"'] \end{tikzcd} \quad \text{we get the situation} \quad \begin{tikzcd} H^2_T(L^nM) \arrow[r, "\alpha"] \arrow[d, phantom, "\cong" rotate=90] & H^2_T(L^{n-1}M) \arrow[d, phantom, "\cong" rotate=90] \\ H^2_T(M) \arrow[r, "x_1"'] & H^2_T(M) \end{tikzcd}$$

As x_1 forms a J, T-pS^+-sequence with respect to M we have $x_1H^2_T(M) = 0$. This shows that α vanishes. By the sequence $0 \to L^nM \to D_J(L^nM) \to H^1_J(L^nM) \to 0$ we obtain exact sequence $H^2_T(L^nM) \xrightarrow{\beta} H^2_T(D_J(L^nM)) \to H^2_T(H^1_J(L^nM)) = 0$, which shows that β is onto. Now the diagram

$$\begin{tikzcd} H^2_T(D_J(L^nM)) \arrow[rr] & & H^2_T(L^{n-1}M) \\ & H^2_T(L^nM) \arrow[ul, "\beta"] \arrow[ur, "\alpha = 0"'] & \end{tikzcd}$$

allows to conclude.

Next we treat the case $n = 1$. By the above we know the result if $r = 1$. Let $r > 1$. Using (2.9) (ii) (iii) and putting $L' = (x_2, \ldots, x_r)R$ and $M' = M/x_1M$ we have the following diagram with exact rows and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ & D_J(M) \arrow[r, "\cong"'] \arrow[d, "x_1"'] & D_J(x_1M) \arrow[d] & & \\ 0 \arrow[r] & D_J(LM) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \overline{M/LM^J} \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & D_J(L'M') \arrow[r] \arrow[d] & \overline{M'^J} \arrow[r] \arrow[d] & \overline{\overline{M'^J}/L'\overline{M'^{J}}^J} \arrow[r] & 0 \\ & 0 & 0 & & \end{tikzcd}$$

Applying $H_T(\cdot)$ to this diagram we obtain the following situation

$$\begin{tikzcd}
H_T^2(D_J(M)) \arrow[r] \arrow[d, phantom, "\cong"] & H_T^2(D_J(LM)) \arrow[r] \arrow[d] & H_T^2(D_J(L'M')) \arrow[d] \\
H_T^2(D_J(x_1M)) \arrow[r] & H_T^2(M) \arrow[r] & H_T^2(\overline{M'^J})
\end{tikzcd} \qquad \text{(rows exact!)}$$

By the case $r=1$ the first map in the second row vanishes, so that the second map in this row is mono. By induction the last vertical map vanishes. So the vertical arrow in the middle is trivial. But this proves our claim.

Finally we treat the case $r, n > 1$ by double induction, starting with the following commutative diagram with exact rows and columns and splitting last column (s. (2.9))

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D_J(L^{n-1}M) \arrow[r] \arrow[d, "x_1^*"] & L^{n-2}M \arrow[r] \arrow[d, "x_1^*"] & L^{n-2}M/D_J(L^{n-1}M) \arrow[r] \arrow[d, "x_1^*"] & 0 \\
0 \arrow[r] & D_J(L^nM) \arrow[r] \arrow[d] & L^{n-1}M \arrow[r] \arrow[d] & L^{n-1}M/D_J(L^nM) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_J(L'^nM') \arrow[r] \arrow[d] & L'^{n-1}M' \arrow[r] \arrow[d] & L'^{n-1}M'/D_J(L'^nM') \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \qquad (*)$$

Applying $H_T(\cdot)$ to (*) we get the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
H_T^2(D_J(L^{n-1}M)) \arrow[r] \arrow[d] & H_T^2(L^{n-2}M) \arrow[r] \arrow[d] & H_T^2(L^{n-2}M/D_J(L^{n-1}M)) \arrow[d] \\
H_T^2(D_J(L^nM)) \arrow[r] \arrow[d] & H_T^2(L^{n-1}M) \arrow[r] \arrow[d] & H_T^2(L^{n-1}M/D_J(L^nM)) \arrow[d] \\
H_T^2(D_J(L'^nM')) \arrow[r] & H_T^2(L'^{n-1}M') \arrow[r] & H_T^2(L'^{n-1}M'/D_J(L'^nM')) \arrow[d] \\
 & & 0
\end{tikzcd}$$

By induction the second map in the first row and the second map in the last row are injective. So the second map in the middle row is injective too, which proves our claim.

(3.7) COROLLARY: *In the notations and under the hypotheses of* (3.6) (*and setting* $L'=(x_2,\ldots,x_r)$, $M'=M/x_1M$) *the natural maps* $D_T(D_J(L^nM)/D_J(L^{n+h}M))\to D_T(D_J(L'^nM')/D_J(L'^{n+h}M'))$ *are onto for all* $n, h>0$ *and all* $r>0$.

Proof. By the sequence (1.13) it suffices to show that the maps $H^1_T(D_J(L^nM)/D_J(L^{n+h}M))\xrightarrow{\pi_n}H^1_T(D_J(L'^nM')/D_J(L'^{n+h}M'))$ are onto. For $r=1$ the right hand term vanishes. So we also may assume that $r>1$. Clearly we only have to consider the case $\Gamma_J(M)=0$.

Consider the diagram (*) of the proof of (3.6) (whose last column splits) and apply H_T. So, making use of (3.6) we obtain the following diagram (for each $n>1$)

$$\begin{tikzcd}
H^1_T(L^{n-1}M/D_J(L^nM)) \arrow[r] \arrow[d] & H^2_T(D_J(L^nM)) \arrow[r] \arrow[d] & 0 \\
H^1_T(L'^{n-1}M'/D_J(L'^nM')) \arrow[r] \arrow[d] & H^2_T(D_J(L'^nM')) \arrow[r] & 0 \\
0 & &
\end{tikzcd}$$

These diagrams show that the induced map $H^2_T(D_J(L^{n+h}M))\xrightarrow{\varepsilon}H^2_T(D_J(L'^{n+h}M'))$ is surjective for all $n, h>0$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & D_J(L^{n+h}M) \arrow[r, "\iota"] \arrow[d] & D_J(L^nM) \arrow[r] \arrow[d] & D_J(L^nM)/D_J(L^{n+h}M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_J(L'^{n+h}M') \arrow[r, "\iota'"] & D_J(L'^nM') \arrow[r] & D_J(L'^nM')/D_J(L'^{n+h}M') \arrow[r] & 0
\end{tikzcd} \tag{**}$$

Making use of the diagram

$$\begin{tikzcd}
D_J(L^{n+h}M) \arrow[rr, "\iota"] \arrow[dr] & & D_J(L^nM) \\
& L^{n+h-1}\overline{M}^J \arrow[ur] &
\end{tikzcd}$$

and of (3.6) we see that $H^2_T(\iota)=0$. Clearly it also holds $H^2_T(\iota')=0$. By (1.12) it holds $D_T(D_J(L^nM))=D_J(L^nM)$, so that (1.13) induces $H^1_T(D_J(L^nM))=0$. By the same argument we have $H^1_T(D_J(L'^nM'))=0$. So, applying H_T to (**) we obtain finally the situation:

$$(3.8)\qquad \begin{array}{l} H^1_T(D_J(L^nM)/(D_J(L^{n+h}M)) \overset{\cong}{\twoheadrightarrow} H^2_T(D_J(L^{n+h}M)) \\ H^1_T(D_J(L'^nM')/D_J(L'^{n+h}M')) \overset{\cong}{\twoheadrightarrow} H^2_T(D_J(L'^{n+h}M')) \end{array}$$

So π_n is an epimorphism.

(3.9) COROLLARY. *In the notations and under the hypotheses of* (3.6) *it holds*

(i) $H^1_T(D_J(L^nM))=0$ *for all* $n>0$,
(ii) $\operatorname{ann}(H^1_T(\overline{M/LM^J}))\cdot H^2_T(D_J(L^nM))=0$ *for all* $n>0$,
(iii) $D_J(L)\cdot H^2_T(D_J(L^nM))=0$ *for all* $n>0$.

Proof. Using the sequences $0\to D_J(L^nM)\to L^{n-1}\overline{M^J}\to L^{n-1}M/D_J(L^nM)\to 0$ and (3.6) we get epimorphism $H^1_T(D_J(L^nM))\to H^2_T(D_J(L^nM))\to 0$. So (2.8) shows that $H^2_T(D_J(L^nM))$ is a homomorphic image of copies of $H^1_T(\overline{M/LM^J})$. This proves (ii). (iii) is implied by the obvious fact $D_J(L)\cdot\overline{M/LM^J}=0$. (i) has been observed in the proof of (3.7).

(3.10) PROPOSITION. *Let* $(R,\mathfrak{m})$ *be local, universally catenary, of pure dimension* $d\geq 2$ *and such that* $R/\mathfrak{m}$ *is algebraically closed. Let* $J\subseteq R$ *be an ideal such that* $\dim(R/J)=1$ *and such that* $\operatorname{Spec}(R)-V(J)$ *is* CM. *Assume that* $e_{J,\mathfrak{m}}(R)=d-1$ *and choose* $\mathfrak{a}\subseteq R$ *according to* (3.4). *Let* $x_1,\dots,x_{d-2}\in\mathfrak{a}$ *be a partial system of parameters and put* $L=(x_1,\dots,x_{d-2})R$ *and* $\tilde{L}=D_J(L)$ $(=\bigcup_j(LR^J:J^j)_{\overline{R^J}})$. *Then* $D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L}))$ *is finitely generated as a* $\operatorname{Gr}(L)$*-module and* $\operatorname{Proj}(D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L})))$ *is a* CM*-scheme.*

Proof. By our hypotheses $x_1,\dots,x_{d-2}$ form a $J,\mathfrak{m}$-pS^+-sequence with respect to R (s. (2.1) (i) and (3.4)). Clearly we may replace R by $\overline{R^J}$ thus assuming that $\Gamma_J(R)=0$. $\operatorname{Gr}(\tilde{L})$ is given by $\mathfrak{R}(\tilde{L})/D_J(L\mathfrak{R}(L))$ (see (2.11)(i)). $\mathfrak{R}(\tilde{L})$ is a $\mathfrak{R}(L)$-submodule of $D_J(\mathfrak{R}(L))=\oplus_n D_J(L^n)$. But $D_J(\mathfrak{R}(L))$ is known to be a finitely generated module over $\mathfrak{R}(L)$ in our situation [4]. So $\operatorname{Gr}(\tilde{L})$ is finite over $\operatorname{Gr}(L)$. Therefore, to show that $D_{\mathfrak{m}}(\operatorname{Gr}(\tilde{L}))$ is a finitely generated module over $\operatorname{Gr}(L)$ it suffices to find an element $a\in R$, which is regular with respect to $Gr(L)$ (thus with respect to $D_{\mathfrak{m}}(Gr(\tilde{L}))$) and which satisfies $a\cdot D_{\mathfrak{m}}(Gr(\tilde{L}))\subseteq Gr(\tilde{L})$. As $x_1,\dots,x_{d-2}$ form a $J,\mathfrak{m}$-pS^+-sequence with respect to R we have $\sqrt{\operatorname{ann}}(H^1_{\mathfrak{m}}(\overline{R/L^J})\supseteq J$. Moreover it holds $\operatorname{Ass}(\overline{R/L^J})\cap V(J)=\emptyset$. These facts allow to find an element

$a \in R$ which is regular with respect to $\overline{R/L^J}$ (thus with respect to $\mathrm{Gr}(\tilde{L})$) and which belongs to $\mathrm{ann}(H^1_{\mathfrak{m}}(\overline{R/L^J}))$. By (3.9)(ii) we therefore $aH^2_{\mathfrak{m}}(D_J(L^{n+1}))=0$ for all $n>0$. By the isomorphism $H^1_{\mathfrak{m}}(D_J(L^n)/D_J(L^{n+1}))=H^2_{\mathfrak{m}}(D_J(L^{n+1}))$, (3.8), we now may conclude that $a \cdot H^1_{\mathfrak{m}}(\mathrm{Gr}(\tilde{L}))=a\cdot[H^1_{\mathfrak{m}}(\overline{R/L^J})\oplus_{n\geqslant} H^1_{\mathfrak{m}}(D_J(L^n)/D_J(L^{n+1}))]=0$. By (1.13) we see that a is of the requested type. It remains to show that $\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L})))$ is a CM-scheme. We do this by induction on d. First let $d=2$. Then (as $\tilde{L}=0$) we have $\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L})))=\mathrm{Spec}\,(D_{\mathfrak{m}}(R))$. By the above arguments the morphism $\mathrm{Spec}\,(D_{\mathfrak{m}}(R))\to\mathrm{Spec}\,(R)=\mathrm{Proj}\,(\mathrm{Gr}\,(L))$ is finite. So it suffices to proof that $D_{\mathfrak{m}}(R)$ (which is a finitely generated R-module) is a CM-module over R. But this is clear as $H^i_{\mathfrak{m}}(D_{\mathfrak{m}}(R))=0$ for $i<2$ (1.14).

Now let $d>2$. Put $X=\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(L)))$ and let $q\in X$ be a closed point. We want to show that $\mathcal{O}_{X,q}$ is a CM-ring. Let $\mathfrak{q}\in\mathrm{Spec}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(L)))$ be the (homogeneous) prime ideal which corresponds to the point q ($\mathfrak{q}$ is essential and satisfied $\dim\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(L))/\mathfrak{q})=1$). Let $\mathfrak{p}\subseteq\mathrm{Gr}\,(L)$ be the retraction of $\mathfrak{q}$. By the previously proved finiteness of $X\to\mathrm{Proj}\,(\mathrm{Gr}\,(L))$, $\mathfrak{p}$ corresponds to a closed point p of the latter scheme. As $\mathfrak{p}\cap R=\mathfrak{m}$ and as $R/\mathfrak{m}$ is algebraically closed, the homogeneous Nullstellensatz guarantees the existence of an element y of $L-\mathfrak{m}L$ such that the induced one-form $\bar{y}\in\mathrm{Gr}\,(L)$ is contained in $\mathfrak{p}$. y clearly is a parameter. So we may assume without loss of generality that $y=x_1$. Let $R'=R/x_1R$ and $L'=(x_2,\ldots,x_{d-2})R'$. Then R' and $x_1'=x_1\cdot 1_{R'},\ldots,x_{d-2}'=x_{d-2}\cdot 1_{R'}$ satisfy again our hypotheses with $d-1$ instead of d. Let $\tilde{L}'=D_J(L')$. We may write $\tilde{L}'=\bigcup_j (L'\overline{R'^J}:J^j)_{\overline{R'^J}}$. According to (2.9) the canonical map $\tilde{L}=D_J(L)\to D_J(L')=\tilde{L}'$ is onto. So $\tilde{L}'$ is the image of $\tilde{L}$ under the canonical map $R\to\overline{R'^J}$. So we have a canonical projection $\psi:\mathrm{Gr}(\tilde{L})\to\mathrm{Gr}\,(\tilde{L}')$ which is given in positive degrees by the maps $D_J(L^n)/D_J(L^{n+1})\to D_J(L'^n)/D_J(L'^{n+1})$ which occur in (2.9)(ii). Using (2.9)(ii), (3.7) and the left-exactness of $D_{\mathfrak{m}}$ we obtain an exact sequence of graded $\mathfrak{R}(L)$-modules: $0\to D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L}))_{>0}\xrightarrow{x_1}D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L}))_{>1}\to D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L}'))_{>1}\to 0$. So $X':=\mathrm{Proj}\,(D_{\mathfrak{m}}(\mathrm{Gr}\,(L'))$ is a closed subscheme of X, which contains q and whose ideal of vanishing sections in $\mathcal{O}_X$ is the invertible ideal defined by the (non-degenerate) one-form $\overline{x_1}$. By induction $\mathcal{O}_{X',q}$ is a CM-ring. So the same holds for $\mathcal{O}_{X,q}$.

(3.11) COROLLARY. *Keep the notations and hypotheses of* (3.10), *assuming moreover that* $d\geqslant 3$. *Let* $X_0=\mathrm{Proj}\,(\mathfrak{R}(\tilde{L}))$ *and let* $y\in\mathfrak{m}$ *be such that* $ht(L,y)=d-1$ *and* $yH^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0})=0$. *Let* $\mathcal{K}_0\subseteq\mathcal{O}_{X_0}$ *be the ideal* $\bigcup_j [(\tilde{L},y)\mathcal{O}_{X_0}:\mathfrak{m}^j]_{\mathcal{O}X_0}$. *Then* (i) $\mathcal{K}_0$ *is normally torsion-free and without embedded component.* (ii) *The blow up* $Y_0=\mathrm{Proj}\,(\mathfrak{R}(\mathcal{K}_0))=\mathrm{Bl}_{X_0}(\mathcal{K}_0)$ *of* X_0 *at* $\mathcal{K}_0$ *is a* CM-scheme.

Proof. Let $p\in X_0$ be a closed point and put $B=\mathcal{O}_{X_0,p}$. Then $\mathfrak{m}B\neq B$. As $\tilde{L}\mathcal{O}_{X_0}$ is invertible (being the exceptional divisor of the blow-up $X_0\to\mathrm{Spec}\,(R)$), there is a regular element $t \in \mathfrak{m}B$ such that $\tilde{L}\mathcal{O}_{X_0,p} = tB$. Gr $(\tilde{L})$ is a finite and torsion-free extension of the ring Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$, which is canonically isomorphic to the polynomial ring $\bar{R}/\tilde{L}[X_1, \ldots, X_{d-2}]$ (for the finiteness statement see (3.10), for the shape of Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$ see (2.7) and (2.8)). As L is generated by a partial system of parameters the embedded members of Ass (R/L) may not belong to CM (R) and so must belong to $V(J)$. Therefore we get by (2.11)(ii) that Ass $(R/\tilde{L}) =$ Min $(R/\tilde{L})$. This shows that y is regular with respect to $R/\tilde{L}$, thus with respect to Gr $(L)/\Gamma_J(\mathrm{Gr}(L))$, thus with respect to Gr $(\tilde{L})$. This shows in particular that t, y form a regular sequence with respect to B. Moreover it holds $(\mathcal{K}_0)_p = \bigcup_i ((t, y)B : \mathfrak{m}^i)_B := \tilde{K} \subseteq B$.

We have to show that $\tilde{K}$ is normally torsion-free, without embedded component and that $Y_1 := \mathrm{Proj}\,(\mathfrak{R}(\tilde{K}))$ is a CM-scheme.

To prove the first claim, observe that $H^0_{\mathfrak{m}B}(B) = H^1_{\mathfrak{m}B}(B) = 0$, as t, y is a regular sequence contained in $\mathfrak{m}B$. Note also, that $D_\mathfrak{m}(B/tB)$ is the semilocal ring of the (finitely many and closed) preimage points of p under the finite morphism Proj $(D_\mathfrak{m}(\mathrm{Gr}(\tilde{L})) \to \bar{X}_0 := \mathrm{Proj}\,(\mathrm{Gr}(\tilde{L}))$. So – by (3.10) – $D_\mathfrak{m}(B/tB)$ is a CM-module over B/tB which is of finite type. In particular we see that $H^1_{\mathfrak{m}B}(B/tB) = H^1_\mathfrak{m}(B/tB)$ is finitely generated (1.13). On the other side the isomorphisms $H^1_\mathfrak{m}(\tilde{L}^n/\tilde{L}^{n+1}) \cong H^2_\mathfrak{m}(\tilde{L}^{n+1})$ (s. (3.8)) give rise to an isomorphism $H^1_\mathfrak{m}(\mathfrak{R}(\tilde{L})/\tilde{L}\mathfrak{R}(\tilde{L})) \cong H^2_\mathfrak{m}(\tilde{L}\mathfrak{R}(\tilde{L}))$, thus to an isomorphism $H^1_{\mathfrak{m}B}(B/tB) \cong H^2_{\mathfrak{m}B}(tB) \cong H^2_{\mathfrak{m}B}(B)$. So $H^2_{\mathfrak{m}B}(B)$ is finitely generated and we obtain $e_{\mathfrak{m}B}(B) \geq 3$. Moreover, by our choice of y, we have $yH^1_{\mathfrak{m}B}(B/tB) = 0$. So t, y form a $\mathfrak{m}B$-S^+-sequence with respect to B. As $tH^1_{\mathfrak{m}B}(B/tB) = 0$ we get by the above isomorphism, that $tH^2_{\mathfrak{m}B}(B) = 0$. Applying $H^{\bullet}_{\mathfrak{m}B}$ to the exact sequence $0 \to B \xrightarrow{y} B \to B/yB \to 0$ we moreover obtain an injection $H^1_{\mathfrak{m}B}(B/yB) \hookrightarrow H^2_{\mathfrak{m}B}(B)$ which induces $tH^1_{\mathfrak{m}B}(B/yB) = 0$. So y, t form a $\mathfrak{m}B$-S^+-sequence with respect to B. This shows that t, y forms a $\mathfrak{m}B$-pS^+-sequence with respect to B. By (2.12) we now get that $\tilde{K}$ is normally torsion-free. Note that there is a canonical embedding $B/\tilde{K} = \overline{(B/tB)/y(B/tB)}^\mathfrak{m} = (B/tB)/(B/tB) \cap yD_\mathfrak{m}(B/tB) \subseteq D_\mathfrak{m}(B/tB)/yD_\mathfrak{m}(B/tB) =: U$. As $D_\mathfrak{m}(B/tB)$ is a CM-module, Ass (U) has no embedded members. So the same is true for Ass $(B/\tilde{K})$.

It remains to show that Y_1 is CM. This is done by induction on d. If $d = 3$, we have $\tilde{L} = x_1 DJ(R)$, hence $X_0 = \mathrm{Proj}\,(\mathfrak{R}(x_1 D_j(R)) = \mathrm{Proj}\,(\bigoplus_{n \geq 0} x_1^n D_J(R)) = \mathrm{Proj}\,(D_J(R)[X]) = \mathrm{Spec}\,(D_J(R))$. As $D_J(R)$ is a finitely generated R-module (observe that $H^1_J(R)$ is, finitely generated and use (1.13)) and as B is a localization of $D_JJ(R)$ in one of its maximal ideals, $\mathfrak{m}B$ is primary to the maximal ideal $\mathfrak{n}$ of B. So t, y is a $\mathfrak{n}$-pS^+-sequence with respect to the 3-dimensional ring B. Therefore we see by (2.13) that Y_1 is CM.

Finally let $d > 3$. Using the homogeneous Nullstellensatz we may assume as in the proof of (3.10) that the one-form $x_1^* \in \mathfrak{R}(L)$ defines a closed subscheme of

Proj $(\mathfrak{R}(L))$, which contains the image point of p. Put $R' = R/x_1R$, $x_i' = x_i \cdot 1_{R'}$, $y' = y \cdot 1_{R'}$, $L' = (x_2', \ldots, x_{d-2}')R'$, $\widetilde{L'} = D_J(L')$ and $X_0' = \mathrm{Proj}\,(\mathfrak{R}(L'))$ and $\mathcal{K}_0' = \bigcup_i ((L', y')\mathcal{O}_{X_0} : \mathfrak{m}^i)\mathcal{O}_{X_0}$. The epimorphisms π_n in the proof of (3.7) give rise to an epimorphism $H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) \to H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/L'\mathcal{O}_{X_0})$, which gives $y' \cdot H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\widetilde{L'}\mathcal{O}_{X_0})$. This shows that the dashed objects satisfy again our hypotheses with $d-1$ instead of d. The exact sequence $0 \to \mathfrak{R}(\tilde{L})_{>0} \xrightarrow{x^*} \mathfrak{R}(\tilde{L})_{>1} \to \mathfrak{R}(\tilde{L})_{>1} \to 0$ (2.9) (ii) shows that X_0' is a closed subscheme of X_0 which contains p and that the ideal sections of vanishing at X_0 is the invertible ideal of $\mathcal{O}_{X_0}$ induced by x_1^*. Let $w \in B$ be a generator of this latter ideal in the point p (wB = stalk at p) and set $B' = \mathcal{O}_{X_0', p}$. Then the above sequence gives rise to an exact sequence $0 \to B \xrightarrow{w} B \to B' \to 0$ and shows that $\widetilde{L'}\mathcal{O}_{X_0, p} = tB'$. Clearly we have $(\mathcal{K}_0')_p = \bigcup_i ((t, y)B' : \mathfrak{m}^i)_{B'} := \widetilde{K'}$. We now want to verify two properties of $\tilde{K}$. The first of them is that $\widetilde{K'} = \tilde{K} \cdot B'$. As t, y form a B-pS^+-sequence with respect to B we may write $\tilde{K} = D_{\mathfrak{m}B}((t, y)B)$ by (2.7) (iii). Similarly we may write $\widetilde{K'} = D_{\mathfrak{m}B'}((t, y)B')$. By (2.9)(ii) there is a canonical epimorphism $\tilde{K} = D_{\mathfrak{m}B}((t, y)B) \overset{\pi}{\twoheadrightarrow} D_{\mathfrak{m}B}(y \cdot (B/tB)) = yD_{\mathfrak{m}}(B/tB)$. Similarly there is a canonical epimorphism $\widetilde{K'} \to yD_{\mathfrak{m}}(B'/tB')$. The epimorphisms $D_{\mathfrak{m}}(\tilde{L}^n/\tilde{L}^{n+1}) \to D_{\mathfrak{m}}(\widetilde{L'}^n/\widetilde{L'}^{n+1})$ (3.7) give rise to an epimorphism $D_{\mathfrak{m}}(\mathrm{Gr}\,(\tilde{L})) \twoheadrightarrow D_{\mathfrak{m}}(\mathrm{Gr}\,(L'))$ hence to an epimorphism $D_{\mathfrak{m}}(B/tB) \twoheadrightarrow D_{\mathfrak{m}}(B'/tB')$, thus finally to a surjection $\tilde{K}/tB \twoheadrightarrow \widetilde{K'}/tB'$. As $tB' \subset \tilde{K} \cdot B'$, the canonical map $B \to B'$ gives rise to an epimorphism $\tilde{K} \to \widetilde{K'}$, which is our claim. The second property of $\tilde{K}$ in which we are interested is that w is regular with respect to $B/\tilde{K}^n$ for all $n > 0$. We have seen above that t, y form a regular sequence with respect to B. By the same argument applied to the dashed objects we see that they are a regular sequence for B'. As w is regular this shows that w is regular with respect to $B/(t, y)$, thus with respect to $B/\tilde{K}$. As $\tilde{K}$ is normally torsion-free, this gives our claim. The second property of $\tilde{K}$ means that $wB \cap \tilde{K}^n = w\tilde{K}^n$ for all $n > 0$. So, together with the first property we obtain an exact sequence $0 \to \mathfrak{R}(\tilde{K})_{>0} \xrightarrow{w} \mathfrak{R}(\tilde{K})_{>0} \to \mathfrak{R}(\widetilde{K'})_{>0} \to 0$, which induces another exact sequence: $0 \to \mathcal{O}_{Y_1} \xrightarrow{w} \mathcal{O}_{Y_1} \to \mathcal{O}_{Y_1'} \to 0$, where $Y_1' = \mathrm{Proj}\,(\mathfrak{R}(K'))$. $\mathcal{O}_{Y_1'}$ is CM. So the same holds for $\mathcal{O}_{Y_1}$. This completes our proof.

4. The Globalization

In this section we shall use the following notations: If $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded algebra over a field k, we write $\mathrm{Reg}_+(A)$, $\mathrm{Nor}_+(A)$, $\mathrm{CM}_+(A)$, $\mathrm{Fac}_+(A)$, $\mathrm{Sing}_+(A)$ for the corresponding loci of Proj (A). If R is noetherian and if $\mathfrak{a} \subseteq R$ is an ideal $V(\mathfrak{a})$ denotes the closed subset $\{\mathfrak{p} \in \mathrm{Spec}\,(R) \mid \mathfrak{p} \supseteq \mathfrak{a}\}$, $U(\mathfrak{a})$ the open set $\mathrm{Spec}\,(R) - V(\mathfrak{a})$. If A is as above and if $\mathfrak{a} \subseteq A$ is a homogeneous ideal, $V_+(\mathfrak{a})$ and

$U_+(\mathfrak{a})$ stand for the closed respectively open subset of Proj (A) which are induced by $V(\mathfrak{a})$ resp. $U(\mathfrak{a})$.

We begin with the following result, which is of Bertini-type:

(4.1) LEMMA. *Let k be an infinite field and let $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded k-algebra which is generated over k by finitely many of its one-forms* (*as an algebra*). *Let $\mathfrak{a} \subseteq A$ be a homogeneous ideal and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in U(\mathfrak{a})$ Then there is a form $f \in \mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$ such that* (i) $\mathrm{Reg}_1(A/fA) \supseteq \mathrm{Proj}(A/fA) \cap \mathrm{Reg}_+(A) \cap U_+(\mathfrak{a})$, (ii) $\mathrm{Nor}_+(A/fA) \supseteq \mathrm{Proj}(A/fA) \cap \mathrm{Nor}_+(A) \cap U_+(\mathfrak{a})$.

Proof. We find forms $f_0, \ldots, f_n \in \mathfrak{a}$ of some degree N such that $\sqrt{\mathfrak{a}} = \sqrt{(f_0, \ldots, f_n)}$. So we may replace $\mathfrak{a}$ by $(f_0, \ldots, f_n)$. Assume first that k is of characteristic 0. Then for general $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^{n+1}$ we have $f_\alpha := \sum_{i=0}^{n} \alpha_i f_i \notin \mathfrak{p}_j$ $(j = 1, \ldots, s)$. Here, general means for all α outside the union of finitely many proper linear subspaces. Combining this with [9, (5.4)] we get the requested statement. So let k be of characteristic $p > 0$. Then we have $\sqrt{\mathfrak{a}} = \sqrt{(f_0^p, \ldots, f_n^p)}$. Therefore we may replace $\mathfrak{a}$ by $(f_0^p, \ldots, f_n^p)$, thus assuming additionally that the generators f_i are pth powers of the same degree. Now consider the local ring $B := A_{A_+}$, where A_+ denotes the homogeneous maximal ideal $A_{>0}$ of A. Embed k into the completion $\hat{B}$ of B and let $\hat{B} \xrightarrow{\hat{d}} \hat{\Omega}_{\hat{B}/k} =: \hat{\Omega}$ be the corresponding universal finite differential [19]. Let $x_0, \ldots, x_t$ be a k-basis of A_1. Then, as $\hat{d}(x_0), \ldots, \hat{d}(x_t)$ generate $\hat{\Omega}$, $\hat{d}(f_0), \ldots, \hat{d}(f_n)$, $\hat{d}(f_0x_0)$, $\hat{d}(f_0x_1), \ldots, \hat{d}(f_ix_j), \ldots, \hat{d}(f_nx_t)$ generate $\hat{\Omega}_{\hat{\mathfrak{p}}}$ for each $\hat{\mathfrak{p}} \in U(\mathfrak{a}\hat{B})$. As f_i is a pth power, we have $\hat{d}(f_i) = 0$ for $i = 0, \ldots, n$. So $\hat{d}(f_ix_j)$ $(0 \leq i \leq n;\ 0 \leq j \leq t)$ generate $\hat{\Omega}_{\hat{\mathfrak{p}}}$ for all $\hat{\mathfrak{p}}$ as above.

Let $\hat{\mathfrak{p}}_i \in \mathrm{Spec}(\hat{B})$ such that $\hat{\mathfrak{p}}_i \cap A = \mathfrak{p}_i$. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_h\}$ be the set of those primes $\mathfrak{p}$ of $U(\mathfrak{a}B)$ for which $B_\mathfrak{p}$ satisfies the second Serre-property S_2 and for which $\mathrm{depth}(B_\mathfrak{p}) = 2 < \dim(B_\mathfrak{p})$ (as B is excellent, we have in fact only finitely many such primes [9, (3.2)]). Let $\hat{\mathfrak{q}}_i \in \mathrm{Spec}(\hat{B})$ be such that $\hat{\mathfrak{q}}_i \cap B = \mathfrak{q}_i$. Finally let $\mathfrak{r}_1, \ldots, \mathfrak{r}_l$ be the minimal primes of $\mathrm{Nor}(B) \cap U(\mathfrak{a}B) \cap \mathrm{Sing}(B)$ (which is closed in $\mathrm{Nor}(B) \cap U(\mathfrak{a}B)$ as B is excellent). Let $\hat{\mathfrak{r}}_i \in \mathrm{Spec}(\hat{B})$ such that $\hat{\mathfrak{r}}_i \cap B = \mathfrak{r}_i$. Applying [9, (1.5)] to $\hat{B}$ with $S = k$, $M = \hat{\Omega}$, $U = U(\mathfrak{a}\hat{B})$ and observing the complement to the quoted result we find elements $\alpha_{ij} \in k$ $(0 \leq i \leq n;\ 0 \leq j \leq t)$ such that $f := \sum_{i,j} \alpha_{ij} f_i x_j$ does not belong to the symbolic square $\hat{\mathfrak{p}}^{(2)}$ for any of the primes $\hat{\mathfrak{p}} \in U(\mathfrak{a}\hat{B})$ and such that $f \notin \hat{\mathfrak{p}}_1, \ldots, \hat{\mathfrak{p}}_s, \hat{\mathfrak{q}}_1, \ldots, \hat{\mathfrak{q}}_h, \hat{\mathfrak{r}}_1, \ldots, \hat{\mathfrak{r}}_l$. By our choice of the elements f_i, f clearly is a form. According to [9] (pg. 103, proof of (2.1) and pg. 105, proof of (3.3)) f satisfies the properties: $\mathrm{Reg}(B/fB) \supseteq \mathrm{Spec}(B/fB) \cap \mathrm{Reg}(B) \cap U(\mathfrak{a}B)$, $\mathrm{Nor}(B/fB) \supseteq \mathrm{Spec}(B/fB) \cap \mathrm{Nor}(B) \cap U(\mathfrak{a}B)$. Noticing that the canonical morphism $\mathrm{Spec}(B) - \{\mathfrak{m} := A_+ \cdot B\} \to \mathrm{Proj}(A)$ transforms Reg (Spec

$(B)-\{\mathfrak{m}\})$ to $\mathrm{Reg}_+(A)$, $\mathrm{Nor}(\mathrm{Spec}(B)-\{\mathfrak{m}\})$ to $\mathrm{Nor}_+(A)$ etc. [9, (5.1)], we are done.

(4.2) COROLLARY. *Let X be a projective variety over the algebraically closed field k. Let $Z\subseteq X$ be a closed subset of codimension $h>0$. Let $\mathscr{I}\subseteq\mathcal{O}_X$ be any coherent ideal such that $Z=V(\mathscr{I})$. Then there is a complete intersection ideal $\mathscr{L}\subseteq\mathscr{I}$ of codimension $h-1$ such that:*

(i) $\mathscr{L}_p\subseteq\mathcal{O}_{X,p}$ *is reduced for all* $p\in(X-Z)\cap\mathrm{CM}(X)$,

(ii) $\mathrm{Reg}(V(\mathscr{L}))\supseteq\mathrm{Reg}(X)\cap V(\mathscr{L})\cap(X-Z)$,

(iii) $\mathrm{Nor}(V(\mathscr{L}))\supseteq\mathrm{Nor}(X)\cap V(\mathscr{L})\cap(X-Z)$.

Proof. Let A be the homogeneous coordinate ring of X and let $\mathfrak{a}\subseteq A$ be the homogeneous ideal which corresponds to $\mathscr{I}$. We have to find forms $f_1,\dots,f_{h-1}\in\mathfrak{a}$ such that:

(i) $(f_1,\dots,f_{h-1})A$ is reduced for any homogeneous prime $\mathfrak{p}\in U(\mathfrak{a})$ for which $A_\mathfrak{p}$ is CM,

(ii) $\mathrm{Reg}_+(A/(f_1,\dots,f_{h-1}))\supseteq\mathrm{Reg}_+(A)\cap\mathrm{Proj}(A/(f_1,\dots,f_{h-1}))\cap U_+(\mathfrak{a})$,

(iii) $\mathrm{Nor}_+(A/(f_1,\dots,f_{h-1}))\supseteq\mathrm{Nor}_+(A)\cap\mathrm{Proj}(A/(f_1,\dots,f_{h-1}))\cap U_+(\mathfrak{a})$.

(iv) $ht(f_1,\dots,f_{h-1})=h-1$.

We construct these forms by induction on h. Thereby we only assume that $A=k\oplus A_1\oplus A_2\oplus\cdots$ is a graded finitely generated algebra over an infinite field k, that $A=k[A_1]$ and that $A_\mathfrak{p}$ is reduced whenver $\mathfrak{p}$ belongs to $\mathrm{CM}(A)\cap U(\mathfrak{a})$.

The case $h=1$ is trivial. So let $h>1$. $\mathrm{Sing}(A)$ is closed. So there is an ideal $\mathfrak{b}\subseteq A$ such that $\mathrm{Reg}(A)=U(\mathfrak{b})$. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ be the minimal primes of A. As $ht(\mathfrak{a})=h>0$, they all belong to $U(\mathfrak{a})$. Clearly they also belong to $\mathrm{CM}(A)$. So $A_\mathfrak{p}$ is reduced for $i=1,\dots,t$. This shows that $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ belong to $\mathrm{Reg}(A)=U(\mathfrak{b})$. So there is an element $c\in\mathfrak{b}-\bigcup_{i=1}^t\mathfrak{p}_i$. In particular $U(cA)$ belong to $\mathrm{Reg}(A)$. Now let $\mathfrak{p}_{t+1},\dots,\mathfrak{p}_s$ be those minimal prime divisors of cA which are homogeneous (We do not exclude the case $s\leq t$ in which there are no such primes). Clearly $ht(\mathfrak{p}_i)=1$ for $t<i\leq s$. This shows that $\mathfrak{p}_{t+1},\dots,\mathfrak{p}_s$ belong to $U(\mathfrak{a})$. So, according to (4.1) there is a form $f\in\mathfrak{a}-\mathfrak{p}_1\cup\cdots\cup\mathfrak{p}_s$ such that:

$$(*)\quad\begin{cases}\mathrm{Reg}_+(A/fA)\supseteq\mathrm{Reg}_+(A)\cap\mathrm{Proj}(A/fA)\cap U_+(\mathfrak{a}),\\ \mathrm{Nor}_+(A/fA)\supseteq\mathrm{Nor}_+(A)\cap\mathrm{Proj}(A/fA)\cap U_+(\mathfrak{a}).\end{cases}$$

Let $\mathfrak{q}$ be a minimal prime divisor of fA. $\mathfrak{q}$ is homogeneous and of height one. This implies that $\mathfrak{q}\in U(\mathfrak{a})$ and that $\mathfrak{q}\in U(cA)$ ($as\notin U(cA)$ would imply $\mathfrak{q}=\mathfrak{p}_i$ for some $i\in\{t+1,\dots,s\}$). So $\mathfrak{q}$ belongs to $\mathrm{Reg}(A)\cap V(fA)\cap U(\mathfrak{a})$, thus corresponds to a generic point p of $\mathrm{Proj}(A/fA)$ which belongs to $\mathrm{Reg}_+(A)\cap U_+(\mathfrak{a})$. By (*) we have $p \in \text{Reg}_+(A/fA)$, thus $\mathfrak{q} \in \text{Reg}(A/fA)$. This shows that the minimal primes of A/fA belong to $\text{Reg}(A/fA)$.

Put $\bar{A} = A/fA$, $\bar{\mathfrak{a}} = \mathfrak{a} \cdot \bar{A}$. As $f \in \mathfrak{a} - \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$ we clearly have $ht(\bar{\mathfrak{a}}) = h - 1$. Let $\bar{\mathfrak{p}} \in U(\bar{\mathfrak{p}})$ be a homogeneous prime such that $\bar{A}_{\bar{\mathfrak{p}}}$ is CM. By the previous remark the minimal primary components of $\bar{A}_{\bar{\mathfrak{p}}}$ are in fact all prime. As $\bar{A}_{\bar{\mathfrak{p}}}$ is CM, it is unmixed. So $\bar{A}_{\bar{\mathfrak{p}}}$ is reduced. Therefore we may apply induction to the pair $\bar{A}, \bar{\mathfrak{a}}$ to find form $\bar{f}_2, \ldots, \bar{f}_{h-1} \in \bar{A}$ such that:

(i) $(\bar{f}_2, \ldots, \bar{f}_{h-1})$ is reduced for any homogeneous prime $\bar{\mathfrak{p}} \in U(\bar{\mathfrak{a}})$ such that $\bar{A}_{\bar{\mathfrak{p}}}$ is CM.

(ii) $\text{Reg}_+(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \supseteq \text{Reg}_+(\bar{A}) \cap \text{Proj}(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \cap U_+(\bar{\mathfrak{a}})$,

(iii) $\text{Nor}_+(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \supseteq \text{Nor}_+(\bar{A}) \cap \text{Proj}(\bar{A}/(\bar{f}_2, \ldots, \bar{f}_{h-1})) \cap U_+(\bar{\mathfrak{a}})$.

(iv) $ht(\bar{f}_2, \ldots, \bar{f}_{h-1}) = h - 2$.

Now, put $f_1 = f$ and let $f_2, \ldots, f_{h-1} \in A$ be forms which respectively lift $\bar{f}_2, \ldots, \bar{f}_{h-1}$. Let $\mathfrak{p} \in \text{U}(\mathfrak{a})$ be a homogeneous prime such that $A_{\mathfrak{p}}$ is CM. If $\bar{f}_1 \notin \mathfrak{p}$, we have $(f_1, \ldots, f_{h-1})A_{\mathfrak{p}} = A_{\mathfrak{p}}$, thus statement (i). If $f_1 \in \mathfrak{p}$, $\bar{\mathfrak{p}} = \mathfrak{p}\bar{A}$ belongs to $U(\bar{\mathfrak{a}})$. As $A_{\mathfrak{p}}$ is reduced $f_1 \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$ implies that f_1 is regular with respect to A. So $\bar{A}_{\bar{\mathfrak{p}}} = A_{\mathfrak{p}}/fA_{\mathfrak{p}}$ is a CM-ring. Therefore by (i) $(f_1, \ldots, f_{h-1})A_{\mathfrak{p}}/f_1A_{\mathfrak{p}} = (\bar{f}_2, \ldots, \bar{f}_{h-1})\bar{A}_{\bar{\mathfrak{p}}}$ is reduced. So $(f_1, \ldots, f_{h-1})A$ is reduced, which proves (i). (iv) follows by (iv), whereas (ii) and (iii) are induced by (ii), (iii) and (*).

(4.3) COROLLARY. *Let X be a quasiprojective variety over the algebraically closed field k. Let $Z \subseteq X$ be a closed subset of codimension $h > 0$ and such that $X - Z$ is* CM. *Let $\tilde{\mathcal{J}} \subseteq \mathcal{O}_X$ be a coherent ideal such that $Z = V(\tilde{\mathcal{J}})$. Then there is a locally complete intersection ideal $\mathcal{L} \subseteq \tilde{\mathcal{J}}$ of codimension $h - 1$ such that*

(i) *$\mathcal{L}_p \subseteq \mathcal{O}_{X,p}$ is reduced for all $p \in X - Z$,*

(ii) $\text{Reg}(V(\mathcal{L})) \supseteq \text{Reg}(X) \cap V(\mathcal{L}) \cap (X - Z)$,

(iii) $\text{Nor}(V(\mathcal{L})) \supseteq \text{Nor}(X) \cap V(\mathcal{L}) \cap (X - Z)$.

Proof. X may be written as an open dense subset of a projective variety $\bar{X}$. Let $\bar{Z} \subseteq \bar{X}$ be the closure of Z. Then $\bar{Z}$ also is of codimension h with respect to $\bar{X}$ (as X is dense in $\bar{X}$). As $\bar{Z} \cap X = Z$ there is a coherent ideal $\bar{\tilde{\mathcal{J}}} \subseteq \mathcal{O}_{\bar{X}}$ such that $V(\bar{\tilde{\mathcal{J}}}) = \bar{Z}$ and such that $\bar{\tilde{\mathcal{J}}}|_X = \tilde{\mathcal{J}}$. Now apply (4.2) to $\bar{X}$, $\bar{\tilde{\mathcal{J}}}$ and $\bar{Z}$ to get a complete intersection $\bar{\mathcal{L}} \subseteq \bar{\tilde{\mathcal{J}}}$ of codimension $h - 1$ such that:

(i) $\bar{\mathcal{L}}_{\bar{p}}$ is reduced for all $\bar{p} \in (\bar{X} - \bar{Z}) \cap \text{CM}(\bar{X})$,

(ii) $\text{Reg}(V(\bar{\mathcal{L}})) \supseteq \text{Reg}(\bar{X}) \cap V(\bar{\mathcal{L}}) \cap (\bar{X} - \bar{Z})$,

(iii) $\text{Nor}(V(\bar{\mathcal{L}})) \supseteq \text{Nor}(\bar{X}) \cap V(\bar{\mathcal{L}})\ (\bar{X} - \bar{Z})$.

Setting $\mathcal{L} = \bar{\mathcal{L}}|_X$ our statement follows as $(\bar{X} - \bar{Z}) \cap X = X - Z$, $\text{CM}(\bar{X}) \cap X = \text{CM}(X) \supseteq X - Z$.

(4.4) COMPLEMENT. *If in* (4.3) $X - Z = \text{CM}(X)$, (ii) *may be replaced by* (ii)′ $\text{Reg}(V(\mathcal{L})) \supseteq \text{Reg}(X) \cap V(\mathcal{L})$.

Proof. In this case we clearly have $\mathrm{Reg}(X) \subseteq X - Z$.

Let Y be a closed subset of a locally noetherian scheme X and let $\mathscr{J} \subseteq \mathcal{O}_X$ be the ideal of sections vanishing at Y. We say that Y is *normally torsion-free with respect to X* if $\mathscr{J}$ is normally torsion-free in $\mathcal{O}_X$.

Let $\mathcal{M} \subseteq \mathcal{O}_X$ be a coherent ideal. We say that $\mathcal{M}$ is *generically a complete intersection ideal* if $\mathcal{M}_p \subseteq \mathcal{O}_{X,p}$ is a complete intersection-ideal for each generic point p of $V(\mathcal{M})$ (which means that $\mathcal{M}_p$ may be generated by h elements, where $h = ht(\mathcal{M}_p)$). We say that Y is generically a complete intersection with respect to X if its ideal $\mathscr{J}$ of vanishing sections is generically a complete intersection. If X is an algebraic variety (or generally an excellent scheme) this is equivalent to saying that each irreducible component of Y leaves the singular locus $\mathrm{Sing}(X)$ of X.

(4.5) LEMMA. *Let X be a noetherian scheme and let $Y \subseteq X$ be a closed subset of pure codimension h which generically is a complete intersection with respect to X. Consider the blow-up* $\mathrm{Bl}_X(Y) = \mathrm{Proj}(\mathfrak{R}(\mathscr{J})) \xrightarrow{\pi} X$, *where $\mathscr{J} \subseteq \mathcal{O}_X$ is the ideal of sections vanishing at Y. Then it holds*

(i) $\mathrm{Reg}(\mathrm{Bl}_X(Y)) \supseteq \pi^{-1}((\mathrm{Reg}(X) \cap \mathrm{Reg}(Y)) \cup \mathrm{Reg}(X - Y))$.

If Y is normally torsion-free with respect to X it holds

(ii) $\mathrm{Nor}(\mathrm{Bl}_X(Y)) \supseteq \pi^{-1}(\mathrm{Nor}(X))$.

(iii) *If Y is moreover irreducible we also have*

$\mathrm{Fac}(\mathrm{Bl}_X(Y)) \supseteq \pi^{-1}\,\mathrm{Fac}(X))$.

Proof. The result is of local nature. So let $X = \mathrm{Spec}(R)$, $I \subseteq R$, where R is a noetherian local ring and where I is a reduced ideal, which is of pure codimension and generically a complete intersection. Now (i) is clear, as $\mathrm{Proj}(\mathfrak{R}(I))$ is regular if R and R/I are. To show (ii) assume that R is normal and that I is normally torsion-free. It suffices to show that $\mathrm{Proj}(\mathfrak{R}(I))$ is normal under these assumptions. In fact it holds even more, namely $\mathfrak{R}(I)$ is normal [6, (6.10)], so that $\mathrm{Proj}(\mathfrak{R}(I))$ is arithmetically normal. To prove (iii) we may assume that R is factorial, that I is prime and restrict ourselves to show that $\mathrm{Proj}(\mathfrak{R}(I))$ is locally factorial. Let $x_1, \ldots, x_t$ be a system of generators of I, which are $\neq 0$. Then $\mathrm{Proj}(\mathfrak{R}(I))$ has an affine open covering by the sets $\mathrm{Spec}\left(A_i := R\left[\frac{x_1}{x_i}, \ldots, \frac{x_t}{x_i}\right]\right)$ $(i = 1, \ldots, t)$. So it suffices to verify that the rings A_i are factorial. As I is normally torsion-free the prime divisors of the ideal x_iA_i retract all to the prime I in R (observe that $x_iA_i = \Gamma(\mathrm{Spec}(A_i), I\mathcal{O}_{\mathrm{Proj}(\mathfrak{R}(I))})$). As R_I is regular $\mathrm{Proj}(\mathfrak{R}(I \cdot R_I)/I\mathfrak{R}(I \cdot R_I))$ is a projective space over the field $K(I) := R_I/I \cdot R_I$. So $x_i(A_i)_I$ is a prime ideal of $(A_i)_I$. In view of the above remark on the prime divisors of x_iA_i this latter ideal is prime. Clearly $A_i\left[\frac{1}{x_i}\right] = R_{x_i}$. So $A_i\left[\frac{1}{x_i}\right]$ is factorial. Thus it is well known that A_i is factorial, too, [17].

(4.6) *Remark.* The previous proof shows that (ii) may be sharpened to $\mathrm{Nor}\,(\mathfrak{R}(\mathcal{J})) \supseteq \lambda^{-1}\,\mathrm{Nor}\,(X)$, where $\lambda : \mathrm{Spec}\,(\mathfrak{R}(\mathcal{J})) \to X$ is defined canonically. So, if $Y \subseteq X$ is normally torsion-free, of pure codimension and generically a complete intersection, $\mathrm{Bl}_X\,(Y)$ is arithemtically normal, if X is normal.

Proof of Theorem (1.1). We use the notations of the introduction and of (2.6), (2.15), assuming that $\dim\,(W) \leq 0$. Choose $\mathcal{J}, \mathfrak{b} \subseteq \mathcal{O}_V$ according to (2.15). Observe that $V(\mathfrak{b}) = W$ is of pure codimension d (or empty). According to (4.3) there is a locally complete intersection ideal $\mathcal{L} \subseteq \mathfrak{b}$ of codimension $d-1$ such that $\mathcal{L}_p \subseteq \mathcal{O}_{V,p}$ is reduced for all $p \in V - W$ and such that $\mathrm{Reg}\,(V(\mathcal{L}) \supseteq \mathrm{Reg}\,(V) \cap V(\mathcal{L})$ (choose $\tilde{\mathcal{J}} = \mathfrak{b}$ and observe (4.4)). In particular $C := V(\mathcal{L})$ is a (reduced) curve. Define $\tilde{\mathcal{L}}$ as in (2.15). We claim that $\tilde{\mathcal{L}}$ is the ideal of sections vanishing at C, thus that $\sqrt{\mathcal{L}} = \tilde{\mathcal{L}}$. As C is of pure dimension one (as $\mathcal{L}$ is of pure codimension $d-1$) we have $\mathcal{L}_p = \tilde{\mathcal{L}}_p = \sqrt{\mathcal{L}_p}$, for all generic points p of C ((4.3)(i)), for these points are not closed, thus outside W. As $\tilde{\mathcal{L}} \subseteq \sqrt{\mathcal{L}}$ this shows that $\sqrt{\mathcal{L}} = \tilde{\mathcal{L}}$ (as both of these ideals have no embedded component and are of pure dimension 1). So our claim is shown. According to (2.15) it follows that C is normally torsion-free with respect to V and that $\mathrm{Bl}_V\,(C) = \mathrm{Proj}\,(\mathfrak{R}(\tilde{\mathcal{L}}))$ is CM. As $\tilde{\mathcal{L}}_p = \mathcal{L}_p$ for all generic points we also see that C is a generic complete intersection. Let p any point in $C \cap \mathrm{Reg}\,(V)$. Then we have $\mathcal{L}_p = \tilde{\mathcal{L}}_p$ (as $\mathcal{L}_p$ is reduced), thus $\mathcal{O}_{C,p} = \mathcal{O}_{V,p}/\mathcal{L}_p$. This shows that $p \in \mathrm{Reg}\,(C)$, and it follows $\mathrm{Reg}\,(C) \supseteq \mathrm{Reg}\,(V) \cap C$. So the canonical map $\pi : \mathrm{Bl}_V\,(C) \to V$ preserves regularity and normality (this latter even arithmetically) (4.5), (4.6). This proves (1.1).

(4.7) *Remark.* (i) If we may choose C irreducible in addition, it follows by (4.5) (iii) that π even preserves local factoriality. (ii) If $\dim\,(V) \leq 3$ and if V is normal we know that $\mathrm{Bl}_V\,(C)$ is arithemtically CM (see (2.15)). We already have remarked above (4.6) that $\mathrm{Bl}_V(C)$ is arithemtically normal. This proves (1.4). (1.3) is also clear by the previous remark.

Proof of Theorem (1.2). We use the notations of the introduction and of (3.5), assuming moreover that $\dim\,(W) = 1$. This latter implies in particular that $d = \dim\,(V) \geq 3$. Let $\mathcal{J}$ be the ideal of sections vanishing at W. Choose $\mathfrak{b} \subseteq \mathcal{O}_V$ according to (3.5). $\mathfrak{b}$ is of codimension $d-1$. Now apply (4.3) (with $\mathfrak{b} = \tilde{\mathcal{J}}$) to obtain a locally complete intersection ideal $\mathcal{L} \subseteq \mathfrak{b}$ of codimension $d-2$ such that $\mathcal{L}_p \subseteq \mathcal{O}_{V,p}$ is reduced for all $p \in V - W$ and such that $\mathrm{Reg}\,(V(\mathcal{L})) \supseteq \mathrm{Reg}\,(V) \cap V(\mathcal{L})$. Define $\tilde{\mathcal{L}}$ as in (3.5). Clearly $S := V(\mathcal{L})$ is a pure surface. In literally the same way as in the proof of (1.1) we may verify that $\tilde{\mathcal{L}}$ is the ideal of sections vanishing at S and that S is a generic complete intersection. So, observing (3.5) and arguing as in the proof of (1.1) we see that the blow-up $X := Bl_V(S) = \text{Proj}(\mathfrak{R}(\tilde{\mathcal{L}})) \xrightarrow{\varphi} V$ preserves regularity and normality. So it remains to define $T \subseteq X$ in the appropriate way. Let Z be the set of closed points of S. $\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X$ is the structure-sheaf of the exceptional fiber $\varphi^{-1}(S)$ of φ. Let $p \in Z$. We claim that the ideal $\mathfrak{m}_p \cdot \mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X$ is of codimension 2. As $\varphi^{-1}(S)$ is of pure dimension $d-1$, it suffices to show that $\varphi^{-1}(p)$ is of dimension $d-3$. But this is a local result, which follows by (2.10) (applied with $M = \mathcal{O}_{V,p}$, $L = \mathcal{L}_p$), as this latter shows that $\mathcal{L}_p$ is a minimal reduction of $\tilde{\mathcal{L}}_p$ (in the sense of Northcott-Rees). This proves our claim.

Our next claim is that $\text{Ass}(\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X)$ has no embedded member. So let q be any of the points associated to $\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X$. As T is normally torsion-free we must have $\varphi(q) \in \text{Ass}(\mathcal{O}_V/\tilde{\mathcal{L}})$. So $q' := \varphi(q)$ must be a generic point of S. As S is a generic complete intersection we see that $q' \in \text{Reg}(V)$. As φ preserves regularity we have $q \in \text{Reg}(X)$. As $(\tilde{\mathcal{L}}\mathcal{O}_X)_q$ is a principal ideal in $\mathcal{O}_{X,q}$ ($\tilde{\mathcal{L}}\mathcal{O}_X$ is invertible), q may not be embedded. This proves the second claim.

Both claims together imply that $H^1_Z(\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X)$ is a coherent sheaf over $\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X$ (s. for example [3, (3.1)] and use an affine open covering of $\varphi^{-1}(S)$). This shows that $\dim(V(\mathfrak{d})) \leq 0$, where $\mathfrak{d} = \text{ann}_{\mathcal{O}_S}(H^1_Z(\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X))$. Therefore clearly $\mathfrak{d}$ is of codimension ≥ 2 in $\mathcal{O}_S$. Applying (4.3) to S and $\mathfrak{d}$ we obtain an invertible $\mathcal{M} \subseteq \mathfrak{d}$ such that $\mathcal{M}_p \subseteq \mathcal{O}_{S,p}$ is reduced for all $p \in S - V(\mathfrak{d})$ and such that $\text{Reg}(V(\mathcal{M})) \supseteq \text{Reg}(S) \cap V(\mathcal{M})$. Finally let $T = V(\mathcal{M} \cdot \mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X = \mathcal{M} \cdot \mathcal{O}_{\varphi^{-1}(S)}) \subseteq \varphi^{-1}(S)$. We claim that the closed subset $T \subseteq X$ is of the requested type. As $\varphi^{-1}(S)$ is of pure dimension $d-1$ and as T is a hypersurface in $\varphi^{-1}(S)$, T is of codimension 2 with respect to X. It remains to show that the blow-up $\psi: \text{Bl}_X(T) \to X$ is a Macaulayfication which preserves normality and that $\varphi \circ \psi$ preserves regularity. To prove this we choose a point $p \in Z$ and put $R = \mathcal{O}_{V,p}$, $\mathfrak{m} = \mathfrak{m}_p$, $L = \mathcal{L}_p$, $\tilde{L} = \tilde{\mathcal{L}}_p$, $X_0 = \text{Proj}(\mathfrak{R}(\tilde{L}))$ and define $\mathcal{N}_0 \subseteq \mathcal{O}_{X_0}$ as the ideal of sections vanishing at the closed set $T_0 := T \cap X_0 \subseteq X_0 = \varphi^{-1}\ (\text{Spec}(R)) = \{q \in X \mid p \in \overline{\{\varphi(q)\}}\}$. Consider the canonical morphisms $\varphi_0: X_0 \to \text{Spec}(R)$ and $Y_0 = \text{Bl}_{X_0}(T_0) = \text{Proj}(\mathfrak{R}(\mathcal{N}_0)) \xrightarrow{\psi_0} X_0$. It suffices to show that Y_0 is CM, that ψ_0 preserves normality and that $\varphi_0 \circ \psi_0$ preserves regularity.

Note that $R, \mathfrak{m}, L, \tilde{L}$ satisfy the hypotheses of (3.10) and that φ_0 preserves normality and regularity. Assume first that $p \notin V(\mathcal{M})$. Then clearly $T \cap X_0 = \emptyset$, so that $Y_0 = X_0$. As φ_0 preserves regularity and normality, it remains to prove that X_0 is CM in this case. By our choice of p we have $\mathfrak{d}_p = \mathcal{O}_{S,p}$ so that $H^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = (H^1_Z(\mathcal{O}_X/\tilde{\mathcal{L}}\mathcal{O}_X))_p = 0$. So we have $\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0} = D_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0})$ (observe that $H^0_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = 0$ by the normal torsion-freeness of $\tilde{L}$). But the righthand term equals $\text{Proj}(D_{\mathfrak{m}}(\text{Gr}(\tilde{L})))$, which latter is CM by (3.10). Therefore the exceptional fiber $\varphi^{-1}(S)$ is CM, which induces that X_0 is CM.

Finally let $p \in V(\mathcal{M})$. Then we find an element $y \in \mathfrak{m}$ such that $\mathcal{M}_p = y \cdot R/\tilde{L}$ (note that $\mathcal{O}_{S,p} = R(\tilde{L})$. By our choice of $\mathcal{M}$ we have $yH^1_{\mathfrak{m}}(\mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}) = 0$, $ht(\tilde{L}, y) =$

$d-1$. Clearly it holds $\mathcal{N}_0=\sqrt{(L, y)\mathcal{O}_{X_0}}$ ($\mathcal{N}_0 \subseteq \mathcal{O}_{X_0}$ is the ideal of sections vanishing at T_0). We claim that $\mathcal{N}_0$ also may be written as $\bigcup_i ((\tilde{L}, y)\mathcal{O}_{X_0} : \mathfrak{m}^i)\mathcal{O}_{X_0} =: \mathcal{K}_0$: As $\varphi^{-1}(p)$ is of dimension $d-3$ and as T is of dimension $d-2$, no generic point of T is mapped to p. This shows that $\mathcal{K}_0 \subseteq \mathcal{N}_0$, thus $\sqrt{\mathcal{K}_0}=\mathcal{N}_0$. So it remains to show that $\mathcal{K}_0$ is reduced. By (3.11) $\mathcal{K}_0$ has no embedded component. So we only have to proof that $(\mathcal{K}_0)_q \subseteq \mathcal{O}_{X_0,q}$ is reduced for any generic point q of T_0. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ be the image of such a point q. As $s \neq p$, $s \in V(y \cdot R/\tilde{L})$, $\dim(R/\tilde{L})=2$, $\mathfrak{p}$ corresponds to a point of codimension one in S. In particular $\mathfrak{p}$ corresponds to a point $s \in S - V(\mathfrak{b})$ so that $\mathcal{M}_s$ is reduced. This means that $(\tilde{L}, y)R_\mathfrak{p}=\mathfrak{p}R_\mathfrak{p}$. In particular we see that $(R/\tilde{L})_\mathfrak{p}$ is a discrete valuation ring. Clearly $x_1, \ldots, x_{d-2}$ (minimal system of generators of L) form a $\mathfrak{p}R_\mathfrak{p}$-$S^+$-sequence with respect to $R_\mathfrak{p}$ and it holds $\tilde{L}_\mathfrak{p}=D_{\mathfrak{p}R_\mathfrak{p}}(L_\mathfrak{p})$. So $\overline{(\operatorname{Gr}(L)^J)_\mathfrak{p}}=\overline{\operatorname{Gr}(LR_\mathfrak{p})\mathfrak{p}R_\mathfrak{p}}$ is a polynomial algebra over $R_\mathfrak{p}/\tilde{L}_\mathfrak{p}=(R/\tilde{L})_\mathfrak{p}$, thus regular. Moreover $\operatorname{Gr}(\tilde{L})_\mathfrak{p}=\operatorname{Gr}(\tilde{L}R_\mathfrak{p})$ is a finite birational extension ring of $\overline{\operatorname{Gr}(L_\mathfrak{p})^{\mathfrak{p}R_\mathfrak{p}}}$ (use (2.13)(i)). So both rings coincide. This shows that $\operatorname{Gr}(\tilde{L})_\mathfrak{p}$ is a polynomial algebra over $(R/\tilde{L})_\mathfrak{p}$. Therefore $y \cdot \operatorname{Gr}(\tilde{L})_\mathfrak{p}=\mathfrak{p}(R/\tilde{L})_\mathfrak{p} \cdot \operatorname{Gr}(\tilde{L})_\mathfrak{p}$ is a prime ideal of $\operatorname{Gr}(\tilde{L})_\mathfrak{p} \cdot \mathcal{O}_{X_0}/\tilde{L}\mathcal{O}_{X_0}$ is the structure sheaf of $\operatorname{Proj}(\operatorname{Gr}(\tilde{L}))$; this shows that $((\tilde{L}, y)\mathcal{O}_{X_0})_q$ is a reduced ideal of $\mathcal{O}_{X_0,q}$. This clearly proves our claim.

The last argument shows that $(\mathcal{K}_0)_q=((\tilde{L}, y)\mathcal{O}_{X_0})_q$ for all generic points q of T_0. As $\tilde{L}\mathcal{O}_{X_0}$ is invertible and as T is of pure codimension 2, it follows that T is a generic complete intersection of pure codimension. $\mathcal{K}_0$ is the ideal of sections vanishing at T. So – by (3.11) – T is also normally torsion-free with respect to X_0. Applying (3.11)(ii) and (4.5)(ii) we see that $Y_0=\operatorname{Proj}(\mathfrak{R}(\mathcal{K}_0)) \xrightarrow{\psi_0} X_0$ is a Macaulayfication which preserves normality (even arithmetically).

It remains to show that $\varphi_0\psi_0$ preserves regularity. So let $r \in Y_0$ be such that $\mathfrak{p}:=\varphi_0 \cdot \psi_0(r) \in \operatorname{Reg}(R)$. $\mathfrak{p}$ corresponds to a regular point s of V, and therefore specializes to a closed point p' of V which is regular. Replace p by p'. This allows to assume that R is regular. Then, by our choice of $\mathcal{L}$ we have $R/L=R/\tilde{L}$ and this ring is regular too. In particular $\operatorname{Gr}(\tilde{L})$ is a polynomial ring over $R/\tilde{L}$ and X_0 is regular, as φ_0 preserves regularity. Moreover – by our choice of $\mathcal{M}$ – we either have $p \notin V(\mathcal{M})$ or we may assume that y is a regular parameter with respect to $R/\tilde{L}$. In the first case we are done by the above. In the second case $\operatorname{Gr}(\tilde{L})/y\operatorname{Gr}(\tilde{L})$ is regular. So T_0 is regular (observe that in particular $\mathcal{K}_0=(\tilde{L}, y)\mathcal{O}_{X_0}$, as this latter is a complete intersection in the regular scheme X_0 and so has no embedded component). Now we see by (4.5)(i) that $Y_0=\operatorname{Bl}_{X_0}(T_0)$ is regular.

Conclusive remark. The Macaulayfications we give in (1.1) and (1.2) clearly also preserve the property of being a complete intersection point or a Gorenstein point. To see this notice that the occurring centers C, S and T of our blow up are scheme-theoretically complete intersections in those points of the ambient variety which are complete intersection points or of Gorenstein type. But blowing up a local scheme at a complete intersection ideal clearly preserves the two properties in question. More generally, any propery P (of local nature) is preserved under our birational models if only it satisfies the following axioms:

If a local ring satisfies P, it is CM.

If a local ring R satisfies P, R/xR does for any regular $x \in R$.

If, for a local ring R and a regular element $x \in R$, R/xR satisfies P then R does too.

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Some existence theorems for closed geodesics

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In the present paper we use elementary methods to prove some results on the existence of closed geodesics. One of the main results is as follows:

THEOREM. *Let g be a metric on $P^n\mathbb{R}$ such that $\frac{1}{4} \leq \delta \leq K \leq 1$, where K denotes the sectional curvature of M. Then g has at least $g(n) = 2n - s - 1$, $0 \leq s = n - 2^k < 2^k$, closed geodesics without self-intersections, with lengths in $[\pi, \pi/\sqrt{\delta}] \subset [\pi, 2\pi]$, and which are not null-homotopic. If all closed geodesics of length $\leq 2\pi$ are non-degenerate (an open and dense condition on the set of metrics with respect to the C^2 topology), then g has at least $n(n+1)/2$ such closed geodesics.*

Notice that this theorem does not follow from the corresponding existence theorem for S^n in [BTZ2]. Moreover the proof is more elementary, since no use is made of loop space methods.

The proof of the Lusternik–Schnirelmann theorem [LS] can be used to show that any metric on $P^2\mathbb{R}$ has three closed geodesics without self-intersections which are not null-homotopic, see [Ba]. The ellipsoid with pairwise different principal axes sufficiently close to one induces a metric on $P^n\mathbb{R}$ with $n(n+1)/2$ closed geodesics without self-intersections which are non-degenerate and not null-homotopic and have length approximately π. One can achieve that the lengths of all other closed geodesics are greater than any given number by choosing the axes sufficiently close to 1.

Some of the other results in this paper are

(i) If M is homeomorphic to S^n and $\frac{1}{4} \leq K \leq 1$, then there exists a closed geodesic without self-intersections, with lengths in $[2\pi, 4\pi]$, and with index $n-1$.

(ii) If M is homeomorphic to S^n and $\frac{4}{9} \leq K \leq 1$, then there exist two closed geodesics c and d without self-intersections, with lengths in $[2\pi, 3\pi]$, such that $\operatorname{ind}(c) = n-1$ and $\operatorname{ind}(d) + \operatorname{null}(d) = 3(n-1)$.

(iii) If M is homeomorphic to S^n and $\frac{1}{4} \leq \delta \leq K \leq 1$, and if K is not constant,

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then there does not exist any closed geodesic with length in $[2\pi/\sqrt{\delta}, 4\pi]$. If $n=2$ and if $\frac{1}{9}<\delta\leq K\leq 1$, then there does not exist any prime closed geodesic with length in $(2\pi/\sqrt{\delta}, 6\pi)$.

(iv) If g_0 is the metric on $P^n\mathbb{R}$ of constant curvature 1 and if g satisfies the Morse condition $g_0<g<9g_0$ then there exist at least $g(n)$ closed geodesics with lengths in $(\pi, 3\pi)$.

(v) A convex hypersurface in E^{n+1} which contains a ball of radius r and is contained in a ball of radius R has at least $g(n)$ closed geodesics with lengths in the interval $[2\pi r, 2\pi R]$ if $2r>R$.

(ii) and (iii) are partially proved in [Ka] and [Ts] respectively [Su].

Some of these elementary results are existence results needed in [BTZ1], where we examined stability properties of closed geodesics. Thus the present paper can be used as an introduction to [BTZ1].

In Chapter 1 we examine closed geodesics on S^n, in Chapter 2 closed geodesics on $P^n\mathbb{R}$, in Chapter 3 the Morse condition, in Chapter 4 closed geodesics on convex surfaces, and in Chapter 5 closed geodesics on convex hypersurfaces.

1. Closed geodesics on spheres

We first review the definitions of a few concepts and some of their properties. M will always denote a compact Riemannian manifold.

Let Λ be the space of closed piecewise C^∞ curves $c: I=[0,1]\to M$, and for $p\in M$ let Ω_p be the subspace of Λ consisting of curves with $c(0)=p$. To a C^∞ map $f:(I^k, \partial I^k)\to(M,p)$ we associate the maps $f_\Omega:(I^{k-1}, \partial I^{k-1})\to(\Omega_p, p)$ and $f_\Lambda:(I^{k-1}, \partial I^{k-1})\to(\Lambda, \Lambda^0)$ defined by $f_\Omega(x_1,\ldots,x_{k-1})(t)=f(x_1,\ldots,x_{k-1},t)$ and $f_\Lambda=j\circ f_\Omega$, where $j:(\Omega_p,p)\to(\Lambda,\Lambda^0)$ is the inclusion, Λ^0 the space of constant curves, and p the constant curve with image p. (We have also used the convention $I^0=\{0\}$ and $\partial I^0=\emptyset$.)

Let $h\in\pi_k(M)$, $k\geq 1$, be a non-trivial homotopy class, and let $f:(I^k,\partial I^k)\to(M,p)$ be a C^∞ representative. We define

$$\alpha_{\Omega_p}(h)=\inf\left\{\max_{x\in I^{k-1}} E(g(x))\mid g \text{ homotopic to } f_\Omega\right\}$$

$$\alpha_\Lambda(h)=\inf\left\{\max_{x\in I^{k-1}} E(g(x))\mid g \text{ homotopic to } f_\Lambda\right\}$$

where E is the energy functional $E(c)=\frac{1}{2}\int_0^1\langle\dot c,\dot c\rangle\,dt$. For a compact manifold M there always exists a $k\geq 1$ such that $\pi_k(M)\neq 0$.

Let $V=V(c)$ denote the space of piecewise C^∞ vector fields along a closed geodesic $c\in\Lambda$ satisfying $\langle X(t), \dot c(t)\rangle=0$ for all $t\in I$ and $X(0)=X(1)$. On V we define the index form of c by

$$H(X, Y)=\int_0^1 (\langle \nabla X, \nabla Y\rangle-\langle R(X, \dot c)\dot c, Y\rangle)\, dt.$$

The index (resp. extended index) of c as a closed geodesic, denoted by $\operatorname{ind}(c)$ (resp. $\operatorname{ind}_0(c)$), is the maximal dimension of a subspace U of V such that $H \mid U$ is negative definite (resp. negative semi-definite). The index (resp. extended index) of c as a geodesic segment, denoted by $\operatorname{ind}_\Omega(c)$ (resp. $\operatorname{ind}_\Omega(c)+\operatorname{null}_\Omega(c)$) is the maximal dimension of a subspace U of V consisting of vector fields X satisfying $X(0)=X(1)=0$ such that $H \mid U$ is negative definite (resp. negative semi-definite). Obviously

$$\operatorname{ind}(c)\geq \operatorname{ind}_\Omega(c) \quad \text{and} \quad \operatorname{ind}_0(c)\geq \operatorname{ind}_\Omega(c)+\operatorname{null}_\Omega(c).$$

In [BTZ1], (1.5) and (1.6), it is shown that

$$\operatorname{ind}(c)\leq \operatorname{ind}_\Omega(c)+n-1 \quad \text{and} \quad \operatorname{ind}_0(c)\leq \operatorname{ind}_\Omega(c)+\operatorname{null}_\Omega(c)+n-1.$$

These inequalities immediately give the following estimates of the index and the extended index of a closed geodesic on a Riemannian manifold whose sectional curvature satisfies $0<\delta\leq K\leq 1$:

$$(1.1)\qquad \begin{aligned} &L(c)>k\frac{\pi}{\sqrt{\delta}}\Rightarrow \operatorname{ind}(c)\geq k(n-1)\\ &L(c)<k\pi\Rightarrow \operatorname{ind}_0(c)\leq k(n-1)\end{aligned}$$

see [BTZ1], (1.8) and (1.9).

We will frequently use the following injectivity radius estimate, see [CE], [CG], [KS]:

(1.2) (Klingenberg). If M^n is simply connected and the sectional curvature K of M satisfies $\frac{1}{4}\leq\delta\leq K\leq 1$, or if n is even and $0<\delta\leq K\leq 1$, then the injectivity radius of M satisfies $i(M)\geq\pi$. In particular, biangles, geodesic loops, and closed geodesics have length $\geq 2\pi$.

The following theorem follows easily from critical point theory (e.g., apply Lemma 22.5 in [Mi] to a finite dimensional approximation of Λ, see [Mi] §16) and the fact that $f\to f_\Lambda$ induces an isomorphism $\pi_k(M)\to\pi_{k-1}(\Lambda, \Lambda^0)$ for $k\geq 2$.

1.3. THEOREM. *Suppose $\pi_k(M) \neq \{0\}$, $k \geq 1$. Then there exists a closed geodesic of index $\leq k-1$ and of length $\alpha_\Lambda(h)$ when $h \in \pi_k(M)$ is non-trivial. If $k \geq 2$, then there exists such a closed geodesic which is null-homotopic.* □

Remark. The existence of a closed geodesic if $\pi_1(M)=0$ is due to Birkhoff [Bi] for $M = S^n$ and Fet–Lusternik [LF] in general.

1.4. THEOREM. *Suppose that the sectional curvature K of M^n satisfies $0 < \delta \leq K \leq 1$.*

(i) *Then there exists a closed geodesic of index $\leq n-1$ and length $\leq 2\pi/\sqrt{\delta}$. If M is not homotopy equivalent to S^n, then there exists a closed geodesic of index $< n/2$ and length $\leq \pi/\sqrt{\delta}$.*

(ii) *If M is homeomorphic to S^n and $\delta > \frac{1}{4}$, then any closed geodesic of index $\leq n-1$ has index $n-1$, length in $[2\pi, 2\pi/\sqrt{\delta}] \subset [2\pi, 4\pi)$, and no self-intersections.*

Proof. (i) Since $\pi_k(M) \neq 0$ for some $1 \leq k \leq n$ there exists a closed geodesic c of index $\leq n-1$ by (1.3). $L(c) > 2\pi/\sqrt{\delta}$ would imply $\operatorname{ind}(c) \geq 2(n-1)$ by (1.1), which is a contradiction. If M is not homotopy equivalent to S^n there exists a k, $1 \leq k \leq n/2$, such that $\pi_k(M) \neq 0$ and therefore a closed geodesic c of index $< n/2 \leq n-1$, hence $L(c) \leq \pi/\sqrt{\delta}$ by (1.1).

(ii) Let c be a closed geodesic of index $\leq n-1$. Then $L(c) \leq 2\pi/\sqrt{\delta} < 4\pi$ by (1.1), and (1.2) implies $L(c) \geq 2\pi$. Since $2\pi > \pi/\sqrt{\delta}$, (1.1) implies $\operatorname{ind}(c) \geq n-1$, hence $\operatorname{ind}(c) = n-1$. If a closed geodesic has a self-intersection it is the union of two loops, each of which has length at least 2π. This contradicts $L(c) < 4\pi$. □

We now prove the existence of a second closed geodesic. Assume that $\pi_1(M^n) = 0$ and $\frac{4}{9} \leq \delta \leq K \leq 1$. Then M is homeomorphic to S^n, and hence $\pi_n(M) = \mathbb{Z}$. Choose a fixed generator $h \in \pi_n(M)$. According to theorem (1.3) there exists a closed geodesic of length $l = \sqrt{2}\alpha_\Lambda(h)$.

1.5. THEOREM. *Suppose that M is homeomorphic to S^n and $\frac{4}{9} \leq \delta \leq K \leq 1$.*

(i) (Karcher [Ka]) *A geodesic loop c of maximal length $L \leq 2\pi/\sqrt{\delta}$ is a closed geodesic without self-intersections and $\operatorname{ind}_0(c) = 3(n-1)$. Furthermore, c is a geodesic triangle of maximal perimeter.*

(ii) *$l \leq L$, and $l = L$ implies that there exists a closed geodesic c of length L through every point p of M with $\operatorname{ind}_0(c) = 3(n-1)$. In particular, there exist two different closed geodesics without self-intersections and lengths in $[2\pi, 3\pi]$.*

Remarks. (a) The claim about $\operatorname{ind}_0(c)$ in (i) is not contained in [Ka] but was communicated to us by Karcher. The proof in [Ka] contains a mistake. In using the triangle inequality one has to make a separate discussion when equality occurs. In particular, it is not correct, as claimed there, that every geodesic triangle of maximal perimeter is a closed geodesic. But by replacing some sides in a triangle of maximal perimeter one gets a closed geodesic among the triangles of maximal perimeter.

(b) $L \geq 2d(M)$. If $L = 2d(M)$, then for every two points $p, q \in M$ at maximal distance $d(M)$ every geodesic through p is closed of length L and meets q. This follows since a geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$, where γ_1 connects p and q, obviously has perimeter $\geq 2d(M)$ and $= 2d(M)$ if $\gamma_1 * \gamma_2$ is a geodesic.

Proof. (i) We only prove the claim about $\operatorname{ind}_0(c)$. A theorem of Toponogov states that the perimeter of every geodesic triangle is $\leq 2\pi/\sqrt{\delta}$, and if there exists a geodesic triangle of perimeter $2\pi/\sqrt{\delta}$, then K is constant. We first assume that K is not constant. Let c be a geodesic loop of maximal length $\leq 2\pi/\sqrt{\delta}$ and set $\gamma_i = c \mid [(i-1)/3, i/3]$, $1 \leq i \leq 3$. The triple $(\gamma_1, \gamma_2, \gamma_3)$ is a geodesic triangle since $L(\gamma_i) \leq \pi$. Since K is not constant the perimeter of $(\gamma_1, \gamma_2, \gamma_3)$ is $< 2\pi/\sqrt{\delta} \leq 3\pi$, hence $L(\gamma_i) < \pi$. Therefore there are no pairs of conjugate points on γ_i. Hence each triple of vectors (X_0, X_1, X_2), $X_i \in T_{c(t_i)}M$ and $X_i \perp \dot{c}(t_i)$, $t_i = i/3$, determines a unique broken Jacobi field J along c such that $J(t_i) = X_i$ and $J(1) = X_0$. $H(J, J) \leq 0$ since J corresponds to a variation of c through geodesic triangles and c is a geodesic triangle of maximal perimeter. Hence there exists a $3(n-1)$-dimensional subspace of $V(c)$ on which H is negative semi-definite which implies $\operatorname{ind}_0(c) \geq 3(n-1)$. Since $L(c) < 2\pi/\sqrt{\delta} \leq 3\pi$, (1.1) implies $\operatorname{ind}_0(c) \leq 3(n-1)$ and hence $\operatorname{ind}_0(c) = 3(n-1)$.

If K is constant, say $K \equiv 1$, it is clear that the great circles are triangles of maximal perimeter and the above arguments show that their extended index ind_0 is equal to $3(n-1)$.

(ii) Let h be a generator of $\pi_n(M)$. Then there exists a closed geodesic c_1 of length $l = \sqrt{2}\alpha_\Lambda(h)$. Theorem (1.3) also applies to geodesic loops, i.e., for every $p \in M$ there exists a geodesic loop c of length $\sqrt{2}\alpha_{\Omega_p}(h)$ and $\operatorname{ind}_\Omega(c) \leq n-1$, and hence $L(c) \leq 2\pi/\sqrt{\delta} \leq 3\pi$. Furthermore by definition $\alpha_\Lambda(h) \leq \alpha_{\Omega_p}(h)$ for every $p \in M$. Hence $2\pi \leq \sqrt{2}\alpha_\Lambda(h) \leq \sqrt{2}\alpha_{\Omega_p}(h) \leq 3\pi$. A geodesic loop of maximal length $\leq 2\pi/\sqrt{\delta}$ is a closed geodesic c_2, and c_1 and c_2 can be geometrically equal only if $L(c_1) = L(c_2) = L$. But then $\sqrt{2}\alpha_{\Omega_p}(h) = L$ for every p which implies by (i) that there exists a closed geodesic c of length L through every p and $\operatorname{ind}_0(c) = 3(n-1)$. □

Remark. We remarked above that there is a closed geodesic among the geodesic triangles of maximal perimeter on a $\frac{4}{9}$-pinched manifold. More generally one can apply Lusternik–Schnirelmann theory to a $\mathbb{Z}_2$-quotient of a space of triangles on a δ-pinched manifold to obtain $n+1$ closed geodesics with lengths in the interval $[2\pi, 2\pi/\sqrt{\delta}]$ if $\delta \geq \frac{4}{9}$. We briefly sketch how this can be done. A similar proof will be carried out in more detail in chapter 2. In [BTZ2], theorem (4.1), using much more complicated methods, the existence of $g(n) \geq n+1$ such geodesics is proved on δ-pinched manifolds if $\delta \geq \frac{1}{4}$.

If there is a geodesic triangle of perimeter $2\pi/\sqrt{\delta}$ on M, then $K \equiv \delta$, and all geodesics are closed of length $2\pi/\sqrt{\delta}$. Hence we may assume that there exists an $\varepsilon > 0$ such that any geodesic triangle has perimeter $\leq (2\pi/\sqrt{\delta}) - 3\varepsilon$. Let v, w be unit tangent vectors with the same foot point, and let γ_v, γ_w be the geodesics determined by $\dot{\gamma}_v(0) = v$, $\dot{\gamma}_w(0) = w$. Then

$$d(\gamma_v(\pi - \varepsilon), \gamma_w(\pi - \varepsilon)) + 2(\pi - \varepsilon) \leq (2\pi/\sqrt{\delta}) - 3\varepsilon \leq 3\pi - 3\varepsilon.$$

Hence $d(\gamma_v(\pi - \varepsilon), \gamma_w(\pi - \varepsilon)) \leq \pi - \varepsilon < i(M)$. Denote by $U^2(M)$ the set $\{(v, w) \mid v, w$ are unit tangent vectors with the same foot point$\}$. The above inequality shows that the function $f: U^2M \to \mathbb{R}$, $(v, w) \to d^2(\gamma_v(\pi - \varepsilon), \gamma_w(\pi - \varepsilon))$ is C^∞. It is also invariant under the $\mathbb{Z}_2$-action $(v, w) \to (w, v)$ of U^2M. (v, w) is a minimum of f if and only if $(v, w) \in U^1(M) = \{(v, w) \in U^2(M) \mid v = w\}$. $U^1(M)$ is the fixed point set of the $\mathbb{Z}_2$-action on $U^2(M)$. Using the first variation formula and the fact that the sides of the geodesic triangles have no conjugate points it is easy to prove that (v, w) is a critical point of f if and only if $v = w$ or $v = -w$ and the triangle corresponding to (v, w) is a closed geodesic. Hence Lusternik–Schnirelmann theory implies that there are at least as many closed geodesics on M with lengths in $[2\pi, 2\pi/\sqrt{\delta})$ as the length of a maximal chain of homology classes $h_1, \ldots, h_s$ in $H_*(U^2(M)/\mathbb{Z}_2, U^1(M)/\mathbb{Z}_2; \mathbb{Z}_2)$ with the following properties: $\xi_i \cap h_i = h_{i-1}$ for some $\xi_i \in H^*(U^2(M)/\mathbb{Z}_2; \mathbb{Z}_2)$, $* > 0$, whose restriction to a sufficiently small neighborhood of $S^1c/\mathbb{Z}_2 = \{(v, w) \mid v = -w = \dot{c}(t)\}/\mathbb{Z}_2$ vanishes for every closed geodesic c which is a geodesic triangle, see [BTZ2], (1.2) and (1.3). The length of such a chain is $n+1$ since $(U^2(M), U^1(M))$ is an $n-1$ bundle over $T_1M/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts by $v \to -v$) and since the cohomology ring of $T_1M/\mathbb{Z}_2$ is easily seen to be generated by $\theta \in H^1$ and $\omega \in H^{n-1}$ with the relations $\theta^n = 0$, $\omega^2 = 0$, and $\theta^{n-1} \cup \omega = [T_1M/\mathbb{Z}_2]$.

We now discuss certain gaps in the length spectra of closed geodesics and geodesic loops. If $\pi_1(M) = 0$ and $\frac{1}{4} \leq K \leq 1$, then we already know from (1.2) that there does not exist any closed geodesic or geodesic loop with length in $[0, 2\pi)$. As was observed by Tsukamoto [Ts] the proof of Berger's rigidity theorem implies the following result:

(1.6) Suppose $\pi_1(M) = 0$ and $\frac{1}{4} \leq K \leq 1$. If there exists a closed geodesic of length 2π, then M is isometric to a sphere with $K \equiv 1$, or to a projective space $P^k\mathbb{C}$, $P^k\mathbb{H}$, P^2Ca equipped with their standard metrics.

Remark. If $\pi_1(M)=0$, $i(M)\geq\pi$, and $0<K\leq 1$, then a closed geodesic of length 2π and of positive index is contained in a totally geodesic, embedded surface of constant curvature 1. This follows easily using arguments as in the proof of [Be], Theorem 4. For dim $M=2$ this was already proved by Klingenberg.

We already know that there exist closed geodesics in the interval $[2\pi, 2\pi/\sqrt{\delta}]$ if M is homeomorphic to S^n and $\frac{1}{4}\leq\delta\leq K\leq 1$. The next gap in the length spectrum is given by

1.7. THEOREM. *If M is homeomorphic to S^n and $\frac{1}{4}\leq\delta\leq K\leq 1$, and if K is not constant, then there does not exist any closed geodesic with length in* $[2\pi/\sqrt{\delta}, 4\pi]$.

Remarks. (a) In [Th] it was proved that there does not exist any closed geodesic with length in $[2\pi/\sqrt{\delta}, 4\pi)$ if $\frac{4}{9}<\delta\leq K\leq 1$ unless $K\equiv\delta$.

(b) Tsukamoto [Ts] claims that a closed geodesic without self-intersections on a simply connected manifold with $\frac{1}{4}\leq\delta\leq K\leq 1$ does not have length $2\pi/\sqrt{\delta}$ unless $K\equiv\delta$. But his proof contains a gap. For even dimensions a complete proof of his result was given by Sugimoto (now Goto) [Su], Theorem B and C. Using the injectivity radius estimate (1.2) the proof in [Su] can be shortened considerably and carries over directly to odd dimensions.

Proof. If there is a closed geodesic of length $\leq 4\pi$ and with self-intersections, then it is the union of loops one of which has length $\leq 2\pi$. Since $i(M)\geq\pi$ by (1.2) a geodesic loop of length $\leq 2\pi$ is a closed geodesic of length 2π. By (1.6) this implies $K\equiv 1$. If there exists a closed geodesic c of length $2\pi/\sqrt{\delta'}\in[2\pi/\sqrt{\delta}, 4\pi]$, $\frac{1}{4}\leq\delta'\leq\delta$, and $K\not\equiv 1$, then c has no self-intersections contradicting the theorem of Tsukamoto–Sugimoto quoted in Remark (b) above. This proves the theorem. □

1.8. COROLLARY. *If $\pi_1(M^n)=0$ and $\frac{1}{4}\leq K\leq 1$, then there exists a closed geodesic c without self-intersections,* ind$(c)\leq n-1$, *and length in* $[2\pi, 4\pi]$. *Unless $K\equiv 1$, or $K\equiv\frac{1}{4}$, or M isometric to $P^k\mathbb{C}$, $P^k\mathbb{H}$, P^2Ca equipped with their standard metrics, we have $2\pi<L(c)<4\pi$ and* ind$(c)=n-1$.

Proof. This follows by combining (1.7) with the proof of (1.4). □

Remark. Notice that on $P^k\mathbb{C}$, $P^k\mathbb{H}$, and P^2Ca, equipped with their standard metrics, the closed geodesics have index 1, 3, and 7 respectively.

2. Closed geodesics on real projective spaces

In this chapter we examine the existence of closed geodesics on $P^n\mathbb{R}$ and manifolds with $\pi_1(M) \cong \mathbb{Z}_2$.

2.1. THEOREM. *Suppose M is diffeomorphic to $P^n\mathbb{R}$ and g is a metric such that $\frac{1}{4} \leq \delta \leq K \leq 1$. Then g has at least $g(n)$ closed geodesics without self-intersections, with lengths in $[\pi, \pi/\sqrt{\delta}] \subset [\pi, 2\pi]$, and which are not null-homotopic. If all closed geodesics of length $\leq 2\pi$ are non-degenerate, then g has at least $n(n+1)/2$ such closed geodesics.*

Remark. The indices of the closed geodesics in (2.1) lie in the interval $[0, 2(n-1)]$. Using the methods developed in [BTZ1] and [BTZ2] one easily obtains stability properties of these closed geodesics.

Proof. Since M is not simply connected, we have $d(M) \leq \pi/2\sqrt{\delta}$ by a result of Shiohama, see [Sh], Proposition 2.1. If $\pi_1(M) = \mathbb{Z}_2$, $\frac{1}{4} \leq \delta \leq K \leq 1$, and $d(M) = \pi/2\sqrt{\delta}$, then a result of Sakai [Sa], p. 428, implies that $\tilde{M}$ is isometric to S^n or $P^k\mathbb{C}$ with their standard metrics. Since the theorem is obvious for such spaces we can assume $d(M) < \pi/2\sqrt{\delta} \leq \pi$.

Let v be any unit tangent vector of M and let $c_v(t)$ be the geodesic determined by $\dot{c}_v(0) = v$. There is a first $t(v) > 0$ such that $c_v \mid [-s, s]$ is not minimizing for any $s > t(v)$. We have $2t(v) \leq d(M)$, and hence $t(v) < \pi/4\sqrt{\delta} \leq \pi/2$. Since there is no conjugate point along $c_v \mid [-t(v), t(v)]$, there is a second geodesic segment d_v such that $d_v(-t(v)) = c_v(-t(v))$ and $d_v(t(v)) = c_v(t(v))$. $(c_v \mid [-t(v), t(v)]) * (d_v^{-1} \mid [-t(v), t(v)])$ is not null-homotopic since $i(\tilde{M}) \geq \pi$. c is uniquely determined since $\pi_1(M) = \mathbb{Z}_2$. Hence $t(v)$ and $\dot{d}_v(-t(v))$ depend continuously on v and satisfy the equation

$$d^2(c_v(t(v)), \exp 2t(v)\dot{d}_v(-t(v))) = 0.$$

Since there are no conjugate points for $t < \pi$, $t(v)$ depends differentiably on v by the implicit function theorem.

Suppose v is a critical point of the function t. Given two vectors $X \perp \dot{c}_v(-t(v))$ and $Y \perp \dot{c}_v(t(v))$ there exists a variation c_s of c_v through geodesics such that $c_0 = c_v$ and

$$\left.\frac{d}{ds}\right|_{s=0} c_s(t(v)) = Y, \qquad \left.\frac{d}{ds}\right|_{s=0} c_s(-t(v)) = X.$$

Set $v(s) = \dot{c}_s(0)/\|\dot{c}_s(0)\|$. Then by the chain rule and the first variation formula we have

$$\begin{aligned}0&=\frac{1}{2}\frac{d}{ds}\bigg|_{s=0} t(v(s))=\frac{d}{ds}\bigg|_{s=0} L(c_s[-t(v(s)), t(v(s))])\\&=\frac{d}{ds}\bigg|_{s=0} L(d_s[-t(v(s)), t(v(s))])\\&=\left\langle\frac{d}{ds}\bigg|_{s=0} d_s(t(v(s))), \dot{d}_v(t(v))\right\rangle-\left\langle\frac{d}{ds}\bigg|_{s=0} d_s(-t(v(s))), \dot{d}_v(-t(v))\right\rangle\\&=\langle Y, \dot{d}_v(t(v))\rangle-\langle X, \dot{d}_v(-t(v))\rangle.\end{aligned}$$

Hence a critical point of t corresponds to a closed geodesic on M of length $4t(v)$. Such a closed geodesic does not have self-intersections since $4t(v)<2\pi$.

The definition of t implies $t(v)=t(-v)$. Hence we obtain a differentiable function on T_1M/θ, where $\theta v=-v$, and the critical points correspond to closed geodesics. Notice though that each closed geodesic c gives rise to a circle of critical points $\dot{c}(t+\alpha)$, $0\leq\alpha\leq 1$. Lusternik–Schnirelmann theory implies that there exist at least as many critical circles as there are homology classes $h_1, \ldots, h_s$ in $H_*(T_1M/\theta, \mathbb{Z}_2)$ such that $\xi_i\cap h_i=h_{i-1}$, $i=2, 3, \ldots, s$, for some cohomology classes $\xi_i\in H^*(T_1M/\theta, \mathbb{Z}_2)$, $*>0$, with the property that ξ_i vanishes on every sufficiently small neighborhood of a critical circle $\dot{c}(t+\alpha)$, $0\leq\alpha\leq 1$, see [BTZ2], (1.2) and (1.3). We now show that there exist $g(n)$ such homology classes. Since the unit tangent bundles with respect to g and the constant curvature 1 metric g_0 are θ equivariantly diffeomorphic, the computation can be done for the case $M=(P^n\mathbb{R}, g_0)$.

The geodesics on $\tilde{M}=(S^n, \tilde{g}_0)$ are the great circles. Each unit tangent vector $v\in T_1\tilde{M}$ determines a unique parametrized great circle γ_v with $\dot{\gamma}_v(0)=v$. This identifies $T_1\tilde{M}$ and the space G of all parametrized great circles. We have an O(2) action on G defined by $\psi\gamma(t)=\gamma(\psi t)$ for $\psi\in O(2)$. Let $\theta=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $\phi=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\in O(2)$. Then T_1M corresponds to G/ϕ and T_1M/θ to G/Γ, where $\Gamma\simeq\mathbb{Z}_2\times\mathbb{Z}_2$ is the subgroup of O(2) generated by θ and ϕ. Let $\bar{G}=G/O(2)$ be the space of unparametrized great circles on S^n. $H_*(\bar{G}, \mathbb{Z}_2)$ has a basis of $n(n+1)/2$ homology classes $[a, b]$, $0\leq a\leq b\leq n-1$, of dimension $a+b$. Denote by (a, b) the corresponding dual basis of $H^*(\bar{G}, \mathbb{Z}_2)$. Then $(0, 1)$ is the first Stiefel–Whitney class of the S^0 bundle $G/SO(2)\to\bar{G}$ and (1.1) is the second Stiefel–Whitney class of the S^1 bundle $G/\theta\to\bar{G}$. $(0, 1)^{2n-2s-2}\cup(1, 1)^s=(n-1, n-1)$ and $(0, 1)^{2n-2s-1}=0$, and hence $\bar{G}$ has a chain of $g(n)$ subordinate homology classes, and each chain of subordinate homology classes has length $\leq g(n)$, see [Kl], p. 49.

The second Stiefel–Whitney class of the S^1 bundle $p: G/\Gamma \to \bar{G}$ is zero since it it twice the second Stiefel–Whitney class of $G/\theta \to \bar{G}$. The Gysin sequence of p then implies that p^* is injective and p_* is surjective. Choose a class $h_{g(n)}$ such that $p_*(h_{g(n)}) = [n-1, n-1]$. From the naturality of cup and cap products it follows that the classes h_i, $1 \le i \le g(n)$, inductively defined by

$$h_{g(n)-1} = p^*(1, 1) \cap h_{g(n)}, \ldots, h_1 = p^*((1, 1)^s \cup (0, 1)^{2n-s-2}) \cap h_{g(n)}$$

are non-zero. Hence they are a chain of $g(n)$ subordinate homology classes in $H_*(G/\Gamma) = H_*(T_1M/\theta)$.

We now show that $p^*(0, 1)$ and $p^*(1, 1)$ vanish on a sufficiently small neighborhood U of a critical circle in T_1M/θ. If U is a tabular neighborhood of the critical circle, then it has the homotopy type of a circle and hence $p^*(1, 1) \mid U = 0$. $p^*(0, 1) \mid U$ vanishes if U is sufficiently small since $p^*(0, 1)$ is the Stiefel–Whitney class of the bundle $q: T_1M \to T_1M/\theta$, and $q^{-1}(U) \to U$ is trivial if U is sufficiently small. This finishes the proof of the existence of $g(n)$ closed geodesics.

Suppose now that all closed geodesics of length $\le 2\pi$ are non-degenerate. We first want to show that this implies that all critical points of $t: T_1M \to \mathbb{R}$ are non-degenerate critical circles. Let c be a closed geodesic such that $v = \dot{c}(0)$ is a critical point of t. T_1M can be viewed as a set of geodesic biangles as in the beginning of the proof and hence $T_1M \subset \Lambda$. Since $E \mid T_1M = \frac{1}{2}t^2$ the Hessian of t is proportional to $H \mid TT_1M$. The tangent space of T_1M in Λ consists of piecewise Jacobi fields with breaks at 0 and $\frac{1}{2}$, and since $\dim T_1M = 2n-1$, it coincides with the set of all such Jacobi fields. $V(c)$ is the direct sum of these Jacobi fields and the set of vector fields vanishing at 0 and $\frac{1}{2}$. This direct sum is orthogonal with respect to H. Hence the nullspace of $E \mid T_1M$ coincides with the nullspace of H. Therefore all critical circles of $t: T_1M \to \mathbb{R}$ and hence also all critical circles of $t: T_1M/\theta \to \mathbb{R}$ are non-degenerate. The local homology of a critical circle vanishes in dimension $\ne \operatorname{ind}(c), \operatorname{ind}(c)+1$ and is equal to $\mathbb{Z}_2$ in dimension $\operatorname{ind}(c)$ and $\operatorname{ind}(c+1$. The Morse inequalities for such functions now imply that there are at least $\frac{1}{2}\sum b_i(G/\Gamma, \mathbb{Z}_2)$ critical circles. But the Gysin sequence of $p: G/\Gamma \to \bar{G}$ implies $\sum b_i(G/\Gamma, \mathbb{Z}_2) = n(n+1)$ since the second Stiefel–Whitney class of p vanishes. $\square$

As we noted in the above proof, it follows by results of Shiohama [Sh] and Sakai [Sa] that $d(M) < \pi/2\sqrt{\delta} \le \pi$ if $\pi_1(M) = \mathbb{Z}_2$, $\frac{1}{4} \le \delta \le K \le 1$, and $\tilde{M}$ not a symmetric space of rank one. Then the proof of Lemma 4.1 in [Sh] applies and shows that there is a closed geodesic c of length $2d(M)$ through p and q if $d(p, q) = d(M)$. c is not null-homotopic since $L(c) < 2\pi$. Clearly c has maximal perimeter in the set of all geodesic biangles with minimizing sides. Note that this also follows from the above proof: the first part of the proof only uses $\pi_1(M)=\mathbb{Z}_2$ and the curvature restrictions; c corresponds to a maximum of t. As in the proof of (1.5) it follows that $\operatorname{ind}_0(c)=2(n-1)$.

Since there is no conjugate point along a geodesic segment of length $<\pi$, there exists a closed geodesic d of length $2i(M)$ by Lemma (5.6) in [CE]. $L(d)\geq\pi$ by (1.2). d is not null-homotopic since $L(d)<2\pi$. Hence d is a shortest curve in its homotopy class and therefore $\operatorname{ind}(d)=0$. $L(c)\geq L(d)$ since $i(M)\leq d(M)$. $L(c)=L(d)$ implies $i(M)=d(M)$. By (5.6) in [CE] this implies that all geodesics are closed of length $2i(M)$. Since $\pi_1(M)=\mathbb{Z}_2$ it follows that the universal covering space of M is a Wiedersehen manifold and the generalized Blaschke conjecture, recently proved by Berger, Kazdan, Weinstein, and Yang, implies that K is constant, see [Bs].

Summarizing the above we obtain

2.2. THEOREM. *Suppose that $\pi_1(M^n)=\mathbb{Z}_2$ and $\frac{1}{4}\leq\delta\leq K\leq 1$. Then there exist two closed geodesics c and d which are not null-homotopic, have no self-intersections and satisfy*

$$\pi\leq 2i(M)=L(d)\leq L(c)=2d(M)\leq\pi/\sqrt{\delta}$$

$$\operatorname{ind}(d)=0 \quad \textit{and} \quad \operatorname{ind}_0(c)=2(n-1).$$

c has maximal perimeter in the set of all biangles with minimizing sides, and $L(d)=L(c)$ implies that K is constant. □

3. The Morse condition

To prove the existence of more than one closed geodesic on S^n, M. Morse [Mo], p. 354, introduced the following condition: Let g_0 be the metric on S^n of constant curvature 1. Then the metric g on S^n satisfies the *Morse condition* if

$$g_0<g<4g_0$$

This condition immediately implies that the critical levels of certain homology classes consisting of circles lie in $(2\pi, 4\pi)$. Hence the closed geodesics on which they remain hanging cannot be iterates of each other. But this does not prove the existence of geometrically different closed geodesics since these closed geodesics could all be iterates of one short closed geodesic. In [Al] Alber stated the theorem that $g_0\leq g<4g_0$ and $0<K\leq 1$ if n even or $\frac{1}{4}<K\leq 1$ if n odd implies the

existence of $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 4\pi)$. Under these conditions there are no closed geodesics of length $<2\pi$ by (1.2). But the topological part of his proof turned out to be incorrect. Correct proofs have been given in [BTZ2], [An], and [Hi]. In [BTZ2] it was also proved that $\frac{1}{4} \le K \le 1$ implies the existence of $g(n)$ closed geodesics, i.e. the Morse condition is not needed for $\frac{1}{4}$-pinched manifolds. In this chapter we give some further theorems involving the Morse condition.

Let

$$d_p = \max_{q \in M} d(p, q)$$

3.1. LEMMA [Sh]. *If $d_p > \pi$ for every $p \in M$ and $K \ge \frac{1}{4}$, then the length of every closed geodesic is $>2\pi$.*

Proof. Let c be a closed geodesic with $L(c) \le 2\pi$ and let $p = c(0)$. Since $d_p > \pi$, there exists a point $q \in M$ with $d(p, q) > \pi$. Let d be a minimal geodesic from p to q. Then we have a generalized triangle whose one side consists of c and the two minimal sides are d. This is a generalized triangle in the sense of the Toponogov comparison theorem since $L(c) \le 2\pi$ and $2L(d) > 2\pi \ge L(c)$. Since the angles of one of the minimal sides with c is $\le \pi/2$, it follows from Toponogov's theorem that $d(p, q) < \pi$, a contradiction. □

Remark. A theorem of Berger and Grove–Shiohama [GS] states that $K \ge \frac{1}{4}$ and $d_p > \pi$ for some $p \in M$ implies that M is homeomorphic to S^n.

As a first consequence we obtain the following theorem.

3.2. THEOREM. *If $d_p > \pi$ for every $p \in M$ and $K \ge \frac{1}{4}$, then there exist at least $n-1$ closed geodesics with lengths in $(2\pi, 4\pi]$.*

Proof. Since a closed geodesic c with $L(c) > 4\pi$ has index $\ge 2(n-1)$, it follows as in the proof of (3.3) in [BTZ2] that there exist $n-1$ closed geodesics of length $\le 4\pi$. By (3.1) every closed geodesic has length $>2\pi$ and hence these $n-1$ closed geodesics are geometrically different. □

Remark. It does not follow that these closed geodesics have no self-intersections. Notice also that $d_p > \pi$ for every $p \in M$ follows from one half of the Morse condition, namely $g > g_0$.

3.3. THEOREM. *If g is a metric on S^n with $g_0 < g < 4g_0$ and $K \ge \frac{1}{4}$, then there exist $g(n)$ closed geodesics with lengths in $(2\pi, 4\pi)$.*

Proof. $g > g_0$ implies that $d_p > \pi$ for every $p \in M$ and hence all closed geodesics have lengths $> 2\pi$. $g < 4g_0$ implies that the homology classes considered in [BTZ2] thm. (2.4), have critical levels $< 4\pi$ and hence the theorem follows from the methods in [BTZ2]. □

Finally we observe that on $P^n\mathbb{R}$ the Morse condition suffices without any curvature assumptions.

3.4. THEOREM. *Let* g *be a metric on* $P^n\mathbb{R}$ *and* g_0 *the metric on* $P^n\mathbb{R}$ *with constant curvature* 1. *If* $g_0 < g < 9g_0$, *then there exists at least* $g(n)$ *closed geodesics which are not null-homotopic and with lengths in* $(\pi, 3\pi)$.

Proof. $g > g_0$ implies that every closed curve which is not null-homotopic, and hence every closed geodesic which is not null-homotopic, has length $> \pi$. We denote by $\bar{\Lambda}_*$ the unparametrized closed curves on $P^n\mathbb{R}$ which are not null-homotopic, by $\bar{\Lambda}_*^\alpha$ those curves in $\bar{\Lambda}_*$ whose energy with respect to the g metric is $\leq \alpha$, and by $\bar{\Lambda}_{*,0}^\alpha$ those in $\bar{\Lambda}_*$ whose energy in the g_0 metric is $\leq \alpha$. Then $g_0 < g < 9g_0$ implies that we have the following inclusions

$$\bar{\Lambda}_{*,0}^{\pi^2/2} \subset \bar{\Lambda}_*^{9\pi^2/2} \subset \bar{\Lambda}_{*,0}^{9\pi^2/2-}. \tag{$*$}$$

Every curve in $\bar{\Lambda}_*$ has odd multiplicity. The closed geodesics in $\bar{\Lambda}_{*,0}^{9\pi^2/2-}$ consist of the ones of minimal length, i.e., the great circles. Hence $\bar{\Lambda}_{*,0}^{9\pi^2/2-}$ is homotopy equivalent to $G(2, n-1)$, the space of unoriented two planes in $\mathbb{R}^{n+1}$. Thus $\bar{\Lambda}_{*,0}^{9\pi^2/2}$ and $\bar{\Lambda}_{*,0}^{\pi^2/2}$ have $g(n)$ subordinate homology classes and the inclusions in $(*)$ together with naturality of cap products implies that $\bar{\Lambda}_*^{9\pi^2/2}$ also has $g(n)$ subordinate homology classes. Standard Lusternik–Schnirelmann theory now implies that there exist $g(n)$ closed geodesics in $\bar{\Lambda}_*^{9\pi^2/2}$, i.e. $g(n)$ closed geodesics with lengths $< 3\pi$. Since every closed geodesic has length $> \pi$ and since two fold iterates do not lie in $\bar{\Lambda}_*$, these closed geodesics are geometrically different. □

4. Closed geodesics on convex surfaces

In this chapter we study metrics of positive curvature on S^2. As is well-known, such metrics can be realized by embeddings into Euclidean space E^3. We first improve Theorem (1.7) on the gap in the length spectrum. We get the following result.

4.1. THEOREM. *Suppose* M *is diffeomorphic to* S^2 *and* $\frac{1}{9} < \delta \leq K \leq 1$. *Then there does not exist any prime closed geodesic with length in* $(2\pi/\sqrt{\delta}, 6\pi)$.

Remark. This implies that there exists no closed geodesic with length in $(4\pi/\sqrt{\delta}, 6\pi)$ if $\delta > \frac{4}{9}$, since a closed geodesic of length $< 6\pi$ which is not prime is a twofold cover of a closed geodesic without self-intersections and hence has length $\leq 4\pi/\sqrt{\delta}$ by [To].

Proof. We say that a prime closed geodesic $c:[0,1] \to M$ has k self-intersections if $k = \sum_{x \in \mathrm{im}(c)} (\#\{0 \leq t \leq 1 \mid c(t) = x\} - 1)$. If a prime closed geodesic has only one self-intersection, it is the boundary of a convex polygon and hence has length $\leq 2\pi/\sqrt{\delta}$ by [To]. If c has more than one self-intersection one has the following cases: Either there exist $0 \leq t_1 < t_1' < t_2 < t_2' \leq 1$ with $c(t_i) = c(t_i')$, or there exist $0 \leq t_1 < t_2 < t_3 < t_1' < t_2' < t_3' < 1$ with $c(t_i) = c(t_i')$. In the first case c consists of at least two geodesic loops and a biangle, in the second case of at least three biangles. In either case $L(c) \geq 6\pi$ by (1.2). $\square$

Remark. It follows from arguments as in the proof that there do not exist prime closed geodesics with 1, 2, or 3 self-intersections if $\delta > \frac{1}{4}, \frac{1}{9}$, or $\frac{4}{9}$ respectively. Furthermore, if c is a prime closed geodesic with k self-intersections, then $L(c) \leq (k+2)\pi/\sqrt{\delta}$ since the complement of c consists of $k+2$ regions with convex polygons as boundary.

Let α_0 be the energy of a shortest closed geodesic. One can ask whether there exists a homotopy class $h \in \pi_\alpha(M)$ with $\alpha_\Lambda(h) = \alpha_0$. In general this is false as the closed geodesic on the equator of an hour glass shows. We can prove:

4.2. THEOREM. *Suppose M is diffeomorphic to S^2 and $\frac{1}{4} \leq K \leq 1$. If h is a generator of $\pi_2(M)$, then $\alpha_0 = \alpha_\Lambda(h)$. Any shortest closed geodesic on M has no self-intersections and index* 1.

Proof. By (1.6) we can assume that a shortest closed geodesic has length $< 4\pi$ and hence no self-intersections. c has index ≥ 1 since a parallel orthogonal vectorfield X along c satisfies $H(X, X) < 0$.

M is the union of two closed balls B_+ and B_- such that $B_+ \cap B_- = \mathrm{im}(c)$. B_+ and B_- are both locally convex since c is a geodesic. Let X be the parallel vector field pointing into B_+. The map $f: S^1 \times [-\varepsilon, \varepsilon] \to M$, $(e^{2\pi i t}, s) \to \exp_{c(t)}(s \cdot X(t))$, $\varepsilon > 0$ sufficiently small, defines a variation of c such that the closed curves f_τ defined by $f_\tau(t) = f(t, \tau)$ lie in B_+ for $\tau > 0$ and in B_- for $\tau < 0$, and $E(f_\tau) < E(c)$ for $\tau \neq 0$. One can now easily extend f to a variation $\tilde{f}: S^1 \times [-1, +1] \to M$ with $E(\tilde{f}_\tau) < E(c)$ for all $\tau \neq 0$ and such that $\tilde{f}_\tau$ lies in B_+ for $\tau > 0$ and in B_- for $\tau < 0$ and $\tilde{f}_1, \tilde{f}_{-1}$ are point curves in B_+ resp. B_-: deform $f_{\pm\varepsilon}$ into geodesic polygons in $B_\pm$ and then apply the negative gradient flow of the energy in a finite dimensional approximation, see [Mi], §16. The curves stay in $B_\pm$ since $B_\pm$ is locally convex and one eventually obtains point curves since there exist no closed geodesics of energy $<\alpha_0$. $\tilde{f}$ can be viewed as a map $g: I^2/\partial I^2 = S^2 \to M$ such that g_Λ (up to homotopy) consists of the curves $\tilde{f}_\tau$. g has degree ± 1 since the inverse image of $c(t)$ consists only of one point. Hence g is a generator of $\pi_2(M)$, and by changing X into $-X$ if necessary, g is in the homotopy class h. Hence $\alpha_\Lambda(h) \leq \alpha_0$ and since α_0 is the energy of a shortest closed geodesic we obtain $\alpha_\Lambda(h) = \alpha_0$.

If $\operatorname{ind}(c) > 1$ one can apply Lemma 2 in [CG] to a finite dimensional approximation to show that $\alpha_\Lambda(h) < \alpha_0$. This is a contradiction. Hence $\operatorname{ind}(c) = 1$. □

Remarks. (a) If one can show that a shortest closed geodesic on a convex surface has no self-intersections the above proof would apply and show that $\alpha_0 = \alpha_\Lambda(h)$ and $\operatorname{ind}(c) = 1$.

(b) One can also show that for an arbitrary metric on S^2 a closed geodesic without self-intersections which is shortest among all closed geodesics without self-intersections has index ≤ 1. This follows as in the above proof if one replaces the negative gradient flow by the Lusternik–Schnirelmann deformation, which leaves the set of closed curves without self-intersections invariant, see [LS] and [Ba].

5. Closed geodesics on convex hypersurfaces

5.1. THEOREM. *Let M be a convex hypersurface in E^{n+1} which contains a ball of radius r and is contained in a ball of radius R. Assume that $2r > R$. Then there are at least $g(n)$ closed geodesics on M with lengths in the interval $[2\pi r, 2\pi R]$.*

Remark. The projection of a convex hypersurface onto a convex hypersurface inside its interior is length-decreasing. Hence the assumption $2r > R$ implies the Morse condition. In addition, we have the estimate of C. Croke [Cr], Theorem 1.5: the closed geodesics on a convex hypersurface containing a ball of radius r are of length $\geq 2\pi r$; equality occurring if and only if the hypersurface is tangential to the ball of radius r along a great circle. Under the hypothesis of the theorem we can prove that M has at least $g(n)$ closed geodesics with lengths in the interval $[2\pi r, 2\pi R)$ unless the ball of radius R is tangential to the hypersurface along a great circle.

Note that the proof below cannot be used to decide whether the $g(n)$ closed geodesics have self-intersections or not.

Proof. The intersections of M with two-planes define a map from the space of parameterized circles on a sphere into the space $P(M)$ (in the notation of

[BTZ2]). The lengths of these curves are $\leq 2\pi R$ since they are contained in the convex hull of circles on the sphere of radius R, and their lengths are $<2\pi R$ unless M is tangential to the sphere of radius R along a great circle. As in Theorem (2.4) in [BTZ2] this gives rise to $g(n)$ subordinate homology classes in $(\bar{P}^{2\pi^2R^2}, (\bar{V}\cap\bar{P})^{2\pi^2R^2})$. The theorem of Croke quoted in the remark above now implies, together with Lemma (1.5)(ii) in [BTZ2], that there are $g(n)$ closed geodesics on M with lengths in $[2\pi r, 2\pi R]$. □

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Explicit resolutions for the binary polyhedral groups and for other central extensions of the triangle groups

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Geometric background and algebraic results

This paper will exhibit two classes of finitely presented groups for which explicit free resolutions can be obtained by direct algebraic calculations. The resolutions will be either periodic of period 4, or of length 3.

1. The groups of the first class are *central extensions of the triangle groups*, admitting a 2-generator 2-relator presentation. Specifically, let l, m, n be integers with $\min(|l|, |m|, |n|) \geqq 2$ and define

$$G = G(l, m, n) = \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n \rangle. \tag{1}$$

In the sequel the canonical images in G of α, resp. β, will be denoted by a, resp. b.

The element $(ab)^n$ of G is central, being a power of either generator, and the central quotient $G(l, m, n)/\langle (ab)^n \rangle$ is the triangle group

$$\begin{aligned} T = T(l, m, n) &= \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n = 1 \rangle \\ &\cong \langle \alpha, \beta, \gamma; \alpha^l = \beta^m = \gamma^n = \alpha\beta\gamma = 1 \rangle. \end{aligned} \tag{2}$$

Every triangle group T can be realized faithfully as a group of isometries of the sphere $S^2 \subset \mathbb{R}^2$ when $|l|^{-1} + |m|^{-1} + |n|^{-1} > 1$ (or, equivalently, if T is finite), of the Euclidean plane if $|l|^{-1} + |m|^{-1} + |n|^{-1} = 1$, or of the hyperbolic plane if $|l|^{-1} + |m|^{-1} + |n|^{-1} < 1$. This action of T leads in all three cases to a tesselation of the space in question by pairs of adjacent triangles – whence the name "triangle group". (See, e.g., [10], and the references cited there for proofs and more details.)

2. Each of the groups $G(l, m, n)$ occurs as the *fundamental group of a suitable Seifert fiber space*. Such a space M is a compact 3-dimensional manifold equipped with a foliation by circles, called fibers. The set of all fibers forms the orbit space, which is a compact surface. The neighbourhoods of a fiber are fibered solid tori with the given fiber as their core, and depending on how these solid tori are fibered, the given fiber is called exceptional or ordinary. From the fact that M is compact it follows that there are only finitely many exceptional fibers.

In the special case where the orbit space of M is the 2-sphere and where there are three exceptional fibers, the fundamental group Γ of M has a presentation of the form

$$\Gamma = \langle \alpha, \beta, \gamma, \zeta;\ \alpha^l = \zeta^{l'},\ \beta^m = \zeta^{m'},\ \gamma^n = \zeta^{n'},\ \alpha\beta\gamma = \zeta^p \text{ and } \zeta \text{ is central}\rangle.$$

Here the pairs (l, l'), (m, m') and (n, n') are relatively prime, $\min(|l|, |m|, |n|) \geqq 2$ and p is an arbitrary integer. For $l' = m' = 1$, $n' = -1$ and $p = 0$, the group Γ is isomorphic with $G(l, m, n)$. (See H. Seifert [15], or [13], for more details and proofs.)

If the fundamental group Γ of a Seifert fiber space M is *finite*, it admits a faithful representation $\rho: \Gamma \rightarrowtail SO_4(\mathbb{R})$ for which the induced action on $S^3 \subset \mathbb{R}^4$ is fixpoint-free and the quotient space $\rho(\Gamma)\backslash S^3$ is homeomorphic to M (W. Threlfall and H. Seifert [20, Part II, p. 568, Hauptsatz]). From this fixpoint-free action of Γ on S^3 one can deduce (see, e.g., [2, p. 154]) that there exists a $\mathbb{Z}\Gamma$-free resolution $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ which has period 4 and is finitely generated in each dimension; in particular, this is true for the finite groups $G(l, m, n)$.

If the fundamental group Γ of a Seifert fiber space M is *infinite*, M is in most cases aspherical, as can be deduced from the sphere theorem (see [13, p. 56, Satz 5] for a precise statement). In particular, the spaces with an infinite $G(l, m, n)$ are aspherical, whence the infinite groups $G(l, m, n)$ must be Poincaré-duality groups of dimension 3.

3. Our first result describes for each $G = G(l, m, n)$ an explicit $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. These resolutions will have the same form up to dimension 3 for all groups, they be finite or infinite, and will permit one to read off many properties known on topological grounds. The proof will be uniform for all groups.

In order to define the resolution, choose the defining relators

$$r = (\alpha\beta)^n\beta^{-m} \quad \text{and} \quad s = (\beta\alpha)^n\alpha^{-l} \tag{3}$$

for $G(l, m, n)$; these relators do define $G(l, m, n)$, as can be checked speedily. Let F be the free group on $\{\alpha, \beta\}$, and let D_α, resp. D_β, denote the composite of the partial derivation $\partial/\partial\alpha: F \to \mathbb{Z}F$, resp. $\partial/\partial\beta: F \to \mathbb{Z}F$, and the canonical ring epimorphism $\mathbb{Z}F \twoheadrightarrow \mathbb{Z}G$.

THEOREM A. *If $G = G(l, m, n)$ and r, s are as before, the sequence of left $\mathbb{Z}G$-modules and $\mathbb{Z}G$-homomorphisms*

$$\mathbb{Z}G \xrightarrow{(1-b,\ 1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b \end{pmatrix}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{4}$$

is exact. (Here $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ *takes* $g \in G$ *to* $1 \in \mathbb{Z}$*.) Furthermore: (i) If* G *is finite, the kernel of the left-most homomorphism is infinite cyclic, generated by the element* $\sum\{g \mid g \in G\}$*; therefore* (4) *leads by splicing to a periodic resolution with period* 4. *(ii) If* G *is infinite, the left-most homomorphism is injective and* (4) *is a* $\mathbb{Z}G$*-free resolution of* $\mathbb{Z}$*; moreover,* G *is an orientable Poincaré-duality group of dimension* 3.

4. The groups of the second class are *central extensions of* 1*-relator groups.* Let F be the free group on $\{\alpha_1, \ldots, \alpha_n\}$ and let ρ be a non-trivial element of F which is not a proper power. Define

$$L = \langle \alpha_1, \ldots, \alpha_n; [\rho^l, \alpha_1], \ldots, [\rho^l, \alpha_n] \rangle, \tag{5}$$

where $l \geqq 1$ and $[\rho^l, \alpha_i] := \rho^l \cdot \alpha_i \cdot \rho^{-l} \cdot \alpha_i^{-1}$. For $i = 1, \ldots, n$, let $a_i \in L$ denote the canonical image of $\alpha_i \in F$, and let D_{α_i} be the composite of the partial derivation $\partial/\partial\alpha_i : F \to \mathbb{Z}F$ and the canonical ring epimorphism $\mathbb{Z}F \twoheadrightarrow \mathbb{Z}L$.

THEOREM B. *If the notation is as before and* $n \geqq 2$*, the sequence of left* $\mathbb{Z}L$*-modules and* $\mathbb{Z}L$*-homomorphisms*

$$\mathbb{Z}L \xrightarrow{(D_{\alpha_1}\rho, \ldots, D_{\alpha_n}\rho)} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} D_{\alpha_1}[\rho^l, \alpha_1] \cdots D_{\alpha_n}[\rho^l, \alpha_1] \\ \vdots \qquad\qquad \vdots \\ D_{\alpha_1}[\rho^l, \alpha_n] \cdots D_{\alpha_n}[\rho^l, \alpha_n] \end{pmatrix}} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} 1 - a_1 \\ \vdots \\ 1 - a_n \end{pmatrix}} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z} \tag{6}$$

is a $\mathbb{Z}L$*-free resolution of* $\mathbb{Z}$.

The groups L are only in special cases Poincaré-duality groups of dimension 3. As we shall prove in Theorem 9 this happens if, and only if, F admits either a basis $\xi_1, \eta_1, \ldots, \xi_g, \eta_g$ such that $\rho = [\xi_1, \eta_1] \cdot \cdots \cdot [\xi_g, \eta_g]$, or a basis $\xi_1, \xi_2, \ldots, \xi_g$ for which $\rho = \xi_1^2 \cdot \cdots \cdot \xi_g^2$. It follows that the Poincaré-duality groups in the second class of groups are fundamental groups of Seifert fiber spaces which have at most one exceptional fiber and an orientable, or non-orientable, closed surface of genus $g > 0$ as their orbit space.

1. Some general facts about the groups $G(l, m, n)$

Let l, m, n be arbitrary integers and set

$$G = G(l, m, n) = \langle \alpha, \beta; \alpha^l = \beta^m = (\alpha\beta)^n \rangle.$$

As before we denote the canonical images in G of α, β by a, b.

Normalization of the parameters. The defining relations of G imply that $(ab)^n=(ba)^n$ and so the assignments $\alpha\mapsto b$, $\beta\mapsto a$ induce an isomorphism $G(l,m,n)\xrightarrow{\sim} G(m,l,n)$. Similarly, the assignments $\alpha\mapsto a^{-1}$, $\beta\mapsto ab$ lead to an isomorphism $G(l,m,n)\xrightarrow{\sim} G(-l,n,m)$. These two types of isomorphisms allow one to make any of $l,-l,m,-m,n$ or $-n$ the third parameter and so we can assume that $n=\min(|l|,|m|,|n|)$. By exchanging l and m, if need be, we arrive at the normalization

$$l\geqq m \quad\text{and}\quad \min(|l|,|m|)\geqq n\geqq 0.$$

The groups $G(l,m,0)$ are free products $\langle a\rangle*\langle b\rangle$, whereas the groups $G(l,m,1)$ are all cyclic; indeed one has

$$\begin{aligned}G(l,m,1)=\langle\alpha,\beta;\alpha^l=\beta^m=\alpha\beta\rangle&=\langle\alpha;\alpha^l=(\alpha^{l-1})^m\rangle\\&=\langle\alpha;\alpha^{(l-1)(m-1)-1}\rangle.\end{aligned}$$

As the results of this paper are of little interest when specialized to cyclic groups or to free products of cyclic groups, we shall henceforth assume, as we did in the introduction, that $\min(|l|,|m|,|n|)\geqq 2$. The previous normalization can then be sharpened to

$$l\geqq m \quad\text{and}\quad \min(|l|,|m|)\geqq n\geqq 2. \tag{7}$$

Isomorphic groups. Different triples satisfying this normalization condition yield in general non-isomorphic groups, as is revealed by our

THEOREM 1. *Let (l_1,m_1,n_1) and (l_2,m_2,n_2) be ordered triples of integers satisfying (7). If $G(l_1,m_1,n_1)$ and $G(l_2,m_2,n_2)$ are isomorphic, then either both ordered triples are equal, or the two triples are related to each other as are (l,n,n) and $(n,-l,n)$.*

For infinite groups $G(l_i,m_i,n_i)$ the assertion of Theorem 1 can be deduced from a far more general result of P. Orlik et al. on the fundamental groups of Seifert fiber spaces [13, p. 53, Satz 4]. However, whereas the proof of this more general result is quite complicated, Theorem 1 can be established by merely comparing the central quotients $G(l_i,m_i,n_i)/\zeta G(l_i,m_i,n_i)$ and the abelianizations $G(l_i,m_i,n_i)_{ab}$; for this reason we give an independent proof. Before embarking on it we determine the abelianization of the group $G(l,m,n)$.

Computation of the abelianized group. The relators $r=(\alpha\beta)^n\beta^{-m}$ and $s=(\beta\alpha)^n\alpha^{-l}$, which can be used to define G, lead to the relation matrix $R=$

$\begin{pmatrix} n & n-m \\ n-l & n \end{pmatrix}$ of G_{ab}. Put another way, G_{ab} is isomorphic to the cokernel of the homomorphism $\mathbb{Z}^2 \to Z^2$ that takes (x, y) to the matrix product $(x, y) \cdot R$. The determinant of R is

$$\det R = (l+m)n - lm = lmn(l^{-1} + m^{-1} - n^{-1}), \tag{8}$$

and so the theory of elementary divisors implies that

$$G_{ab} \cong Z/\mathbb{Z}e \times \mathbb{Z}/\mathbb{Z}e' \tag{9}$$

where $e = gcd(l, m, n)$ and $e' = \det R/e$.

Proof of Theorem 1. Set $G_i = G(l_i, m_i, n_i)$ for $i = 1$ or 2. We contend the element $(ab)^{n_i}$ generates the centre of G_i. To see this, let (l, m, n) be a triple satisfying (7) and let $T = T(l, m, n)$ be the corresponding triangle group. If T is infinite, its centre is trivial (see, e.g., [22, p. 126, **4.8.1**]). If T is finite, it is either dihedral, or it is polyhedral, i.e. isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$, and ζT will be trivial except in case T is a dihedral group the order of which is divisible by 4. It follows that $\zeta G(l, m, n) = \langle (ab)^n \rangle$, except possibly if G is isomorphic to $G(k, \pm 2, 2)$ where $|k| \geqq 2$ is even. These exceptional groups have the alternative presentation

$$\left.\begin{aligned} G(k, 2, 2) &= \langle \alpha, \beta; \alpha^k = \beta^2, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\ &= \langle \alpha, \beta; \alpha^{2k} = 1, \alpha^k = \beta^2, \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \end{aligned}\right. \quad (k \geqq 2)$$

and

$$\begin{aligned} G(k, -2, 2) &= \langle \alpha, \beta; \alpha^k = \beta^{-2}, \beta\alpha\beta^{-1} = \alpha^{-1}\beta^{-4} \rangle \\ &= \langle \alpha, \beta; \alpha^k = \beta^{-2}, \beta\alpha\beta^{-1} = \alpha^{2k-1} \rangle. \end{aligned}$$

Every element $g \in G(k, \pm 2, 2)$ is of one of the forms a^h or $a^h b$; the centrality condition $bgb^{-1} = g$ implies that $a^{2h} = 1$ in the first and that $a^{2h(k-1)} = 1$ in the second case. Now the order of a in $G(k, 2, 2)$ is $2k$, while the order of a in $G(k, -2, 2)$ is $2k(k-1)$; see, e.g., [5, §6.5, pp. 68–70], or the discussion below. So h is a multiple of k in either case. Since b is not central, we conclude that $\zeta G(k, \pm 2, 2) = \langle a^k \rangle = \langle (ab)^2 \rangle$.

Assume now that $G_1 = G(l_1, m_1, n_1)$ and $G_2 = G(l_2, m_2, n_2)$ are isomorphic. Then so are $T_1 = T(l_1, m_1, n_1) \cong G_1/\zeta G_1$ and $T_2 = T(l_2, m_2, n_2)$. But T_1 and T_2 can only be isomorphic if the unordered triples $\{|l_1|, |m_1|, |n_1|\}$ and $\{|l_2|, |m_2|, |n_2|\}$ coincide; this assertion is obvious if the T_i are finite and it follows for the infinite triangle groups from the fact that the elements of finite order of T_i have order dividing $|l_i|, |m_i|, |n_i|$ and that the orders $|l_i|$, $|m_i|$ and $|n_i|$ occur (cf. [22, p. 126, Thm. 4.8.1.a)]). The normalization (7) now implies that $n_1 = n_2 = n$. In addition, we have that

$$\{|l_1|, |m_1|\} = \{|l_2|, |m_2|\}. \tag{10}$$

Next we exploit the fact that $(G_1)_{ab}$ and $(G_2)_{ab}$ are isomorphic. In view of (8), (9) and (10) this fact leads to the equation

$$l_1^{-1} + m_1^{-1} - n^{-1} = \varepsilon(l_2^{-1} + m_2^{-1} - n^{-1}),$$

where $\varepsilon = \pm 1$. Suppose first that $\varepsilon = 1$ and set $\mu := l_1^{-1} + m_1^{-1} = l_2^{-1} + m_2^{-1}$. Clearly

$$\mu = 0 \Leftrightarrow l_i = -m_i$$

$$\mu > 0 \Leftrightarrow l_i \geqq m_i > 0, \quad \text{or} \quad l_i > 0, m_i < 0 \quad \text{and} \quad l_i < |m_i|$$

$$\mu < 0 \Leftrightarrow 0 > l_i \geqq m_i, \quad \text{or} \quad l_i > 0, m_i < 0 \quad \text{and} \quad l_i > |m_i|$$

Making use of these case distinctions and of (7), (10) one verifies quickly that $l_1^{-1} + m_1^{-1} = l_2^{-1} + m_2^{-1}$ implies that $l_1 = l_2$ and $m_1 = m_2$.

Finally let $\varepsilon = -1$. Then

$$l_1^{-1} + l_2^{-1} + m_1^{-1} + m_2^{-1} = 2n^{-1} > 0.$$

If all four summands of the left hand side are positive, (7) and (10) imply that $l_1 = l_2$ and $m_1 = m_2$. If one summand of the left hand side is negative, it follows from (10) and (7) that two summands must be the negative of each other, whence (7) implies that the two remaining summands are equal to n^{-1}. So we are in the special case where the two ordered triples are of the form (l, n, n) and $(n, -l, n)$. □

We proceed to determine when the abelianized group G_{ab} is infinite and when it is trivial.

Groups with infinite abelianization. Assume the parameters are normalized as in (7) and G_{ab} is infinite. Then $l^{-1} + m^{-1} = n^{-1}$ by (8) and (9), and l and m will be positive. Set $d = gcd(l, m) > 0$ and $f = l/d$, resp. $g = m/d$. From $n^{-1} = l^{-1} + m^{-1} = (f+g)/dfg$ one sees that n is a multiple of fg, say $n = e \cdot fg$, and so d becomes $e(f+g)$. This shows that l, m, n are given by

$$\begin{cases} l = e \cdot f(f+g), \qquad m = e \cdot (f+g)g, \qquad n = e \cdot fg, \\ \text{where } e \geqq 1 \text{ and } f, g \text{ are relatively prime positive integers.} \end{cases} \tag{11}$$

Conversely, every triple (l, m, n) given by (11) satisfies $l^{-1}+m^{-1}=n^{-1}$. Therefore (11) characterizes the normalized triples leading to an infinite abelianization G_{ab}. Note that $G_{ab} \cong \mathbb{Z}/\mathbb{Z}e \times \mathbb{Z}$ by (9).

Perfect groups. If G_{ab} is trivial, i.e. if G is *perfect*, the parameters l, m, n satisfy by (8) the equation $(l+m)n-lm=\pm 1$, which can be rewritten as

$$(l-n)(m-n)=n^2 \mp 1. \tag{12}$$

The integral solutions of (12) are easily surveyed; an infinite sequence of solutions is, e.g., given by the formula

$$(l, m, n)=(2n+1, 2n-1, n) \quad \text{where} \quad n \geqq 2.$$

For $n=2$ equation (12) has the normalized solutions $(5, 3, 2)$ and $(7, 3, 2)$. The first gives the binary icosahedral group, the second the infinite group

$$\begin{aligned} G(7,3,2) &= \langle \alpha, \beta; \alpha^7=\beta^3=(\alpha\beta)^2\rangle \cong \langle \alpha, \beta, \gamma; \alpha^7=\beta^3=(\alpha\beta)^2, \gamma:=\alpha\beta\rangle \\ &= \langle \alpha, \beta, \gamma;\ \alpha^7=\beta^3=\gamma^2=\alpha\beta\gamma\rangle \end{aligned}$$

discussed in [7].

Analysis of the finite groups $G(l, m, n)$. We begin with the question which triangle groups $T(l, m, n)$ are finite. The answer is that this happens if, and only if, $|l|^{-1}+|m|^{-1}+|n|^{-1}>1$. This answer is usually justified by letting T act on a suitable space in the way indicated in **1** of the introduction. There is also a little-known algebraic argument due to P. M. Curran [6]; it is in the spirit of the proofs of this paper and runs briefly like this: Verify first that the canonical images of α, β, γ in

$$T(l, m, n)=\langle \alpha, \beta, \gamma; \alpha^l=\beta^m=\gamma^n=\alpha\beta\gamma=1\rangle \tag{14}$$

have orders l, m and n by constructing suitable quotient groups in which the canonical images of α, β, γ have the desired order (see, e.g. [22, p. 135]). Then use the following

LEMMA 2 (P. M. Curran [6, p. 620]). *Let l, m, n be positive integers and set*

$$S=\langle \alpha, \beta, \gamma; \alpha^l=\beta^m=\gamma^n=\rho=1\rangle, \tag{15}$$

where ρ is an arbitrary element of the free group on α, β, γ. Denote the canonical images of α, β, γ by a, b, c, and let $\bar{l}, \bar{m}, \bar{n}$ be their orders. If S is finite then $\bar{l}^{-1}+\bar{m}^{-1}+\bar{n}^{-1}>1$.

Proof. Since S is finite, $H^1(S, \mathbb{Z}S) = 0$. On the other hand, $H^1(S, \mathbb{Z}S) = \ker \partial_2^* / \operatorname{im} \partial_1^*$ can be computed by means of the exact complex of left $\mathbb{Z}S$-modules and $\mathbb{Z}S$-homomorphisms associated with the presenta-

$$\mathbb{Z}S^4 \xrightarrow{\begin{pmatrix} D_\alpha \alpha^l & 0 & 0 \\ 0 & D_\beta \beta^m & 0 \\ 0 & 0 & D_\gamma \gamma^n \\ D_\alpha \rho & D_\beta \rho & D_\gamma \rho \end{pmatrix}} \mathbb{Z}S^3 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b \\ 1-c \end{pmatrix}} \mathbb{Z}S \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

tion (15); cf. [2, p. 45, Ex. 3(d), or p. 90, Ex. 4(c) and (d)]). Clearly $\operatorname{im} \partial_1^* \cong \mathbb{Z}S / \sum \{s \mid s \in S\} \cdot \mathbb{Z}S$ and so $\operatorname{rank} (\operatorname{im} \partial_1^*) = |S| - 1$. Next $\ker (\mathbb{Z}S \xleftarrow{1+a+\cdots+a^{l-1}} \mathbb{Z}S) = (1-a) \cdot \mathbb{Z}S$ and hence $\operatorname{rank} ((1-a) \cdot \mathbb{Z}S) = |S|(1 - 1/\bar{l})$; similar statements hold for the two other, analogous maps. It follows that $\ker \partial_2^*$ equals the kernel of

$$\mathbb{Z}S \xleftarrow{(D_\alpha \rho,\, D_\beta \rho,\, D_\gamma \rho)} ((1-a) \cdot \mathbb{Z}S \oplus (1-b) \cdot \mathbb{Z}S \oplus (1-c) \cdot \mathbb{Z}S)^{\text{transposed}}$$

and thus $\operatorname{rank} (\ker \partial_2^*) \geqq |S| \, ((1 - 1/\bar{l}) + (1 - 1/\bar{m}) + (1 + 1/\bar{n}) - 1)$. The claim then follows from $|S| - 1 = \operatorname{rank} (\operatorname{im} \partial_1^*) = \operatorname{rank} (\ker \partial_2^*)$. $\square$

To complete the determination of the finite triangle groups $T(l, m, n)$, use that a change of the order or the signs of the parameters l, m, n does not influence the isomorphism type of the group; so one can assume that $l \geqq m \geqq n = 2$. The solutions $(l, 2, 2)$ of $l^{-1} + m^{-1} + n^{-1} > 1$ yield the dihedral groups of order $2l$; the remaining three solutions $(l, 3, 2)$, where $l = 3, 4, 5$, give the polyhedral groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ and $\mathfrak{A}_5$ (cf. [5, p. 7 and p. 67]).

We now pass to the finite groups $G(l, m, n)$ and begin with the simple

LEMMA 3. *If* $\min (|l|, |m|, |n|) \geqq 2$ *and* $T = T(l, m, n)$ *is finite then* $G = G(l, m, n)$ *is likewise finite.*

Proof. Assume the parameters are normalized as in (7). As T is finite one has $n = 2$ and $|l|^{-1} + |m|^{-1} > 1/2$; a comparison with (11) discloses that G_{ab} is finite. The central extension $\langle a^l \rangle \lhd G \twoheadrightarrow T$ gives rise to the exact sequence

$$H_2 G \to H_2 T \to \langle a^l \rangle \to G_{ab} \to T_{ab} \to 0. \tag{16}$$

Since T is finite, $H_2 T$ is so, and then (16) shows that $\langle a^l \rangle$ and hence G are finite. $\square$

The determination of the finite groups $G(l, m, n)$ can be completed as follows: By (7) and the previous reasonings n can be assumed to be 2. Since G is a finite

2-generator, 2-relator group, its multiplicator H_2G is trivial (see, e.g., [2, p. 46, Ex. 5]). The central extension $\langle a^l\rangle \lhd G \twoheadrightarrow T$ gives rise to the exact sequence (16), the exactness of which implies that

$$|\langle a^l\rangle| = |G_{ab}| \cdot |H_2T| \cdot |T_{ab}|^{-1}. \tag{17}$$

The quotient $|H_2T| \cdot |T_{ab}|^{-1}$ can be shown to be 2/4, resp. 2/3, 2/2 or 2/1 for $T(l, 2, 2)$, resp. $T(l, 3, 2)$. On the other hand, G_{ab} is by (8) and (9) equal to 4, resp. 3, 2 or 1 for the special cases $G(l, 2, 2)$, resp. $G(l, 3, 2)$, the parameter l being positive.

It follows that $|\langle a^l\rangle| = 2$ in all these special cases and so these groups are the binary dihedral, resp. polyhedral groups (cf. [5, §6.5]).

Now let $G(l, m, 2)$ be an arbitrary finite group and set

$$G_0 := G(|l|, |m|, 2) = \langle \alpha_0, \beta_0;\ \alpha_0^{|l|} = \beta_0^{|m|} = (\alpha_0\beta_0)^2\rangle$$

Because $a_0^{|l|} = b_0^{|m|} = (a_0b_0)^2$ has order 2, the assignments $\alpha \mapsto a_0$, $\beta \mapsto b_0$ extend to an epimorphism $G \twoheadrightarrow G_0$ and give rise to a central extension

$$\langle a^{2l}\rangle \lhd G \twoheadrightarrow G_0. \tag{18}$$

By (17) and (8), (9) the kernel $\langle a^{2l}\rangle$ has order

$$u(l, m) = |l^{-1} + m^{-1} - \tfrac{1}{2}| \cdot (|l|^{-1} + |m|^{-1} - \tfrac{1}{2})^{-1}.$$

The sequence (18) will split whenever $u(l, m)$ and the order of $G_0 = G(|l|, |m|, 2)$ are relatively prime. Upon computation one finds that the values of $u(l, m)$ corresponding to the sequence of signs $--$, $+-$, $-+$, $++$ are

31, 19, 11 and 1 for the triple (5, 3, 2) with $|G_0| = 120$

13, 7, 5 and 1 for the triple (4, 3, 2) with $|G_0| = 48$

7, 3, 3 and 1 for the triple (3, 3, 2) with $|G_0| = 24$

$l+1$, $l-1$, 1 and 1 for the triples $(l, 2, 2)$ with $|G_0| = 4.l$

Hence the only extensions (18) which may not split correspond to the groups $G(-3, 3, 2) \cong G(3, -3, 2)$ and $G(l, -2, 2)$ for odd $l \geqq 3$. These extensions do in fact not split, as can be seen from the 5-term sequence

$$0 \to 0 \to \langle a^{2l}\rangle \to G_{ab} \to (G_0)_{ab} \to 0$$

induced by (18) and the facts that G_{ab} is cyclic in all these cases, while $|\langle a^{2l}\rangle|$ and $|(G_0)_{ab}|$ are not relatively prime.

2. Proof of Theorem A

The proof relies on two auxiliary results. The first of them asserts that sequence (4), occurring in the statement of Thm. A is always a complex and that it is exact if $H^1(G, \mathbb{Z}G)$ is trivial; it can be established by an easy calculation. The second result shows that $H^1(G, \mathbb{Z}G)$ is trivial for our groups G; its proof makes use of an argument of Serre's and Stallings' characterization of finitely generated groups with infinitely many ends.

Let $F = F(\{\alpha, \beta\})$ be the free group on α and β. As in the introduction set $r = (\alpha\beta)^n\beta^{-m}$ and $s = (\beta\alpha)^n\alpha^{-l}$, and define $G = G(l, m, n) := \langle \alpha, \beta; r, s\rangle$. Assume $lmn \neq 0$.

LEMMA 4. *Let $\partial_3 : \mathbb{Z}G \to \mathbb{Z}G^2$ and $\partial_2 : \mathbb{Z}G^2 \to \mathbb{Z}G^2$ be the homomorphisms of left $\mathbb{Z}G$-modules given by multiplying on the right by the matrices*

$$(1-b, 1-a) \quad \textit{and} \quad \begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}.$$

Then $\operatorname{im}\partial_3 \subseteq \ker\partial_2$ *and the homology group* $\ker\partial_2/\operatorname{im}\partial_3$ *is $\mathbb{Z}$-isomorphic with the first cohomology group $H^1(G, \mathbb{Z}G)$. Moreover, if $H^1(G, \mathbb{Z}G)$ is trivial so is $H^2(G, \mathbb{Z}G)$.*

Proof. We shall compute $H^1(G, \mathbb{Z}G)$ by means of the well-known exact sequence

$$\mathbb{Z}G^2 \xrightarrow{\partial_2} \mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{19}$$

where ∂_2 is as above, $\partial_1(\lambda, \mu) = \lambda(1-a) + \mu(1-\mathrm{b})$ and $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the unit augmentation (cf. [2, p. 90, Ex. 4]). For reasons that will become clear at a later stage of the proof we extend (19) by adding ∂_3 to the left ending up with the sequence

$$\mathbf{P} \twoheadrightarrow \mathbb{Z} : \mathbb{Z}G \xrightarrow{(1-b,\, 1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix} 1-a \\ 1-b \end{pmatrix}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0. \tag{20}$$

Note that at this stage the sequence $\mathbf{P}$ is not known to be a complex.

The dual sequence $\mathbf{P}^* = \operatorname{Hom}_{\mathbb{Z}G}(\mathbf{P}, \mathbb{Z}G)$ can be described in dual bases as

$$\mathbb{Z}G \xleftarrow{(1-b,\, 1-a)} \mathbb{Z}G^2 \xleftarrow{\begin{pmatrix} D_\alpha r & D_\beta r \\ D_\alpha s & D_\beta s \end{pmatrix}} \mathbb{Z}G^2 \xleftarrow{\begin{pmatrix} 1-a \\ 1-b \end{pmatrix}} \mathbb{Z}G, \tag{21}$$

where all modules are right $\mathbb{Z}G$-modules and the matrices describe $\mathbb{Z}G$-linear homomorphisms by multiplication on the left. In order to revert to left $\mathbb{Z}G$-modules we use the ring antiautomorphism $\tau:\mathbb{Z}G \xrightarrow{\sim} \mathbb{Z}G$, obtained by extending the inversion $g \mapsto g^{-1}$ in the group $\mathbb{Z}$-additively. This transforms (21) into the sequence of left $\mathbb{Z}G$-modules

$$\mathbb{Z}G \xrightarrow{(1-a^{-1},\,1-b^{-1})} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}\tau(D_\alpha r) & \tau(D_\alpha s)\\ \tau(D_\beta r) & \tau(D_\beta s)\end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}1-b^{-1}\\ 1-a^{-1}\end{pmatrix}} \mathbb{Z}G. \tag{22}$$

Also $H^1(G,\mathbb{Z}G) \cong \ker \partial_2^*/\mathrm{im}\, \partial_1^*$ is $\mathbb{Z}$-isomorphic to the homology of (22) at the left middle module $\mathbb{Z}G^2$.

So far only the fact that G is given by a 2-generator 2-relator presentation has been used. We now bring into play the pecularity of G that the assignments $\alpha \mapsto a^{-1}$, $\beta \mapsto b^{-1}$ induce an automorphism $\sigma: G \xrightarrow{\sim} G$; indeed

$$(a^{-1}b^{-1})^n \cdot (b^{-1})^{-m} = (ba)^{-n}b^m = b^m(ab)^{-n} = ((ab)^n b^{-m})^{-1} = 1$$

and, similarly, $(b^{-1}a^{-1})^n \cdot (a^{-1})^{-l} = 1$. By means of this isomorphism σ and by exchanging the two summands of both middle modules $\mathbb{Z}G^2$ in (22) we arrive at the isomorphic sequence

$$\mathbb{Z}G \xrightarrow{(1-b,\,1-a)} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}\sigma\tau(D_\beta s) & \sigma\tau(D_\beta r)\\ \sigma\tau(D_\alpha s) & \sigma\tau(D_\alpha r)\end{pmatrix}} \mathbb{Z}G^2 \xrightarrow{\begin{pmatrix}1-a\\ 1-b\end{pmatrix}} \mathbb{Z}G \tag{23}$$

Observe that we know at this stage that the homology is defined at the middle left term and that it is isomorphic to $H^1(G,\mathbb{Z}G)$. Our aim is to verify that the (2×2)-matrix displayed in (23) is identical with the (2×2)-matrix of the sequence $\mathbf{P}$ defined in (20). Once this is known, $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ must be a complex and its homology in dimension 2 equals $H^1(G,\mathbb{Z}G)$. In addition, if $H^1(G,\mathbb{Z}G)=0$ the complex $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ is exact and can be used to compute $H^2(G,\mathbb{Z}G)$. If this is done, the above manipulations show that $H^2(G,\mathbb{Z}G)$ is isomorphic to the homology of (23) at the right middle term, i.e. to the homology of $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ in dimension 1. As this homology is a priori known to be trivial, $H^2(G,\mathbb{Z}G)$ must be trivial and all assertions of Lemma 4 will be established.

The entries $D_\alpha r$ and $D_\beta r$ of the (2×2)-matrix displayed in (20) are:

$$D_\alpha r = D_\alpha((\alpha\beta)^n\beta^{-m}) = (\mathrm{sign}\, n)\cdot(1+ab+\cdots+(ab)^{|n|-1})\cdot(ab)^{(n-|n|)/2}$$

and

$$D_\beta r = D_\beta((\alpha\beta)^n\beta^{-m}) = (\text{sign } n)\cdot(a + aba + \cdots + (ab)^{|n|-1}a)\cdot(ab)^{(n-|n|)/2}$$
$$-(\text{sign } m)\cdot(1 + b + \cdots + b^{|m|-1})\cdot b^{(m-|m|)/2}.$$

The entry $D_\beta s$ arises from $D_\alpha r$ by exchanging throughout a and b, while $D_\alpha s$ arises from $D_\beta r$ by exchanging a and b, and replacing m by l. The composite $\sigma \circ \tau$ transforms a product $x_1x_2 \cdots x_k$, where each factor is one of a, a^{-1}, b or b^{-1}, into the reversed product $x_k \cdots x_2x_1$, and it is $\mathbb{Z}$-linear. Using these facts one checks easily that $\sigma \circ \tau$ exchanges $D_\alpha r$ and $D_\beta s$, while it fixes $D_\beta r$ and $D_\alpha s$, these group ring elements being $\mathbb{Z}$-linear combinations of palindroms. It follows that the (2×2)-matrices displayed in (20) and (23) are identical. □

LEMMA 5. *If* $\min(|l|, |m|, |n|) \geqq 2$ *then* $H^1(G, \mathbb{Z}G) = 0$.

Proof. The claim is true for finite groups on general grounds; but the following argument takes care of them at no extra expense.

Let l_1, m_1, n_1 be integers greater than 1 and let $T(l_1, m_1, n_1)$ be the corresponding triangle group. By a result of Serre's [16, p. 85, **6.3.5**] every inversion-free action of $T(l_1, m_1, n_1)$ on a tree has a fixed point; in particular, no homomorphic image of $T(l_1, m_1, n_1)$ can be a non-trivial amalgam $A *_C B$.

Let us go back to the given group G. If $a^l = b^m$ has finite order, say k, then G is a homomorphic image of $T(l_1, m_1, n_1)$, where $l_1 = k\,|l|$, $m_1 = k\,|m|$ and $n_1 = k\,|n|$, and so it is not a non-trivial amalgam. Moreover, G_{ab} is finite. Stallings' structure theorem (e.g. [17, p. 38, **4.A.6.5** and p. 57, **5.A.10**]) therefore implies that $H^1(G, \mathbb{Z}G)$ is trivial.

If $a^l = b^m$ has infinite order, consider the central extension $Z = \langle a^l\rangle \lhd G \twoheadrightarrow T$. It leads in cohomology to the exact sequence

$$H^1(T, H^0(Z, \mathbb{Z}G)) \rightarrowtail H^1(G, \mathbb{Z}G) \to H^0(T, H^1(Z, \mathbb{Z}G)) \to H^2(T, H^0(Z, \mathbb{Z}G)) \to \cdots. \quad (24)$$

Since Z is infinite, $H^0(Z, \mathbb{Z}G) = (\mathbb{Z}G)^Z$ is trivial; because Z is an orientable Poincare-duality group of dimension one, $H^1(Z, \mathbb{Z}G) = \mathbb{Z} \otimes_{\mathbb{Z}Z} \mathbb{Z}G \cong \mathbb{Z}T$. Finally, since T is infinite by Lemma 3, the exactness of (24) and the previous reasoning imply that $H^1(G, \mathbb{Z}G) \xrightarrow{\sim} H^0(T, \mathbb{Z}T) = 0$, as asserted. □

The *proof of Theorem A* is now quickly completed. By assumption $\min(|l|, |m|, |n|) \geqq 2$ and so Lemmata 3, 4 and 5 apply. They show that the sequence (4), which is identical with (20), is an exact complex and that $H^1(G, \mathbb{Z}G)$

and $H^2(G, \mathbb{Z}G)$ are both trivial. The kernel of $\partial_3: \mathbb{Z}G \to \mathbb{Z}G^2$, taking $\lambda \in \mathbb{Z}G$ to $(\lambda(1-b), \lambda(1-a))$, consists of all $\lambda \in \mathbb{Z}G$ which are fixed by the generators b and a, hence by all of G.

If G is finite, $\ker \partial_3$ is therefore generated by $\sum \{g \mid g \in G\}$ and the sequence

$$\mathbb{Z}G^2 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\sum\{g \mid g\in G\}} \mathbb{Z}G \xrightarrow{\partial_3} \mathbb{Z}G^2$$

is exact. This proves that (4) leads by splicing to a periodic $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ having period 4.

If, on the other hand, G is infinite, $\ker \partial_3$ is trivial and (4) is a finite $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. Moreover, G is an orientable Poincaré-duality group of dimension 3. Indeed, $H^1(G, \mathbb{Z}G)$ and $H^2(G, \mathbb{Z}G)$ are trivial by the previous remarks. Next, if $H^3(G, \mathbb{Z}G)$ is computed by means of (4), one obtains the following chain of isomorphisms of right $\mathbb{Z}G$-modules:

$$H^3(G, \mathbb{Z}G) \cong \mathbb{Z}G/(1-b)\mathbb{Z}G + (1-a)\mathbb{Z}G = \mathbb{Z}G/IG \xrightarrow{\varepsilon_\sim} \mathbb{Z}.$$

The claim then follows from well-known results about duality groups (see, e.g., [1, p. 140, Thm. 9.2 and p. 173], or [2, p. 220, Thm. 10.1, and definition on p. 221]). □

Remark 6. If G is a binary dihedral group $G(l, 2, 2)$, the periodic resolution $\mathbf{P} \twoheadrightarrow \mathbb{Z}$ obtained from sequence (20) by splicing, is isomorphic to the resolution $\mathbf{P}' \twoheadrightarrow \mathbb{Z}$ described by Cartan-Eilenberg [3, p. 252]. In order to see this, identify $G(l, 2, 2)$ and $\pi = \langle x, y;\, x^l = y^2 = (xy)^2 \rangle$ in the obvious way, and note that

$$D_\beta r = D_\beta(\alpha\beta\alpha \cdot \beta^{-1}) = a - 1.$$

The function $\Phi: \mathbf{P} \to \mathbf{P}'$ which respects the dimensions of the chain groups, is the identity in dimensions different from $2+4p$ and sends $(\lambda, \mu) \in P_{2+4p}$ to $(-\mu, \lambda + \mu b) \in P'_{2+4p}$, can then easily be verified to be a chain isomorphism. □

3. Proof of Theorem B

Let F be free on $\{\alpha_1, \ldots, \alpha_n\}$, let ρ be a non-trivial element of F which is not a proper power, let $l \geqq 1$ and set

$$L = \langle \alpha_1, \ldots, \alpha_n;\, [\rho^l, \alpha_1], \ldots, [\rho^l, \alpha_n] \rangle,$$

where $[\rho^l, \alpha_i] := \rho^l \cdot \alpha_i \cdot \rho^{-l} \cdot \alpha_i^{-1}$. If $n=1$, the group L is infinite cyclic and $\mathbb{Z}L \xrightarrow{1-a_1} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z}$ is an explicit $\mathbb{Z}L$-free resolution of $\mathbb{Z}$. So assume $n \geqq 2$. Denote the canonical image in L of α_i by a_i, that of ρ by r and write D_{α_i} for the composite $F \xrightarrow{\partial/\partial_{\alpha_i}} \mathbb{Z}F \xrightarrow{\text{can}} \mathbb{Z}G$. We aim at verifying that the sequence of left $\mathbb{Z}L$-modules and $\mathbb{Z}L$-homomorphisms

$$\mathbb{Z}L \xrightarrow{(D_{\alpha_1}\rho, \ldots, D_{\alpha_n}\rho)} \mathbb{Z}L^n \xrightarrow{\begin{pmatrix} D_{\alpha_1}[\rho^l, \alpha_1] \cdots D_{\alpha_n}[\rho^l, \alpha_1] \\ \vdots \qquad\qquad \vdots \\ D_{\alpha_1}[\rho^l, \alpha_n] \cdots D_{\alpha_n}[\rho^l, \alpha_n] \end{pmatrix}} \mathbb{Z}L \xrightarrow{\begin{pmatrix} 1-a_1 \\ \vdots \\ 1-a_n \end{pmatrix}} \mathbb{Z}L \xrightarrow{\varepsilon} \mathbb{Z} \tag{25}$$

is an exact complex. Our verification will be based on Lemma 7 below and Lyndon's Identity Theorem (cf. [9, pp. 158+161]) which asserts that

$$\mathbb{Z}\bar{L} \xrightarrow{1-\bar{r}} \mathbb{Z}\bar{L} \xrightarrow{(D_{\alpha_1}\rho^l, \ldots, D_{\alpha}\rho^l)} \mathbb{Z}\bar{L} \xrightarrow{\begin{pmatrix} 1-a_1 \\ 1-a_n \end{pmatrix}} \mathbb{Z}\bar{L} \xrightarrow{\varepsilon} \mathbb{Z} \tag{26}$$

is exact; here $\bar{L}$ denotes the 1-relator group $\langle \alpha_1, \ldots, \alpha_n; \rho^l \rangle$.

LEMMA 7. *The canonical image $r \in L$ of $\rho \in F \setminus \{1\}$ has infinite order.*

Proof. Since free groups are residually nilpotent there exists $c \geqq 1$ such that $\rho \in \gamma_c F \setminus \gamma_{c+1} F$. The obvious epimorphism $L \twoheadrightarrow F/\gamma_{c+1}F$ sends r to a non-trivial element of the central subgroup $\gamma_c F/\gamma_{c+1}F$, which is known to be free abelian (cf. [11, p. 341, Cor. 5.12(iv)]). $\square$

We are now ready to prove that (25) is an exact complex. We begin by verifying that the left-most homomorphism $\partial_3 : \mathbb{Z}L \to \mathbb{Z}L^n$ is injective. If $\lambda \in \mathbb{Z}L$ and $\lambda \cdot D_{\alpha_j}\rho = 0$ for all j, then

$$0 = \sum_j (\lambda \cdot D_{\alpha_j}\rho)(1-a_j) = \lambda \sum_j (D_{\alpha_j}\rho \cdot (1-a_j)) = \lambda \cdot (1-r).$$

As r has infinite order by Lemma 7, $(1-r)$ is not a zero-divisor and hence $\lambda = 0$.

Let $\partial_2 : \mathbb{Z}L^n \to \mathbb{Z}L^n$ be the differential of (25) given by the $(n \times n)$-matrix with entries $D_{\alpha_i}[\rho^l, \alpha_j]$. These entries can be described more explicitly; indeed:

$$\begin{aligned} D_{\alpha_i}[\rho^l, \alpha_j] &= D_{\alpha_i}(\rho^l \cdot \alpha_j \cdot \rho^{-l} \cdot \alpha_j^{-1}) = D_{\alpha_i}\rho^l + r^l \cdot D_{\alpha_i}\alpha_j - a_j D_{\alpha_i}\rho^l - D_{\alpha_i}\alpha_j \\ &= (1-a_j) \cdot D_{\alpha_i}\rho^l + (r^l - 1) \cdot \delta_{ij}. \end{aligned}$$

Here δ_{ij} is the Kronecker symbol. A row vector $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}L^n$ lies in the kernel of ∂_2 if, and only if,

$$0 = \sum_j \lambda_j \cdot \{(1-a_j) \cdot D_{\alpha_i}\rho^l + (r^l - 1)\delta_{ij}\}$$

or, equivalently, if

$$\lambda_i \cdot (1-r^l) = \left(\sum_j \lambda_j(1-a_j)\right) \cdot D_{\alpha_i}\rho^l \tag{27}$$

for every $i = 1, 2, \ldots, n$.

Assume $(\lambda_1, \ldots, \lambda_n) \in \ker \partial_2$ and set $\mu := \sum_j \lambda_j(1-a_j)$. By applying the canonical ring epimorphism

$$^{-} : \mathbb{Z}L \twoheadrightarrow \mathbb{Z}\bar{L} := \mathbb{Z}\langle \alpha_1, \ldots, \alpha_n ; \rho^l \rangle$$

to (27) one obtains the equations $0 = \bar{\mu} \cdot \bar{D}_{\alpha_i}\rho^l$, where i ranges over $1, 2, \ldots, n$; they show that $\bar{\mu}$ is in the kernel of the second differential $\bar{\partial}_2$ of the exact sequence (26). There exists therefore $\nu \in \mathbb{Z}L$ with $\bar{\mu} = \bar{\nu}(1-\bar{r})$. The kernel of the canonical projection $L \twoheadrightarrow \bar{L}$ is the cyclic, *central* subgroup generated by r^l, whence $\ker(\mathbb{Z}L \twoheadrightarrow \mathbb{Z}\bar{L})$ is the ideal generated by the central element $1-r^l$. So there will exist $\nu_1 \in \mathbb{Z}L$ with $\mu = \nu_1(1-r)$. The equations (27) now imply that

$$\begin{aligned}\lambda_i \cdot (1-r^l) &= \mu \cdot D_{\alpha_i}\rho^l = \nu_1(1-r)(1+r+\cdots+r^{l-1})D_{\alpha_i}\rho \\ &= \nu_1(1-r^l)D_{\alpha_i}\rho = \nu_1 \cdot D_{\alpha_i}\rho \cdot (1-r^l)\end{aligned} \tag{28}$$

for $i = 1, 2, \ldots, n$. Since r^l has infinite order by Lemma 7, $1-r^l$ is not a zero-divisor of $\mathbb{Z}L$. Therefore (28) implies that $\lambda_i = \nu_1 \cdot D_{\alpha_i}\rho$ for all i, i.e. that $(\lambda_1, \ldots, \lambda_n)$ belongs to $\operatorname{im} \partial_3$. Conversely, if $(\lambda_1, \ldots, \lambda_n)$ is in $\operatorname{im} \partial_3$, i.e., if $\lambda_i = \nu_1 D_{\alpha_i}\rho$ for some $\nu_1 \in \mathbb{Z}L$, one sees by reversing part of (28) that

$$(\nu_1 D_{\alpha_i}\rho)(1-r^l) = \nu_1 \cdot (1-r) \cdot D_{\alpha_i}\rho^l,$$

while it is quite generally true that

$$\nu_1 \cdot (1-r) = \nu_1 \sum_j (D_{\alpha_j}\rho(1-a_j)) = \sum_j (\nu_1 D_{\alpha_j}\rho)(1-a_j).$$

Put together, these equations give

$$(\nu_1 D_{\alpha_i}\rho)(1-r^l) = \sum_j ((\nu_1 D_{\alpha_j}\rho)(1-a_j)) \cdot D_{\alpha_i}\rho^l$$

for every i; they show that the row vector $\partial_3(\nu_1)$ satisfies the equations (27) or, equivalently, that $\partial_2(\partial_3(\nu_1))=0$. The verification that (25) is exact at the left middle module $\mathbb{Z}L^n$ is now complete.

As the right half of (25) is exact on general grounds, the exactness of the entire sequence (25) is established. □

4. Poincaré-duality groups among the groups L

In this section we shall determine which of the groups

$$L=\langle\alpha_1,\ldots,\alpha_n;[\rho^l,\alpha_1],\ldots,[\rho^l,\alpha_n]\rangle$$

are 2-dimensional and which of them are Poincaré-duality-groups of dimension 3. The notation will be as in Section 3; in particular, $\rho\in F=F(\{\alpha_1,\ldots,\alpha_n\})$ is a non-trivial element which is not a proper power, $l\geqq 1$ and $n\geqq 2$. Let $\bar{L}$ and $\bar{\bar{L}}$ designate the 1-relator groups

$$\bar{L}=\langle\alpha_1,\ldots,\alpha_n;\rho^l\rangle \quad\text{and}\quad \bar{\bar{L}}=\langle\alpha_1,\ldots,\alpha_n;\rho\rangle.$$

We begin with the easy

LEMMA 8. (i) $H^1(L,\mathbb{Z}L)=0$.
(ii) $H^{j+1}(L,\mathbb{Z}L)$ *and* $H^j(\bar{L},\mathbb{Z}\bar{L})$ *are* $\mathbb{Z}$*-isomorphic for all* $j\geqq 1$.
(iii) $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ *is a* $\mathbb{Z}L$*-homomorphic image of* $H^3(L,\mathbb{Z}L)$.

Proof. Spectral sequences, applied to the central extension $\langle r^l\rangle\triangleleft L\twoheadrightarrow\bar{L}$, and the fact that $\langle r^l\rangle$ is infinite cyclic by Lemma 7, imply that

$$H^{j+1}(L,\mathbb{Z}L)\cong H^j(\bar{L},H^1(\langle r^l\rangle,\mathbb{Z}L))\cong H^j(\bar{L},\mathbb{Z}\bar{L})$$

for all $j\geqq 0$. These isomorphisms, together with the fact that $\bar{L}$ is infinite and hence $H^0(\bar{L},\mathbb{Z}\bar{L})=0$, establish (i) and (ii). Claim (iii) is a consequence of the facts that the second cohomology group $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ of the torsion-free 1-relator group $\bar{\bar{L}}$ is by Lyndon's Identity Theorem isomorphic with $\mathbb{Z}\bar{\bar{L}}/\sum_j(\bar{\bar{D}}_{\alpha_j}\rho\cdot\mathbb{Z}\bar{\bar{L}})$, whereas $H^3(L,\mathbb{Z}L)$, when computed by the resolution (25), is given by $\mathbb{Z}L/\sum_j(D_{\alpha_j}\rho\cdot\mathbb{Z}L)$. □

2-dimensional groups. Assume the cohomological dimension of L is less than 3. Then $H^3(L,\mathbb{Z}L)=0$ and so $H^2(\bar{\bar{L}},\mathbb{Z}\bar{\bar{L}})$ will be trivial by Lemma 8(iii). But $\bar{\bar{L}}$ is a torsion-free 1-relator group, hence of cohomological dimension at most 2; it is also of type (FP). Therefore the vanishing of $H^2(\bar{\bar{L}}, \mathbb{Z}\bar{\bar{L}})$ implies that $\bar{\bar{L}}$ is at most 1-dimensional (cf. [1, p. 137, Lemma 9.1]), and so $\bar{\bar{L}}$ is free by Stallings' theorem. By a result of J. H. C. Whitehead's (see, e.g., [9, p. 106, Prop. 5.10]) a 1-relator group can only be free if the defining relator is either trivial or primitive. Since $\rho \neq 1$, we conclude that ρ must be a member of some basis of $F = F(\{\alpha_1, \ldots, \alpha_n\})$.

After a change of notation, we will have $\rho = \alpha_1$ and

$$L = \langle \alpha_1, \ldots, \alpha_n; [\alpha_1^l, \alpha_2], \ldots, [\alpha_1^l, \alpha_n] \rangle. \tag{29}$$

This group can ve viewed as an HNN-extension with base group $\langle a_1 \rangle$, associated subgroups both equal to $\langle a_1^l \rangle$ and stable letters $\alpha_2, \ldots, \alpha_n$; it can also be obtained from the direct product $\mathbb{Z} \times F(\{\alpha_2, \ldots, \alpha_n\})$ by adjoining an l-th root. Both descriptions allow to infer that every group of the form (29) has cohomological dimension precisely 2 (recall that $n \geqq 2$). Incidentally, the second description of L reveals also that if $l > 1$ or $n > 2$, the group L has a subgroup of infinite index which is free abelian of rank 2, hence, in particular, 2-dimensional, and therefore L cannnot be a Poincaré-duality group of dimension 2 (by [18]). So the only Poincaré-duality group of the form (29) is $L = \langle \alpha_1, \alpha_2; [\alpha_1, \alpha_2] \rangle$.

3-dimensional Poincaré-duality groups. A second application of Lemma 8 will be made in the proof of

THEOREM 9. *The following statements are equivalent:*

(i) *There exists either a basis* $\xi_1, \eta_1, \ldots, \xi_g, \eta_g$ *of* $F(\{\alpha_1, \ldots, \alpha_n\})$ *with* $\rho = [\xi_1, \eta_1] \cdot \cdots \cdot [\xi_g, \eta_g]$, *or a basis* $\xi_1, \xi_2, \ldots, \xi_g$ *with* $\rho = \xi_1^2 \cdot \xi_2^2 \cdot \cdots \cdot \xi_g^2$.

(ii) $H^3(L, \mathbb{Z}L)$ *is infinite cyclic.*

(II) L *is a* 3-*dimensional Poincaré-duality group.*

Proof. If (i) is true L has a presentation

$$\left\langle \xi_1, \eta_1, \ldots, \xi_g, \eta_g; \left(\prod_i [\xi_i, \eta_i]\right)^l \text{ commutes with all } \xi_j, \eta_j \right\rangle \tag{30}$$

or a presentation

$$\langle \xi_1, \ldots, \xi_g; (\xi_1^2 \cdot \xi_2^2 \cdot \cdots \cdot \xi_g^2)^l \text{ commutes with all } \xi_j \rangle. \tag{31}$$

If $H^3(L, \mathbb{Z}L)$ is computed by means of the resolution (25) one finds that $H^3(L, \mathbb{Z}L) \cong \mathbb{Z}L / \sum_\beta (D_\beta \rho \cdot \mathbb{Z}L)$, where β ranges over the basis displayed in (30) or (31).

Assume first L has a presentation of the form (30); we contend that $\sum_\beta (D_\beta\rho \cdot \mathbb{Z}L)$ is the augmentation ideal IL of $\mathbb{Z}L$. In order to verify this we shall prove more, namely that

$$I := \sum_j (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}F + \sum_j (\partial/\partial\eta_j(\rho)) \cdot \mathbb{Z}F$$

is the augmentation ideal IF of $\mathbb{Z}F$. The key to this is the fact that the partial derivatives of $\rho' := [\xi_1, \eta_1] \cdots [\xi_{g-1}, \eta_{g-1}]$ with respect to $\xi_1, \eta_1, \ldots, \xi_{g-1}$ and η_{g-1} agree with those of ρ. Hence we can assume inductively that

$$J := \sum_{1=j}^{g-1} \frac{\partial\rho}{\partial\xi_j} \cdot \mathbb{Z}F + \sum_{1=j}^{g-1} \frac{\partial\rho}{\partial\eta_j} \cdot \mathbb{Z}F = \sum_{1=j}^{g-1} (1-\xi_j) \cdot \mathbb{Z}F + \sum_{1=j}^{g-1} (1-\eta_j) \cdot \mathbb{Z}F.$$

In particular, $(1-\rho')$ belongs to J. Let ξ, resp. η be short for ξ_g, resp. η_g. From

$$\partial/\partial\xi(\rho) = \rho' \cdot (1-\xi\eta\xi^{-1}) \quad \text{and} \quad \partial/\partial\eta(\rho) = \rho' \cdot (\xi - [\xi, \eta])$$

one deduces that $I \subseteq IF$, and that $(1-\xi\eta\xi^{-1})$ and $(\xi-[\xi,\eta])$ belong to I. Hence so do

$$(1-\xi\eta\xi^{-1}) \cdot (1-\xi) \qquad\qquad -(\xi-[\xi,\eta]) \cdot \eta = 1-\xi$$
$$(1-\xi\eta\xi^{-1}) \cdot (\xi\eta-\eta+1) + (\xi-[\xi,\eta]) \cdot (\eta^2-\eta) = 1-\eta,$$

and thus $I = IF$, as contended.

Assume next L is of the form (31). We assert that $\sum_j (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}L = \sum_j (1+x_j) \cdot \mathbb{Z}L$; it will suffice to establish the corresponding statement for $\mathbb{Z}F$. Set $\rho' := \xi_1^2 \cdot \cdots \cdot \xi_{g-1}^2$. Then $\partial/\partial\xi_j(\rho') = \partial/\partial\xi_j(\rho)$ for $j = 1, \ldots, g-1$ and thus

$$J := \sum_{1 \leq j < g} (\partial/\partial\xi_j(\rho)) \cdot \mathbb{Z}F = \sum_{1 \leq j < g} (1+\xi_j) \cdot \mathbb{Z}F.$$

Since $1-\rho' = \sum_{1\leq j<g} (\partial/\partial\xi_j(\rho))\ (1-\xi_j)$ is in J and as $(\partial/\partial\xi_g(\rho)) = \rho' \cdot (1+\xi_g)$, the assertion is established. Finally the quotients $\mathbb{Z}F/\sum_j (1+\xi_j) \cdot \mathbb{Z}F$ and $\mathbb{Z}L/\sum_j (1+x_j) \cdot \mathbb{Z}L$ are infinite cyclic, as can easily be verified and so (ii) holds also for the groups of the form (31).

Conversely, assume (ii) is true. Then $I := \sum_j D_{\alpha_j}\rho \cdot \mathbb{Z}L$ contains the element $1-r$ and I is a two-sided ideal. Therefore

$$H^2(\bar{\bar{L}}, \mathbb{Z}\bar{\bar{L}}) \cong \mathbb{Z}\bar{\bar{L}} \Big/ \sum_j (\bar{\bar{D}}_{\alpha_j}\rho) \cdot \bar{\bar{\mathbb{Z}}}L \cong \mathbb{Z}L/(I + \mathbb{Z}L(1-r)\mathbb{Z}L) \cong \mathbb{Z}L/I$$

is infinite cyclic. As in [1, p. 155, Remark] one sees next that $\bar{\bar{L}}$ is actually a Poincaré-duality group of dimension 2, and hence a surface group by a result of B. Eckmann and H. Müller [8, Thm. 1]. Work of H. Zieschang and N. Peczynski ([21], [12], cf. [22, p. 58, **2.11.9**]) now guarantees the existence of a basis of $F(\{\alpha_1, \ldots, \alpha_n\})$ with either $\rho = \prod_j [\xi_j, \eta_j]$ or $\rho = \prod_j \xi_j^2$.

Finally, we establish that (ii) implies (iii), the converse being evident. It suffices to show that (ii) implies that $H^1(L, \mathbb{Z}L) = 0 = H^2(L, \mathbb{Z}L)$. From Lemma 8 one sees that $H^1(L, \mathbb{Z}L) = 0$, and that

$$H^2(\bar{L}, \mathbb{Z}\bar{L}) \cong H^3(L, \mathbb{Z}L) \cong \mathbb{Z} \quad \text{and that} \quad H^2(L, \mathbb{Z}L) \cong H^1(\bar{L}, \mathbb{Z}\bar{L}).$$

Much as in [1, p. 155, Remark] one deduces from Stallings' theory of groups with infinitely many ends and from $H^2(\bar{L}, \mathbb{Z}\bar{L}) \cong \mathbb{Z}$ that $H^1(\bar{L}, \mathbb{Z}\bar{L}) = 0$, whence $H^2(L, \mathbb{Z}L) = 0$. $\square$

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Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2

MICHAEL R. HERMAN

1. Introduction et notations

1.1. Pour $r>0$ on considère le polydisque

$$D_r^n=\{(z_1,\ldots,z_n)\in\mathbb{C}^n \mid |z_i|\le r,\ 1\le i\le n\},$$

où, si $z\in\mathbb{C}$, $|z|^2=z\bar z$, ainsi que le polycercle

$$\mathbb{T}_r^n=\{(z_1,\ldots,z_n)\in\mathbb{C}^n \mid |z_i|=r,\ 1\le i\le n\},$$

qui est difféomorphe à $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$, et on désigne la mesure de Haar normalisée sur $\mathbb{T}_r^n$ par m ou $d\theta$ ($\mathbb{T}^1$ sera toujours identifié au cercle unité $\mathbb{T}_1^1$ de $\mathbb{C}$ par le difféomorphisme $t\to e^{2\pi it}$).

1.2. Soit X un espace compact métrisable non vide et m une mesure de probabilité sur X. On se donne une application continue g de X dans X et une application continue A de X dans une algèbre de Banach $(\mathcal{B},\|\ \|)$ où $\|\ \|$ est une norme d'algèbre de $\mathcal{B}$ sur $\mathbb{R}$.

Au couple (g,A) on associe l'application fibrée (aussi appelée produit gauche, produit croisé ou skew produit)

$$\bar G:(x,y):X\times\mathcal{B}\to(g(x),A(x)y)\in X\times\mathcal{B}$$

au dessus de $g:X\to X$. On a, si $k\in\mathbb{N}^*$,

$$\bar G^k(x,y)=(g^k(x),A_g^k(x)y),\qquad A_g^k(x)=A(g^{k-1}(x))\cdots A(x).$$

On pose

$$\lambda_+(g,A)=\liminf_{k\to+\infty}\frac{a_k}{k}\quad\text{où}\quad a_k=\int_X \operatorname{Log}\|A_g^k(x)\|\,dm(x),\qquad \lambda_+(g,A)\in\mathbb{R}\cup\{-\infty\}.$$

(L'intégrale étant ici et dans la suite l'intégrale inférieure.) (on peut avoir $\lambda_+(g, A) = -\infty$). On note aussi $\lambda_+(g, A) = \lambda_+(X, g, A)$ pour indiquer que $g : X \to X$.

La valeur de $\lambda_+(g, A)$ ne change pas si on remplace la norme de $\mathcal{B}$ par une norme d'espace de Banach équivalente. Ceci implique $\lambda_+(g, A) = \lambda_+(g, A \otimes 1)$, où $(A \otimes 1)(x) = A(x) \otimes 1 \in \mathcal{B} \otimes_{\mathbb{R}} \mathbb{C}$ et $\mathcal{B} \otimes_{\mathbb{R}} \mathbb{C}$ est l'algèbre de Banach compléxifié de $\mathcal{B}$, sur laquelle on peut par exemple choisir pour norme sur $\mathbb{C}$ $\|x \otimes \lambda\| = \|x\| \, |\lambda|$.

1.3. Si l'application g préserve la mesure m (i.e. $g_* m = m$) alors la suite (a_k) est sous additive (i.e. $a_{k+p} \leq a_k + a_p$ pour tout entier $k \geq 1$ et $p \geq 1$) et on a

$$\lim_{k \to +\infty} \frac{a_k}{k} = \operatorname{Inf}_{k \geq 1} \frac{a_k}{k}$$

et de plus la suite $(2^{-p} a_{2^p})_{p \geq 1}$ est décroissante. Sous la même hypothèse, par le théorème ergodique sous-additif [3] la suite de fonction $((1/k) \operatorname{Log} \|A_g^k(x)\|)_{k \geq 1}$ converge, si $k \to +\infty$, m-presque partout vers une fonction Ψ, presque partout invariante par g et vérifiant $\int_X \Psi(x)\, dm = \lambda_+(g, A)$.

Si $g_* m = m$, l'application $(g, A) \to \lambda_+(g, A) \in \mathbb{R} \cup \{-\infty\}$ est semi-continue supérieurement si sur les couples (g, A) on met la topològie de la convergence uniforme. Dans la littérature $\lambda_+(g, A)$ s'*appelle l'exposant de Lyapounov maximal.*

1.4. Il y a un cas que nour allons décrire où, pratiquement par définition, on a $\lambda_+(g, A) > 0$.

On se place dans la situation suivante: on suppose que $\mathcal{B}_1$ est un espace de Banach de norme notée $\| \ \|_1$ et A une application continue de X dans l'algèbre $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_1)$ des opérateurs $\mathbb{R}$-linéaires continus de $\mathcal{B}_1$ dans $\mathcal{B}_1$ avec la topologie de la norme. On fait agir (g, A) sur $X \times \mathcal{B}_1$ par $G(x, y) = (g(x), A(x)y)$. Si m est une mesure de probabilité sur X on définit $\lambda_+(G) = \lambda_+(\bar{G})$.

DÉFINITION. *On dit que l'application fibrée G a une structure partiellement hyperbolique si le fibré trivial $X \times \mathcal{B}_1 \to X$ est la somme directe de 2 fibrés continus E^S et E^u (si $x \in X, E_x^s$ désigne la fibre en $x \in X$ de E^s) et s'il existe des nombres l_1 et l_2 vérifiant, $l_1 > 1$, $l_1 > l_2 > 0$, et $C > 1$ tels que, quels que soient $x \in X$ et $n \in \mathbb{N}^*$, on ait*:

- *pour tout* $x \in X$, $\dim(E_x^u) \neq 0 \neq \dim(E_x^s)$;
- *si* $v_x \in E_x^u$, $\|G^n(v_x)\|_1 \geq C^{-1} l_1^n \|v_x\|_1$,
- *si* $w_x \in E_x^s$, $\|G^n(w_x)\|_1 \leq C l_2^n \|w_x\|_1$.

On dit que l'application fibrée G a une structure hyperbolique quand de plus on peut choisir $l_2 < 1$.

Remarques. 1) Cette définition implique que les fibrés E^u et E^s sont sous-invariants par G (i.e. $G(E^u) \subset E^u$ $G(E^s) \subset E^s$).

2) Si on se restreint aux G qui sont des *homéomorphismes fibrés*, le fait que G ait une *structure partiellement hyperbolique* (resp. *hyperbolique*) est une propriété *stable* par perturbation de G dans la topologie la convergence uniforme [18, p. 100–1].

1.5. *Résumé de l'article*

Nous nous proposons de donner une méthode pour constuire sur $\mathbb{T}^n_{r_0} \times \mathbb{C}^2$ ou $\mathbb{T}^n_{r_0} \times \mathbb{R}^2$ des examples explicites d'applications fibrées G $\mathbb{R}$-analytiques vérifiant $\lambda_+(\mathbb{T}^n_{r_0}, G) > 0$ mais n'ayant pas de structure hyperbolique. Millionščikov a suggéré en 1969 la possibilité d'exemples au-dessus d'une rotation irrationnelle de $\mathbb{T}^1$ [13] (voir aussi [10] et [19]) (mais de tels exemples ne sont ni explicites, ni précis en ce qui concerne les rotations qu'on peut choisir).

Au §2 nous donnons une méthode abstraite pour minorer, sur des exemples d'applications fibrées $G : D^n_{r_0} \times \mathbb{C}^p \to D^n_{r_0} \times \mathbb{C}^p$ holomorphes l'exposant $\lambda_+(G)$. Cette méthode est basée sur l'utilisation des propriétés des fonctions plurisousharmoniques. En 2.8 et 2.9 nous inclurons une généralisation aux groupes compacts abéliens dont les groupes duaux sont totalement ordonnables.

Au §3 nous étudions des exemples où la méthode du §2 s'applique immédiatement.

En 3.2 nous démontrons un corollaire (immédiat) du théorème de C. L. C. Siegel, tel qu'il a été généralisé par E. Zehnder [20], sur les formes normales d'une application holmorphe au voisinage d'un point fixe: le théorème de Siegel fibré au-dessus d'une rotation.

Les exemples 3.3 et 3.5 montrent que le théorème de Siegel fibré est un théorème local et ceci indépendamment de toute condition arithmétique (il est immédiat de voir que le théorème de Siegel est local ainsi que le montre l'exemple 3.4).

En 3.7 nous donnons un exemple au-dessus d'un difféomorphisme d'Anosov ayant des exposants mais pas de structure hyperbolique.

Au §4 nous donnons des exemples de difféomorphismes fibrés $\mathbb{R}$-analytiques F de $\mathbb{T}^1 \times \mathbb{R}^2$ $F = (R_\alpha, A)$ où $R_\alpha(\theta) = \theta + \alpha$ et $A : \theta \in \mathbb{T}^1 \to A(\theta) \in SL(2, \mathbb{R})$. Ces exemples sont encore une application de la méthode abstraite du §2.

En 4.1 nous retrouvons l'exemple que nous avons construit dans [5] et je pense que le §2 est la version abstraite induite par cet exemple, qui n'a pas de structure hyperbolique puisque $\theta \to A(\theta)$ n'est pas homotope à une matrice constante unité (cf 4.2).

En 4.5 et 4.7 nous donnons des exemples où l'application $\theta \in \mathbb{T}^1 \to A(\theta) \in SL(2, \mathbb{R})$ est homotope à la matrice constante unité.

Pour s'assurer qu'on peut choisir, dans l'exemple 4.5, F sans structure hyperbolique on montre que $\lambda_+(R_\alpha, \tilde{R}_\beta \cdot A) > 0$ pour tout $\alpha \in \mathbb{T}^1$ et $\beta \in \mathbb{T}^1$, où

$$\tilde{R}_\beta = \begin{pmatrix} \cos 2\pi\beta & -\sin 2\pi\beta \\ \sin 2\pi\beta & \cos 2\pi\beta \end{pmatrix}$$

et on choisit β de façon ad hoc en utilisant la théorie du nombre de rotation fibrée. On peut même fixer arbitrairement dans $(\mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})) \times \mathbb{T}^1$ le vecteur de rotation du difféomorphisme induit par F sur $\mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2)$ ayant des exposants et pas de structure hyperbolique (voir 4.6).

Nos exemples one l'avantage d'être $\mathbb{R}$-analytiques et de ne dépendre d'aucune condition arithmétique sur le vecteur de rotation.

4.13 montre que ces exemples n'ont pas en général des propriétés analogues à celles des contre-exemples de Denjoy sur le cercle.

L'exemple 4.14 a des propriétés analogues à celles des exemples suggérés par Millionščikov [13], exemples à propos desquels le lecteur se rapportera à R. A. Johnson [10, 3.13 et §5] pour des démonstrations et quelques propriétés. Le lecteur consultera aussi [8] et [9].

Les §§4.14 à 4.16 sont très semblables à certains des résultats de R. Johnson [10] bien que l'auteur de ces lignes les ait obtenus indépendamment. Dans l'annexe 4.17 nous avons, pour la commodité du lecteur, inclus une proposition essentiellement due à R. Johnson.

Au §5 nous définissons et démontrons quelques propriétés du nombre de rotation fibré pour des homéomorphismes de la forme $F : (x, \theta) \in X \times \mathbb{T}^1 \to (g(x), h(x)(\theta)) \in X \times \mathbb{T}^1$, où $x \to h(x) \in \mathrm{Homeo}_+(\mathbb{T}^1)$ est homotope à l'application constante identité et X est un espace compact métrique. La raison de l'esistence en est presque la même que pour le nombre de rotation d'un homéomorphisme du cercle et la démonstration que nous en proposons est presque celle que nous avons donnée pour les homéomorphismes du cercle [4, II]. Nous étudions aussi les propriétés analogues à celles qu'on a pour le cercle [4, II et III]. Pour d'autres généralisations aux homéomorphismes de $\mathbb{T}^n$ homotope à l'Id le lecteur se rapportera [4, XIII] (la situation est infiniment plus compliquée).

Le théorème d'Arnold et de Moser [4, Appendice] a des corollaires fibrés 5.11, 5.12 et 5.14. Le corollaire 5.12 est l'analogue de la proposition [4, A.2.3] pour les difféomorphismes du cercle. Le corollaire 5.14 (presque immédiat) du théorème d'Arnold et de Moser affirme que, pour les matrices fibrées à valeurs dans $PSL(2, \mathbb{R})$ au-dessus de translations diophantiennes de $\mathbb{T}^{n-1}$, la conjugaison fibrée de 5.12 se fait par des matrices fibrées. Pour des généralisations à des matrices de plus de 2 variables et un affaiblissement des conditions diophantiennes, le lecteur se rapportera à J. Moser [14] et en classe C^∞ à [6]. Le lecteur peut aussi consulter H. Rüssmann [17].

Les exemples 4.6 et 4.12 montrent que le théorème d'Arnold et de Moser sur $\mathbb{T}^n$, $n \geq 2$ est en un certain sens un théorème local, et ceci indépendamment de toute condition d'analycité ou d'approximation par les rationnels du vecteur de rotation, ce qui contraste avec le cas du cercle [4, IX].

Au §6.1 nous étudions la dépendance plurisousharmonique en fonction de paramètres complexes de $\lambda_+(g, A)$. En 6.2 et 6.3 nous donnons des applications dont 6.2 nous semble inattendue.

Dans l'exposé des exemples nous avons évité une trop grande généralité et un caractère exhaustif bien que nos méthodes soient tout à fait générales. Le principe est qu'ayant la minoration du §2, on peut ensuite faire des modifications tout en gardant la minoration. La minoration du Scolie de 4.1 est en général instable par perturbation, et on peut montrer dans l'exemple 4.1 par une perturbation C^0 que l'exposant tombe à 0.

L'existence du nombre de rotation fibré pour certains homéomorphismes fibrés de $X \times \mathbb{T}^1$ a été trouvée indépendamment de l'auteur de cet article par R. Johnson (un peu avant) [11], par une méthode très semblable. En fait la méthode est la même que celle de [2] et [4, II]. Comme notre démonstration est plus générale et que nous avons besoin de certaines des propriétés de 5.9 pour 5.12 et 4.12 nous avons inclus notre démonstration.

Les exemples de 4.7 confirment très simplement et généralisent une conjecture de G. André et S. Aubry [1, 4.4]. J. Avron et B. Simon annoncent dans [A] une démonstration rigoureuse de l'argument esquissé par G. André et S. Aubry [1, 4.4].

Une partie des résultats a été annoncée au séminaire de théorie Ergodique de l'Université Paris VI en Janv. et Février 1980 (i.e. essentiellement le §2 et 4.1) ainsi qu'au séminaire de théorie ergodique tenu aux Plans sur Bex en Mars 1980.

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1.6. *Notations*

Si X est un espace compact métrique, on note $C^0(X) = C^0(X, \mathbb{R})$ l'espace des fonctions continues sur X à valeurs réelles avec la norme $\|\varphi\|_{C^0} = \sup_{x \in X} |\varphi(x)|$. Si

Y est un espace topologique, $C^0(X, Y)$ désigne les applications continues de X dans Y avec la topologie compacte ouverte. Si Ψ est une application de X dans Y le graphe de Ψ est l'esemble $\{(x, \Psi(x)) \in X \times Y \mid x \in X\}$.

Pour $K = \mathbb{R}$ ou $\mathbb{C}$, $SL(2, K)$ désigne le groupe des matrices 2×2 sur K déterminant 1.

Pour p un entier positif on désigne par $\mathscr{L}(\mathbb{R}^p, \mathbb{R}^p)$ (resp. $\mathscr{L}_{\mathbb{C}}(\mathbb{C}^p, \mathbb{C}^p)$) les applications $\mathbb{R}$-linéaires (resp. $\mathbb{C}$ linéaires) de $\mathbb{R}^p$ dans $\mathbb{R}^p$ (resp. $\mathbb{C}^p$ dans $\mathbb{C}^p$). On supposera toujours que les espaces vectoriels $\mathbb{R}^p$ (resp. $\mathbb{C}^p$) sont munis de leurs bases canoniques, et on identifiera les espaces $\mathscr{L}(\mathbb{R}^p, \mathbb{R}^p)$ et $\mathscr{L}_{\mathbb{C}}(\mathbb{C}^p, \mathbb{C}^p)$ à des espaces de matrices. On considérera toujours $\mathscr{L}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2)$ comme espace vectoriel sur $\mathbb{C}$.

On désignera par $\mathbb{P}(\mathbb{R}^2)$ l'espace projectif sur $\mathbb{R}$ de dimension 1 (i.e. les droites de $\mathbb{R}^2$ passant l'origine). Le groupe $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/_{\{-e, e\}}$ où e est la matrice unité, agit canoniquement par transformations projectives sur $\mathbb{P}(\mathbb{R}^2)$. On supposera toujours que $SL(2, \mathbb{R})$ ou $PSL(2, \mathbb{R})$ agit sur $\mathbb{P}(\mathbb{R}^2)$ par cette action. L'espace $\mathbb{P}(\mathbb{R}^2)$ est identifié à $\mathbb{T}^1$ par $\mathbb{P}(\mathbb{R}^2) = \mathbb{T}^1/_{x \sim -x}$.

2. Théorème de minoration des exposants

2.1. On considère une application f holomorphe d'un voisinage de $0 \in \mathbb{C}^n$ dans $\mathbb{C}^n$ vérifiant la condition suivante.

$$*_{r_0}\begin{cases} \text{a) Il existe } r_0 > 0 \text{ tel que } f \text{ soit holomorphe sur un voisinage de } D^n_{r_0}. \\ \text{b) } f(D^n_{r_0}) \subset D^n_{r_0} \text{ et } f(\mathbb{T}^n_{r_0}) \subset \mathbb{T}^n_{r_0}. \\ \text{c) } f(0) = 0. \end{cases}$$

Si f vérifie la condition $*'_{r_0}$ et

d) f laisse invariant la mesure de Haar sur $\mathbb{T}^n_{r_0}$ (i.e. $f_* m = m$),[1]

on dit que f vérifie la condition $*'_{r_0}$.

2.2. *Exemples d'applications vérifiant* $*'$

a) Pour tout n, l'application

$$(z_1, \ldots, z_n) \to (\beta_1 z_1, \ldots, \beta_n z_n) \qquad |\beta_i| = 1, 1 \le i \le n,$$

vérifie la condition $*'_r$ pour tout $r > 0$.

b) Si $n = 1$, l'application $z \to z^2$ vérifie $*'_1$.

[1] $f_* m$ désigne la mesure image directe de la mesure m par l'application continue f.

c) Si $n=2$, $(z_1, z_2)\to(z_1^2 z_2, z_1 z_2)$ vérifie $*_1'$, cette application sur $\mathbb{T}_1^2$ étant conjugúee à l'automorphisme de $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ défini par la matrice $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

d) Si $n=2$, $(z_1, z_2)\to(z_1(z_2-b)/(1-\bar{b}z_2), \beta z_2)$, où b et $\beta\in\mathbb{C}$, $|b|<1$, $|\beta|=1$, vérifie $*_1'$.

e) Si $n=2$, $b\in\mathbb{C}$, $0<|b|<1$ alors $(z_1, z_2)\to(z_1^2(z_2-b)/(1-\bar{b}z_2), z_1 z_2)$ vérifie $*_1$ mais pas $*_1'$.

2.3. On se donne f vérifiant la condition $*_{r_0}$ et une application holomorphe A d'un voisinage de $D_{r_0}^n$ dans une algèbre Banach sur $\mathbb{C}$, $\mathcal{B}$, $\| \;\|$ étant une norme sur $\mathbb{C}$ de l'algèbre $\mathcal{B}$. Sur $\mathbb{T}_{r_0}^n$ on met la mesure de Haar m. On note encore f la restriction de f à $\mathbb{T}_{r_0}^n$. On considère l'application fibrée (f, A) de $\mathbb{T}_{r_0}^n\times\mathcal{B}$. Si $a\in\mathcal{B}$ on note le rayon spectral de a par

$$\text{Rspec}\,(a)=\lim_{n\to+\infty}\|a^n\|^{1/n}=\operatorname*{Inf}_{n\geq 1}\|a^n\|^{1/n}.$$

THÉORÈME. *Sous les hypothèses ci-dessus on a*

$$\lambda_+(f, A)\geq \text{Log}\,(\text{Rspec}\,(A(0))).$$

Démonstration. La fonction $z\to(1/k)\,\text{Log}\,\|A_f^k(z)\|$ est plurisousharmonique, voir [7, 2.6.1] (on se ramène au cas $n=1$ et on fait la même démonstration que [7, 1.6.6]). On a (cf. par exemple [12]):

$$\frac{a_k}{k}=\int_{\mathbb{T}_{r_0}^n}\frac{1}{k}\,\text{Log}\,\|A_f^k(z)\|dm\geq\frac{1}{k}\,\text{Log}\,\|A_f^k(0)\|=\frac{1}{k}\,\text{Log}\,\|A^k(0)\|;$$

or

$$\operatorname*{Inf}_{k\geq 1}\frac{1}{k}\,\text{Log}\,\|A^k(0)\|=\text{Log}\,(\text{Rspec}(A(0)))$$

et donc

$$\operatorname*{Inf}_{k\geq 1} a_k/k\geq\text{Log}\,(\text{Rspec}(A(0))).\quad\blacksquare$$

2.4. *Remarque.* Soit f l'exemple 2.2 a) et A une application holomorphe d'un voisinage de $D_{r_0}^n$ dans une algèbre de Banach $\mathcal{B}$. L'application (f, A) définit une famille d'applications fibrées dépendant d'un paramètre $0\leq r\leq r_0$ de $\mathbb{T}_r^n\times\mathcal{B}$ dans lui-même. Soit l'application $r\in[0, r_0[\to\lambda_+(\mathbb{T}_r^n, f, A)$ aussi notée $\lambda_+(r, f, A)$. On a les propriétés suivantes (pour f et A fixés):

a) $r \to \lambda_+(r, f, A)$ est monotone non décroissante;

b) $r \to \lambda_+(r, f, A)$ est convexe en $\operatorname{Log} r$;

c) $r \in [0, r_0[\to \lambda_+(r, f, A)$ est continue;

d) $\lambda_+(0, f, A) = \operatorname{Log}(\operatorname{Rspec}(A(0)))$;

e) si $\lambda_+(r, f, A) = -\infty$ pour un $r > 0$ alors pour tout r, $\lambda_+(r, f, A) = -\infty$.

En effet, par la même démonstration que celle de 6.1, la fonction $z \in \mathbb{C} \to \lambda_+(|z|, f, A)$ est sousharmonique et il suffit d'appliquer [12.2.3].

2.5. Si dans 2.3 on suppose que $f : \mathbb{T}^n_{r_0} \to \mathbb{T}^n_{r_0}$ est totalement uniquement ergodique (i.e. pour $n \neq 0$, f^n est uniquement ergodique, pour un exemple cf. 2.2. a)), alors, par une démonstration analogue à celle de [5], on peut prouver 2.3 en n'utilisant que le principe du maximum.

2.6. La minoration de 2.3 est stable par parturbation de (f, A) vérifiant les conditions de 2.3 et en supposant de plus que $A(0)$ est un point de continuité de la fonction $\operatorname{Rspec} : \mathcal{B} \to \mathbb{R}_+$ (cette fonction est continue en tout point de $\mathcal{B}$ si $\mathcal{B}$ est une algèbre unitaire de dimension finie). On peut montrer que l'on n'a pas, en général, de minoration stable si l'on perturbe l'application fibrée (f, A) de $\mathbb{T}^n_{r_0} \times \mathcal{B}$ dans la topologie de la convergence uniforme.

2.7. Il serait intéressant de savoir si, dans le théorème 2.3, on peut évaluer la différence $\lambda_+(f, A) - \operatorname{Log}(\operatorname{Rspec}(A(0)))$ en utilisant la théorie du potentiel. Si $n = 1$, cette différence peut être envisagée, en un sens à préciser, comme une généralisation de la formule de Jensen (cf. 6.1).

2.8. *Groupes abéliens compacts dont les groupes duaux sont totalement ordonnables*

On suppose que $X = G$ est un groupe abélien compact métrique dont le groupe dual $\hat{G}$ soit sans torsion. Ceci équivaut à dire que G est connexe. On suppose la loi de groupe de $\hat{G}$ notée additivement.

Il suit de [16, 8.1.2] que $\hat{G}$ peut être considéré comme un sous-groupe du groupe additif $\mathbb{R}$ (i.e. G est un groupe solénoïdal).

$\hat{G}$ peut donc être muni d'un ordre total P compatible avec sa structure de groupe: il existe un monoïde $P \subset \hat{G}$ vérifiant:

$$0 \in P, \qquad P + P \subset P, \qquad P \cap (-P) = \{0\}, \qquad P \cup (-P) = \hat{G}.$$

Ce que nous allons voir dépend de l'ordre P choisi sur $\hat{G}$ et on a souvent intérêt dans les exemples à choisir P de différentes façons.

On met sur G la mesure de Haar normalisée m. On suppose que $g : G \to G$ est une affinité continue P-positive: g est la composition d'une translation de G et d'un endomorphisme continu P-positif: $\hat{g}(P) \subset P$, où $\hat{g}: \hat{G} \to \hat{G}$ est l'endomorphisme dual de g.

Si l'affinité g est surjective, alors g préserve la mesure de Haar m de G (un endomorphisme continu est surjectif si et seulement si $\hat{g}$ est injective).

EXEMPLES. · On convient que toute translation de G est P-positive.

· Soient $G = \mathbb{T}^2$, $\hat{g} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ et $(1, \lambda)$ une direction propre de $\hat{g}$.

On choisit $P = \{(m, n) \in \mathbb{Z}^2 \mid m + \lambda n \geq 0\}$, voir [16. 8.1.7]. L'automorphisme g de $\mathbb{T}^2$ est P-positif.

2.9. Soit $\mathscr{B}$ un algèbre de Banach sur $\mathbb{C}$ avec la norme $\| \ \|$. On définit $H^\infty_{P,\mathscr{B}} \equiv H^\infty_P(G, m, \mathscr{B}) = \{\varphi \in L^\infty(G, m, \mathscr{B}) \mid \hat{\varphi}(x) = 0 \text{ si } x \notin P\}$. On vérifie que $H^\infty_{P,\mathscr{B}}$ est une algèbre de Banach et si φ et $\psi \in H^\infty_{P,\mathscr{B}}$, on a

$$\int \varphi\psi \, dm = \left(\int \varphi \, dm\right) \cdot \left(\int \psi \, dm\right).$$

(Il suffit d'utiliser la densité dans $H^\infty_{P,\mathscr{B}}$ des polynômes trigonométriques de $H^\infty_{P,\mathscr{B}}$, pour la topologie de L^2, propriété résultant d'arguments standards en considérant la convolution de $\varphi \in H^\infty_{P,\mathscr{B}}$ par des polynômes trigonométriques de G à valeurs réelles.)

On définit de façon analogue les espaces $H^q_{P,\mathscr{B}}, q \geq 1$. Pour plus de détails sur cette généralisation, due à Helson et Lowdenslager, des espaces d'Hardy le lecteur consultera [16, chap. 8].

PROPOSITION. *Si $\varphi \in H^1_{P,\mathscr{B}}$ alors on a*

$$\operatorname{Log} \|\hat{\varphi}(0)\| \leq \int_G \operatorname{Log} \|\varphi(x)\| \, dm(x).$$

Démonstration. Par [16, p. 205], si l'on pose $\Delta(\varphi) = \exp \int \operatorname{Log} \|\varphi(x)\| \, dm(x)$ alors

$$\Delta(\varphi) = \operatorname*{Inf}_{Q \in \Omega} \int |e^{Q(x)}|^2 \, \|\varphi(x)\| \, dm(x),$$

où Ω désigne l'ensemble des polynômes trigonométriques Q vérifiant $\hat{Q}(0) = 0$ et

$\hat{Q}(x)=0$ si $x \not\leq P$. On a pour tout $Q \in \Omega$

$$\|\hat{\varphi}(0)\| = \left\| \int e^{2Q}\varphi \, dm \right\| \leq \int |e^{Q}|^2 \|\varphi\| \, dm$$

et le résultat suit. ■

2.10. On se donne G un groupe abélien compact métrisable avec un ordre total P sur $\hat{G}$, $g: G \to G$ une affinité P-positive surjective et $A \in H^\infty_P(G, m, \mathcal{B})$. On a $A \circ g \in H^\infty_{P,\mathcal{B}}$ et $\widehat{A \circ g}(0) = \hat{A}(0)$.

Pour l'application fibrée (mesurable) (g, A) de $G \times \mathcal{B}$, la même démonstration que 2.3 donne en utilisant 2.9:

PROPOSITION. *Avec les hypothèses ci-dessus on a*

$$\lambda_+(g, A) \geq \operatorname{Log}(\operatorname{Rspec}(\hat{A}(0))).$$

Remarque. La condition $*'_1$ de 2.1 est satisfaite par des transormations f de $\mathbb{T}^n_1$ qui ne sont pas nécessairement des affinités du groupe $\mathbb{T}^n_1$, voir 2.2 d).

3. Exemples avec des matrices holomorphes

3.1. On se place sur $\mathbb{C}$ et on considère $f_\beta(z) = \beta z$, $|\beta| = 1$.

1) Soit $A: z \in \mathbb{C} \to e^z \in \mathbb{C}^* = GL(1, \mathbb{C})$ alors par la formule de Jensen on a pour tout $r \geq 0$, $\lambda_+(\mathbb{T}^1_r, f_\beta, A) = 0$.

2) On met sur l'algèbre de Banach $\mathcal{B}$ sur $\mathbb{C}$ une norme $\| \ \|$ d'algèbre sur $\mathbb{C}$. Par exemple $\mathcal{B} = \mathcal{L}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2)$. On considère une application polynomiale

$$A: \mathbb{C} \to \beta, \ A(z) = A_0 + zA_1 + \cdots + z^p A_p \quad \text{où les} \quad A_i \in \mathcal{B}$$

pour $i = 0, 1, \ldots, p$ et $p \geq 1$. *On Suppose que* A_p *vérifie* $\operatorname{Rspec}(A_p) \neq 0$.

PROPOSITION. *Sour les hypothèses ci-dessus, si* $r \to +\infty$, *alors on a*

$$\lambda_+(\mathbb{T}^1_r, f_\beta, A) \to +\infty.$$

Démonstration. On pose $C(Z)=Z^pA_0+\cdots+A_p$ (i.e. on pose $Z=1/z$ pour se placer au voisinage de $+\infty$) et on a $C(Z)=A(z)/z^p$). On a si $|Z|=r$, $|Z|=1/r$ et

$$\mathrm{Log}\,\|A(\beta^{n-1}z)\cdots A(z)\|=\mathrm{Log}\,\|C(\bar\beta^{n-1}Z)\cdots C(Z)\|+np\,\mathrm{Log}\,r$$

et donc

$$\lambda_+(\mathbb{T}^1_r, f_\beta, A)=\lambda_+(\mathbb{T}^1_{1/r}, f_\beta, C)+p\,\mathrm{Log}\,r.$$

Par 2.3 $\lambda_+(\mathbb{T}^1_r, f_\beta, C)\geq \mathrm{Log}\,(\mathrm{Rspec}\,(A_p))>-\infty$ et donc, si $r\to+\infty$, $\lambda_+(\mathbb{T}^1_r, f_\beta, A)\to +\infty$. ■

3) *Exemples de $A:\mathbb{C}\to SL(2,\mathbb{C})\subset\mathscr{L}_{\mathbb{C}}(\mathbb{C}^2,\mathbb{C}^2)$ vérifiant* 2)

· $z\to\begin{pmatrix}P(z) & -1\\ 1 & 0\end{pmatrix}$, où P est un polynôme de degré $p\geq 1$.

· En composant des matrices de la forme $z\to\begin{pmatrix}1 & P(z)\\ 0 & 1\end{pmatrix}$ et des matrices $B\in SL(2,\mathbb{C})$ on obtient des applications polynomiales $A(z)=A_0+\cdots+A_pz^p$ et quitte à considérer $BA(z)$ avec $B\in SL(2,\mathbb{C})$ choisi de façon adhoc, on peut supposer que la condition de 2) est vérifiée.

3.2. *Une application du théorème de Siegel: le théorème de Siegel fibré*

On se donne des entiers positifs k et p. On considère

$$f_\beta(z_1,\ldots,z_k)=(\beta_1z_1,\ldots,\beta_kz_k),\qquad \beta=(\beta_1,\ldots,\beta_k),\qquad |\beta_i|=1\quad\text{si}\quad 1\leq i\leq k,$$

une application $A:D^k_r\to\mathscr{L}_{\mathbb{C}}(\mathbb{C}^p,\mathbb{C}^p)$ holomorphe sur l'intérieur de D^k_r, $r>0$, telle que $A(0)\in GL(p,\mathbb{C})$ soit une matrice diagonale

$$A(0)=\begin{pmatrix}\tilde\lambda_1 & & 0\\ & \ddots & \\ 0 & & \tilde\lambda_p\end{pmatrix}$$

et on suppose qu'il existe $\lambda\in\mathbb{C}^*$ tel que

$$\lambda A(0)=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_p\end{pmatrix}$$

satisfasse à la *condition diophantienne suivante*: il existe $C>0$, $\gamma>0$ tels que l'on ait

$$|\lambda_1^{i_1}\cdots\lambda_p^{i_p}\beta_1^{i_{p+1}}\cdots\beta_k^{i_{p+k}}-\lambda_j|\geq C\,|i|^{-\gamma}$$
$$|\lambda_1^{i_1}\cdots\lambda_p^{i_p}\beta_1^{i_{p+1}}\cdots\beta_k^{i_{p+k}}-\beta_l|\geq C\,|i|^{-\gamma}$$

pour $1\leq j\leq p$, $1\leq l\leq k$ et $i=(i_1,\ldots,i_{p+k})\in\mathbb{N}^{p+k}$ vérifiant $|i|=\Sigma i_j\geq 2$.

PROPOSITION. *Sous les conditions ci-dessus il existe* $0<R_0<r$ *dépendant seulement de* γ, C, *et* r, *une application holomorphe* $B:D^k_{R_0}\to GL(p,\mathbb{C})$, $B(0)=e=$ *la matrice unité, tel que si* $z\in D^k_{R_0}$ *on ait*

$$B(f_\beta(z))^{-1}A(z)B(z)=A(0).$$

Démonstration. On considère le $\mathbb{C}$-difféomorphisme local

$$F:(z,\eta)\in(\mathbb{C}^k\times\mathbb{C}^p,0)\to(f_\beta(z),\lambda A(z)\eta)\in(\mathbb{C}^k\times\mathbb{C}^p,0)$$

Par le théorème de C. L. C. Siegel, généralisé par E. Zehnder [20], il existe un unique germe de difféomorphisme holomorphe $h:(\mathbb{C}^{k+p},0)\to(\mathbb{C}^{k+p},0)$ vérifiant $Dh(0)=e$ et

$$h^{-1}\circ F\circ h(z,\eta)=(f_\beta(z),\lambda A(0)\eta)$$

si $\|\eta\|+\|z\|$ est assez petit. La série formelle de h est aussi unique (moyennant la condition $Dh(0)=e$), et on vérifie sans peine que, formellement,

$$h(z,\eta)=(z,B(z)\eta),\qquad B(0)=e.$$

Par l'unicité des séries formelles il suit qu'il existe R_0 tel que $B:D^k_{R_0}\to GL(p,\mathbb{C})$ soit une application holomorphe et vérifie les conclusions de la proposition. ■

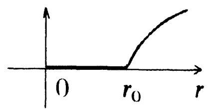
3.3. *Un exemple*

Soit $z\in\mathbb{C}\to A(z)=\begin{pmatrix}E+P(z) & -1\\ 1 & 0\end{pmatrix}\in SL(2,\mathbb{C})$ où $P(z)=a_1z+\cdots+a_pz^p$, $a_p\neq 0$, $E\in\mathbb{R}$ et $|E|<2$, $A(0)$ est conjugué à la matrice $\begin{pmatrix}\alpha & 0\\ 0 & \bar\alpha\end{pmatrix}$ avec $\alpha=e^{2\pi ia}$, où $a\in\mathbb{T}^1$ vérifie $2\cos 2\pi a=E$. On vérifie que pour Lebesgue presque tout $E\in{]-2,2[}$ et $\beta\in\mathbb{T}^1_1$ (i.e. $|\beta|=1$) l'application fibrée (f_β,A) de $\mathbb{C}\times\mathbb{C}^2$ satisfait à 3.2. Il en résulte, pour un tel choix et par 3.2, que la fonction $r\to\lambda_+(r,f_\beta,A)$ (*cf.* 2.4) a la propriété suivante: il existe $r_0 > 0$ tel que

$$\lambda_+(r, f_\beta, A) = 0, \quad \text{si} \quad 0 \leq r \leq r_0. \tag{1}$$

Par 3.1.2) *on a*

$$\lambda_+(r, f, A) \to +\infty, \quad \text{si} \quad r \to +\infty. \tag{2}$$



graphe de $\lambda_+(r, f_\beta, A)$

3.4. *Remarque.* La propriété (2) montre que le théorème de Siegel n'est pas un théorème global, ce qui *n'est pas étonnant du tout*:

EXEMPLE. Soient $F_1(z_1, z_2) = (\alpha_1 z_1 + z_2^2, \alpha_2 z_2)$, $F_2(z_1, z_2) = (z_1, z_2 + z_1^2)$ avec $\alpha_i \in \mathbb{C}$, $|\alpha_i| = 1$, $\alpha_1 \neq 1 \neq \alpha_2$. On pose $G = F_1 \circ F_2^{-1}$ qui est un difféomorphisme de $\mathbb{C}^2$ tel que $G(0) = 0$, $DG(0) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. G possède 3 autres points fixes que 0 solutions de

$$F_1(z) = F_2(z) \quad \text{où} \quad z = (z_1, z_2) \Leftrightarrow \begin{Bmatrix} z_2^2 = (1-\alpha_1) z_1 \\ z_1^2 = (-1+\alpha_2) z_2 \end{Bmatrix}$$
$$\Rightarrow z_2 = 0 \quad \text{et} \quad z_2^3 = (\alpha_2 - 1)(1-\alpha_1)^2.$$

Il en résulte que le difféomorphisme G n'est pas conjugué sur tout $\mathbb{C}^2$ à sa partie linéaire en 0!

3.5. *Nous allons donner un autre exemple où* 3.3(1) *se produit.*

1) *Soient* $f_\beta : \mathbb{C} \to \mathbb{C}$ de la forme $f_\beta(z) = \beta z$, $|\beta| = 1$, et $A(z) = A_0 + zA_1 \in \mathcal{L}_{\mathbb{C}}(\mathbb{C}^p, \mathbb{C}^p)$ où A_0 et A_1 sont des matrices constantes. Si β est une racine primitive $q^{\text{ème}}$ de l'unité on a

$$\operatorname{Tr}(A^q_{f_\beta}(z)) = \operatorname{Tr}(A_0^q) + \operatorname{Tr}(A_1^q) z^q \beta^{q(q-1)/2};$$

où Tr désigne la trace et $\beta^{q(q-1)/2} = \pm 1$. ($\operatorname{Tr}(A^q_{f_\beta}(z)) = P(z)$ où P est un polynôme de degré q dont le terme constant et celui de degré q sont ceux proposés; or $P(\beta z) = P(z)$ (puisque $\operatorname{Tr}(B_1 B_2) = \operatorname{Tr}(B_2 B_1)$), donc les autres termes du polynôme P sont nuls.)

2) On suppose que

$$A_{a,\lambda}(z)=\begin{pmatrix}2\cos 2\pi a+\lambda z & -1\\ 1 & 0\end{pmatrix},$$

$a\in\mathbb{R}/\mathbb{Z}$, $\lambda\in\mathbb{C}-\{0\}$, $|\lambda|\leq 1$. Si $\beta=e^{2\pi ip/q}$ où p et q sont premiers entre eux avec q impair, on a

$$\operatorname{Tr}(A^q_{f_\beta}(z))=2\cos 2q\pi a+\lambda^q z^q. \tag{+}$$

3) PROPOSITION. *On fixe $\lambda\in\mathbb{C}$, $\lambda\neq 0$, $|\lambda|\leq 1$. Il existe un G_δ dense $G_1\subset\mathbb{T}^1\times\mathbb{T}^1_1$ tel que, si $(a,\beta)\in G_1$, alors $\lambda_+(1,f_b,A_{a,\lambda})=0$.*

Démonstration. Par 1.3, la fonction $(a,\beta)\to\lambda_+(1,f_\beta,A_{a,\lambda})$ est semi-continue supérieurement. On veut montrer que, pour tout $\varepsilon>0$, l'ouvert $U_\varepsilon=\{(a,\beta)\mid\lambda_+(1,f_\beta,A_{a,\lambda})<\varepsilon\}$ est dense. Il suivra que $\bigcap_{\substack{n\geq 1\\ n\in\mathbb{N}}}U_{1/n}$ est un G_δ dense. Pour voir que U_ε est dense il suffit de montrer qu'il existe un entier $N>0$ tel que si $a=p_1/q$, $\beta=e^{2\pi ip/q}$, où $(p_1,q)=1$, $(p,q)=1$, q est impair et $q\geq N$, alors

$$0\leq\lambda_+(1,f_\beta,A_{p_1/q,\lambda})<\varepsilon.$$

On a

$$\operatorname{Tr}(A^q_{f_\beta}(z))=2+\lambda^q z^q,$$

donc le maximum des modules des valeurs propres de $A^q_{f_\beta}(z)$ sur $\mathbb{T}^1_1$ est majoré par $c_q=1+(|\lambda|^q)/2+(|\lambda|^q+\frac{1}{4}|\lambda|^{2q})^{1/2}$ (puisque l'équation des valeurs propres de $A^q_{f_\beta}(z)$ est $Y^2-(2+\lambda^q z^q)Y+1=0$). Or

$$\lambda_+(1,f_\beta,A_{p_1/q,\lambda})\leq\frac{1}{q}\operatorname{Log} c_q.$$

Donc, si $q\to+\infty$, comme $|\lambda|\leq 1$, $1/q\operatorname{Log} c_q\to 0$. ∎

4) PROPOSITION. *Il existe un G_δ dense $G_2\subset\mathbb{T}^1\times\mathbb{T}^1_1$ tel que si $(a,\beta)\in G_2$ alors pour tout $r>0$*

$$\Phi_r(a,\beta)=\sup_{n\in\mathbb{N}}\left(\operatorname{Max}_{|z|\leq r}\|(A_{a,\lambda})^n_{f_\beta}(z)\|\right)=+\infty.$$

5) *Remarque.* Si $(a, \beta) \in G_2$ et $a \neq 0$ ou $\frac{1}{2}$ alors, sur tout voisinage de 0, le difféomorphisme holomorphe de $(\mathbb{C} \times \mathbb{C}^2, 0)$ $F(z, \eta) = (\beta z, A(z)\eta)$ n'est pas holomorphiquement équivalent à sa partie linéaire, qui est, pour $a \neq 0$ ou $\frac{1}{2}$, conjuguée à une matrice unitaire.

Démonstration de 4). Par la monotonie de $\Phi_r(a, \beta)$ (voir 2.4a)), il suffit de montrer que, pour tout $n \in \mathbb{N}^*$, $H_{1/n} = \{(a, \beta) \mid \Phi_{1/n}(a, \beta) = +\infty\}$ est un G_δ dense et de poser $G_2 = \bigcap_{n \in \mathbb{N}^*} H_{1/n}$. L'application $(a, \beta) \to \Phi_{1/n}(a, \beta)$ est semi-continue inférieurement et donc $H_{1/n}$ est un G_δ. $H_{1/n}$ est dense puisqu'il contient tous les $a = p_1/q$, $\beta = e^{2\pi i p/q}$, $(p_1, q) = 1$, $(p, q) = 1$, q impair, car pour un tel couple il existe un z_0 arbitrairement petit tel $(A_{a,\lambda})^q_{f_\beta}(z_0)$ soit une matrice hyperbolique. ■

6) Si on choisit $(a, \beta) \in G_1 \cap G_2$, alors la fonction $r \to \lambda_+(r, f_\beta, A_{a,\lambda})$ vérifie pour tout $0 \leq r \leq 1$, $\lambda_+(r, f_\beta, A_{a,\lambda}) = 0$. Si $r \to +\infty$, comme $\lambda \neq 0$ par 3.1.2), $\lambda_+(r, f_\beta, A_{a,\lambda}) \to +\infty$. Comme $(a, \beta) \in G_2$, si $a \neq 0$, ou $\frac{1}{2}$, alors la remarque 5) s'applique.

7) Ce que nous venons de faire reste valable sur $\mathbb{T}^1_r$ tant $r \leq 1/|\lambda|$. Si $r > 1/|\lambda|$ il existe un ouvert U dense de (a, λ, β) tel que si $(a, \lambda, \beta) \in U$ alors $(f_\beta, A_{a,\lambda})$ agissant sur $\mathbb{T}^1_r \times \mathbb{C}^2$ ait une structure hyperbolique. (En effet, si $z \in \mathbb{C}$ et $N > 0$ vérifie $|\lambda z|^N > 4$ et si β est une racine primitive $q^{\text{ème}}$ de l'unité avec $q \geq N$, alors par $(+)$ on a, pour tout $a \in \mathbb{T}^1$, $|\mathrm{Tr}\,(A_{a,\lambda})^q_{f_\beta}(z)| > 2$. Ceci implique que la matrice $(A_{a,\lambda})^q_{f_\beta}(z)$ est une matrice hyperbolique sur $\mathbb{C}$ et de plus ses directions invariantes dépendent holomorphiquement de z pour $|\lambda z|^N > 4$. Il suffit alors d'appliquer la remarque 2) de 1.4.)

9) On peut démontrer le théorème de Siegel fibré pour $(f_\beta, A_{a,\lambda})$, très simplement de la façon suivante. On pose $\alpha = e^{2\pi i a}$ et on cherche une application holomorphe

$$z \to v(z) = \begin{pmatrix} \eta_1(z) \\ \eta_2(z) \end{pmatrix} \in \mathbb{C}^2$$

telle que l'on ait

$$\begin{pmatrix} \eta_1(\beta z) \\ \eta_2(\beta z) \end{pmatrix} = \alpha \begin{pmatrix} 2\cos 2\pi a + \lambda z & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1(z) \\ \eta_2(z) \end{pmatrix}$$

avec $\eta_1(0) = \bar{\alpha}$ et $\eta_2(0) = 1$. Soit

$$\eta_2(\beta z) = \alpha \eta_1(z)$$

et

$$\bar{\alpha}\eta_1(\beta z)+\alpha\eta_1(\bar{\beta}z)-(\alpha+\bar{\alpha})\eta_1(z)=\lambda z\eta_1(z).$$

Si on écrit $\eta_1(z)=\sum_{k\geq 0} b_k z^k$ on a les relations de récurrences:

$$b_0=\bar{\alpha}$$

$$\lambda b_{n-1}=P_\alpha(\beta^n)b_n, \quad \text{so} \quad n\geq 1$$

avec $P_\alpha(z)=(z/\alpha)+(\alpha/z)-(\alpha+(1/\alpha))$. Si pour tout $n\geq 1$, $\beta^n\neq 1$ et $\beta^n\neq\alpha^2$ on peut résoudre et on obtient:

$$b_n=\frac{\lambda^n\bar{\alpha}}{P_\alpha(\beta)\cdots P_\alpha(\beta^n)}.$$

Pour étudier la convergence on remarque, que par la formule de Jensen, on a $\int_0^1 \operatorname{Log}|P_\alpha(e^{2\pi i\theta})|\,d\theta=0$ et il suit d'un théorème de Koksma $[K]$ (on peut aussi adapter l'article d'Hardy et Littlewood $[H]$), que si $a\in\mathbb{T}^1$ est fixé, alors pour Lebesgue presque tout $\beta\in\mathbb{T}_1^1$, si $n\to+\infty$, $(1/n)\operatorname{Log}|P_\alpha(\beta)\cdots P_\alpha(\beta^n)|\to 0$.

Il en résulte que pour, Lebesgue presque tout $(a,\beta)\in\mathbb{T}^1\times\mathbb{T}_1^1$, (avec $\alpha\neq\pm 1$ et β n'est pas une racine de l'unité) il existe 2 applications holomorphes sur $\{z\mid |z|<|\lambda|^{-1}\}$, $z\to v_i(z)\in\mathbb{C}^2$, $i=1,2$, telles que a matrice $H(z)=(v_1(z),v_2(z))$ (i.e. ayant les vecteurs colonnes $v_1(z)$ et $v_2(z)$) vérifie

$$A_{a,\lambda}(z)H(z)=H(\beta z)\begin{pmatrix}\bar{\alpha} & 0\\ 0 & \alpha\end{pmatrix} \quad \text{et} \quad H(0)=\begin{pmatrix}\bar{\alpha} & \alpha\\ 1 & 1\end{pmatrix}.$$

Il en résulte que det $H(z)=\det H(\beta z)$ et donc comme $\alpha\neq\pm 1$ et que β n'est pas une racine de l'unité, on a $\det H(z)=\det H(0)\neq 0$. On obtient finalement, si $|z|<|\lambda|^{-1}$, $H^{-1}(\beta z)A_{a,\lambda}(z)H(z)=\begin{pmatrix}\bar{\alpha} & 0\\ 0 & \alpha\end{pmatrix}$.

Remarques. 1. Si $a=0$ ou $\frac{1}{2}$ (i.e. si $\alpha=\pm 1$) alors pour Lebesgue presque tout $\beta\in\mathbb{T}_1^1$ il existe un nombre $0<R_0<|\lambda|^{-1}$ et des matrices $H_0(z)$ et $H_{1/2}(z)$ holomorphes inversibles sur $\{z\mid |z|<R_0\}$, et $c\in\mathbb{C}^*$ telles que l'on ait

$$H_a^{-1}(\beta z)A_{a,\lambda}(z)H_a(z)=\begin{pmatrix}\alpha & c\\ 0 & \alpha\end{pmatrix} \quad \text{si} \quad a=0,\tfrac{1}{2}.$$

(La démonstration est presque la même. On détermine d'abord un vecteur holomorphe $v_1(z)=\begin{pmatrix}\eta_1(z)\\ \eta_2(z)\end{pmatrix}$ comme ci-dessus, puis on considère la matrice holomorphe $H_a(z)=(v_1(z), v_2(z))$ où $v_2(z)=\begin{pmatrix}0\\ 1/\eta_1(z)\end{pmatrix}$. On a

$$H_a^{-1}(\beta z)A_{a,\lambda}(z)H_a(z)=\begin{pmatrix}\alpha & c_a(z)\\ 0 & \alpha\end{pmatrix}$$

pour $|z|<R_0$ et $a=0, \frac{1}{2}$ où $c_a(z)$ est une fonction holomorphe sur $\{z \mid |z|<R_0\}$. La remarque suit facilement en conjugant par des matrices de la forme $z\longrightarrow \begin{pmatrix}1 & d_a(z)\\ 0 & 1\end{pmatrix}$.)

2. Pour Lebesgue presque tout a et β, par la démonstration ci-dessus le théorème de Siegel est valable sur $\{z \mid |z|<|\lambda|^{-1}\}$. Par 3.1, si $r>1/|\lambda|$, on a $\lambda_+(\mathbb{T}_r^1, f_\beta, A_{a,\lambda})>0$.

3.6. *Une exemple avec une matrice non inversible en un point*

On considère $f_\beta(z)=\beta(z)$, $|\beta|=1$, $z\in\mathbb{C}$ et

$$z\longrightarrow A(z)=\begin{pmatrix}E & -1+z\\ 1 & 0\end{pmatrix},\quad \text{où}\quad E\in\mathbb{R},\quad E>2.$$

On a par 2.3, pour tout $r>0$, $\lambda_+(\mathbb{T}_r^1, f_\beta, A)>0$. Si on considère l'application fibrée (f_β, A) sur $\mathbb{T}_1^1\times\mathbb{C}^2$ alors $\det(A(z))=1-z$ s'annule en $z=1$. Par la formule de Jensen on a, si $0\leq r\leq 1$, $\int_{\mathbb{T}_r^1}\operatorname{Log}|1-z|\,dm=0$ et donc si $0<r\leq 1$ l'application fibrée (f_β, A) sur $\mathbb{T}_r^1\times\mathbb{C}^2$ a 4 exposants de Lyapounov sur $\mathbb{R}$ non nuls: 2 sont égaux à $\lambda_+(\mathbb{T}_r^1, f_\beta, A)$ et 2 *à* $-\lambda_+(\mathbb{T}_r^1, f_\beta, A)$. (Il suffit de considérer

$$z\longrightarrow\frac{1}{|\det A(z)|^{1/2}}A(z)=B(z),\qquad z\neq 1,$$

de noter que $B(z)\in SL(2,\mathbb{C})$, $\operatorname{Log}\|B^{\pm 1}\mid\mathbb{T}_1^1\|\in L^1(m)$ et d'appliquer [15] en utilisant le fait que $B(z)$ est une matrice définie sur $\mathbb{C}$.)

3.7. *Un exemple avec un difféomorphisme d'Anosov*

Soit $f:\mathbb{C}^2\longrightarrow\mathbb{C}^2$, $f(z_1, z_2)=(z_1^2z_2, z_1z_2)$ (voir 2.2. c)) $f(1,1)=(1,1)$. Soit

$$A(z_1, z_2)=\begin{pmatrix}E-\lambda z_1 & -1\\ 1 & 0\end{pmatrix}\in SL(2,\mathbb{C}),\quad \text{où}\quad E\in\mathbb{R},\qquad E>2.$$

On a $\operatorname{Log}(\operatorname{Rspec}(A(0)))>0$. On suppose que E est fixé, et que $\lambda\in\mathbb{R}$ vérifie $|E-\lambda|\leq 2$. La matrice

$$A(1,1)=\begin{pmatrix} E-\lambda & -1 \\ 1 & 0\end{pmatrix}$$

est donc elliptique (i.e. conjugué à une matrice unitaire) si $|\lambda-E|<2$ et parabolique si $E-\lambda=\pm 2$.

Le difféomorphisme fibré (f,A) sur $\mathbb{T}_1^2\times\mathbb{C}^2$ par 2.2a) et 2.3 vérifie $\lambda_+(f,A)>0$ mais le difémorphisme fibré n'a pas de structure hyperbolique puisque $f(1,1)=(1,1)$ et que $A(1,1)$ est une matrice de $SL(2,\mathbb{R})$ elliptique ou parabolique.

Remarque. Pour $E-\lambda=\pm 2$, le difféomorphisme fibré (f,A) ne laisse pas invariant un scindement continu (non trivial) du fibré (trivial) $\mathbb{T}_1^2\times\mathbb{C}^2$. On peut construire d'autres exemples (f,B_λ) de difféomorphismes fibrés de $\mathbb{T}_1^2\times\mathbb{R}^2$ qui ne laissent invariant aucun scindement continu (non trivial) du fibré $\mathbb{T}_1^2\times\mathbb{R}^2$ et qui vérifient $\lambda_+(f,B_\lambda)>0$. Pour cela on choisit f un difféomorphisme d'Anosov comme ci-dessus, B_λ comme en 4.1 avec $\lambda>1$ et on raisonne comme en 4.2.

3.8. *Un exemple d'application dans $SL(2,\mathbb{C})$ sans structure hyperbolique*

On considère $f_\beta:\mathbb{C}\to\mathbb{C}$, $f_\beta(z)=\beta z$ avec $\beta=e^{2\pi i\alpha}$, $\alpha\in\mathbb{R}-\mathbb{Q}$, et

$$z\in\mathbb{C}\to A(z)=\begin{pmatrix} E+\lambda z & -1 \\ 1 & 0\end{pmatrix}\in SL(2,\mathbb{C})$$

où $E\in\mathbb{R}$, $E>2$ et $\lambda\in\mathbb{C}^*$.

On suppose que β est choisit pour que 3.2 s'applique à (f_β,A), (pour Lebesgue presque tout α cela sera le cas). Il en résulte qu'il existe $r_0>0$, tel que pour $0<r\leq r_0$, on ait:

$$\lambda_+(\mathbb{T}_r^1,f_\beta,A)=\operatorname{Log}\mu>0 \quad \text{avec} \quad \mu=\operatorname{Rspec}(A(0));$$

$f_\beta\times A$ agissant sur $D_{r_0}^1\times\mathbb{C}^2$ a une structure hyperbolique.

Par 3.1, si $r\to+\infty$, $\lambda_+(\mathbb{T}_r^1,f,A)\to+\infty$.

PROPOSITION. *Sous les hypothèses ci-dessus il existe $r_1>0$, tel que le difféomorphisme fibré $f_\beta\times A$ de $\mathbb{T}_{r_1}^1\times\mathbb{C}^2$ n'ait pas de structure hyperbolique mais vérifie par 2.3:* $\lambda_+(\mathbb{T}_{r_1}^1,f,A)\geq\operatorname{Log}\mu$.

Démonstration. Raisonnons par l'absurde. Si pour tout $r>0$, le difféomorphisme fibré $f\times A$ de $\mathbb{T}^1_r\times\mathbb{C}^2$ avait une structure hyperbolique cette structure dépendrait continument de r et même analytiquement. Cette structure hyperbolique sur $\mathbb{T}^1_r\times\mathbb{C}^2=E^s_r\oplus E^u_r$ serait complexe (i.e. les fibrés continues E^s_r et E^u_s seraient des fibrés complexes, puisque A est une matrice définie sur $\mathbb{C}$, dans la définition de 1.4, on peut multiplier les vecteurs v_x et w_x par $\lambda\in\mathbb{C}^*$). Les fibrés E^s_r et E^u_r comme fibrés continues complexes dépendraient $\mathbb{R}$-analytiquement de $r>0$ (cela résulte de la démonstration [18 p. 100–101] en complexifiant le paramètre r). Il en résulterait que la fonction $r\to\lambda_+(r, f_\beta, A)$ est $\mathbb{R}$-analytique ce qui est absrude. ■

4. Exemples avec $SL(2,\mathbb{R})$

4.1. a) on se place sur $\mathbb{T}^1$ et on considère la rotation (ou translation) $R_\alpha(\theta)=\theta+\alpha$. Soit $B_\lambda:\mathbb{T}^1\to SL(2,\mathbb{R})$ la matrice

$$B_\lambda(\theta)=\begin{pmatrix}\cos 2\pi\theta & -\sin 2\pi\theta\\ \sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \dfrac{1}{\lambda}\end{pmatrix},\quad \text{où}\quad \lambda\in\mathbb{R}\quad\text{et}\quad \lambda\geq 1.$$

On considère sur $\mathbb{T}^1\times\mathbb{R}^2$ le difféomorphisme fibré associé à (R_α, B_λ). Nous allons démontrer à nouveau le théorème 3.1. de [5]. On met sur $\mathbb{T}^1$ la mesure de Haar notée m ou $d\theta$.

PROPOSITION. *On a* $\lambda_+(R_\alpha, B_\lambda)\geq \mathrm{Log}\,((\lambda/2)+(1/2\lambda))$.

Démonstration. On pose $\beta=e^{2\pi i\alpha}$, $\cos 2\pi\theta=\frac{1}{2}(z+z^{-1})$, $\sin 2\pi\theta=\dfrac{1}{2i}(z-z^{-1})$, pour $z=e^{2\pi i\theta}$, $|z|=1$. On se place sur $\mathbb{T}^1_1$ dans $\mathbb{C}$. Soit

$$\frac{A_\lambda(z)}{z}=\begin{pmatrix}\dfrac{1}{2}(z+z^{-1}) & -\dfrac{1}{2i}(z-z^{-1})\\ \dfrac{1}{2i}(z-z^{-1}) & \dfrac{1}{2}(z+z^{-1})\end{pmatrix}\Lambda=\frac{1}{z}\begin{pmatrix}\dfrac{1}{2}(z^2+1) & -\dfrac{1}{2i}(z^2-1)\\ \dfrac{1}{2i}(z^2-1) & \dfrac{1}{2}(z^2+1)\end{pmatrix}\Lambda$$

avec $\Lambda=\begin{pmatrix}\lambda & 0\\ 0 & \dfrac{1}{\lambda}\end{pmatrix}$.

On considère l'application fibrée (f_β, A_λ) de $\mathbb{T}_1^1 \times \mathbb{C}^2$, où $f_\beta(z) = \beta z$. On met sur $\mathscr{L}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2)$ une norme d'algèbre sur $\mathbb{C}$. Pour $z = e^{2\pi i\theta}$, on a

$$\|A_\lambda(\beta^{n-1}z) \cdots A_\lambda(z)\| = \|B_\lambda(\theta + (n-1)\alpha) \cdots B_\lambda(\theta)\|$$

d'où

$$\lambda_+(R_\alpha, B_\lambda) = \lambda_+(\mathbb{T}_1^1, f_\beta, A_\lambda).$$

Par 2.3 on a $\lambda_+(\mathbb{T}_1^1, f_\beta, A_\lambda) \geq \operatorname{Log}(\operatorname{Rspec}(A_\lambda(0))) = \operatorname{Log}(\lambda/2 + 1/2\lambda)$. ■

b) Par la même démonstration que ci-dessus et en se plaçant aussi au voisinage de $+\infty$ comme en 3.1.2) on a le

Scolie. Soit $\mathscr{B}$ une algèbre de Banach sur $\mathbb{C}$ (par exemple $\mathscr{L}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2)$) et $A : \mathbb{T}^1 \to \mathscr{B}$ un polynôme trigonométrique de la forme

$$A(\theta) = \sum_{|k| \leq n} A_k e^{2\pi i k\theta} \quad ou \quad A_k \in \mathscr{B}.$$

Pour l'application fibré (R_α, A) de $\mathbb{T}^1 \times \mathscr{B}$, on a

$$\lambda_+(R_\alpha, A) \geq \operatorname{Log} a_n,$$

où

$$a_n = \operatorname{Max}(\operatorname{Rspec}(A_n), \operatorname{Rspec}(A_{-n})).$$

c) Une généralisation de b) est la suivante: on se place sous les hypothèses de 2.10, et l'on considère un polynôme trigonométrique $A : G \to \mathscr{B}$ de la forme

$$A(g) = \sum_{k=0}^{n} A_k \chi_k(g) \quad \text{si} \quad g \in G, \qquad A_k \in \mathscr{B} \quad \text{et pour} \quad 0 \leq k \leq n, \qquad \chi_k \in \hat{G}.$$

On suppose que χ_0 est le minimum de l'esemble $\{\chi_0, \ldots, \chi_n\}$ pour l'ordre total P sur $\hat{G}$. Comme $\bar{\chi}_0 A \in H^\infty_{p,\mathscr{B}}$, on a, par 2.10, la

PROPOSITION. $\lambda_+(g, A) \geq \operatorname{Log}(\operatorname{Rspec}(A_0))$.

4.2. PROPOSITION. *Soit $B : \mathbb{T}^1 \to SL(2, \mathbb{R})$ une application continue non homotope à la matrice constante unité alors l'homéomorphisme $F = (R_\alpha, B) : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2$ n'a pas de structure hyperbolique (voir partiellement hyperbolique).*

Démonstration. Si F laisse invariant 2 fibrés E^u et E^s sur $\mathbb{T}^1$ de some directe le fibré trivial, alors F agit comme homéomorphisme sur $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$, le fibré en espaces projectifs associé un fibré trivial $\mathbb{T}^1\times\mathbb{R}^2$, et laisse invariant les fibrés projectifs associés à E^u et E^s. Comme les fibrés projectifs associés aux fibrés de rang 1 E^u ou E^s sont triviaux, il en résulte que le difféomorphisme fibré

$$\begin{array}{ccc} F:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) & \longrightarrow & \mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_1} \\ R_\alpha:\ \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \end{array}$$

laisse invariante l'image d'une section continue du fibré $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)\to\mathbb{T}^1$, p_1 étant la première projection. Ceci n'est pas possible car le difféomorphisme F de $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)\approx\mathbb{T}^1\times\mathbb{T}^1$ est homotope à $(\theta_1,\theta_2)\in\mathbb{T}^2\to(\theta_1,\theta_2+2k\theta_1)\in\mathbb{T}^2$ avec $k\in\mathbb{Z}-\{0\}$. L'existence d'une section continue invariante par F n'est pas compatible avec l'action de $F_*:H_1(\mathbb{T}^2,\mathbb{Z})\circlearrowleft$ en homologie $\left(\text{i.e. }\begin{pmatrix}1 & 0\\ 2k & 1\end{pmatrix}\right)$. ∎

4.3 *Remarques.* 1) Les fibrés E^u et E^s ne sont pas nécessairement triviaux puisque, si ξ est le fibré de rang 1 non trivial sur $\mathbb{T}^1$, (i.e. un ruban de Möbius), alors $\xi\oplus\xi=\mathbb{T}^1\times\mathbb{R}^2$. Pour tout $\theta\in\mathbb{T}^1$, on définit $B_\lambda(v_\theta,w_\theta)=(\lambda v_\theta,(1/\lambda)w_\theta)$ si $(v_\theta,w_\theta)\in\xi_\theta\oplus\xi_\theta$, où ξ_θ est la fibre de ξ en $\theta\in\mathbb{T}^1$ et $\lambda\neq0$ un nombre fixé. B_λ est homotope à une matrice constante.

2) Si l'homéomorphisme fibré $F:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)\circlearrowleft$ est de la forme $F=(R_\alpha,B)$ où B est une application continue de $\mathbb{T}^1$ dans $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{-e,e\}$, laisse invariant les graphes de 2 fonctions continues distinctes $\Psi_i:\mathbb{T}^1\to\mathbb{P}(\mathbb{R}^2)$ pour $i=1,2$, alors il existe $H(\theta,y)=(\theta,l(\theta)y)$, où $l:\mathbb{T}^1\to PSL(2,\mathbb{R})$ est une application continue, tel que l'on ait

$$H^{-1}\circ F\circ H(\theta,y)=(\theta+\alpha,K(\theta)y),\ \text{où}$$

$$K(\theta)=\begin{pmatrix}\varphi(\theta) & 0\\ 0 & \dfrac{1}{\varphi(\theta)}\end{pmatrix}\quad\text{et}\quad \varphi\in C^0(\mathbb{T}^1,\mathbb{R}_+^*).$$

En effet, si Ψ_1 est homotope à une constante, alors il en va de même de Ψ_2 et il suffit d'amener pour tout θ les éléments $\Psi_1(\theta)$ et $\Psi_2(\theta)$ de $\mathbb{P}(\mathbb{R}^2)$ sur respectivement les points de coordonnées projectives $(1,0)$ et $(0,1)$ et si nécessaire multiplier les matrices par $-e$. Si Ψ_1 est de degré $k\neq0$, il suffit de remplacer F par $S_k^{-1}\circ F\circ S_k$, où S_k est défini en 5.16, pour se ramener au cas ou Ψ_1 est homotope à une constante. Si F est de classe C^r ainsi que les fonctions Ψ_i, alors on peut supposer que H et K sont aussi de classe C^r.

4.5. *Une exemple d'application dans $Sl(2,\mathbb{R})$ homotope à l'identité*

On se donne $\varepsilon>0$ petit, $\eta>0$ et λ tel que $\lambda\varepsilon>2+\eta$. Soit

$$B(\theta)=\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\varepsilon\begin{pmatrix}\cos 2\pi\theta & -\sin 2\pi\theta\\ \sin 2\pi\theta & \cos 2\pi\theta\end{pmatrix}\right]\begin{pmatrix}\lambda & 0\\ 0 & \dfrac{1}{\lambda}\end{pmatrix}.$$

On considère (R_α, B) agissant sur $\mathbb{T}^1\times\mathbb{R}^2$. Par le scolie de 4.1 on a, pour tout $\alpha\in\mathbb{T}^1$, $\lambda_+(R_\alpha, B)\geq \operatorname{Log}(\varepsilon\lambda/2+\varepsilon/2\lambda)$. Si $\varepsilon>0$ est assez petit, l'application $\theta\in\mathbb{T}^1\to C(\theta)=B(\theta)/(\det B(\theta))^{1/2}\in SL(2,\mathbb{R})$ est $\mathbb{R}$-analytique et homotope à la matrice constante unité. On a

$$\lambda_+(R_\alpha, C)=\lambda_+(R_\alpha, B)-\frac{1}{2}\int_0^1 \operatorname{Log}(\det B(\theta))\,d\theta,$$

si $\varepsilon\to 0$, $\frac{1}{2}\int_0^1 \operatorname{Log}(\det B(\theta))\,d\theta\to 0$ et donc si ε est assez petit $\lambda_+(R_\alpha, C)>0$. On a mieux: si $\beta\in\mathbb{T}^1$, on pose

$$\tilde{R}_\beta=\begin{pmatrix}\cos 2\pi\beta & -\sin 2\pi\beta\\ \sin 2\pi\beta & \cos 2\pi\beta\end{pmatrix}\quad\text{et}\quad (\tilde{R}_\beta B)(\theta)=\tilde{R}_\beta B(\theta),$$

on vérifie comme en 4.1 que $\lambda_+(R_\alpha, \tilde{R}_\beta B)\geq\operatorname{Log}(\varepsilon\lambda/2+\varepsilon/2\lambda)$. Si $\varepsilon\lambda>2+\eta$, pour $\varepsilon>0$ assez petit, l'application $\mathbb{T}^1\ni\theta\to\tilde{R}_\beta C(\theta)\in SL(2,\mathbb{R})$ est de classe C^ω, homotope à la matrice constante $\theta\to e$ et on a pour tout $\alpha\in\mathbb{T}^1$ et $\beta\in\mathbb{T}^1$

$$\lambda_+(R_\alpha, \tilde{R}_\beta C)>0 \tag{1}$$

4.6. PROPOSITION. *Soit $\alpha\in\mathbb{T}^1-(\mathbb{Q}/\mathbb{Z})$; pour tout $\beta\in\mathbb{T}^1$, il existe une matrice $\mathbb{T}^1\ni\theta\to C_\beta(\theta)\in SL(2,\mathbb{R})$ $\mathbb{R}$-analytique, homotope à la matrice constante unité et vérifiant*:

a) $\lambda_+(R_\alpha, C_\beta)>0$.

b) *Le difféomorphisme induit sur $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ par le difféomorphisme fibré (R_α, C_β) de $\mathbb{T}^1\times\mathbb{R}^2$ a pour vecteur de rotation* (α, β) (cf. 5.16).

c) *Le difféomorphisme de $\mathbb{T}^1\times\mathbb{R}^2$, $F=(R_\alpha, C_\beta)$ n'a pas de structure hyperbolique.*

Démonstration. On se fixe α. On a par définition (voir 5.16) $\rho(R_\alpha, \tilde{R}_b C) = (\alpha, \rho_f(R_\alpha, \tilde{R}_b C))$. Comme l'application $b \in \mathbb{T}^1 \to \rho_f(\tilde{R}_b C)$ est continue et monotone croissante de degré 1, elle est surjective (voir 5.9.3)). Il existe donc b_1 tel que $\rho_f(R_\alpha, \tilde{R}_{b_1} C) = \beta$. On choisit de façon plus précise b,: on pose $I_\beta = \{b_1, \rho_f(\tilde{R}_{b_1} C) = \beta\}$; I_β est un intervalle $[a_1, a_2](a_1 \leq a_2)$. Soit

$$b_1 \in I_\beta \quad \text{si} \quad \beta \notin \mathbb{Z}\alpha \bmod 1; \qquad b_1 = a_1 \text{ ou } a_2 \quad \text{si} \quad \beta \in \mathbb{Z}\alpha \bmod 1.$$

On pose $C_\beta = \tilde{R}_{b_1} C$.

La propriété a) est vérifiée par (1), b) par construction. Si $\beta \notin \mathbb{Z}\alpha \bmod 1$ la propriété c) résulte de 5.17. Si $\beta \in \mathbb{Z}\alpha (\bmod 1)$ alors (R_α, C_β) n'a pas de structure hyperbolique car l'ensemble des b tels que le difféomorphisme fibré $(R_\alpha, \tilde{R}_b C)$ ait une structure hyperbolique est ouvert et vérifie $\rho_f(R_\alpha, \tilde{R}_b C) \in \mathbb{Z}\alpha \bmod 1$. Or nous avons choisi $b_1 \notin \operatorname{Int} I_\beta$. ■

4.7. *Un autre exemple d'application dans $SL(2, \mathbb{R})$ homotope à l'identité*

Soit $p(\theta) = \sum_{|k| \leq n} a_k e^{2\pi i k\theta}$ un polynôme trigométrique de degré fixé n, $n \geq 1$ et $a_k \in \mathbb{C}$. On suppose que $|a_{-n}| > 1$. Soit l'application

$$\theta \in \mathbb{T}^1 \to A(\theta) = \begin{pmatrix} p(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{C}).$$

Si p est à valeurs réelles (i.e. si $\bar{a}_k = a_{-k}$ pour tout entier k) alors $A(\theta) \in SL(2, \mathbb{R})$ et réciproquement. Par le scolie de 4.1, on a pour le difféomorphisme fibré (R_α, A) de $\mathbb{T}^1 \times \mathbb{C}^2$ $\lambda_+(R_\alpha, A) \geq \operatorname{Log} |a_{-n}|$.

(Si $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$ on peut donner de ce fait la démonstration encore plus élémentaire, suivante: on vérifie san peine que, si $p \in \mathbb{N}$,

$$\int_0^1 \|A^p_{R_\alpha}(\theta)\| \, d\theta \geq \left\| \begin{pmatrix} a^p_{-n} & 0 \\ 0 & 0 \end{pmatrix} \right\| \quad \text{d'où} \quad \lim_{p \to +\infty} \frac{1}{p} \operatorname{Log} \|A^p_{R_\alpha}\|_{C^0(\mathbb{T}^1)} \geq \operatorname{Log} |a_{-n}|$$

et donc, par [5, 2.5] $\lambda_+(\mathbb{T}^1, R_\alpha, A) \geq \operatorname{Log} |a_{-n}|$. Si $\alpha \in \mathbb{Q}/\mathbb{Z}$ on peut raisonner comme dans [5, 3.2].)

4.8. Soit une fonction continue $\varphi \in C^0(\mathbb{T}^1, \mathbb{R})$ et $E \in \mathbb{R}$ alors on pose

$$A_E : \theta \in \mathbb{T}^1 \to A_E(\theta) = \begin{pmatrix} E + \varphi(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{R}).$$

On considère l'homéomorphisme fibré (R_α, A_E) de $\mathbb{T}^1 \times \mathbb{R}^2$. On fixe $\alpha \in \mathbb{T}^1$.

PROPOSITION. *Il existe $E_0 > 0$ tel que, si $|E| > E_0$, (R_α, A_E) ait une structure hyperbolique.*

Démonstration. On suppose que $E > 0$, le cas $E < 0$ étant analogue. On cherche les variétés invariantes de (R_α, A_E) en coordonnées projectives $\begin{pmatrix} \Psi(\theta) \\ 1 \end{pmatrix}$, $\Psi \in C^0(\mathbb{T}^1, \mathbb{R})$, $\Psi > 0$ (si $E < 0$ on suppose que $\Psi < 0$). On a l'equation

$$\Psi \circ R_\alpha + \frac{1}{\Psi} = E + \varphi. \tag{2}$$

On pose $\lambda_E + 1/\lambda_E = E$ pour E grand>2. On cherche $\Psi_1 = \lambda_E(1 + \eta_1)$ avec $\|\eta_1\|_{C^0} \leq \frac{1}{2}$ (et dépendant de E). L'équation que vérifie η_1 est:

$$\eta_1 \circ R_\alpha - \frac{\eta_1}{\lambda_E^2(1 + \eta_1)} = \frac{1}{\lambda_E} \varphi.$$

Si $E \to +\infty$, $\lambda_E \to +\infty$, il suit pour E assez grand que l'application

$$\Phi_E : \eta : \to \frac{1}{\lambda_E} \varphi \circ R_{-\alpha} + \frac{\eta \circ R_{-\alpha}}{\lambda_E^2(1 + \eta \circ R_{-\alpha})}$$

envoie la boule $\{\|\eta\|_{C^0} \leq \frac{1}{2}\}$ dans elle même et est une contraction lipschitzienne: on a $\|\Phi_E(a_1) - \Phi_E(a_2)\|_{C^0} \leq k\, \|a_1 - a_2\|$ avec $k < 1$. Il existe donc un point fixe η_1 de Φ_E. On détermine ainsi Ψ_1 vérifiant (2) où $\Psi_1 = \lambda_E(1 + \eta_1)$ avec $\|\eta_1\|_{C^0} < \frac{1}{2}$.

Par la même méthode on détermine une autre solution Ψ_2 de l'équation (2)

$$\Psi_2 = \frac{1}{\lambda_E}(1 + \eta_2) \quad \text{avec} \quad \|\eta_2\|_{C^0} \leq \frac{1}{2}.$$

Si on pose $v_i(\theta) = \begin{pmatrix} \Psi_i(\theta) \\ 1 \end{pmatrix}$ pour $i = 1, 2$, on a

$$A_E(\theta) v_i(\theta) = \Psi_i(\theta) v_i(\theta + \alpha).$$

Soient $H(\theta, y) = (\theta, l(\theta)y)$, où $l(\theta) = (v_1(\theta), v_2(\theta))$ (matrice ayant pour vecteurs colonnes v_1 et v_2; si $|E|$ est assez grand on a $\det(l(\theta)) \neq 0$ pour tout θ), et $F_E(\theta, y) = (\theta + \alpha, A_E(\theta)y)$. On a pour $|E|$ assez grand

$$H^{-1} \circ F_E \circ H(\theta, y) = (\theta + \alpha, K_E(\theta)y),$$

où

$$K_E(\theta)=\begin{pmatrix}\Psi_1(\theta) & 0\\ 0 & \Psi_2(\theta)\end{pmatrix} \tag{3}$$

avec, si $E>E_0$, $\Psi_i>0$ et si $E<-E_0$, $\Psi_i<0$ pour $i=1, 2$. ■

4.9. PROPOSITION. *Si $E>0$ et si $\Psi>0$ vérifie* (2) *alors on pose*

$$M=\operatorname{Max}\left(\|\Psi\|_{C^0}, \left\|\frac{1}{\Psi}\right\|_{C^0}\right) \quad \textit{et on a} \quad \frac{1}{M}+M\leq E+\|\varphi\|_{C^0}.$$

Démonstration. On suppose que $M=\|\Psi\|_{C^0}$, l'autre cas étant analogue. Soit θ_0 tel que $M=\Psi\circ R_\alpha(\theta_0)$. On a l'inégalité

$$E+\varphi(\theta_0)=\Psi\circ R_\alpha(\theta_0)+\frac{1}{\Psi(\theta_0)}\geq M+\frac{1}{M},$$

et donc

$$M+\frac{1}{M}\leq E+\operatorname{Max}_\theta \varphi(\theta). \quad ■$$

4.10. *Remarque.* 4.8 et 4.9 restent valables si l'on considère

$$A_E(x)=\begin{pmatrix}E+\varphi(x) & -1\\ 1 & 0\end{pmatrix}$$

avec $\varphi\in C^0(X,\mathbb{R})$, où X est un espace compact métrique, et si l'on remplace $R_\alpha:\mathbb{T}^1\to\mathbb{T}^1$ par un homéomorphisme $g:X\to X$.

4.11. On considère (R_α, A_E) comme en 4.8 et on suppose de plus $\alpha\notin\mathbb{Q}/\mathbb{Z}$. On relève $E\to A_E\in C^0(\mathbb{T}^1, SL(2,\mathbb{R}))$ en $E\to\tilde{A}_E\in C^0(\mathbb{T}^1, D^\omega(\mathbb{T}^1))$. On suppose que $SL(2,\mathbb{R})$ agit sur $\mathbb{P}(\mathbb{R}^2)$ par l'action standard, (voir 1.6). Soit $\rho_f(R_\alpha, \tilde{A}_E)\in\mathbb{R}$ le nombre de rotation fibré (voir §5); on a, si φ et $\alpha\in\mathbb{T}^1-(\mathbb{Q}/\mathbb{Z})$ sont fixés:

a) $E\to\rho_f(R_\alpha, \tilde{A}_E)$ est une fonction continue et non-décroissante (elle est non décroissante puisque $E_1<E_2\Rightarrow\tilde{A}_{E_1}(x)<\tilde{A}_{E_2}(x)$ pour tout $x\in\mathbb{T}^1$).

b) La fonction $\rho_f(R_\alpha, \tilde{A}_E)$ est constante si $|E|>E_0$ (où E_0 est défini en 4.8), et on a

$$\rho_f(R_\alpha, \tilde{A}_E) = p \in \mathbb{Z} \quad \text{so} \quad E \geq E_0$$

$$\rho_f(R_\alpha, \tilde{A}_E) = p - 1 \quad \text{si} \quad E \leq -E_0.$$

(Ceci résulte de 4.8, en utilisant 5.9.3) et en remarquant que le chemin $E \to A_E$ est homotope, les extrémitées restant la même composante connexe par arc de l'esemble $\{B \in C^0(\mathbb{T}^1, SL(2, \mathbb{R})) \mid (R_\alpha, B)$ agissant sur $\mathbb{T}^1 \times \mathbb{R}^2$ a une structure hyperbolique$\}$ au voisinage de $E = +\infty$ et $E = -\infty$, au chemin $E \to \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$.

4.12. Soit

$$A_E(\theta) = \begin{pmatrix} E + p(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

où $E \in \mathbb{R}$, $p(\theta) = \sum_{|k| \leq n} a_k e^{2\pi i k \theta}$ est un polynôme trigonométrique réel, de degré $n \geq 1$ et vérifiant $\operatorname{Log} |a_{-n}| > 0$ $(n \geq 1)$; par 4.7, pour tout E, on a

$$\lambda_+(R_\alpha, A_E) \geq \operatorname{Log} |a_{-n}|.$$

L'application $\theta \in \mathbb{T}^1 \to \begin{pmatrix} E + p(\theta) & -1 \\ 1 & 0 \end{pmatrix}$ est évidemment $\mathbb{R}$-analytique et homotope à la matrice constante $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ et donc à e.

En utilisant 4.11 on démontre une proposition analogue à 4.6. Ces exemples montrent que les corollaires du théorème d'Arnold et de Moser 5.12 et 5.14 ne sont pas globaux et ceci nonobstant des conditions d'analycité ou d'approximations par les rationnels du vecteur de rotation contrairement au théorème fondamental de [4] pour les difféomorphismes du cercle. Ces exemples sont à rapprocher du caractère local du théorème de Siegel (cf. 3.3). Le lecteur se rapportera aussi à 5.19.

Ces exemples ne sont pas analogues aux contre-exemples de Denjoy sur le cercle (cf. [4, X]):

4.13. PROPOSITION. *Soit* (R_α, C_β) *vérifiant les conditions* a) *et* b) *de* 4.6 *et tel que* $\rho(R_\alpha, C_\beta) = (\alpha, \beta)$ *ou* α *et* β *sont irrationnels et rationnellement indépendants. On fait agir* $F = (R_\alpha, C_\beta)$ *sur* $\mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2)$. *Alors il n'existe pas d'application continue* $H : \mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2) \to \mathbb{T}^2$ *homotope à l'Id telle que le diagramme* *suivant soit commutatif*:

$$\begin{tikzcd} F:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[r] \arrow[d, "H"'] & \mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[d, "H"] \\ R_{(\alpha,\beta)}:\ \mathbb{T}^2 \arrow[r] & \mathbb{T}^2 \end{tikzcd}$$

Remarque. Ceci implique que le difféomorphisme F de $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ n'est pas topologiquement conjugué à une translation de $\mathbb{T}^2$.

Démonstration. Supposons, que H existe. Par le théorème d'Osedelec (voir par exemple [15]), puisque $\lambda_+(R_\alpha, C_\beta)>0$, il existe une application $d\theta$-mesurable $s_+:\mathbb{T}^1\to\mathbb{P}(\mathbb{R}^2)$ telle que le diagramme suivant soit commutatif $d\theta$-presque partout (p_1 désigne la 1 ère projection)

$$\begin{tikzcd} F:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[r] \arrow[d, shift right, "p_1"'] & \mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[d, shift right, "p_1"'] \\ R_\alpha:\ \mathbb{T}^1 \arrow[r] \arrow[u, shift right, "Id\times s_+"'] & \mathbb{T}^1 \arrow[u, shift right, "Id\times s_+"'] \end{tikzcd}$$

(Le graphe de s_+ est la direction invariante ($d\theta$ presque partout) associée à l'exposant de Lyapounov maximal $\lambda_+(R_\alpha, C_\beta)$; il existe une section $Id\times s_-$ associée à $-\lambda_+(R_\alpha, C_\beta)$, et on a $d\theta$-presque partout $s_+\neq s_-$ (voir [5.6] et [8]).)

Si on relève $H:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)\to\mathbb{T}^1\times\mathbb{T}^1$ en $\tilde{H}:\mathbb{R}^2\to\mathbb{R}^2$ de la forme $\tilde{H}(\theta_1,\theta_2)=(\theta_1+\eta_1(\theta_1,\theta_2),\ \theta_2+\eta_2(\theta_1,\theta_2))$ avec $\eta_i\in C^0(\mathbb{T}^2,\mathbb{R})$, on doit avoir

$$\eta_1\circ\tilde{F}=\eta_1.$$

Ceci implique que $\eta_1=$constante. En effet par [8] ou 4.17, F laisse invariant un unique ensemble minimal $M\neq\emptyset$. Ceci force η_1 à être égale à une constante. (On a $\eta_{1|M}=$constante qu'on peut supposer égale à 0. Soient $\varepsilon>0$ et $\mathcal{V}$ un voisinage ouvert de M tel que tout $x\in\mathcal{V}$ vérifie $|\eta_1(x)|<\varepsilon$; puisque M est l'unique ensemble minimal de F, on a $\bigcup_{i\in\mathbb{N}} F^i(\mathcal{V})=\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$. Donc $\|\eta_1\|_{C^0}<\varepsilon$ mais comme $\varepsilon>0$ est arbitraire le résultat suit.)

Il en résulte que H est fibré: il existe $c\in\mathbb{T}^1$ tel qu'on ait le diagramme commutatif:

$$\begin{tikzcd} H:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[r] \arrow[d, "p_1"'] & \mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \arrow[d, "p_1"] \\ R_c:\ \mathbb{T}^1 \arrow[r] & \mathbb{T}^1 \end{tikzcd}$$

Si H existe, il existe donc une application $\tilde{s}$, $d\theta$-mesurable, telle que le diagramme suivant soit commutatif, $d\theta$-presque partout:

$$\begin{array}{ccc} R_{(\alpha,\beta)}:\mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \\ p_1\big\downarrow\big\uparrow Id\times\tilde{s} & & Id\times\tilde{s}\big\uparrow\big\downarrow p_1 \\ R_\alpha:\mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \end{array}$$

d'où

$$e^{2\pi i\tilde{s}(\theta+\alpha)} \underset{d\theta-\text{p.p.}}{=} e^{2\pi i\beta}e^{2\pi i\tilde{s}(\theta)};$$

mais ceci implique que $\beta\in\mathbb{Z}\alpha(\text{mod } 1)$; or, nour avons supposé que α et β sont irrationnels et rationnellement indépendants, et nous aboutissons ainsi à une absurdité. ■

4.14. On se donne (R_α, A_E) satisfaisant aux conditions de 4.7 et on suppose de plus que $\alpha\in\mathbb{T}^1-(\mathbb{Q}/\mathbb{Z})$. Soit E_1 le plus grand nombre réel tel que:

a) $\rho_f(R_\alpha, \tilde{A}_{E_1})\in\mathbb{Z}$ où $\tilde{A}_{E_1}$ est un relèvement de A_{E_1} à $D^0(\mathbb{T}^1)$;

b) le difféomorphisme fibré $\bar{F}_{E_1}=(R_\alpha, A_{E_1})$ de $\mathbb{T}^1\times\mathbb{R}^2$ n'ait pas de structure hyperbolique. On peut aussi définir E_1 ainsi: si $E\in\,]E_1, +\infty[$, $\bar{F}_E$ a une structure hyperbolique et $\bar{F}_{E_1}$ vérifie b).

Un tel nombre E_1 existe par 4.8, 4.11 et par le fait que l'ensemble des nombres $E\in\mathbb{R}$ vérifiant b) est fermé (cf. 1.4 remarque 2). On note F_E le difféomorphisme induit par $\bar{F}_E$ sur $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$.

PROPOSITION. *Le difféomorphisme F_{E_1} possède un unique ensemble minimal $M\neq\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ ($M\neq\emptyset$); M est l'adhérence du graphe d'une fonction semi-continue $s:\mathbb{T}^1\to\mathbb{P}(\mathbb{R}^2)$ telle que le diagramme suivant soit commutatif:*

$$\begin{array}{ccc} F_{E_1}:\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) & \longrightarrow & \mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2) \\ p_1\big\downarrow\big\uparrow Id\times s & & Id\times s\big\uparrow\big\downarrow p_1 \\ R_\alpha:\quad \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \end{array}$$

Démonstration. Soit E_2 très grand ($E_2>2\,\|p\|_{C^0}$), pour que le difféomorphisme fibré $\bar{F}_{E_2}=(R_\alpha, A_{E_2})$ ait une structure hyperbolique telle que les directions invariantes $\begin{pmatrix}(\Psi_{E_2})_1\\1\end{pmatrix}$ $\begin{pmatrix}(\Psi_{E_2})_2\\1\end{pmatrix}$ vérifient $(\Psi_{E_2})_i>0$ (voir 4.8). On a donc l'inégalité

4.9, $M_{E_2}+1/M_{E_2}\le E_2+\|p\|_{C^0}$, où

$$M_{E_2}=\operatorname*{Max}_{i=1,2}\left(\operatorname{Max}\left(\|(\Psi_{E_2})_i\|_{C^0},\left\|\frac{1}{(\Psi_{E_2})_i}\right\|_{C^0}\right)\right).$$

Soit E_3 le plus grand nombre réel $\le E_2$ tel que (R_α, A_{E_3}) possède une structure hyperbolique dont les directions invariantes ne soient pas de la forme $\begin{pmatrix}(\Psi_{E_3})_i\\ 1\end{pmatrix}$ avec $0<(\Psi_{E_3})_i<+\infty$.

Je dis que le nombre E_3 (*s'il existe*) *vérifie* $E_3<E_1$. En effet, pour $E>E_3$, les directions invariantes existene et sont de la forme $\begin{pmatrix}(\Psi_E)_i\\ 1\end{pmatrix}$, $0<(\Psi_E)_i<+\infty$, $i=1,2$, et vérifient $M_E+1/M_E\le E_2+\|p\|_{C^0}$. Ceci implique que $M_E\le C_2$, $1/M_E\ge C_2^{-1}$, où $C_2\ge 1$ est une constante et en particulier $C_2^{-1}\le(\Psi_E)_i\le C_2$. On arrive a une contradiction puisque les directions invariantes varient continument avec E que le difféomorphisme fibré (R_α, A_E) a une structure hyperbolique (voir [18]). On a bien montré $E_3<E_1$.

Si $E_1<E\le E_2$, on conclut que les directions invariantes de $\bar{F}_E$ restent dans le cône projectif positif $C=\{(a,1)\in\mathbb{P}(\mathbb{R}^2)\mid 1/C_2\le a\le C_2\}$ (on utilise sur $\mathbb{P}(\mathbb{R}^2)$ les coordonnées projectives).

Par 4.8 et 4.17 (voir aussi [8]) puisque $\bar{F}_{E_1}$ vérifie $\lambda_+(\bar{F}_{E_1})>0$ et, par le choix de E_1, n'a pas de structure hyperbolique, F_{E_1} laisse invariant un unique ensemble minimal M et toute mesure de probabilité ν de $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ invariante par F_{E_1} vérifie support $(\nu)=M$.

On a $M\subset\mathbb{T}^1\times C$.

En effet, soit μ_E une mesure de probabilité de $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ invariante par F_E $(E_1<E\le E_2)$ et ergodique. Puisque $\alpha\in\mathbb{T}^1-(\mathbb{Q}/\mathbb{Z})$, support$(\mu_E)$ est une direction invariante de F_E dans $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$ et donc support$(\mu_E)\subset\mathbb{T}^1\times C$. Soit $(e_i)_{i\ge1}$ une suite de nombres réels telle que $E_1<e_i\le E_2$, $e_i\to E_1$ si $i\to+\infty$ et que la suite $(\mu_{e_i})_i$ tende vaguement vers la mesure de probabilité μ. Si $i\to+\infty$, $F_{e_i}\to F_{E_1}$ uniformément, et la mesure μ est donc invariante par F_{E_1} (voir 5.6). Soit φ une fonction ≥0 de classe C^∞, nulle sur un voisinage $\mathcal{V}$ de $\mathbb{T}^1\times C$ dans $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)$. Comme $\mu_{e_i}(\varphi)=0$, si $e_i\to E_1$, on a $\mu(\varphi)=0$ et done support$(\mu)\subset\mathcal{V}$; comme $\mathcal{V}$ est arbitraire, il en résulte que support$(\mu)\subset\mathbb{T}^1\times C$. Or, support$(\mu)=M$ et on a bien démontré que $M\subset\mathbb{T}^1\times C$.

On définit, si $\theta\in\mathbb{T}^1$, $l_-(\theta)=\operatorname{Inf}\{a>0\mid(\theta,(a,1))\in M\}$ et $l_+(\theta)=\sup\{a>0\mid(\theta,(a,1))\in M\}$. On pose $s:\theta\to(l_-(\theta),1)$.

Comme M est fermé, la fonction l_- est semi-continue inférieurement, et l_+ est semi-continue supérieurement. Puisque F_E préserve l'ordre sur chaque fibre séparément de la fibration $\mathbb{T}^1\times\mathbb{P}(\mathbb{R}^2)\to\mathbb{T}^1$, et que $\bar{F}_E(\theta,(0,1))=(\theta+\alpha,(-1,0))$ l'ensemble $gr(s)=\{(\theta,s(\theta))\mid\theta\in\mathbb{T}^1\}$ est invariant par F_{E_1} et on a $gr(s)\subset M$.

L'unique ensemble minimal M de F_E est donc l'adhérence de $gr(s)$. L'ensemble M est aussi l'adhérence de $\{\theta, (l_+(\theta), 1) \mid \theta \in \mathbb{T}^1\}$. ■

Remarque. On a $M \subset M_1 = \{(\theta, (b, 1)) \mid l_-(\theta) \leq b \leq l_+(\theta)\}$. L'ensemble M_1 est fermé, sans point intérieur dans $\mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2)$ et est invariant par F_{E_1}.

La mesure de Lebesgue de M_1 est positive. En effet, par le théorème d'Osedelec il existe, puisque $\lambda_+(F_{E_1}) > 0$, 2 directions invariantes $d\theta$-mesurables distinctes ($d\theta$-presque partout) graphes de s_+ et s_-, supports des deux mesures de probabilités de $\mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2)$ μ_+ et μ_- invariantes par F_{E_1} et ergodiques [5]. On a $\text{support}(\mu_+) = \text{support}(\mu_-) = M$, voir 4.17. Pour $d\theta$-presque tout $\theta \in \mathbb{T}^1$ $s_+(\theta)$ et $s_-(\theta)$ limitent sur $\{\theta\} \times \mathbb{P}(\mathbb{R}^2)$, 2 intervalles invariants par F_{E_1} donc l'un est contenu dans M_1.

La mesure de Lebesgue de M_1 est bien positive par le théorème de Fubini, car si M_1 était de mesure de Lebesgue nulle pour presque tout $\theta \in \mathbb{T}^1$, l'ensemble $M_1 \cap (\{\theta\} \times \mathbb{P}(\mathbb{R}^2))$ serait de mesure de Lebesgue nulle et donc ne contiendrait pas un intervalle presque partout.

Question. Est ce que $M = M_1$?

4.15. *Remarques.* 1. On vérifie sans peine que:

- les fonctions $l_\pm$ satisfont $1/C_2 \leq l_\pm \leq C_2$ (ce qui implique que $l_\pm^{\pm 1} \in L^\infty(d\theta)$);
- $l_+(\theta) \neq l_-(\theta)$ $d\theta$-presque partout;
- les fonctions $l_\pm$ sont solutions de l'équation

$$l_\pm(\theta + \alpha) + \frac{1}{l_\pm(\theta)} = \varphi(\theta) + E_1, \quad \text{pour tout} \quad \theta \in \mathbb{T}^1. \tag{2}$$

Si $E > E_1$ l'équation (2) à 2 solutions continues mais pour $E = E_1$ les solutions $l_\pm$ ne sont pas continues.

2. Chacun de 2 ensembles $\{(\theta, (l_\pm(\theta), 1)) \mid \theta \in \mathbb{T}^1\}$ est $d\theta$-presque partout égal à une des directions invariantes données par le théorème d'Osedelec (resp. aux graphes de s_+ et s_-) et on a

$$\int_0^1 \operatorname{Log} l_+(\theta)\, d\theta = -\int_0^1 \operatorname{Log} l_-(\theta)\, d\theta = \lambda_+(R_\alpha, A_{E_1}) > 0.$$

(Cela résulte de ce que pour F_{E_1} les seules directions invariantes $d\theta$-mesurables sont presque partout égales aux graphes de s_+ ou s_-; voir aussi [10. §3.6].)

4.16. On se place dans les mêmes conditions que 4.10 et on suppose de plus que l'espace X est connexe et que l'homéomorphisme g de X est minimal et uniquement ergodique. On définit pour $\bar{F}_E = (g, A_E)$ de façon analogue à 4.14 un unique nombre E_1.

PROPOSITION. *L'homéomorphisme F_{E_1} induit par (g, A_{E_1}) sur $X \times \mathbb{P}(\mathbb{R}^2)$ laisse invariant un ensemble minimal $M \neq \emptyset$ possédant les propriétés suivantes:*

· *$M \subset X \times C$, où C est le cône $\{(a, 1) \in \mathbb{P}(\mathbb{R}^2) \mid 1/C_2 \leq a \leq C_2\}$ avec un $C_2 > 1$;*

· *M est l'adhérence du graphe d'une application $\theta \in X \to s(\theta) = (l_-(\theta), 1) \in \mathbb{P}(\mathbb{R}^2)$, où $1/C_2 \leq l_- \leq C_2$, et la fonction l_- est semi-continue inférieurement;*

· *la fonction l_- vérifie*

$$l_-(g(\theta)) + \frac{1}{l_-(\theta)} = \varphi(\theta) + E_1 \quad \textit{pour tout} \quad \theta \in X; \tag{2}$$

· *si $E > E_1$ l'équation*

$$\Psi \circ g + \frac{1}{\Psi} = \varphi + E \tag{2}$$

possède 2 solutions continues strictement positives.

Remarque. La fonction l_- peut être continue ainsi que le montre l'exemple $A_{E_1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

Démonstration. En utilisant 4.10, la démonstration est presque identique à celle de 4.14. Comme on ne suppose pas $\lambda_+(g, A_{E_1}) > 0$, on n'est pas sûr que l'homéomorphisme F_{E_1} laisse invariant un unique ensemble minimal; néanmoins, par la même démonstration que 4.14, on obtient une mesure de probabilité μ invariante par F_{E_1} et vérifiant $\text{support}(\mu) \subset X \times C$. L'ensemble fermé $\text{support}(\mu)$ est invariant par F_{E_1}, et il suffit de considérer un ensemble minimal $M \neq \emptyset$, $M \subset \text{support}(\mu)$ le reste du raisonnement étant analogue. ∎

4.17. *Annexe*

Dans cet annexe nous allons démontrer des résultats essentiellement dûs à R. Johnson [8]:

PROPOSITION. *Soit g un homéomorphisme minimal et uniquement ergodique de l'espace compact métrique X. On suppose que $A \in C^0(X, SL(2, \mathbb{R}))$ vérifie*

· $\lambda_+(g, A) > 0$;

· *L'homéomorphisme fibré* (g, A) *agissant sur* $X \times \mathbb{R}^2$ *n'a pas de structure hyperbolique. Il en résulte que l'homéomorphisme F induit par* (g, A) *sur* $X \times \mathbb{P}(\mathbb{R}^2)$ *laisse invariant un unique ensemble minimal* $M \neq \emptyset$ *et que toute mesure de probabilité* μ *de* $X \times \mathbb{P}(\mathbb{R}^2)$ *invariante par F vérifie support*$(\mu) = M$.

Remarque. L'homéomorphisme F peut être minimal; pour la construction d'un exemple voir [5].

Démonstration. Puisque $\lambda_+(g, A) > 0$, par [5], l'homéomorphisme F laisse invariant seulement 2 mesures de probabilités ergodiques μ_+ et μ_-. On pose $\text{support}(\mu_\pm) = K_\pm$. Soit $M = \emptyset$ un unsemble minimal invariant par F. Par le théorème de Markov–Kakutani, il en résulte qu'il existe une mesure de probabilité ν invariante par F, ergodique et vérifiant $\text{support}(\nu) \subset M$ (et donc $\text{support}\,(\nu) = M$ puisque M est un ensemble minimal de F). On a $M = K_+$ ou k_- puisque $\nu = \mu_+$ ou μ_-. On suppose que $M = K_+$, l'autre cas étant analogue, on veut montrer que $K_+ = K_-$.

On raisonne par l'absurde. Si $K_+ \neq K_-$ on a $\mu_-(K_+) = 0$ (puisque F est μ_--ergodique et que les ensembles compacts K_+ et K_- sont invariants par F). Il en résulte que l'homéomorphisme $F \mid K_+$ est uniquement ergodique l'unique mesure de probabilité invariante étant μ_+.

Par [5] ou [8], si $(x, v) \in K_+ \subset X \times \mathbb{P}(\mathbb{R}^2)$, alors, si $n \to \pm\infty$, $(1/n) \operatorname{Log} \|A_g^n(x)v\| \to \lambda_+(g, A)$ en posant, pour $n < 0$, $A_g^n = (\tilde{A}^{-1})_{g^{-1}}^n$ et $\tilde{A}^{-1}(x) = A^{-1}(g^{-1}(x))$. Puisque l'homéomorphisme g de X est minimal, pour tout $x \in X$, il existe $v \in \mathbb{P}(\mathbb{R}^2)$ tel que $(x, v) \in K_+$, et on a donc

① si $\quad n \to -\infty, \qquad \|A_g^n(x)v\| \to 0.$

Maintenant on utilise le résultat suivant[1] (du à R. Mañé, J. Selgrade, R. J. Sacker et G. R. Sell): puisque (g, A) n'a pas de structure hyperbolique et que l'homéomorphisme g est minimal il existe $(y, u) \in X \times \mathbb{P}(\mathbb{R}^2)$ tel que

② $\sup_{n \in \mathbb{Z}} \|A_g^n(u)u\| < +\infty$

et donc $(y, u) \notin K_+$.

LEMME. *Quels que soient* $B \in SL(2, \mathbb{R})$, *et* $u, v \in \mathbb{R}^2$ *vérifiant* $\|u\| = \|v\| = 1$ *et* $\|u \wedge v\| \neq 0$, *si* $\|Bv\| < 1/C$, *avec* $C > 0$, *alors on a* $\|Bu\| \geq C\,\|u \wedge v\|$.

Démonstration du lemme. Il suffit d'écrire $\|Bu\|\,\|Bv\| \geq \|Bu \wedge Bv\| = \|u \wedge v\|$. ■

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[1] Voir par exemple, R. Mañé, [M].

Fin de la démonstration de la proposition. Par ①, ② et le lemme, on arrive à une absurdité si $K_+ \neq K_-$. La proposition résulte facilement de ce fait. ■

5. Nombre de rotation fibré et quelques propriétés; application à des corollaires du théorème d'Arnold et de Moser

5.1. Soient X un espace compact métrique ($\neq \emptyset$) et g un homéomorphisme de X. Soit $x \in X \to h(x) \in \text{Homéo}_+(\mathbb{T}^1)$ une application continue, où $\text{Homéo}_+(\mathbb{T}^1)$ désigne le groupe topologique des homéomorphismes de $\mathbb{T}^1$ préservant l'orientation avec la topologie compacte ouverte.

On définit l'homéomorphisme F de $X \times \mathbb{T}^1$ par $F(x, \theta) = (g(x), h(x)(\theta))$. F est un homéomorphisme fibré, le diagramme suivant étant commutatif:

$$\begin{array}{ccc} F: X \times \mathbb{T}^1 & \longrightarrow & X \times \mathbb{T}^1 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_1} \\ g: \quad X & \longrightarrow & X \end{array}$$

$(p_1(x, \theta) = x)$.

On veut définir le nombre de rotation fibré; pour cela on suppose que l'application $x \in X \to h(x) \in \text{Homéo}_+ (\mathbb{T}^1)$ est homotope à l'application constante égale à l'idendité de $\mathbb{T}^1$. On peut donc relever l'application $x \to h(x)$ à $x \in X \to \tilde{h}(x) \in D^0(\mathbb{T}^1)$ avec $D^0(\mathbb{T}^1) = \{h \in \text{Homéo}_+ (\mathbb{R}) \mid h(x+1) = h(x) + 1, \text{ si } x \in \mathbb{R}\}$. On définit l'homéomorphisme $\tilde{F}$ de $X \times \mathbb{R}$ par $\tilde{F}(x, \theta) = (g(x), \tilde{h}(x)(\theta))$. Si $\lambda \in \mathbb{R}$, $Id \times R_\lambda$ est l'application de $X \times \mathbb{R}$ définie par $(Id \times R_\lambda)(x, \theta) = (x, \theta + \lambda)$.

Si $\tilde{F}_1$ et $\tilde{F}_2$ sont 2 relèvements de F à $X \times \mathbb{R}$ alors $\tilde{F}_1 \circ \tilde{F}_2^{-1}(x, \theta) = (x, \theta + \chi(x))$ où $\chi : X \to \mathbb{Z}$, est une application continue. Il en résulte que si l'espace X est connexe alors $\tilde{F}_1 = (Id \times R_p) \circ \tilde{F}_2$, avec $p \in \mathbb{Z}$.

5.2. On définit le nombre de rotation dans la direction de la fibre comme une fonction de $(x, \theta) \in X \times \mathbb{R}$ par

$$\bar{\rho}_f(\tilde{F})(x, \theta) = \limsup_{n \to +\infty} \frac{1}{n} (p_2 \circ \tilde{F}^n(x, \theta) - \theta) \in \mathbb{R}$$

avec $p_2(x, \theta) = \theta$. (On note ρ_f pour ρ_{fibre}). La fonction $\bar{\rho}_f(\tilde{F})$ a les propriétés:

a) Elle est $\mathbb{Z}$-périodique en θ.

b) Si $p \in \mathbb{Z}$, $\bar{\rho}_f((Id \times R_p) \circ \tilde{F}) = p + \bar{\rho}_f(\tilde{F})$.

Il suit que si l'espace X est connexe alors la foction $(x, \theta) \to \bar{\rho}_f(F)(x, \theta)$ (mod 1) ne dépend pas du relèvement $\tilde{F}$ de F.

c) Si pour tout x, $\tilde{h}_1(x) \le \tilde{h}_2(x)$ (i.e. pour tout θ $\tilde{h}_1(x)(\theta) \le \tilde{h}_2(x)(\theta)$), où $\tilde{h}_i \in C^0(X, D^0(\mathbb{T}^1))$ pour $i = 1, 2$) alors, pour tout (x, θ), on a

$$\bar{\rho}_f(g, \tilde{h}_1)(x, \theta) \le \bar{\rho}_f(g, \tilde{h}_2)(x, \theta).$$

d) $\bar{\rho}_f(\tilde{F})(x, \theta) = \bar{\rho}_f(\tilde{F})(\tilde{F}(x, \theta))$.

Remarques. 1) On peut remplacer $\mathbb{T}^1$ par $\mathbb{T}^n$ en supposant que l'application $x \in X \to h(x) \in \text{Homéo}(\mathbb{T}^n)$ est homotope à l'application constante égale à l'Identité de $\mathbb{T}^n$ et en posant si $x \to \tilde{h}(x) \in D^0(\mathbb{T}^n)$ est un relèvement de h et $\tilde{F} : (x, \theta) \in X \times \mathbb{R}^n \to (g(x), \tilde{h}(x)(\theta)) \in X \times \mathbb{R}^n$,

$$\bar{\rho}_f(\tilde{F})(x, \theta) = \limsup_{k \to +\infty} \frac{1}{k} (p_2 \circ \tilde{F}^k(x, \theta) - \theta) \in \mathbb{R}^n$$

avec $p_2(x, \theta) = \theta$ et la lim sup étant la limite supérieure de chaque composante.

Le lecteur peut se rapporter à [4, XIII 1 et 2].

2) Il est nécessaire de supposer que $x \to h(x)$ est homotope à l'application constante identité de $\mathbb{T}^n$ ainsi que le montrent les exemples suivants:

a) $X = \{1 \text{ point}\}$, $x \to h(x) = A \in \text{Homéo}(\mathbb{T}^2)$ où $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

b) $X = \mathbb{T}^1$, $x \in \mathbb{T}^1 \to h(x) \in \text{Homéo}_+(\mathbb{T}^1)$ avec $g(x) = x + \alpha$, $\alpha \ne 0$ et $h(x)(\theta) = \theta + x$.

Ces exemples montrent que le facteur $1/k$ est ridicule.

5.3. Soit $\tilde{F} : X \times \mathbb{R} \to X \times \mathbb{R}$ comme en 5.1

PROPOSITION. *Soit $x \in X$ fixé.*

a) *Quels que soient θ_1 et θ_2, on a $\bar{\rho}_f(\tilde{F})(x, \theta_1) = \bar{\rho}_f(\tilde{F})(x, \theta_2)$.*

b) *Si pour un $\theta_1 \in \mathbb{R}$, la limite* $\lim_{n \to \infty} (1/n)(p_2 \circ \tilde{F}^n(x, \theta_1) - \theta_1)$ $(= \bar{\rho}_f(\tilde{F})(x, \theta_1))$ *existe, alors pour tout $\theta \in \mathbb{R}$, la limite* $\lim_{n \to +\infty} (1/n)(p_2 \circ \tilde{F}(x, \theta) - \theta)$ *existe et elle est* $= \bar{\rho}_f(\tilde{F})(x, \theta) = \bar{\rho}_f(\tilde{F})(x, \theta_1)$.

Démonstration. Soit $p_2 \circ \tilde{F}^n(x, \theta) = h_g^n(x)(\theta)$, où $h_g^n(x) = h(g^{n-1}(x)) \circ \cdots \circ h \in D^0(\mathbb{T}^1)$; si $l \in D^0(\mathbb{T}^1)$ on a $l(\theta) = \theta + \varphi(\theta)$ $\varphi \in C^0(\mathbb{T}^1, \mathbb{R})$ avec

$$\operatorname{Max} \varphi - \operatorname{Min} \varphi < 1 \quad [4, \text{II}.2.2],$$

d'où

$$\frac{1}{n} |h_g^n(x)(\theta_1) - h_g^n(x)(\theta_2) - (\theta_1 - \theta_2)| < \frac{1}{n}. \quad \blacksquare$$

Remarque. Cette proposition n'est pas correcte si on remplace $\mathbb{T}^1$ par $\mathbb{T}^n$ (voir [4, chap XIII 1.3]).

5.4. Le théorème suivant ne serait pas correct si l'homéomorphisme g n'est pas uniquement ergodique mais seulement minimal (voir à ce propos [4, XIII 1.3]).

THÉORÈME. *On suppose que l'homéomorphisme* g *de* X *est uniquement ergodique, d'une unique mesure de probabilité invariante* μ *sur* X. *Soit* $\tilde{F}=(g, \tilde{h})$ *un homéomorphisme de* $X\times\mathbb{R}$ *comme en* 5.1. *Alors, si* $n\to+\infty$, *la suite de fonction* $((1/n)(p_2\circ\tilde{F}^n(x,\theta)-\theta))_{n\geq1}$ *converge uniformément vers une fonction constante; cette constante est notée* $\rho_f(\tilde{F})$.

Rappels d'exemples d'homéomorphismes uniquement ergodiques:

Si $X=\mathbb{T}^{n-1}$ et si $g=R_\alpha$ est une translation minimale de $\mathbb{T}^{n-1}$, alors g est un homéomorphisme uniquement ergodique de $\mathbb{T}^{n-1}$, l'unique mesure de probabilité invariante étant la mesure de Haar de $\mathbb{T}^{n-1}$. Plus généralement pour un groupe abélien compact, une translation est minimale si et seulement si elle est uniquement ergodique.

Pour démontrer le théorème nous avons besoin du lemme suivant:

LEMME. *Soit* Y *un espace compact,* G *un homéomorphisme de* Y. *Soit* ψ *une fonction continue de* Y, $\psi\in C^0(Y,\mathbb{R})$ *telle qu'il existe* $\lambda\in\mathbb{R}$ *tel que, pour toute mesure de probabilité* v *de* Y *invariante par* G, *on ait* $\int_Y \psi\, dv=\lambda$. *Alors, si* $n\to+\infty$, *la suite*

$$\left(\frac{1}{n}\sum_{i=0}^{n-1}\psi\circ G^i\right)_{n\geq1}$$

converge uniformément vers λ.

Démonstration du lemme. Il suffit de voir que ψ appartient à la fermeture pour la topologie de la convergence uniforme de l'ensemble des fonctions $\{\lambda+\eta-\eta\circ G\mid\lambda\in\mathbb{R},\ \eta\in C^0(Y)\}$, Or cela résulte du théorème de Hahn–Banach, en utilisant le fait qu'une mesure de Radon sur Y est invariante par G si et seulement si elle est nulle sur l'espace $\{\eta-\eta\circ G\mid\eta\in C^0(Y)\}$ et que toute mesure de Radon v invariante par G s'écrit de facon unique $v=v_+-v_-$, où v_+ et v_- sont des mesures positives étrangères invariantes par G (l'invariance venant de l'unicité de la décomposition de Jordan). ■

Démonstration du théorème de 5.4. On écrit $\tilde{F}(x,\theta)=(g(x),\theta+\varphi(x,\theta))$ avec

$\varphi(x, \theta) \in C^0(X \times \mathbb{T}^1)$. On a

$$\tilde{F}(x, \theta) = \left(g^n(x), \theta + \sum_{i=0}^{n-1} \varphi \circ F^i(x, \theta)\right)$$

et donc

$$\frac{1}{n}(p_2 \circ \tilde{F}^n(x, \theta) - \theta) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ F^i(x, \theta).$$

Soient v_i, $i = 1, 2$, 2 mesures de probabilitiés invariantes par F sur $X \times \mathbb{T}^1$. Chaque mesure v_i se projecte par p_1 sur l'unique mesure μ invariante par g.

Il résulte du théorème ergodique de Birkhoff et de 5.3 qu'il existe un ensemble $B \subset \mathbb{T}^1$ de μ-mesure 1 tel que, quels que soient $x \in B$ et $\theta \in \mathbb{T}^1$, on ait

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ F^i(x, \theta) \to c \in \mathbb{R}$$

où c est indépendant de $(x, \theta) \in B \times \mathbb{T}^1$. Comme l'ensemble $B \times \mathbb{T}^1$ est de v_1 et v_2 mesure 1 on a

$$\int_{X \times \mathbb{T}^1} \varphi \, dv_1 = \int_{X \times \mathbb{T}^1} \varphi \, dv_2 = c$$

et le théorème résulte du lemme. ∎

5.5. *Topologies*

Sur l'espace des applications continues de X dans $D^0(\mathbb{T}^1)$ (noté par $C^0(X, D^0(\mathbb{T}^1)))$ et sur le groupe des homéomorphismes de X (noté Homéo (X)) on met la topologie de convergence uniforme. Puisque X est compact métrique ces espaces sont métrisables. Sur le sous ensemble de Homéo (X) $Ue(X) = \{g \in \text{Homéo}\,(X) \mid g \text{ est uniquement ergodique}\}$ on met la topologie induite.

5.6. Soit Y un espace compact métrique; rappelons que, si l'on munit l'espace des fonctions continues sur Y, $C^0(Y)$, de la topologie de la convergence uniforme et l'espace des mesures de probabilités sur Y (noté $M(Y)$) de la topologie vague (ou topologie faible induite par la dualité $\sigma(M(Y), C^0(Y))$), alors l'espace $M(Y)$ est compact, métrisable, et on a le lemme immédiat:

LEMME. *Les applications suivantes sont continues*:

$$(\mu, \varphi) \in M(Y) \times C^0(Y) \to \int_X \varphi \, d\mu \in \mathbb{R};$$

$$(\mu, g) \in M(Y) \times \text{Homéo}\,(Y) \to g_*\mu \in M(Y).$$

5.7. *Continuité du nombre de rotation fibré*

PROPOSITION. *L'application*

$$\tilde{F} = (g, \tilde{h}) \in Ue(X) \times C^0(X, D^0(\mathbb{T}^1)) \to \rho_f(\tilde{F}) \in \mathbb{R}$$

est continue.

Démonstration. Il suffit de montrer que, toute suite $(\tilde{F}_i)_{i \geq 1}$, convergeant vers $\tilde{F}$ a une sous-suite $(F_{n_i})_{i \geq 0}$, $0 < n_i < n_{i+1}$, telle que si $i \to +\infty$, $\rho_f(\tilde{F}_{n_i}) \to \rho_f(\tilde{F})$. On écrit $\tilde{F}_i = (g_i, Id + \varphi_i)$ et $\tilde{F} = (g, Id + \varphi)$, où φ_i et φ sont dans $C^0(X \times \mathbb{T}^1, \mathbb{R})$. On note F_i(resp. F) l'homéomorphisme induit sur $X \times \mathbb{T}^1$ par $\tilde{F}_i$ (resp. $X \times \mathbb{T}^1$), et ν_i une mesure de probabilité sur $X \times \mathbb{T}^1$ invariante par F_i. Soit $(\nu_{n_i})_{i \geq n}$ une sous-suite de la suite $(\nu_i)_i \subset M(X \times \mathbb{T}^1)$ convergeant vaguement vers $\nu \in M(Y)$; on a, par 5.6, $F_*\nu = \nu$ puisque $F_{n_i} \to F$ uniformément. Par la démonstration de 5.4,

$$\rho_f(\tilde{F}_{n_i}) = \int_{X \times \mathbb{T}^1} \varphi_{n_i} \, d\nu_{n_i}, \qquad \rho_f(\tilde{F}) = \int \varphi \, d\nu$$

mais, par le lemme de 5.6, si $i \to +\infty$,

$$\int \varphi_{n_i} \, d\nu_{n_i} \to \int \varphi \, d\nu. \quad \blacksquare$$

5.8. *Remarques.* 1. Il n'est pas difficile de voir (par des arguments similaires à 5.7) que l'application $\tilde{F} \to \bar{\rho}_f(\tilde{F})$ est continue pour la topologie uniforme au point $(g_1, \tilde{h}_1) \in Ue(X) \times C^0(X, D^0(\mathbb{T}^1))$ si on définit $\bar{\rho}_f(\tilde{F})$ comme un fonction de $F = (g, \tilde{h}) \in \text{Homéo}\,(X) \times C^0(X, D^0(\mathbb{T}^1))$ ainsi que nous l'avons fait en 5.2 (on peut aussi remplacer la lim sup par la lim inf).

2. Si l'homéomorphisme g préserve une mesure de probabilité fixée μ de X (mais on ne suppose pas que g est uniquement ergodique) si $\tilde{h} = Id + \varphi \in C^0(X, D^0(\mathbb{T}^1))$ et si $\tilde{F} = (g, h)$, alors, quand $n \to +\infty$, $(1/n)(p_2 \circ \tilde{F}^n(x)(\theta) - \theta)$ tend pour μ-presque tout x et tout θ vers une fonction $\Psi \in L^\infty(X, \mu)$, g invariante (cf.

5.4). De plus l'application

$$\tilde{F} = (g, \tilde{h}) \in \{f \in \text{Homéo}\,(X) \mid f_*\mu = \mu\} \times C^0(X, D^0(\mathbb{T}^1)) \to \int_X \Psi\, d\mu \in \mathbb{R}$$

est continue par la même démonstration que 5.7.

5.9. *Propriétés*

On se donne $\tilde{F} = (g, \tilde{h}) \in Ue(X) \times C^0(X, D^0(\mathbb{T}^1))$ On écrit $\tilde{F}(x, \theta) = (g(x), \tilde{h}(x)(\theta)) = (g(x), \ \theta + \varphi(x, \theta))$ où $\varphi \in C^0(X \times \mathbb{T}^1)$. On désigne par F l'homéomorphisme induit par $\tilde{F}$ sur $X \times \mathbb{T}^1$.

On a les propriétés suivantes pour la fonction continue $\tilde{F} \to \rho_f(\tilde{F}) \in \mathbb{R}$:

1) Si $p \in \mathbb{Z}$, $\rho_f((Id \times R_p) \circ \tilde{F}) = p + \rho_f(\tilde{F})$.

Si l'espace X est connexe, on pose $\rho_f(F) = \rho_f(\tilde{F}) (\text{mod } 1)$ et cela ne dépend pas du relèvement $\tilde{F}$ de F. Les propriétés suivantes ont alors des analogues immédiats pour $\rho_f(F) \in \mathbb{T}^1$.

Si l'espace X n'est pas connexe on pose $\rho_f(F) = \rho_f(\tilde{F}) \bmod D$, où $D \subset \mathbb{R}$ est le sous-groupe $\rho_f(g \times N)$ avec $N = \{Id_{\mathbb{R}} + \chi \mid \chi \in C^0(X, \mathbb{Z})\}$. Si $H = g \times (Id + \chi)$, avec $Id + \chi \in N$ on a $\rho_f(H) = \int_X \chi(x)\, d\mu(x)$.

Par la démonstration de 5.4 on vérifie que $\rho_f(\tilde{F}) \bmod D$ ne dépend pas du relèvement $\tilde{F}$ de F.

Le groupe D est dénombrable car l'esemble N l'est (puisque avec la topologie de la convergence uniforme N est discret et séparable l'espace X étant compact métrisable). On a toujours $\mathbb{Z} \subset D$ et D est le $\mathbb{Z}$-module de $\mathbb{R}$ engendré les valeurs $\mu(U_i)$ où U_i sont les ensembles compacts ouverts de X.

2) Si pour tout $x \in X$ on a $\tilde{h}_1(x) \leq \tilde{h}_2(x)$, où $h_i \in C^0(X, D^0(\mathbb{T}^1))$ pour $i = 1, 2$, alors $\rho_f(g, \tilde{h}_1) \leq \rho_f(g, \tilde{h}_2)$.

3) Il en résulte que la fonction suivante

$$\lambda \to \rho_f((Id \times R_\lambda) \circ \tilde{F}) = k(\lambda) \in \mathbb{R}$$

est continue, monotone non décroissante, et vérifie $k(\lambda + 1) = k(\lambda) + 1$.

4) Si $\rho_f(\tilde{F}) = \alpha \in \mathbb{R}$, alors l'homéomorphisme de $X \times \mathbb{R}$ $(g, R_\alpha)^{-1} \circ \tilde{F}$ a un point fixe. (Cela résulte de ce que $\int_{X \times \mathbb{T}^1} \varphi\, d\nu = \alpha$, où ν est une mesure de probabilité invariante par l'homéomorphisme F de $\mathbb{T}^1 \times \mathbb{P}(\mathbb{R}^2)$. La fonction φ s'annule donc en au moins un point.)

5) Soit $H(x, \theta) = (x, l(x)(\theta))$, où $l \in C^0(X, D^0(\mathbb{T}^1))$. La même démonstration que [4, II et XIII. 1] donne $\rho_f(H \circ (g \times R_\alpha) \circ H^{-1}) = \alpha$.

6) Soit g un homéomorphisme de X totalement uniquement ergodique (i.e.

pour tout $n \in \mathbb{Z}-\{0\}$, g^n est uniquement ergodique) alors

$$\rho_f(\tilde{F}^n) = n\rho_f(\tilde{F}).$$

7) Soit $\tilde{F} = H \circ (g \times R_\alpha) \circ H^{-1}$ avec $H(x, \theta) = (x, l(x)(\theta))$. Par 5) $\rho_f(\tilde{F}) = \alpha$. On a la

PROPOSITION. $\rho_f((Id \times R_\lambda) \circ \tilde{F}) = \alpha \Leftrightarrow \lambda = 0$.

Démonstration. Si $\tilde{F}_1 = (g, \tilde{h}_1) \in Ue(X) \times C^0(X, D^0(\mathbb{T}^1))$ vérifie $\rho_f(\tilde{F}_1) = \alpha$ et si $\tilde{F} = H \circ (g \times R_\alpha) \circ H^{-1}$, alors $\tilde{F}_1 \circ \tilde{F}^{-1}$ a un point fixe. En effet par 4) l'homéomorphisme $H^{-1} \circ \tilde{F}_1 \circ \tilde{F}^{-1} \circ H$ de $X \times \mathbb{R}$ a un point fixe et donc aussi $\tilde{F}_1 \circ \tilde{F}^{-1}$. Si l'on avait $\rho_f((Id \times R_\lambda) \circ \tilde{F}) = \alpha$ pour un $\lambda \neq 0$, l'homéomorphisme $Id \times R_\lambda$ de $X \times \mathbb{R}$ aurait un point fixe et donc $\lambda = 0$. ■

8) Soient $g \in Ue(X)$ et $\alpha \in \mathbb{R}$ fixé. On pose $\bar{0}^0(g \times R_\alpha)$ = la fermeture pour la C^0-topologie dans $Ue(X) \times C^0(X, D^0(\mathbb{T}^1))$ de l'ensemble $\{H^{-1} \circ (g \times R_\alpha) \circ H \mid H(x, \theta) = (x, l(x)(\theta)),\ l \in C^0(X, D^0(\mathbb{T}^1))\}$.

PROPOSITION. *Pour tout* $\tilde{F}_1 = (g, \tilde{h}) \in Ue(X) \times C^0(X, D^0(\mathbb{T}^1))$ *vérifiant* $\rho_f(\tilde{F}_1) = \alpha$ *et tout* $\tilde{F} \in \bar{0}^0(g \times R_\alpha)$, *l'homéomorphisme* $\tilde{F}_1 \circ F^{-1}$ *de* $X \times \mathbb{R}$ *a un point fixe.*

Démonstration. Si l'on fixe $\tilde{F}_1$, alors pour tout H, par 4) $\tilde{F}_1 \circ (H \circ (g \times R_\alpha) \circ H^{-1})^{-1}$ a un point fixe; or, l'ensemble des $F = (g, \tilde{h})$ avec g fixé et $\tilde{h} \in C^0(X, D^0(\mathbb{T}^1))$ tel que $\tilde{F}_1 \circ \tilde{F}^{-1}$ n'ait pas de point fixe sur $X \times \mathbb{R}$ est ouvert pour la C^0-topologie. ■

COROLLAIRE. *Si* $\tilde{F} \in \bar{0}^0(g \times R_\alpha)$ *alors* $\rho_f((Id \times R_\lambda) \circ \tilde{F}) = \alpha \Leftrightarrow \lambda = 0$.

Remarque. La proposition ci-dessus n'est pas valable si on remplace $\mathbb{T}^1$ par $\mathbb{T}^n$.

EXEMPLE. $X = \mathbb{T}^1$, $g = R_\alpha$ où $\alpha \in \mathbb{T}^1 - (\mathbb{Q}/\mathbb{Z})$, $\tilde{h}(x) \in \text{Homéo}(\mathbb{T}^2)$, $\tilde{h}(x)(\theta_1, \theta_2) = (\theta_1 + \varphi_1(z),\ \theta_2 + \varphi_2(x))$, où $\varphi_i \in C^\infty(\mathbb{T}^1, \mathbb{R})$, $\int_0^1 \varphi_i(x)\, dx = 0$, pour $i = 1, 2$ et les fonctions φ_1 et φ_2 n'ont pas de 0 commun.

5.10. *Le groupe* $G^\infty(\mathbb{T}^n)$

On considère le sous-groupe $G^\infty(\mathbb{T}^n)$ de groupe des difféomorphismes de $\mathbb{T}^n$, $n \geq 2$, défini par $G^\infty(\mathbb{T}^n) = \{(R_\alpha \times h) \mid \alpha \in \mathbb{T}^{n-1},\ h \in C_0^\infty(\mathbb{T}^{n-1}, \text{Diff}_+^\infty(\mathbb{T}^1))\}$, où, si

$\alpha \in \mathbb{T}^{n-1}$, $R_\alpha : \theta \in \mathbb{T}^{n-1} \to \theta + \alpha \in \mathbb{T}^{n-1}$, et où $C_0^\infty(\mathbb{T}^{n-1}\,\mathrm{Diff}_+^\infty(\mathbb{T}^1))$ désigne l'ensemble des applications de classe C^∞ de $\mathbb{T}^{n-1}$ dans $\mathrm{Diff}_+^\infty(\mathbb{T}^1)$ homotopes à l'application constante $\theta \in \mathbb{T}^{n-1} \to Id_{\mathbb{T}^1}$ (i.e. $h \in C^\infty$ veut dire que $(x, \theta) \in \mathbb{T}^{n-1} \times \mathbb{T}^1 \to h(x, \theta) \in \mathbb{T}^1$ est de classe C^∞). $\mathrm{Diff}_+^\infty(\mathbb{T}^1)$ désigne le groupe topologique des difféomorphismes de classe C^∞ préservant l'orientation avec la C^∞-topologie. On met sur $G^\infty(\mathbb{T}^n)$ la topologie C^∞.

On définit aussi le sous-groupe de Homéo $(\mathbb{T}^n)$ $G^0(\mathbb{T}^n) = \{(R_\alpha \times h) \mid \alpha \in \mathbb{T}^{n-1}, h \in C_0^0(\mathbb{T}^{n-1}, \text{Homéo}_+(\mathbb{T}^1))\}$ et on met sur $G^0(\mathbb{T}^n)$ la C^0-topologie. On rappelle la définition:

DÉFINITION. $(\alpha, \beta) \in \mathbb{T}^{n-1} \times \mathbb{T}^1$ *satisfait à une condition diophantienne s'il existe* $C > 0$, $\gamma > 0$ *tel que, pour tout* $(k_0, k_1, \ldots, k_n) \in \mathbb{Z} \times (\mathbb{Z}^n - \{0\})$, *on ait* $|k_0 + k_n\tilde{\beta} + \sum_{i=1}^{n-1} k_i\tilde{\alpha}_i| \geq C\,|k|^{-\gamma}$, *où* $|k| = \sup_{1 \leq i \leq n} |k_i|$ *et* $(\tilde{\alpha}, \tilde{\beta}) \in \mathbb{R}^n$ *est un relèvement de* $(\alpha, \beta) \in \mathbb{T}^n$ *à* $\mathbb{R}^n$.

On rapelle que Lebesgue-presque-tout $(\alpha, \beta) \in \mathbb{T}^n$ satisfait à une condition diophantienne.

Si $(\alpha, \beta) \in \mathbb{T}^n$ satisfait à une condition diophatienne, alors la translation R_α de $\mathbb{T}^{n-1}$ est minimale et donc uniquement ergodique.

5.11. *Corollaire due théorème d'Arnold et de Moser*

On a le corollaire suivant du théorème d'Arnold et de Moser, théorème qui est démontré dans [4, Appendice] (voir aussi [6]).

COROLLAIRE 1. *Soit* $(\alpha, \beta) \in \mathbb{T}^{n-1} \times \mathbb{T}^1$ *satisfaisant à une condition diophantienne. Il existe un voisinage* $\mathcal{V}_{\alpha,\beta}$ *de* (R_α, R_β) *dans* $G^\infty(\mathbb{T}^n)$ *et une application continue pour la* C^∞*-topologie (et même de classe* C^∞ *au sens d'Hamilton)* $S_{\alpha,\beta}$: $\mathcal{V}_{\alpha,\beta} \to \mathbb{T}^1 \times G^\infty(\mathbb{T}^n)$ *telle que* $S_{\alpha,\beta}(F) = (\lambda, H)$ *vérifie* $H(0) = 0$ *et* $F = R_{(0,\lambda)} \circ H \circ (R_\alpha \times R_\beta) \circ H^{-1}$.

Démonstration. Il suffit d'appliquer [4, A] et [4, IV.5.1] pour s'assurer que $H \in G^\infty(\mathbb{T}^n)$. Cela résulte aussi de la démonstration de [4, A] ou de [6]. (On peut aussi raisonner directement et montrer que H est fibré exactement comme dans la démonstration de 4.13.) ■

Remarque. Si F définit un difféomorphisme $\mathbb{R}$-analytique de $\mathbb{T}^n$ il en est de même de H (voir [4, A]).

5.12. COROLLAIRE 2. *Soient* (α, β) *satisfaisant à une condition diophantienne et* $F = (R_\alpha, h) \in \mathcal{V}_{\alpha,\beta}$ *(voisinage du corollaire* 1). *On suppose que* $\rho_f(F) = \beta$ *(ici* $X = \mathbb{T}^{n-1}$, $g = R_\alpha$); *il existe alors* $H = Id_{\mathbb{T}^{n-1}} \times l$ *avec* $l \in C_0^\infty(\mathbb{T}^{n-1}, \mathrm{Diff}_+^\infty(\mathbb{T}^1))$ *tel que l'on ait* $F = H \circ (R_\alpha \times R_\beta) \circ H^{-1}$.

Démonstration. Il suffit d'appliquer 5.11 et 5.9.7). ■

5.13. Soit $G_L^\infty(\mathbb{T}^n)=\mathbb{T}^{n-1}\times C_0^\infty(\mathbb{T}^{n-1}, PSL(2,\mathbb{R}))$: on suppose $PSL(2,\mathbb{R})\hookrightarrow \mathrm{Diff}_+^\infty(\mathbb{P}(\mathbb{R}^2))\cong \mathrm{Diff}_+^\infty(\mathbb{T}^1)$, cette inclusion venant de l'action canonique de $PSL(2,\mathbb{R})$ sur $\mathbb{P}(\mathbb{R}^2)$. L'indice 0 dans C_0^∞ indique que l'on ne considère que les applications de classe C^∞ homotopes à l'application constante $x\in\mathbb{T}^{n-1}\to e$.

L'application

$$\beta\in\mathbb{T}^1\to\begin{pmatrix}\cos 2\pi\beta & -\sin 2\pi\beta\\ \sin 2\pi\beta & \cos 2\pi\beta\end{pmatrix}\in SL(2,\mathbb{R})$$

donne l'application $\beta\to R_{2\beta}\in\mathbb{T}^1\subset PSL(2,\mathbb{R})\subset \mathrm{Diff}_+^\infty(\mathbb{T}^1)$. $G_L^\infty(\mathbb{T}^n)$ est canoniquement un sous-groupe de $G^\infty(\mathbb{T}^n)$, et on utilise l'indice L pour linéaire. On définit aussi le sous-groupe de $G^0(\mathbb{T}^n)$, $G_L^0(\mathbb{T}^n)=\mathbb{T}^{n-1}\times C_0^0(\mathbb{T}^{n-1}, PSL(2,\mathbb{R}))$.

5.14. COROLLAIRE 3. *Soit $(\alpha,\beta)\in\mathbb{T}^{n-1}\times\mathbb{T}^1$ satisfaisant à une condition diophantienne et $(R_\alpha, A)\in\mathcal{V}_{\alpha,\beta}\cap G_L^\infty(\mathbb{T}^n)$ (voisinage du corollaire* 1) *vérifiant $\rho_f(R_\alpha, A)=\beta$. Alors il existe $H\in G_L^\infty(\mathbb{T}^n)$ tel que l'on ait*
(+) $F=H\circ(R_\alpha\times R_\beta)\circ H^{-1}$.

1*ère Démonstration.* Par 5.12, il existe $H_1\in G^\infty(\mathbb{T}^n)$ vérifiant (+); on veut voir que $H_1\in G_L^\infty(\mathbb{T}^n)$; or H_1 est unique si on impose que: $H_1(0)=0$ et il suffit de voir qu'il existe $H_2\in G_L^0(\mathbb{T}^n)$ (i.e. de classe C^0) vérifiant (+). Il existe $C>1$ tel que, pour tout $x\in\mathbb{T}^{n-1}$ et tout entier $n\geq 0$, on ait

$$\|A_{R_\alpha}^n(x)\|\leq C,$$

où $A_{R_\alpha}^n(x)=A(R_{(n-1)\alpha}(x))\cdots A(x)$ et $\|\ \|$ est la fonction induite sur $PSL(2,\mathbb{R})$ par la norme sur $\mathscr{L}(\mathbb{R}^2,\mathbb{R}^2)$, elle même induite par la norme euclidienne de $\mathbb{R}^2$: si $v=(v_1,v_2)\in\mathbb{R}^2$, $\|v\|^2=v_1^2+v_2^2$. En effet le difféomorphisme H_1 est de classe C^1 et donc

$$\sup_n\left\|\frac{1}{\det DF^n}\right\|_{C^0(\mathbb{T}^{n-1}\times\mathbb{P}(\mathbb{R}^2))}<+\infty$$

puisque $\tilde{F}=H_1\circ(R_\alpha\times R_\beta)\circ H_1^{-1}$ (cf. [4, IV 1]). DF^n désigne la dérivée de F^n. Or

$$\left\|\frac{1}{\det DF^n}\right\|_{C^0}=\|A_{R_\alpha}^n\|_{C^0(\mathbb{T}^{n-1})}^2.$$

(On utilise le fait que, si $v \in \mathbb{R}^2$, $\|v\| = 1$, $v = (\cos 2\pi\theta, \sin 2\pi\theta)$ pour un $\theta \in \mathbb{T}^1$ et

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),$$

alors

$$① \quad \frac{d}{d\theta}\left(\frac{1}{2\pi} \operatorname{Arctg}\left(\frac{a \cos 2\pi\theta + b \sin 2\pi\theta}{c \cos 2\pi\theta + d \operatorname{din} 2\pi\theta}\right)\right) = \frac{1}{\|Bv\|^2}.\Big)$$

Par un théorème de Cameron (la démonstration étant analogue à celle de [9-§2.5]) il existe $H_3 \in G_L^0(\mathbb{T}^n)$ (on peut supposer que H_3 est homotope à l'identité) tel que $H_3 \circ F \circ H_3^{-1} = (R_\alpha, A_1)$, où $A_1(x) = e^{2\pi i\varphi(x)}$ et $\varphi \in C^0(\mathbb{T}^{n-1}, \mathbb{R})$. Par l'invariance du nombre de rotation fibré (cf. 5.9.5)), on a $\int_{\mathbb{T}^{n-1}} \varphi(x)\, dx = \beta$. Par [4, XIII 5.3], il existe $\eta \in C^0(\mathbb{T}^{n-1}, \mathbb{R})$ tel que $H_4(x, \theta) = (x, e^{2\pi i(\theta + \eta(x))})$ vérifie

$$H_4 \circ H_3 \circ F \circ H_3^{-1} \circ H_4^{-1}(x, \theta) = (x + \alpha, e^{2\pi i(\theta + \beta)})$$

et

$$H_2 = H_4 \circ H_3 \in G_L^0(\mathbb{T}^n) \quad \text{vérifie donc} \quad (+). \quad \blacksquare$$

Remarque. On a montré en plus: *si* $F \in G_L^\infty(\mathbb{T}^n)$ *vérifie* $(+)$ *avec un* $H \in G^\infty(\mathbb{T}^n)$ *alors* $H \in G_L^\infty(\mathbb{T}^n)$ (et on a seulement utilisé le fait que $R_\alpha \times R_\beta$ est une translation minimale de $\mathbb{T}^n$).

2ème démonstration si le voisinage $\mathcal{V}_{\alpha,\beta}$ *est assez petit.* Pour tout $\eta > 0$, si le voisinage $\mathcal{V}_{\alpha,\beta}$ est assez petit, on peut supposer qu'il existe $H_1 \in G^\infty(\mathbb{T}^n)$ vérifiant $(+)$ et $\|H_1 - Id_{\mathbb{T}^n}\|_{C^1} < \eta$. (Cela résulte de la continuité de l'applications $S_{\alpha,\beta}$ de 5.11). Puisque H_1 est de classe C^∞, on a:

a) La suite $(\bar{R}_{-n\beta} \cdot \bar{A}^n_{R_\alpha})_{n \in \mathbb{N}}$ d'éléments de $C^\infty(\mathbb{T}^{n-1}, SL(2, \mathbb{R}))$ est bornée dans la C^∞-topologie, où

$$\bar{R}_{n\beta} = \begin{pmatrix} \cos 2\pi n\bar{\beta} & -\sin 2\pi n\bar{\beta} \\ \sin 2\pi n\bar{\beta} & \cos 2\pi N\bar{\beta} \end{pmatrix},$$

avec $\bar{\beta} \in \mathbb{T}^1$ vérifiant $2\bar{\beta} = \beta$, $\bar{A}^n_{R_\alpha}(x) = \bar{A}(R_{(n-1)\alpha}(x)) \cdots \bar{A}(x)$ et $x \to \bar{A}(x)$ est un relèvement de $x \to A(x) \in PSL(2, \mathbb{R})$ voisin de $\bar{R}_\beta$. (C'est possible puisque $x \to A(x)$ est homotope à l'application $x \to e$.) Pour voir ceci, il suffit d'écrire la matrice $A^n_{R_\alpha}$ en coordonnées polaires, en utilisant ① et [4, IV 1 et XIII 1.4].

b) Si η est assez petit, on a

$$\sup_{n\geq 1} \|\bar{R}_{-n\beta}\bar{A}^n_{R_\alpha} - e\|_{C^0(\mathbb{T}^{n-1})} < \tfrac{1}{2},$$

où e est la matrice unité de $SL(2,\mathbb{R})$.

Soient $\bar{B}_n = (1/n)\sum_{i=0}^{n-1} \bar{R}_{-n\beta} \circ \bar{A}^n_{R_\alpha}$, $n \geq 1$; par a) la suite $(\bar{B}_n)_{n\in\mathbb{N}^*}$ est d'adhérence compacte pour la C^∞-topologie dans l'espace de Fréchet-Montel $C^\infty(\mathbb{T}^{n-1}, \mathscr{L}(\mathbb{R}^2, \mathbb{R}^2))$; soit $\bar{B}$ une valeur d'adhérence de cette suite (i.e. telle qu'il existe une suite $0 < n_i < n_{i+1}$ d'entiers telle que, si $i \to +\infty$, $\bar{B}_{n_i} \to \bar{B}$ dans la C^∞-topologie). Par b), on a $\|\bar{B} - e\|_{C^0} < \frac{1}{2}$ et donc $\bar{B} \in C^\infty(\mathbb{T}^{n-1}, GL_+(2,\mathbb{R}))$. Si $x \in \mathbb{T}^n$,

$$\bar{B}_n(R_\alpha(x)) \cdot A(x) = \bar{R}_\beta \cdot \bar{B}_n(x) + \frac{1}{n}\bar{R}_\beta(\bar{R}_{+n\beta} \cdot \bar{A}^n_{R_\alpha}(x) - e)$$

et donc, si $n_i \to +\infty$,

$$\bar{B}(R_\alpha(x)) \cdot \bar{A}(x) = \bar{R}_\beta \cdot \bar{B}(x).$$

Il en résulte qu'il existe $c > 0$ telle que pour tout $x \in \mathbb{T}^{n-1}$, on ait $\det \bar{B}(x) = c$. Si B désigne l'image de $\bar{B}$ dans $C^\infty(\mathbb{T}^{n-1}, PSL(2,\mathbb{R}))$, alors $H = Id \times B \in G^\infty_L(\mathbb{T}^n)$ vérifie $(+)$. ■

5.15 ***Remarques.*** 1. Le voisinage $\mathscr{V}_{\alpha,\beta}$ est induit par un voisinage dans la $C^{2\gamma+\varepsilon}$-topologie et sa taille ne dépend que de la constante $C > 0$, où γ et C sont les constantes de la condition diophantienne 5.10 (cf. [4, A.2.5]).

2. On peut affaiblir la condition diophantienne sur (α, β) (cf. 3.2 [14] et [6]).

5.16. *Vecteur de rotation*

Soit $F = (R_\alpha, h) \in G^0(\mathbb{T}^n)$, agissant sur $\mathbb{T}^{n-1} \times \mathbb{T}^1$. Si R_α est une translation minimale de $\mathbb{T}^{n-1}$, *on définit le vecteur de rotation de F* par

$$\rho(F) = (\alpha, \rho_f(F)) \in \mathbb{T}^n.$$

Cette définition est compatible avec celle de 5.2 remarques 1) et de [4, XIII.1]. L'existence et la valeur de $\rho(F)$ sont invariantes par conjugaison par un homéomorphisme de $\mathbb{T}^n$ homotope à l'identité [4, XIII 1].

Si $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{T}^{n-1}$, on définit le module $\mathscr{M}_\alpha$ des fréquences comme le $\mathbb{Z}$-module de $\mathbb{T}^1$ engendré par $\alpha_1, \ldots, \alpha_{n-1}$.

Si $k=(k_1,\ldots,k_{n-1})\in\mathbb{Z}^{n-1}$, on définit le difféomorphisme $S_k:(x,\theta)\in\mathbb{T}^{n-1}\times\mathbb{T}^1\to(x,\theta+\langle k,x\rangle)\in\mathbb{T}^n$ avec $\langle k,x\rangle=\sum k_ix_i$ si $x=(x_1,\ldots,x_{n-1})$. On a $S_k\in\mathbb{T}^{n-1}\times C^\omega(\mathbb{T}^{n-1},PSL(2,\mathbb{R}))$, mais pour $k\neq 0$, $S_k\notin G^0(\mathbb{T}^n)$. On définit des automorphismes extérieurs des groupes $G^0(\mathbb{T}^n)$ et $G^0_L(\mathbb{T}^n)$ par $F\to S_k\circ F\circ S_k^{-1}$, et on a

$$\rho(S_k\circ F\circ S_k^{-1})=\left(\alpha,\rho_f(F)+\sum_{i=1}^{n-1}k_i\alpha_i\right)$$

et un isomorphisme de $\mathbb{Z}$-modules

$$k\in\mathbb{Z}^{n-1}\to\rho_f(S_k\circ F\circ S_k^{-1})-\rho_f(F)\in\mathcal{M}_\alpha.$$

5.17. *Vecteur de rotation et structure hyperbolique*

Soit $F=(R_\alpha,f)\in G^0(\mathbb{T}^n)$, où R_α est une translation minimale. *On suppose que l'homéomorphisme F de $\mathbb{T}^{n-1}\times\mathbb{T}^1$ laisse invariant le graphe d'une application continue $\Psi:\mathbb{T}^{n-1}\to\mathbb{T}^1$.* Par exemple, F est l'homéomorphisme induit sur $\mathbb{T}^{n-1}\times\mathbb{P}(\mathbb{R}^2)$ par l'homéomorphisme fibré $(R_\alpha,\bar{A})\in\mathbb{T}^{n-1}\times C^0(\mathbb{T}^{n-1},SL(2,\mathbb{R}))$ agissant sur $\mathbb{T}^{n-1}\times\mathbb{R}^2$ et possédant une structure hyperbolique (cf. 4.2 et 4.3).

PROPOSITION. *Soit F comme ci-dessus Si $\rho(F)=(\alpha,\beta)\in\mathbb{T}^n$ alors $\beta\in\mathcal{M}_\alpha$.*

Démonstration. Si Ψ est homotope à une application constante alors $\beta=0$. On se ramène à ce cas en considérant $S_k\circ F\circ S_k^{-1}$ (cf. 5.16). ∎

5.18. Soient R_α une translation minimale de $\mathbb{T}^{n-1}$ et $\beta\in\mathbb{T}^1$. On pose $F^0_{\alpha,\beta}=\{F\in G^0(\mathbb{T}^n)\mid\rho(F)=(\alpha,\beta)\}$ et $F^0_{\alpha,\beta}=F^\infty_{\alpha,\beta}\cap G^\infty(\mathbb{T}^1)$. Par la continuité de la fonction ρ_f les ensembles $F^0_{\alpha,\beta}$ et $F^\infty_{\alpha,\beta}$ sont fermés (pour la C^0-topologie).

Il n'est pas difficile de voir que les groupes topologiques $G^0(\mathbb{T}^n)$, $G^0_L(\mathbb{T}^n)$, $G^0_L(\mathbb{T}^n)$ et $G^\infty_L(\mathbb{T}^n)$ sont connexes par arcs et métrisables.

PROPOSITION. *L'ensemble $F^\infty_{\alpha,\beta}$ est connexe pour la C^∞-topologie.*

Démonstration. Soient $L^\infty_\alpha=\{(R_\gamma\times f)\in G^\infty(\mathbb{T}^n)\mid\gamma=\alpha,\ f(0)(0)=0\}$ et $H^\infty_\alpha=\{(R_\gamma\times f)\in G^\infty(\mathbb{T}^n)\mid\gamma=\alpha\}$, on a $L^\infty_\alpha\subset H^\infty_\alpha$.

En remontant à $\mathbb{T}^{n-1}\times\mathbb{R}$ on vérifie que L^∞_α est connexe par arcs. Soit l'application continue, surjective (cf. 5.9.3); $p_2:H^\infty_\alpha\to L^\infty_\alpha$, $(R_\alpha\times f)\to(Id\times R_{-f(0)})\circ(R_\alpha\times f)$. Si on identifie H^∞_α à $\mathbb{T}^1\times L^\infty_\alpha$, par l'application $(\lambda,F)\to(Id\times R_\lambda)\circ F$ alors p_2 est la 2ème projection. Il suit, que puisque $\mathbb{T}^1$ est un espace compact métrique, que si F est un ensemble fermé de H_α^∞ alors l'ensemble $p_2(F)$ est fermé dans L_α^∞.

Si l'ensemble fermé $F_{\alpha,\beta}^\infty$ n'est pas connexe alors $F_{\alpha,\beta}^\infty = F_1 \coprod F_2$ où F_i, $i = 1,2$, sont des ensembles fermés disjoints de H_α^∞. Or, les ensembles $p_2(F_i)$ $i = 1, 2$, sont fermés et vérifient $p_2(F_1) \cup p_2(F_2) = L_\alpha^\infty$ et donc $p_2(F_1) \cap p_2(F_2) \neq \emptyset$. Soit $y \in p_2(F_1) \cap p_2(F_2)$, par 5.9.3), $p_2^{-1}(y)$ est un segment (pouvant être réduit à un point) et il en résulte par l'absurde que $F_{\alpha,\beta}^\infty$ est connexe. ■

Remarque. Par la même démonstration les ensembles $F_{\alpha,\beta}^0$, $F_{\alpha,\beta}^\infty \cap G_L^\infty(\mathbb{T}^n)$ sont connexes.

QUESTIONS. *Si* $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ mod 1 (*resp. si* $\beta \notin \mathbb{Z}\alpha$ mod 1) *et si* $f \in F_{\alpha,\beta}^0$ (*resp.* $f \in F_{\alpha,\beta}^0 \cap G_L^0(\mathbb{T}^n)$) *est-ce-que* $\rho_f((Id \times R_\lambda) \circ F) = \beta$ *implique* $\lambda = 0$?

Pour de réponses partielles positives à ces questions cf. 5.9. 7) et 8). Des résponses positives à ces questions impliquent respectivement que si $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ mod 1 (resp. $\beta \notin \mathbb{Z}\alpha$ mod 1) alors $F_{\alpha,\beta}^0 \subset H_\alpha^0$ (resp. $F_{\alpha,\beta}^0 \cap G_L^0(\mathbb{T}^n) \subset H_\alpha^0 \cap G_L^0(\mathbb{T}^n)$) est le graphe d'une fonction continue de L_α^0 dans $\mathbb{R}$ (resp. de $L_\alpha^0 \cap G_L^0(\mathbb{T}^n)$ dans $\mathbb{R}$) (cf. [4, III]).

Il suit de 5.17, que si $\beta \in \mathbb{Z}\alpha$ mod 1 alors $G_L^0(\mathbb{T}^n) \cap F_{\alpha,\beta}^0$ a un intérieur non vide dans $G_L^0(\mathbb{T}^n)$ (i.e. il contient l'ouvert de ceux qui agissant sur $\mathbb{T}^{n-1} \times \mathbb{R}^2$ ont une structure hyperbolique). Il en résulte que $F_{\alpha,\beta}^\infty$ a aussi un intérieur non vide dans $G^\infty(\mathbb{T}^n)$ (en utilisant le fait qu'un tore invariant par un difféomorphisme de classe C^∞, normalement hyperbolique, est stable par perturbation C^∞ du difféomorphisme).

En utilisant les revêtements finis il en résulte que si $\beta \in \mathbb{Q} + \mathbb{Q}\alpha$ mod 1 alors $F_{\alpha,\beta}^\infty$ a un intérieur non vide dans $G^\infty(\mathbb{T}^n)$ (si $(R_\alpha, f) \in G^\infty(\mathbb{T}^n)$, et $(R_\alpha, \tilde{f})$ est un relèvement à $\mathbb{T}^{n-1} \times \mathbb{R}$, pour $q \in \mathbb{N}^*$ et $p \in \mathbb{Z}$ les revêtements d'ordre q s'obtiennent par $(R_\alpha, \tilde{f}_q)$ où

$$(R_\alpha, \tilde{f}_q)(\theta, y) = \left(\theta + \alpha, \frac{1}{q} f(\theta)(qy) + \frac{p}{q}\right)\Big).$$

5.19. On reprend les notations de 5.18. On pose

$$0_{\alpha,\beta}^\infty = \{F \circ (R_\alpha \times R_\beta) \circ F^{-1} \mid F \in G^\infty(\mathbb{T}^n)\} \quad \text{et} \quad 0_{\alpha,\beta,L}^\infty = 0_{\alpha,\beta}^\infty \cap G_L^\infty(\mathbb{T}^n).$$

Par 5.9.5), on a $0_{\alpha,\beta}^\infty \subset F_{\alpha,\beta}^\infty$ (de plus $0_{\alpha,\beta}^\infty$ est connexe).

Si (α, β) satisfait à une condition diophantienne il suit de 5.11, par conjugaison C^∞, que l'ensemble $0_{\alpha,\beta}^\infty$ est ouvert dans $F_{\alpha,\beta}^\infty$ pour la C^∞-topologie.

PROPOSITION. *Si* (α, β) *satisfait à une condition diophantienne alors l'ensemble* $0^{\infty}_{\alpha,\beta}$ *n'est pas fermé dans* $F^{\infty}_{\alpha,\beta}$ *pour la* C^{∞}*-topologie.*

Démonstration. Par l'absurde. Si $0^{\infty}_{\alpha,\beta}$ était fermé dans $F^{\infty}_{\alpha,\beta}$ alors il serait ouvert et fermé et donc par 5.18, $0^{\infty}_{\alpha,\beta}=F^{\infty}_{\alpha,\beta}$. Ceci contredit 4.6 et 4.13. ∎

Par la remarque de 5.14, si $R_\alpha \times R_\beta$ est une translation minimale de $\mathbb{T}^n$, alors on a

$$0^{\infty}_{\alpha,\beta,L}=\{F\circ(R_\alpha\times R_\beta)\circ F^{-1}\mid F\in G^{\infty}_{L}(\mathbb{T}^n)\}.$$

Par la même démonstration on a la proposition.

PROPOSITION. *Si* (α, β) *satisfait à une condition diophantienne alors l'ensemble* $0^{\infty}_{\alpha,\beta,L}$ *n'est pas fermé dans* $F^{\infty}_{\alpha,\beta}\cap G^{\infty}_{L}(\mathbb{T}^n)$ *pour la* C^{∞}*-topologie.*

Remarque. Si α ne satisfait pas à une condition diophantienne alors l'ensemble $0^{\infty}_{\alpha,\beta,L}$ n'est pas fermé ni ouvert dans $F^{\infty}_{\alpha,\beta}\cap G^{\infty}_{L}(\mathbb{T}^n)$ (cf. [4, XIII 5]).

6. Complément: dépendance plurisousharmonique de paramètres complexes

6.1. Soient X est un espace compact métrique, μ une mesure de probabilité sur X, et $g: X\to X$ une application borélienne préservant la mesure μ.

On suppose que $r>0$, $p\in\mathbb{N}^*$ et que l'application $A: D^p_r\times X\to\mathcal{B}$ est borélienne, où $\mathcal{B}$ est une algèbre de Banach sur $\mathbb{C}$ avec la norme $\|\ \|$. On suppose que la fonction $(\eta, x)\to\|A(\eta, x)\|$ est bornée sur $D^p_r\times X$ et que, pour tout $x\in X$, l'application $\eta\to A(\eta, x)$ est holomorphe sur l'intérieur de D^p_r (notée Int (D^p_r)). Pour η fixé on note par $A_\eta: X\to\mathcal{B}$ l'application $x\to A(\eta, x)$.

La mesure μ sur X étant donnée et η fixé pour l'application fibré (g, A_η) de $X\times\mathcal{B}$ comme en 1.2 et 1.3 on définit $\lambda_+(g, A_\eta)\in\mathbb{R}\cup\{-\infty\}$.

PROPOSITION. *Sour les hypothèses ci-dessus la fonction* $\eta\in \mathrm{Int}\, D^p_r\to\lambda_+(g, A_\eta)\in\mathbb{R}\cup\{-\infty\}$ *est plurisousharmonique.*

Démonstration. On pose $b_k(\eta, x)=1/2^k \operatorname{Log}\|(A_\eta)^{2^k}_g(x)\|$ pour $k\in\mathbb{N}$. Pour $x\in X$ fixé, $\eta\to b_k(\eta, x)$ est une fonction plurisousharmonique. La fonction $\eta\to a_k(\eta)=\int_X b_k(\eta, x)\,d\mu(x)$ est aussi plurisousharmonique: soient $N\in\mathbb{N}$ et $x\in X$ fixé la fonction $\eta\to b_{k,N}(\eta, x)=\sup(b_k(\eta, x), -N)$ est une fonction plurisousharmonique bornée en module, et donc, par [12, 2.2.1], $\eta\to a_{k,N}(\eta)=\int_X b_{k,N}(\eta, x)\,d\mu$ est une fonction plurisousharmonique. La suite de fonctions $(b_{k,N})_{N\geq 0}$ est décroissante et donc $\eta \to a_k(\eta) = \mathrm{Inf}_{N\geq 0}\,(a_{k,N}(\eta))$ est une fonction plurisousharmonique (voir [7, 1.6.2]). Il en est finalement de même par 1.3 de la fonction $\eta \to \mathrm{Inf}_k\, a_k(\eta) = \lambda_+(g, A_\eta)$. ■

Remarque. Si $p = 1$, par la décomposition de Riesz la fonction $\eta \to \lambda_+(g, A_\eta)$ est sur $\mathrm{Int}\, D_r^1$ la somme d'une fonction harmonique et de la fonction

$$\int \mathrm{Log}\, |\eta - x|\, dv(x)$$

où v est une mesure de Radon positive ou nulle. Cette décomposition de Riesz peut, en un certain sens, être considérée comme une version "abstraite" de la formule de Thouless.

6.2. *Exemple d'application*

On se donne X, g, μ comme en 6.1 et une application borélienne $B: X \to SL(2, \mathbb{R})$ telle que les fonctions $x \in X \to \|B(x)\|$ et $x \to \|B^{-1}(x)\|$ soient bornées où $\|\ \|$ est une norme de $\mathscr{L}(\mathbb{R}^2, \mathbb{R}^2)$.

Comme la décomposition d'Iwasawa est un difféomorphisme $\mathbb{R}$-analytique de $SL(2, \mathbb{R})$ sur NAK on peut écrire de façon unique

$$B(x) = \begin{pmatrix} \lambda(x) & b(x) \\ 0 & 1/\lambda(x) \end{pmatrix} \begin{pmatrix} \cos(2\pi\varphi(x)) & -\sin(2\pi\varphi(x)) \\ \sin(2\pi\varphi(x)) & \cos(2\pi\varphi(x)) \end{pmatrix}$$

où $\varphi: X \to \mathbb{T}^1$ est une application borélienne, les fonctions $x \to \lambda(x)$, $x \to 1/\lambda(x)$, $x \to b(x)$ sont boréliennes bornées et pour tout $x \in X$, $\lambda(x) > 0$. De plus, si X est une variété $\mathbb{R}$-analytique, les fonctions φ, λ, b sont aussi dérivables que l'est l'application $x \to B(x)$. On écrit

$$T(x) = \begin{pmatrix} \lambda(x) & b(x) \\ 0 & 1/\lambda(x) \end{pmatrix}$$

et si $\alpha \in \mathbb{T}^1$,

$$B_\alpha(x) = B(x) \begin{pmatrix} \cos 2\pi\alpha & -\sin 2\pi\alpha \\ \sin 2\pi\alpha & \cos 2\pi\alpha \end{pmatrix}.$$

Si $T(x) \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ sur un ensemble de μ-mesure positive, on a

$$\delta(B) = \int_X \tfrac{1}{2}\,\mathrm{Log}\,\left(\tfrac{1}{4}[(\lambda(x) + 1/\lambda(x))^2 + b^2(x)]\right) d\mu(x) > 0.$$

On considère l'application fibré (g, B_α) de $X \times \mathscr{L}(\mathbb{R}^2, \mathbb{R}^2)$, on met sur X la mesure μ et on a:

PROPOSITION. *Il existe un ensemble de $\alpha \in \mathbb{T}^1$ de mesure de Haar positive tel que l'on ait*:

$$\lambda_+(g, B_\alpha) \geq \delta(B).$$

Démonstration. On pose

$$A_\eta(x) = (T(x)C + \eta^2 e^{4\pi i\varphi(x)} T(x)\bar{C}) \in \mathscr{L}_{\mathbb{C}}(\mathbb{C}^2, \mathbb{C}^2)$$

où $C = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, $\bar{C}$ est la matrice complexe conjuguée et $\eta \in \mathbb{C}$. On a si $\eta_\alpha = e^{2\pi i\alpha}$ (i.e. si $|\eta_\alpha| = 1$)

$$B_\alpha(x) = A_{\eta_\alpha}(x)/(\eta_\alpha e^{2\pi i\varphi(x)}) \quad \lambda_+(g, B_\alpha) = \lambda_+(g, A_{\eta_\alpha}).$$

Par 6.1 la fonction $\eta \in \mathbb{C} \to \lambda_+(g, A_\eta) \in \mathbb{R}$ est sousharmonique et donc pour η appartenant à un ensemble de mesure de Haar positive de $\mathbb{T}^1_1$ (i.e. $|\eta| = 1$) on a

$$\lambda_+(g, A_\eta) \geq \lambda_+(g, A_0) \equiv \lambda_+(g, TC).$$

On veut montrer $\lambda_+(g, TC) \geq \delta(B)$, pour cela on pose $L = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$ et on a

$$L^{-1}T(x)CL = \begin{pmatrix} 0 & c(x) \\ 0 & d(x) \end{pmatrix}$$

où $d(x) = \frac{1}{2}(\lambda(x) + 1/\lambda(x) + ib(x)) = \mathrm{Tr}(T(x)C)$. Puisque L est une matrice constante on a

$$\lambda_+(g, L^{-1}TCL) = \lambda_+(g, TC)$$

et on vérifie que

$$\lambda_+(g, L^{-1}TCL) \geq \int_X \mathrm{Log}\,|d(x)|\, d\mu(x) = \delta(B). \quad \blacksquare$$

6.3. *Un autre exemple*

Soient X, μ, g et $A : X \to \mathscr{L}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n)$ vérifiant les conditions de 6.1.

Soit a un élément de l'algèbre de Lie de $SL(n, \mathbb{C})$. La proposition suivante résulte facilement de 6.1.

PROPOSITION. *Pour tout* $r > 0$ *il existe* $\eta_r \in \mathbb{C}$ *tel que l'on ait* $|\eta_r| = r$ *et* $\lambda_+(g, A \cdot \exp(\eta_r a)) \geq \lambda_+(g, A)$.

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Wall's obstructions and Whitehead torsion

SŁAWOMIR KWASIK

In this note we show that the Wall-type obstruction defined by S. Ferry in [4] is in fact the original Wall's one. As a consequence we obtain the geometric proof of the Product Formula (see [5]) for the Wall finiteness obstructions.

1. Introduction

Let X be a topological space which is homotopy dominated by a finite CW complex. In [9] C. T. C. Wall introduced the obstruction $w(X)$ which is an element of $\tilde{K}_0(Z(\pi_1(X)))$ to decide when X has the homotopy type of some finite CW complex. Alternatively in [4] S. Ferry has found, in a geometric manner, an analogous obstruction $\sigma(X)$ in $Wh(X\times S^1)$. The natural question about the relation between these two obstructions was not considered in [4] (note that this question was explicitly asked by H. J. Munkholm in [10]). The purpose of this note is to fill this gap. We prove a rather expected result that these two obstructions are the same. To be more precise; we prove that $w(X)$ is the image of $\sigma(X)$ under the Bass–Heller–Swan isomorphism, thus answering the question from [10].

As a consequence we obtain the geometric proof of the Product Formula for the Wall finiteness obstructions. Originally the Product Formula was proved by S. Gersten in [5] in a purely algebraic manner. This note does not pretend to the orginality, but we hope that it will a little bit clarify the geometry of the Wall finiteness obstruction.

We will assume some familiarity with the simple homotopy theory. An excellent reference is [3].

2. Wall's obstruction and simple types

In our note we will consider the Whitehead group of an arbitrary topological space following [8].

Let us recall the construction of the obstruction to the finiteness given by S. Ferry in [4].

Let X be a topological space which is homotopy dominated by a finite CW complex K, i.e. there exist maps $g: X \to K$, $f: K \to X$ such that $fg \simeq id_X$. By the theorem of M. Mather (see [6]) $X \times S^1$ has a homotopy type of a finite CW complex. To see it we repeat his beautiful geometric argument. Namely, consider the mapping torus $T(\alpha)$ of the map $\alpha = gf: K \to K$; recall that $T(\alpha)$ is the space obtained from the mapping cylinder $M(\alpha)$ by identifying the top and bottom of $M(\alpha)$ using the identity map. Of course we can assume that up to homotopy type $T(\alpha)$ is a finite CW complex. Now the following picture shows that $X \times S^1 \simeq T(\alpha)$.

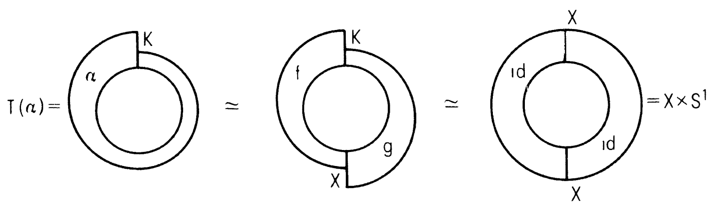

Figure 1

We will denote this homotopy equivalence by $\Phi: T(\alpha) \to X \times S^1$ and its inverse by $\Phi^{-1}: X \times S^1 \to T(\alpha)$.

DEFINITION 2.1 (S. Ferry [4]). Let $\mathrm{T}: \mathrm{X} \times \mathrm{S}^1 \to \mathrm{X} \times \mathrm{S}^1$ be a homeomorphism given by $T(x, \theta) = (x, \bar{\theta})$. We define $\sigma(X) = \Phi_*(\tau(\Phi^{-1} T \Phi)) \in Wh(X \times S^1)$, where $\tau(\Phi^{-1} T \Phi)$ is a torsion of the homotopy equivalence $\Phi^{-1} T \Phi: T(\alpha) \to T(\alpha)$.

It turns out (see [4]) that $\sigma(X)$ is well-defined (does not depend from f, g and K) and $\sigma(X) = 0$ if and only if X is a homotopy equivalent to some finite CW complex.

The crucial role in our considerations plays the following Bass-Heller-Swan decomposition of the Wh functor (see [1], [2]).

Let X be a topological space. Then there exists a functorial direct sum decomposition

$$Wh(X \times S^1) = Wh(X) \oplus \mathrm{Nil}\,(X) \oplus \mathrm{Nil}\,(X) \oplus \tilde{K}_0(X)$$

where by $\mathrm{Nil}\,(X)$, $\tilde{K}_0(X)$ we mean $\mathrm{Nil}\,(Z(\pi_1(X)))$, $\tilde{K}_0(Z(\pi_1(X)))$ respectively. Using this we prove:

THEOREM 2.2. *Let X be a topological space which is homotopy dominated by a finite CW complex. Then the Wall finiteness obstruction $w(X)$ is a image of $\sigma(X)$ under the Bass-Heller-Swan decomposition of Wh $(X \times S^1)$.*

Proof. Let K be a finite CW complex and let $g : X \to K$, $f : K \to X$ be maps such that $fg \simeq id_X$. As previous by $T(\alpha)$ we denote the mapping torus of the map $\alpha = gf : K \to K$.

Let $\Phi : T(\alpha) \to X \times S^1$ be a homotopy equivalence. The natural infinite cyclic covering of $X \times S^1$ induces an infinite cyclic covering $I(\alpha)$ of $T(\alpha)$.

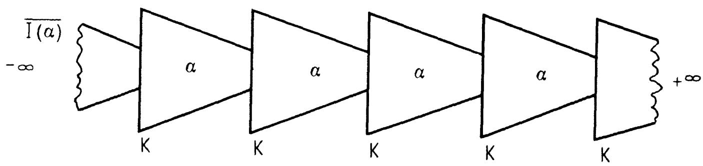

Figure 2

The space $I(\alpha)$ is an infinite CW complex with two ends ϵ_+, ϵ_- which correspond to the two ends of the real line.

Observe that the homotopy equivalence $h = \Phi^{-1}T\Phi : T(\alpha) \to T(\alpha)$ induces a proper homotopy equivalence $\tilde{h}$ between $I(\alpha)$ and its reversed copy $\overline{I(\alpha)}$ (reversed with respect to the ends).

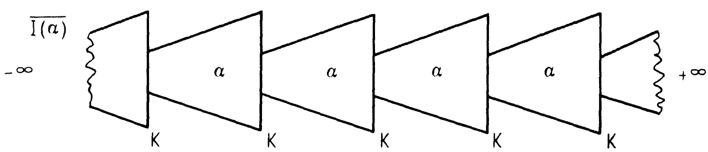

Figure 3

Without loss of generality we can assume that $\tilde{h}$ is a strong proper deformation retraction of $I(\alpha)$.

Now we proceed as in [7]. In $I(\alpha)$ consider a subcomplex L such that L is a neighborhood of ϵ_+ and $(I(\alpha) - L) \cup \overline{I(\alpha)}$ is a neighborhood of ϵ_-. Put $L_1 = \overline{I(\alpha)} \cap L$ and consider the pair (L, L_1). It can be easly proved (see Lemma 4.5 in [7]) that the pair (L, L_1) is homotopy dominated by a pair $(L_0 \cup L_1, L_1)$, where L_0 is a finite subcomplex of L. Then the cellular chain complex $C_*(\tilde{L}, \tilde{L}_1)$ of the universal covering $p : \tilde{L} \to L$ of the pair (L, L_1), which is a free $Z(\pi_1(I(\alpha))$-complex is chain homotopy dominated by the free $Z(\pi_1(I(\alpha))$-complex $C_*(\tilde{L}_0 \cup \tilde{L}_1, \tilde{L}_1)$; we used the notation: for every $B \subset L$, $\tilde{B} = p^{-1}(B)$. Hence we can define

$w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = w(C_*(\tilde{L}, \tilde{L}_1) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$, where $w(C_*(\tilde{L}, \tilde{L}_1))$ is the Wall obstruction. It is not difficult to see that $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+)$ is well-defined i.e. does not depend of the choice of L_1.

Now let L_-, L_+ be a neighborhoods of ϵ_-, ϵ_+ so that $I(\alpha)-L_+$, $I(\alpha)-L_-$ are again neighborhoods of ϵ_-, ϵ_+ respectively and $L_- \cup L_+ = I(\alpha)$. Then $L_- \cap L_+$ is a finite CW complex and since $I(\alpha)$ is homotopy dominated by a finite CW comples (in fact by K) then from the Mayer-Vietoris sequence

$$0 \to C_*(\tilde{L}_- \cap \tilde{L}_+) \to C_*(\tilde{L}_-) \oplus C_*(\tilde{L}_+) \to C_*(I(\alpha)) \to 0$$

we infer that $C_*(\tilde{L}_+)$ is chain homotopy dominated by a finitely generated free complex. This gives us the well defined element $w(I(\alpha), \epsilon_+) = w(C_*(\tilde{L}_+)) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. Analogously we can define $w(\overline{I(\alpha)}, \epsilon_+) \in \tilde{K}_0(Z(\pi_1(I(\alpha))))$. An elementary property of the Wall obstructions yields:

$$w(I(\alpha), \epsilon_+) = w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) + w(\overline{I(\alpha)}, \epsilon_+)$$

Observe (see Fig. 4) that in our situation $w(\overline{I(\alpha)}, \epsilon_+) = 0$ and $\tilde{h}_*(w(I(\alpha)) = w(I(\alpha), \epsilon_+)$ by a homotopy type invariance of the Wall obstruction.

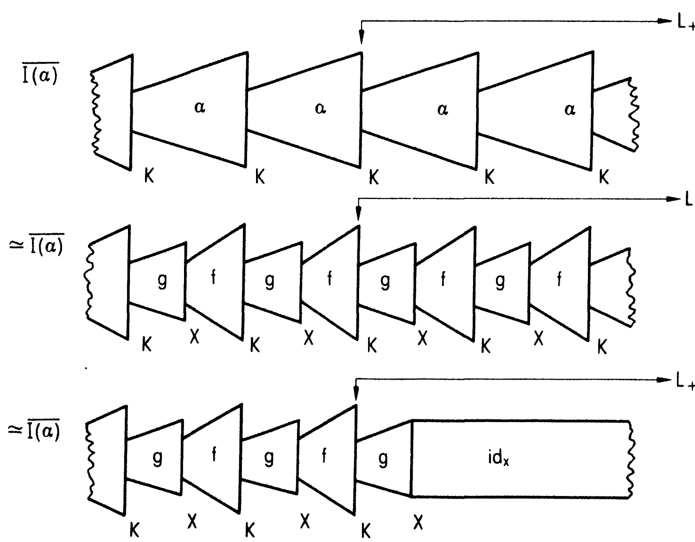

Figure 4

Hence $w(I(\alpha), \overline{I(\alpha)}, \epsilon_+) = \tilde{h}_*(w(I(\alpha)))$. The Bass–Heller–Swan projection (B–H–S) : $Wh(X \times S^1) \to \tilde{K}_0(X)$ induces a natural projection $p : Wh(T(\alpha)) \to \tilde{K}_0(I(\alpha))$.

This gives the following commutative diagram:

$$\begin{array}{ccc} Wh(T(\alpha)) & \xrightarrow{\Phi_*} & Wh(X\times S^1) \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \text{(B-H-S)}} \\ \tilde{K}_0(I(\alpha)) & \xrightarrow{\tilde{\Phi}_*} & \tilde{K}_0(X) \end{array}$$

where the map $\tilde{\Phi}: I(\alpha)\to X\simeq X\times R$ is induced by Φ. So we have:

$$\tilde{\Phi}_* p(\tau(\Phi^{-1}T\Phi) = \text{(B-H-S)}\Phi_*(\tau(\Phi^{-1}T\Phi) = \text{(B-H-S)}(\sigma(X)).$$

But $\tilde{\Phi}_* p(\tau(\Phi^{-1}T\Phi) = \tilde{\Phi}_*(w(I(\alpha), \overline{I(\alpha)}, \epsilon_+))$ by the Proposition 4.7 in [7], hence:

$$\text{(B-H-S)}(\sigma(X)) = \tilde{\Phi}\tilde{h}_*(w(I(\alpha))) = w(X)$$

by the homotopy type invariance of the Wall obstruction.

COROLLARY 2.3 (*Product Formula*). *Let X be a topological space which is homotopy dominated by a finite CW complex, and let L be a finite CW complex. Then*:

$$w(L\times X) = \chi(L)\cdot i_*(w(X))$$

where $i: X\to L\times X$ is given by $i(x)=(1_0, x)$ for some $1_0 \varepsilon L$, and $\chi(L)$ denotes the Euler characteristic of L.

Proof. Let K be a finite CW complex and $g: X\to K$, $f: K\to X$ be maps such that $fg\simeq id_X$. Let $T(\alpha)$ be the mapping torus of the map $\alpha = gf: K\to K$ and let $\Phi: T(\alpha)\to X\times S^1$ be a homotopy equivalence. The space $L\times X$ is a homotopy dominated by the finite CW complex $L\times K$ using the maps $id\times g: L\times X\to L\times K$, $id\times f: L\times K\to L\times X$. Hence we have the homotopy equivalence $\bar{\Phi}: T(id\times\alpha)\to L\times X\times S^1$. But $T(id\times\alpha) = L\times T(\alpha)$ and without loss of the generality we can write $\bar{\Phi} = id\times\Phi: L\times T(\alpha)\to L\times X\times S^1$. Now our finiteness obstruction is given by:

$$\sigma(L\times X) = (id\times\Phi)_*(\tau(id\times\Phi^{-1}T\Phi)) \in Wh(L\times X\times S^1).$$

By the product theorem for Whitehead torsion (see [3] for the nice and short geometric proof) we have:

$$\tau(id \times \Phi^{-1}T\Phi) = \chi(L) \cdot j_*(\tau(\Phi^{-1}T\Phi)$$

where $j: T(\alpha) \to L \times T(\alpha)$ is given by $j(t) = (1_0, t)$, for $t \in T(\alpha)$. Hence $\sigma(L \times X) = \chi(L) \cdot i_*(\sigma(X))$, where $i_*: Wh(X \times S^1) \to Wh(L \times X \times S^1)$. Now the formula $w(L \times X) = \chi(L) \cdot i_*(w(X))$ follows from the naturality of the Bass–Heller–Swan decomposition of $Wh(X \times S^1)$.

This work was done while the author was visiting the University of Heidelberg. I am grateful to Professor Dieter Puppe for the opportunity to work there.

Note added in proof:

In fact $\sigma(X) \in \tilde{K}_0(X)$. This can be deduced from T. Chapman, Approximation results in Hilbert cube manifolds, Trans. Amer. Math. Soc. 262 (1980), 303–334, in particular, see p. 321 of this paper.

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Quasilinear elliptic eigenvalue problems

MICHAEL STRUWE[(1)]

Summary. The generalized Palais–Smale condition introduced in [26] is applied to obtain multiple solutions of variational eigen-value problems with quasilinear principal part, thereby extending some well-known existence results for semilinear elliptic problems.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $a=(a^{\alpha\beta})_{1\leq\alpha,\beta\leq n}$ be a uniformly elliptic, symmetric, and bounded matrix function $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^{n\times n}$. For $u\in H_0^{1,2}(\Omega;\mathbb{R}^N)$ define the energy integral

$$E(u)=\frac{1}{2}\int a^{\alpha\beta}(x,u)\,\partial_\alpha u^i\,\partial_\beta u^i\,dx. \tag{1.1}$$

By convention, repeated Greek indices are summed from 1 to n, Latin indices from 1 to N. Unless otherwise stated all integrals are taken over Ω. Let $G:H_0^{1,2}(\Omega;\mathbb{R}^N)\to\mathbb{R}$ be a sufficiently regular function (cp. conditions (3.4)–(3.7)).

In this note we present existence results for the nonlinear eigen-value problem in $H_0^{1,2}(\Omega;\mathbb{R}^N)$:

$$\nabla E(u)=\mu\nabla G(u),\qquad G(u)=1,\qquad \mu\in\mathbb{R}. \tag{1.2}$$

The results obtained generalize results of Amann [1], [2], Ambrosetti [3], Berger [6], Browder [7], Clark [8], Coffman [9], [10], Hempel [12], [13], Pohožaev [20], Rabinowitz [21], [22] and others for semilinear eigenvalue problems. In the above setting such problems correspond to functionals E with coefficients $a=a(x)$ independent of u.

In contrast to this latter situation under the hypotheses made here the functional E need not be differentiable in $H_0^{1,2}$. Thus the classical Palais–Smale

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condition (cp. [19], [24]) cannot hold in this space and the standard Lusternik–Schnirelman theory of critical points (cp. [16], [23]) cannot be applied to obtain solutions of (1.2). Instead the existence results presented will be deduced from the generalization of the Lusternik–Schnirelman method that was introduced in [26].

In the next section this method will be briefly recalled; in particular, the compactness Criterion A* will be restated which replaces the Palais–Smale condition in applications of Lusternik–Schnirelman theory to irregular functionals. Precise formulations and proofs of existence results for problem (1.2) are given in Section 3.

These results are not presented in the most conceivable generality. Instead they have been selected as models to illustrate the application of Criterion A* in a "non-reflexive" situation. Possible extensions and generalizations are mentioned without proof in Section 4.

Remark that we retain a compactness assumption on the term $\nabla G(u)$ in (1.2), cp. condition (3.7). However, even with coefficients a_{ij} independent of u, the Palais–Smale condition may no longer hold true for (1.2) if ∇G is only continuous, i.e. if in assumption (3.7) $p=2n/(n-2)$ is admitted. In this case (and with coefficients independent of u) important progress has recently been made by Brezis and Nirenberg [28], [29], cp. also [30].

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2. The compactness criterion

Throughout this section we shall make the following *Assumption A*:

B is a reflexive Banach space with norm $\|\cdot\|_B$. $T\subset B$ is a dense subspace of B (in $\|\cdot\|_B$) given by $T=\bigcup_{\iota\in I} T_\iota$, where $\{T_\iota\}$ is family of Banach spaces T_ι with norms $\|\cdot\|_{T_\iota}$. (2.1)

$E: B\to \mathbb{R}\{\pm\infty\}$ for any $\iota\in I$ is continuously Fréchet differentiable with respect to T_ι in the following sense: For any $u\in B$ such that $|E(u)|<\infty$, any $\iota\in I$, the derivative $\nabla E(u)\in T_\iota^*$ exists and the mapping $u\to \nabla E(u)\in T_\iota^*$ is continuous on its domain. (2.2)

Set $T'=\{\xi: T\to\mathbb{R} \mid \xi_{|T_\iota}\in T_\iota^* \text{ for all } \iota\in I\}$. By (2.2) it is meaningful to define a critical point of E as an element $u\in B$ such that $|E(u)|<\infty$ and $\nabla E(u)=0\in T'$. The value $E(u)$ at such a critical point will be called a critical value of E.

DEFINITION. A sequence $\{\xi_m\}$ in T' converges to $0 \in T'$ T_ι-uniformly iff for any $\iota \in I$

$$\|\xi_m\|^*_{T_\iota} \to 0 \quad (m \to \infty).$$

As an illustration of this definition consider the extreme cases:

Ex. 1. Let $B = T = T_\iota$. Then for any $\xi \in T' = B^*$, any $\iota : \|\xi\|^*_{T_\iota} = \|\xi\|_{B^*}$, $\|\cdot\|_{B^*}$ denoting the norm in the dual B^*. Thus, T_ι-uniform convergence is equivalent to norm-convergence in B^*.

Ex. 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $B = H_0^{1,2}(\Omega)$, $T = C_0^\infty(\Omega)$, and $T_\iota = \{\lambda_\iota \mid \lambda \in \mathbb{R}\} \cong \mathbb{R}$, $\iota \in T$. In this case, T_ι-uniform convergence is equivalent to weak convergence in the sense of distributions.

In the application given below, a nontrivial choice of B, T, T_ι will be presented.

Let M be a subset of B such that E is finite on M. Assuming (2.1), (2.2) we may then state our

CRITERION A*. *If $\{u_m\}$ is any sequence in M such that $u_m \rightharpoonup u$ weakly in B and $\nabla E(u_m) \to 0$ T_ι-uniformly as $m \to \infty$, then we may extract a subsequence that converges strongly to a critical point of E in M.*

Remark. In Ex. 1 Criterion A* reduces to a variant of the classical Palais–Smale condition. Similarly, imposing a coerciveness condition on E (with respect to B) and assuming M to be regular (with respect to T, $\{T_\iota\}$), from Criterion A* the existence of saddle-type critical points characterized by "mountain-pass" or "minimax" conditions may be derived for functionals which may be irregular in B (cp. [26]).

Note that since we are exhausting the testing space T by a family of Banach spaces $\{T_\iota\}$, the "limit space" T itself may be very badly behaved; in particular, it need not be a reflexive Banach space.

In the applications given in Section 3 (and in those presented in [26]) Criterion A* can be verified by introducing a "necessary constraint", i.e. by restricting E to a set M of admissible functions given by

$$M = \{u \in B \mid \langle \nabla E(u), \Psi(u) \rangle = 0\} \tag{2.3}$$

for some mapping Ψ. Necessary conditions of this kind seem to have been first introduced by Nehari [18] in 1957 as a means to improve properties of badly behaved functions. Working with the constraint $\langle \nabla E(u), u\rangle = 0$ he and other authors were able to derive existence and multiplicity results for superlinear elliptic boundary value problems (cp. [4], [5], [7], [9], [12], [18], [25]). For such problems the functional in variation is neither bounded from above nor from below on the whole space, whereas it was found to be coercive on the set of functions satisfying the above constraint.

3. Quasilinear eigenvalue problems

In this section we apply Criterion A* to prove the existence of multiple solutions for problem (1.2) mentioned in the introduction. The following assumptions (3.1)–(3.7) will be made throughout this section.

The symmetric matrix function $a = (a^{\alpha\beta}) : \Omega \times \mathbb{R}^N \to \mathbb{R}^{n\times n}$, $a^{\alpha\beta} = a^{\beta\alpha}$, is measurable in $x \in \Omega$, differentiable in $u \in \mathbb{R}^N$, and for a.e. x, $\partial_u a^{\alpha\beta}(x, \cdot)$ is uniformly continuous in u, uniformly in x. Moreover, there is a constant c such that $|a|$, $|\partial_u a|$, $|u \cdot \partial_u a(\cdot, u)| \le c$ a.e. in $\Omega \times \mathbb{R}^N$. (3.1)

There exists $\lambda > 0$ such that

$$a^{\alpha\beta}(x, u)\xi^i_\alpha \xi^i_\beta \ge \lambda\, |\xi|^2 \tag{3.2}$$

for a.e. $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$.

Also we shall need the one-sided condition:

There exists a constant $\lambda^* < \lambda$ such that

$$-u \cdot \partial_u a^{\alpha\beta}(\cdot, u)\xi^i_\alpha \xi^i_\beta \le 2\lambda^*\, |\xi|^2 \tag{3.3}$$

for a.e. $(x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$.

With respect to the function G we suppose:

$G : H_0^{1,2}(\Omega; \mathbb{R}^N) \to \mathbb{R}$[1] is continuous with respect to weak convergence in $H_0^{1,2}$. $G(u) \ge 0$, $G(0) = 0$. (3.4)

[1] $H_0^{1,2}(\Omega; \mathbb{R}^N)$ is the completion of the set of C^∞-functions $u : \Omega \to \mathbb{R}^N$ having compact support in Ω with respect to the norm $\|u\|^2_{1,2} = \int_\Omega |\nabla u|^2\, dx$. For brevity in the following we simply write $H_0^{1,2}$.

G is continuously Fréchet differentiable in $H_0^{1,2}$ with compact derivative $\nabla G: H_0^{1,2} \to H^{-1} := (H_0^{1,2})^*$. (3.5)

Moreover, we require the non-degeneracy condition:

At any point $u \in H_0^{1,2}$ such that $G(u) > 0$ we have $\langle \nabla G(u), u \rangle > 0$. (3.6)

Finally, we also need a regularity condition:

For any $u \in H_0^{1,2}$ $\nabla G(u) \in H^{-1}$ is represented by a function $g(x, u)$ such that $\langle \nabla G(u), \varphi \rangle = \int g(u)\varphi\, dx$ for all $\varphi \in H_0^{1,2}$, and $g: \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the estimate for some $p < 2^* = \dfrac{2n}{n-2}$[2]: (3.7)

$$|g(x, u)| \leq c(1 + |u|^{p-1}) \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N.$$

EXAMPLE 3.1. The assumptions (3.4)–(3.7) are satisfied with $G(u) = \int |u|^p\, dx$, for some $p \in]1, 2^*[$. The assumptions (3.1)–(3.3) are satisfied with $a^{\alpha\beta} = (\lambda + \operatorname{arctg}(|u|^2))\delta_{\alpha\beta}$, where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta, = 0$ else.

We will show that the general assumptions (2.1), (2.2) are satisfied for the functional E given by (1.1), if we let $B = H_0^{1,2}(\Omega; \mathbb{R}^N)$, $T = H_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N)$, and for any $L \in \mathbb{N}$ let T_L be the set T equipped with the norm $\|\cdot\|_L = \|\cdot\|_{1,2} + L^{-1}\|\cdot\|_\infty$.

LEMMA 3.1. *Assumption A is satisfied with B, T_L, T, and E as above. Moreover, for any $L \in \mathbb{N}$ the function $\nabla E: H_0^{1,2} \to T_L^*$ is uniformly bounded on $H_0^{1,2}$-bounded sets U, and for any such U the mappings $\nabla E(u + \cdot): T_L \to T_L^*$ are continuous at $0 \in T_L$, uniformly with respect to $u \in U$.*

Proof. (2.1) is trivially verified. Similarly, it is easy to check that E is finite on $H_0^{1,2}$ and differentiable with respect to T_L, for any $L \in \mathbb{N}$. Indeed, for any $u \in H_0^{1,2}$, any $\varphi \in H_0^{1,2} \cap L^\infty$

$$\langle \nabla E(u), \varphi \rangle = \int a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta \varphi^i + \tfrac{1}{2}\varphi^j\, \partial_{u^j} a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta u^i\, dx, \tag{3.8}$$

and the stated continuity properties may readily be derived from condition (3.1) and standard convergence theorems. By (3.1) and (3.8) also the uniform boundedness on $H_0^{1,2}$-bounded sets of $\nabla E: H_0^{1,2} \to T_L^*$ is immediate. q.e.d.

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[2] $2^* = \infty$, if $n = 2$.

Thus it is meaningful to define a critical point u of E subject to the constraint $G(u)=1$.

DEFINITION 3.1. $u \in H_0^{1,2}$ is called a critical point of E on the set $\{u \in H_0^{1,2} \mid G(u)=1\}$ iff there exists a number $\mu \in \mathbb{R}$ such that

$$\nabla E(u) + \mu \nabla G(u) = 0 \in T'.$$

If u is a critical point of E the value $E(u)$ is called critical.

Of course, in the semilinear case we may take $T=B=H_0^{1,2}$ and the above definition reduces to the standard definition of a constrained critical point. Set

$$M=\{u \in H_0^{1,2} \mid G(u)=1\}. \tag{3.9}$$

Now we can formulate our first result:

THEOREM 3.1. *Suppose conditions* (3.1)–(3.7) *are satisfied and that* $M \neq \varnothing$. *Then there exists a constant* $\rho_1 > 0$ *such that whenever the condition*

$$|\partial_u a| < \rho_1 \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N$$

is satisfied there exists a solution $(u, \mu) \in H_0^{1,2} \cap L^\infty(\Omega; \mathbb{R}^N) \times \mathbb{R}$ *of problem* (1.2), *characterized by the condition*

$$u \in M : E(u) = \inf_{v \in M} E(v).$$

In the symmetric case a much stronger existence result may be obtained. Assume that

$$a^{\alpha\beta}(x, u) = a^{\alpha\beta}(x, -u) \quad \text{a.e. in} \quad \Omega \times \mathbb{R}^N \tag{3.10}$$

and

$$G(u) = G(-u) \quad \text{for all} \quad u \in H_0^{1,2}. \tag{3.11}$$

Let

$$\Sigma = \{A \subset H_0^{1,2} \backslash \{0\} \mid A \text{ is closed and symmetric}\}$$

and define the Krasnoselskii "genus" $\gamma:\Sigma\to\mathbb{N}_0\cup\{\infty\}$ on Σ by letting $\gamma(\emptyset)=0$, and for $A\neq\emptyset$:

$$\gamma(A)=\min\{m\in\mathbb{N}\mid \exists h:A\to\mathbb{R}^m\setminus\{0\},\ h \text{ is continuous, } h(-u)=-h(u)\}$$

if $\{\cdots\}\neq\emptyset$, $\gamma(A)=\infty$ else (cp. [9], [14]). The genus has the following properties (cp. [21, Lemma 1.1]):

PROPOSITION 3.1. *Let A, $B\in\Sigma$:*
i) *If there exists an odd continuous mapping $h:A\to B$ then $\gamma(A)\le\gamma(B)$.*
ii) $\gamma(A\cup B)\le\gamma(A)+\gamma(B)$
iii) *If A is compact, then $\gamma(A)<\infty$ and there exists a neighborhood N of A in $H_0^{1,2}$ such that $\bar{N}\in\Sigma$ and $\gamma(\bar{N})=\gamma(A)$.*

For any $l\in\mathbb{N}$ set

$$\Sigma_l=\{A\in\Sigma\mid A\subset M,\ \gamma(A)\ge l,\ A \text{ is compact}\},$$

and for any $\beta\in\mathbb{R}$ let

$$K_\beta=\{u\in M\mid E(u)=\beta,\ \exists\mu:\nabla E(u)+\mu\nabla G(u)=0\}$$

be the set of constrained critical points of E of energy β. We then obtain:

THEOREM 3.2. *Suppose conditions* (3.1)–(3.7), (3.10), (3.11) *are satisfied and that $\Sigma_l\neq\emptyset$ for $l\le m$. Then for any $k\le m$ there is a constant ρ_k such that if the condition*

$$|\partial_u a|<\rho_k \quad \text{a.e. in} \quad \Omega\times\mathbb{R}^N$$

is satisfied there exist solutions $(u_l,\mu_l)\in H_0^{1,2}\cap L^\infty(\Omega;\mathbb{R}^N)\times\mathbb{R}$, $1\le l\le k$, of problem (1.2) *characterized by the generalized minimax-principle*

$$u_l\in M: E(u_l)=\beta_l:=\inf_{A\in\Sigma_l}\sup_{u\in A}E(u).$$

If for some numbers k, $l\in\mathbb{N}$ we have

$$\beta=\beta_l=\cdots=\beta_{l+k}$$

then

$$\gamma(K_\beta)\ge k+1$$

and, in particular, K_β *must be infinite. Finally, if* $\Sigma_l \neq \emptyset$ *for all* $l \in \mathbb{N}$

$$\beta_l \to \infty \qquad (l \to \infty).$$

EXAMPLE. If the coefficients $a^{\alpha\beta}$ and the function G are given as in Example 3.1 all the assumptions (3.1)–(3.7), (3.10), (3.11) are satisfied, $M \neq \emptyset$ and $\Sigma_l \neq \emptyset$ for all $l \in \mathbb{N}$. Moreover, for a scalar equation ($N = 1$) or a plane system ($n = 2$) the parameters ρ_k may be chosen as $\rho_k = \infty$ (cp. Section 4).

Remarks. For coefficients $a^{\alpha\beta} = a^{\alpha\beta}(x)$ independent of u as a special case of Theorem 3.2 we obtain the well-known existence results cited in the Introduction.

For general elliptic eigenvalue problems of the type

$$-\Delta u + f(x, u, \nabla u) = 0 \tag{3.12}$$

only the results of Browder [7] seem to have been available. However, by using a standard Palais-Smale type condition Browder was forced to impose the "unnatural" growth condition on the free term in (3.12):

$$|f(x, u, \eta)| \leq c(1 + |u|^p + |\eta|^q)$$

with $p < 2^* - 1$ and $q \leq (n+2)/n$ if $n > 2$, resp. $p < \infty$, $q < 2$ if $n = 2$ (cp. [11]). In the case of quadratic growth ($q = 2$) which naturally arises from variational problems like the problems considered here the question of existence of non-minimum critical points seems to have been completely open.

Bounds for the numbers ρ_k in terms of the structure parameters of the system (1.2) can be obtained from the proof of Lemma 3.4 ii) (cp. (3.16)).

To prove Theorems 3.1, 3.2 we first need an additional information on the set of admissible functions.

LEMMA 3.2. i) *The mapping* $u \mapsto \langle \nabla E(u), u \rangle$ *from* $H_0^{1,2} \cap L^\infty$ *into* $\mathbb{R}$ *continuously extends to* $u \in H_0^{1,2}$.

ii) *For any* $u \in M$ *there exists a unique number* $\mu = \mu(u)$ *such that* $\langle \nabla E(u) + \mu \nabla G(u), u \rangle = 0$.

iii) *The mapping* $u \to \mu(u)$ *from* M *into* $\mathbb{R}$ *is bounded on* ($H_0^{1,2}$–) *bounded sets, hence continuous.*

Proof. i) Given any $u \in H_0^{1,2}$ and any sequence $\{\varphi_m\}$, $\varphi_m \in H_0^{1,2} \cap L^\infty$, such that $\varphi_m \to u$ in $H_0^{1,2}$ as $m \to \infty$ we show that $\lim_{m\to\infty} \langle \nabla E(\varphi_m), \varphi_m \rangle$ exists and is independent of the approximating sequence $\{\varphi_m\}$. Indeed,

$$\langle \nabla E(\varphi_m), \varphi_m\rangle = \int a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx.$$

By Lusin's theorem, given any number $\delta > 0$ there exists a set $E_\delta \subset \Omega$ such that meas $(E_\delta) < \delta$ and $\varphi_m \to u$ uniformly on $\Omega \backslash E_\delta$ as $m \to \infty$. Hence also $a(\cdot, \varphi_m) \to a(\cdot, u)$ and $\varphi_m \cdot \partial_u a(\cdot, \varphi_m) \to u \cdot \partial_u a(\cdot, u)$ uniformly on $\Omega \backslash E_\delta$ as $m \to \infty$; and since $\varphi_m \to u$ in $H^{1,2}_0 (m \to \infty)$ we obtain

$$\int_{\Omega\backslash E_\delta} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx \to$$

$$\int_{\Omega\backslash E_\delta} a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta u^i + \tfrac{1}{2}u^j\, \partial_{u^j} a^{\alpha\beta}(x, u)\, \partial_\alpha u^i\, \partial_\beta u^i\, dx \quad (m \to \infty).$$

On the remainder set E_δ by uniform boundedness of a, $u \cdot \partial_u a(\cdot, u)$ and uniform absolute continuity of $\int |\nabla \varphi_m|^2\, dx$ we may estimate

$$\left| \int_{E_\delta} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m + \tfrac{1}{2}\varphi^j_m\, \partial_{u^j} a^{\alpha\beta}(x, \varphi_m)\, \partial_\alpha \varphi^i_m\, \partial_\beta \varphi^i_m\, dx \right| \leq c(\delta),$$

where $c(\delta) \to 0$ $(\delta \to 0)$, uniformly in m. Hence the statement follows on first letting $m \to \infty$ and then passing to the limit $\delta \to 0$.

ii) By condition (3.6) statement ii) is immediate from i).

iii) To show iii) note that the mapping $u \mapsto \langle \nabla E(u), u\rangle$ is bounded on bounded sets. Moreover, by compactness of ∇G and weak continuity of G from condition (3.6) it follows that $|\langle \nabla G(u), u\rangle|$ is uniformly bounded from below by some positive constant if u ranges in any given bounded subset of M. This proves boundedness. From the boundedness continuity follows by uniqueness, part ii). q.e.d.

In conclusion the set M may equivalently be expressed

$$M = \{u \in H^{1,2}_0 \backslash \{0\} \mid G(u) = 1, \exists \mu : \langle \nabla E(u) + \mu \nabla G(u), u\rangle = 0\} \tag{3.13}$$

with an additional constraint reminiscent of (2.3). In order to verify Criterion A* for E on M the following regularity result based on a device of Moser [17] will be needed. For this lemma we also assume the following auxiliary condition which will later be removed again:

$$\exists \nu_0 : u \cdot \partial_u a(\cdot, u) \geq 0 \quad \text{for} \quad |u| \geq \nu_0, \text{ a.e. in } \Omega. \tag{3.14}$$

LEMMA 3.3. *Assume in addition to the general assumptions* (3.1)–(3.7) *that condition* (3.14) *is satisfied for some number* ν_0. *Then, if* $\{u_m\}$ *is a sequence in* M *such that* $\|u_m\|_{1,2} \leq M$ *uniformly in* m, *and such that* $u_m \rightharpoonup u$ *weakly in* $H_0^{1,2}$ *while* $\nabla E(u_m) + \mu_m \nabla G(u_m) \to 0$ T_L*-uniformly as* $m \to \infty$ *with* $\mu_m = \mu(u_m)$, *it follows that* $u \in H_0^{1,2} \cap L^\infty$ *and, moreover,* $\|u\|_\infty \leq c = c(\lambda, \lambda^*, n, N, \Omega, p, M)$ *for some function* c *which is non-decreasing in* M.

Proof. First note that by Lemma 3.2 the numbers $\mu_m = \mu(u_m)$ are uniformly bounded by a constant depending only on the parameters of the system and the number M.

For arbitrary numbers $r \geq 0$, $\nu \geq \nu_0$ define test vectors $\varphi_m^{r,\nu} = u_m |u_m|^{-1} \times \min\{|u_m|, \nu\}^{r+1}$. Note that for any r, ν, and L_0 the sequence $\varphi_m^{r,\nu}$ is uniformly bounded in T_L for any $L \geq L_0$. By T_L-uniform convergence we thus obtain, writing $\varphi_m^{r,\nu} = \varphi_m$ for brevity:

$$\begin{aligned} 0 &= \lim_{m\to\infty} \langle \nabla E(u_m) + \mu_m \nabla G(u_m), \varphi_m \rangle \\ &= \lim_{m\to\infty} \int a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta \varphi_m^i + \tfrac{1}{2} \varphi_m^j\, \partial_{u^j} a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i \\ &\qquad + \mu_m g^i(x, u_m) \varphi_m^i \, dx. \end{aligned}$$

Let

$$A_m^1 = \int a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta \varphi_m^i + \tfrac{1}{2} \varphi_m^j\, \partial_{u^j} a^{\alpha\beta}(x, u_m)\, \partial_\alpha u_m^i\, \partial_\beta u_m^i \, dx$$

$$A_m^2 = \int \mu_m g^i(x, u_m) \varphi_m^i \, dx.$$

By conditions (3.2), (3.3), (3.14) we estimate

$$\begin{aligned} A_m^1 &= \int (a^{\alpha\beta}(x, u_m) + \tfrac{1}{2} u_m \cdot \partial_u a^{\alpha\beta}(x, u_m))\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, |u_m|^{-1} \min\{|u_m|, \nu\}^{r+1} \, dx \\ &\quad + r \int_{\{x \,|\, |u_m(x)| < \nu\}} a^{\alpha\beta}(x, u_m) u_m^i\, \partial_\alpha u_m^i u_m^j\, \partial_\beta u_m^j\, |u_m|^{r-2} \, dx \\ &\quad - \nu^{r+1} \int_{\{x \,|\, |u_m(x)| \geq \nu\}} a^{\alpha\beta}(x, u_m) u_m^i\, \partial_\alpha u_m^i u_m^j\, \partial_\beta u_m^j\, |u_m|^{-3} \, dx \end{aligned}$$

$$\geq \int_{\{x \mid |u_m(x)|<\nu\}} (a^{\alpha\beta}(x, u_m)+\tfrac{1}{2}u_m \cdot \partial_u a^{\alpha\beta}(x, u_m))\, \partial_\alpha u_m^i\, \partial_\beta u_m^i\, |u_m|^r\, dx$$

$$+\nu^{r+1}\int_{\{x \mid |u_m(x)|\geq\nu\}} a^{\alpha\beta}(x, u_m)\left(\delta_{ij}-\frac{u_m^i u_m^j}{|u_m|^2}\right)\partial_\alpha u_m^i\, \partial_\beta u_m^j\, |u_m|^{-1}\, dx$$

$$\geq(\lambda-\lambda^*)\int_{\{x \mid |u_m(x)|<\nu\}} |\nabla u_m|^2\, |u_m|^r\, dx.$$

Also note that by (3.2) for a.e. $(x, u, \xi)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn}$

$$a^{\alpha\beta}(x, u)\left(\delta_{ij}-\frac{u^i u^j}{|u|^2}\right)\xi_\alpha^i\xi_\beta^j\geq 0,$$

as may be verified by transforming the positively semidefinite and symmetric matrices $a^{\alpha\beta}$, $\left(\delta_{ij}-\frac{u^i u^j}{|u|^2}\right)$, resp. into diagonal form through orthogonal transformations of the independent and dependent variables $x\in\mathbb{R}^n$, $u\in\mathbb{R}^N$, resp. Thus, letting $u_m^\nu=\min\{|u_m|, \nu\}$ and noting that by a well-known theorem of Stampacchia $\nabla u_m^\nu=0$ a.e. on $\{x \mid u_m^\nu(x)=\nu\}$ we obtain

$$A_m^1\geq 4(\lambda-\lambda^*)(r+2)^{-2}\, \|(u_m^\nu)^{r+2/2}\|_{1,2}^2.$$

On the other hand

$$A_m^2\leq c\mu_m\int (1+|u_m|^{p-1})(u_m^\nu)^{r+1}\, dx.$$

Passing to the limit $m\to\infty$, from the Sobolev embedding theorem we thus obtain with constants $c=c(\lambda, \lambda^*, n, N, \Omega, p)$

$$\|u^\nu\|_{(r+2)n/(n-2)}^{r+2}\leq c\, \|(u^\nu)^{r+2/2}\|_{1,2}^2\leq c(r+2)^2 \lim_{m\to\infty} A_m^1$$

$$\leq c(r+2)^2 \lim_{m\to\infty} |A_m^2|\leq c(r+2)^2\int (1+|u|^{p-1})\, |u^\nu|^{r+1}\, dx. \qquad (3.15)$$

Choosing $r_0=0$ and letting $\nu\to\infty$ in (3.15) we obtain

$$\|u\|_{2^*}^2\leq c\int 1+|u|^p\, dx\leq c\, \|u\|_p^p.\text{[1]}$$

---

[1] Note that by condition (3.6) there exists a constant $c>0$ such that $\|u\|_p\geq c$ for $\|u\|_{1,2}\leq M$.

Similarly, letting $r_{i+1}=\dfrac{n}{n-2}(r_i+2)-p$ for $i\in\mathbb{N}$ by induction it results from (3.15) that

$$\|u\|_{r_{i+1}+p}\le c^{1/(r_i+2)}(r_i+2)^{2/(r_i+2)}\|u\|_{r_i+p}^{(r_i+p)/(r_i+2)}$$
$$\le c^{\sum_{j\le i}\sigma_j^i}\prod_{j\le i} r_j^{2\sigma_j^i}\|u\|_p^{\prod_{j\le i}\tau_j},$$

with

$$\sigma_j^i=\frac{1}{r_j+2}\prod_{j\le k\le i}\tau_k;\qquad \tau_k=\frac{r_k+p}{r_k+2}.$$

From the asymptotic behavior $r_{i+1}/r_i\sim n/n-2$ as $i\to\infty$ it is elementary to verify that the products converge. By a standard estimate we thus obtain

$$\|u\|_\infty\le\limsup_{i\to\infty}\|u\|_{r_i}\le c\,\|u\|_p^c\le cM^c=:c(\lambda,\lambda^*,n,N,\Omega,p,M)\quad\text{q.e.d.}$$

For $\beta\in\mathbb{R}$, $L\in\mathbb{N}$ define

$$M_\beta=\{u\in M\mid E(u)<\beta\},$$
$$N_{\beta,L}=\{u\in M\mid |E(u)-\beta|<L^{-1},\ \|\nabla E(u)+\mu(u)\nabla G(u)\|_{T_L^*}^2<5L^{-1}\}.$$

LEMMA 3.4. *i) The functional $E:M\to\mathbb{R}$ is coercive with respect to $\|\cdot\|_{1,2}: E(u)\to\infty$ if $\|u\|_{1,2}\to\infty$.*

ii) Assume that condition (3.14) *holds for some $\nu_0\ge 0$. Then for any $\beta\ge 0$ there exists a number $\rho>0$ such that the functional E satisfies Criterion A^* on the set M_β provided that $|\partial_u a|<\rho$ a.e. on $\Omega\times\mathbb{R}^N$.*

iii) Under the assumptions of ii) for any $\beta\ge 0$ there exists a number ρ such that whenever $|\partial_u a|<\rho$ a.e. on $\Omega\times\mathbb{R}^N$ then the following is true:

If $K_\beta=\varnothing$ there exists $L\in\mathbb{N}$ such that $N_{\beta,L}=\varnothing$.

Proof. i) By assumption (3.2) the first statement is immediate.

ii) Let $\{u_m\}$ be a sequence in M_β such that $u_m\rightharpoonup u$ weakly in $H_0^{1,2}$ and $\nabla E\,|_M(u_m)=\nabla E(u_m)+\mu(u_m)\nabla G(u_m)\to 0$ T_L-uniformly as $m\to\infty$. By i) of this proof there exists a constant $M=M(\beta)$ such that $\|u_m\|_{1,2}\le M$, uniformly in m. Applying Lemma 3.3 we obtain that $u\in H_0^{1,2}\cap L^\infty$ and that $\|u\|_\infty<c(\lambda,\lambda^*,n,N,\Omega,p,M)$. Fix $\rho>0$ such that

$$\|u\|_\infty\,\rho<\lambda-\lambda^*.\tag{3.16}$$

For $m\in\mathbb{N}$ let A_m be the quadratic form $A_m(\varphi)=\frac{1}{2}\int a^{\alpha\beta}(x,u_m)\,\partial_\alpha\varphi^i\,\partial_\beta\varphi^i\,dy$. By convexity of A_m we then obtain for every $m\in\mathbb{N}$:

$$\begin{aligned}A_m(u_m)-A_m(u)&\le\langle\nabla A_m(u_m),u_m-u\rangle\\&=\langle\nabla E(u_m)+\mu(u_m)\nabla G(u_m),u_m-u\rangle-\langle\mu(u_m)\nabla G(u_m),u_m-u\rangle\\&\quad-\frac{1}{2}\int(u_m-u)\,\partial_u a^{\alpha\beta}(x,u_m)\,\partial_\alpha u_m^i\,\partial_\beta u_m^i\,dx.\end{aligned}\tag{3.17}$$

By weak convergence $u_m\to u$ $(m\to\infty)$

$$\limsup_{m\to\infty}A_m(u_m-u)=\limsup_{m\to\infty}\,[A_m(u_m)-A_m(u)],$$

and by compactness of ∇G and boundedness of $\mu(u_m)$

$$\langle\mu(u_m)\nabla G(u_m),u_m-u\rangle\to0\quad(m\to\infty).$$

By Lusin's theorem for any $\delta>0$ there exists $E_\delta\subset\Omega$, meas $(E_\delta)<\delta$, such that $u_m\to u$ uniformly on $\Omega\backslash E_\delta$. Denoting by $c(\delta)$ any constant such that $c(\delta)\to0$ $(\delta\to0)$ we hence obtain

$$\begin{aligned}&\limsup_{m\to\infty}-\frac{1}{2}\int(u_m-u)\,\partial_u a^{\alpha\beta}(x,u_m)\,\partial_\alpha u_m^i\,\partial_\beta u_m^i\,dx\\&\quad=\limsup_{m\to\infty}-\frac{1}{2}\int_{E_\delta}(u_m-u)\,\partial_u a^{\alpha\beta}(x,u_m)\,\partial_\alpha u_m^i\,\partial_\beta u_m^i\,dx\\&\quad=\limsup_{m\to\infty}-\frac{1}{2}\int_{E_\delta}(u_m-u)\,\partial_u a^{\alpha\beta}(x,u_m)\,\partial_\alpha(u_m-u)^i\,\partial_\beta(u_m-u)^i\,dx+c(\delta)\\&\quad\le\limsup_{m\to\infty}(\lambda^*+\|u\|_\infty\,\rho)\int|\nabla(u_m-u)|^2\,dx+c(\delta).\end{aligned}$$

Letting $\delta\to0$ and inserting our estimates into (3.17) we thus find that

$$\begin{aligned}\lambda\cdot\limsup_{m\to\infty}\int|\nabla(u_m-u)|^2\,dx&\le\limsup_{m\to\infty}A_m(u_m-u)\\&\le(\lambda^*+\|u\|_\infty\,\rho)\limsup_{m\to\infty}\int|\nabla(u_m-u)|^2\,dx.\end{aligned}$$

By our choice of ρ this implies that $u_m\to u$ strongly in $H_0^{1,2}$ as $m\to\infty$, and hence ii).

iii) Given an arbitrary number $\beta \in \mathbb{R}$ let ρ be chosen such that Criterion A* is satisfied for the functional E on the set $M_{\beta+1}$. Suppose $K_\beta = \emptyset$ and assume by contradiction that $N_{\beta,L} \neq \emptyset$ for all $L \in \mathbb{N}$. Select a "diagonal" sequence $u_m \in N_{\beta,m}$, $m \in \mathbb{N}$. Note that $u_m \in M_{\beta+1}$ for all m, hence by part i) of this proof $\{u_m\}$ is bounded in $H_0^{1,2}$ and by the Banach-Saks theorem we may assume that $u_m \rightharpoonup u$ weakly in $H_0^{1,2}$ as $m \to \infty$. Moreover, by continuous embedding $T_L \hookrightarrow T_N$, $\|u\|_N \leq \|u\|_L$ for all $u \in T$, $L \leq N$, we have that for any $L \in \mathbb{N}$

$$\limsup_{m \to \infty} \|\nabla E|_M(u_m)\|_{T_L^*} \leq \limsup_{m \to \infty} \|\nabla E|_M(u_m)\|_{T_m^*} = 0.$$

Thus $\nabla E(u_m) + \mu(u_m)\nabla G(u_m) \to 0$ T_L-uniformly as $m \to \infty$. By part ii) now we may select a strongly convergent subsequence $u_m \to u$ in $H_0^{1,2}$ $(m \to \infty)$, whence $u \in K_\beta$, contrary to the assumption that $K_\beta = \emptyset$. This concludes the proof. q.e.d.

LEMMA 3.5. *For any numbers $\beta \in \mathbb{R}$, $L \in \mathbb{N}$ there exists a number $\delta > 0$ and a continuous mapping $\phi_L : M \to M$ such that*

$$\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta} \cup N_{\beta,L}.$$

If conditions (3.10), (3.11) *are satisfied, ϕ_L may be chosen to be odd, i.e. $\phi_L(-u) = -\phi_L(u)$.*

Proof. Fix an arbitrary number $\beta \in \mathbb{R}$ and any $L \in \mathbb{N}$. The mapping ϕ_L will be constructed from a "gradient-line" deformation in direction of a gradient-like vector field related to $\nabla E|_M \in T_L^*$. (For brevity we again write $\nabla E|_M(u) = \nabla E(u) + \mu(u)\nabla G(u)$.)

i) For any $u \in M$ let $\xi_L(u) \in T_L$ be a vector satisfying

$$\|\xi_L(u)\|_{T_L} = \|\nabla E|_M(u)\|_{T_L^*}, \qquad \langle \nabla E|_M(u), \xi_L(u)\rangle \geq \|\nabla E|_M(u)\|_{T_L^*}^2 - L^{-1}.$$

By continuity of $\nabla E : H_0^{1,2} \to T_L^*$ for any $u \in M$ there exists a neighborhood $V(u)$ such that for all $v \in V(u)$

$$\langle \nabla E|_M(v), \xi_L(u)\rangle \geq \|\nabla E|_M(v)\|_{T_L^*}^2 - 2L^{-1}.$$

Since M is a subset of a Hilbert space there exists a locally finite refinement $\{V(u_\iota)\}_{\iota \in I}$ of the covering $\{V(u)\}$. Letting $\{\psi_\iota\}_{\iota \in I}$ be a partition of unity subordinate to $\{\tilde{V}(u_\iota)\}$ with continuous functions ψ_ι having support in $\tilde{V}(u_\iota)$ and such that

$$0 \leq \psi_i \leq 1, \qquad \sum_{\iota \in I} \psi_\iota = 1 \quad \text{on } M,$$

we define

$$\tilde{e}_L(u)=\sum_{\iota\in I}\psi_\iota(u)\xi_L(u_\iota).$$

In the general case we let $e_L(u)=\tilde{e}_L(u)$. In the symmetric case $E(u)=E(-u)$, $G(u)=G(-u)$, corresponding to assumptions (3.10), (3.11), we define

$$e_L(u)=\tfrac{1}{2}(\tilde{e}_L(u)-\tilde{e}_L(-u))=-e_L(-u).$$

In both cases $e_L: M\to T_L$ is a continuous vector field with the property that for any $u\in M$ there holds

$$\langle \nabla E|_M(u), e_L(u)\rangle \geq \|\nabla E|_M(u)\|^2_{T_L^*}-2L^{-1}.$$

Moreover, since $\nabla E|_M: M\to T_L^*$ is bounded on $H_0^{1,2}$-bounded subsets of M the same is true for e_L.

ii) Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $0\leq\alpha\leq 1$, $\alpha(t)=1$ if $t\leq\beta+1$, $\alpha(t)=0$ if $t\geq\beta+2$. For $\varepsilon>0$, $u\in M$ let

$$u^\varepsilon=u-\varepsilon\alpha(E(u))e_L(u).$$

Note that $u^\varepsilon\equiv u$ for $u\in M\backslash M_{\beta+2}$. Also, by part i) and Lemma 3.4 i) the vectors $\alpha(E(u))e_L(u)\in T_L$ are uniformly bounded on $M_{\beta+2}$.

By weak compactness of $M_{\beta+2}$ and assumptions (3.4), (3.6) there exists a constant $c>0$ such that $\langle\nabla G(u), u\rangle\geq c$ on $M_{\beta+2}$. Hence by the implicit function theorem there exists a number $\varepsilon_0>0$ and a function $\tau(\varepsilon, u)$ of class C^1 in ε and depending continuously on u such that

$$u_\varepsilon=\tau(\varepsilon, u)u^\varepsilon\in M$$

for all $u\in M$, $0\leq\varepsilon<\varepsilon_0$. Calculating $E(u_\varepsilon)$ by Lemma 3.2 we obtain

$$\begin{aligned}
E(u_\varepsilon)-E(u)&=E(u_\varepsilon)+\mu(u)G(u_\varepsilon)-(E(u)+\mu(u)G(u))\\
&=\int_0^\varepsilon \frac{d}{d\varepsilon}(E(u_\varepsilon)+\mu(u)G(u_\varepsilon)\,d\varepsilon\\
&=-\int_0^\varepsilon \langle\nabla E(u_\varepsilon)+\mu(u)\nabla G(u_\varepsilon), \tau(\varepsilon, u)\alpha(E(u))e_L(u)\rangle\,d\varepsilon \qquad (3.18)\\
&\quad+\int_0^\varepsilon \langle\nabla E(u_\varepsilon)+\mu(u)\nabla G(u_\varepsilon), u_\varepsilon\rangle\frac{\partial}{\partial\varepsilon}\ln\tau(\varepsilon, u)\,d\varepsilon\\
&\leq-\varepsilon\langle\nabla E(u)+\mu(u)\nabla G(u), \alpha(E(u))e_L(u)\rangle+o(\varepsilon),
\end{aligned}$$

with a Landau function $o(\varepsilon)$ such that $o(\varepsilon)/\varepsilon \to 0$ $(\varepsilon \to 0)$ uniformly with respect to $u \in M$. In particular, choosing ε_0 sufficiently small we may assume that $o(\varepsilon) \leq \varepsilon L^{-1}$. Moreover, by locally uniform continuity of $\nabla E|_M \in T_L^*$ we may assume that $\|\nabla E|_M(u_\varepsilon)\|^2_{T_L^*} < 5L^{-1}$ whenever $\varepsilon \leq \varepsilon_0$ and $\|\nabla E|_M(u)\|^2_{T_L^*} < 4L^{-1}$.

Fix such a number $0 < \varepsilon \leq \frac{1}{4}$ and let $\phi_L(u) = u_\varepsilon$. Set $\delta = \varepsilon/2L \leq 1/L$. Then, if $u \in M_{\beta+\delta}$, either $\|\nabla E|_M(u)\|^2_{T_L^*} \geq 4L^{-1}$ and hence by (3.18)

$$E(\phi_L(u)) \leq E(u) - 2\varepsilon L^{-1} + \varepsilon L^{-1} = E(u) - 2\delta,$$

i.e. $\phi_L(u) \in M_{\beta-\delta}$; or $\|\nabla E|_M(u)\|^2_{T_L^*} < 4L^{-1}$ whence $\|\nabla E|_M(\phi_L(u))\|^2_{T_L^*} < 5L^{-1}$ while

$$E(\phi_L(u)) \leq E(u) + 2\varepsilon L^{-1} + \varepsilon L^{-1} < \beta + 4\varepsilon L^{-1} \leq \beta + L^{-1},$$

i.e. $\phi_L(u) \in M_{\beta-\delta} \cup N_{\beta,L}$.

Finally, in the case of assumptions (3.10), (3.11) $\phi_L(-u) = -\phi_L(u)$. The proof is complete. q.e.d.

Proof of Theorems 3.1 *and* 3.2 *for coefficients satisfying* (3.14).

i) Let $\beta = \inf_{u \in M} E(u)$. Choose $\rho_1 > 0$ corresponding to Lemma 3.4 iii) and assume that $|\partial_u a| < \rho_1$ a.e. in $\Omega \times \mathbb{R}^N$. Then if $K_\beta = \emptyset$ by Lemma 3.4 iii) there exists L such that $N_{\beta,L} = \emptyset$, and by Lemma 3.5 we can find $\delta > 0$ and a continuous mapping $\phi_L : M \to M$, such that $\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta}$. But $M_{\beta-\delta} = \emptyset$ and a contradiction results, proving Theorem 3.1.

ii) In the case of Theorem 3.2 fix a number $l \leq m$, and let $\beta = \beta_l$. Choose $\rho_l > 0$ corresponding to Lemma 3.4 iii) and assume that $|\partial_u a| < \rho_l$ a.e. in $\Omega \times \mathbb{R}^N$. Then, if $K_\beta = \emptyset$ as in part i) of this proof there exists a continuous mapping $\phi_L : M \to M$ such that $\phi_L(M_{\beta+\delta}) \subset M_{\beta-\delta}$ for some number $\delta > 0$. Moreover, by assumptions (3.10), (3.11) ϕ_L can be chosen to be odd. Letting $A \in \Sigma_l$ be such that $A \subset M_{\beta+\delta}$, we thus obtain that $\phi_L(A) \in \Sigma_l$ (cp. Proposition 3.1 i)) and satisfies $\phi_L(A) \subset M_{\beta-\delta}$. But this contradicts the definition of $\beta = \beta_l$, concluding the existence proof.

In the case of degeneracy $\beta = \beta_l = \cdots = \beta_{l+k}$ for some numbers $k, l \in \mathbb{N}$ we estimate $\gamma(K_\beta)$ as follows. By Criterion A* the set K_β is compact. Hence from Proposition 3.1 iii) we can find a symmetric neighborhood N of K_β, $N \subset H_0^{1,2} \backslash \{0\}$, such that the closure $\bar{N} \in \Sigma$ and $\gamma(\bar{N}) = \gamma(K_\beta)$. Again by Criterion A* there exists L such that $N_{\beta,L} \subset \bar{N}$. Let $\delta > 0$, ϕ_L be chosen according to Lemma 3.5 corresponding to this number L and β and let $A \in \Sigma_{l+k}$ be such that $A \subset M_{\beta+\delta}$. Then by

Proposition 3.1.

$$l+k \leq \gamma(A) \leq \gamma(\phi_L(A)) \leq \gamma(\phi_L(A)\backslash N_{\beta,L}) + \gamma(\bar{N})$$
$$= \gamma(\phi_L(A)\backslash N_{\beta,L}) + \gamma(K_\beta).$$

But $\phi_L(A)\backslash N_{\beta,L} \subset M_{\beta-\delta}$, whence $\gamma(\phi_L(A)\backslash N_{\beta,L}) < l$, and $\gamma(K_\beta) \geq k+1$.

(The proof of the asymptotic behavior of the sequence β_l will be postponed to the full proof of Theorems 3.1, 3.2 without assumption (3.14).)

Proof of Theorems 3.1, 3.2 (completed). To remove assumption (3.14) approximate the functional E by functions E_ν with coefficients $a_\nu^{\alpha\beta}$ satisfying condition (3.14) and coinciding with $a^{\alpha\beta}$ for $|u| \leq \nu$. To construct these coefficients let $\varphi: \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \leq \varphi \leq 1$, $\varphi' \leq 0$, $\varphi(0) = 1$, $\varphi(1) = 0$. Then given any number $\nu \in \mathbb{N}$, for $u \in \mathbb{R}^N$, $|u| = 1$, define

$$a_\nu^{\alpha\beta}(x, u \cdot t) = \varphi(t-\nu) a^{\alpha\beta}(x, u \cdot t) + (1-\varphi(t-\nu))\lambda\delta_{\alpha\beta}.$$

Clearly, assumptions (3.1), (3.2) are satisfied for the coefficients $a_\nu^{\alpha\beta}$ with the uniform ellipticity constant λ. Moreover, for any u

$$-u \cdot \partial_u a_\nu^{\alpha\beta}(x, u) = -\varphi(|u|-\nu) u \cdot \partial_u a^{\alpha\beta}(x, u) - |u|\, |\varphi'(|u|-\nu)| (a^{\alpha\beta}(x, u) - \lambda \cdot \partial_{\alpha\beta}).$$

Hence, by conditions (3.2), (3.3) on $a^{\alpha\beta}$, condition (3.3) is satisfied by the coefficients $a_\nu^{\alpha\beta}$ with the uniform constant $\lambda^* < \lambda$.

By the preceding proof there exist solutions (u_l, μ_l) of (1.2) for the functionals E_ν characterized by the condition $E_\nu(u_l) = \beta_{l,\nu} = \inf_{A \in \Sigma_l} \sup_{u \in A} E_\nu(u)$. Now, for any $u \in M$ $E_\nu(u) \to E(u)$ $(\nu \to \infty)$, and similarly, for any $A \in \Sigma_l$ $\sup_{u \in A} E_\nu(u) \to \sup_{u \in A} E(u)$ $(\nu \to \infty)$. Hence $\beta_{l,\nu} \to \beta_l (\nu \to \infty)$, for any $l \leq m$. By Lemma 3.3 therefore $\|u_l\|_\infty \leq c(\lambda, \lambda^*, n, N, \Omega, p, \beta_l + 1)$ for ν sufficiently large, and u_l is in fact a critical point of E with energy $E(u_l) = \beta_l = \beta_{l,\nu}$ for such ν. (This also justifies writing u_l instead of $u_{l,\nu}$.) This shows existence and by the preceding proof also the assertion about K_β in the case of degeneracy now follows.

To show that $\beta_l \to \infty$ $(l \to \infty)$ assume by contradiction that $\beta_l \leq c$ for all $l \in \mathbb{N}$ with a uniform constant c. Then there exist sets $A_l \in \Sigma_l$, $l \in \mathbb{N}$, such that $A = \bigcup_l A_l \subset M_{c+1}$. By [21, proof of Lemma 2.21] or [25, proof of Lemma 10] A contains a sequence of mutually orthogonal vectors $u_m \in A$, and by weak compactness of M_{c+1} we may assume that $u_m \rightharpoonup u$ weakly as $m \to \infty$. But by mutual orthogonality of $\{u_m\}$, $u=0$. Hence from $0=G(u)=\lim_{m\to\infty} G(u_m)=1$ a contradiction results. q.e.d.

4. Extensions and generalizations

Without proof we remark that in the subquadratic case corresponding e.g. to functionals of the type

$$E_\delta(u)=\int |\nabla u|^2\,dx+\int (a^{\alpha\beta}\,\partial_\alpha u^i\,\partial_\beta u^i)^{2-\delta/2}\,dx, \qquad \delta>0 \tag{4.1}$$

existence results similar to Theorems 3.1, 3.2 hold without any smallness assumption on $|\partial_u a|$ in addition to (3.3).[1] By a different kind of argument than used in Section 3 also in case of quadratic growth such a smallness assumption can be removed if either $N=1$ (scalar equation) or $n=2$ (plane system). This may be shown by approximating the functional E by functionals E_δ of the form (4.1) and using the regularity results of Ladyshenskaya–Ural'tseva [15] for example, resp. the results of Wiegner [28] for plane diagonal systems to pass to the limit $\delta\to 0$ in the Euler equations which are satisfied at the critical points of E_δ.

By the same techniques as presented above also existence results for the boundary value problem

$$u\in H_0^{1,2}: \nabla E(u)-\nabla G(u)=0\in T'$$

can be given under hypotheses similar to those of Theorems 3.1 and 3.2 plus some additional hypotheses to ensure the regularity (with respect to T_L, for any $L\in\mathbb{N}$) of the set of admissible functions

$$M=\{u\in H_0^{1,2}\mid \langle\nabla E(u)-\nabla G(u),\,u\rangle=0\}.$$

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I. The birational geometry of degenerations: an overview by R. Friedman and D. R. Morrison – Appendix: On the classification of surfaces with Kodaira number zero by R. Friedman – II. Degenerations with numerically effective canonical divisor by N. I. Shepherd-Barron-III. Smooth models for elliptic threefolds by Rick Miranda-IV. Extending polarizations on families of K3 surfaces by N. I. Shepherd-Barron-Appendix: Linear systems on anticanonical pairs by R. Friedman-V. The minus one theorem by R. Miranda and D. R. Morrison-VI. Degenerations of Kodaira surfaces by R. Friedman and N. I. Shepherd-Barron – VII. Base change, automorphisms, and stable reduction for type III K3 surfaces by R. Friedman – VIII. Threefold birational morphisms and degenerations without triple points by B. Crauder – IX. Triple-point-free degenerations of surfaces with Kodaira number zero by B. Crauder and D. R. Morrision

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Vorwort von Pál Révész – Ueber den Begriff der Information in der Mathematik – Gedanken zum Lehren der Wahrscheinlichkeitsrechnung – Glücksspiele und Wahrscheinlichkeitsrechnung – Variationen über ein Thema von Fibonacci – Ueber die mathematische Theorie der Bäume

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Group actions on fibered three-manifolds

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1. Introduction

In this paper we present several results about finite group actions on three-dimensional manifolds. The results are primarily directed toward a geometric understanding of periodic knots in the 3-sphere, that is, knots left invariant by periodic homeomorphism of S^3 which fix a simple closed curve in the complement of the knot.

One noteworthy application of the present techniques is that nontrivial periodic knots have "Property R," that is, surgery on such a knot cannot produce $S^1 \times S^2$.

Let G be a finite group acting effectively and smoothly on an orientable 3-manifold M, preserving orientation. The first result is that if the orbit manifold M^* contains an incompressible surface F^*, then (after suitably adjusting the embedding) the preimage of F^* in M is an incompressible surface. The proof of this makes use of the Equivariant Loop Theorem of Meeks and Yau [12, 13] and is given in Section 2.

An immediate corollary of this result is that a periodic knot has an invariant incompressible Seifert surface. If there is a bound on the possible genera of the incompressible Seifert surfaces of a given knot K (as is the case for fibered knots), then the Riemann-Hurwitz formula places nontrivial bounds on the possible periods of K. See Section 2. There is an extensive literature devoted to the problem of determining the possible periods of a given knot [1, 5, 6, 7, 11, 15, 16, 18, 22]. Most previous work on this problem has been heavily algebraic in nature, in contrast to the present more geometric approach.

In Section 3 the preceding work is applied to prove that periodic knots have Property R.

Now suppose that F is a compact, orientable surface and that G acts on $F \times [0, 1]$ preserving orientation and leaving both $F \times \{0\}$ and $F \times \{1\}$ invariant. We show that the action is equivalent to the level-preserving action which is the product of the action of $F \times \{0\}$ with the trivial action on the interval [0, 1], except

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possibly when the action on $F\times\{0\}$ has fewer than four exceptional orbits and orbit space the 2-sphere. The proof is given in Section 4 and makes essential use of the Equivariant Dehn Lemma of Meeks and Yau [12, 13] and of the solution of the classical Smith Conjecture [19], which corresponds to the case here that F is a 2-disk. After the first version of this paper was written we learned that W. Meeks and G. P. Scott have proved the above result in the more difficult excluded case as well.

An easy application of the preceding result, given in Section 5, is that quite generally a group action on a surface bundle over the circle *with unique fibering* is equivalent to a fiber-preserving action. In particular, any periodic fibered knot in S^3 admits a fibering preserved by the corresponding group action.

The final section contains further applications of the preceding results to group actions preserving fibered knots. In particular one has a very short list of fibered knots of a given genus g which admit periods m which are maximal $(m=2g+1)$ or nearly maximal $(m=g+1)$. Several instructive examples are given as well. Finally we show that the present techniques give a partial answer to a question of D. Goldsmith in the Kirby problem set [9; 1.28].

The techniques used in this paper are similar to those used by Gordon and Litherland [4]. That work is primarily concerned with invariant incompressible surfaces in the complement of the exceptional set, while our main interest is in incompressible surfaces which intersect the exceptional set.

Quite similar and more complete results in the case $G=\mathbb{Z}_2$ are due to Tollefson [21] and Kim and Tollefson [8].

2. Lifting incompressible surfaces

Throughout this section let G be a finite group acting effectively and smoothly on an orientable, differentiable 3-manifold M preserving orientation. The *exceptional set* E for the action is the set of all points in M having nontrivial isotropy group. If nonempty, E is a one-dimensional CW complex with all vertices of valence one lying in ∂M. The exceptional set is a 1-manifold precisely if all isotropy groups are cyclic. Let M^* denote the orbit manifold and let $p: M\to M^*$ be the orbit map. The set $B=p(E)$ is called the *branch set.*

Two embedded surfaces F and F' in a 3-manifold N are said to be *disk equivalent* if there is a sequence $F_1=F, F_2,\ldots,F_n=F'$ of surfaces in N such that for each k, $1<k\le n$, there exist disks $D_k\subset \operatorname{int} F_k$ and $D'_k\subset \operatorname{int} F_{k-1}$ such that $F_k-D_k=F_{k-1}-D'_k$. If N is irreducible, then standard techniques show that disk equivalent surfaces are isotopic.

Let F be a surface in a 3-manifold N such that $F\cap\partial N=\partial F$. A *compressing* *disk* for F is a 2-disk $D \subset \text{int } N$ such that $D \cap F = \partial D$ and ∂D is a homotopically nontrivial simple loop on F. If F has a compressing disk or is a nullhomotopic 2-sphere or is a 2-disk which can be deformed rel ∂F into ∂N, then F will be called *compressible*; otherwise F is said to be *incompressible.*

A surface F in the orbit manifold M^* is said to be *transverse* to the branch set B provided that F only meets B in the subset B_0 of points where B is a 1-manifold and F is transverse to B_0 in the usual sense.

As indicated previously we shall need the following basic result of Meeks-Yau [12, 13]. We quote from [4].

EQUIVARIANT LOOP THEOREM. *Let N be a 3-manifold, and G a finite group acting on N. Suppose F is a compressible component of ∂N. Then there exists a compressing disk D for F such that for all $g \in G$, either $g(D) = D$ or $g(D) \cap D = \varnothing$.*

Although Meeks and Yau only assert the result when N is compact and orientable, the theorem is true in the generality given. Further one may assume G acts freely on $G(\partial D)$.

THEOREM 2.1. *Any two-sided incompressible surface in the orbit manifold M^* is disk equivalent to an incompressible surface F^* which meets the branch set B transversely and such that $F = p^{-1}(F^*)$ is a two-sided incompressible surface in M.*

Proof. It may be assumed that F^* is an incompressible surface in M^* which is transverse to B and meets the non-singular part of B in the minimum number of points possible for incompressible surfaces in its disk equivalence class. Transversality implies that $F = p^{-1}(F^*)$ is a surface in M.

Suppose that F is compressible in M. If a component F_1 of F is a null-homotopic 2-sphere, then $F_1 = \partial \Delta$ where Δ is a homotopy 3-cell in M. Let H be the subgroup of G leaving Δ invariant. Then $F = \partial H(\Delta)$ and $F^* = \partial p(\Delta)$, a 2-sphere. Moreover $p(\Delta)$ is a simply connected 3-manifold with boundary F^*, and this implies that F^* is compressible.

Similarly, if some component F_1 of F is a disk homotopic into ∂M, it follows that F^* is a disk homotopic into ∂M^*, contradicting incompressibility of F^*.

Finally suppose F contains no spheres or disks. Let N be an invariant tubular neighborhood of F in M. Then the manifold $W = (M - \partial M) - \text{int } N$ has as boundary two copies of int F, and has compressible boundary. By the Equivariant Loop Theorem there is a compressing disk D_1 for ∂W in W such that for any $g \in G$ either $g(D_1) = D_1$ or $g(D_1) \cap D_1 = \varnothing$. By using an equivariant product structure in N one may expand D_1 to a compressing disk D for F such that for $g \in G$ either $g(D) = D$ or $g(D) \cap D = \varnothing$.

Let $D^* = p(D)$ in M^*. Then D^* is a disk transverse to the branch set B, $D^* \cap B$ consists of at most one point and $D^* \cap F^* = \partial D^*$. Let F_0^* be the component of F^* meeting D^*. First suppose F_0^* is neither a 2-sphere nor a 2-disk. Since F_0^* is incompressible there is a 2-disk $D_0^* \subset F_0^*$ with $\partial D_0^* = \partial D^*$. The minimality condition on F^* implies that D_0^* meets B in at most one point. Therefore each component of $p^{-1}(D_0^*)$ is a disk in F. Since the boundary of some component of $p^{-1}(D_0^*)$ is ∂D, it follows that D is not a compressing disk for F after all. If F_0^* is a 2-sphere, then ∂D^* divides F_0^* into two 2-disks F_1^* and F_2^*. Since F contains no 2-sphere and ∂D is nontrivial on F, each F_i^* must contain at least two branch points, while D^* contains at most one. One of the two 2-spheres $D^* \cup F_1^*$ or $D^* \cup F_2^*$ must be incompressible and meet B in fewer points than F_0^* did. This contradicts the choice of F. If F_0^* is a 2-disk, then ∂D^* divides F_0^* into a 2-disk F_1^* and an annulus F_2^*. Since no component of F is a 2-disk, and ∂D is nontrivial on F, $p^{-1}(F_1^*)$ does not consist of disks, and so F_1^* meets B at least twice. But D^* meets B at most once. Thus a disk move would reduce $F^* \cap B$, a contradiction. This completes the proof. □

Remark. It follows that if the orbit manifold M^* is sufficiently large, then so is M.

COROLLARY 2.2. *Let K be a knot in an integral homology 3-sphere Σ invariant under a semifree orientation-preserving action of the cyclic group C_m of order m, with fixed set A (the "axis") disjoint from K. Then K bounds an incompressible Seifert surface invariant under C_m.*

Proof. Since C_m has fixed points the quotient map $p : \Sigma \to \Sigma^*$ induces a surjection $H_1(\Sigma; \mathbb{Z}) \to H_1(\Sigma^*; \mathbb{Z})$. It follows that Σ^* is also an integral homology 3-sphere containing the knot $K^* = p(K)$. Let F^* be an incompressible Seifert surface for K^* meeting the branch set transversely a minimum number of times. Then by Theorem 2.1 $p^{-1}(F^*)$ is the required incompressible Seifert surface for K. □

Note that in the situation of Corollary 2.2 the linking number $lk(K, A) = lk(K^*, A^*)$ is relatively prime to m. Otherwise $p^{-1}(K^*) = \partial p^{-1}(F^*)$ would be disconnected.

The following well known lemma is the basis for further restrictions on the periods of knots.

LEMMA 2.3. *Let F be a compact, connected, orientable surface of genus $g > 0$ with one boundary component. If F admits a semifree, orientation-preserving action of the cyclic group C_m with nonempty fixed set, then $m \le 2g + 1$.*

Proof. The Riemann-Hurwitz formula for the regular branched cyclic covering $p: F \to F^*$ says that

$$1-2g = m(1-2g^*) - k(m-1)$$

where g^* is the genus of the orbit surface F^* and k is the number of branch points. If $k=1$, then $mg^* = g$; so $m = g/g^* \leq g$. If $k > 1$, then

$$\begin{aligned} m &= (2g-1+k)/(2g^*-1+k) \\ &\leq (2g-1+k)/(k-1) \\ &\leq 2g+1. \quad \square \end{aligned}$$

Remark. The maximum value $m = 2g+1$ occurs only when $k=2$. Otherwise $m \leq g+1$. If $m = g+1$, then $k \leq 3$. For all other values of $k \geq 4$, $m \leq g$.

COROLLARY 2.4. (cf. [15]). *If K is a fibered knot in S^3 of genus g and period m, then $m \leq 2g+1$.*

Proof. It is well known that a fibered knot has a unique incompressible Seifert surface, up to isotopy. See Lemma 5.1. By Corollary 2.2 K has an invariant Seifert surface of genus g. By Lemma 2.3 $m \leq 2g+1$. $\square$

See Section 6 for more precise results in this direction, where it will be shown that *very* few fibered knots of genus g admit periods greater than g.

3. Periodic knots have Property R

Let K be a knot in S^3 which is invariant under a semifree, orientation-preserving action of a cyclic group C_m, with nonempty fixed point set A disjoint from K. The manifold obtained by 0-surgery on K is defined to be $M(K,0) = (S^3 - \operatorname{int} N(K)) \cup B^2 \times S^1$ where $N(K)$ is a tubular neighborhood of K and the image of the meridian $\partial B^2 \times pt$ is a longitude of K (nullhomologous in $S^3 - K$).

THEOREM 3.1. *If K is a periodic knot in S^3 and $\pi_1(M(K,0)) \approx \mathbb{Z}$, then K is unknotted (and $M(K,0) \cong S^1 \times S^2$).*

Proof. One may assume that the tubular neighborhood $N(K)$ is invariant under a given C_m action. As in the proof of Theorem 2.1 K has an invariant

Seifert surface. Therefore the group action leaves a preferred longitude invariant, and it is possible to extend the action on $S^3 - \operatorname{int} N(K)$ to $M(K, 0)$. The fixed point set of the extended action is $A \cup (0 \times S^1)$. The quotient of $M(K, 0)$ by the induced C_m action may be described as the result of 0-surgery on the quotient of K under the original action. Put succinctly, $M(K, 0)^* = M(K^*, 0)$. In particular then $H_1(M(K, 0)^*; \mathbb{Z}) \approx \mathbb{Z}$, generated by the image of the core $0 \times S^1$. Therefore $M(K, 0)^*$ contains a nonseparating incompressible surface S^* (in fact a 2-sphere). It may be assumed that among all incompressible surfaces disk equivalent to S^*, S^* meets the branch set transversely in $M(K, 0)^*$ a minimum number of points.

By Theorem 2.1, and its proof, the inverse image S of S^* in $M(K, 0)$ is an invariant, incompressible surface in $M(K, 0)$. Because $\pi_1(M(K, 0)) \approx \mathbb{Z}$, S must be a 2-sphere or a collection of m 2-spheres.

If S is not connected, then $M(K, 0) - S$ consists of m components cyclically permuted by the action. In this case the action would have empty fixed point set. This contradiction implies that S is a single invariant 2-sphere, which must therefore meet the fixed point set in exactly two points.

Since the core $0 \times S^1$ of $B^2 \times S^1$ represents a generator of $H_1(M(K, 0))$, the intersection number of $0 \times S^1$ with S must be ± 1. But S meets $0 \times S^1$ at most twice. Therefore S meets $0 \times S^1$ once and the original axis A once.

Finally, the manifold obtained by removing from $M(K, 0)$ a small tubular neighborhood of $0 \times S^1$ is just $S^3 - \operatorname{int} N(K)$. But in this space S becomes a disk with boundary a longitude of K. Hence K is unknotted. $\square$

4. Actions on $F^2 \times [0, 1]$

As the second step toward standardizing actions on fibered knots and other fibered 3-manifolds we show how to straighten actions on the product of a surface with the unit interval.

Let F be a compact surface and G be a finite group of diffeomorphisms of $F \times [0, 1]$, leaving $F \times \{0\}$ and $F \times \{1\}$ invariant. The restriction of G to $F \times \{0\}$ induces an action of G on $F: g(x) = \pi_F g(x, 0)$ where $\pi_F: F \times [0, 1] \to F$ is the projection. The *associated straight action* of G on $F \times [0, 1]$ is given by $g(x, t) = (g(x), t)$.

THEOREM 4.1. *Let F be a compact, orientable surface and let G be a finite group acting smoothly and effectively on $F \times [0, 1]$, preserving orientation and leaving $F \times \{0\}$ and $F \times \{1\}$ invariant. Assume that the orbit space F/G is not a 2-sphere with less than 4 branch points. Then the given action is equivalent to its associated straight action, by a diffeomorphism of $F \times [0, 1]$ which is the identity on $F \times \{0\}$.*

Proof. It is easy to see that one may assume F is connected: Separate orbits of components may clearly be handled separately; if F_1 is component of F such that $G(F_1)=F$, let H be the subgroup of G leaving $F_1\times[0,1]$ invariant and suppose the action of H can be straightened, so there is a diffeomorphism $\phi : F_1\times[0,1]\to F_1\times[0,1]$ such that $\phi \mid F_1\times\{0\}$ is the identity and $\phi(h(x),t)=h\phi(x,t)$ for $h\in H$, $x\in F_1$, $t\in[0,1]$; define $\theta : F\times[0,1]\to F\times[0,1]$ by $\theta(g(x),t)=g\phi(x,t)$ for $g\in G$, $x\in F_1$, $t\in[0,1]$; then θ is the required equivariant diffeomorphism satisfying $\theta(g(x),t)=g\theta(x,t)$ for all $g\in G$, $x\in F$, $t\in[0,1]$.

So suppose F is connected. We proceed by induction on the ordered pairs $(r,b)=$(genus of F, number of components of ∂F), ordered lexicographically.

If $r=0$, then $b\geq 1$ since $(0,0)$ is an excluded case. If $b=1$, then F is a disk. In this case G must be cyclic. The solution of the Smith Conjecture [19] then shows that the action on $F\times[0,1]$ can be straightened. Note that if one assumes the action is already straight on $\partial F\times[0,1]$, then the action can be straightened rel $\partial F\times[0,1]$.

Inductively, first consider the cases where F has nonempty boundary ($b\geq 1$). Covering space theory shows that we may assume the action is already straight on $\partial F\times[0,1]$. We show inductively that the action on $F\times[0,1]$ can be straightened by a diffeomorphism which is the identity on $F\times\{0\}\cup\partial F\times[0,1]$. Choose an invariant family of disjoint, properly embedded arcs $A_1,\ldots,A_m\subset F$, where m is the order of G, such that if F is cut open along $A_1,\ldots,A_m$, then the complexity (r,b) is reduced. Such a family is easily constructed as the preimage of an arc A^* in the orbit surface F/G which, if homotopic into $\partial F/G$, does not cut off a disk containing less than two branch points.

Let $B_i=A_i\times\{1\}$ in $F\times\{1\}, 1\leq i\leq m$. This family of arcs is probably not G-invariant. We may isotope $\{B_i\}$, rel end points, to an invariant collection as follows: Impose a G-invariant hyperbolic metric on $F\times\{1\}$ with totally geodesic boundary curves. (If S is an annulus, one must use a euclidean metric instead.) Each B_i is then homotopic, rel end points, to a unique geodesic arc C_i with the same end points as B_i. By the uniqueness of the choice of these geodesics the collection $\{C_i\}$ is G-invariant.

Moreover $\{C_i\}$ consists of embedded, pairwise disjoint arcs. The essential facts here are that the minimum number of self-intersections in an arc representing a given relative homotopy class is realized by its geodesic representative and that the minimum number of intersections between two arcs representing two given relative homotopy classes is also realized by their geodesic representatives. A proof of these assertions follows the lines of the proof of the analogue of the second statement in the context of closed curves as given in [3; Exposé 3, Proposition 10]. Since the B_i are disjoint and embedded, it follows that the same is true of the C_i.

The closed curves $\alpha_i = A_i \times \{0\} \cup \partial A_i \times [0, 1] \cup C_i$ form a G-invariant family of pairwise disjoint simple loops in $\partial(F \times [0, 1])$. Each α_i is nullhomotopic in $F \times [0, 1]$, since $\alpha_i \simeq A_i \times \{0\} \cup \partial A_i \times [0, 1] \cup A_i \times \{1\}$. By the Equivariant Dehn Lemma [13], there is a G-invariant family of disjoint, embedded disks $D_i \subset F \times [0, 1]$ with $\partial D_i = \alpha_i$. Since $F \times [0, 1]$ is irreducible, $\bigcup D_i$ is isotopic to $\bigcup A_i \times [0, 1]$. Therefore, cutting $F \times [0, 1]$ open along $\bigcup D_i$ results in a new G-manifold of the form $F' \times [0, 1]$ where the product structure imposed on the copies of D_i extends that on $\partial F \times [0, 1]$. One may apply covering space theory to straighten the action on the copies of D_i prior to cutting open along $\bigcup D_i$. Now the inductive hypothesis says that the action on $F' \times [0, 1]$ is equivalent rel $\partial F' \times [0, 1]$ to the associated straight action. Gluing $F' \times [0, 1]$ back together along the cuts provides an equivalence of the given action on $F \times [0, 1]$ with its associated straight action.

Finally suppose $\partial F = \emptyset$ and either the orbit space F^* has positive genus or is a sphere with at least four branch points. Choose an invariant family of disjoint embedded simple loops $A_1, \ldots, A_m \subset F$, $m \leq |G|$, which are homotopically nontrivial in F. Such a family is constructed as the preimage of a suitable closed loop in F^* which does not bound a disk containing less than two branch points.

Let $B_i = A_i \times \{1\}$, $1 \leq i \leq m$. This family of loops is not in general G-invariant. But by replacing them with a corresponding set of geodesics for a G-invariant hyperbolic metric on $F \times \{1\}$ we obtain an invariant family of simple, closed loops $\{C_i\}$ such that $C_i \simeq B_i$ for each i.

Now the Equivariant Dehn Lemma (for planar domains) of Meeks-Yau [13] implies that there is a G-invariant family of disjoint embedded annuli $V_i \subset F \times [0, 1]$ with $\partial V_i = A_i \times \{0\} \cup C_i$. Since $C_i \simeq A_i \times \{1\}$ and $F \times [0, 1]$ is irreducible, V_i is isotopic rel $A_i \times 0$ to $A_i \times [0, 1]$. Thus the given action is equivalent to one which preserves the family $\{A_i \times [0, 1]\}$ and is straight there. Now cut open along $\bigcup A_i \times [0, 1]$, apply the inductive hypothesis, and glue back together to complete the argument. □

Remark. We have been informed that W. Meeks and G. P. Scott have proved Theorem 4.1 without the hypothesis that F/G is not a 2-sphere with fewer than four branch points. The proofs involve singular incompressible surfaces.

5. Actions on bundles over the circle

Let M denote a compact, connected, orientable 3-manifold which fibers over the circle S^1, in such a way that ∂M, if nonempty, is a torus which inherits an induced fibering by restriction, and satisfies $H_1(M; \mathbb{Q}) \approx \mathbb{Q}$. For example M might be the complement of an open tubular neighborhood of a fibered knot in a homology sphere. The condition on homology is used to insure that M admits an essentially unique fibering. The following lemma is well known.

LEMMA 5.1. *Any two connected, properly embedded, non-separating, two-sided, incompressible surfaces* F_1, $F_2 \subset M$ *are isotopic.*

Proof sketch. One may suppose that F_1 is the fiber of a fibering $M \to S^1$. Let $\bar{M} \cong F_1 \times \mathbb{R}$ be the corresponding infinite cyclic covering. Because F_2 in non-separating, connected, and two-sided, intersection with F_2 defines a surjective homomorphism $\pi_1(M) \to \mathbb{Z}$, which must coincide, up to sign, with the corresponding homomorphism associated with the fibration since $H_1(M; \mathbb{Z})/\text{torsion} \approx \mathbb{Z}$. Thus F_2 lifts homeomorphically to an incompressible surface $\bar{F}_2 \subset \bar{M}$. If $\pi_1(\bar{F}_2) \to \pi_1(\bar{M})$ were not also surjective, then an argument with Van Kampen's theorem would imply that $\pi_1(\bar{M})$ is not finitely generated. (Compare [20; p. 97].)

THEOREM 5.2. *Let G be a finite group acting by orientation-preserving diffeomorphisms on the fibered* 3*-manifold M as above, with exceptional set E disjoint from ∂M. Assume* (i) *that the orbit manifold M^* is not $S^2 \times S^1$ where $S^2 \times$ point meets the branch set in less than four points, and* (ii) *that $H_1(M; \mathbb{Q})^G \approx \mathbb{Q}$. Then the given fibering of M is isotopic to a fibering in which G maps fibers into fibers.*

Proof. The quotient M^* is a manifold with ∂M^*, if nonempty, a torus. If $\partial M \neq \varnothing$, this immediately implies $H_1(M^*; \mathbb{Q}) \neq 0$. In general $H_1(M^*; \mathbb{Q}) \approx H_1(M; \mathbb{Q})^G$, nonzero by assumption. Therefore M^* contains a nonseparating, connected, two-sided incompressible surface F^*. We may assume that F^* meets the branch set B transversely in a minimal number of points. Then by Theorem 2.1 the preimage F of F^* in M is an incompressible surface. Let $F_1, \ldots, F_r$ be the components of F. By Lemma 5.1 each component F_i is isotopic to a fiber of M. Cutting M open along F one obtains a group action on $F \times I$. By Theorem 4.1 this action is equivalent to a straight action on $F \times I$, each component of $F \times \{t\}$ being a fiber. Reidentifying $F \times \{0\}$ with $F \times \{1\}$ one obtains the required equivariant fibering. □

Remarks. If the action has fixed points then F is connected, and each fiber is G-invariant. The recent result of Meeks-Scott mentioned earlier shows that the hypothesis (i) above can be dropped.

COROLLARY 5.3. *The quotient M^* fibers over S^1.* □

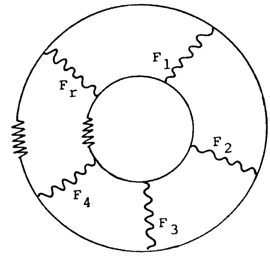

Figure 1

An alternative elementary proof of the corollary invokes Stallings' fibering theorem [20]. Compare [16].

6. Applications to fibered knots

Let $K \subset S^3$ be a fibered knot invariant under a smooth, orientation-preserving, semifree action of the cyclic group C_m of order m, having fixed axis A, a knot disjoint from K. The solution of the Smith Conjecture [19] implies that A is unknotted, and hence that the orbit space is again S^3 with branch set $B = A^*$ also an unknotted circle. The following simply interprets Theorem 5.2 in this setting.

PROPOSITION 6.1. *The fibering of K is isotopic to a fibering preserved by the action of C_m so that the axis A is transverse to the fibers; thus the quotient knot K^* inherits a fibering with all fibers transverse to the branch set B.* □

Let $F \subset S^3$ denote a typical fiber for the knot K transverse to A and let F^* denote its image in the orbit space. Note that since the axis is connected the local two-dimensional representations of C_m in the tangent space of F at the points of $F \cap A$ are all equivalent to rotation by $\pm 2\pi k/m$ for some fixed integer k.

Our first application of Proposition 6.1 is based on combining Corollary 2.4 with the fact that a genus 0 fibered knot is unknotted.

PROPOSITION 6.2. *For $g \geq 1$ the only fibered knot of genus g which is invariant under an action of C_m as above, with $m = 2g+1$, is the $(2, 2g+1)$ torus knot, up to orientation.*

Proof. One can check using the Riemann-Hurwitz formula that C_m, $m = 2g+1$, acts semifreely on a surface F of genus g with one boundary component,

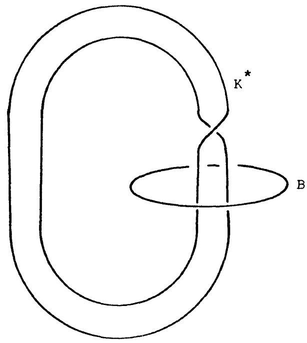

Figure 2

only so that the quotient F^* is disk with two branch points. Therefore the quotient of an m-periodic knot K is an unknotted circle K^* in S^3, and the branch set B is transverse to each disk fiber for K^*, meeting each fiber in exactly two points. The only such link $\{K^*, B\}$ up to orientation is shown in Figure 2. One easily checks that the preimage of K^* must be the $(2, \pm m)$ torus knot. □

PROPOSITION 6.3. *For* $g \geq 1$ *the only fibered knots of genus* g *which are invariant under an action of* C_m *as above, with* $m = g + 1$, *are the* $(3, \pm m)$ *torus knot and the* $(3, \pm m)$ *Turk's head knot.*

Proof. The group C_m must act on the typical fiber F of genus $g = m - 1$ with orbit surface a disk and exactly three branch points. According to [10] the only three-stranded braids (up to conjugation in the braid group, and reversal of orientation) closing to unknotted circles, are the two depicted in Figure 3. The preimages of K^* in the two cases are precisely, the $(3, \pm m)$ torus knot and the $(3, \pm m)$ Turk's head knot. □

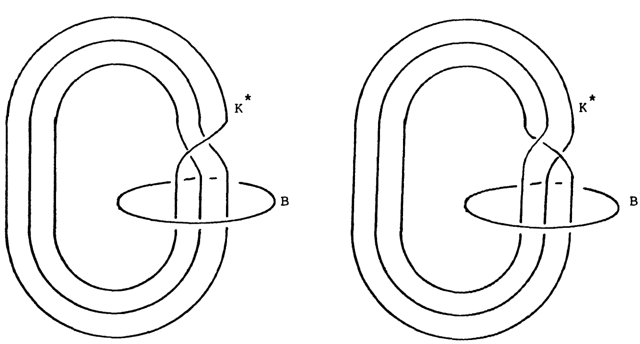

Figure 3

Remark. In working with specific examples the following observation, based on a fundamental result of Murasugi [15], is often helpful. If a knot K in S^3 is invariant under an action of the cyclic group C_m and the linking number of the axis with K is λ, then λ is completely determined by m. In the case of fibered knots this readily implies that the genus of the quotient fibered knot K^* is determined by m as well.

To see this assertion about λ, let p be any prime divisor of m, of order r in m. According to Murasugi

$$\Delta_K(t) \equiv [\delta_\lambda(t)]^{p^r-1}[f(t)]^{p^r} \bmod p,$$

where $\delta_\lambda(t)=(t^\lambda-1)/(t-1)$. Notice that $\delta_\lambda(t)$ has no repeated factors as a polynomial with coefficients in $\mathbb{Z}_p$. This is because the derivative of $t^\lambda-1$ has no nontrivial roots, since λ is relatively prime to p.

If there were two C_m actions on (S^3, K) with axes having different linking numbers with K, say λ_1 and λ_2, then we would have

$$[\delta_{\lambda_1}(t)]^{p^r-1}[f_1(t)]^{p^r} \equiv [\delta_{\lambda_2}(t)]^{p^r-1}[f_2(t)]^{p^r} \bmod p.$$

Now suppose $g(t)$ is an irreducible factor of this product, of order n. Then one has

$$n=\varepsilon_1(p^r-1)+\delta_1 p^r=\varepsilon_2(p^r-1)+\delta_2 p^r,$$

where ε_i is the order of g in δ_{λ_i} and δ_i is the order of g in f_i. Since δ_{λ_i} has no repeated roots, ε_i is 0 or 1. One concludes that $\varepsilon_1=\varepsilon_2$, and hence that the factorization of δ_{λ_1} is the same as that of δ_{λ_2}. In particular they have the same degree, so $\lambda_1=\lambda_2$.

The preceding uniqueness results do not hold in the presence of larger numbers of branch points, as the following example (inspired by a similar construction of Morton [14]) shows.

EXAMPLE. There exist infinitely many distinct genus two fibered knots with C_2 actions. Consider the link $\{K^*, B_n\}$ in Figure 4, where n denotes the number of half twists. For all n, B_n is unknotted. Let K_n be the preimage of K^* in S^3 under the two-fold cover of S^3 branched along B_n. Then K_n is a genus two fibered knot with a C_2 action.

The Alexander polynomial of K_n can be readily computed to be

$$\Delta_{K_n}(t)=t^4+(n^2+n-1)t^3+(-2n^2-2n+1)t^2+(n^2+n-1)t+1.$$

Thus the knots K_n are all distinct, for $n>0$.

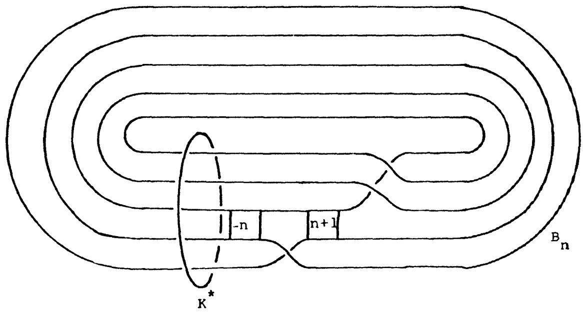

Figure 4

To compute $\Delta_{K_n}(t)$ proceed as follows: $S^3 - K^*$ is fibered by disks and hence $S^3-(K^* \cup B_n)$ is fibered by 5 times punctured disks. The monodromy for that fibration is the product of twists σ_i with each σ_i corresponding to one half twist in the braid B_n. The monodromy for the genus 2 fibration of K_n is the product of the lifts of the σ_i to the branched covering space. The Alexander polynomial of K_n is the characteristic polynomial of the monodromy. Details of the calculation are left to the reader.

In cases where the quotient knot has positive genus further complexities arise.

EXAMPLE. Consider the two unknotted curves B_1 and B_2 in the complement of the trefoil knot K^* as depicted in Figure 5. In each case it is easy to check that B_i can be made transverse to the standard fibration of the trefoil complement, meeting each fiber exactly once. The preimage K_1 of K^* under the two-fold cover of S^3 branched along B_1 is the granny knot. The preimage K_2 of K^* under the two-fold cover of S^3 branched along B_2 is the knot 8_{21} in the standard knot tables [17].

In [9; 1.28] D. Goldsmith posed the following question. Suppose $p: M \to S^3$ is a cyclic cover of degree m branched over a link B; suppose K^* is an unknotted

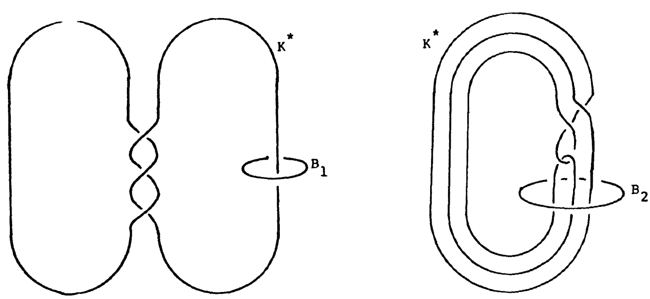

Figure 5

simple closed curve such that $K = p^{-1}(D^*)$ is a fibered knot in M. Is B isotopic to a braid about K^*? (The same question makes sense if K^* is just a fibered knot.) When M is a rational homology sphere, this is an immediate consequence of Theorem 5.2. If the degree m is a prime power and B is connected, then M is necessarily a rational homology sphere (see [2; §4]), so the answer is also yes in this case.

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Classification des feuilletages totalement géodésiques de codimension un

ETIENNE GHYS

Introduction

Soit $\mathcal{F}$ un feuilletage de codimension un sur une variété M. Pour étudier $\mathcal{F}$, il peut être utile de construire une métrique riemannienne sur M adaptée au feuilletage. Par exemple, on peut chercher une métrique telle que les feuilles de $\mathcal{F}$ soient des sous-variétés minimales. Ce problème a été résolu dans [Sul], où il est montré qu'une telle métrique existe relativement fréquemment (précisément lorsqu'aucune réunion finie de feuilles compactes orientées ne borde un domaine de M. Voir aussi [Hae], [Rum].)

De manière analogue, on peut chercher une métrique telle que $\mathcal{F}$ soit totalement géodésique, c'est à dire telle que toute géodésique de M tangente à $\mathcal{F}$ en un point soit entièrement contenue dans une feuille. Cela revient à imposer l'annulation en tout point de la seconde forme quadratique fondamentale des feuilles. Divers auteurs ont considéré le problème de l'existence d'une telle métrique ainsi que le problème inverse d'existence d'un feuilletage totalement géodésique sur une variété riemannienne fixée. Ceux-ci obtiennent certaines obstructions portant sur la variété ([Abe], [Bri], [Dom], [Fer]), sur les classes caractéristiques ([Joh–Nav]) ou encore sur le comportement qualitatif de $\mathcal{F}$([Blu–Heb], [Joh–Whi]).

Dans un travail précédent, en collaboration avec Yves Carrière ([Car–Ghy]), nous avions étudié ce même problème en dimension trois. Rappelons le résultat obtenu.

Soit A une matrice diagonalisable de $SL_2(\mathbb{Z})$. Le feuilletage du plan $\mathbb{R}^2$ par droites parallèles à l'une des directions propres de A définit un feuilletage sur le tore $T^2=\mathbb{R}^2/\mathbb{Z}^2$. Par produit, nous obtenons un feuilletage de codimension 1 sur $T^2\times\mathbb{R}$ clairement invariant par l'application:

$$(m, t)\in T^2\times\mathbb{R}\mapsto(A(m), t+1)\in T^2\times\mathbb{R}.$$

La variété quotient, notée T^3_A est donc munie d'un feuilletage de codimension

1. (cf [Ghy–Ser] pour une description plus précise de ce feuilletage). Le résultat du travail précédemment cité est le

THEOREME. *Soit $\mathcal{F}$ un feuilletage de codimension* 1 (*orientable, de classe* C^∞) *sur une* 3-*variété fermée orientable. Alors, il existe une métrique riemannienne telle que $\mathcal{F}$ soit totalement géodésique si et seulement si*

· *soit $\mathcal{F}$ est transverse à une action localement libre du cercle* (*i.e. une fibration de Seifert*)

· *soit la variété ambiante est difféomorphe à T^3_A* (*pour une certaine matrice* A) *et le feuilletage $\mathcal{F}$ est différentiablement conjugué au feuilletage décrit ci-dessus.*

En un certain sens, il y a donc peu de feuilletages totalement géodésiques en dimension 3. Le but de ce travail est de montrer que cette pauvreté relative subsiste en dimension supérieure. Plus précisément, nous nous proposons de *décrire explicitement tous les feuilletages totalement géodésiques de codimension* 1 *sur les variétés riemanniennes compactes et sur certaines variétés non compactes.*

Commençons par imiter la construction précédente, de façon à obtenir un certain nombre d'exemples de feuilletages de codimension 1. Pour cela, supposons donnés

1°) un entier $n \geqslant 2$

2°) un vecteur v de $\mathbb{R}^n$ dont les coordonnées sont linéairement indépendantes sur $\mathbb{Q}$.

3°) une forme linéaire ω sur $\mathbb{R}^n$ telle que $\omega(v) \neq 0$.

A ces trois premières données, nous pouvons associer le groupe $G(v, \omega)$ formé des matrices A de $SL(n, \mathbb{Z})$ telles que v soit vecteur propre de A et ω vecteur propre de ${}^tA$. De plus, nous pouvons considérer le groupe $\tilde{G}(v, \omega)$ formé des transformations affines du tore T^n de la forme

$$x \in \mathbb{R}^n/\mathbb{Z}^n \to Ax + b \in \mathbb{R}^n/\mathbb{Z}^n$$

où

$$A \in G(v, \omega) \text{ et } b \in \mathbb{R}^n/\mathbb{Z}^n.$$

Supposons de plus que nous disposons

4°) d'une variété B

5°) d'un morphisme φ du groupe fondamental de B dans $\tilde{G}(v, \omega)$. Pour simplifier les notations, nous noterons (D) le quintuplet $(n, v, \omega, B, \varphi)$. A l'aide de (D), nous pouvons faire la construction naturelle suivante. Le tore T^n est muni du feuilletage linéaire de codimension 1 défini par la forme ω. Soit $\tilde{B}$ le revêtement universel de B. Par produit, nous obtenons un feuilletage de codimension 1 sur

$\tilde{B} \times T^n$ évidemment invariant par toutes les transformations

$$(x, y) \in \tilde{B} \times T^n \mapsto (\gamma \cdot x, \varphi(\gamma)(y)) \in \tilde{B} \times T^n$$

(Ici γ représente un élément du groupe fondamental de B et $\gamma \cdot x$ l'action correspondante sur $\tilde{B}$.)

On obtient, par passage au quotient, une variété feuilletée $(M_D, \mathcal{F}_D)$ que nous appellerons "le feuilletage modèle associé à (D)." La variété M_D fibre sur B avec une fibre difféomorphe à T^n. Remarquons que le vecteur v définit sur T^n un champ de directions invariant par l'action de $\tilde{G}(v, \omega)$. On obtient donc un feuilletage canonique $\mathcal{G}_D$, de dimension 1, transverse à $\mathcal{F}_D$.

Nous pouvons maintenant formuler notre résultat.

THEOREME 1. *Soit $\mathcal{F}$ un feuilletage de codimension* 1, *transversalement orientable, de classe C^∞, sur une variété compacte orientable M. Il existe une métrique riemannienne sur M telle que $\mathcal{F}$ soit totalement géodésique si et seulement si:*

I) *soit $\mathcal{F}$ est transverse à une action localement libre du cercle (c'est à dire aux fibres d'une fibration de Seifert généralisée sur M).*

II) *soit M est difféomorphe à M_D et $\mathcal{F}$ est différentiablement conjugué à un "feuilletage modèle" associé à un certain quintuplet $(D) = (n, v, \omega, B, \varphi)$.*

Lorsque la variété M n'est pas compacte mais complète, nous obtenons le résultat partiel suivant:

THEOREME 2. *Soit $\mathcal{F}$ un feuilletage de codimension* 1, *transversalement orientable sur une variété orientable et non compacte M. Supposons que l'une des conditions suivantes est réalisée:*

1) *le groupe fondamental de M est de type fini et le feuilletage est de classe C^∞.*

2) *le feuilletage $\mathcal{F}$ est analytique.*

Alors, il existe une métrique riemannienne complète sur M telle que $\mathcal{F}$ soit totalement géodésique si et seulement si:

I) *soit $\mathcal{F}$ est transverse à une action localement libre du cercle.*

II) *soit M est difféomorphe à M_D et $\mathcal{F}$ est différentiablement conjugué à un feuilletage modèle associé à un certain quintuplet $(n, v, \omega, B, \varphi)$*

III) *soit $\mathcal{F}$ est transverse à une fibration (triviale) de M de fibre $\mathbb{R}$ et de base B dont la restriction à chaque feuille est un revêtement.*

Les étapes essentielles de la démonstration sont les suivantes: si $\mathcal{F}$ est totalement géodésique, le flot orthogonal est riemannien (partie 1), ce qui impose une structure très rigide pour ce flot. Dans un certain fibré principal $\hat{M}$ au-dessus de M, les adhérences des orbites du flot orthogonal (relevé dans $\hat{M}$) fibrent $\hat{M}$. Ces fibres sont des tores. En étudiant la trace sur ces tores du feuilletage $\mathcal{F}$ relevé dans $\hat{M}$, on voit apparaître une distinction entre les cas I et II (partie 2). Dans la partie 3, nous traitons le cas des feuilletages de type I. Pour les feuilletages de type II, nous montrons (partie 4) que la structure induite par le feuilletage sur un tore est essentiellement affine, ce qui nous permet de réduire le groupe structural de la fibration étudiée à un groupe dont l'homotopie est simple (seul le Π_1 est non nul). Il s'agit ensuite de déformer la métrique riemannienne à travers des métriques rendant $\mathcal{F}$ géodésique de manière à ce que le groupe structural de la nouvelle fibration en tores devienne discret (partie 5). Les parties 6 et 7 permettent de "redescendre" les résultats obtenus de $\hat{M}$ dans M et règlent le cas où la variété est non compacte. On donne enfin (partie 8) quelques corollaires et remarques.

C'est grâce à de nombreuses discussions avec Yves Carrière que ce travail a pu être réalisé. Sans son étude très détaillée des flots riemanniens, il aurait été impossible d'aborder ce problème. Je le remercie pour son intérêt et son amitié.

I. Feuilletages totalement géodésiques et flots riemanniens

Soit donc $\mathcal{F}$ un feuilletage de codimension 1, de classe C^∞, transversalement orientable sur une variété orientable M que nous supposerons compacte pour commencer. Soit g une métrique riemannienne sur M et $\mathcal{F}^\perp$ le feuilletage de dimension 1 orthogonal à $\mathcal{F}$. Dans [Car–Ghy], nous remarquions que:

PROPOSITION 1-1. *$\mathcal{F}$ est totalement géodésique pour la métrique* g *si et seulement si* g *est quasi-fibrée pour $\mathcal{F}^\perp$. Il existe une métrique riemannienne telle que $\mathcal{F}$ soit totalement géodésique si et seulement si $\mathcal{F}$ est transverse à un flot riemannien.*

En ce qui concerne les notions de métriques quasi-fibrées, de flots riemanniens, introduites par [Rei], nous référons à [Car 1–2] qui utilise le même language que le notre. En particulier, nous appellerons fréquemment "flot" un feuilletage de dimension 1 même si aucun paramétrage de ce feuilletage n'est donné.

La proposition 1-1 signifie que l'étude des feuilletages totalement géodésiques se ramène à celle des flots riemanniens qui admettent un feuilletage transverse de codimension 1. Or la structure des flots riemanniens est assez bien connue grâce à [Mol] et surtout grâce à [Car 1–2]. Résumons les résultats essentiels.

THEOREME 1-2 ([Mol]). *Soit $\mathcal{G}$ un feuilletage riemannien sur une variété compacte M, alors les adhérences des feuilles de $\mathcal{G}$ sont des sous-variétés et forment une partition de M.*

THEOREME 1-3 ([Car–Car], [Car 1–2]). *Soit $\mathcal{G}$ un flot riemannien* (ie $\dim \mathcal{G} = 1$) *sur une variété compacte M, alors les adhérences des orbites de $\mathcal{G}$ sont des tores T^n et $\mathcal{G}$ restreint à chaque adhérence est conjugué à un flot linéaire de T^n.*

Malheureusement les adhérences des orbites de $\mathcal{G}$ ne fibrent pas toujours M (leur dimension peut varier). Un procédé efficace pour éliminer ce problème est le suivant: soit $\hat{M}$ la variété fibrée au dessus de M dont la fibre au dessus du point x de M est constituée des repères orthonormés de l'orthogonal de $T_x(\mathcal{G})$ dans $T_x(M)$. Il est clair que $\hat{M}$ est un $SO(p)$ fibré principal où $p = \dim M - \dim \mathcal{G}$.

THEOREME 1-4 ([Mol]). *Il existe un feuilletage naturel $\hat{\mathcal{G}}$ sur $\hat{M}$ de même dimension que $\mathcal{G}$ tel que:*

1°) *$\hat{\mathcal{G}}$ est invariant par l'action de $SO(p)$ sur $\hat{M}$*

2°) *$\hat{\mathcal{G}}$ se projette sur $\mathcal{G}$ dans M*

3°) *$\hat{\mathcal{G}}$ est transversalement parallélisable complet; en particulier les adhérences des feuilles de $\hat{\mathcal{G}}$ fibrent $\hat{M}$ et la restriction de $\hat{\mathcal{G}}$ à l'une de ces adhérences admet une structure transverse de Lie.*

Rappelons qu'un feuilletage admet une structure transverse de Lie si son pseudo-groupe transverse est un pseudo-groupe de translations sur un groupe de Lie. Un feuilletage est transversalement parallélisable complet s'il existe p champs de vecteurs complets et transverses $X_1, \ldots, X_p$, tels que, d'une part, ils forment une base du fibré normal en chaque point et que, d'autre part, les flots associés soient des automorphismes du feuilletage.

La fibration de $\hat{M}$ ainsi obtenue s'appelle "la fibration basique."

Soit maintenant $\mathcal{F}$ un feuilletage totalement géodésique sur M. Au flot riemannien $\mathcal{F}^\perp$ correspond le fibré principal $\hat{M}$ muni du flot riemannien $\hat{\mathcal{F}}^\perp$. Bien entendu, nous pouvons considérer l'image réciproque du feuilletage $\mathcal{F}$ dans $\hat{M}$. Le feuilletage $\hat{\mathcal{F}}$ (de codimension 1) ainsi obtenu est évidemment totalement géodésique, son flot orthogonal étant $\hat{\mathcal{F}}^\perp$. L'adhérence d'une orbite de $\hat{\mathcal{F}}^\perp$ est un tore T^n et ces adhérences fibrent $\hat{M}$. Enfin, sur chacune de ces adhérences, le flot $\hat{\mathcal{F}}^\perp$ est conjugué à un flot linéaire et il est donc transversalement de Lie modelé sur le groupe $\mathbb{R}^{n-1}$. Le couple $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ peut donc être considéré comme une "désingularisation" de $(\mathcal{F}, \mathcal{F}^\perp)$. Nous étudierons d'abord $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ pour redescendre ensuite dans M.

Nous sommes déjà en mesure de démontrer la partie facile du théorème que nous avons en vue. La proposition suivante est déjà dans [Car–Ghy].

PROPOSITION 1-5. *Tout feuilletage transverse à une action localement libre de S^1 est totalement géodésique pour une certaine métrique riemannienne complète.*

DEMONSTRATION. Une action de S^1 peut être rendue isométrique, donc riemannienne.

PROPOSITION 1-6. *Soit $\pi : M \to B$ une fibration (triviale) de fibre $\mathbb{R}$. Soit $\mathcal{F}$ un feuilletage transverse à π tel que la restriction de π à toute feuille de $\mathcal{F}$ est un revêtement. Alors il existe une métrique riemannienne complète telle que $\mathcal{F}$ soit totalement géodésique.*

DEMONSTRATION. On part d'une métrique complète sur B que l'on transporte sur les feuilles de $\mathcal{F}$ à l'aide de π. On étend la métrique ainsi construite en imposant aux fibres de π d'être orthogonales à $\mathcal{F}$. Puisque le choix de la métrique sur les fibres de π est arbitraire, on peut imposer à ces fibres d'avoir une longueur infinie de façon à obtenir une métrique complète sur M.

PROPOSITION 1-7. *Les feuilletages modèles $\mathcal{F}_D$ sont totalement géodésiques (pour une certaine métrique complète de M_D).*

DEMONSTRATION. Il suffit de montrer que le supplémentaire canonique $\mathcal{G}_D$ à $\mathcal{F}_D$ correspondant au vecteur v est un flot riemannien. On considère tout d'abord une métrique complète sur B et l'on munit M d'une métrique telle que la projection p de M_D sur B soit riemannienne. Soit U un ouvert trivialisant pour p. Les fibres de p au dessus de U peuvent être définies par une action de T^n sur $p^{-1}(U)$. En considérant la moyenne de al métrique dont nous disposons sur $p^{-1}(U)$ sous l'action de T^n, nous obtenons une métrique quasi-fibrée pour la restriction de $\mathcal{G}_D$ à $p^{-1}(U)$. En utilisant une partition de l'unité sur B, on construit une métrique quasi-fibrée complète sur M_D.

Tous les feuilletages cités dans le théorème sont donc bien totalement géodésiques.

Peut-être est-il bon de donner un exemple simple illustrant les constructions qui vont suivre. Soit X un champ de Killing sur S^2; ce champ possède deux singularités et toutes ses orbites sont fermées (de période 1 par exemple). Munissons $M = S^2 \times S^1$ d'une métrique riemannienne induisant sur chaque facteur $S^2 \times \{*\}$ la métrique usuelle et telle que le champ de vecteurs unitaires orthogonal à ces sphères $S^2\times\{*\}$ soit $N=\varepsilon X+\partial/\partial\theta$ où $\varepsilon\in\mathbb{R}$ et $\partial/\partial\theta$ représente le champ habituel sur S^1.

Le feuilletage $\mathcal{F}$ de M par sphères S^2 est totalement géodésique, l'adhérence d'une orbite de N est un tore T^2 sauf pour deux orbites fermées si ε est irrationnel. Si ε est rationnel toutes les orbites de N sont fermées et munissent M d'une fibration de Seifert. Le passage de M à $\hat{M}$ est ici le passage de $S^2\times S^1$ à $T_1(S^2)\times S^1\simeq SO(3)\times S^1$, le feuilletage $\hat{\mathcal{F}}$ est le feuilletage par fibres $SO(3)\times\{*\}$ et les adhérences des orbites de $\hat{\mathcal{F}}^\perp$ définissent maintenant une véritable fibration (en cercles si $\varepsilon\in\mathbb{Q}$, en tores T^2 sinon). La déformation de métrique que nous ferons plus loin consistera dans ce cas à approcher le réel ε par un rationnel.

II. La trace du feuilletage sur une fibre

Comme nous l'avons indiqué au paragraphe précédent, nous commençons par étudier la situation relevée dans $\hat{M}$. D'une manière générale, nous allons nous intéresser au groupe structural de la fibration basique. Soit F une fibre de cette fibration. Nous savons que F est difféomorphe à T^n et que la restriction de $\hat{\mathcal{F}}^\perp$ à F est linéaire. Etudions maintenant la trace de $\hat{\mathcal{F}}$ sur F.

PROPOSITION 2-1. *La restriction de $\hat{\mathcal{F}}$ à F peut être définie par une action localement libre de $\mathbb{R}^{n-1}$. Cette action préserve $\hat{\mathcal{F}}^\perp_{|F}$ et elle est définie à un automorphisme de $\mathbb{R}^{n-1}$ près. Trois cas sont possibles:*

type I_a: Toutes les feuilles de $\hat{\mathcal{F}}_{|F}$ sont compactes et $\hat{\mathcal{F}}_{|F}$ peut être défini par une action libre de T^{n-1}.

type I_b: Le feuilletage $\hat{\mathcal{F}}_{|F}$ possède au moins une feuille compacte et une feuille non compacte.

type II: Les feuilles de $\hat{\mathcal{F}}_{|F}$ sont denses. Il existe un homéomorphisme de T^n sur F linéarisant $\hat{\mathcal{F}}$ et $\hat{\mathcal{F}}^\perp$, ie tel que, dans ces coordonnées $\hat{\mathcal{F}}^\perp_{|F}$ soit constitué des droites parallèles à un vecteur v fixe de $\mathbb{R}^n$ (à coordonnées linéairement indépendantes sur $\mathbb{Q}$) et $\hat{\mathcal{F}}_{|F}$ soit constitué des hyperplans parallèles à un hyperplan fixe d'équation $\omega=0$ où ω est une forme linéaire sur $\mathbb{R}^n$.

DEMONSTRATION. La première partie résulte du fait que la restriction de $\hat{\mathcal{F}}^\perp$ à une fibre F est transversalement de Lie $\mathbb{R}^{n-1}$. Cela signifie que $\hat{\mathcal{F}}^\perp_{|F}$ peut être défini par des submersions locales sur $\mathbb{R}^{n-1}$, les changements de cartes opérant par translations sur $\mathbb{R}^{n-1}$. On peut donc définir sans ambiguïté $n-1$ champs de vecteurs sur F, tangents à $\hat{\mathcal{F}}_{|F}$, commutant deux à deux, et se projetant localement sur les champs $\dfrac{\partial}{\partial x_1},\ldots,\dfrac{\partial}{\partial x_{n-1}}$ de $\mathbb{R}^{n-1}$. L'action localement libre ainsi construite préserve $\hat{\mathscr{F}}^{\perp}_{|F}$ puisque ces champs de vecteurs sont localement projetables sur $\mathbb{R}^{n-1}$.

La deuxième partie est une propriété bien connue des actions localement libres de $\mathbb{R}^{n-1}$ sur T^n. Approchant $\hat{\mathscr{F}}^{\perp}_{|F}$ par une fibration en cercles, on voit que $\hat{\mathscr{F}}_{|F}$ peut être défini par $(n-1)$ difféomorphismes du cercle commutant deux à deux. Ces difféomorphismes ont donc un point périodique en commun ou bien sont simultanément topologiquement conjugués à des rotations (cf. [Mor–Tsu] par exemple).

Supposons que toutes les feuilles de $\hat{\mathscr{F}}_{|F}$ sont fermées. Le stabilisateur d'un point m de F sour l'action de $\mathbb{R}^{n-1}$ est constant le long des feuilles de $\hat{F}_{|F}$ (car $\mathbb{R}^{n-1}$ est abélien). Ce stabilisateur est par ailleurs constant le long des orbites de $\hat{\mathscr{F}}^{\perp}_{|F}$ puisque l'action de $\mathbb{R}^{n-1}$ préserve $\hat{\mathscr{F}}^{\perp}_{|F}$. Par conséquent le stabilisateur est indépendant du point m et $\hat{\mathscr{F}}_{|F}$ peut être défini par une action libre de T^{n-1}.

Apres avoir étudié une fibre F de la fibration basique, étudions un voisinage de cette fibre.

LEMME 2-2. *Soit $\Pi:\hat{M}\to\hat{B}$ la fibration basique et x un point de $\hat{B}$. Soit F la fibre $\Pi^{-1}(x)$. Il existe un voisinage U de x et une trivialisation Ψ de Π au dessus de U:*

$$\Psi : F\times U \to \Pi^{-1}(U)$$

telle que:

1°) *pour toute feuille L de $\hat{\mathscr{F}}^{\perp}_{|F}$, le difféomorphisme Ψ envoie $L\times\{*\}$ sur une feuille de $\hat{\mathscr{F}}^{\perp}_{|\Pi^{-1}(U)}$.*

2°) *pour toute feuille L' de $\hat{\mathscr{F}}_{|F}$, le difféomorphisme Ψ envoie $L'\times U$ sur une feuille de $\hat{\mathscr{F}}_{|\Pi^{-1}(U)}$.*

DEMONSTRATION. Nous savons que $\hat{\mathscr{F}}^{\perp}$ est un feuilletage transversalement parallélisable complet. Si la dimension de $\hat{B}$ est p, il est donc facile de construire p champs de vecteurs $X_1,\dots,X_p$ sur un voisinage de F dans $\hat{M}$ tels que

1°) les X_i sont tangents à $\hat{\mathscr{F}}$

2°) les flots locaux X_i^t associés aux X_i sont des automorphismes de $\hat{\mathscr{F}}^{\perp}$.

3°) les champs induits sur $\hat{B}$ forment une base de $T(\hat{B})$ au point x.

Dans ces conditions, la fonction

$$[(t_1,\dots,t_p),m]\in\mathbb{R}^p\times U \mapsto X_1^{t_1}X_2^{t_2}\cdots X_p^{t_p}(m)$$

est définie pour $(t_1,\dots,t_p)$ appartenant à un voisinage de $(0,\dots,0)$. Ce voisinage peut être identifié à un voisinage U de x et l'on obtient la trivialisation cherchée.

COROLLAIRE 2-3. *Soit* Diff $(F, \hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}_{|F}^{\perp})$ *le groupe des difféomorphismes de F préservant $\hat{\mathcal{F}}_{|F}$ et $\hat{\mathcal{F}}_{|F}^{\perp}$ muni de la topologie C^∞. Alors le groupe structural de la fibration basique peut se réduire à* Diff $(F, \hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}_{|F}^{\perp})$.

III. Les feuilletages de type I_a et I_b

Grâce à 2-1 et 2-2, nous avons obtenu certaines cartes locales pour la fibration basique Π. Pour étudier les changements de cartes, il nous faut étudier les homéomorphismes d'une fibre préservant la structure $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$.

LEMME 3-1. *Supposons que $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ est de type I_a ou I_b. Soit h un homéomorphisme de $F = \Pi^{-1}(x)$ préservant $\hat{\mathcal{F}}_{|F}$ et $\hat{\mathcal{F}}_{|F}^{\perp}$. Alors h commute avec l'action de $\mathbb{R}^{n-1}$ sur F.*

DEMONSTRATION. Sot L une feuille compacte de $\hat{\mathcal{F}}_{|F}$. Celle ci s'identifie à T^{n-1} à l'aide de l'action de $\mathbb{R}^{n-1}$ nous disposons. Cette identification est unique à une translation près de T^{n-1}. Le flot $\hat{\mathcal{F}}_{|F}^{\perp}$ définit une application de premier retour de L dans L donc de T^{n-1} dans T^{n-1}. Cette application est clairement une translation à orbites denses.

Soit $L' = h(L)$. L'application de premier retour correspondant à L' est évidemment la même translation que celle de L. Par conséquent, puisque h préserve $\hat{\mathcal{F}}^{\perp}$, la restriction de h à L considérée comme homéomorphisme de L sur L' et donc de T^{n-1} sur T^{n-1} doit commuter avec une translation à orbites denses de T^{n-1}. Il est bien connu (et facile de vérifier) que cela implique que h/L est une translation. Dans le cas I_a, toutes les feuilles de $\hat{\mathcal{F}}_{|F}$ sont compactes et h commute avec l'action de $\mathbb{R}^{n-1}$ sur F. Dans le cas I_b, nous n'avons obtenu cette commutation que sur les feuilles compactes de $\hat{\mathcal{F}}_{|F}$. Notons $s.m$ l'action de l'élément s de $\mathbb{R}^{n-1}$ sur le point m de F. Pour tout s de $\mathbb{R}^{n-1}$, l'homéomorphisme

$$g_s : m \in F \to h^{-1}(s \cdot h(s^{-1} \cdot m)) \in F$$

préserve $(\hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}_{|F}^{\perp})$. Lorsque s varie et m reste fixe, le point $g_s(m)$ décrit une courbe tangente à $\hat{\mathcal{F}}_{|F}$. Lorsque m appartient à une feuille compacte, nous venons de voir que ce point ne dépend pas de s. Si maintenant m n'est pas situé sur une feuille compacte on peut construire un segment $[m, m']$ contenu dans $\hat{\mathcal{F}}_{|F}^{\perp}$ et tel que m' soit dans une feuille compacte de $\hat{\mathcal{F}}_{|F}$. En considérant la variation de ce segment par les g_s (s variable), on voit que le point m doit rester fixe, c'est à dire que h commute avec l'action de $\mathbb{R}^{n-1}$.

Nous pouvons maintenant commencer la démonstration du résultat principal de ce paragraphe.

PROPOSITION 3.2. *Si $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ est de type I_a ou I_b, alors $\mathcal{F}$ est transverse à une action localement libre du cercle.*

DEMONSTRATION DANS LE CAS I_a: D'après le lemme 3-1 et le lemme 2-2 on peut définir sans ambiguïté une action de T^{n-1} sur $\hat{M}$ notée $(s, x) \mapsto s.x$ dont les orbites sont exactement les composantes connexes des traces des feuilles de $\hat{\mathcal{F}}$ sur les fibres de la fibration basique. Par ailleurs l'action naturelle de $SO(p)$ sur $\hat{M}$ préservant $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$, elle commute avec l'action de T^{n-1}. Nous disposons donc d'une action de $T^{n-1} \times SO(p)$ sur $\hat{M}$.

Nous nous proposons de construire une fibration en cercles de $\hat{M}$ dont les fibres soient tangentes à la fibration basique et transverses à $\hat{\mathcal{F}}$. Cette fibration doit être $SO(p)$ équivariante de façon à ce qu'elle induise dans M une fibration de Seifert, c'est à dire une action localement libre de S^1 transverse à $\mathcal{F}$.

Munissons $\hat{M}$ d'une métrique invariante par l'action de $T^{n-1} \times SO(p)$. Celle-ci nous permet de paramétrer $\hat{\mathcal{F}}^{\perp}$ et définit donc un flot Ψ_τ transverse à $\hat{\mathcal{F}}$ et commutant avec l'action de $T^{n-1} \times SO(p)$. Considérons, pour chaque m de $\hat{M}$, le temps de premier retour $T(m)$ de l'orbite de Ψ_τ passant par m sur la feuille de $\hat{\mathcal{F}}_{|\Pi^{-1}(\Pi(m))}$ passant par m. Puisque $\Psi_{T(m)}(m)$ et m appartiennent à la même orbite de l'action de T^{n-1}, il existe $s(m)$ de T^{n-1} tel que

$$\Psi_{T(m)}(m) = s(m) \cdot m$$

Il est clair que $T(m)$ et $s(m)$ ne dépendent en fait que de la projection $\Pi(m)$ de m dans la base de la fibration basique. En fait, le lemme de trivialisation 2-2 et le lemme 3-1 montrent que $s(m)$ ne dépend pas de m. Nous pouvons alors construire une action de S^1 sur M par:

$$\begin{aligned} &\mathbb{R}/\mathbb{Z} \times \hat{M} \to \hat{M} \\ &(\theta, m) \mapsto (-\theta s(m)) \cdot \Psi_{\theta T(m)}(m) \end{aligned}$$

(A priori $\theta s(m)$ ne signifie rien puisque $\theta \in \mathbb{R}$ et $s \in T^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, mais nous choisissons un représentant quelconque de $s(m)$ dans $\mathbb{R}^{n-1}$, ce qui ne pose aucune difficulté puisque $s(m)$ est en fait constant).

Ceci est bien une action de S^1 telle que nous la souhaitions.

Avant d'étudier le cas I_b, rappelons un résultat de [Car–Ghy], obtenu aussi sous une forme un peu différente dans [Joh–Whi].

PROPOSITION 3-3. *Soit $\mathcal{F}$ un feuilletage totalement géodésique de codimension 1. Si $\mathcal{F}$ a une feuille compacte, alors $\mathcal{F}$ est transverse à une action localement libre de S^1.*

(Il suffit de remarquer que la feuille compacte est une section globale pour le flot $\mathcal{F}^\perp$ et celui ci est donc défini par la suspension d'une isométrie d'une variété compacte donc approchable par une isométrie périodique.)

Pour démontrer la proposition 3–2 dans le cas I_b, il nous suffit donc de démontrer le

LEMME 3-4. *Si $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ est de type I_b, alors $\hat{\mathcal{F}}$ (et donc $\mathcal{F}$) possède une feuille compacte.*

Fixons une fibre F de la fibration basique et soient m_1 et m_2 deux points de F. Commençons par montrer le résultat suivant:

LEMME 3-5. *Si m_1 et m_2 appartiennent à la même feuille de $\hat{\mathcal{F}}$, alors il existe un difféomorphisme h de F tel que $h(m_1) = m_2$ et h préserve $\hat{\mathcal{F}}_{|F}$ et $\hat{\mathcal{F}}^\perp_{|F}$.*

DEMONSTRATION DU LEMME 3-5. Si $\gamma : [0, 1] \to \hat{M}$ est un chemin de $\hat{M}$ tangent à $\hat{\mathcal{F}}$ et reliant m_1 et m_2, la projection de γ dans $\hat{B}$ nous fournit un lacet $\gamma' : S^1 \to \hat{B}$. L'image réciproque de la fibration basique par γ' nous donne un T^n-fibré au dessus de S^1 dont la monodromie est le difféomorphisme souhaité. (On peut supposer cette monodromie dans Diff $(F, \hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}^\perp_{|F})$ d'après le corollaire 2.3).

Si l'on fait la même construction que dans le lemme précédent dans le cas où $m_1 = m_2$ et γ un lacet tangent à $\hat{\mathcal{F}}$, et si l'on utilise le lemme 3-1, on obtient le

LEMME 3-6. *Si c est un lacet de F tangent à $\hat{\mathcal{F}}_{|F}$, alors c considéré comme lacet d'une feuille de $\hat{\mathcal{F}}$, est dans le centre du groupe fondamental de cette feuille de $\hat{\mathcal{F}}$.*

DEMONSTRATION DU LEMME 3-4. (et donc de la proposition 3-2 dans le cas I_b via la proposition 3-3).

Rappelons tout d'abord que $\hat{\mathcal{F}}_{|F}$ peut être défini par la suspension de $(n-1)$ difféomorphismes du cercle de classe C^∞ et commutant deux à deux.

Soit $K \subset F$ la réunion des feuilles compactes de $\hat{\mathcal{F}}_{|F}$. D'après les hypothèses, K est un fermé qui n'est ni vide ni égal à F. Soit K_1 la frontière de K. Tout difféomorphisme de F préservant $\hat{\mathcal{F}}_{|F}$ préserve K et donc K_1. D'après le lemme 3-5, K_1 est la trace sur F d'un fermé de $\hat{M}$ qui est saturé par $\hat{\mathcal{F}}$. Ce fermé doit contenir un ensemble minimal de $\hat{\mathcal{F}}$. Si l'on suppose que $\hat{\mathcal{F}}$ ne possède pas de feuilles compactes, c'est donc que K_1 contient la trace sur F d'un minimal exceptionnel de $\hat{\mathcal{F}}$.

D'après le théorème de Sacksteder ([Sac]) une feuille de ce minimal contient un lacet dont l'holonomie est hyperbolique (c'est à dire dont la dérivée au point fixe est différente de 1). D'après [Ste], on peut supposer que ce germe d'holonomie est linéaire. Rappelons par ailleurs que tout germe de difféomorphisme qui commute avec une contraction linéaire est lui même linéaire et donc hyperbolique s'il n'est pas l'identité.

Observons que l'holonomie des feuilles de $K_1 \subset F$ est non triviale car arbitrairement près d'une feuille de K_1, il y a des feuilles non compactes. En combinant l'observation faite précédemment avec le lemme 3-6, on en déduit qu'il existe un lacet c tangent à une feuille de K_1 dont l'holonomie est hyperbolique. Soit λ la dérivée de cette holonomie au point fixe. Remarquons que toutes les feuilles compactes de $\hat{\mathcal{F}}_{|F}$ sont des tores T^{n-1} et que leurs groupes fondamentaux sont tous canoniquement isomorphes. Dans chaque feuille compacte de $\hat{\mathcal{F}}_{|F}$, nous pouvons construire un lacet homotope à c et considérer l'holonomie de ce lacet. On obtient ainsi un germe de difféomorphisme de $\mathbb{R}$ ayant un point fixe. Soit $K_2 \subset K_1$ la réunion des feuilles de K_1 telles que la dérivée de ce germe soit égale à λ en son point fixe. Il est clair que K_2 est invariant par tout difféomorphisme de F préservant $(\hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}^{\perp}_{|F})$ (On utilise ici encore le lemme 3-1 impliquant qu'un tel difféomorphisme doit préserver la classe d'homotopie de c.)

Mais K_2 ne peut contenir qu'un nombre fini de feuilles compactes de $\hat{\mathcal{F}}_{|F}$. En effet, le difféomorphisme C^∞ du cercle associé à c ne peut avoir qu'un nombre fini de points fixes où sa dérivée est égale à λ. (La variation totale de la dérivée doit être finie.)

Nous avons donc trouvé un nombre fini de feuilles compactes de $\hat{\mathcal{F}}_{|F}$ invariantes par tous les difféomorphismes de F préservant $(\hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}^{\perp}_{|F})$. Cette réunion finie contient la trace sur F d'une feuille compacte de $\hat{\mathcal{F}}$ d'après 3.5.

IV. Réduction du groupe structural dans le cas II

De même que pour le cas I, il nous faut obtenir des informations sur les homéomorphismes des fibres basiques préservant le couple de feuilletages $(\hat{\mathcal{F}}_{|F}, \hat{\mathcal{F}}^{\perp}_{|F})$.

LEMME 4-1. *Soit h un homéomorphisme du tore T^n tel que*

1°) *h préserve les orbites du flot linéaire parallèle au vecteur v à coordonnées linéairement indépendantes sur $\mathbb{Q}$.*

2°) *h préserve le feuilletage linéaire de codimension 1 transverse à v, à feuilles denses, défini par la forme linéaire $\omega = 0$.*

Alors h est en fait un difféomorphisme affine du type $h(x) = Ax + b$ où $A \in G(v, \omega) \subset SL(n, \mathbb{Z})$ et $b \in T^n = \mathbb{R}^n/\mathbb{Z}^n$.

DEMONSTRATION. Il est bien connu que le feuilletage de T^n d'équation $\omega = 0$ possède, à une constante multiplicative près, une unique mesure transverse invariante (au sens de [Pla]). Cette mesure n'est autre que celle obtenue par intégration de ω. Par conséquent, cette mesure transverse est multipliée par une constante λ sous l'action de l'homéomorphisme h. C'est-à-dire que h est "transversalement affine." Puisque h préserve les orbites du flot linéaire parallèle à v, on en déduit que la restriction de h à toute droite parallèle à v est affine de pente λ.

Soit d'autre part A la matrice de $SL(n, \mathbb{Z})$ induite par h sur $H_1(T^n, \mathbb{Z}) \simeq \mathbb{Z}^n$. Comme h préserve deux feuilletages linéaires, il est facile de voir que A préserve les "nombres de rotation" de ces feuilletages. Plus précisément v est vecteur propre de A (car v est le "vecteur de rotation" du feuilletage linéaire de dimension 1 parallèle à v). De même ω est vecteur propre de ${}^tA$ (car ω est la "forme de rotation" du feuilletage défini par ω). Il est même clair que ${}^tA(\omega) = \lambda\omega$ d'après l'interprétation de ω comme mesure transverse faite plus haut.

Considérons maintenant l'homéomorphisme $h' = A^{-1} \circ h$. Il préserve le couple de feuilletages et, restreint à chaque orbite du flot parallèle à v, c'est une translation. En composant h' avec une translation adéquate, on obtient un homéomorphisme h'' ayant un point fixe, donc toute une droite dense parallèle à v formée de points fixes, donc h'' est l'identité. Par conséquent h était affine.

Introduisons maintenant un entier k dont le rôle sera important par la suite. Les feuilles de $\omega = 0$ sont obtenues par une action localement libre de $\mathbb{R}^{n-1}$, ce sont donc des "cylindres" du type $T^k \times \mathbb{R}^{n-1-k}$. Nous appellerons k l'"invariant de $(\hat{\mathscr{F}}, \hat{\mathscr{F}}^\perp)$." De manière équivalente si ω s'écrit $\sum_{i=1}^n a_i\, dx_i$ dans une base rationnelle de $\mathbb{R}^n$, alors $n-k$ est le rang sur $\mathbb{Q}$ du système $\{a_i\}$.

Résumons maintenant les lemmes 2-1, 2-2 et 4-1 en indiquant ce qu'ils signifient pour la fibration basique $\Pi : \hat{M} \to \hat{B}$. Tout point x de $\hat{B}$ admet un voisinage trivialisant V et une carte (lemme 2-2)

$$V \times F \to \Pi^{-1}(V)$$

Identifiant F à T^n à l'aide de l'homéomorphisme "linéarisant" du lemme 2-1, nous obtenons une carte

$$V \times T^n \to \Pi^{-1}(V)$$

Sur l'intersection de deux ouverts trivialisants V_1 et V_2, les changements de cartes doivent être du type:

$$(x, m) \in V_1 \times T^n \mapsto (x, \psi_x(m)) \in V_2 \times T^n$$

pour x dans $V_1 \cap V_2$. D'après 4-1, $\psi_x(m)$ s'écrit $\psi_x(m) = A_x(m) + b(x)$, où $b(x) \in \mathbb{R}^n/\mathbb{Z}^n$ et $A_x \in G(v, \omega)$. Puisque le changement de carte doit préserver le feuilletage $\hat{\mathcal{F}}$, on voit que la courbe

$$x \in V \mapsto b(x) \in \mathbb{R}^n/\mathbb{Z}^n$$

doit rester tangente à un hyperplan parallèle à $\omega = 0$.

Nous allons munir $\tilde{G}(v, \omega)$ d'une topologie de la manière suivante. Le groupe de Lie T^n est feuilleté par $\omega = 0$, il peut donc être muni de la topologie des feuilles. On peut alors munir $\tilde{G}(v, \omega)$ (qui ensemblistement est $G(v, \omega) \times T^n$) d'une topologie en donnant à $G(v, \omega)$ la topologie discrète et à T^n la topologie des feuilles. Nous appellerons cette topologie la "topologie des feuilles de $\tilde{G}(v, \omega)$."

Ce paragraphe peut maintenant se résumer:

PROPOSITION 4.2. *Le groupe structural de la fibration basique associée à un feuilletage de type II peut se réduire à $\tilde{G}(v, \omega)$ muni de la "topologie des feuilles."*

REMARQUE 4-3. A priori, les cartes que nous avons construites pour la fibration basique ne sont que continues. Cependant, nous pouvons les utiliser pour transporter la structure différentiable de $V \times T^n$ dans $\hat{M}$. Nous obtenons ainsi une nouvelle structure différentiable dans $\hat{M}$ pour laquelle $\hat{\mathcal{F}}$ et $\hat{\mathcal{F}}^\perp$ restent différentiables. De même l'action naturelle de $SO(p)$ sur $\hat{M}$ préservant $\hat{\mathcal{F}}$ et $\hat{\mathcal{F}}^\perp$, le lemme 4-1 nous dit que cette action reste elle aussi différentiable dans cette nouvelle structure. Dorénavant, nous supposerons $\hat{M}$ muni de cette structure. Nous verrons cependant dans la partie VI que ce changement de structure était en fait inutile.

REMARQUE 4-4. La composante connexe de l'identité de $\tilde{G}(v, \omega)$ munie de la topologie des feuilles est homéomorphe à $T^k \times \mathbb{R}^{n-1-k}$. Par conséquent le groupe fondamental de $\tilde{G}(v, \omega)$ est isomorphe à $\mathbb{Z}^k$ et les groupes d'homotopie d'ordres supérieurs sont nuls. L'invariant k est donc l'unique obstruction à ce que $\tilde{G}(v, \omega)$ ait le type d'homotopie d'un groupe discret. Notre but est de ramener l'étude des feuilletages de type II aux feuilletages modèles pour lesquels le groupe structural de la fibration basique est discret. Ceci explique l'importance de l'invariant k dans ce qui suit.

V. Déformation du flot orthogonal

Le but de ce paragraphe est de montrer que si l'invariant k est non nul, on peut déformer $\hat{\mathcal{F}}^{\perp}$ à travers les flots riemanniens pour obtenir un flot $\hat{\mathcal{G}}$ tel que l'invariant k associé à $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ soit nul. En d'autres termes, nous allons perturber la métrique sur la variété de telle sorte que l'adhérence d'une orbite du flot orthogonal baisse de dimension et que la trace de $\hat{\mathcal{F}}$ sur ces nouveaux tores soit par plans (et non par cylindres $T^k \times \mathbb{R}^{n-k-1}$). Evidemment cette déformation de métrique se fait à travers des métriques rendant $\hat{\mathcal{F}}$ totalement géodésique.

Le groupe $\tilde{G}(v, \omega)$ se surjecte sur $\pi_0(\tilde{G}(v, \omega))$ et donc sur $G(v, \omega)$. Ceci permet de construire un morphisme "d'holonomie" de la fibration basique:

$$H: \pi_1(\hat{B}) \rightarrow G(v, \omega)$$

Remarquons que si w est un vecteur propre pour toutes les matrices $H(\gamma)$, nous pouvons sans difficulté construire sur les fibres de la fibration basique, un feuilletage de dimension 1 dont les feuilles sont les droites parallèles à w ce qui a un sens intrinsèque puisque $\mathbb{R} \cdot w$ est fixe par $H(\gamma)$. Notre but est de montrer qu'il existe effectivement de tels vecteurs dès que k est non nul, ce qui nous permettra d'effectuer la déformation souhaitée de $\hat{\mathcal{F}}^{\perp}$. Pour cela, nous utiliserons quelques lemmes d'algèbre linéaire.

LEMME 5-1. *Si $k \neq 0$ et si $A \in G(v, \omega)$, soit λ le réel tel que $A(v) = \lambda v$. Alors λ est un nombre algébrique de degré strictement inférieur à n.*

DEMONSTRATION. Par définition de l'entier k, la forme ω ne dépend que de $n-k$ coordonnées dans une certaine base rationnelle $(e_i)_{i=1,n}$ de $\mathbb{R}^n$. C'est-à-dire que ω s'écrit

$$\omega = \sum_{i=1}^{n-k} \alpha_i e_i^*$$

où e_i^* représente la base duale de e_i et α_i est un réel.

Puisque ω est vecteur propre de ${}^tA$ et $\omega(v) \neq 0$, il est clair que ${}^tA\ \omega = \lambda\omega$. Si l'on note (a_{ij}) les coefficients de ${}^tA$ dans la base e_i^*, cette dernière égalité ainsi que l'écriture de ω montrent que λ est aussi valeur propre de la sous-matrice de ${}^tA$ formée des a_{ij} avec i et j compris entre 1 et $n-k$. Par conséquent, λ est algébrique de degré inférieur à $n-k$.

LEMME 5–2. *Si $k \neq 0$ et si $A \in G(v, \omega)$, alors v n'est pas un vecteur propre simple de A.*

DEMONSTRATION. On suppose toujours que les coordonnées de v sont linéairement indépendantes sur $\mathbb{Q}$. Si v était un vecteur propre simple, les $(n-1)$ dernières lignes du système

$$A(v_1, \ldots, v_n) = \lambda(v_1, \ldots, v_n)$$

avec v_1 fixé et $v_2, \ldots, v_n$ inconnus, formeraient un système de Cramer. Par conséquent les coordonnées $v_2, \ldots, v_n$ pourraient être calculées rationnellement à l'aide de v_1 et de λ. C'est-à-dire que les rapports $\frac{v_2}{v_1}, \ldots, \frac{v_n}{v_1}$ appartiendraient au corps $\mathbb{Q}(\lambda)$. Celui-ci étant de dimension strictement inférieure à n, il existe des rationnels α_i tels que

$$\alpha_1 + \sum_{i=2}^{n} \alpha_i \frac{v_i}{v_1} = 0.$$

Ceci contredit le fait que les v_i sont linéairement indépendants sur $\mathbb{Q}$.

Les deux lemmes précédents nous ont permis de trouver d'autres vecteurs propres pour chaque A de $G(v, \omega)$. Notre but cependant est de trouver un vecteur propre *commun* à tous les $H(\gamma)$ qui soit différent de v. La situation est en fait très simple grâce au

LEMME 5-3. *$G(v, \omega)$ est un groupe abélien libre de rang au plus $n-1$.*

DEMONSTRATION. Considérons le morphisme θ de $G(v, \omega)$ dans $\mathbb{R}^*$ associant à chaque matrice A de $G(v, \omega)$ la valeur propre λ telle que $A(v) = \lambda v$. Ce morphisme est injectif; en effet, si $A(v) = v$, alors le noyau de $A\text{-}Id$ est un sous-espace rationnel de $\mathbb{R}^n$ contenant v, c'est donc $\mathbb{R}^n$ tout entier (nous disons qu'un sous-espace de $\mathbb{R}^n$ est rationnel s'il possède une base formée de vecteurs rationnels). Ceci montre que $G(v, \omega)$ est abélien. On montre de même que si $\theta(A)$ est rationnel, alors $A = id$.

Ecrivons l'égalité $A(v) = \lambda v$ sous la forme

$$\begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ \cdot & \cdots & \cdot \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Ceci permet d'écrire

$$\lambda = a_1 + a_2 \frac{v_2}{v_1} + \cdots + a_n \frac{v_n}{v_1}.$$

L'image de θ est donc contenue dans un $\mathbb{Q}$-sous-espace vectoriel de $\mathbb{R}$ de dimension au plus n. Le groupe $G(v, \omega)$ est donc abélien libre de rang au plus n. Si ce rang était exactement n, on trouverait une matrice A de $G(v, \omega)$ non triviale avec $\theta(A)$ rationnel.

REMARQUE 5-4. Pour v et ω "génériques," le groupe $G(v, \omega)$ est trivial. Lorsqu'il est non trivial, le rang de $G(v, \omega)$ peut effectivement atteindre $n-1$. En effet, soit A une matrice entière dont le polynôme caractéristique est $(-1)^n X(X - b_1) \cdots (X - b_{n-1}) + 1$ où les b_i sont des entiers arbitraires. Alors, les matrices A, $A - b_1 I, \ldots, A - b_{n-1} I$ sont de déterminant 1 et commutent deux à deux. Le produit de ces n matrices est $(-1)^{n+1} I$, mais on peut choisir les entiers b_i de telle sorte qu'elles engendrent un groupe abélien libre de rang $n-1$. On peut par ailleurs choisir les b_i de telle sorte que A soit diagonalisable. On appelle alors v l'un des vecteurs propres et ω la forme linéaire valant 1 sur v et s'annulant sur tous les autres vecteurs propres. On obtient ainsi un exemple où $G(v, \omega)$ est de rang $n-1$.

Les trois lemmes précédents nous mènent alors au

LEMME 5-5. *Si $k \geq 1$, il existe w non multiple de v tel que w soit un vecteur propre commun à tous les éléments de $G(v, \omega)$.*

DEMONSTRATION. Le groupe $G(v, \omega)$ opère sur $\mathbb{R}^n$; par hypothèse la droite $\mathbb{R} \cdot v$ est fixe ainsi que le noyau de ω. Nous avons donc une action

$$G(v, \omega) \times \operatorname{Ker} \omega \to \operatorname{Ker} \omega$$

D'après le lemme 5-2, le vecteur propre v n'est pas simple. Chaque élément de $G(v, \omega)$ admet donc dans Ker ω un vecteur propre correspondant à la même valeur propre que v. Puisque $G(v, \omega)$ est abélien, le théorème de Lie permet de conclure.

Nous en arrivons au résultat principal de ce paragraphe.

PROPOSITION 5-6. *Soit $\mathcal{F}$ un feuilletage totalement géodésique tel que $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ soit du type II. Alors, il existe une (autre) métrique riemannienne sur M telle que si $\mathcal{G}$ est le nouveau flot orthogonal à $\mathcal{F}$ et $\hat{\mathcal{G}}$ le nouveau flot orthogonal à $\hat{\mathcal{F}}$,* *on ait:*

1°) *ou bien $\mathcal{G}$ est défini par une action localement libre du cercle.*

2°) *ou bien la fibration basique correspondante à $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ a un groupe structural réductible à un groupe discret.*

DEMONSTRATION. Le groupe structural de la fibration basique peut se réduire à $\tilde{G}(v, \omega)$ muni de la topologie des feuilles. Ce groupe a le type d'homotopie d'un groupe discret dès que k est nul.

Si $k(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ est non nul, le lemme 5-5 nous fournit un plan de dimension deux formé de vecteurs propres communs à toutes les matrices $H(\gamma)$. Ce plan contient, arbitrairement près de v, un vecteur w tel que les adhérences des orbites du flot linéaire de T^n parallèle à w sont des tores T^{n-1}. A ce vecteur correspond un flot riemannien $\hat{\mathcal{G}}$ transverse à $\hat{\mathcal{F}}$. Remarquons que ce flot est invariant par l'action de $SO(p)$ sur $\hat{M}$ car cette action préserve $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^{\perp})$ et elle est donc affine sur les fibres (lemme 4-1).

Le feuilletage $\hat{\mathcal{G}}$ provient donc d'un feuilletage $\mathcal{G}$ riemannien sur M, transverse à $\mathcal{F}$. Les fibres de la fibration basique de $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ sont maintenant de dimension $n-1$.

Si $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ est de type I_a ou I_b, on utilise la proposition 3-3 et $\mathcal{F}$ est donc transverse à une action localement libre de S^1.

Si $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ est de type II, on itère le procédé jusqu'à obtenir un couple $(\mathcal{F}, \mathcal{G})$ pour lequel l'invariant $k(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ est nul.

Une fois cette déformation effectuée, le groupe structural de la fibration basique est réduit à un groupe ayant le type d'homotopie d'un groupe discret. Soit π_0 le quotient de $\tilde{G}(v, \omega)$ par la composante connexe de l'identité (isomorphe à $\mathbb{R}^{n-1}$). Si nous possédons une section s à la projection naturelle de $\tilde{G}(v, \omega)$ sur π_0, qui soit un homomorphisme de groupes, nous pouvons réduire le groupe structural au sous groupe discret $s(\pi_0)$ de $\tilde{G}(v, \omega)$ (En effet $\tilde{G}(v, \omega)/s(\pi_0)$ est contractile et homéomorphe à $\mathbb{R}^{n-1}$). Pour terminer la démonstration de la proposition, il nous suffit donc de montrer le

LEMME 5-8. *La projection naturelle de $\tilde{G}(v, \omega)$ sur π_0 admet une section qui est un homomorphisme de groupes.*

DEMONSTRATION. Notons τ le groupe $T^n/\operatorname{Ker}\omega$. Le groupe $G(v, \omega)$ opère sur T^n tout en préservant $\operatorname{Ker}\omega$, il opère donc sur τ. Un élément de π_0 s'écrit donc comme un couple (A, β) où A appartient à $G(v, \omega)$ et β à τ. Le produit de (A, β) et de (A', β') est $(AA', A\beta' + \beta)$ où $A\beta'$ désigne l'action de A sur β'. Soit pr la projection naturelle de T^n sur τ.

Trouver une section de π_0 dans $\tilde{G}(v, \omega)$ revient donc à trouver une section de

pr, de τ dans T^n, qui soit équivariante sous les actions de $G(v, \omega)$ sur T^n et τ.

Remarquons tout d'abord que la restriction de pr au sous-groupe de torsion de T^n (noté $\mathrm{Tor}(T^n)$) est un isomorphisme sur le sous-groupe de torsion de τ (noté $\mathrm{Tor}(\tau)$). En effet, l'élément $(x_1, \ldots, x_n) \bmod \mathbb{Z}^n \oplus \mathrm{Ker}\, \omega$ est de torsion dans τ s'il existe un entier p tel que:

$$p(x_1, \ldots, x_n) = (k_1, \ldots, k_n) + (\alpha_1, \ldots, \alpha_n)$$

avec

$$(k_1, \ldots, k_n) \in \mathbb{Z}^n \text{ et } (\alpha_1, \ldots, \alpha_n) \in \mathrm{Ker}\, \omega.$$

Dans l'image réciproque de $(x_1, \ldots, x_n) \bmod \mathbb{Z}^n \oplus \mathrm{Ker}\, \omega$ par pr, il y a un unique élément de torsion, en l'occurrence

$$\left(x_1 - \frac{\alpha_1}{p}, \ldots, x_n - \frac{\alpha_n}{p}\right) \bmod \mathbb{Z}^n$$

La section s que nous cherchons est donc parfaitement définie sur $\mathrm{Tor}(\tau)$.

D'autre part, la droite $\mathbb{R} \cdot v$ se plonge dans T^n. Ce plongement noté i, suivi de la projection pr de T^n sur τ donne une application surjective $pr \circ i$ de $\mathbb{R}$ sur τ. Le noyau de $pr \circ i$ est un groupe abélien libre engendré par n réels $\xi_1, \ldots, \xi_n$, linéairement indépendants sur $\mathbb{Q}$.

Le groupe $G(v, \omega)$ opère sur $\mathbb{R} \cdot v$ et sur τ. Bien entendu, l'application $pr \circ i$ est équivariante sous ces actions. Soient $A_1, \ldots, A_k$ un système de générateurs de $G(v, \omega)$ et $\lambda_1, \ldots, \lambda_k$ les valeurs propres correspondantes, i.e. $A_i v = \lambda_i v$. Il est clair que $\bigoplus_{i=1}^n \mathbb{Z} \cdot \xi_i$ est invariant par multiplication par λ_i puisque c'est le noyau de l'application équivariante $pr \circ i$. Par conséquent, le $\mathbb{Q}$-espace vectoriel $\bigoplus_{i=1}^n \mathbb{Q} \cdot \xi_i$ est aussi invariant par multiplication par λ_i. Soit K le sous-corps de $\mathbb{R}$ engendré par les réels λ_i. La droite $\mathbb{R}$ apparaît comme un K-espace vectoriel et $\bigoplus_{i=1}^n \mathbb{Q} \cdot \xi_i$ comme un K-sous-espace vectoriel de $\mathbb{R}$. Soit E un supplémentaire du K-sous-espace $\bigoplus_{i=1}^n \mathbb{Q} \cdot \xi_i$ dans $\mathbb{R}$. Nous avons alors:

$$\mathbb{R} = E \oplus \left(\bigoplus_{i=1}^n \mathbb{Q} \xi_i \right)$$

$$\tau = \mathbb{R}/\mathrm{Ker}\,(pr \circ i) \simeq E \oplus \left(\bigoplus_{i=1}^n \mathbb{Q}/\mathbb{Z} \cdot \xi_i \right)$$

C'est à dire

$$\tau = pr \circ i(E) \oplus \mathrm{Tor}\,(\tau)$$

La section équivariante s que nous cherchons de τ dans T^n est maintenant définie par:

$$s(pr \circ i(e)) = i(e) \quad \text{si} \quad e \in E$$
$$s(\tau) = (pr \circ i_{|\mathrm{Tor}(T^n)})^{-1}(\tau) \quad \text{si} \quad \tau \in \mathrm{Tor}\,(\tau)$$

Cette section est clairement équivariante par l'action de $G(v, \omega)$ car E est un K-espace vectoriel. Ceci termine la démonstration du lemme 5-8 et donc de la proposition 5-7.

VI. Interprétation des résultats dans M

Partant du feuilletage $\mathcal{F}$ sur M, nous supposerons effectuée la déformation de la métrique dont il était question au paragraphe précédent. Nous nous placerons de plus dans le seul cas qui nous reste à étudier, c'est à dire celui où la fibration basique possède un groupe structural discret (cas 2 de la proposition 5-6). Nous noterons de nouveau $\hat{\mathcal{F}}^\perp$ pour $\hat{\mathcal{G}}$ car nous n'utiliserons plus l'ancienne métrique (non perturbée).

Puisque le groupe structural est discret, nous pouvons décrire $(\hat{\mathcal{F}}, \hat{\mathcal{F}}^\perp)$ par:

$$\hat{M} = \tilde{\hat{B}} \times T^n/(x, m) \sim (\gamma \cdot x, \varphi(\gamma)(m))$$

où $\tilde{\hat{B}}$ est le revêtement universel de la base $\hat{B}$ de la fibration basique, γ un élément quelconque de $\pi_1(\hat{B})$ et φ un morphisme de $\pi_1(\hat{B})$ dans $\tilde{G}(v, \omega)$. Cet élément $\varphi(\gamma)$ s'écrit:

$$\varphi(\gamma)(m) = H(\gamma)(m) + b(\gamma)$$

où $b(\gamma) \in T^n$ et H est le "morphisme d'holonomie" déjà considéré. (Ici encore v et ω ne sont plus les mêmes qu'avant la perturbation de la métrique).

Dans cette description, le feuilletage $\hat{\mathcal{F}}$ est décrit par l'équation $\omega = 0$ et le feuilletage $\hat{\mathcal{F}}^\perp$ est donné, dans chaque $\{*\} \times T^n$ par la direction v.

On peut toujours supposer qu'il existe γ tel que $\varphi(\gamma)$ n'est pas une translation de T^n car sinon la fibration basique serait principale et l'on trouverait une action localement libre de S^1 transverse à $\mathcal{F}$.

Nous disposons donc d'une description explicite de $\hat{\mathcal{F}}$. Pour étudier $\mathcal{F}$, il faut étudier l'action de $SO(p)$ sur $\hat{M}$. Bien entendu cette action en induit une autre sur $\hat{B}$ puisqu'elle préserve la fibration basique. Nous nous proposons de montrer que $SO(p)$ opère librement sur $\hat{B}$ ce qui fera apparaître $(M, \mathcal{F})$ comme un modèle dont la base B est le quotient de $\hat{B}$ par $SO(p)$.

L'action de $SO(p)$ sur $\hat{M}$ se relève en une action de Spin (p) (le revêtement universel de $SO(p)$) sur $\tilde{\hat{B}} \times T^n$. Notons $R(x, m)$ l'image de l'élément (x, m) de $\tilde{\hat{B}} \times T^n$ sous l'action de l'élément R de Spin (p).

LEMME 6-1. *L'action de* Spin (p) *sur* $\tilde{\hat{B}} \times T^n$ *s'écrit sous la forme*:

$$R(x, m) = (R(x), m + u(R, x))$$

où $R(x)$ *désigne l'action de* Spin (p) *sur* $\tilde{\hat{B}}$ *et* $u(R, x)$ *est un vecteur du noyau de* ω.

DEMONSTRATION. Ecrivons tout d'abord que l'action étudiée préserve la fibration basique

$$R(x, m) = (R(x), f(R, x, m))$$

Pour x et R fixés, l'application $m \mapsto f(R, x, m)$ préserve le feuilletage, donc

$$R(x, m) = (R(x), A(R, x)m + u(R, x))$$

où $A(R, x) \in G(v, \omega)$ et $u(R, x) \in T^n$.

Par continuité $A(R, x) = id$. Si m et R sont fixés et x varie, $R(x, m)$ doit rester sur une même feuille de $\hat{\mathcal{F}}$. De même, si m et x sont fixés et R varie, $R(x, m)$ doit rester sur une même feuille de $\hat{\mathcal{F}}$. Par conséquent, lorsque R et x varient, $u(R, x)$ reste sur une feuille du feuilletage linéaire de T^n défini par ω. Puisque $u(id, x) = 0$, $u(R, x)$ reste sur la feuille passant par 0 et peut donc être identifié à un vecteur du noyau de ω. On a donc bien la description souhaitée de l'action:

$$R(x, m) = (R(x), m + u(R, x)) \quad \text{où} \quad u(R, x) \in \operatorname{Ker} \omega.$$

Nous nous proposons de simplifier encore cette écriture en montrant que l'on peut toujours supposer $u(R, x) \equiv 0$. La description que nous avons donnée de $(\hat{M}, \hat{\mathcal{F}})$ comme quotient de $\tilde{\hat{B}} \times T^n$ n'est évidemment pas unique. Si h est un difféomorphisme de $\tilde{\hat{B}} \times T^n$ préservant les deux feuilletages et commutant avec l'action de $\pi_1(\hat{B})$ sur $\tilde{\hat{B}} \times T^n$, nous pouvons considérer h comme un "changement de coordonnées" sur $\tilde{\hat{B}} \times T^n$ compatible avec nos données.

LEMME 6-2: *Il existe un tel difféomorphisme h, envoyant le point (x, m) sur le point de coordonnées (x_1, m_1) tel que, dans ces nouvelles coordonnées, l'action de* Spin (p) *s'écrive*

$$R(x_1, m_1) = (R(x_1), m_1).$$

Autrement dit, quitte à changer les coordonnées dans $\tilde{\hat{B}} \times T^n$, on peut toujours supposer que $u(R, x)$ est identiquement nul.

DEMONSTRATION. Il est clair que $u(R, x)$ vérifie les relations suivantes:

$$u(R_1R_2, x) = u(R_2, x) + u(R_1, R_2(x))$$
$$u(R, \gamma \cdot x) = H(\gamma)u(R, x)$$

exprimant le fait que $R(x, m)$ définit effectivement une action et que cette action commute à celle de $\pi_1(\hat{B})$. Si l'on pose

$$u(x) = \int_{\mathrm{Spin}\,(p)} u(R, x)\, dR \in \operatorname{Ker} \omega$$

et si l'on définit h par:

$$h : (x, m) \mapsto (x_1, m_1) = (x, m + u(x))$$

on obtient évidemment le difféomorphisme cherché.

Nous sommes maintenant en mesure de démontrer une version presque complète du théorème principal:

PROPOSITION 6-3. *Si $\mathcal{F}$ est un feuilletage totalement géodésique de codimension* 1, *orientable sur une variété compacte orientable M, alors*

1°) *soit $\mathcal{F}$ est transverse à une action localement libre de S^1*

2°) *soit $\mathcal{F}$ est topologiquement conjugué à un feuilletage modèle $(M_D, \mathcal{F}_D)$.*

DEMONSTRATION. D'après le lemme 6-2, les points fixes de l'action de Spin (p) sur $\tilde{\hat{B}}$ correspondent aux points fixes de l'action de Spin (p) sur $\hat{B} \times T^n$. Remarquant que $\hat{M}$ est un $SO(p)$ fibré principal, on en déduit que l'action de $SO(p)$ sur $\hat{B}$ est elle aussi libre. Si B est le quotient de $\hat{B}$ sous cette action, on en déduit la description de $(M, \mathcal{F})$ sous la forme

$$M = \tilde{B} \times T^n / (x, m) \sim (\gamma \cdot x, \varphi(\gamma)(m))$$

(Remarquons que $\varphi: \pi_1(\hat{B}) \to \tilde{G}(v, \omega)$ se factorise à travers $\pi_1(B)$ car, d'après 6-2, le groupe $SO(p)$ opère trivialement sur T^n.) Ceci est précisément un modèle $(M_D, \mathcal{F}_D)$.

VII. Fin de la démonstration du théorème principal

Il nous reste essentiellement à nous débarrasser du changement de structure différentiable (Remarque 4-3) et à étudier le cas où M n'est pas compacte.

Observant qu'une action continue et localement libre du cercle, peut être lissée sans difficulté, il nous faut étudier le cas des feuilletages modèles.

LEMME 7-1. *On peut toujours se limiter à l'étude des modèles pour lesquels les coordonnées $\omega_1, \ldots, \omega_n$ de ω dans la base canonique de $(\mathbb{R}^n)^*$ sont telles que $\dfrac{\omega_i}{\omega_1}$ est algébrique.*

DEMONSTRATION. Si ω n'est pas vecteur propre simple de ${}^tA$ avec $A \in G(v, \omega)$, alors v n'est pas non plus vecteur propre simple de A. Nous avons déjà observé au paragraphe 5 que si v est vecteur propre multiple de tous les éléments de $G(v, \omega)$, nous pouvons déformer la métrique et faire baisser la dimension de la fibration basique. Cette déformation étant faite, ω peut être supposé vecteur propre simple. Les réels $\dfrac{\omega_2}{\omega_1}, \ldots, \dfrac{\omega_n}{\omega_1}$ peuvent alors être calculés rationnellement en fonction des coefficients d'une matrice A de $G(v, \omega)$ et de la valeur propre λ correspondante. Ils sont donc algébriques.

PROPOSITION 7-2. *Le changement de structure différentiable effectué en* 4-3 *était en fait inutile.*

DEMONSTRATION. Le feuilletage $\hat{\mathcal{F}}_{|F}$ initial était obtenu par $n_0 - 1$ difféomorphismes du cercle commutant deux à deux. Lorsque nous avons effectué la déformation de la métrique, la dimension des fibres est passée de n_0 à n et la trace de $\hat{\mathcal{F}}$ sur ces nouvelles fibres est obtenue par la suspension de $n - 1$ difféomorphismes du cercle dont les nombres de rotation sont $\dfrac{\omega_2}{\omega_1}, \ldots, \dfrac{\omega_n}{\omega_1}$. Par conséquent parmi les $n_0 - 1$ difféomorphismes initiaux, certains avaient un nombre de rotation algébrique. On conclut à l'aide de [Her] qui montre qu'un groupe abélien de difféomorphismes C^∞ du cercle dont un élément a un nombre de rotation irrationnel algébrique, est C^∞-conjugué à un groupe de rotations.

Pour terminer, il nous faut traiter le cas où M est non compacte mais complète. Le feuilletage $\mathscr{F}^{\perp}$ reste riemannien, et $\hat{\mathscr{F}}^{\perp}$ reste transversalement parallélisable complet, les théorèmes 1-2 et 1-4 restent valables. Nous devons trouver l'équivalent de 1-3 lorsque M n'est pas compacte.

PROPOSITION 7-3. *Soit $\mathscr{G}$ un flot transversalement de Lie modelé sur le groupe de Lie G, sur une variété N (orientable mais éventuellement non compacte). On suppose que les orbites de $\mathscr{G}$ sont denses et que la structure transverse est complète. Alors deux cas sont possibles.*

1) *soit N est compacte.*
2) *soit $N=\mathbb{R}$ et le flot est de codimension zéro.*

DEMONSTRATION. Supposons que la dimension de N est supérieure à deux et montrons que N est compacte. On suppose $\mathscr{G}$ engendré par le groupe à un paramètre φ_t. Pour chaque point x de N, l'orbite positive ou l'orbite négative de x est dense dans N. En remarquant que le groupe des automorphismes de $\mathscr{G}$ agit transitivement sur N (la structure est complète), et en inversant au besoin l'orientation des orbites, on peut donc supposer que, pour tout x de N, l'orbite positive de x est dense dans N.

Soit D_1 un disque fermé, plongé dans N et transverse à $\mathscr{G}$, et soit x un point de l'intérieur de D_1. Ce disque s'identifie à un voisinage de l'élément neutre du groupe transverse G. Soit $D_2 \subset D_1$ le disque de centre x et de rayon ε où la métrique utilisée est une métrique invariante à gauche sur G et où ε est choisi de telle sorte que le disque de centre x et de rayon 2ε soit entièrement contenu dans D_1. Soit $D_3 \subset D_2$ le disque de centre x et de rayon $\varepsilon/3$. Enfin soient $x_1, \ldots, x_k$ k points de D_2 tels que les disques Δ_i de centre x_i et de rayon $\varepsilon/3$ recouvrent D_2 (remarquons que $\Delta_i \subset D_1$). Pour chaque i, il existe un réel t_i positif, tel que $\varphi_{t_i}(x_i)$ appartienne à D_3 car les orbites de φ_t sont positivement denses. Considérons l'holonomie du chemin joignant x_i à $\varphi_{t_i}(x_i)$. C'est un germe de translation à gauche de G et donc un germe d'isométrie. De par le choix des rayons de D_2, D_3 et Δ_i, cette isométrie se prolonge en une isométrie h_i définie sur Δ_i tout entier et à valeurs dans D_2. Puisque la structure transverse à $\mathscr{G}$ est complète, il est clair que pour tout y de Δ_i, les points y et $h_i(y)$ appartiennent à la même feuille de $\mathscr{G}$. De manière plus précise, il existe k fonctions continues $t_i(y)$ définies sur Δ_i telles que:

$$t_i(x_i) = t_i$$

$$\varphi_{ti(y)}(y) = h_i(y) \in D_2$$

Soit

$$T_i = \operatorname*{Max}_{y \in \Delta_i} |t_i(y)| \quad \text{et} \quad T > \operatorname*{Max}_{i} T_i$$

Pour tout y de D_2, la portion de l'orbite de y située entre y et $\varphi_T(y)$ recoupe au moins une fois D_2. Si l'on considère l'ensemble des points de N de la forme $\varphi_t(y)$ avec $y \in D_2$ et $0 \leqslant t \leqslant T$, on obtient un compact invariant par les φ_t avec $t \geqslant 0$. Puisque toutes les orbites positives de φ_t sont denses, ce compact est N tout entier et, en particulier, N est compact.

Soit $(M, \mathscr{F})$ un feuilletage totalement géodésique sur une variété non compacte. Si l'adhérence d'une orbite de $\mathscr{F}^\perp$ est compacte, on est tenté de reproduire intégralement la démonstration qui vient d'être donnée dans le cas compact. La difficulté est alors de démontrer la proposition 3-2. Cette proposition est elle même basée sur la proposition 3-3 et sur le lemme 3-4. Ces deux derniers résultats sont les seuls qui ne s'étendent pas clairement lorsque M est non compacte.

PROPOSITION 3-3 (cas non compact). *Si l'adhérence d'une orbite de $\mathscr{F}^\perp$ est compacte et si $\mathscr{F}$ a une feuille fermée, alors $\mathscr{F}$ est transverse à une action localement libre du cercle.*

DEMONSTRATION: Soit F cette feuille fermée. Par hypothèse chaque orbite de $\mathscr{F}^\perp$ rencontre F une infinité de fois; soit Ψ: $F \to F$ l'application de premier retour. C'est une isométrie de F. L'adhérence H du groupe engendré par Ψ est un sous-groupe de Lie abélien du groupe des isométries de F. Par conséquent H est isomorphe à $T^k \times R^l \times \Gamma$ où Γ est abélien discret. Puisque H contient un sous-groupe monogène dense, on en déduit que H est isomorphe à 0, $\mathbb{Z}$ ou à $T^k \times \Gamma$ où Γ est un groupe fini. Le premier cas signifie que $\Psi = id$ c'est à dire que les orbites de $\mathscr{F}^\perp$ sont des cercles. Dans le second cas le groupe engendré par Ψ est fermé ce qui impliquerait que les orbites de $\mathscr{F}^\perp$ seraient fermées mais non compactes, et nous avons exclu ce cas. Enfin, dans le dernier cas, Ψ est approchable par un élément de torsion de $T^k \times \Gamma$, et donc par une isométrie périodique. $\mathscr{F}^\perp$ est alors approchable par une action localement libre du cercle.

En ce qui concerne le lemme 3-4, nous ne sommes parvenus à le généraliser au cas où M est non compacte que sous certaines conditions.

LEMME 3-4 (cas non compact). *Supposons que l'adhérence d'une orbite de*

$\mathcal{F}^{\perp}$ est compacte et que l'une des conditions suivantes est réalisée

1) *le groupe fondamental de M est de type fini.*

2) *le feuilletage $\mathcal{F}$ est analytique.*

alors, si $\mathcal{F}$ est de type I_b, $\mathcal{F}$ possède une feuille fermée.

DEMONSTRATION. Si le groupe fondamental de M est de type fini, la démonstration du lemme 3-4 donnée au paragraphe III est valable puisque le théorème de Sacksteder s'applique aux pseudo-groupes de type fini. Si le feuilletage est analytique réel, la réunion des feuilles compactes de $\hat{\mathcal{F}}_{|F}$ ne peut contenir qu'un nombre fini de feuilles compactes. Cette réunion finie contient donc la trace sur F d'une feuille fermée de $\hat{\mathcal{F}}$ d'après 3-5.

Nous pouvons donc décrire les feuilletages $\mathcal{F}$ totalement géodésiques sur les variétés riemanniennes complètes non compactes en imposant l'une des deux conditions 1) et 2) du lemme précédent.

Si l'adhérence d'une orbite de $\mathcal{F}^{\perp}$ est compacte, le problème se traite exactement comme nous l'avons fait dans le cas où M est compacte. Sinon toutes les orbites sont fermées et celles-ci définissent une fibration de M de fibre $\mathbb{R}$ et de base B, transverse à $\mathcal{F}$. Puisque M est supposée complète, toute feuille de $\mathcal{F}$ apparaît comme un revêtement de B. On peut donc écrire

$$M = \tilde{B} \times \mathbb{R}/(x, y) \sim (\gamma \cdot x, \varphi(\gamma)(y))$$

où

$$\varphi : \pi_1(B) \to \mathrm{Diff}^+(\mathbb{R})$$

est le morphisme d'holonomie. Les feuilles de $\mathcal{F}$ sont définies par l'équation $y = \mathrm{Cst}$. Ceci achève la démonstration du théorème 2.

VIII. Remarques finales

Les corollaires qui suivent sont des conséquences immédiates du théorème principal. Certains d'entre eux peuvent d'ailleurs se démontrer directement. Rappelons tout d'abord un résultat de [Car–Ghy].

PROPOSITION 8-1. *Soit $\mathcal{F}$ un feuilletage totalement géodésique, de codimension 1, transversalement orientable, sur une variété riemannienne complète M. Soit $\tilde{M}$ le revêtement universel de M, $\tilde{\mathcal{F}}$ le relevé de $\mathcal{F}$ dans $\tilde{M}$ et $\tilde{\mathcal{F}}^{\perp}$ le flot orthogonal à $\tilde{\mathcal{F}}$. Alors $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^{\perp})$ est un produit, c'est à dire qu'il existe un difféomorphisme de $\tilde{M}$ sur $L \times \mathbb{R}$ envoyant les feuilles de $\tilde{\mathcal{F}}$ sur $L \times \{*\}$ et celles de $\tilde{\mathcal{F}}^{\perp}$ sur $\{*\} \times \mathbb{R}$.*

COROLLAIRE 8-2. *Soit $\mathcal{F}$ un feuilletage totalement géodésique de codimension* 1 *sur la variété riemannienne complète M, alors*

- *ou le groupe fondamental de M contient un sous-groupe distingué abélien libre non trivial*
- *ou M est un produit $B \times \mathbb{R}$ et le feuilletage est transverse aux fibres $\{*\} \times \mathbb{R}$.*

Ce corollaire, ainsi que celui qui suit est déjà dans [Car 1], lorsque M est supposée compacte.

DEMONSTRATION. Dans le cas où les adhérences des orbites de $\mathcal{F}^\perp$ sont non compactes, nous savons que M est un produit $B \times \mathbb{R}$. Sinon, l'adhérence d'une orbite de $\mathcal{F}^\perp$ est un tore T^n $(n > 1)$. L'image du groupe fondamental de ce tore dans celui de M est alors un sous-groupe distingué abélien. Ce sous-groupe est non trivial car certaines classes d'homotopie de T^n correspondent à des transversales fermées à $\mathcal{F}$ et celles ci ne peuvent être triviales d'après 8-1.

COROLLAIRE 8-3. *Si M admet une métrique riemannienne à courbure strictement positive ou si M est compacte et admet une métrique à courbure strictement négative, alors, il n'existe aucun feuilletage totalement géodésique sur M (même pour une autre métrique de M).*

DEMONSTRATION. Une variété à courbure strictement positive est compacte et possède un groupe fondamental fini. Le corollaire 8-2 exclut donc la possibilité d'existence d'un feuilletage totalement géodésique sur une telle variété.

Le groupe fondamental d'une variété compacte à courbure strictement négative ne peut contenir de sous-groupe abélien de rang 2 et son centre est trivial. Les modèles ainsi que les fibrés de Seifert ne peuvent donc pas être des variétés compactes à courbure strictement négative.

En ce qui concerne le comportement qualitatif des feuilles, nous avons le

COROLLAIRE 8-4. *Si $\mathcal{F}$ est totalement géodésique sur une variété compacte M et si $\mathcal{F}$ possède une feuille compacte ou un minimal exceptionnel, alors $\mathcal{F}$ est transverse à une action du cercle. Sinon, toutes les feuilles sont denses et le feuilletage possède une structure transverse affine (c'est-à-dire que le pseudo-groupe transverse peut se réduire à un pseudo-groupe de transformations affines de $\mathbb{R}$). En particulier, dans ce dernier cas, le premier nombre de Betti de M est non nul* (cf. [Fed–Fur]).

Il est bien connu que la classe des feuilletages transverses à des fibrations en cercles est très diversifiée; presque tous les phénomènes qualitatifs rencontrés en codimension 1 se rencontrent dans cette classe. Les feuilletages modèles possèdent par contre une remarquable propriété de rigidité:

COROLLAIRE 8-5. *Soit* $(M_D, \mathcal{F}_D)$ *un feuilletage modèle compact pour lequel* v *est vecteur propre simple* (*d'après* 7-1, *on peut toujours se limiter à ces modèles*). *Alors* $(M_D, \mathcal{F}_D)$ *possède un "module de stabilité" fini, c'est à dire que l'on peut décrire les feuilletages voisins de* $\mathcal{F}_D$ *à l'aide d'un nombre fini de paramètres* (*à conjugaison* C^∞ *près*).

DEMONSTRATION. Un feuilletage proche d'un feuilletage totalement géodésique est encore totalement géodésique (pour une autre métrique). Si $\mathcal{F}'$ est proche de $\mathcal{F}_D$, grâce à l'hypothèse faite sur v, on voit facilement que $\mathcal{F}'$ doit aussi être conjugué à un modèle associé à $(D') = (n, v', \omega', B, \varphi')$ correspondant au même entier n et à la même base B que (D). Le morphisme φ' s'écrit

$$\varphi'(\gamma)(m) = H'(\gamma)(m) + b'(\gamma)$$

avec

$$H'(\gamma) \in SL(n, \mathbb{Z}) \quad \text{et} \quad b'(\gamma) \in T^n.$$

De la proximité de $\mathcal{F}'$ et $\mathcal{F}_D$, on déduit que $H' = H$, $\omega = \omega'$ et $v = v'$. Par conséquent, φ' et φ ne différent que par le terme $b'(\gamma)$. Les valeurs de $b'(\gamma)$ pour γ décrivant un système de générateurs de $\pi_1(B)$ fournissent un nombre fini de paramètres décrivant les feuilletages voisins de $\mathcal{F}_D$.

Sans vouloir faire une étude détaillée des déformations des feuilletages modèles, donnons un exemple typique. Supposons que le groupe fondamental de B soit le groupe libre à deux générateurs α, β noté $L(\alpha, \beta)$ et soit $(M, \mathcal{F})$ le feuilletage modèle correspondant au morphisme φ défini par

$$\varphi(\alpha) = (A, 0) \in \tilde{G}(v, \omega)$$
$$\varphi(\beta) = (id, 0) \in \tilde{G}(v, \omega)$$

Les feuilletages proches peuvent être décrits par deux paramètres u_1, u_2 de T^n. Le feuilletage $\mathcal{F}_{u_1,u_2}$ est associé au morphisme φ_{u_1,u_2} défini par:

$$\varphi_{u_1,u_2}(\alpha) = (A, u_1) \in \tilde{G}(v, \omega)$$
$$\varphi_{u_1,u_2}(\beta) = (id, u_2) \in \tilde{G}(v, \omega)$$

Bien entendu, deux couples (u_1, u_2) et (u_1', u_2') peuvent correspondre à des feuilletages conjugués. Par exemple, soit x_0 tel que $Ax_0 + u_1 = x_0$, si l'on conjugue φ_{u_1,u_2} par la translation (id, x_0), on obtient φ_{0,u_2}. On peut donc se limiter aux déformations pour lesquelles $u_1 = 0$. De même, si u_2 est un petit élément de $\mathrm{Ker}\,(\omega) \subset T^n$, le feuilletage $\mathcal{F}_{0,u_2}$ est conjugué à $\mathcal{F}_{0,0}$. Cependant, il existe effectivement des déformations non triviales. Pour le constater, calculons le groupe fondamental de la feuille de $\mathcal{F}_{u_1,u_2}$ passant par le point x de T^n. Ce groupe est le sous-groupe de $L(\alpha, \beta)$ défini par:

$$\{\gamma \in L(\alpha, \beta), \varphi_{u_1,u_2}(\gamma)(x) - x \in \mathrm{Ker}\,(\omega)\}$$

Si $(u_1, u_2) = (0, 0)$ et $x = 0$, ce groupe est $L(\alpha, \beta)$ tout entier.

C'est à dire que la feuille de $\mathcal{F}_{0,0}$ passant par $0 \in T^n$ est un fibré en $\mathbb{R}^{n-1}$ au dessus de B. Si b désigne la dimension de B, le b-ème nombre de Betti de cette feuille est donc non nul. Si $u_1 = 0$ et $u_2 \notin \mathrm{Ker}\,(\omega)$, pour tout x de T^n, le groupe fondamental de la feuille de $\mathcal{F}_{u_1,u_2}$ passant par x ne contient pas β, c'est donc un sous-groupe strict de $L(\alpha, \beta)$. Cette dernière feuille est donc un fibré en $\mathbb{R}^{n-1}$ au dessus d'une variété B' qui est un revêtement non trivial de B. Si B' est non compacte, le b-ème nombre de Betti de cette feuille est nul; si B' est compacte, le groupe fondamental de cette feuille est un sous-groupe strict d'indice fini de $L(\alpha, \beta)$, c'est donc un groupe libre ayant au moins 3 générateurs. Quoiqu'il en soit, si $u_2 \notin \mathrm{Ker}\,(\omega)$, aucune feuille de $\mathcal{F}_{0,u_2}$ n'est homéomorphe à la feuille de $\mathcal{F}_{0,0}$ passant par 0. Les feuilletages $\mathcal{F}_{0,0}$ et $\mathcal{F}_{0,u_2}$ ne sont donc pas conjugués.

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The integral homology of SL_2 and PSL_2 of euclidean imaginary quadratic integers

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Introduction

One way to study the cohomology of a group Γ of finite virtual cohomological dimension is to find a finite dimensional contractible space X on which Γ acts properly (such a space X always exists by [18], 1–7), and to then analyze the action. In this paper we want to consider the arithmetic groups $SL_2(\mathcal{O})$ and $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/\pm I$ where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field k; the classical choice of X in this case is hyperbolic three-space H, i.e., the associated symmetric space $SL_2(\mathbb{C})/SU(2)$. As early as 1892 Bianchi [2] exhibited fundamental domains for the action of $PSL_2(\mathcal{O})$ on H for some small values of the discriminant. The space H has also turned out to be very useful in studying the relation between automorphic forms associated to $SL_2(\mathcal{O})$ and the cohomology of $SL_2(\mathcal{O})$ (cf. [12], [10]), and in studying the topology of certain hyperbolic 3-manifolds (cf. [25]).

However, this choice of X is inconvenient for actual explicit computations of the cohomology of $\Gamma = (P)SL_2(\mathcal{O})$ with integral coefficients because the dimension of H is three, whereas the virtual cohomological dimension of Γ is two, indicating that it may be possible for Γ to act properly on a contractible space of dimension two; in addition, the quotient $\Gamma\backslash H$ is not compact. A more useful space X for our purposes is given by work of Mendoza [14], which we recall in §3; using Minkowski's reduction theory (cf. §2), he constructs a Γ-invariant 2-dimensional deformation retract $I(k)$ of H such that the quotient of $I(k)$ by any subgroup of Γ of finite index is compact; $I(k)$ is endowed with a natural CW structure such that the action of Γ is cellular and the quotient $\Gamma\backslash I(k)$ is a finite CW-complex.

The main object of this paper is to show how this construction can be used to completely determine the integral homology groups of $PSL_2(\mathcal{O})$. This is done by analyzing a spectral sequence which relates the homology of $PSL_2(\mathcal{O})$ to the homology of the quotient space $PSL_2(\mathcal{O})\backslash I(k)$ and the homology of the stabilizers of the cells (cf. [5], VII). We will confine our computations to the cases where $\mathcal{O}$ is a euclidean ring, i.e., $\mathcal{O} = \mathcal{O}_{-d}$ is the ring of integers in $k = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7$ and 11. We will write out in detail the case $d = 2$ (cf. §5), which contains all the essential features in the computations. In the other cases, we indicate briefly any necessary modifications in the analysis of the spectral sequence and list the results.

The complex $I(k)$ and spectral sequence may be used to compute homology and cohomology groups for the groups $SL_2(\mathcal{O})$, $GL_2(\mathcal{O})$ and $PGL_2(\mathcal{O})$ as well as $PSL_2(\mathcal{O})$. The homology of these groups with coefficients in the Steinberg module has particular interest in algebraic K-theory ([15]). We indicate here how to do this computation for $SL_2(\mathcal{O}_{-2})$ and list the results in the other euclidean cases (cf. §6).

We conclude the paper with some observations on torsion classes in the cohomology of subgroups of finite index in $SL_2(\mathcal{O})$ which are not detected by the torsion in the stabilizers of cells in $I(k)$.

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Notation

(1) $\mathbb{Z}/m$ denotes the cyclic group $\mathbb{Z}/m\mathbb{Z}$, $m \in \mathbb{N}$

(2) If a is an element of $SL_2(\mathbb{C})$, we denote by $\bar{a}$ the matrix whose entries are complex conjugates of the entries of a.

(3) For a group Γ, Γ^{ab} denotes the abelianization of Γ, i.e., the quotient of Γ by its commutator subgroup.

§1. A spectral sequence

1.1 Let Γ be a group which acts cellularly on a contractible CW-complex X of dimension $\dim X = n$. If σ is a cell of X we let Γ_σ denote the stabilizer of σ in Γ, i.e. $\Gamma_\sigma = \{\gamma \in \Gamma \mid \gamma\sigma = \sigma\}$. If Σ (resp. Σ_p) is a set of representatives for Γ-orbits of cells (resp. p-cells) of X, then there exists a natural spectral sequence (cf. [5], VII, [18], p. 93ff.) whose E^1-term is given by the homology groups $H_*(\Gamma_\sigma, \mathbb{Z}_\sigma)$, $\sigma \in \Sigma$, and which converges to the homology $H_*(\Gamma, \mathbb{Z})$ of Γ with trivial coefficients $\mathbb{Z}$. (Here $\mathbb{Z}_\sigma$ denotes the Γ_σ-module $\mathbb{Z}$ given by the homomorphism $\varepsilon : \Gamma_\sigma \to \{\pm 1\}$ where $\varepsilon(\gamma) = 1$ (resp. -1) if γ fixes (resp. reverses) the orientation of σ.) This spectral sequence can be constructed as follows: Let $C_q(X)$ be the group of cellular q-chains of X. If X_q denotes the q-skeleton of X, one has $C_q(X) =$

$H_q(X_q, X_{q-1}; \mathbb{Z})$. Then the action of Γ on X gives rise to a natural action of the group algebra $\mathbb{Z}[\Gamma]$ on the cellular q-chains $C_q(X)$. Let E_* be a $\mathbb{Z}[\Gamma]$-free resolution of $\mathbb{Z}$. Then we can form the double complex $C_*(X) \otimes_{\mathbb{Z}[\Gamma]} E_*$. Since X is contractible, C_* is exact except at $C_0(X)$, where $\operatorname{cok}(\partial_1 : C_1(X) \to C_0(X)) \cong \mathbb{Z}$. Since E_p is free, $C_*(X) \otimes_{\mathbb{Z}[\Gamma]} E_p$ is still exact, except that $\operatorname{cok}(\partial_1 \otimes 1) \cong \mathbb{Z} \otimes_{Z[\Gamma]} E_p$. Thus the horizontal filtration of the double complex gives us a spectral sequence with

$$E^1_{pq} = \begin{cases} 0 & q > 0 \\ \mathbb{Z} \otimes_\Gamma E_p & q = 0 \end{cases} \tag{1}$$

The differential d^1 is given by $d^1 = 1 \otimes \partial_{E_*}$, where ∂_{E_*} is the boundary map for E_*. Thus the spectral sequence converges to the homology $H_*(\Gamma, \mathbb{Z})$ with trivial $\mathbb{Z}$-coefficients, i.e., $E^2 = E^\infty = H_*(\Gamma, \mathbb{Z})$.

The vertical filtration of the double complex $C_*(X) \otimes_{\mathbb{Z}[\Gamma]} E_*$ gives us

$$E^1_{pq} = H_q(\Gamma, C_p(X)) \tag{2}$$

with the differential d^1 induced by $\partial_{C_*(X)}$. We note that $C_p(X)$ can be identified with the direct sum $\oplus_\sigma \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbb{Z}_\sigma$, where the sum is taken over $\sigma \in \Sigma_p$. We assume from now on that Γ_σ fixes σ pointwise for every cell σ. In this case the orbit space $\Gamma \backslash X$ inherits a CW-structure. One can then orient each cell of X in such a way that the Γ-action preserves orientations. In particular, it follows the action of Γ_σ on $\mathbb{Z}$ is trivial. Thus we get

$$\begin{aligned} E_p \otimes_{\mathbb{Z}[\Gamma]} C_q(X) &= \bigoplus_{\sigma \in \Sigma_q} (E_p \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbb{Z}) \\ &= \bigoplus_{\sigma \in \Sigma_q} (E_p \otimes_{\mathbb{Z}[\Gamma_\sigma]} \mathbb{Z}) \end{aligned} \tag{3}$$

and the spectral sequence has

$$E^1_{pq} = H_q(\Gamma, C_p(X)) \cong \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathbb{Z}). \tag{4}$$

The differential d^1 in E^1 can be described in terms of the homology of the stabilizers as follows. If $\sigma \in \Sigma$ is a p-cell in the orbit complex $\Gamma \backslash X$, and $\partial\sigma = \Sigma_\tau \pm g_\tau \tau$, where $g_\tau \in \Gamma$ and $\tau \in \Sigma$ are $(p-1)$-cells in $\Gamma \backslash X$, then there are maps

$$(i_{\sigma\tau})_* : H_*(\Gamma_\sigma, \mathbb{Z}) \to H_*(\Gamma_{g_\tau \tau}, \mathbb{Z}) \tag{5}$$

induced by the inclusion $\Gamma_\sigma \to \Gamma_{g_\tau\tau}$, resp.

$$(g_\tau)_*: H_*(\Gamma_{g_\tau\tau}, \mathbb{Z}) \to H_*(\Gamma_\tau, \mathbb{Z}), \tag{6}$$

induced by the conjugation isomorphism $a \mapsto g_\tau^{-1} a g_\tau$. The restriction of the differential $d^1: E^1_{p,*} \to E^1_{p-1,*}$ to $H_*(\Gamma_\sigma, \mathbb{Z})$ is then given by

$$d^1_{|H_*(\Gamma_\sigma,\mathbb{Z})} = \Sigma(g_\tau)_* \circ (i_{\sigma\tau})_* \tag{7}$$

where we sum over all $\tau \in \Sigma$ which occur in $\partial\sigma$. In other words, the d^1-maps are boundary maps 'twisted' by the identifications of $\partial\sigma$ in the orbit complex $\Gamma\backslash X$.

§2. Reduction theory

Let k be an imaginary quadratic number field and $\mathcal{O}$ its ring of integers. In this section we recall briefly reduction theory for the action of $PSL_2(\mathcal{O})$ on the Poincaré upper half space $H = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}$. The version we use is based on the notion 'distance from a cusp' and is a special case of Harder's results [11]. These were inspired by ideas of Siegel obtained in the case SL_2 (resp. SL_n) over the ring of integers of a totally real number field [20]. In fact, the proofs given by Siegel there can be easily generalized to the case we are considering.

2.1 We denote by $\bar{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta \geq 0\} \cup \{(\infty, \infty)\}$ the extended upper half space. The usual action of $SL_2(k)$ on H extends then to one on $\bar{H}$. An element $g = \begin{pmatrix} ab \\ cd \end{pmatrix}$ of $SL_2(k)$ acts on $\bar{H}$ by

$$g(z, \zeta) = (z', \zeta') \tag{1}$$

where

$$z' = \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}\zeta^2}{(cz+d)(\bar{c}\bar{z}+\bar{d}) + c\bar{c}\zeta^2}$$

resp.

$$\zeta' = \frac{\zeta}{(cz+d)(\bar{c}\bar{z}+\bar{d}) + c\bar{c}\zeta^2}.$$

We call an element $(z, \zeta) \in \bar{H}$ a *cusp* if either $(z, \zeta) = (\infty, \infty)$ or $\zeta = 0$ and $z \in k$. The set of cusps will be identified with $k \cup \{\infty\}$, the projective space over k. Let $\lambda = \alpha/\beta$, α, $\beta \in k$, be a cusp; in the case $\lambda = \infty$ we write $\lambda = 1/0$. The distance from a point $(z, \zeta) \in H$ to the cusp λ is then defined by

$$n_\lambda(z, \zeta) = \frac{|z-\lambda|^2 + \zeta^2}{\zeta N_\lambda}, \qquad \lambda \neq \infty \tag{2}$$

(and $n_\infty(z, \zeta) = 1/\zeta$) where N_λ is the norm of the fractional ideal (α, β) (in $\mathcal{O}$) divided by $\beta\bar{\beta}$; note that N_λ depends only on λ and not on the choice of α and β.

Remarks. (1) Note that each level set of the smooth function $n_\lambda : H \to \mathbb{R}_+$ is a horosphere at λ. For $\lambda = \infty$, $n_\infty^{-1}(r)$, $r > 0$, is a horizontal plane of height $1/r$.

(2) The definition is intuitively motivated by the following alternative description. We can identify H with the set $\underline{H}$ of binary positive definite hermitian forms on $\mathbb{C}^2$ with determinant equal to one; a cusp λ is identified with the line L_λ in $\mathbb{C}^2$ with slope λ. The distance from a point (z, ζ) to a cusp λ is then the area of a fundamental domain for $L_\lambda \cap \mathcal{O}^2$ in $\mathbb{C}^2$ measured using the hermitian form in $\underline{H}$ corresponding to (z, ζ), and normalized so that $n_\infty(0, 1) = 1$.

The distance function to a cusp has the following invariance property which follows easily from the reinterpretation in remark (2): Let (z, ζ) be a point in H and λ a cusp of k; then for any $g \in SL_2(\mathcal{O})$

$$n_{g\lambda}(g(z, \zeta)) = n_\lambda(z, \zeta) \tag{3}$$

The main results in reduction theory can then be formulated in terms of the n_λ as follows (cf. [11], §1).

(i) For a given point $(z, \zeta) \in H$ and a fixed constant C there are only finitely many cusps μ such that $n_\mu(z, \zeta) \leq C$.

(ii) There is a constant C_1 (depending only on k) such that for every point (z, ζ) in H there exists at least one cusp λ of k such that $n_\lambda(z, \zeta) \leq C_1$.

(iii) There exists a constant C_2 with the following property: For each $(z, \zeta) \in H$ there is at most one cusp μ of k such that $n_\mu(z, \zeta) < C_2$ i.e. if $n_\mu(z, \zeta) < C_2$ and $n_{\mu'}(z, \zeta) < C_2$ then $\mu = \mu'$.

For later use we fix such a constant C_2. Elementary proofs of (i)–(iii) which also yield actual values for C_1 and C_2 are given in [14], (§1).

2.2 *Fundamental domain for $PSL_2(\mathcal{O})$.* We conclude this section by reviewing the construction of a strict fundamental domain for the action of $PSL_2(\mathcal{O}) = SL_2(\mathcal{O})/\pm I$ on H. To each cusp λ of k we associate the set

$$H(\lambda) = \{(z, \zeta) \in H \mid n_\lambda(z, \zeta) \leq n_\mu(z, \zeta) \text{ for all cusps } \mu \neq \lambda\};$$

this is called the minimal set of λ. By property 2.1 (iii) of the distance function, the set $H(\lambda)$ is non-empty. Moreover, each $H(\lambda)$ is a closed subset of H, and we denote its boundary by $I(\lambda)$. The main facts in reduction theory imply that the sets $H(\lambda)$ make up a locally finite closed covering of H. Note that the minimal sets transform under an element of $PSL_2(\mathcal{O})$ in the following way

$$gH(\lambda) = H(g\lambda), \qquad g \in PSL_2(\mathcal{O}). \tag{1}$$

We can now begin to construct the fundamental domain. One knows ([20], p. 242) that there are exactly h_k $PSL_2(\mathcal{O})$-orbits of cusps of k, where $h = h_k$ denotes the class number of k. Let $\lambda_1, \ldots, \lambda_h$ be representatives of these $PSL_2(\mathcal{O})$-orbits. Furthermore, denote by Γ_{λ_i}, $i = 1, \ldots, h$, the isotropy group of λ_i in $PSL_2(\mathcal{O})$, and let $T(\lambda_i)$ be a fundamental domain for the action of Γ_{λ_i} on H. Now let $F_i = H(\lambda_i) \cap T(\lambda_i)$, $i = 1, \ldots, h_k$. Then

$$F = \bigcup_{i=1}^{h} F_i \tag{2}$$

is a fundamental domain for the action of $PSL_2(\mathcal{O})$ on H. We refer to Siegel's notes [20], p. 261–269, for a detailed proof.

Finally, we have as a consequence the following compactness criterion (cf. [20], p. 270): Let $r_1, \ldots, r_h$ be positive real numbers. Then the set

$$F(r_1, \ldots, r_h) = \{(z, \zeta) \in F \mid n_{\lambda_i}(z, \zeta) \geq r_i \text{ for all } 1 \leq i \leq h\} \tag{3}$$

is compact.

§3. The minimal incidence set

We now review Mendoza's construction of a contractible CW-complex $I(k)$ with a $PSL_2(\mathcal{O})$-action. $I(k)$ is a 2-dimensional closed subspace of H with $PSL_2(\mathcal{O})$-equivariant deformation retraction from H to $I(k)$. The quotient $\Gamma \backslash I(k)$ by a subgroup Γ of finite index of $PSL_2(\mathcal{O})$ is a compact finite CW complex, with cell structure inherited naturally from $I(k)$. In [14] he uses extensively the main facts in reduction theory.

3.1 The minimal incidence set $I(k)$ of the given imaginary quadratic number field k is defined as the set of all points (z, ζ) in H which lie in the minimal sets of (at least) two different cusps, i.e. $I(k)$ is formally given as

$$I(k) = \bigcup_{\substack{(\lambda,\mu) \\ \lambda,\mu \text{ distinct cusps}}} H(\lambda) \cap H(\mu). \tag{1}$$

By 2.2(1) one sees that $I(k)$ is stable under the action of $PSL_2(\mathcal{O})$. Moreover, $I(k)$ is closed since the sets $H(\lambda) \cap H(\mu)$ form a locally finite closed covering of $I(k)$.

Remark. If one does the same construction on the upper half plane (substitute the real variable x for the complex variable z, and the projective space $P_1(\mathbb{Q})$ for the cusps of k) one obtains the tree for $SL_2(\mathbb{Z})$ studied by Serre [19], p. 52.

3.2 THEOREM (Mendoza [14]). (i) *The minimal incidence set $I(k)$ in H associated to an imaginary quadratic field k is a closed subspace of the symmetric space H, such that $I(k)$ is invariant under the proper action of $PSL_2(\mathcal{O})$ and the quotient $PSL_2(\mathcal{O})\backslash I(k)$ is compact. Moreover, $I(k)$ is a $PSL_2(\mathcal{O})$-equivariant deformation retract of H, and hence connected and contractible.*

(ii) *The set $I(k)$ is naturally endowed with the structure of a 2-dimensional locally finite regular CW-complex. The action of $PSL_2(\mathcal{O})$ on $I(k)$ is cellular, and so, $PSL_2(\mathcal{O})\backslash I(k)$ is a finite CW-complex.*

Since the thesis [14] of Mendoza is not easily at hand everywhere we sketch his proof with his kind permission.

Ad(i). We observed already that $I(k)$ is closed. To show that $PSL_2(\mathcal{O})\backslash I(k)$ is compact it suffices to exhibit a compact set $K \subset H$ such that $I(k) = \Gamma \cdot K$. We take $K = I(k) \cap F$ where F is the fundamental domain described in 2.2. This set is non-empty, closed, and by 2.1(iii) contained in $F' = F \cap \{(z, \zeta) \in H \mid n_{\lambda_i}(z, \zeta) \geq C_2\}$, where $\lambda_r, \ldots, \lambda_h$ are representatives of $PSL_2(\mathcal{O})$-orbits of cusps. The compactness criterion (cf. 2.2.(3)) implies that F' is compact. To show that $I(k)$ is a deformation retract of H it suffices to prove that for each cusp λ the boundary $I(\lambda)$ is a deformation retract of $H(\lambda)$, since the sets $H(\lambda)$ make up a locally finite closed covering of $I(k)$ as pointed out before.

This latter assertion follows from the fact that the distance function n_λ is a Morse function without critical points in $H(\lambda) - I(\lambda)$. (The retraction is perpendicular to the level sets of n_λ, i.e. we retract along a geodesic. For $\lambda \neq \infty$ this is a vertical semicircle, for $\lambda = \infty$ a vertical half line). The $PSL_2(\mathcal{O})$-equivariance is a direct consequence of 2.1.(3).

Ad(ii). The main ingredient here is to prove that each set $H(\lambda)\cap H(\mu)$ with λ, μ distinct cusps can be obtained by intersecting a finite number of hemispheres with center in the plane $\zeta=0$ and vertical half planes. Hence $H(\lambda)\cap H(\mu)$ is in a natural way a regular cell. For the rather technical complete proof we refer to [14], p. 26–38.

Remarks. (1) The theorem is clearly true for any subgroup Γ in $PSL_2(\mathcal{O})$ of finite index. Furthermore, for any such Γ one can refine the natural cell structure such that the stabilizers Γ_x in Γ remain the same for all points x in an open cell.

(2) Note that the dimension of $I(k)$ is exactly the same as the virtual cohomological dimension of Γ.

3.3 If the class number of k is one, the reduction theory described in §2 coincides with classical reduction theory as initiated by Bianchi [2] and pursued by Humbert [13] (for an account of the latter one see Swan [24]). Therefore, in this case, the minimal incidence set $I(k)$ can be obtained as the translation by Γ of the 'bottom'-boundary $B(k)$ of the classical fundamental domains determined by Bianchi and Humbert. Indeed, for someone familiar with the geometry of the examples in [2] it is not too difficult to work out separately in each case of class number one a fundamental domain for the action of $PSL_2(\mathcal{O})$ on H which is good enough for actual cohomological computations. Part of this is done by Flöge in his unpublished thesis [8], where he also constructs fundamental domains in some cases of class number two. He has used this to give group theoretical descriptions of $PSL_2(\mathcal{O}_{-d})$ for $d=1, 2, 3, 5, 6, 7, 10, 11$ as amalgamated products of suitable subgroups or as HNN-extensions of such products. But we preferred to use Mendoza's conceptual and systematic approach in order to have some general foundations for later considerations in [9], [26].

§4. Finite subgroups of $PSL_2(\mathcal{O})$

In this section we list the finite groups which may occur as subgroups of $PSL_2(\mathcal{O})$, together with their integral homology. These homology groups appear in the spectral sequence described in §1 which we will use to compute the homology of $PSL_2(\mathcal{O})$.

4.1 The only finite subgroups which occur in $SL_2(\mathbb{C})$ are the binary polyhedral groups (see, e.g. [22], §4.4). Let x be an element of $SL_2(\mathcal{O})\leq SL_2(\mathbb{C})$ of finite order n. Then x has eigenvalues ρ and $\bar{\rho}$, where ρ is a primitive nth root of unity. Since $\operatorname{tr} x=\rho+\bar{\rho}$ is in $\mathcal{O}\cap\mathbb{R}=\mathbb{Z}$, we must have $n=1, 2, 3, 4$ or 6. Thus the only finite subgroups of $SL_2(\mathcal{O})$ which can occur are cyclic of the above orders, the quaternion group, binary tetrahedral group and binary octahedral group. Since the stabilizer of any point in H is finite and contains $\pm I$, the possible stabilizers in $PSL_2(\mathcal{O})$ are the cyclic groups of orders two and three, the Klein four-group $D_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group S_3 and the alternating group A_4. The integral homology of these groups can be computed by elementary means (see, e.g. [5]) and is given in the following tables:

4.2 LEMMA. *The integral homology of the finite groups $G = \mathbb{Z}/n$, $(n>0)$, D_2, S_3, A_4 respectively is given as follows (for simplicity we neglect to write the trivial coefficients $\mathbb{Z}$)*

$$H_q(\mathbb{Z}/n) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/n & q \text{ odd} \\ 0 & q \text{ even, } q>0 \end{cases} \tag{1}$$

$$H_q(D_2) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^{(q+3)/2} & q \text{ odd} \\ (\mathbb{Z}/2)^{q/2} & q \text{ even, } q>0 \end{cases} \tag{2}$$

$$H_q(S_3) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/2 & q \equiv 1\ (4) \\ 0 & q \equiv 2\ (4) \\ \mathbb{Z}/6 & q \equiv 3\ (4) \\ 0 & q \equiv 0\ (4),\ q>0. \end{cases} \tag{3}$$

$$H_q(A_4) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3 & q=6k+1 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 & q=6k+2 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6 & q=6k+3 \\ (\mathbb{Z}/2)^k & q=6k+4 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q=6k+5 \\ (\mathbb{Z}/2)^k & q=6(k+1) \end{cases} \tag{4}$$

In order to compute in the spectral sequence, we must also determine the maps induced on homology by the inclusions of cyclic groups into D_2, S_3 and A_4; these are given in the following series of lemmas.

4.3 LEMMA. (1) *Any inclusion* $i:\mathbb{Z}/2 \to S_3$ *induces an injection on homology.*

(2) *An inclusion* $i:\mathbb{Z}/3 \to S_3$ *induces an injection on homology in degrees congruent to* 3 *mod* 4 (*and is otherwise zero*).

Proof. One can directly compute the Leray spectral sequence of the extension

$$1 \to \mathbb{Z}/3 \to S_3 \to \mathbb{Z}/2 \to 1$$

The action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ is the non-trivial action, and the result is that $E^2_{pq} = 0$ for p, $q > 0$, $E^2_{p,0} = H_p(\mathbb{Z}/2)$ and $E^2_{0,q} = \mathbb{Z}/3$ in dimensions congruent to 3 mod 4, zero otherwise. The diagrams

$$\begin{tikzcd} & & \mathbb{Z}/2 \arrow[r, phantom, "="] \arrow[d, "i"'] & \mathbb{Z}/2 \arrow[d] & \\ 1 \arrow[r] & \mathbb{Z}/3 \arrow[r] & S_3 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

and

$$\begin{tikzcd} & \mathbb{Z}/3 \arrow[r, phantom, "="] \arrow[d] & \mathbb{Z}/3 \arrow[d] & & \\ 1 \arrow[r] & \mathbb{Z}/3 \arrow[r, "i"] & S_3 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd}$$

induce maps of $H_*(\mathbb{Z}/2)$ onto the bottom row of the spectral sequence, and of $H_*(\mathbb{Z}/3)$ onto the left-hand column of the spectral sequence, $H_q(\mathbb{Z}/3) \to H_0(\mathbb{Z}/2, H_q(\mathbb{Z}/3))$. Since all differentials are zero ($\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/3) = 0$), these maps induce maps on the abutment ($H_*(S_3)$) as claimed.

4.4 LEMMA. *Any inclusion* $i:\mathbb{Z}/2 \to D_2$ *induces an injection on homology in all dimensions.*

Proof. This is clear from the trivial extension

$$1 \to \mathbb{Z}/2 \xrightarrow{i} D_2 \to \mathbb{Z}/2 \to 1.$$

4.5 LEMMA. (1) *An inclusion* $i:\mathbb{Z}/3 \to A_4$ *induces injections on homology in all dimensions.*

(2) *An inclusion* $i:\mathbb{Z}/2 \to A_4$ *induces injections on homology in dimensions greater than* 1, *and is zero on* H_1.

Proof. We consider the spectral sequence of the extension

$$1 \to D_2 \to A_4 \to \mathbb{Z}/3 \to 1. \tag{3}$$

The homology of $\mathbb{Z}/3$ appears in the bottom row of the E^2-term, and every differential which originates in this row must be zero, since it lands in the 2-torsion group $H_p(\mathbb{Z}/3; H_q(D_2))$ for some $q>0$. Thus $E^2_{p,0}=E^\infty_{p,0}$, and the homology of $\mathbb{Z}/3$ injects into the homology of A_4.

For a map $i:\mathbb{Z}/2\to A_4$, factor through the (unique) 2-sylow subgroup D_2:

$$\begin{tikzcd} & \mathbb{Z}/2 \arrow[d, "\alpha"'] \arrow[dr, "i"] & & & \\ 1 \arrow[r] & D_2 \arrow[r, "\beta"] & A_4 \arrow[r] & \mathbb{Z}/3 \arrow[r] & 1 \end{tikzcd} \tag{4}$$

Then the homology of $\mathbb{Z}/2$ maps to the left-hand column of the spectral sequence by the maps

$$H_q(\mathbb{Z}/2) \xrightarrow{\alpha_*} H_q(D_2) \xrightarrow{\pi} H_0(\mathbb{Z}/3; H_q(D_2)). \tag{5}$$

The action of $\mathbb{Z}/3$ on D_2 is non-trivial, and one can compute that the composition $\pi\circ\alpha_*$ in (5) is injective for $q\geq 2$, and zero for $q=1$; also $E^2_{pq}=0$ for p and $q>0$, so all the differentials are zero, and $E^2=E^\infty$. Thus the injectivity properties extend to the abutment, and i_* is injective for $q>1$.

§5. Integral homology of $PSL_2(\mathcal{O}_{-d})$

We denote by $\mathcal{O}_{-d}$ the ring of integers of the imaginary quadratic number field $k=\mathbb{Q}(\sqrt{-d})$, $d\in\mathbb{N}$, d squarefree. In this section we will give a reasonably detailed explanation of the calculations for the integral homology of $PSL_2(\mathcal{O}_{-d})$ for $d=2$, then list the results of our computations for the other cases where $\mathcal{O}_{-d}$ is a euclidean ring, i.e. $d=1,3,7,11$.

For simplicity, we leave out the coefficients $\mathbb{Z}$ when we mean trivial $\mathbb{Z}$ coefficients.

5.1 We begin with a notion of fundamental domain which takes the cell structure on $I(k)$ into account. A finite sub-complex F of $I(k)$ is called a *fundamental cellular domain* for $PSL_2(\mathcal{O})=\Gamma$ if $I(k)=\Gamma\cdot F$ and if points in open induced 2-cells are not $PSL_2(\mathcal{O})$-equivalent. If we denote by "$\sim$" the cellular equivalence relation on F induced by identification of 0-cells or 1-cells, then it follows easily that $\sim\backslash F$ and $PSL_2(\mathcal{O})\backslash I(k)$ are isomorphic CW-complexes.

5.2 *The case $d=2$.*

Let $\omega=\sqrt{-2}$; then ω and 1 generate $\mathcal{O}_{-2}$ as a free $\mathbb{Z}$-module (lattice) in

$\mathbb{Q}(\sqrt{-2}) = k$. A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-2})$ on the complex $I(k)$ is the area on the unit hemisphere centered at $(0,0) \subseteq \mathbb{C} \times \mathbb{R}^+$ lying above the rectangle in $\mathbb{C}$ with vertices $\pm(\sqrt{2}/2)i$ and $\frac{1}{2} \pm (\sqrt{2}/2)i$ (cf. 4.2.5 in [14]). We label the vertices of this two-cell as follows:

$$P_1 = \left(-\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}\right), \quad P_2 = \left(\frac{1}{2} - \frac{\sqrt{2}}{2}i, \frac{1}{2}\right), \quad P_3 = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}i, \frac{1}{2}\right), \quad P_4 = \left(\frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2}\right).$$

If we let Γ_i denote the stabilizer in $PSL_2(\mathcal{O}_{-2})$ of P_i, and Γ_{ij} the stabilizer of the one-cell P_iP_j, then we have specific descriptions of these stabilizers as follows: Let $a = \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then:

$$\begin{aligned}
\Gamma_{12} &= \langle a \rangle \cong \mathbb{Z}/2; & \Gamma_1 &= \langle a, c \rangle \cong D_2 \\
\Gamma_{23} &= \langle b \rangle \cong \mathbb{Z}/3; & \Gamma_2 &= \langle a, b \rangle \cong A_4 \\
\Gamma_{34} &= \langle \bar{a} \rangle \cong \mathbb{Z}/2; & \Gamma_3 &= \langle \bar{a}, b \rangle \cong A_4 \\
\Gamma_{41} &= \langle c \rangle \cong \mathbb{Z}/2; & \Gamma_4 &= \langle \bar{a}, c \rangle \cong D_2
\end{aligned} \tag{2}$$

Note that the stabilizer of the only 2-cell is trivial. In pictorial form, the fundamental domain and stabilizers are

$$\begin{tikzcd}
P_4 \arrow[r, "\langle \bar{a} \rangle = \mathbb{Z}/2"] \arrow[d, dash, "\mathbb{Z}/2 = \langle c \rangle"'] & P_3 \arrow[d, dash, "\langle b \rangle = \mathbb{Z}/3"] \\
P_1 \arrow[r, "\langle a \rangle = \mathbb{Z}/2"'] & P_2
\end{tikzcd}$$

The top and bottom edges of the rectangle are identified by the element $g = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ of $PSL_2(\mathcal{O}_{-2})$: $gP_1P_2 = P_4P_3$. These are the only identifications, so the quotient by $PSL_2(\mathcal{O}_{-2})$ is a cylinder.

We now feed this information into the spectral sequence described in §1, with

$$E^1_{pq} = \bigoplus_{p\text{-cells } \sigma} H_q(\Gamma_\sigma) \Rightarrow H_{p+q}(PSL_2(\mathcal{O}_{-2})).$$

There are only three non-zero columns, corresponding to the 0−, 1− and 2− cells of the complex. In fact, since the stabilizer of the 2-cell is trivial, the third column is zero except that $E^1_{2,0}=\mathbb{Z}$. For $q>0$, the qth row is

$$H_q(\Gamma_1)\oplus H_q(\Gamma_2) \xleftarrow{d^1} H_q(\Gamma_{12})\oplus H_q(\Gamma_{23})\oplus H_q(\Gamma_{14}) \tag{3}$$

i.e.

$$\begin{array}{ccc} H_q(D_2)\oplus H_q(A_4) & \xleftarrow{d^1} & H_q(\mathbb{Z}/2)\oplus H_q(\mathbb{Z}/3)\oplus H_q(\mathbb{Z}/2) \\ (-i_*a+g_*c-i_*c,\ i_*a+g_*b-i_*b) & \xleftarrow{d^1} & (a,b,c) \end{array} \tag{4}$$

where the maps i_* are induced by inclusion, and g_* by conjugation by g. In particular, $g_*: H_q(\Gamma_{14})\to H_q(\Gamma_1)$ is induced by the map from Γ_{14} to Γ_1 sending $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ to $\begin{pmatrix}\omega & -1\\ -1 & -\omega\end{pmatrix}$, and $g_*: H_q(\Gamma_{23})\to H_q(\Gamma_2)$ is induced by the map from Γ_{23} to Γ_2 sending $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$ to $\begin{pmatrix}1-\omega & \omega+1\\ 1 & \omega\end{pmatrix}$.

The bottom row of the E^1-term is just a $\mathbb{Z}$-chain complex giving the homology of the quotient; thus $E^2_{0,0}=E^2_{1,0}=\mathbb{Z}$, and $E^2_{p,0}=0$ for $p\geq 2$. Since the third column has disappeared entirely by E^2, we have $E^2=E^\infty$. Note that for q even and bigger than zero 4.2.(1), implies that $H_q(\mathbb{Z}/2)=H_q(\mathbb{Z}/3)=0$, so $E^\infty_{1,q}=E^1_{1,q}=0$ and $E^\infty_{0,q}=E^1_{0,q}=H_q(D_2)\oplus H_q(A_4)$. For q odd, we must calculate using the explicit description of the d^1-map given in 1.1.(7). For example, to calculate $g_*: H_q(\Gamma_{14})\to H_q(\Gamma_1)$, we write a resolution for $\Gamma_{14}=\mathbb{Z}/2=\langle t\rangle$, a resolution for $\Gamma_1=D_2$ and calculate the chain map induced by the map $g:\Gamma_{14}\to\Gamma_1$ given by $t\mapsto ac$:

$$\begin{tikzcd} {} \arrow[r, "t-1"] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "t+1"] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r, "t-1"] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[r] \arrow[d, "g"] & 0 \\ {} \arrow[r, "\partial"'] & (\mathbb{Z}[D_2])^3 \arrow[r, "\begin{pmatrix} a+1 & 1-c & 0 \\ 0 & a-1 & c+1 \end{pmatrix}"'] & \mathbb{Z}[D_2]^2 \arrow[r, "{(a-1,\ c-1)}"'] & \mathbb{Z}[D_2] \arrow[r] & 0. \end{tikzcd} \tag{5}$$

With $\mathbb{Z}$-coefficients, this becomes

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "\begin{pmatrix}1\\1\\1\\1\end{pmatrix}"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "\begin{pmatrix}1\\1\\1\end{pmatrix}"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, "\begin{pmatrix}1\\1\end{pmatrix}"'] & \mathbb{Z} \arrow[r] \arrow[d, "(1)"'] & 0 \\ \mathbb{Z}^5 \arrow[r, "\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}"'] & \mathbb{Z}^4 \arrow[r, "\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}"'] & \mathbb{Z}^3 \arrow[r, "\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}"'] & \mathbb{Z}^2 \arrow[r, "(0\ \ 0)"'] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{6}$$

The effect on homology is now easily calculated, the map $g_*:\mathbb{Z}/2\to(\mathbb{Z}/2)^{(q+3)/2}$ sends x to $(x, x, \ldots, x)$. In a similar manner, we make use of lemmas 4.3.–4.5. to calculate the other maps i_* and g_*. The end result is that the d^1-map is given by

$$\begin{array}{llcl} (\mathbb{Z}/2)^{(q+3)/2} & \oplus(\mathbb{Z}/2)^k\oplus(\mathbb{Z}/3) & \xleftarrow{d^1} & \mathbb{Z}/2\oplus\mathbb{Z}/3\oplus\mathbb{Z}/2 \\ (x-z, x, \ldots, x, x-x)\oplus(z,0) & & \longleftarrow & (z, \quad y, \quad x) \end{array} \tag{7}$$

We now have $E^2_{1q}=\operatorname{Ker} d_1=\mathbb{Z}/3$ and $E^2_{0,q}=\operatorname{coker}\ d_1=(H_q(D_2)\oplus H_q(A_4))/(\operatorname{im} d_1\cong(\mathbb{Z}/2\oplus\mathbb{Z}/2))$. Since $E^2=E^\infty$, the spectral sequence has E^∞-term

$$\begin{array}{l|ll} q & & \\ & \vdots & \vdots \\ q \text{ even} & H_q(D_2)\oplus H_q(A_4) & 0 \\ q \text{ odd} & (H_q(D_2)\oplus H_q(A_4))/(\mathbb{Z}/2\oplus\mathbb{Z}/2) & \mathbb{Z}/3 \\ & \vdots & \vdots \\ & \mathbb{Z}/2\oplus\mathbb{Z}/2 & 0 \\ & \mathbb{Z}/2\oplus\mathbb{Z}/3 & \mathbb{Z}/2\oplus\mathbb{Z}/3 \\ & \mathbb{Z} & \mathbb{Z} \\ \hline & & \qquad\qquad p. \end{array} \tag{8}$$

We can now state the theorem.

5.3 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-2})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-2}))\cong\begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}\oplus\mathbb{Z}/6 & q=1 \\ \mathbb{Z}/4\oplus\mathbb{Z}/6 & q=2 \\ (\mathbb{Z}/2)^{2q/3}\oplus\mathbb{Z}/3 & q\equiv 0(3), q>0 \\ (\mathbb{Z}/2)^{2(q-1)/3}\oplus\mathbb{Z}/3 & q\equiv 1(3), q>1 \\ (\mathbb{Z}/2)^{2(q+1)/3}\oplus\mathbb{Z}/3 & q\equiv 2(3), q>2 \end{cases}$$

For $q\neq 2$ these results follow directly from the above computation 5.2.(8) of the E^∞-term of the spectral sequence together with the descriptions of the homology of A_4 and D_2 given in §4. For $q=2$, the spectral sequence gives us an exact sequence

$$1\to\mathbb{Z}/2\oplus\mathbb{Z}/2\to H_2(PSL_s(\mathcal{O}_{-2}))\to\mathbb{Z}/2\oplus\mathbb{Z}/3\to 1.$$

To resolve the ambiguity in the 2-torsion, we consider our spectral sequence with $\mathbb{Z}/2$-coefficients; we find that for $0 \le q \le 2$,

$$E^1 = \begin{array}{r|llll} q & (\mathbb{Z}/2)^3 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus 0 \oplus \mathbb{Z}/2 & & \\ & (c-a, c, 0, a) & \leftarrow\!\shortmid (a, 0, c) & & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus 0 \oplus \mathbb{Z}/2 & & \\ & (c-a, 0) & \leftarrow\!\shortmid (a, 0, c) & & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \leftarrow \mathbb{Z}/2 & \\ & (-a, a) & \leftarrow\!\shortmid (a, b, c) & & \\ \hline & & & & p \end{array}$$

giving

$$E^\infty = E^2 = \begin{array}{r|lll} q & \vdots & \vdots & \\ & \mathbb{Z}/2 \oplus \mathbb{Z}/2 & 0 & \\ & \mathbb{Z}/2 & \mathbb{Z}/2 & \\ & \mathbb{Z}/2 & \mathbb{Z}/2 & 0 \\ \hline & & & p \end{array}$$

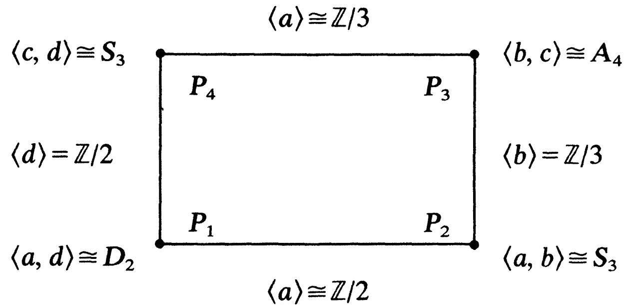
Thus $H_2(PSL_2(\mathcal{O}_{-2}); \mathbb{Z}/2) \cong (\mathbb{Z}/2)^3$. By the universal coefficient theorem, this is isomorphic to $H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}(H_1(PSL_2(\mathcal{O}_{-2})), \mathbb{Z}/2) = H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Therefore $H_2(PSL_2(\mathcal{O}_{-2})) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^2$, so $H_2(PSL_2(\mathcal{O}_{-2})) \cong \mathbb{Z}/6 \oplus \mathbb{Z}/4$.

We will now list the theorems for the other euclidean cases, indicating briefly any necessary modifications in the computations. We denote by $S = \{(z, \zeta) \in H \mid |z|^2 + \zeta^2 = 1\}$ the hemisphere with center $(0, 0)$ and radius 1. For points in S we will sometimes give only the first coordinate.

5.4 *The case* $d = 1$. A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-1})$ on the complex $I(\mathbb{Q}(\sqrt{-1}))$ is the set $F = \{(z, \zeta) \in S \mid 0 \le \operatorname{Re} z \le \frac{1}{2}, 0 \le \operatorname{Im} z \le \frac{1}{2}\}$; the vertices are the points $P_1 = (0, 1)$, $P_2 = (\frac{1}{2}, \sqrt{3}/2)$, $P_3 = (\frac{1}{2} + \frac{1}{2}i, \sqrt{2})$, $P_4 = (\frac{1}{2}i, \sqrt{3}/2)$. There are no further identifications (cf. [14], 4.1.9 or [2], §12). Let

$$a = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then the cellular domain and the stabilizers of the cells are given in pictorial form as follows:



By 4.3–4.5 the inclusion maps $\Gamma_{ij} \to \Gamma_i$, $1 \le i, j \le 4$, of the stabilizers induce injections on homology, except for $\Gamma_{34} \to \Gamma_4$ and $\Gamma_{23} \to \Gamma_2$. These induce (cf. 4.2.) injections on homology of degree n if $n \equiv 3 \bmod 4$, and otherwise induce the zero map.

5.5 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-1})$ is given by*

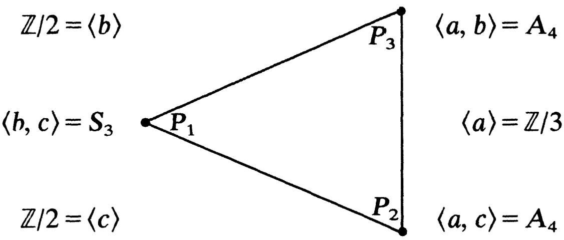
$$H_q(PSL_2\mathcal{O}_{-1}) \cong \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 & q=12k+1 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & q=12k+2 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3 & q=12k+3 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^2 & q=12k+4 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^5 & q=12k+5 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/3 & q=12k+6 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/3 & q=12k+7 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 & q=12k+8 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^7 & q=12k+9 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^6 \oplus \mathbb{Z}/3 & q=12k+10 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^{10} \oplus \mathbb{Z}/3 & q=12k+11 \\ (\mathbb{Z}/2)^{8k} \oplus (\mathbb{Z}/2)^8 & q=12(k+1) \end{cases}$$

where $k \in \mathbb{N}$.

5.6 *The case $d=3$.* A fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-3})$ on the complex $I(\mathbb{Q}(\sqrt{-3}))$ is a subset of the unit hemisphere S; the following picture contains all the information we need. The exact coordinates of the points P_i can be easily worked out if the reader so desires. (cf. [14], 4.2.3 or [2], §13). Let

$$a=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix},\qquad b=\begin{pmatrix}0 & \omega\\ \omega^2 & 0\end{pmatrix},\qquad c=\begin{pmatrix}0 & \omega^2\\ \omega & 0\end{pmatrix} \tag{1}$$

where $\omega=(\frac{1}{2})(1+\sqrt{3}\,i)$. We have in pictorial form



(2)

The map $d_1: E^1_{1,q}\to E^1_{0,q}$ is injective for $q\geq 2$; for $q=1$, $d_1: H_1(\mathbb{Z}/2)\oplus H_1(\mathbb{Z}/3)\oplus H_1(\mathbb{Z}/2)\to H_1(S_3)\oplus H_1(A_4)\oplus H_1(A_4)$ sends (a,b,c) to $(-a-c,-b,b)$, and for $q=0$, the d_1-maps are the boundary maps for the integral homology of the (contractible) fundamental domain.

5.7 THEOREM. *The integral homology of* $PSL_2(\mathcal{O}_{-3})$ *is given by*

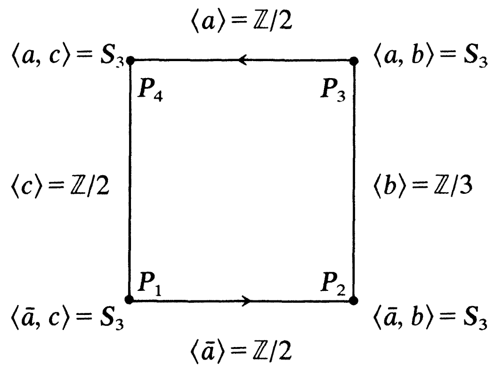
$$H_q(PSL_2(\mathcal{O}_{-3}))=\begin{cases}\mathbb{Z} & q=0\\ (\mathbb{Z}/2)^{4k} & q=12k>0\\ \mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+1\\ \mathbb{Z}/4\oplus\mathbb{Z}/2\oplus(\mathbb{Z}/2)^{4k} & q=12k+2\\ \mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+3\\ (\mathbb{Z}/2)^{4k} & q=12k+4\\ \mathbb{Z}/6\oplus(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+5\\ (\mathbb{Z}/2)^4\oplus(\mathbb{Z}/2)^{4k} & q=12k+6\\ \mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+7\\ (\mathbb{Z}/2)^4\oplus(\mathbb{Z}/2)^{4k} & q=12k+8\\ \mathbb{Z}/6\oplus(\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+9\\ (\mathbb{Z}/2)^2\oplus(\mathbb{Z}/2)^{4k} & q=12k+10\\ (\mathbb{Z}/2)^4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/3\oplus(\mathbb{Z}/2)^{4k} & q=12k+11\end{cases}$$

where $k\in\mathbb{N}$.

5.8 *The case* $d=7$. The fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-7})$ on the complex $I(\mathbb{Q}(\sqrt{-7}))$ is again a subset of the hemisphere S; the domain and the stabilizers of the cells are given in pictorial form as follows, if we let

$$a=\begin{pmatrix}1 & -\bar{\omega}\\ \bar{\omega} & -1\end{pmatrix},\qquad b=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix},\qquad c=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

where $\omega=(\frac{1}{2})(1+\sqrt{-7})$. (cf. [14], 4.2.11. or [2], §16).



The element $g=\begin{pmatrix}0 & -1\\ 1 & \omega-1\end{pmatrix}$ sends P_1P_2 to P_3P_4, so the quotient of $I(\mathbb{Q}(\sqrt{-7}))$ by $PSL_2(\mathcal{O}_{-7})$ is topologically a Möbius band. The homology of each stabilizer is periodic of period 2 or 4 by 4.2. so we need only compute the induced maps on H_1 and H_3 to calculate the $E^2=E^\infty$-term of the spectral sequence. The result is:

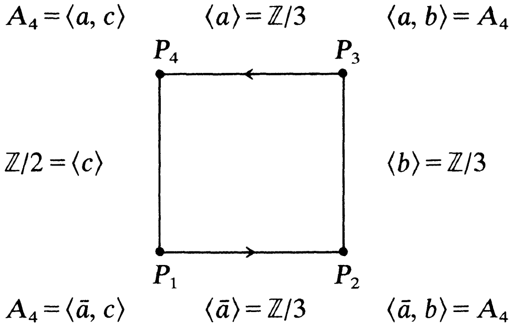
5.9 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-7})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-7}))\cong\begin{cases}\mathbb{Z} & q=0\\ \mathbb{Z}\oplus\mathbb{Z}/2 & q=1\\ \mathbb{Z}/6 & q\equiv 2,3(4)\\ \mathbb{Z}/2 & q\equiv 0,1(4)\quad q\geq 4\end{cases}$$

5.10 *The case* $d=11$. From the topological point of view the situation is quite similar to the case $d=7$. Let $\omega=(\frac{1}{2})(1+\sqrt{-11})$, and put

$$a=\begin{pmatrix}-2 & \omega\\ -\bar{\omega} & 1\end{pmatrix},\qquad b=\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix},\qquad c=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The fundamental cellular domain for the action of $PSL_2(\mathcal{O}_{-11})$ on the complex $I(\mathbb{Q}(\sqrt{-11}))$ is again a subset of the hemisphere S; the domain and the stabilizers of the cells can be visualized as follows: (cf. [14], 4.2.12 or [2], §16).



The element $g = \begin{pmatrix} -\omega & 1 \\ -1 & 0 \end{pmatrix}$ sends P_1P_2 to P_3P_4, and there are no further identifications, so the quotient $PSL_2(\mathcal{O}_{-11})\backslash I(\mathbb{Q}(\sqrt{-11}))$ is topologically a Möbius band. The d^1-maps in odd dimensions $q \geq 3$ are $(x, y) \mapsto (x-y, y-x)$ on 3-torsion and injective on 2-torsion; for $q=1$, the d^1-map is the same on 3-torsion but zero on $H_1(\mathbb{Z}/2)$. There is ambiguity in the 2-torsion of $H_2(PSL_2(\mathcal{O}_{-11}))$ which can be resolved, as in the case $d=2$, by looking at the spectral sequence with $\mathbb{Z}/2$-coefficients. The result is:

5.11 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-11})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-11})) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/3 & q=1 \\ \mathbb{Z}/6 \oplus \mathbb{Z}/4 & q=2 \\ (\mathbb{Z}/2)^{k+1} \oplus \mathbb{Z}/3 & q=3k-1>2 \\ (\mathbb{Z}/2)^{k} \oplus \mathbb{Z}/3 & q=3k>0 \\ (\mathbb{Z}/2)^{k-1} \oplus \mathbb{Z}/3 & q=3k+1>1 \end{cases}$$

where $k \in \mathbb{N}$.

§6. Integral cohomology of $SL_2(\mathcal{O}_{-d})$ and homology with Steinberg coefficients

The complex $I(k)$ and spectral sequence in §1 (resp. a cohomological analogue of it) may be used to compute homology and cohomology groups for the groups

$SL_2(\mathcal{O}_{-d})$, $PGL_2(\mathcal{O}_{-d})$ and $GL_2(\mathcal{O}_{-d})$ as well as $PSL_2(\mathcal{O}_{-d})$. The homology of $SL_2(\mathcal{O}_{-d})$ with coefficients in the Steinberg module $\mathrm{St}\,(2)_{-d}$ of $(\mathbb{Q}(\sqrt{-d}))^2$ has particular interest for algebraic K-theory, (cf. [15]); we indicate here how to do this computation for $d=2$ and list the results for the other euclidean cases. Since homology with Steinberg coefficients is dual to cohomology in degree >2 (see 6.4.) we begin by computing the integral cohomology of $SL_2(\mathcal{O}_{-d})$.

6.1 As noted in §5, the groups which may possibly appear as finite subgroups of $SL_2(\mathcal{O}_{-d})$ are the cyclic groups of orders 2, 3, 4 and 6, the quaternion group Q, the binary octahedral group D and the binary tetrahedral group Te. Any finite subgroup of $SL_2(\mathbb{C})$ acts freely on the maximal compact subgroup $SU_2 \subset SL_2(\mathbb{C})$, which is a 3-sphere. Such a subgroup must therefore have periodic cohomology of period dividing 4, so we need only compute four cohomology groups to obtain each of the following cohomologies:

$$H^q(Q)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/8 & q\equiv 0(4),\ q>0\end{cases} \tag{1}$$

$$H^q(D)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/4 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/12 & q\equiv 0(4),\ q>0\end{cases} \tag{2}$$

$$H^q(Te)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q\equiv 1(4)\\ \mathbb{Z}/3 & q\equiv 2(4)\\ 0 & q\equiv 3(4)\\ \mathbb{Z}/24 & q\equiv 0(4),\ q>0\end{cases} \tag{3}$$

$$H^q(\mathbb{Z}/n)=\begin{cases}\mathbb{Z} & q=0\\ 0 & q \text{ odd}\\ \mathbb{Z}/n & q \text{ even},\ q>0\end{cases} \tag{4}$$

6.2 We now specialize to the case $d=2$, and look at the spectral sequence of

§1 in cohomology (cf. [5], VII), we have

$$E_1^{pq} = \bigoplus_{\substack{\text{orbits of} \\ p\text{-cells } \sigma_p}} H^q(\Gamma_{\sigma_p}) \Rightarrow H^{p+q}(SL_2(\mathcal{O}_{-2})). \tag{1}$$

Note that the stabilizer of the 2-cell in the fundamental cellular domain for the action of $SL_2(\mathcal{O}_{-2})$ on $I(\mathbb{Q}(\sqrt{-2}))$ is now $\mathbb{Z}/2$, and not trivial as it was for $PSL_2(\mathcal{O}_{-2})$. The E_1-term and d_1-maps are as follows:

$$
\begin{array}{c|ccccc}
q & & & & & \\
4 & \mathbb{Z}/8\oplus\mathbb{Z}/24 & \longrightarrow & \mathbb{Z}/4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \\
 & (x, y) & \longmapsto & (y-x, 0, 0) & & \\
3 & 0 & & 0 & & 0 \\
2 & (\mathbb{Z}/2)^2\oplus\mathbb{Z}/3 & \longrightarrow & \mathbb{Z}/4\oplus\mathbb{Z}/6\oplus\mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \\
 & (x_1, x_2, y) & \longmapsto & (2x_2, 0, 2x_1-2x_2) & & \\
1 & 0 & & 0 & & 0 \\
0 & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & (x, y) & \longmapsto & (y-x, 0, 0) & & \\
 & & & (a, b, c) & \longmapsto & (c-b) \\
\hline
 & 1 & & 2 & & 3 \quad p
\end{array} \tag{2}
$$

The result of these computations is

6.3 THEOREM. *The integral cohomology of $SL_2(\mathcal{O}_{-2})$ is given by*

$$H^q(SL_2(\mathcal{O}_{-2})) \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} & q=1 \\ \mathbb{Z}/3 & q\equiv 2(4) \\ \mathbb{Z}/2\oplus\mathbb{Z}/6 & q\equiv 3(4) \\ \mathbb{Z}/2\oplus\mathbb{Z}/24 & q\equiv 0(4),\, q>0 \\ \mathbb{Z}/12 & q\equiv 1(4),\, q>1 \end{cases}$$

6.4 To obtain the homology of $SL_2(\mathcal{O}_{-2})$ with coefficients in the Steinberg module $\mathrm{St}\,(2)_{-2}$, we use Farrell–Tate cohomology theory $\hat{H}^*$ for $SL_2(\mathcal{O}_{-2})$ (cf. [7] or [5], chap. X). By a general result of Borel–Serre ([4], 11.4) the group $SL_2(\mathcal{O}_{-2})$ is a virtual duality group of dimension 2 (in the sense of [3] or [6], §3) whose dualizing module is the Steinberg module $\mathrm{St}\,(2)_{-2} = H^2(SL_2(\mathcal{O}_{-2}), \mathbb{Z}[SL_2(\mathcal{O}_{-2})])$ with its natural $SL_2(\mathcal{O}_{-2})$-action (cf. [4], 11.4. and 8.6). Thus, for $q>2$, there is an isomorphism ([6], 11.7.):

$$H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \cong \hat{H}^{1-q}(SL_2(\mathcal{O}_{-2})). \tag{1}$$

For $0 \leq q \leq 2$, there is an exact sequence relating Farrell–Tate cohomology, the regular cohomology and homology with Steinberg coefficients (cf. [6], 11.8.); for simplicity we abbreviate $\hat{H}^* = \hat{H}^*(SL_2(\mathcal{O}_{-2}))$ resp. $H^* = H^*(SL_2(\mathcal{O}_{-2}))$:

$$\begin{aligned} 0 \to \hat{H}^{-1} \to H_2(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) & \\ \to H^0 \xrightarrow{\alpha_0} \hat{H}^0 \to H_1(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) & \\ \to H^1 \xrightarrow{\alpha_1} \hat{H}^1 \to H_0(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \to H^2 \xrightarrow{\alpha_2} \hat{H}^2 \to 0. & \end{aligned} \tag{2}$$

The spectral sequence 6.2(1) can be used to compute $\hat{H}^*(SL_2(\mathcal{O}_{-2}))$; we have ([5], X, 4.1.)

$$\hat{E}_1^{p,q} = \bigoplus_{\sigma \in \Sigma_p} \hat{H}^q(\Gamma_\sigma) \Rightarrow \hat{H}^{p+q}(SL_2(\mathcal{O}_{-2})). \tag{3}$$

Note that for the finite groups Γ_σ, the Farrell–Tate groups $\hat{H}^q(\Gamma_\sigma)$ coincide with the standard Tate cohomology groups. The maps $\alpha_i: H^i \to \hat{H}^i$, in (2) are then induced by the maps on the cohomology of the stabilizers Γ_σ in the spectral sequences; these are the standard maps from cohomology to Tate cohomology, i.e. for a finite group Γ_σ of order $|\Gamma_\sigma|$ the map $H^i(\Gamma_\sigma) \to \hat{H}^i(\Gamma_\sigma)$ is an isomorphism for $i>0$ and $H^0(\Gamma_\sigma) \to \hat{H}^0(\Gamma_\sigma)$ is the morphism $\mathbb{Z} \to \mathbb{Z}/|\Gamma_\sigma|$. We use these remarks to prove

6.5 THEOREM *The cohomology of $SL_2(\mathcal{O}_{-2})$ with coefficients in the Steinberg module* $\mathrm{St}\,(2)_{-2}$ *is given by*

$$H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) = \begin{cases} 0 & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q=1 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q=2 \\ \mathbb{Z}/3 & q \equiv 3(4) \\ \mathbb{Z}/12 & q \equiv 0(4), q>0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/24 & q \equiv 1\,(4), q>1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q \equiv 2(4), q>2 \end{cases}$$

Proof. For $q \geq 3$, 6.4.(1) says that $H_q(SL_2(\mathcal{O}_{-2}), \mathrm{St}\,(2)_{-2}) \cong \hat{H}^{1-q}(SL_2(\mathcal{O}_{-2}))$; since $\hat{H}^*(SL_2(\mathcal{O}_{-2}))$ is periodic of period 4, and $\hat{H}^i(SL_2(\mathcal{O}_{-2})) \cong H^i(SL_2(\mathcal{O}_{-2}))$ for $i \geq 3$ (cf. [6], 11.4), the result follows from our calculation of $H^*(SL_2(\mathcal{O}_{-2}))$ in Theorem 6.3. For $0 \leq q \leq 2$, the result follows from the calculations of the maps α_i in 6.4.(2) as outlined above.

We now state the results of our computations of $H_q(SL_2(\mathcal{O}_{-d}), \mathrm{St}\,(2)_{-d})$ in the other euclidean cases:

6.6 THEOREM. *The cohomology of $SL_2(\mathcal{O}_{-d})$ for $d = 1, 3, 7, 11$ with coefficients in the Steinberg module* $\mathrm{St}\,(2)_{-d}$ *is given by*

$$H_q(SL_2(\mathcal{O}_{-1}), \mathrm{St}\,(2)_{-1}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z}/4 & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/6 & q = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & q \equiv 3(4) \\ 0 & q \equiv 0(4),\ q > 0 \\ \mathbb{Z}/12 \oplus \mathbb{Z}/8 & q \equiv 1(4),\ q > 1 \\ \mathbb{Z}/6 & q \equiv 2(4),\ q > 2 \end{cases} \tag{1}$$

$$H_q(SL_2(\mathcal{O}_{-3}), \mathrm{St}\,(2)_{-3}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z}/6 & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/4 & q = 2 \\ \mathbb{Z}/3 & q \equiv 3(4) \\ 0 & q \equiv 0(4),\ q \geq 4 \\ \mathbb{Z}/24 \oplus \mathbb{Z}/6 & q \equiv 1(4),\ q \geq 5 \\ \mathbb{Z}/4 & q \equiv 2(4),\ q \geq 6 \end{cases} \tag{2}$$

$$H_q(SL_2(\mathcal{O}_{-7}), \mathrm{St}\,(2)_{-7}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z} & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/12 & q = 2 \\ \mathbb{Z}/4 & q \equiv 3(4) \\ \mathbb{Z}/4 & q \equiv 0(4),\ q \geq 4 \\ \mathbb{Z}/12 & q \equiv 1(4),\ q \geq 5 \\ \mathbb{Z}/12 & q \equiv 2(4),\ q \geq 6 \end{cases} \tag{3}$$

$$H_q(SL_2(\mathcal{O}_{-11}), \mathrm{St}\,(2)_{-11}) = \begin{cases} 0 & q=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & q=1 \\ \mathbb{Z}\oplus\mathbb{Z}/12\oplus\mathbb{Z}/2 & q=2 \\ \mathbb{Z}/3 & q\equiv 3(4) \\ \mathbb{Z}/6 & q\equiv 0(4),\ q\geq 4 \\ \mathbb{Z}/24\oplus\mathbb{Z}/2 & q\equiv 1(4),\ q\geq 5 \\ \mathbb{Z}/12\oplus\mathbb{Z}/2 & q\equiv 2(4),\ q\geq 6 \end{cases} \tag{4}$$

Remark. Up to 2-torsion $H_q(SL_2(\mathcal{O}_{-1}), \mathrm{St}\,(2)_{-1})$ was determined by Staffeldt ([23], Thm. IV.1.3.).

6.7 *Torsion classes in* $H^2(PSL_2(\mathcal{O}_{-d}))$. We conclude this paper with some observations concerning torsion classes in $H^*(PSL_2(\mathcal{O}_{-d}))$. There is a natural map between the usual cohomology and the Farrell cohomology of a subgroup Γ of finite index in $PSL_2(\mathcal{O}_{-d})$

$$H^*(\Gamma) \to \hat{H}^*(\Gamma)$$

which is an isomorphism for $q > vcd(\Gamma)$, and one has $\hat{H}^*(\Gamma) = 0$ if Γ is torsionfree. It is shown in [6], §15 that a great deal of information about $\hat{H}^*(\Gamma)$ can be extracted from the finite subgroups of Γ. The arguments there are of a general nature. As pointed out in §4 there is only 2- and 3-torsion in $\hat{H}^*(PSL_2(\mathcal{O}_{-d}))$, and in the euclidean cases $d = 1, 2, 3, 7, 11$ we considered this is also true for the usual cohomology $H^*(PSL_2(\mathcal{O}_{-d}))$ (cf. §5).

We will give now some examples of subgroups Γ of small index in $PSL_2(\mathcal{O}_{-d})$, $d = 1, 3$, where one has torsion classes in the low-dimensional cohomology of Γ, whose order p is different from 2 and 3. It would be of great interest to have an arithmetic explanation for these phenomena. For more details on this subject see [9].

(1) The group $PSL_2(\mathcal{O}_{-3})$ is generated by the matrices

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

where $\omega = -1/2 + i\sqrt{3}/2$. In [9] it will be shown that there are seven conjugacy classes of subgroups of index 12 in $PSL_2(\mathcal{O}_{-3})$. One of these classes can be represented by the torsionfree group

$$\Gamma_2 = \langle x, y \mid xyx^{-1}yxy^{-1}x^{-1}yx^{-1}y^{-1} = 1\rangle$$

where $x = a$ resp. $y = bcb$. We note that the manifold $\Gamma_2\backslash H$ is homeomorphic to the complement of the figure-eight-knot in the three sphere S^3 (cf. [16], [17]); in particular one has $H_1(\Gamma_2) = \Gamma_2^{ab} \cong \mathbb{Z}$.

Another class can be represented by

$$\Gamma_7 = \langle u, v \mid uv\, uv\, uv^2\, uv^{-1}uv^2 = 1\rangle$$

where $u = a^2$, $v = abc\ aba\ c^{-1}bc^{-1}ba^{-1}$. One sees easily that

$$H_1(\Gamma_7) = \Gamma_7^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/5$$

and uv is indeed an element of order 5 in Γ_7^{ab}. This implies that there is a torsion class of order 5 in $H^2(\Gamma_7)$.

(2) For a given prime ideal f of degree 1 in the ring of integers $\mathcal{O}_{-1}$ of $\mathbb{Q}(\sqrt{-1})$ we consider the group

$$\Gamma_0(f) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in PSL_2(\mathcal{O}_{-1}) \mid c \in f\right\}$$

Denote by $p = N(f)$ the norm of f. Then machine computation (cf. [9]) shows, for example, if $p = 101$ that

$$\Gamma_0(f)^{ab} \cong \mathbb{Z}/4 \oplus \mathbb{Z}/25 \oplus \mathbb{Z}/17.$$

There are other examples of this type due to Grunewald.

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Correction to 'The Boolean algebra of spectra'

Commentarii Mathematici Helvetici 54/3 (1979) pp. 368–377

A. K. Bousfield

J. M. Boardman and others have pointed out an error in my proof of Proposition 1.5 [1]. Namely, the presentation of the CW-spectrum B_λ/A as the homotopy cofibre of $1-g$ on p. 371 is incorrect for a general limit ordinal λ. The error arose when I wrongly simplified an earlier proof, and the proposition remains valid. As suggested by Boardman, the proof can be repaired by using the equivalence of B_λ/A with the homotopy colimit C of the transfinite sequence $\{B_s/A\}_{s<\lambda}$. The required theory of homotopy colimits can be found in [3], [4], [5]. In more detail, C can be obtained by imposing appropriate face identifications on the wedge of the $B_{s_0}/A \wedge (\Delta^n \cup {}^*)$ running over all $(n+1)$-tuples of ordinals $s_0 < s_1 < \cdots < s_n < \lambda$ for all $n \geq 0$. Thus, C is a CW-spectrum with an increasing filtration by closed subspectra $\{F_nC\}$ such that $F_nC/F_{n-1}C$ is the wedge of the $B_{s_0}/A \wedge S^n$ running over all $(n+1)$-tuples of ordinals $s_0 < s_1 < \cdots < s_n < \lambda$. The associated spectral sequence for π_*C has $E^2_{n,t} \approx \operatorname{colim}^n \{\pi_t B_s/A\}$, and this derived colimit vanishes for $n>0$ because it is indexed by a directed set. Thus there is an edge isomorphism $\operatorname{colim}_{s<\lambda} \pi_* B_s/A \approx \pi_* C$ and the canonical map $C \to B_\lambda/A$ is a weak equivalence of CW-spectra. Consequently $C \simeq B_\lambda/A$. This equivalence can also be shown by using the isomorphisms $\operatorname{colim}_T \pi_* C_T \approx \pi_* C$ and $\pi_* C_T \approx \pi_* B_{m(T)}/A$ where T runs over all finite nonempty sets of ordinals less than λ, where $C_T \subset C$ is the homotopy colimit of the finite sequence $\{B_t/A\}_{t\in T}$, and where $m(T)$ is the largest ordinal in T. Having shown $C \simeq B_\lambda/A$ one uses the Milnor cofibering $\bigvee_{n\geq 0} F_nC \to \bigvee_{n\geq 0} F_nC \to C$ together with the above wedge decomposition of $F_nC/F_{n-1}C$ to deduce that B_λ/A is $[E,\ ]_*$-colocal and belongs to Class-E as required for the proof of Proposition 1.5 [1] and for subsequent applications. A similar error appeared in the proof of Lemma 1.13 [2] and can be repaired similarly.

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Knot cobordism and amphicheirality

DANIEL CORAY and FRANÇOISE MICHEL

Introduction

Let C_n denote the cobordism group of n-dimensional knots. Cameron Gordon has asked the following question ([Ha], problem 16):

Can every element of order 2 *in* C_1 *be represented by a* (-1)-*amphicheiral knot*?

A knot is called (-1)-amphicheiral if it is isotopic to its obvious cobordism inverse (see §1 for a precise definition). Hence it is clear that the cobordism class of any (-1)-amphicheiral knot has order two. Gordon's question is about a partial converse of this statement.

Actually the problem makes sense in any odd dimension. (We recall that $C_n = 0$ for n even [K].) But, for $n = 2q-1$, we show:

STATEMENT 1. *The answer is negative for every* $q \geq 2$. *More precisely, some Alexander polynomials* γ *have the following property: the cobordism class of every knot whose Alexander polynomial is* γ *has order two, but contains no* (-1)-*amphicheiral knot.*

STATEMENT 2. *For* $q = 1$ *the same polynomials provide many examples of algebraic cobordism classes of order* 2 *which contain no* (-1)-*amphicheiral knot. Since they are exceedingly numerous, it seems reasonable to expect that Gordon's question should have a negative answer also in the classical case.*

For the proof we work with the algebraic invariants already used in [T], [Mic] and [Hi]. One of the main features is a new (-1)-amphicheirality criterion, which is considerably more general than those previously obtained. In particular it is invariant under cobordism and applies to knots of any odd dimension.

We thank J. Hillman, who pointed out the interest of studying Gordon's problem in higher dimensions.

§1. Statements of the results

We begin with some definitions:

1. An *n-knot* Σ is a smooth, oriented submanifold of S^{n+2} which is homeomorphic to S^n.

2. Let $\sigma: S^{n+2} \to S^{n+2}$ be the reflection in some equatorial plane, $(\sigma(\Sigma))^-$ the image of Σ with the opposite orientation. By $-\Sigma$ we shall denote $(\sigma(\Sigma))^-$ regarded as a submanifold of S^{n+2}. We call it the *inverse of* Σ. As $\Sigma \# -\Sigma$ is null-cobordant, the cobordism class of $-\Sigma$ is the inverse of the cobordism class of Σ.

3. Σ is said to be (-1)-*amphicheiral* ("involutory" in the terminology of J. Conway) if it is isotopic to $-\Sigma$.

4. For $\varepsilon = \pm 1$, let $C^\varepsilon(\mathbb{Z})$ be the cobordism group of ε-forms (cf. [L₁] or [K]). Associating a Seifert form to a $(2q-1)$-knot induces a homomorphism φ_{2q-1} from C_{2q-1} to $C^{(-1)^q}(\mathbb{Z})$. The *algebraic cobordism class* of a $(2q-1)$-knot Σ is the image by φ_{2q-1} of the cobordism class of Σ. We recall that φ_{2q-1} is injective if and only if $q \geqslant 2$ ([L₁] and [C–G]). It is the reason why our results do not answer Gordon's question when $q = 1$.

5. For any polynomial $\Delta \in \mathbb{Z}[X]$, of degree d (say), we define $\Delta^* \in \mathbb{Z}[X]$ by the formula:

$$\Delta^*(X) = X^d \Delta(X^{-1}).$$

We recall that Δ is *reciprocal* if $\Delta = \Delta^*$.

6. Given an irreducible reciprocal polynomial $\gamma \in \mathbb{Z}[X]$, we define K to be the number field $\mathbb{Q}[X]/(\gamma)$, and $\mathcal{O}_K$ its ring of algebraic integers. As γ is reciprocal, mapping X into X^{-1} induces an involution on K. We write $\bar{\alpha}$ for the image of $\alpha \in K$ under this involution. Finally we set $a = \gamma(0)$ and adopt the following terminology:

γ *has property* P_1 if $\alpha\bar{\alpha} = -1$ for some α in K;

γ *has property* P_2 if $\alpha\bar{\alpha} = -1$ for some α in the ring $\mathcal{O}_K[1/a]$;

γ *has property* P_3 if $\eta\bar{\eta} = -1$ for some unit η in $\mathcal{O}_K$.

We are now in a position to give the precise statements that we shall prove. In what follows, q is any positive integer, and Σ a $(2q-1)$-knot. If Δ is the Alexander polynomial of Σ, we have $\Delta = \Delta^*$. Hence we can write:

$$\Delta = \delta\delta^* \prod_{i=1}^{l} \gamma_i,$$

where the γ_i are distinct irreducible reciprocal polynomials. (The γ_i are those reciprocal polynomials which appear with odd multiplicity among the irreducible factors of Δ.)

THEOREM 1. *If Σ is (-1)-amphicheiral then γ_i has property P_2, for every $i \leq l$.*

This (-1)-amphicheirality criterion is proved in §2. In practice, property P_3 is a lot more convenient to work with than P_2. This makes the interest of the following two propositions, where γ *is assumed to be an irreducible reciprocal polynomial such that* $\gamma(1) = \pm 1$.

PROPOSITION 1. *Suppose $|\gamma(0)|$ is a prime p and $\mathbb{Z}[X, X^{-1}]/(\gamma) = \mathcal{O}_K[1/p]$. Then γ has property P_2 if and only if it has P_3.*

PROPOSITION 2. *Suppose $|\gamma(0)|$ is a prime p and K is a Galois extension of $\mathbb{Q}$. Then γ has property P_2 if and only if it has P_3.*

For Proposition 1 we give a topological argument, while Proposition 2 is established by purely algebraic means.

Remark. Proposition 2 will be used in §4 in constructing the appropriate examples.

In §3 we prove:

THEOREM 2. *Let Σ be a $(2q-1)$-knot whose Alexander polynomial γ is irreducible. Then:*

(1) *γ has property P_1 if and only if the algebraic cobordism class of Σ has order two;*

(2) *if Σ is cobordant to some (-1)-amphicheiral knot then γ has property P_2.*

SCHOLIUM. *If $q \geq 2$ and γ has property P_1 then the geometric cobordism class of Σ is also of order two, as follows from* [L$_1$].

COROLLARY. *To prove statements 1 and 2 of the introduction it is enough to produce some irreducible, reciprocal polynomials γ with the following properties:*

(1) *γ is the Alexander polynomial of some $(2q-1)$-knot;*

(2) *γ has property P_1;*

(3) *γ fails to have property P_2.*

In §4 we show how to construct infinitely many irreducible Alexander polynomials having property P_1 but not P_2. As a matter of fact there are some examples already in degree 2, but recall:

Levine's criterion [L_1]. *A reciprocal polynomial* $\gamma \in \mathbb{Z}[X]$, *with degree d, is the Alexander polynomial of some* $(2q-1)$*-knot if and only if* $\gamma(1)=\varepsilon^{d/2}$ *and* $\gamma(\varepsilon)$ *is a perfect square, where* $\varepsilon=(-1)^{q+1}$.

Now, if γ is any reciprocal polynomial of degree 2 such that $\gamma(1)=-1$, we observe that $\gamma(-1)$, being the discriminant of γ, can be a perfect square only if γ is reducible! Therefore, by Levine's criterion, no example with q even can be obtained in degree 2. That is why we shall give two series of examples:

I. *The quadratic case* (*which occurs only for odd values of* q)

$$\gamma(X)=-pX^2+(2p+1)X-p,$$

where p runs through a certain set of primes: $p=367, 379, 461, 751, 991, \cdots$ (61 examples for $p<10{,}000$).

Remark. In [T], H. F. Trotter already observed that the knots with Alexander polynomial $\gamma(X)=-367X^2+735X-367$ are not (-1)-amphicheiral.

II. *The biquadratic case* (*which occurs for any* q)

In §4 we prove the following theorem:

THEOREM 3. *Let p be an odd prime and*

$$\gamma(X)=-pX^4+(2p+1)X^2-p.$$

Then γ *is irreducible. Moreover:*

(1) γ *has property* P_1;

(2) γ *fails to have property* P_2 *if and only if p is congruent to* 3 *modulo* 4 *and the fundamental unit of* $\mathbb{Q}(\sqrt{4p+1})$ *has norm* $+1$.

Remark. This yields infinitely many examples. Indeed the fundamental unit of $\mathbb{Q}(\sqrt{4p+1})$ has norm $+1$ whenever $4p+1$ has a prime factor with odd multiplicity which is congruent to 3 modulo 4 (e.g. $p=19, 23$, etc.), and also in certain other cases, like $p=367, 379, 751, 991, \cdots$ etc.

Other examples. The following polynomials:

$$\gamma(X)=X^4-2\lambda X^3+(4\lambda-1)X^2-2\lambda X+1,$$

with $\lambda=36, 45, \cdots$ (an infinity of examples), satisfy all three properties of the above corollary. This is proved in [C] with the techniques that we use in the proof of Theorem 3. The particular interest of these examples is that they can be realized as Alexander polynomials of some *fibered* knots.

§2. An amphicheirality criterion

Proof of Theorem 1. We recall that Σ is a (-1)-amphicheiral $(2q-1)$-knot with Alexander polynomial Δ. To prove Theorem 1, we must show: if $\Delta = \gamma^{2l+1}\cdot\mu$, where γ is reciprocal, irreducible and prime to μ, then γ has property P_2.

Let $\tilde{X}$ be the infinite cyclic covering of the complement of Σ. Put $M = H_q(\tilde{X})$; this is a torsion module over $\mathbb{Z}[X, X^{-1}]$. Let $B : M\times M \to \mathbb{Q}(X)/\mathbb{Z}[X,X^{-1}]$ be the Blanchfield pairing associated with Σ (cf. [L_3], p. 15). If we write ε for $(-1)^{q+1}$, the Blanchfield pairing is ε-hermitian and unimodular (i.e. the adjoint of B yields a $\mathbb{Z}[X, X^{-1}]$-isomorphism between M and $\mathrm{Hom}_{\mathbb{Z}[X,X^{-1}]}(M, \mathbb{Q}(X)/\mathbb{Z}[X,X^{-1}])$) (cf. [$L_3$]). We recall that B can be constructed as follows ([L_3], Proposition 14.3, p. 44):

Let A be an $r\times r$ matrix which represents a Seifert pairing of Σ (see, for example, [K]). We denote by A^t the transpose of A. Now M is isomorphic to

$$(\mathbb{Z}[X, X^{-1}])^r/(AX-\varepsilon A^t)$$

and, with this presentation of M, the form B corresponds to $(1-X)(AX-\varepsilon A^t)^{-1}$.

As $-A$ is a Seifert matrix for $-\Sigma$, it follows from the above that $(M, -B)$ is the Blanchfield pairing of $-\Sigma$. Now the isomorphism class of (M, B) is an invariant of the isotopy class of Σ. Hence the (-1)-amphicheirality of Σ yields a $\mathbb{Z}[X, X^{-1}]$-automorphism F of M such that $B(F(\alpha), F(\beta)) = -B(\alpha, \beta)$ for all α and β in M.

Let A_0 be any non-degenerate Seifert matrix in the S-equivalence class of A (cf. [T]). Then $\Delta(X) = \det(A_0X-\varepsilon A_0^t)$ is independent of the choice of A_0, and $\Delta(0) = \det(A_0) \neq 0$.

By assumption, $\Delta = \gamma^{2l+1}\cdot\mu$, with coprime γ and μ. Let us define:

$$M_\gamma = \mu(X)M \subset \mathrm{Ker}\ \gamma(X)^{2l+1} \quad \text{and} \quad M_\mu = \gamma^{2l+1}(X)M \subset \mathrm{Ker}\ \mu(X).$$

Clearly $M_\gamma \cap M_\mu = 0$. Since μ and γ are both reciprocal, the Blanchfield pairing B splits orthogonally on $M_\gamma \oplus M_\mu$. Moreover, the index of $M_\gamma \oplus M_\mu$ in M is finite; therefore the restriction, B_γ, of B to M_γ is non-degenerate. Furthermore the restriction, F_γ, of F to M_γ yields an isomorphism from (M_γ, B_γ) to $(M_\gamma, -B_\gamma)$.

We now define:

$$M^i = \{\alpha \in M_\gamma \mid \gamma^i(X)\alpha = 0\}$$
$$H^i = M^i/(M^{i-1} + \gamma(X)M^{i+1}).$$

Put $R = \mathbb{Z}[X, X^{-1}]/(\gamma)$. Then H^i is an R-module of finite rank e_i (say) and the $\mathbb{Q}[X, X^{-1}]$-module $M_\gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to:

$$\bigoplus_{i=1}^{\infty} (\mathbb{Q}[X, X^{-1}]/(\gamma)^i)^{e_i}.$$

As $\sum_{i=1}^{\infty} i \cdot e_i = 2l+1$, there is only a finite number of non-zero e_i; and one of them, say e_{i_0}, must be odd. We write: $n = e_{i_0}$, $H = H^{i_0}$, and denote by $[\alpha]$ the class in H of an element $\alpha \in M^{i_0}$. One can define a non-degenerate, ε-hermitian form $b: H \times H \to \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}]$ by setting, for any α and β in M^{i_0}:

$$b([\alpha], [\beta]) = B_\gamma(\gamma^{i_0-1}(X)\alpha, \beta).$$

That the form b is well-defined is proved in [Mil], where it is also shown that b is non-degenerate provided B_γ is.

As $\gamma(X)b(\alpha, \beta)$ is in $\mathbb{Z}[X, X^{-1}]$ for all α and β in H, it follows that $b(\alpha, \beta) = P(X)/\gamma(X)$, where $P(X)$ is some polynomial in $\mathbb{Z}[X, X^{-1}]$. Setting $b'(\alpha, \beta) = P(X)$ defines a non-degenerate ε-hermitian form $b': H \times H \to R$, and F_γ induces an R-isomorphism from (H, b') to $(H, -b')$. Since H is of rank n over R, we see that $\Lambda^n H$, the n-th exterior power of H, can be identified with an R-ideal I. In [B] (§1, no 9, p. 31) the n-th exterior power of b' is defined, and it is shown that $\Lambda^n b'$ is non-degenerate provided b' is. Let f be the isomorphism from $(I, \Lambda^n b')$ to $(I, \Lambda^n(-b'))$ which is induced by F_γ. We write R_I for the ring of coefficients of the R-ideal I, i.e. $R_I = \{\alpha \in K \mid \alpha I \subset I\}$. We recall that $a = \gamma(0)$; so $R \subset \mathcal{O}_K[1/a]$.

LEMMA 1. $R_I \subset \mathcal{O}_K[1/a]$.

Proof. Put $S = \mathcal{O}_K[1/a]$ and $J = I \cdot S$. Clearly the ring of coefficients, S_J, of J contains R_I. Hence it is enough to show that $S_J \subset S$. But S is a Dedekind ring; hence the ring of coefficients of any non-zero S-ideal is S itself. $\square$

As f is an R-automorphism of I, there exists u in R_I, hence in $\mathcal{O}_K[1/a]$ (by the lemma), such that $f(\alpha) = u\alpha$ for all α in I. Now n is odd, hence $\Lambda^n(-b') = -\Lambda^n b'$.

Let us take α and β in I, both non-zero; then $(\Lambda^n b')(\alpha, \beta) \neq 0$; so the relations

$$(\Lambda^n b')(f(\alpha), f(\beta)) = u\bar{u}(\Lambda^n b')(\alpha, \beta) = -(\Lambda^n b')(\alpha, \beta)$$

imply $u\bar{u} = -1$. This completes the proof of Theorem 1. □

Proof of Proposition 1. Suppose γ has property P_2. Under the assumptions of Proposition 1, we show that γ also has property P_3. Let $M = \mathbb{Z}[X, X^{-1}]/(\gamma)$; we define a unimodular hermitian form $B : M \times M \to \mathbb{Q}(X)/\mathbb{Z}[X, X^{-1}]$ by setting $B(\alpha, \beta) = \alpha\bar{\beta}/\gamma(X)$ for any α and β in M.

As γ has property P_2 and $\mathcal{O}_K[1/\gamma(0)] = \mathbb{Z}[X, X^{-1}]/(\gamma)$, multiplication by an element u in $\mathcal{O}_K[1/\gamma(0)]$ such that $u\bar{u} = -1$ yields an isomorphism from (M, B) to $(M, -B)$. Now the form (M, B) is always the Blanchfield pairing of some $(2q-1)$-knot, provided we choose q odd (see Theorem 12.1 in [L3]). Let A be a non-degenerate Seifert matrix associated with such a knot. Assuming $|\gamma(0)|$ is a prime number p, Trotter [T] (Corollary 4.7, p. 196) shows that (M, B) is isomorphic to $(M, -B)$ if and only if A is isomorphic to $-A$. (A word of caution: Trotter calls Seifert form what is usually called Blanchfield pairing, as here.)

On the other hand, since γ is irreducible, the isomorphism between A and $-A$ implies the existence of u in $\mathcal{O}_K$ such that $u\bar{u} = -1$ (cf. [Mic]). This completes the proof of Proposition 1. □

Proof of Proposition 2. We begin by showing that Proposition 2 can be deduced from the following lemma:

LEMMA 2. *Let F be a number field, $\mathcal{O}_F$ its ring of algebraic integers and $a \in \mathbb{N}^*$. Suppose there exists a Galois automorphism $\sigma : F \to F$ such that $\sigma^2 = \mathrm{id}$ (σ is an involution of F), and an element α in $\mathcal{O}_F[1/a]$ such that $\alpha \cdot \sigma(\alpha) = -1$. If, for some odd integer $\lambda \in \mathbb{N}^*$, every prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ dividing a and distinct from $\sigma(\mathfrak{p})$ is such that $\mathfrak{p}^\lambda$ is principal, then there exists η in $\mathcal{O}_F$ such that $\eta\sigma(\eta) = -1$.*

Lemma 2 *implies Proposition* 2:

We recall that $p = |\gamma(0)|$ is prime. Consider the following polynomial:

$$\varphi(X) = \gamma(1)X^d\gamma\left(1 - \frac{1}{X}\right) = \gamma(1)(\gamma(1)X^d + \cdots + (-1)^d\gamma(0)),$$

where d is the degree of γ. As $\gamma(1) = \pm 1$, the polynomial φ is monic, and $\varphi(0) = \pm p$.

Let $\xi_1, \ldots, \xi_d$ be the roots of φ. Since φ is irreducible, they are all distinct. Moreover, K is Galois; hence they all lie in K. For every i, the ideal $\mathfrak{p}_i = (\xi_i)$ is prime (with degree one), since $N_{K/\mathbf{Q}}(\xi_i) = \pm p$. By construction it is principal and

$$(p) = \prod_{i=1}^{d} \mathfrak{p}_i.$$

(We do not claim that the $\mathfrak{p}_i$ are all distinct!) Therefore all prime ideals dividing p are principal. Thus Proposition 2 is a consequence of Lemma 2 (with $F = K$, $\sigma(\alpha) = \bar{\alpha}$, $a = p$ and $\lambda = 1$).

Proof of Lemma 2. Suppose $\alpha \cdot \sigma(\alpha) = -1$ for some α in $\mathcal{O}_F[1/a]$. We may write the fractional ideal (α) as a product of prime ideals:

$$(\alpha) = \prod \mathfrak{p}^{v_{\mathfrak{p}}(\alpha)} \tag{2.1}$$

Since $\alpha\sigma(\alpha) = -1$, we have:

$$v_{\mathfrak{p}}(\alpha) + v_{\sigma(\mathfrak{p})}(\alpha) = 0 \quad \forall \mathfrak{p}. \tag{2.2}$$

If $v_{\mathfrak{p}}(\alpha) \neq 0$, it follows from (2.2) that either $v_{\mathfrak{p}}(\alpha)$ or $v_{\sigma(\mathfrak{p})}(\alpha)$ is negative; hence $\mathfrak{p}$ divides a. The relation (2.2) shows also that $v_{\mathfrak{p}}(\alpha) = 0$ if $\mathfrak{p} = \sigma(\mathfrak{p})$.

Let us now consider the prime ideals $\mathfrak{p}_i \neq \sigma(\mathfrak{p}_i)$ which divide a. By assumption, we may write $\mathfrak{p}_i^{\lambda} = (\pi_i)$ for some $\pi_i \in F$. Then the relations (2.1) and (2.2) imply:

$$\alpha^{\lambda} = \eta \prod_i \left(\frac{\sigma(\pi_i)}{\pi_i} \right)^{\mu_i}, \tag{2.3}$$

with $\mu_i \in \mathbb{Z}$ and η a unit in $\mathcal{O}_F$. We see that $\alpha^{\lambda}\sigma(\alpha)^{\lambda} = \eta \cdot \sigma(\eta)$. Now λ is *odd*; hence $\eta \cdot \sigma(\eta) = -1$. This completes the proof of Lemma 2. □

§3. Knot cobordism classes of order two

Proof of the first assertion of Theorem 2. Let Σ be a $(2q-1)$-knot whose Alexander polynomial γ is irreducible. Put $\varepsilon = (-1)^q$. We recall some definitions and basic facts about algebraic cobordism (for more details see [K]).

DEFINITION. An $n \times n$ integral matrix B *represents an* ε*-form* if the matrix $B + \varepsilon B^t$ is invertible over $\mathbb{Z}$.

If A is a Seifert matrix associated with Σ, then $A + \varepsilon A^t$ is the matrix of the intersection form on a Seifert surface of Σ. Since Σ is a sphere, this intersection form is unimodular [K]. Hence A represents an ε-form.

DEFINITION. An ε-form is *null-cobordant* if it is represented by a matrix of the form $\begin{pmatrix} 0 & A_1 \\ A_2 & A_3 \end{pmatrix}$, where the A_i are all square integral matrices.

Let $C^\varepsilon(\mathbb{Z})$ be the group of cobordism classes of ε-forms. On tensoring with $\mathbb{Q}$, we obtain an injective map from $C^\varepsilon(\mathbb{Z})$ to the group of cobordism classes of rational ε-forms, say, $C^\varepsilon(\mathbb{Q})$.

The first assertion of Theorem 2 can therefore be stated as follows: *Given a Seifert matrix A of Σ, then $A \oplus A$ is null-cobordant if and only if γ has property P_1.* This fact can be deduced from Levine's description of $C^\varepsilon(\mathbb{Q})$ [L_2] or from Stoltzfus's computation of $C^\varepsilon(\mathbb{Z})$ [St], but we shall give here a direct and elementary proof.

As in §1, K is the number field $\mathbb{Q}[X]/(\gamma)$. Let $H^\varepsilon(K)$ be the Witt group of non-degenerate ε-hermitian forms $B: M \times M \to K$, where M runs through the finite-dimensional vector spaces over K.

LEMMA 3. *Suppose M is a one-dimensional vector space over K. Then the class of B in $H^\varepsilon(K)$ has order two if and only if γ has property P_1.*

Proof. If γ has property P_1, then $\alpha\bar{\alpha} = -1$ for some α in K. Multiplication by α yields an automorphism of M that carries B into $-B$. Thus $B \oplus B$ is isomorphic to $B \oplus (-B)$; therefore its Witt class is zero.

As B has rank one, if the Witt class of $B \oplus B$ is zero, this form is represented by a matrix of the form $\begin{pmatrix} 0 & \beta_1 \\ \varepsilon\bar{\beta}_1 & \beta_2 \end{pmatrix}$ with $\beta_i \in K$. If $\beta \in K^*$ is the determinant of B, then:

$$\beta = \varepsilon\bar{\beta}. \tag{3.1}$$

As the determinant is defined up to an element of K^* of the form $\eta \cdot \bar{\eta}$, we obtain the relation:

$$\det(B \oplus B) = \beta^2 = -\varepsilon\beta_1\bar{\beta}_1\eta\bar{\eta}. \tag{3.2}$$

If we write $\alpha = \beta^{-1}\beta_1\eta$, the relations (3.1) and (3.2) show that $\alpha \cdot \bar{\alpha} = -1$. This completes the proof of Lemma 3. $\square$

In the S-equivalence class of Seifert matrices corresponding to Σ, we can choose one which is non-degenerate [T]. Call it A. Then the rank of A is equal to the degree, d, of γ. Let M be a d-dimensional vector space over $\mathbb{Q}$. The matrix $T=-\varepsilon A^{-1}A^t$ represents an automorphism of M. Put $X \cdot \alpha = T(\alpha)$. This action of $\mathbb{Q}[X]$ induces on M the structure of a one-dimensional K-vector space. There exists an ε-hermitian form $B: M \times M \to K$ such that the relation:

$$(a\alpha)^t(A+\varepsilon A^t)(\beta)=\mathrm{trace}_{K/\mathbb{Q}}\, aB(\alpha, \beta) \tag{3.3}$$

is satisfied for all a in K and α, β in M (see [Mil]). Now, using (3.3), a direct computation shows that $A \oplus A$ is null-cobordant if and only if the Witt class of B has order two. By Lemma 3 this completes the proof of assertion (1).

Proof of the second assertion of Theorem 2. Suppose Σ is cobordant to Σ'. Then the Fox–Milnor relation shows that the Alexander polynomial of Σ' is of the form $\delta \cdot \delta^* \cdot \gamma$ for some integral polynomial δ (for a proof see [L_1], p. 237). If, moreover, Σ is (-1)-amphicheiral, it follows from Theorem 1 that γ has property P_2. This completes the proof of Theorem 2. □

§4. Explicit examples

In this section, which is purely number-theoretical, we show that there exist infinitely many irreducible Alexander polynomials of low degree having property P_1 but not P_2.

I. *The quadratic case*

PROPOSITION 3. *Let p be an odd prime, D the square-free part of* $4p+1$, *and*

$$\gamma(X)=-pX^2+(2p+1)X-p.$$

Then γ is irreducible. Moreover:

(1) *γ has property P_1 if and only if all prime factors of D are congruent to* 1 *modulo* 4;

(2) *γ fails to have property P_2 if and only if the fundamental unit of $\mathbb{Q}(\sqrt{D})$ has norm* $+1$.

Proof. The discriminant $4p+1$ of γ is not a square, since it is congruent to 5 modulo 8. Hence γ is irreducible.

(1) As $K=\mathbb{Q}(\sqrt{D})$, it is clear that P_1 holds if and only if the equation

$$x^2-Dy^2=-1 \tag{4.1}$$

can be solved with $x, y \in \mathbb{Q}$. A local calculation and the Hasse–Minkowski theorem show that this is the case if and only if all prime factors of D are congruent to 1 modulo 4. (In fact this is a well-known result on sums of two squares.)

(2) This is an immediate consequence of Proposition 2, since $|\gamma(0)|=p$ and $K/\mathbb{Q}$ is Galois. $\square$

EXAMPLES. As is well-known, the fundamental unit of $\mathbb{Q}(\sqrt{D})$ has norm $+1$ if and only if the period of the continued fraction expansion of $\sqrt{D}$ is even.

There is a very efficient algorithm for determining that period (see [Si], p. 296; and [P], §26, pp. 102–103, for a useful refinement). In point of fact the fundamental unit itself is detected by this procedure, which involves a computer calculation whose only difficulty is the number of digits to be handled (for $D=991$, already thirty digits are required!). The two smallest examples(1) illustrating Proposition 3 are:

$$p=367; \qquad D=13\cdot 113; \qquad \eta=56+3\delta$$

$$p=379; \qquad D=37\cdot 41; \qquad \eta=19+\delta$$

(We denote by η the fundamental unit of $\mathbb{Q}(\sqrt{D})$, and $\delta=(1+\sqrt{D})/2$.)

Remark. In these examples, D is never a prime. This follows from an elementary result, which will be used again later:

LEMMA 4. *Suppose D is a prime congruent to* 1 *modulo* 4. *Then equation* (4.1) *can be solved with $x, y \in \mathbb{Z}$. Hence the fundamental unit of $\mathbb{Q}(\sqrt{D})$ has norm* -1.

A proof can be found in [Mo], Chap. 8. The idea is to start from the fundamental solution of the Pell equation $t^2-Du^2=1$. The assumptions on D enable one to write

$$\frac{t-1}{2}=x^2 \quad \text{and} \quad \frac{t+1}{2}=Dy^2,$$

with $x, y \in \mathbb{Z}$. Then (x, y) is a solution of (4.1).

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(1) A complete list with $p \leq 50{,}000$ is available on request.

II. *Proof of Theorem* 3

Let τ be any root of the polynomial

$$\gamma(X) = -pX^4 + (2p+1)X^2 - p.$$

As the other roots are $-\tau$ and $\pm 1/\tau$, we see that $K = \mathbb{Q}(\tau)$ is a Galois extension of $\mathbb{Q}$. Moreover, K contains $\mathbb{Q}(\tau^2) = \mathbb{Q}(\sqrt{D})$, where as above *we denote by D the square-free part of* $4p+1$. Since p is odd, $4p+1$ is congruent to 5 modulo 8, hence $D \neq 1$. the fixed field of the involution $\tau \mapsto 1/\tau$ is the field $\mathbb{Q}(\sigma)$, with $\sigma = \tau + 1/\tau = \sqrt{(4p+1)/p}$. From this we see that $K/\mathbb{Q}$ is an extension of degree 4 (whence γ is irreducible), with Galois group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Therefore K contains three quadratic subfields:

$$\begin{tikzcd}[column sep=small]
 & K = \mathbb{Q}(\tau) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
k_1 = \mathbb{Q}(\sqrt{p}) \arrow[dr, dash] & k_2 = \mathbb{Q}(\sqrt{D}) \arrow[d, dash] & k_3 = \mathbb{Q}(\sigma) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

It is useful to observe that the involution $\tau \mapsto 1/\tau$ induces the ordinary conjugation on k_1 and on k_2: $\sqrt{p} \mapsto -\sqrt{p}$, resp. $\sqrt{D} \mapsto -\sqrt{D}$.

(1) *We wish to prove that γ has property P_1.* Now an element $\alpha \in K$ can be written in the form $\alpha = x + y\sqrt{p}$ with $x, y \in \mathbb{Q}(\sigma)$. Therefore we have to show that the equation

$$x^2 - py^2 = -1 \tag{4.2}$$

can be solved with $x, y \in \mathbb{Q}(\sigma)$. Equivalently, we are reduced to showing that the homogeneous equation

$$x^2 - py^2 + z^2 = 0 \tag{4.3}$$

has a non-trivial solution in $\mathbb{Q}(\sigma)$.

By the Hasse–Minkowski theorem for the number field $\mathbb{Q}(\sigma)$ (see for example [C–F], ex. 4.8, p. 359), it will suffice to show that (4.3) can be solved non-trivially in all completions of $\mathbb{Q}(\sigma)$. Since the quadratic form in (4.3) is defined over $\mathbb{Q}$, it is indefinite for each of the two real embeddings of $\mathbb{Q}(\sigma)$. Therefore it suffices to consider the non-archimedean valuations.

If $p \equiv 1 \pmod 4$, we know that p is a sum of two squares; hence (4.3) is already solvable over $\mathbb{Q}$. (By Lemma 4 we know that (4.2) is even solvable over $\mathbb{Z}$.) Thus we may assume without loss of generality that $p \equiv 3 \pmod 4$. Then $pD \equiv 3 \pmod 4$; hence the ideal (2) ramifies in $\mathbb{Q}(\sigma) = \mathbb{Q}(\sqrt{pD})$. *Therefore it is enough to prove that* (4.3) *has a non-trivial solution in all non-archimedean completions of* $\mathbb{Q}(\sigma)$ *whose residue field is of characteristic* $\neq 2$. Indeed, by the product formula ([C–F], ex. 4.5, p. 358), the number of places where a quadratic form in three variables does not represent zero is even; but there is only one prime ideal above (2).

By a well-known result (a special case of the Chevalley–Warning theorem), (4.3) has non-trivial solutions in every finite field. By Hensel's lemma (cf. [C–F], p. 83), these solutions can be lifted over the corresponding completions, provided the characteristic of the residue field is not equal to 2 or p. In addition, the ideal (p) ramifies in $\mathbb{Q}(\sigma) = \mathbb{Q}(\sqrt{pD})$; therefore all we have to show is that (4.3) can be solved $\mathfrak{p}$-adically, where $\mathfrak{p}$ denotes the unique ideal above (p).

Since $(p) = \mathfrak{p}^2$, locally we can write $p = \pi^2\eta$, where π is a uniformizing element and η a $\mathfrak{p}$-adic unit. Now, if we write $Y = \pi y$, we are reduced to showing that

$$x^2 - \eta Y^2 + z^2 = 0$$

has a non-trivial $\mathfrak{p}$-adic solution. Since now η is a unit, the above argument with Hensel's lemma applies. This completes the proof that γ has property P_1.

(2) *Let us examine under what conditions γ has property P_2.* In each quadratic subfield k_i of K there is a fundamental unit ε_i. Now since the involution $\tau \mapsto 1/\tau$ acts as the ordinary conjugation on k_1 and k_2, it is clear that γ has property P_3 (a fortiori P_2) if either ε_1 or ε_2 has norm -1. As we saw in Lemma 4, ε_1 has norm -1 if $p \equiv 1 \pmod 4$; and only then, since obviously (4.2) has no rational solution for $p \equiv 3 \pmod 4$. This proves one of the implications in the second assertion of Theorem 3. In order to establish the converse, we first note that the two properties P_2 and P_3 are in fact equivalent in our case, as follows from Proposition 2. Therefore we are reduced to proving the following lemma:

LEMMA 5. *Suppose ε_1 and ε_2 have norm $+1$. Then γ fails to have property P_3.*

Proof. The general theory of units in biquadratic fields is fairly well understood (cf. [Kur], [Kub], [N]); but the special shape of the polynomial γ yields some further information, which will be needed. Let U_K be the group of units in the ring of integers $\mathcal{O}_K$. As K is totally real, we choose once for all a real embedding and denote by U_K^+ the free $\mathbb{Z}$-module of rank 3 consisting of all positive units. Correspondingly, we agree that ε_1, ε_2 and ε_3 are elements of U_K^+.

(**a**) A classical argument [Kur] shows that the sub-$\mathbb{Z}$-module $R \subset U_K^+$ generated by ε_1, ε_2 and ε_3 is of finite index in U_K^+. *For if $\eta \in U_K$ is any unit then $\eta^2 \in R$.* Indeed, let η' be the conjugate of η above k_1. Then

$$N_{K/k_1}(\eta) = \eta\eta' = \pm\varepsilon_1^a,$$

and similarly:

$$\eta\bar{\eta}' = \pm\varepsilon_2^b, \qquad \eta\bar{\eta} = \pm\varepsilon_3^c \quad (a, b, c \in \mathbb{Z}).$$

Hence $\eta^2 = \pm\,\eta^2(\eta\eta'\bar{\eta}\bar{\eta}') = \; = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c \in R$.

Remark. This argument shows that the index $J = [U_K^+ : R]$ is in fact a divisor of 8. Kuroda [Kur] has shown that, in the general case, there are seven essentially distinct possibilities and that every divisor of 8 can occur. In our present case, however, J is always equal to 2, since we prove below that *U_K^+ is generated by ε_1, ε_2 and $\sqrt{\varepsilon_3}$.*

(**b**) *Suppose now $\eta\bar{\eta} = \pm 1$ for some unit η; then η^2 belongs to the submodule $R' \subset R$ which is generated by ε_1 and ε_2.* Indeed, we have just seen that $\eta^2 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^c$; in addition $\varepsilon_3 = \bar{\varepsilon}_3$. From the assumption $\eta\bar{\eta} = \pm 1$ we therefore get: $1 = \eta^2\bar{\eta}^2 = \varepsilon_3^{2c}$, which is possible only if $c = 0$.

(**c**) Suppose γ has property P_3, i.e. there exists a unit $\eta \in U_K^+$ such that $\eta\bar{\eta} = -1$. Then $\eta \notin R'$, since by assumption $\varepsilon_1\bar{\varepsilon}_1 = \varepsilon_2\bar{\varepsilon}_2 = +1$. Further we know, by (**b**) above, that η^2 is of the form $\varepsilon_1^a\varepsilon_2^b$ with $a, b \in \mathbb{Z}$. Since $\eta \notin R'$, we see that a and b are not both even. This implies that at least one of the numbers ε_1, ε_2 and $\varepsilon_1\varepsilon_2$ is a square in K. *Therefore the lemma will be proved once we show that none of the numbers ε_1, ε_2 and $\varepsilon_1\varepsilon_2$ is a square in K.*

(**d**) We consider first ε_2. If it were a square in K, there would exist $\alpha, \beta \in k_2$ such that $\varepsilon_2 = (\alpha + \beta\sqrt{p})^2 = (\alpha^2 + p\beta^2) + 2\alpha\beta\sqrt{p}$. Of course $\beta \neq 0$, since ε_2 is not a square in k_2. But the coefficient of $\sqrt{p}$ must vanish, hence $\alpha = 0$. Thus we get: $\varepsilon_2 = p\beta^2$, with $\beta \in k_2$. This is impossible, since p does not ramify in $k_2 = \mathbb{Q}(\sqrt{4p+1})$. (One can also proceed as in (**g**) below: β is in fact an element of $\mathcal{O}_{k_2}$, and p does not divide the unit ε_2.) We have shown that ε_2 *is not a square in* K.

(**e**) Let us examine ε_1. As $p \equiv 3 \pmod 4$, we can write $\varepsilon_1 = a_1 + b_1\sqrt{p}$ with $a_1, b_1 \in \mathbb{Z}$. *We claim that b_1 is odd.* To see that, it suffices to repeat the argument by which one proves Lemma 4: if b_1 were even, the equality $a_1^2 - pb_1^2 = 1$ would imply

$$uv = p\left(\frac{b_1}{2}\right)^2,$$

where

$$u = \frac{|a_1| - 1}{2} \quad \text{and} \quad v = \frac{|a_1| + 1}{2}$$

are coprime integers. Since p is a prime, either u or v is a square. In either case we get a contradiction: if $u=s^2$ and $v=pt^2$, then $s^2-pt^2=u-v=-1$; hence $s+t\sqrt{p}\in k_1$ would be a unit of norm -1. If $v=s^2$ and $u=pt^2$, then $s^2-pt^2=v-u=1$, and $1<|s|<|a_1|$. This is impossible, for the fundamental unit ε_1 corresponds to a solution of the Pell equation $s^2-pt^2=1$ for which $|s|>1$ is minimal.

(f) We put $\rho=\sqrt{p}$, $\delta=(1+\sqrt{D})/2$. As $D\equiv 1 \pmod 4$, one checks easily that $\mathcal{O}_K$ is the free $\mathbb{Z}$-module with basis $\{1, \rho, \delta, \rho\delta\}$. (This follows also from [L], chap. 3, §3, prop. 17.) Thus any element $\xi\in\mathcal{O}_K$ can be written in the form $\xi=\alpha+\beta\rho$ with $\alpha, \beta\in\mathcal{O}_{k_2}$. Then ξ^2 takes the form $(\alpha^2+p\beta^2)+2\alpha\beta\rho$. On writing $\alpha\beta=a+b\delta$ with $a, b\in\mathbb{Z}$, we reach the following conclusion: *when ξ^2 is expressed in the $\mathbb{Z}$-base $\{1, \rho, \delta, \rho\delta\}$, the coefficients of ρ and $\rho\delta$ are even.*

(g) Putting **(e)** and **(f)** together, it is immediate that $\varepsilon_1=a_1+b_1\rho$ *is not a square in K*, since b_1 is odd. Finally, let us write $\varepsilon_2=a_2+b_2\delta$, with $a_2, b_2\in\mathbb{Z}$; the coefficients of ρ and $\rho\delta$ in the product $\varepsilon_1\varepsilon_2$ are then respectively b_1a_2 and b_1b_2. If $\varepsilon_1\varepsilon_2$ were a square in K, these integers would have to be even. But b_1 is odd; hence both a_2 and b_2 should be even. This is clearly not the case, since ε_2 is not divisible by 2. This shows that $\varepsilon_1\varepsilon_2$ *is not a square in* K and completes the proof of the lemma. □

Remark. In **(a)** above it is claimed that U_K^+ is generated by ε_1, ε_2 and $\sqrt{\varepsilon_3}$. In view of the results gathered so far, it is enough to prove that ε_2 is a square. Now the situation we are in is quite exceptional in that the fundamental unit ε_3 is given by an explicit formula! Indeed let

$$\eta_3=(8p+1)+4\sqrt{p(4p+1)}. \tag{4.4}$$

It is a simple exercise to show that $(8p+1, 4)$ is the fundamental solution of the Pell equation $x^2-p(4p+1)y^2=1$. Hence $\eta_3=\varepsilon_3$ if $4p+1$ is square-free; otherwise $\eta_3=\varepsilon_3^{\nu}$ for some $\nu\in\mathbb{N}$. Moreover, η_3 is the square of

$$\sqrt{\eta_3}=2\sqrt{p}+\sqrt{4p+1}\in K \tag{4.5}$$

Furthermore, ν is necessarily odd, since $\sqrt{\eta_3}$ does not lie in k_3. Hence in all cases ε_3 is a square, and $J=2$.

It is a firmly established tradition that unit computations in a number field culminate in the determination of the class number. As $J=2$, one has the following formula, ([Kub], Satz 5, p. 80):

$$H=\tfrac{1}{2}h_1h_2h_3, \tag{4.6}$$

in which h_i (resp. H) denotes the class number of the field k_1 (resp. K). We see that the product $h_1h_2h_3$ is always even. This is not surprising; indeed, using (4.4) or (4.5), one shows easily that every prime factor of D is the square of a non-principal ideal of k_3, and therefore accounts for a factor 2 in h_3.

Note. The proof of Theorem 3 shows that there exist infinitely many polynomials of degree four having the required properties. For the quadratic case we do not know whether the constructed family of polynomials is infinite (but we believe so).

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Closed forms on symplectic fibre bundles

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A *bundle of symplectic manifolds* is a differentiable fibre bundle $F \to E \xrightarrow{\pi} B$ whose structure group (not necessarily a Lie group) preserves a symplectic structure on F. The vertical subbundle $\mathcal{V} = \operatorname{Ker}(T\pi) \subseteq TE$ carries a field of bilinear forms which we call the *symplectic structure along the fibres* and denote by ω. Any 2-form Ω on E has a restriction to $\mathcal{V}$; if this restriction is ω, we call Ω an *extension* of ω.

In this note, we discuss the problem of finding a closed extension of the symplectic structure along the fibres. This is the first step toward finding a symplectic extension – a problem already considered in special cases in [Th] and [Wn].

The first theorem shows that the existence of a closed extension is a purely topological problem.

THEOREM 1. *Let $F \to E \to B$ be a differentiable fibre bundle carrying a field ω of p-forms on the vertical bundle $\mathcal{V}$, defining a closed form on each fibre. Then there is a closed p-form Ω on E extending ω if and only if there is a de Rham cohomology class c on E whose restriction to each fibre is the class determined by ω.*

Theorem 1 is tacitly assumed in the usual identification of the E_1 term in the Leray spectral sequence for de Rham cohomology as forms on the base with values in the cohomology of the fibres [Gr–Ha]. No proof is given in the cited reference, and indeed the only proof we have found in the literature [Ha] applies only when the fibres have finite-dimensional cohomology groups.

Using a partition of unity on B, it is easy to reduce Theorem 1 to the following lemma.

LEMMA. *Let $\{\omega_u\}$ be a family of p-forms on the manifold F, depending*

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smoothly on a parameter u ranging over an open set $\mathcal{U}$ *in* $\mathbb{R}^n$. *If* ω_u *is exact for each u, then there is a smooth family* $\{\eta_u\}$ *of* $p-1$ *forms on F such that* $d\eta_u = \omega_u$ *for each u in* $\mathcal{U}$.

For compact F, the lemma can be proven by Hodge theory. For general F, the first step in our proof is to pass from de Rham to Čech cohomology by the usual sheaf-theoretic argument [Gr–Ha] [W1] using smooth dependence on parameters in the Poincaré lemma. Now one applies the universal coefficient theorem for Čech cohomology to the case of coefficients in $C^\infty(\mathcal{U})$. (See Thm. 5.4.13(c) in [Hi–Wy].)

From now on, we confine our attention to the case $p=2$. The following theorem seems to be something of a surprise.

THEOREM 2. *Let* $F \to E \to B$ *be a fibre space with F and B* 1*-connected. If the restriction map* $H^2(E;\mathbb{R}) \to H^2(F;\mathbb{R})$ *is not surjective, then* $H^{2k}(F;\mathbb{R})$ *is non-trivial for all* $k \geqslant 0$.

Proof. 1. There is a map $\alpha: S^3 \to B$ such that $\partial_*[\alpha] \in \pi_2(F)$ has infinite order: Since F is 1-connected, $\pi_2(F) \cong H_2(F)$ and so $\pi_2(F) \otimes \mathbb{R} \cong H_2(F) \otimes \mathbb{R} \cong H_2(F;\mathbb{R})$. Also, $H^2(F;\mathbb{R}) \cong \operatorname{Hom}(H_2(F;\mathbb{R}),\mathbb{R})$. Since $\pi_1(B)$ is trivial, so is $\pi_1(E)$. Thus $H^2(E;\mathbb{R}) \to H^2(F;\mathbb{R})$ not surjective is equivalent to $H_2(F;\mathbb{R}) \to H_2(E;\mathbb{R})$ not injective, and hence equivalent to $\pi_2(F) \otimes \mathbb{R} \to \pi_2(E) \otimes \mathbb{R}$ not injective. Thus α exists.

2. There is a map $\psi: \Omega S^3 \to F$ such that $\psi^*: H^2(F;\mathbb{R}) \to H^2(\Omega S^3;\mathbb{R})$ is non-trivial: Consider the commutative diagram

$$\begin{tikzcd}
\Omega S^3 \arrow[r, "\psi"] \arrow[d] & F \arrow[r, "="] \arrow[d] & F \arrow[d] \\
PS^3 \arrow[r, "\varphi"] \arrow[d, "q"'] & \alpha^* E \arrow[r] \arrow[d] & E \arrow[d] \\
S^3 \arrow[r, "="] & S^3 \arrow[r, "\alpha"'] & B
\end{tikzcd}$$

where PS^3 is the space of paths beginning at the base point of S^3, q is the endpoint projection, φ is any lift of q, and ψ is the restriction of φ. From the homomorphism of exact homotopy sequences we see that $\psi_*: \pi_2(\Omega S^3) \to \pi_2(F)$ sends the generator onto $\partial_*[\alpha]$. Hence $\psi_*: H_2(\Omega S^3;\mathbb{R}) \to H_2(F;\mathbb{R})$ is non-trivial, and so ψ^* is non-trivial.

3. $H^{2k}(;\mathbb{R})$ is non-trivial for all $k \geqslant 0$: $H^*(\Omega S^3;\mathbb{R}) = \mathbb{R}[x]$, where $x \in H^2(\Omega S^3;\mathbb{R})$ is the generator. By step 2, $H^*(F;\mathbb{R}) \to H^*(\Omega S^3;\mathbb{R}) = \mathbb{R}[x]$ is surjective, and the theorem follows. ■

Remark. If $F \to E \to B$ has as its structure group a Lie group G, then $\pi_2(G) = 0$ implies $\alpha^* E = S^3 \times F$. Hence we may choose φ so that ψ is the constant map and ψ^* is trivial. Thus $H^2(E; \mathbb{R}) \to H^2(F; \mathbb{R})$ must be surjective. Also note that $\partial_*: \pi_3(B) \to \pi_2(F)$ is trivial since it factors through $\pi_2(G)$.

More generally, there are smooth finite dimensional manifolds F with $\pi_2(\operatorname{Diff} F) \neq 0$, and hence smooth non-trivial bundles E over S^3 with fibre F [Hr] [La]. But we must have $\partial_*: \pi_3(S^3) \to \pi_2(F)$ of finite order since the conclusions of the theorem cannot hold for finite dimensional F. Such a bundle looks like a product when viewed with real (or rational) coefficients; i.e., $\pi_i(E) \otimes \mathbb{R} = [\pi_i(S^3) \otimes \mathbb{R}] \oplus [\pi_i(F) \otimes \mathbb{R}]$, and in some sense the action of Diff F on F resembles the action of a Lie group.

Combining Theorems 1 and 2 leads to the following conclusion:

THEOREM 3. *Let $F \to E \to B$ be a bundle of symplectic manifolds with 1-connected fibre and base. Unless $H^{2k}(F; \mathbb{R}) \neq 0$ for all $k \geq 0$, the symplectic structure along the fibres has a closed extension.*

Theorem 3 was originally proven for compact and 1-connected F in [Wn]. The following two examples show the necessity of the hypotheses on the fibre in Theorem 3. It is possible that the assumption of simple connectivity of B could be dropped or weakened (compare [B1]).

EXAMPLE 1. Begin with the Hopf bundle $S^1 \to S^3 \to S^2$ and cross the fibre with S^1 to make a torus bundle $S^1 \times S^1 \to S^3 \times S^1 \to S^2$. The structure group consists of translations of the torus, which preserve the standard area element; thus this is a bundle of symplectic manifolds, but $H^2(S^3 \times S^1; \mathbb{R}) = 0$.

EXAMPLE 2. The loop space fibration $K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 3)$ can be pulled back to S^3 to give a bundle $\mathbb{C}P^\infty \to E \to S^3$ with $H^2(E; \mathbb{R}) = 0$. This bundle can be constructed with a gluing map from S^2 to the projective unitary group $PU(\mathbb{H}) = U(\mathbb{H})/S^1$ acting on the Hilbert space $\mathbb{H}$ for which $\mathbb{C}P^\infty$ is the space of lines. Since $PU(\mathbb{H})$ acts symplectically on $\mathbb{C}P^\infty$, we have a bundle of symplectic manifolds. It would be interesting to have an explicit geometric or analytic construction of this bundle.

Theorem 3 has a nice interpretation involving connections and reductions of the structure group for bundles of symplectic manifolds. If $F \to E \to B$ is any bundle of symplectic manifolds, there always exists an extension Ω (not necessarily closed) of the symplectic structure along the fibres. Since the restriction of Ω to the vertical bundle $\mathcal{V}$ is nondegenerate, the horizontal sub-bundle $\mathcal{H} = \{v \in TE \mid v \rfloor \Omega$ annihilates vertical vectors$\}$ is a complement to $\mathcal{V}$ and so defines a connection on E in the sense of [Eh]. The form Ω is determined by this connection together with the restriction of Ω to $\mathscr{H}$.

We shall call Ω *r-closed* if $(v_1 \wedge \cdots \wedge v_r)\rfloor\, d\Omega = 0$ whenever $v_1, \ldots, v_r$ belong to $\mathscr{V}$. Since Ω is closed on fibres, it is 3-closed. Ω is 0-closed if and only if it is closed. The intermediate steps in this (Leray) filtration have the following interpretations.

THEOREM 4. (i) *Ω is 2-closed if and only if parallel translations preserve the symplectic structure.*

(ii) *If Ω is 2-closed, the curvature of the connection takes values in the Lie algebra of locally hamiltonian vector fields along the fibres. The curvature takes values in the globally hamiltonian vector fields if and only if Ω is 1-closed. Thus there exists a 1-closed extension if and only if the structure group can be reduced to one which admits a momentum mapping. (Compare Example 1).*

(iii) *There exists a closed extension if and only if the structure group can be lifted to one which admits an* Ad*-*equivariant momentum mapping. (Compare Example* 2.)

Our results can also be interpreted in the language of geometric quantization. The bundle $F \to E \to B$ of symplectic manifolds may be considered as a family of classical mechanical systems depending on a parameter in B. If the symplectic cohomology class of F is integral, then it is the Chern class of a complex line bundle over F, i.e. a prequantization. One might wish to prequantize the whole family at once by finding a line bundle over E which has the right restriction to each fibre. Theorem 2 shows that, if we multiply the symplectic form on F by a suitably chosen integer (in fact, this might not be necessary), then the line bundle over E can be found if F is simply connected and finite dimensional. Examples 1 and 2 show that these conditions on the fibre cannot be omitted.

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Applications de la p-induction en analyse harmonique*

JEAN-PHILIPPE ANKER

A la mémoire de Serge Reichenthal

La p-induction est la version L^p de l'induction unitaire de G. W. Mackey [28; part I]. Certains aspects du cas unitaire classique laissaient supposer qu'elle puisse fournir un cadre naturel à des phénomènes liés à l'analyse L^p sur les groupes localement compacts et permettre d'en donner des démonstrations simples et conceptuelles.

Parmi ceux que nous abordons dans ce travail, mentionnons ici les principes de majoration de C. Herz (théorème 7 et exemple 11), ainsi que des techniques de transferts de convoluteurs, notamment l'induction, à partir d'un sous-groupe (§ 2) et le passage au quotient (exemples 10 et 13). L'outil principal que nous avons eu à développer est la continuité de la p-induction par rapport à une topologie de Fell généralisée (théorème 13). Signalons à ce propos qu'une difficulté à laquelle nous nous sommes heurtés à plusieurs reprises a consisté à trouver le bon substitut aux techniques hilbertiennes classiques.

Comme autre type d'application de la p-induction, nour verrons qu'un procédé itéré fournit un tube de représentations isométriques autour de chaque série principale unitaire d'un groupe de Lie semi-simple (§ 4). Signalons que, dans le cas sphérique, l'existence d'un tel tube avait été constatée par M. G. Cowling et exploitée avec succès dans sa démonstration du phénomène de Kunze–Stein [7].

1. Généralités sur la p-induction

Nous reprenons maintenant les aspects généraux de la p-induction, qui ont fait l'objet de [24], [17], [31], et fixons à cette occasion les notations.

Soient G un groupe localement compact, H un sous-groupe fermé et $1 \leqq p \leqq \infty$. La *p-induction* associe à toute représentation π de H – isométrique, fortement

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* Extrait d'un travail de thèse à l'Université de Lausanne [3]

continue, sur un espace de Banach B – une représentation de G du même type, notée $\mathrm{Ind}_H^G(p, \pi)$. L'espace sur lequel opère $\mathrm{Ind}_H^G(p, \pi)$ est un complété de l'espace $C_c^p(G, H; \pi)$ des fonctions $f: G \to B$ vérifiant:

i) la condition d'*homogénéité*:

$$f(xh) = \delta(h)^{1/p}\pi(h)^{-1}f(x) \quad (x \in G, h \in H),$$

où $\delta(h) = \Delta_H(h)/\Delta_G(h)$ est le rapport des modules de H et de G,

ii) les conditions de *régularité*: f est continue, à support compact modulo H.

Avant de passer à la complétion de $C_c^p(G, H; \pi)$, mentionnons *l'application de Mackey* $f \mapsto [f]$ de $C_c(G; B)$ (fonctions continues, à support compact, de G dans B) dans $C_c^p(G, H; \pi)$ définie par le procédé d'*homogénéisation* $[f](x) = \int_H dh\, \delta(h)^{-1/p}\pi(h)f(xh)$.

Remarques. 1) C'est une application surjective, des sections étant données par les applications $f \mapsto \beta f$ associées aux *fonctions de Bruhat* β de la paire $H \subset G$. Rappelons qu'il s'agit là des fonctions continues $\geqq 0$ sur G vérifiant:

i) supp $\beta \cap KH$ est compact, pour tout compact K de G,

ii) $\int_H dh\, \beta(xh) = 1$, pour tout $x \in G$,

(dont l'existence est prouvée, par exemple, dans [29; ch. 8, 1.7–1.8]).

2) Du point de vue topologique, l'application de Mackey est une *application quotient* dans la catégorie des espaces localement convexes, $C_c(G; B)$ et $C_c^p(G, H; \pi)$ étant munis des topologies *limite inductive stricte* évidentes.

Introduisons maintenant, pour $p < \infty$, le complété $L^p(G, H; \pi)$ de $C_c^p(G, H; \pi)$ pour la *norme* intrinsèque

$$\|f\|_p = \left[\int_{G/H} d_q(xH) \frac{|f(x)|^p}{q(x)}\right]^{1/p} = \left[\int_G dx\, \beta(x)\, |f(x)|^p\right]^{1/p},$$

où q est une fonction continue >0 sur G vérifiant la condition d'homogénéité $q(xh) = q(x)\,\delta(h)$ $(x \in G, h \in H)$ et $d_q\xi$ la mesure quasi-invariante sur G/H qui lui est associée par

$$\int_{G/H} d_q(xH) \int_H dh \frac{f(xh)}{q(xh)} = \int_G dx\, f(x) \qquad (f \in C_c(G))$$

(cf par exemple [28; ch. 8, 1.1–1.3] à ce sujet), β étant une fonction de Bruhat de la paire $H \subset G$.

Il serait pénible de donner un sens raisonnable à $L^\infty(G, H; \pi)$. Nous nous contenterons, pour $p=\infty$, de l'espace $C_0(G, H; \pi)$ des fonctions $f: G \to B$ vérifiant

i) $f(xh) = \pi(h)^{-1} f(x) \qquad (x \in G, h \in H)$,

ii) f est continue et s'annule à l'infini modulo H,

complete de $C_c^\infty(G, H, \pi)$ pour la norme $\|f\|_\infty = \sup_{x \in G} |f(x)|$.

La *représentation* $\mathrm{Ind}_H^G(p, \pi)$ opère sur $L^p(G, H; \pi)$, resp. $C_0(G, H; \pi)$ par translations à gauche: $\{\mathrm{Ind}_H^G(p, \pi)(g)f\}(x) = f(g^{-1}x)$.

EXEMPLE 1. La représentation p-induite à partir de la représentation triviale de $\{e\}$ sur B est la *représentation régulière à gauche*

$\{\rho_{G,B}^p(g)f\}(x) = f(g^{-1}x)$ de G sur $L^p(G, B)$, resp. $C_0(G, B)$.

La *version* dite de *Mackey* de la p-induction se déduit de la précédente par la transformation $f \mapsto f/q^{1/p}$. Pour $p=\infty$, il n'y a donc aucune différence. Pour $p<\infty$, l'espace initial est ici $C_c^\infty(G, H; \pi)$, la norme

$$\|f\|_p = \left[\int_{G/H} d_q(xH)\, |f(x)|^p \right]^{1/p}$$

et la représentation

$$\{\mathrm{Ind}_H^G(p, \pi)(g)f\}(x) = m(g, xH)^{1/p} f(g^{-1}x),$$

où $m(g, xH) = \dfrac{q(g^{-1}x)}{q(x)}$ est le module de quasi-invariance de la mesure $d_q\zeta$.

EXEMPLE 2. La représentation p-induite à partir de la représentation triviale de H sur B est la représentation *quasi-régulière*

$\{\pi^p(g)f\}(\zeta) = m(g, \zeta)^{1/p} f(g^{-1} . \zeta)$ de G sur $L^p(G/H; B)$, resp. $C_0(G/H; B)$.

Remarque. Dans la version de Mackey, l'espace $L^p(G, H; \pi)$, resp. $C_0(G, H; \pi)$ s'identifie naturellement à l'espace des *sections* L^p, resp. C_0 du *fibré induit* $G \times_H B = G \times B/(x, \xi) \sim (xh, \pi(h)^{-1}\xi)$ sur G/H. La théorie de l'intégration pour les sections d'un tel fibré, qui se fait sur le modèle classique [5], fournit des réalisations concrètes des espaces $L^p(G, H; \pi)$.

On peut réaliser la représentation $\mathrm{Ind}_H^G(p, \pi)$ sur $L^p(G/H; B)$, resp. $C_0(G/H; B)$ lorsqu'il existe une section – admettons continue – s de G/H dans G:

$\{\mathrm{Ind}_H^G(p,\pi)(g)f\}(\zeta)=c(g,\zeta)f(g^{-1}\,.\,\zeta)$, où $c(g,\zeta)=\delta(h)^{1/p}\pi(h)^{-1}$ (1-cocycle en g), h étant l'élément de H déterminé par $g^{-1}s(\zeta)=s(g^{-1}\,.\,\zeta)h$.

EXEMPLE 3. Supposons qu'il existe un sous-groupe fermé K de G pour lequel l'application $(k,h)\mapsto kh$ soit un homéomorphisme de $K\times H$ sur G. La représentation $\mathrm{Ind}_H^G(p,\pi)$ est alors réalisée sur $L^p(K;B)$, resp. $C_0(K;B)$ par $\{\mathrm{Ind}_H^G(p,\pi)(g)f\}(k)=c(g,k)f(g^{-1}\,.\,k)$, où $c(g,k)=\delta(h)^{1/p}\pi(h)^{-1}$, $g^{-1}\,.\,k$ et h étant les éléments de K et de H déterminés par $g^{-1}k=(g^{-1}\,.\,k)h$.

Ceci reste valable, pour $p<\infty$, lorsque KH est un ouvert de G dont le complémentaire est localement négligeable.

Terminons par une propriété classique de transitivité.

THEOREME 1 (induction par étages). *Soit K un sous-groupe fermé intercalé entre H et G. Les représentations* $\mathrm{Ind}_K^G(p,\mathrm{Ind}_H^K(p,\pi))$ *et* $\mathrm{Ind}_H^G(p,\pi)$ *sont alors équivalentes.*

Démonstration. En effectuant successivement les applications de Mackey correspondant aux représentations $\sigma=\mathrm{Ind}_H^K(p,\pi)$ et $\mathrm{Ind}_K^G(p,\sigma)$, on construit, à partir d'une fonction $F\in C_c(G\times K;B)$, la fonction

$$[\![F]\!](x,k)=\int_K dl\left[\frac{\Delta_K(l)}{\Delta_G(l)}\right]^{-1/b}\int_H dh\left[\frac{\Delta_H(h)}{\Delta_K(h)}\right]^{-1/p}\pi(h)F(xl,l^{-1}kh),$$

qui est continue sur $G\times K$ et fournit un élément de $C_c^p(G,K;\sigma)$. L'image E de $C_c(G\times K;B)$ par cette *double application de Mackey* est un sous-espace dense de $L^p(G,K;\sigma)$, resp. $C_0(G,K;\sigma)$. Il contient en effet les éléments de la forme $[\![f_1\otimes f_2\otimes\xi]\!]=[f_1\otimes[f_2\otimes\xi]]=(f_1\in C_c(G),f_2\in C_c(K),\xi\in B)$. Considérons maintenant les applications $C_c(G;B)\underset{A'}{\overset{A}{\rightleftarrows}}C_c(G\times K;B)$ définies par

$$Af(x,k)=\alpha(k^{-1})\left[\frac{\Delta_K(k)}{\Delta_G(k)}\right]^{-1/p}f(xk)$$

(où $\alpha\in C_c(K)$ vérifie $\alpha\geqq 0,\int_K dk\ \alpha(k)=1$ et a été fixée une fois pour toutes) et

$$A'F(x)=\int_K dk\left[\frac{\Delta_K(k)}{\Delta_G(k)}\right]^{-1/p}F(xk,k^{-1}).$$

Elles induisent, par les diagrammes commutatifs

$$\begin{tikzcd} C_c(G,B) \arrow[r, "A"] \arrow[d, "{[\,]}"'] & C_c(G\times K;B) \arrow[d, "\llbracket\,\rrbracket"] \\ C_c^p(G,H;\pi) \arrow[r, "{[A]}"] & E \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} C_c(G,B) \arrow[d, "{[\,]}"'] & C_c(G\times K;B) \arrow[l, "A'"'] \arrow[d, "\llbracket\,\rrbracket"] \\ C_c^p(G,H;\pi) & E \arrow[l, "{[A']}"'] \end{tikzcd}$$

les applications $C_c^p(G,H;\pi) \underset{[A']}{\overset{[A]}{\rightleftarrows}} E$ données par

$$[A]f(x,k)=\left[\frac{\Delta_K(k)}{\Delta_G(k)}\right]^{-1/p} f(xk) \quad \text{et} \quad [A']F(x)=F(x,e).$$

Les applications $[A]$ et $[A']$ sont inverses l'une de l'autre et commutent aux translations à gauche. Elles sont enfin isométriques. En effet, si β et γ sont des fonctions de Bruhat des paires $H\subset G$ et $K\subset G$, on a, pour $p<\infty$,

$$\begin{aligned} \|[A]f\|_p^p &= \int_G dx\,\gamma(x)\int_K dk\,\beta(xk)\,|[A]f(x,k)|^p \\ &= \int_G dx\,\gamma(x)\int_K dk\left[\frac{\Delta_K(k)}{\Delta_G(k)}\right]^{-1/p}\beta(xk)\,|f(xk)|^p \\ &= \int_G dx\,\beta(x)\,|f(x)|^p = \|f\|_p^p \end{aligned}$$

étant donné que, chaque $x\in G$, la fonction $k\mapsto\beta(xk)$ est de Bruhat pour la paire $H\subset K$.

EXEMPLE 4. La représentation régulière (à gauche) $\rho_{G,B}^p$ est équivalente à la représentation p-induite à partir de la représentation régulière (à gauche) $\rho_{H,B}^p$. Explicitement, le diagramme commutatif qui est au centre de la démonstration précédente se résume ici à

$$\begin{tikzcd} & C_c(G\times H;B) \arrow[dl, "A'"'] \arrow[d, "\llbracket\,\rrbracket"] \\ C_c(G;B) \arrow[r, shift left, "{[A]}"] & E \arrow[l, shift left, "{[A']}"] \end{tikzcd}$$

où

$$[\![F]\!](x, k) = \int_H dh \left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{-1/p} F(xh, h^{-1}k),$$

où

$$[A]f(x, h) = \left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{-1/p} f(xh),$$

où

$$A'F(x) = \int_H dh \left[\frac{\Delta_H(h)}{\Delta_G(h)}\right]^{-1/p} F(xh, h^{-1})$$

et où $[A']F(x) = F(x, e)$. Notons qu'on a le même diagramme pour la version de Mackey de la représentation $\mathrm{Ind}_H^G(p, \rho_{H,B}^p)$, avec, dans ce cas,

$$[A]f(x) = \left\{\frac{f}{q^{1/p}}\right\}_{x,H} : h \mapsto \frac{f(xh)}{q(xh)^{1/p}}.$$

Remarquons que l'identification de l'espace $L^p(G; B)$, resp. $C_0(G; B)$ à l'espace des sections L^p, resp. C_0 du fibré induit par $\rho_{H,B}^p$ fait apparaître la restriction à H de la *représentation régulière à droite* $\{\lambda_{G,B}^p(g)f\}(x) = \Delta_G(g)^{1/p}f(xg)$ de G sur $L^p(G; B)$ comme *intégrale directe* L^p, resp. *somme directe continue* sur G/H de la représentation régulière à droite $\lambda_{H,B}^p$.

2. Entrelacements et convoluteurs induits*

Nous faisons apparaître la notion de convoluteur induit, considérée précédemment par plusieurs auteurs comme un cas particulier de la notion naturelle d'entrelacement induit et obtenons, grâce à ce point de vue, des démonstrations simples de ses propriétés.

Nous commençons par étudier la *p-induction des entrelacements* (classique dans le cas unitaire). Donnons-nous deux représentations π_1 et π_2 de H (isométriques, fortement continues, sur des espaces de Banach B_1 et B_2) et considérons les représentations correspondantes $\mathrm{Ind}_H^G(p, \pi_1)$ et $\mathrm{Ind}_H^G(p, \pi_2)$ de G, dans la version de Mackey. On obtient, à partir d'un *entrelacement* T de π_1 avec π_2 (rappelons qu'il s'agit là des opérateurs bornés T de B_1 dans B_2 pour lesquels

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* Les résultats de ce paragraphe ont fait l'objet de l'annonce [2].

$T \circ \pi_1(h) = \pi_2(h) \circ T$ pout tout $h \in H$), un entrelacement $\mathrm{Ind}_H^G(p, T)$ de $\mathrm{Ind}_H^G(p, \pi_1)$ avec $\mathrm{Ind}_H^G(p, \pi_2)$ en posant $\mathrm{Ind}_H^G(p, T)f = T \circ f$, i.e. en faisant opérer T *fibre par fibre.* Notons que $\|\mathrm{Ind}_H^G(p, T)\| \leqq \|T\|$ et que $\mathrm{Ind}_H^G(p, T)$ commute à la multiplication ponctuelle par $L^\infty(G/H)$, resp. $C^b(G/H)$.

PROPOSITION 2. *La p-induction* $T \mapsto \mathrm{Ind}_H^G(p, T)$ *des entrelacements est une isométrie.*

Démonstration. Montrons $\|T\| \leqq \|\mathrm{Ind}_H^G(p, T)\|$ pour $p < \infty$ (le cas $p = \infty$ se traitant de même). L'inégalité $\|\psi \cdot \mathrm{Ind}_H^G(p, T)f\|_p \leqq \|\mathrm{Ind}_H^G(p, T)\| \, \|\psi \cdot f\|_p$, valable pour tout $f \in C_c^\infty(G, H; \pi)$, $\psi \in L^\infty(G/H)$, s'écrit

$$\int_{G/H} d_q(xH)\, |\psi(xH)|^p\, |Tf(x)|^p \leqq \|\mathrm{Ind}_H^G(p, T)\|^p \int_{G/H} d_q(xH)\, |\psi(xH)|^p\, |f(x)|^p.$$

On en tire facilement $\|T\| \leqq \|\mathrm{Ind}_H^G(p, T)\|$.

Relevons que la p-induction jouit de toutes les propriétés fonctorielles souhaitables. Nous allons encore donner deux caractérisations des entrelacements p-induits. Dans le cas unitaire et lorsque $\pi_1 = \pi_2$, la première caractérisation est classique (cf. par exemple [4; ch. 16, § 3.B, theorem 3]); on la rencontre dans les questions liées au théorème d'imprimitivité de Mackey.

THEOREME 3. *Un entrelacement S de* $\mathrm{Ind}_H^G(p, \pi_1)$ *avec* $\mathrm{Ind}_H^G(p, \pi_2)$ *est induit à partir de H si et seulement s'il vérifie une des deux conditions équivalentes suivantes:*

1) *S commute à la multiplication ponctuelle par* $L^\infty(G/H)$, *resp.* $C^b(G/H)$,
2) $\mathrm{supp}\,(Sf) \subset \mathrm{supp}\, f$ *pour tout* $f \in L^p(G, H; \pi_1)$, *resp.* $C_0(G, H; \pi_1)$.

Précisons la notion de *support* d'une fonction $f \in L^p(G, H; \pi_i)$: il s'agit du plus petit fermé de G/H en dehors duquel $|f| = 0$ (pp). On peut également le définir par dualité avec $C_c^\infty(G, H; \pi_i^*)$, π_i^* étant la contragrédiente de π_i ou toute autre représentation formant avec π_i une paire duale de H (cf § 3): $\zeta \notin \mathrm{supp}\, f$ si et seulement s'il existe un voisinage V de ζ dans G/H tel que $\langle f, \phi \rangle = 0$ pout tout $\phi \in C_c^\infty(G, H; \pi_i^*)$ avec $\mathrm{supp}\, \phi \subset V$.

Démonstration. Il est clair que les deux conditions sont nécessaires et il est facile de déduire la seconde de la première. Le point délicat consiste à montrer que tout entrelacement S de $\mathrm{Ind}_H^G(p, \pi_1)$ avec $\mathrm{Ind}_H^G(p, \pi_2)$ vérifiant la seconde condition est induit par un entrelacement T de π_1 avec π_2. L'idée de la preuve est simple: on définit T en *isolant* l'action de S sur la fibre au-dessus de H.

Commençons par le cas $p=\infty$ et par établir l'inégalité

$$(*) \quad |Sf(e)| \leqq \|S\| \, |f(e)| \qquad (f \in C_0(G, H; \pi_1)).$$

Supposons, par l'absurde, qu'il existe $f \in C_0(G, H; \pi_1)$ avec $|Sf(e)| - \|S\| \, |f(e)| = \varepsilon > 0$. Il existe par suite un voisinage V de H dans G/H sur lequel $\|S\| \, |f(x)| \leqq |Sf(e)| - \varepsilon/2$. Choisissons une fonction auxiliaire $\psi : G/H \to [0, 1]$ continue, à support dans V et valant 1 sur un voisinage de H dans G/H. Comme, par hypothèse, $S(\psi \cdot f)(e) = Sf(e)$, on obtient $\|S\| \, \|\psi \cdot f\|_\infty \leqq |S(\psi \cdot f)(e)| - \varepsilon/2$, ce qui est absurde.

Maintenant, l'inégalité $(*)$ et le fait que les vecteurs $f(e)$, pour $f \in C_0(G, H; \pi_1)$, décrivent B_1 montrent que $f(e) \mapsto Sf(e)$ définit un opérateur borné T de B_1 dans B_2. On vérifie facilement que T est un entrelacement de π_1 avec π_2, qui induit S.

Dans le cas $p<\infty$, considérons le sous-espace E_0 de $C_0^\infty(G, H; \pi_1)$ engendré par les fonctions de la forme $f_0 = \mathrm{Ind}_H^G(p, \pi_1)(u)f$, où $u \in C_c(G)$ et $f \in C_c^\infty(G, H; \pi_1)$. Il jouit des propriétés suivantes:

— $\{f_0(e) \mid f_0 \in E_0\}$ est dense dans B_1,

— $S(E_0)$ est composé de fonctions continues (cf. lemme ci-dessous), ce qui permet de reprendre la démonstration du cas $p=\infty$.

LEMME 4. *Toute fonction de la forme* $f_0 = \mathrm{Ind}_H^G(p, \pi_i)(u)f$, *où* $u \in C_c(G)$ *et* $f \in L^p(G, H; \pi_i)$, *est (égale pp modulo H à une fonction) continue.*

Démonstration. Commençons par exprimer l'opérateur intégral $\mathrm{Ind}_H^G(p, \pi_i)(u)$ à l'aide d'un noyau:

$$\{\mathrm{Ind}_H^G(p, \pi_i)(u)f\}(x) = \int_{G/H} d_q(yH) \, K_u(x, y) f(y) \; (f \in C_c^\infty(G, H; \pi_i)),$$

où

$$K_u(x, y) = q(x)^{-1/p} q(y)^{-1/p'} \Delta_G(y)^{-1} \int_H dh \, \delta(h)^{-1/p} u(xhy^{-1}) \pi_i(h).$$

On en déduit une majoration d'ordre technique:
pour tout compact K de G/H il existe une constante $C \geqq 0$ telle que

$$|\{\mathrm{Ind}_H^G(p, \pi_i)(u)f\}(x)| \leqq C \, \|f\|_p \qquad (f \in C_c^\infty(G, H, \pi_i), \, xH \in K).$$

Considérons maintenant une suite $(f_n) \subset C_c^\infty(G, H; \pi_i)$ convergeant (en norme) vers f. D'après la majoration précédente, la suite des fonctions $\mathrm{Ind}_H^G(p, \pi_i)(u)f_n$

converge uniformément sur la préimage de tout compact de G/H vers une fonction homogène continue f_∞. Comme, d'autre part, une sous-suite converge (pp modulo H) vers f_0, on a $f_0 = f_\infty$ (pp modulo H).

Nous explicitons maintenant ce qui précède dans le cas particulier où $\pi_1 = \pi_2$ est la représentation régulière (à gauche) ρ_H^p de H sur $L^p(H)$, $1 < p < \infty$. Rappelons que nous avons identifié au § 1 les représentations $\mathrm{Ind}_H^G(p, \rho_H^p)$ et ρ_G^p. La p-induction des entrelacements correspond donc ici à un procédé *d'induction des convoluteurs à droite*. Explicitement,

$$\langle \mathrm{Ind}_H^G(p, T)f, \phi\rangle = \int_{G/H} d_q(xH) \left\langle T\left\{\frac{f}{q^{1/p}}\right\}_{x,H}, \left\{\frac{\phi}{q^{1/p'}}\right\}_{x,H}\right\rangle \qquad (f, \phi \in C_c(G)).$$

EXEMPLE 5. $\mathrm{Ind}_H^G(p, \lambda_H^p(\mu)) = \lambda_G^p(\mu)$ pour tout $\mu \in M^1(H)$ (mesure bornée sur H).

EXEMPLE 6. Plus généralement, si T est défini par convolution à droite par une distribution $\tau: T\xi(k) = \xi * \tau(k) = \int_H d\tau(h)\, \Delta_h(h)^{-1} \xi(kh^{-1})$, alors $\mathrm{Ind}_H^G(p, T)$ est défini par convolution à droite par la distribution $\delta^{-1/p'}\tau$ (à support dans H):

$$\{\mathrm{Ind}_H^G(p, T)f\}(x) = f * (\delta^{-1/p'}\tau)(x) = \int_H d\tau(h)\, \Delta_H(h)^{-1/p'} \Delta_G(h)^{-1/p} f(xh^{-1}).$$

EXEMPLE 7. Dans le contexte abélien, l'inclusion $CV^p(H) \to CV^p(G)$ des convoluteurs correspond au *relèvement* $m^p(\hat{G}/H^\perp) \to m^p(\hat{G})$ *des multiplicateurs.*

PROPOSITION 5. *L'induction des convoluteurs à droite est un homomorphisme d'algèbres isométrique.*

Notons qu'on passe facilement aux *convoluteurs à gauche*, au moyen des involutions $\iota(\xi)(h) = \Delta_H(h)^{-1/p}\xi(h^{-1})$ de $L^p(H)$ et $\iota(f)(x) = \Delta_G(x)^{-1/p} f(x^{-1})$ de $L^p(G)$.

Remarque. L'induction des convoluteurs coïncide avec les notions apparues dans [8; § 4] ($G = \mathbb{R}, H = \mathbb{Z}$), puis [30; section 3] et [25; théorème I.2] (cas abélien), [23] (cas des pseudo-mesures), et finalement [26], [27; appendice], [9] (cas général).

La traduction de la seconde caractérisation du théorème 3 nous amène à définir la notion de *support* d'un convoluteur S de $L^p(G): x \notin \operatorname{supp} S$ si et seulement s'il existe dans G des voisinages U de e et V de x tels que $\langle Su, v\rangle = 0$ pour tout $u, v \in C_c(G)$ avec $\operatorname{supp} u \subset U$, $\operatorname{supp} v \subset V$.

Remarques. 1) Cette notion coïncide avec

— le support de la mesure – ou plus généralement de la distribution – σ, lorque S est défini par convolution (à droite ou à gauche) par σ,

— la définition [23; p. 117],

— le support défini par dualité avec $A^p(G)$, lorsque S est une pseudo-mesure [23; p. 101],

— le spectre du multiplicateur m, lorsque, dans le contexte abélien, $S = \hat{m}$.

2) Le support d'un convoluteur est conservé par induction.

THEOREME 6. *Un convoluteur – à droite, resp. à gauche – S de $L^p(G)$ est induit à partir de H si seulement s'il vérifie une des deux conditions équivalentes suivantes:*

1) *S commute à la multiplication ponctuelle par $L^\infty(G/H)$, resp. $L^\infty(H\backslash G)$,*

2) *S a son support dans H.*

C'est un corollaire immédiat du théorème 3, le support de S étant, dans le cas d'un convoluteur à droite, le plus petit fermé F de G tel que $\operatorname{supp}(Sf) \subset (\operatorname{supp} f)F$ pout tout $f \in C_c(G)$ (le support de Sf étant pris au sens des mesures).

Remarque. Le seconde caractérisation, au moyen du support, a été prouvée dans [30; lemma 3.2] (cas abélien), [23; theorem B] (cas des pseudo-mesures, H moyennable ou normal dans G) et finalement [26], [27; appendice] (cas général). Notre démonstration du cas général, techniquement assez voisine de [30; loc. cit.], présente l'avantage d'être directe et relativement simple.

3. Coefficients de représentations p-induites et transferts de convoluteurs.

Dans ce paragraphe nous établissons des propriétés fonctionnelles des coefficients de certaines représentations p-induites, découlant principalement de la continuité de la p-induction, et en déduisons par dualité des transferts d'opérateurs, notamment de convoluteurs. Signalons à ce propos que, sauf mention explicite, on travaillera ici avec des convoluteurs *à gauche.*

Dans ce paragraphe nous considérons toutes les représentations par paires duales. Une *paire duale* d'un groupe localement compact G est composée de deux représentations π et π^* de G – isométriques, fortement continues, sur des espaces de Banach B et B^* – mises en dualité par un *couplage*, i.e. une forme bilinéaire $\langle\,,\rangle$ sur $B \times B^*$ vérifiant

i) $\langle \pi(g)\xi, \pi^*(g)\xi^*\rangle = \langle \xi, \xi^*\rangle \qquad (g \in G, \xi \in B, \xi^* \in B^*)$,

ii) $|\xi| = \sup\limits_{|\eta^*|=1} |\langle \xi, \eta^*\rangle|, \quad |\xi^*| = \sup\limits_{|\eta|=1} |\langle \eta, \xi^*\rangle| \qquad (\xi \in B, \xi^* \in B^*)$.

EXEMPLE 8. Soit π une représentation de G – isométrique, fortement continue, sur un espace de Banach B. Les vecteurs ξ^* du dual B^* pour lesquels la fonction $g \mapsto \pi^*(g)\xi^* = {}^t\pi(g)^{-1}\xi^*$ est continue composent un sous-espace fermé, faiblement dense B_0^*. La représentation π^* de G sur B_0^*, appelée *contragrédiente de* π, forme avec π une paire duale. Toute représentation de G en dualité avec π est contenue dans π^*, et coincide même avec π^* lorsque B est réflexif.

EXEMPLE 9. Les représentations $\mathrm{Ind}_H^G(p, \pi)$ et $\mathrm{Ind}_H^G(p', \pi^*)$, induites à partir d'une paire duale (π, π^*) de H suivant des indices p et p' conjugués $\left(\text{i.e. } \frac{1}{p}+\frac{1}{p'}=1\right)$, forment une paire duale de G pour le couplage

$$\langle f, \phi\rangle = \int_{G/H} d_q(xH)\frac{\langle f(x), \phi(x)\rangle}{q(x)} = \int_G dx\, \beta(x)\langle f(x), \phi(x)\rangle,$$

resp.

$$\langle f, \phi\rangle = \int_{G/H} d_q(xH)\langle f(x), \phi(x)\rangle$$

dans la version de Mackey.

Le premier principe de majoration de Herz permet de contrôler les coefficients des représentations p-induites de H à G à l'aide des coefficients de la représentation quasi-régulière π^p de G sur $L^p(G/H)$, resp. $C_0(G/H)$. Considérons une paire duale induite $(\mathrm{Ind}_H^G(p, \pi), \mathrm{Ind}_H^G(p', \pi^*))$ – dans la version de Mackey.

THEOREME 7 (premier principe de majoration de Herz en p-induction). *En posant* $|f|(xH) = |f(x)|$, $|\phi|(xH) = |\phi(x)|$ $(x \in G)$, *on a* $|\langle \mathrm{Ind}_H^G(p, \pi)(g)f, \phi\rangle| \leqq \langle \pi^p(g)\,|f|, |\phi|\rangle$ *pout tout* $g \in G$ *avec* $\|f\|_p = \|\,|f|\,\|_p$, $\|\phi\|_{p'} = \|\,|\phi|\,\|_{p'}$.

Remarque. Ce principe a été énoncé par C. Herz pour la représentation régulière de G sur $L^p(G)$ [20]. La version dans le cadre de l'induction unitaire est due à G. Arsac [13; § 3].

Il fournit par dualité un transfert d'opérateurs. Supposons $1 < p < \infty$ et soit μ une mesure positive sur G définissant un opérateur borné $\pi^p(\mu)$ sur $L^p(G/H)$ par $\langle \pi^p(\mu)f, \phi\rangle = \int_G d\mu(g)\langle \pi^p(g)f, \phi\rangle$.

COROLLAIRE 8. *Si la paire duale induite* $(\mathrm{Ind}_H^G(p, \pi)\,\mathrm{Ind}_H^G(p', \pi^*))$ *est* *réflexive,* μ *définit un opérateur borné* $\mathrm{Ind}_H^G(p,\pi)(\mu)$ *sur* $L^p(G,H;\pi)$ *par* $\langle \mathrm{Ind}_H^G(p,\pi)(\mu)f, \phi\rangle = \int_G d\mu(g)\langle \mathrm{Ind}_H^G(p,\pi)(g)f,\phi\rangle$ *avec* $\|\mathrm{Ind}_H^G(p,\pi)(\mu)\| \leqq \|\pi^p(\mu)\|$.

Remarque. Dans le cas particulier de la représentation régulière de G sur $L^p(G)$, ce résultat est contenu dans [27; prop. 1].

Définissons maintenant une *topologie* en terme de voisinages sur les paires duales d'un groupe localement compact G. Etant donné une paire duale (π,π^*) de G, $\xi_1,\ldots,\xi_m \in B_\pi$, $\xi_1^*,\ldots,\xi_n^* \in B_{\pi^*}$, $\mu_1,\ldots,\mu_1 \in M^1(G)$ et $\varepsilon>0$, le voisinage correspondant $V_{(\pi,\pi^*)}((\xi_i);(\xi_j^*);(\mu_k);\varepsilon)$ de (π,π^*) est composé des paires duales (ω,ω^*) pour lesquelles il exists $\eta_1,\ldots,\eta_m \in B_\omega$, $\eta_1^*,\ldots,\eta_n^* \in B_{\omega^*}$ avec

1°) $|\langle \pi(\mu_k)\xi_i, \xi_j^*\rangle - \langle \omega(\mu_k)\eta_i, \eta_j^*\rangle| < \varepsilon$

$(i=1,\ldots,m,\ j=1,\ldots,n,\ k=1,\ldots,r)$,

2°) $\left\| \left|\sum_{(i,k)\in I} \pi(\mu_k)\xi_i\right| - \left|\sum_{(i,k)\in I} \omega(\mu_k)\eta_i\right| \right\| < \varepsilon,$

$\left\| \left|\sum_{(j,k)\in J} \pi^*(\mu_k)\xi_j^*\right| - \left|\sum_{(j,k)\in J} \omega^*(\mu_k)\eta_j^*\right| \right\| < \varepsilon$

$(I\subset\{1,\ldots,m\}\times\{1,\ldots,r\},\ J\subset\{1,\ldots,n\}\times\{1,\ldots,r\})$.

Cette topologie possède toutes les propriétés souhaitables. Elle coïncide en effet avec la *topologie régionale* [16; § 6] sur les représentations unitaires continues de G, et par conséquent avec la *topologie de Fell–Jacobson* [14] sur le dual unitaire G. Elle fournit également les analogues L^p des phénomènes d'adhérence caractérisant la *moyennabilité.*

PROPOSITION 9. *Soit* $1<p<\infty$. *G est moyennable si et seulement si la représentation triviale* $\mathbb{1}$ *de G sur* $\mathbb{C}$ *adhère à la représentation régulière de G sur* $L^p(G)$.

La *démonstration* repose sur l'expression de la moyennabilité par les conditions de Reiter.

Remarque. On peut caractériser de la même manière la moyennabilité d'un espace homogène G/H [12], ou plus généralement encore la moyennabilité d'une action de G sur une espace localement compact X [1; première partie].

PROPOSITION 10. *Soient* (π,π^*) *une paire duale de G opérant dans des espaces de Banach B,* B^* *et* $1<p<\infty$. *Si G est moyennable,* (π,π^*) *adhère alors à la paire duale* $(\rho^p_{G,B}, \rho^{p'}_{G,B^*})$.

Démonstration. La moyennabilité de G se traduit par l'existence d'une *suite pout la propriété P_1 de Reiter*, i.e. d'une suite généralisée $(s_{K,\varepsilon})_{(K,\varepsilon)\in\mathscr{K}\times\mathscr{E}}$, où $\mathscr{K}$ est une famille fondamentale de compacts de G (fondamentale signifiant que tout compact de G est contenu dans un compact $K\in\mathscr{K}$), $\mathscr{E}$ un ensemble de nombres réels >0 admettant 0 comme borne inférieure $-\mathscr{K}\times\mathscr{E}$ étant muni de l'ordre "$(K,\varepsilon)\leqq(K',\varepsilon')\Leftrightarrow K\subset K'$ et $\varepsilon\geqq\varepsilon'$" $-$ et $s_{K,\varepsilon}\in C_c(G)$ avec $s_{K,\varepsilon}\geqq 0$, $\int_G dx\, s_{K,\varepsilon}(x)=1$, $\|\rho(g)s_{K,\varepsilon}-s_{K,\varepsilon}\|_1<\varepsilon$ pout tout $g\in K$. Notons que $(s_{K,\varepsilon}^{1/p})$ et $(s_{K,\varepsilon}^{1/p'})$ sont alors des suites pour les propriétés P_p et $P_{p'}$.

Considérons les fonctions (à valeurs vectorielles) de la forme

$$\Delta(s^{1/p}\otimes\xi)(x)=s(x)^{1/p}\pi(x)^{-1}\xi,\quad \Delta(s^{1/p'}\otimes\xi^*)(x)=s(x)^{1/p'}\pi^*(x)^{-1}\xi^*$$

où $s=s_{K,\varepsilon}$, $\xi\in B$, $\xi^*\in B^*$. L'identité

$$\langle\rho^p_{G,B}(g)\,\Delta(s^{1/p}\otimes\xi),\Delta(s^{1/p'}\otimes\xi^*)\rangle=\langle\rho(g)s^{1/p},s^{1/p'}\rangle\langle\pi(g)\xi,\xi^*\rangle$$

fournit l'estimation sur les coefficients

$$|\langle\pi(\mu)\xi;\xi^*\rangle-\langle\rho^p_{G,B}(\mu)\,\Delta(s^{1/p}\otimes\xi),\Delta(s^{1/p'}\otimes\xi^*)\rangle|<\|\mu\|\,|\xi|\,|\xi^*|\,\varepsilon^{1/p}$$

pour toute mesure μ à support dans K. On *a*, d'autre part, les estimations suivantes sur les normes:

$$\left|\left|\sum_i\pi(\mu_i)\xi_i\right|-\left\|\sum_i\rho^p_{G,B}(\mu_i)\,\Delta(s^{1/p}\otimes\xi_i)\right\|_p\right|<\left(\sum_i\|\mu_i\|\,|\xi_i|\right)\varepsilon^{1/p},$$
$$\left|\left|\sum_i\pi^*(\mu_i)\xi_i^*\right|-\left\|\sum_i\rho^{p'}_{G,B^*}(\mu_i)\,\Delta(s^{1/p'}\otimes\xi_i^*)\right\|_{p'}\right|<\left(\sum_i\|\mu_i\|\,|\xi_i^*|\right)\varepsilon^{1/p'}$$

pour toute famille finie de mesures μ_i à support dans K. On en tire facilement que (π,π^*) adhère $(\rho^p_{G,B},\rho^{p'}_{G,B^*})$.

Avant de passer à la continuité de la p-induction, déduisons de la proposition précédente des transferts de convoluteurs. Nous supposerons à cet effet que B est un *p-espace* [21], i.e. essentiellement un (sous-espace fermé d'un) espace L^q avec $p\leqq q\leqq 2$ ou $p\geqq q\geqq 2$. Rappelons qu'alors tout opérateur borné T sur $L^p(G)$ s'étend, par $T_B(f\otimes\xi)=Tf\otimes\xi$, en un opérateur borné T_B sur $L^p(G,B)$, de même norme.

COROLLAIRE 11. *Toujours dans le cas où G est moyennable, on a $\|\pi(\mu)\|\leqslant\|\rho^p_B(\mu)\|$ pour tout $\mu\in M^1(G)$.*

Remarque. Dans le cas particulier où B est un espace L^p, il s'agit de la *transférence de Coifman et Weiss* [6; § 2]. Relevons que la propriété d'extension est ici une conséquence banale du théorème de Fubini.

Avec l'hypothèse supplémentaire que la paire duale (π, π^*) est réflexive, nous prolongeons maitenant à $\mathscr{L}(L^p(G))$ (opérateurs bornés sur $L^p(G)$) la *contraction* $\rho_G^p(\mu) \mapsto \pi(\mu)$ du corollaire 11. Etant donnée une suite (s) pour la propriété P_1, considérons les formes trilinéaires $F_s(T, \xi, \xi^*) = \langle T_B \Delta(s^{1/p} \otimes \xi), \Delta(s^{1/p'} \otimes \xi^*)\rangle$ sur $\mathscr{L}(L^p(G)) \times B \times B^*$. Par compacité faible de la boule-unité, la suite (F_s) possède un point d'accumulation faible F_∞, qui correspond à une contraction $t_{(s)}$ de $\mathscr{L}(L^p(G))$ dans $\mathscr{L}(B)$ par $\langle t_{(s)}(T)\xi, \xi^*\rangle = F_\infty(T, \xi, \xi^*)$. Les cas intéressants sont ceux où $t_{(s)}(T)$ est *intrinsèque*, i.e. indépendant des choix de la suite (s) et du point d'accumulation F_∞; on le note alors $t(T)$.

PROPOSITION 12. *On suppose toujours G moyennable.*

i) *$t_{(s)}$ applique $\mathscr{L}(L^p(G))$ dans le bicommutant de $\pi(G)$.*

ii) *$t_{(s)}$ est intrinsèque sur les convoluteurs associés aux mesures bornées: $t(\rho_G^p(\mu)) = \pi(\mu)$ pour tout $\mu \in M^1(G)$. Elle l'est plus généralement sur la fermeture (normique) $cv^p(G)$ des convoluteurs à support compact: si T est un tel convoluteur et $\alpha \in A^p(G)$ vaut identiquement 1 au voisinage de son support, on a $\langle t(T)\xi, \xi^*\rangle = \langle T, \alpha\langle \pi(\cdot)\xi, \xi^*\rangle\rangle$ dans la dualité avec $A^p(G)^*$.*

Démonstration de ii). Dans la dualité entre $CV^p(G) = PM^p(G)$ et $A^p(G)$, $F_s(T, \xi, \xi^*)$ s'écrit $\langle T, \langle \rho(\cdot)s^{1/p}, s^{1/p'}\rangle\langle \pi(\cdot)\xi, \xi^*\rangle\rangle$. Comme $T = \alpha T$ et que les fonctions $u_s = \langle \rho(\cdot)s^{1/p}, s^{1/p'}\rangle$ forment une unité approchée (bornée) dans $A^p(G)$ (cf. par exemple [22, lemma 5, p. 121]) $F_s(T, \xi, \xi^*)$ converge vers $\langle T, \alpha\langle \pi(\cdot)\xi, \xi^*\rangle\rangle$. Par conséquent,

$$\langle t(T)\xi, \xi^*\rangle = \langle T, \alpha\langle \pi(\cdot)\xi, \xi^*\rangle\rangle.$$

EXEMPLE 10. π est la représentation quasi-régulière de G sur $L^p(G/H)$, H étant un sous-groupe fermé normal dans G. Rappelons que l'application de Mackey $[f](xH) = \int_H dh\, f(xh)$ $(f \in C_c(G))$ est dans ce cas prolongée par *l'application de Reiter* T_H de $M^1(G)$ sur $M^1(G/H)$, définie par

$$\int_{G/H} d(T_H\mu)(\zeta)\, \psi(\zeta) = \int_G d\mu(x)\, \psi(xH) \qquad (\psi \in C_c(G/H)).$$

---

* Pour tout renseignement sur les espaces $A^p(G)$, $PM^p(G)$, ... nous renvoyons aux travaux de C. Herz (par exemple [21], [23]), ainsi qu'à l'exposé [11] de P. Eymard.

La proposition précédente peut être légèrement précisée dans ce contexte:

i) $t_{(s)}$ applique $\mathscr{L}(L^p(G))$ dens $CV^p(G/H)$;

ii) t est un homomorphisme de $cv^p(G)$ dans $cv^p(G/H)$; explicitement,

— $t(\rho^p_G(\mu)) = \rho^p_{G/H}(T_H\mu)$ pour $\mu \in M^1(G)$,

— $t(T)T_Hf = T_H(Tf)$ pour T à support compact, $f \in C_c(G)$.

Signalons que la surjectivité de $t: cv^p(G) \to cv^p(G/H)$ vient d'être démontrée [10].

Dans le contexte abélien, on peut interpéter la contraction $m^p(\hat{G}) \to m^p(H^\perp)$ correspondant à $t_{(s)}$ comme un *procédé abstrait de restriction des multiplicateurs à* $H^\perp$. Elle est en effet donnée par la restriction à $H^\perp$ pour les multiplicateurs correspondant à $cv^p(G)$ (ce sont les multiplicateurs m pour lesquels la translation $\hat{x} \to \rho_{\hat{G}}(\hat{x})m$ est continue de $\hat{G}$ dans $m^p(\hat{G})$); c'est encore le cas pour les multiplicateurs uniformément continus lorsque la suite (s) est *suffisamment régulière*, au sens où les fonctions $u_s = \langle \rho(.)\ s^{1/p}, s^{1/p'} \rangle$ forment déjà une unité approchée (bornée) dans $A(G)$ – l'existence de telles suites étant assurée par la *propriété de Følner ponctuelle* [18; §3.6]. Nous ne savons pas si c'est vrai pour tous les multiplicateurs continus.

Venons-en à la *continuité de la p-induction* ($1 \leqq p \leqq \infty$).

THEOREME 13. *Soit* (π, π^*) *une paire duale de H adhérant à une famille* Ω. *La paire duale induite* $(\mathrm{Ind}_H^G(p, \pi), \mathrm{Ind}_H^G(p', \pi^*))$ *adhère alors aux paires duales* $(\mathrm{Ind}_H^G(p, \omega), \mathrm{Ind}_H^G(p', \omega^*))$ *induites à partir de* Ω.

La *démonstration* se lit sur les expressions suivantes de coefficients et de normes:

1°) $$\langle \mathrm{Ind}_H^G(p, \pi)(\mu)[f \otimes \xi], [\phi \otimes \xi^*] \rangle = \int_H dh\, \delta(h)^{-1/p} \phi^\times * \mu * f(h) \langle \pi(h)\xi, \xi^* \rangle$$

pour tout $\mu \in M_c(G)$ (mesure sur G à support compact), f, $\phi \in C_c(G)$, $\xi \in B_\pi$, $\xi^* \in B_{\pi^*}$, où on a posé $\phi^\times(x) = \Delta_G(x)^{-1}\phi(x^{-1})$;

2°) $$\Big\| \sum_i \mathrm{Ind}_H^G(p, \pi)(\mu_i)[f_i \otimes \xi_i] \Big\|_p = \left[\int_G dx\, \beta(x) \Big| \sum_i \pi\{\delta^{-1/p}(\mu_i * f_i)_{x,H}\}\xi_i \Big|^p \right]^{1/p},$$

$$\text{resp.}\quad \sup_{x \in G} \Big| \sum_i \pi\{(\mu_i * f_i)_{x,H}\}\xi_i \Big| \quad \text{lorsque} \quad p = \infty$$

pour toute famille finie de $\mu_i \in M_c(G)$, $f_i \in C_c(G)$, $\xi_i \in B_\pi$, où on a posé $(\mu_i * f_i)_{x,H}(h) = \mu_i * f_i(xh)$.

Remarque. La continuité de l'induction unitaire est due à J. M. G. Fell ([15; theorem 4.1], [16; §6]).

Supposons dorénavant H moyennable. Le phénomène d'adhérence décrit à la proposition 10 se transmet alors par continuité aux paires duales induites à partir de H.

COROLLAIRE 14. *Soient (π, π^*) une paire duale de H opérant dans des espaces de Banach B, B^* et $1<p<\infty$. La paire duale induite $(\mathrm{Ind}_H^G(p, \pi), \mathrm{Ind}_H^G(p', \pi^*))$ adhère alors à la paire duale $(\rho^p_{G,B}, \rho^{p'}_{G,B^*})$.*

En particulier, étant donné une suite (s) pour la propriété P_1 sur H, chaque coefficient $\langle \mathrm{Ind}_H^G(p, \pi)(.)[f], [\phi]\rangle$ est limite uniforme sur tout compact de G des coefficients $\langle \rho^p_{G,B}(.)\Delta'(s^{1/p}\otimes f), \Delta'(s^{1/p'}\otimes\phi)\rangle$, avec convergence des normes:

$$\|[f]\|_p = \lim_s \|\Delta'(s^{1/p}\otimes f)\|_p, \qquad \|[\phi]\|_{p'} = \lim_s \|\Delta'(s^{1/p'}\otimes \phi)\|_{p'},$$

où on a posé

$$\Delta'(s^{1/p}\otimes f)(x) = \int_H dh\, \delta(h)^{-1/p} s(h^{-1})^{1/p}\pi(h)f(xh),$$

$$\Delta'(s^{1/p'}\otimes \phi)(x) = \int_H dh\, \delta(h)^{-1/p'} s(h^{-1})^{1/p'}\pi^*(h)\phi(xh).$$

EXEMPLE 11. On retrouve ainsi le *second principe de majoration de Herz*, tel que l'a énoncé N. Lohoué [27; lemme 2]:
chaque coefficient $\langle \pi^p(.)f, \phi\rangle$ de la représentation quasi-régulière de G sur $L^p(G/H)$ est limite uniforme sur tout compact de G d'une suite (dénombrable lorsque H est σ-compact) de coefficients $\langle \rho(\cdot)f_i, \phi_i\rangle$ de la représentation régulière de G sur $L^p(G)$, avec contrôle des normes: $\|f_i\|_p = \|f\|_p, \|\phi_i\|_{p'} = \|\phi\|_{p'}$.

EXEMPLE 12. Les deux séries principales p-induites de $G = SL(2, \mathbb{R})$ (cf §4) adhèrent à la représentation régulière de G sur $L^p(G)$.

Pour terminer, tirons du corollaire précédent des transferts de convoluteurs, tout comme nous l'avons fait à partir de la proposition 10. A cet effet, nous supposerons à nouveau que B est un p-espace et que la paire duale $(\mathrm{Ind}_H^G(p, \pi), \mathrm{Ind}_H^G(p', \pi^*))$ considérée est réflexive.

Etant donné une suite (s) pour la propriété P_1 sur H et une fonction de Bruhat β de la paire $H \subset G$, on obtient une contraction $t'_{(s)}$ de $\mathscr{L}(L^p(G))$ dans

$\mathscr{L}(L^p(G,H;\pi))$ comme point d'accumulation faible des formes trilinéaires $F'_{s,\beta}(T,f,\phi)=\langle T_B\Delta'(s^{1/p}\otimes\beta f),\Delta'(s^{1/p'}\otimes\beta\phi)\rangle$ sur $\mathscr{L}(L^p(G))\times L^p(G,H;\pi)\times L^{p'}(G,H;\pi^*)$.

Remarques. 1) L'opérateur $t'_{(s)}$ dépend en général de la suite (s) et du point d'accumulation choisi, mais pas de β. En effet, si β_1 et β_2 sont des fonctions de Bruhat de la paire $H\subset G$, la suite $(F'_{s,\beta_1}-F'_{s,\beta_2})$ converge faiblement vers 0.
2) Nous ne savons pas si $t'_{(s)}$ applique $CV^p(G)$ dans le bicommutant de la représentation $\mathrm{Ind}_H^G(p,\pi)$. Il est par contre facile de montrer que l'image de tout convoluteur à droite est un entrelacement.

PROPOSITION 15. i) *$t'_{(s)}$ est intrinsèque sur $cv^p(G)$. Explicitement:*
— *$t'(\rho_G^p(\mu))=\mathrm{Ind}_H^G(p,\pi)(\mu)$ pour $\mu\in M^1(G)$,*
— *$t'(T)[f]=[T_Bf]$ pour T à support compact, $f\in C_c(G,B)$.*
ii) *$t'_{(s)}$ est également intrinsèque sur $CV^p(G)_+$ (convoluteurs positifs)*:*

$$t'(\rho_G^p(\mu))=\mathrm{Ind}_H^G\,(p,\pi)(\mu),\quad \text{où}\quad \langle \mathrm{Ind}_H^G\,(p,\pi)(\mu)f,\phi\rangle=$$
$$=\int_G d\mu(g)\,\langle \mathrm{Ind}_H^G\,(p,\pi)(g)f,\phi\rangle.$$

Démonstration. i) La première identité résulte directement du corollaire 14, la seconde de la convergence de

$$\langle T_B\Delta'(s^{1/p}\otimes f_0),\Delta'(s^{1/p'}\otimes\phi)\rangle$$
$$=\int_G dx\int_H dh\,\delta(h)^{-1/p}\langle\pi(h)T_Bf_0(xh),\phi(x)\rangle\langle\rho(h)s^{1/p},s^{1/p'}\rangle$$

vers

$$\langle[T_Bf_0],[\phi]\rangle=\int_G dx\int_H dh\,\delta(h)^{-1/p}\langle\pi(h)T_Bf_0(xh),\phi(x)\rangle$$

pour tout $f_0=\rho_{G,B}^p(u)f$ $(u\in C_c(G), f\in C_c(G,B))$, $\phi\in C_c(G,B^*))$.

ii) Chaque mesure $\langle \mathrm{Ind}_H^G\,(p,\pi)(g)[f],[\phi]\rangle\,d\mu(g)$, étant limite faible des mesures $\langle\rho_{G,B}^p(g)\Delta'(s^{1/p}\otimes f),\Delta'(s^{1/p'}\otimes\phi)\rangle\,d\mu(g)$, est bornée (compacité faible des boules fermées dans $M^1(G)$). Par conséquent, les expressions

$$\langle\rho_{G,B}^p(\mu)\Delta'(s^{1/p}\otimes f),\Delta'(s^{1/p'}\otimes\phi)\rangle=\int_G d\mu(g)\,\langle\rho_{G,B}^p(g)\Delta'(s^{1/p}\otimes f),\Delta'(s^{1/p'}\otimes\phi)\rangle$$

* Rappelons que tout convoluteur *positif* T de $L^p(G)$ est défini par une mesure positive μ sur G, i.e. $T=\rho_G^p(\mu):f\mapsto\mu*f$.

convergent vers

$$\langle \mathrm{Ind}_H^G (p, \pi)(\mu)[f], [\phi]\rangle = \int_G d\mu(g)\langle \mathrm{Ind}_H^G (p, \pi)(g)[f], [\phi]\rangle.$$

EXEMPLE 13. Dans le cas particulier de la représentation quasi-régulière π^p de G sur $L^p(G/H)$, l'assertion i) est le théorème 9 de [22]. L'assertion ii), combinée avec le corollaire 8, fournit quant à elle la proposition 1 de [27]: *une mesure positive μ sur G définit un convoluteur $\rho_G^p(\mu)$ de $L^p(G)$ si et seulement si μ définit un opérateur borné $\pi^p(\mu)$ sur $L^p(G/H)$ par $\langle \pi^p(\mu)f, \phi\rangle = \int_G d\mu(g)\langle \pi^p(g)f, \phi\rangle$; dans ce cas $\|\rho_G^p(\mu)\| = \|\pi^p(\mu)\|$.*

Résumons maintenant les propriétés de $t'_{(s)}$, lorsque H est de plus normal dans G.

a) $t'_{(s)}$ est une contraction de $\mathscr{L}(L^p(G))$ dans $\mathscr{L}(L^p(G/H))$.

b) L'image de tout convoluteur à droite de $L^p(G)$ est un convoluteur à droite de $L^p(G/H)$.

c) t' est un homomorphisme de $cv^p(G)$ dans $cv^p(G/H)$. Explicitement:

— $t'(\rho_G^p(\mu)) = \rho_{G/H}^p(T_H\mu)$ pour $\mu \in M^1(G)$,

—$t'(T)T_Hf = T_H(Tf)$ pour T à support compact, $f \in C_c(G)$.

d) t' applique $CV^p(G)_+$ sur $CV^p(G/H)_+$, en préservant les normes:

$$t'(\rho_G^p(\mu)) = \pi^p(\mu), \quad \text{où} \quad \langle \pi^p(\mu)f, \phi\rangle = \int_G d\mu(g)\, \langle \rho_{G/H}^p(gH)f, \phi\rangle,$$

avec $\|\rho_G^p(\mu)\| = \|\pi^p(\mu)\|$.

Signalons pour terminer le bon comportement de $t'_{(s)}$ dans le contexte abélien. La contraction $m^p(\hat{G}) \to m^p(H^\perp)$ correspondante est en effet donnée par la *restriction à $H^\perp$* pour tous les multiplicateurs continus, lorsque la suite (s) est *suffisamment régulière* (au sens vu précédemment). On retrouve ainsi un résultat connu ([30; cor. 4.6, part (b)] ou [25; théorème I.1]).

4. Séries principales p_*-induites

Nous terminons par un exemple de représentations p-induites. Nous considérons tout d'abord le cas particulier de $SL(2, \mathbb{R})$ et passerons ensuite au cas général d'un GLSS (groupe de Lie semi-simple, connexe, non compact, de centre fini).

Les *séries principales unitaires* de $G = SL(2, \mathbb{R})$ sont composées des représentations induites à partir des caractères $\chi_{\varepsilon,\lambda}\begin{pmatrix} a & * \\ 0 & 1/a \end{pmatrix} = \mathrm{sgn}\,(a)^\varepsilon\, |a|^{i\lambda}$

($\varepsilon=0,1$; $\lambda\in\mathbb{R}$) du sous-groupe *triangulaire* P. Nous nous intéressons ici aux représentations p-induites $\pi^p_{\varepsilon,\lambda}$ à partir de $\chi_{\varepsilon,\lambda}$.

Les décompositions univoques $G=KAN$ et $P=MAN$, où

$$K=\left\{k_\theta=\begin{pmatrix}\cos(\theta/2) & \sin(\theta/2)\\ -\sin(\theta/2) & \cos(\theta/2)\end{pmatrix}\;\middle|\;\theta\in\mathbb{R}/4\pi\mathbb{Z}\right\},\qquad M=\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\},$$

$$A=\left\{\begin{pmatrix}a&0\\0&1/a\end{pmatrix}\;\middle|\;a>0\right\},\qquad N=\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\;\middle|\;x\in\mathbb{R}\right\},$$

conduisent à la *réalisation compacte* de $\pi^p_{\varepsilon,\lambda}$ sur le sous-espace $L^p_\varepsilon(K)$, resp. $C_\varepsilon(K)$ des fonctions de parité ε dans $L^p(K)$, resp. $C(K)$:

$$\{\pi^p_{\varepsilon,\lambda}(g)f\}(k_\theta)=m(g,k_\theta)^{1/p+i\lambda/2}f(g^{-1}\cdot k_\theta),$$

où

$$g^{-1}\cdot k_\theta=\sqrt{m(g,k_\theta)}\begin{pmatrix}d\cos(\theta/2)+b\sin(\theta/2) & c\cos(\theta/2)+a\sin(\theta/2)\\ -c\cos(\theta/2)-a\sin(\theta/2) & d\cos(\theta/2)+b\sin(\theta/2)\end{pmatrix},$$

$$g=\begin{pmatrix}a&b\\c&d\end{pmatrix},$$

et

$$m(g,k_\theta)=\left[(d\cos(\theta/2)+b\sin(\theta/2))^2+(c\cos(\theta/2)+a\sin(\theta/2))^2\right]^{-1}$$

est le module de quasi-invariance de la mesure $d\theta$.

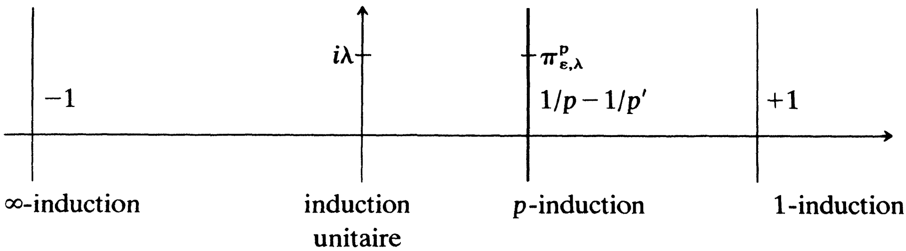
La décomposition univoque $G=VP\cup wP$, où $V=\left\{\begin{pmatrix}1&0\\-x&1\end{pmatrix}\;\middle|\;x\in\mathbb{R}\right\}$ et $w=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, conduit quant à elle à la *réalisation nilpotente* de $\pi^p_{\varepsilon,\lambda}$ sur $L^p(\mathbb{R})$, $1\leqq p<\infty$:

$$\{\pi^p_{\varepsilon,\lambda}(g)f\}(x)=\operatorname{sgn}(bx+d)^\varepsilon\,|bx+d|^{-2/p-i\lambda}f\left(\frac{ax+c}{bx+d}\right),\qquad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

Rappelons qu'on obtient le *prolongement analytique* $(\tilde\pi_{\varepsilon,z})_{z\in\mathbb{C}}$ de la série principale unitaire $(\pi^2_{\varepsilon,\lambda})_{\lambda\in\mathbb{R}}$ en remplaçant le paramètre $i\lambda\in i\mathbb{R}$ par un paramètre $z\in\mathbb{C}$ dans la réalisation compacte. On produit de la sorte des représentations fortement continues sur $L^2_\varepsilon(K)$ [32; 8.3].

LEMME 16. *Les représentations $\pi^p_{\varepsilon,\lambda}$ et $\tilde\pi_{\varepsilon,z}$ coïncident – du moins sur $C_\varepsilon(K)$ – lorsque $z=1/p-1/p'+i\lambda$.*

La *série principale p-induite* $(\pi^p_{\varepsilon,\lambda})_{\lambda\in\mathbb{R}}$ s'identifie donc à la droite d'équation $\operatorname{Re} z = 1/p - 1/p'$ dans le prolongement analytique $(\tilde{\pi}_{\varepsilon,z})_{z\in\mathbb{C}}$ de la série principale unitaire correspondante.



Remarques. 1) Le fait que les représentations sphériques $\tilde{\pi}_{0,z}$, dans la réalisation nilpotente, sont isométriques sur $L^p(\mathbb{R})$ lorsque $\operatorname{Re} z = 1/p - 1/p'$ – facile à vérifier au demeurant: $\dfrac{d}{dx}\left(\dfrac{ax+c}{bx+d}\right) = (bx+d)^{-2}$ – est à la base de la démonstration de M. G. Cowling du phénomène de Kunze–Stein pour $SL(2,\mathbb{R})$ [7; section 3]. La p-induction en donne une interprétation naturelle.
2) Les différentes séries principales p-induites sphériques $(\pi^p_{0,\lambda})_{\lambda\in\mathbb{R}}$ décrivent toute la bande $-1 \leqq \operatorname{Re} z \leqq 1$ correspondant aux fonctions sphériques $\phi_z(g) = \langle\tilde{\pi}_{0,z}(g)1, 1\rangle_K$ *bornées.*

Le cas de $SL(2,\mathbb{R})$ est typique d'un GLSS de rang 1. Pour traiter le cas général d'un GLSS G de rang n, nous ferons appel à des résultats classiques sur las structure des sous-groupes *paraboliques* de G [33; 1.2]. Fixons une décomposition d'Iwasawa $G = KAN$ et notons, comme de coutume, M le centralisateur de A dans K et $P = MAN$ le sous-groupe parabolique minimal correspondant. Les *séries principales unitaires* de G sont composées des représentations induites à partir des représentations $\sigma \times e^{i\lambda}$ $(m \exp Hn) = \sigma(m)e^{i\lambda(H)}$ de P, où $\sigma \in \hat{M}$ (dual unitaire de M), $\lambda \in \mathfrak{a}^*$ (dual de l'algèbre de Lie $\mathfrak{a}$ de A). Nous allons décrire un procédé permettant d'intercaler $(n-1)$ sous-groupes fermés entre P et $G: P^0 = P \subset P^1 \subset \cdots \subset P^n = G$, et nous intéresser, pour un multi-indice $p_* = (p_1, \ldots, p_n)$ donné, à la représentation $\pi^{p_*}_{\sigma,\lambda}$ obtenues à partir de $\sigma \times e^{i\lambda}$ par p_1-induction de P à $P^1, \ldots, p_n$-induction de P^{n-1} à G.

Suivant la coutume, notons $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ et $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ les décompositions correspondantes des algèbres de Lie de G et de P, Σ l'ensemble des racines restreintes de $(\mathfrak{g}, \mathfrak{a})$, Σ_+ le sous-ensemble des racines positives – si bien que $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma_+} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha$ étant l'espace-poids correspondant à α – et $\alpha_1, \ldots, \alpha_n$ les racines simples. Posons $\mathfrak{a}^j = \bigoplus_{k=1}^j \mathbb{R}\alpha_k$ ($\mathfrak{a}$ ayant été identifié à $\mathfrak{a}^*$ au moyen de la forme de Killing de $\mathfrak{g}$, $\Sigma^j = \Sigma_+ \cap \mathfrak{a}^j, \Sigma_j = \Sigma^j \setminus \Sigma^{j-1}$ (avec la convention $\Sigma^0 = \varnothing$), $\rho_j =$

$\frac{1}{2}\sum_{\alpha\in\Sigma_j}(\dim\mathfrak{g}_\alpha)\alpha$ et introduisons les sous-algèbres $\mathfrak{a}_j=\mathbb{R}\rho_j$,

$$\mathfrak{n}^j=\bigoplus_{\alpha\in\Sigma^j}\mathfrak{g}_\alpha,\qquad \mathfrak{n}_j=\bigoplus_{\alpha\in\Sigma_j}\mathfrak{g}_\alpha,\qquad \mathfrak{b}^j=\bigoplus_{\alpha\in-\Sigma^j}\mathfrak{g}_\alpha,\qquad \mathfrak{b}_j=\bigoplus_{\alpha\in-\Sigma_j}\mathfrak{g}_\alpha,$$

$$\mathfrak{m}^j=\mathfrak{b}^j\oplus\mathfrak{m}\oplus\mathfrak{a}^j\oplus\mathfrak{n}^j,\qquad \mathfrak{p}^j=\mathfrak{b}^j\oplus\mathfrak{p}=\mathfrak{m}^j\oplus\Big(\bigoplus_{k>j}\mathfrak{a}_k\Big)\oplus\Big(\bigoplus_{k>j}\mathfrak{n}_k\Big)$$

de $\mathfrak{g}$. Les sous-groupes analytiques A^j, A_j, N^j, N_j, V^j, V_j, M_0^j, P_0^j de G correspondants sont fermés. $M^j=MM_0^j$ et $P^j=MP_0^j$ sont encore des sous-groupes fermés d'algèbre de Lie $\mathfrak{m}^j$ et $\mathfrak{p}^j$. Le sous-groupe parabolique P^j admet la décomposition de Langlands $P^j=M^j(A_{j+1}\cdots A_n)(N_{j+1}\cdots N_n)$. Son module est trivial sur M^j et sur N; il est donné par $\Delta_j(\exp H)=\exp[-2\sum_{k>j}\rho_k(H)]$ sur A.

Remarque. En variant la numérotation des racines simples on obtient ainsi toutes les familles maximales de sous-groupes emboîtés entre P et G.

La génération suivante du théorème d'induction par étages permettra de réaliser simplement les représentations $\pi^{p_*}_{\sigma,\lambda}$. Soient $H_*: H_0\subset H_1\subset\cdots\subset H_n$ une famille de groupes localement compacts emboîtés, π une représentation de H_0 (isométrique, fortement continue, sur un espace de Banach B) et $p_*=(p_1,\ldots,p_n)$ un multi-indice. Fixons sur chaque quotient H_j/H_{j-1} une mesure quasi-invariante $d_{q_j}\zeta_j$ associée à une fonction homogène continue $q_j: H_j\to(0,\infty)$ et posons $q^j(h_j)=q_j(h_j)^{1/p_j}\cdots q_n(h_j)^{1/p_n}$ $(h_j\in H_j)$. Notons $C_c^{p_*}(H_*;\pi)$ l'espace des fonctions $f: H_n\to B$ vérifiant

i) $f(h_nh_0)=\left[\dfrac{\Delta_{H_0}(h_0)}{\Delta_{H_1}(h_0)}\right]^{1/p_1}\cdots\left[\dfrac{\Delta_{H_{n-1}}(h_0)}{\Delta_{H_n}(h_0)}\right]^{1/p_n}\pi(h_0)^{-1}f(h_n)\quad(h_n\in H_n,\ h_0\in H_0),$

ii) f est continue, à support compact modulo H_0,

et $L^{p_*}(H_*;\pi)$ son complété pour la norme (intrinsèque)

$$\|f\|_{p_*}=\Bigg[\int_{H_n/H_{n-1}}d_{q_n}(h_nH_{n-1})\,q^n(h_n)^{-p_n}\cdots$$
$$\cdots\Bigg[\int_{H_1/H_0}d_{q_1}(h_1H_0)\,q^1(h_1)^{-p_1}\,|f(h_n\cdots h_1)|^{p_1}\Bigg]^{p_2/p_1}\cdots\Bigg]^{1/p_n}$$

(avec les modifications habituelles lorsque $p_j=\infty$).

LEMME 17. *La représentation* $\operatorname{Ind}_{H_{n-1}}^{H_n}(p_n,\ldots\operatorname{Ind}_{H_0}^{H_1}(p_1,\pi)\cdots)$ *est équivalente à la représentation* $\{\operatorname{Ind}_{H_*}(p_*,\pi)(g)f\}(h_n)=f(g^{-1}h_n)$ *de* H_n *sur* $L^{p_*}(H_*;\pi)$.

La *démonstration* est semblable à celle du théorème 1.

Les décompositions d'Iwasawa généralisées $P^j = K^jAN$, où $K^j = K \cap P^j$ (c'est le normalisateur de $A_{j+1} \cdots A_n$ dans K), conduisent à la *réalisation compacte* de $\pi^{p_*}_{\sigma,\lambda}$ sur $L^{p_*}(K_*; \sigma)$:

$$\{\pi^{p_*}_{\sigma,\lambda}(g)f\}(k) = \exp\left[-\left(\sum_j (2/p_j)\rho_j + i\lambda\right)(H(g^{-1}k))\right] f(g^{-1} . k),$$

où $g^{-1}k = (g^{-1} . k) \exp H(g^{-1}k)\, n$ dans la décomposition $G = KAN$.

Lorsque tous les indices p_j sont finis, on peut également donner une *réalisation nilpotente* de $\pi^{p_*}_{\sigma,\lambda}$, basée sur des décompositions de Bruhat généralisées.

Pour commencer, rappelons la décomposition de Bruhat $G = \bigsqcup_{w \in W} PwP$, où W est le groupe de Weyl de Σ, identifié à M'/M, M' étant le normalisateur de A dans K. La double classe $P\omega P$ de l'élément $\omega \in W$ échangeant Σ_+ avec $-\Sigma_+$ est un ouvert de G, dont le complémentaire est une sous-variété de dimension inférieure. Comme $\omega^{-1}N\omega = V(=V^n)$, il en est de même de VP; de plus, l'application $(v, p) \mapsto vp$ est un difféomorphisme de $V \times P$ sur VP.

Le groupe de Weyl W^j de Σ^j s'identifie quant à lui à $M' \cap P^j/M$ ($M' \cap P^j$ est le normalisateur de A dans K^j); il admet la décomposition univoque $W^j = W_jW^{j-1}$, où $W_j = \{w \in W^j \mid w . \Sigma^{j-1} \cap (-\Sigma_+) = \varnothing\}$. La décomposition de Bruhat généralisée $P^j = \bigsqcup_{w \in W^j} N_jwP^{j-1}$ fait apparaître à son tour V_jP^{j-1} comme un ouvert de P^j, dont le complémentaire est une sous-variété de dimension inférieure, la paramétrisation $(v_j, p^{j-1}) \mapsto v_jp^{j-1}$ étant un difféomorphisme.

Ces décompositions permettent de réaliser (pp) $\pi^{p_*}_{\sigma,\lambda}$ sur l'espace $L^{p_*}(V; \mathfrak{H}_\sigma)$, complété de $C_c(V; \mathfrak{H}_\sigma)$ pour la norme

$$\|f\|_{p_*} = \left[\int_{V_n} dv_n \cdots \left[\int_{V_1} dv_1 \, |f(v_n \cdots v_1)|^{p_1}\right]^{p_2/p_1} \cdots\right]^{1/p_n},$$

par

$$\{\pi^{p_*}_{\sigma,\lambda}(g)f\}(v) = \sigma(m(g^{-1}v))^{-1} \exp\left[-\left(\sum_j (2/p_j)\rho_j + i\lambda\right)(H(g^{-1}v))\right] f(g^{-1} . v),$$

où $g^{-1}v = (g^{-1} . v)m(g^{-1}v) \exp H(g^{-1}v)n$ dans la carte $VMAN$.

On obtient à nouveau le *prolongement analytique* $(\tilde{\pi}_{\sigma,z})_{z \in \mathfrak{a}^*_{\mathbb{C}}}$ de la série principale unitaire $(\pi^2_{\sigma,\lambda})_{\lambda \in \mathfrak{a}^*}$ en complexifiant le paramètre $i\lambda \in i\mathfrak{a}^*$ dans la réalisation compacte.

LEMME 18. *Les représentations* $\tilde{\pi}_{\sigma,z}$ *et* $\pi^{p_*}_{\sigma,\lambda}$ *coïncident – du moins sur* $C(K, M; \sigma)$ *– lorsque* $z = \sum_j (2/p_j)\rho_j + i\lambda$.

La *série principale p_*-induite* $(\pi^{p_*}_{\sigma,\lambda})_{\lambda\in\mathfrak{a}^*}$ s'identifie donc au plan d'équation $\operatorname{Re} z = \sum_j (1/p_j - 1/p_j')\rho_j$ dans le prolongement analytique $(\tilde{\pi}_{\sigma,z})_{z\in\mathfrak{a}^*_{\mathbf{C}}}$ de la série principale unitaire correspondante.

Remarques. 1) Le fait que les représentations sphériques $\tilde{\pi}_{\mathbb{1},z}$, dans la réalisation nilpotente, sont isométriques sur $L^{p_*}(V)$ est à la base de la démonstration de M. G. Cowling du phénomène de Kunze–Stein dans le cas général [7; sections 5 et 6]. La p_*-induction en donne une interprétation naturelle.

2) Les coefficients sphériques $\langle\pi^{p_*}_{\mathbb{1},\lambda}(\cdot)1, 1\rangle_K$ fournis par les différentes séries principales p_*-induites sphériques décrivent toutes les fonctions sphériques *bornées* de (G, K) [19].

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Higher dimensional simple knots and minimal Seifert surfaces

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Introduction

A knot $K^{2q-1} \subset S^{2q+1}$, $q \geq 2$, is said to be *simple* if K^{2q-1} has a $(q-1)$-connected Seifert surface. Such a Seifert surface is said to be minimal if the associated Seifert matrix is non-singular. Levine has given an isotopy classification of simple $(2q-1)$-knots and their minimal Seifert surfaces in terms of S-equivalence and congruence of Seifert matrices (cf. [8]). Another algebraic classification of simple $(2q-1)$-knots can be obtained via the isometry classification of Blanchfield forms (cf. Trotter [14] or Kearton [7]) which is usually easier to handle than the S-equivalence relation.

In the first section of the present paper we define a $(-1)^{q+1}$-hermitian form which gives an isotopy classification of minimal Seifert surfaces. The Blanchfield form can be obtained from this form by an extension of the scalars. This is inspired by Trotter's papers [14] and [15].

The main purpose of this paper is to apply the algebraic results of [1] and [3] to the classification of a special type of simple knots, called Dedekind knots, which are defined as follows. Let L be the knot module of K^{2q-1} and let $\lambda \in \mathbb{Z}[X]$, $\lambda(1) = \pm 1$, be a generator of the annihilator ideal of the $\mathbb{Z}[X, X^{-1}]$-module L (cf. [9], [10] §7). We shall say that K^{2q-1} is a Dedekind knot if λ is irreducible and $\mathbb{Z}[X, X^{-1}]/(\lambda)$ is Dedekind.

Non-fibered Dedekind $(2q-1)$-knots, $q \geq 3$, are always easy to classify (see Theorem 3). For fibered Dedekind knots we have two quite different cases: if the Blanchfield form is indefinite, then we have the same kind of classification theorem as for non-fibered knots. On the other hand, the classification of fibered Dedekind knots with definite Blanchfield pairing seems very difficult.

In Sections 2 and 3 we give applications to the cancellation problem, to the number of minimal Seifert surfaces, and to the symmetries of Dedekind knots. For instance we shall give a complete criterion for a Dedekind knot to be (-1)-amphicheiral.

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1. An algebraic classification of the minimal Seifert surfaces of a given simple $(2q-1)$-knot, $q \geq 3$.

Let $\Gamma_0 = \mathbb{Z}[z]$, $\Lambda_0 = \mathbb{Z}[z, z^{-1}, (1-z)^{-1}]$ and let E_0 be the field of quotients of Λ_0. These rings have involutions induced by $\bar{z} = 1 - z$.

Let N be a $\mathbb{Z}$-torsion free, finitely generated Γ_0-torsion Γ_0-module. We shall say that an ε-hermitian ($\varepsilon = \pm 1$) form $h: N \times N \to E_0/\Gamma_0$ is *unimodular* if the adjoint map from N to $\operatorname{Hom}_{\Gamma_0}(N, E_0/\Gamma_0)$ which sends x to f_x, defined by $f_x(y) = h(y, x)$, is a conjugate-linear isomorphism.

The following is a consequence of results of Levine [8] and Trotter [14], [15]:

THEOREM 1. *Let K^{2q-1} be a simple knot, $q \geq 3$, and let $b: L \times L \to E_0/\Lambda_0$ be the associated Blanchfield form. The isotopy classes of minimal Seifert surfaces of K^{2q-1} are in bijection with the isometry classes of unimodular $(-1)^{q+1}$-hermitian forms*

$$h: N \times N \to E_0/\Gamma_0$$

such that $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$.

DEFINITION. A form (N, h) as in Theorem 1 will be called a *Trotter form.*

Proof. Let A be a non-singular Seifert matrix associated with a minimal Seifert surface of K^{2q-1}, and let $M_A = A - z(A + (-1)^q A')$ where A' denotes the transpose of A. Let $\Lambda_0^n/M_A\Lambda_0^n$, $N = \Gamma_0^n/M_A\Gamma_0^n$ (A is an $n \times n$-matrix) and let

$$b: L \times L \to E_0/\Lambda_0$$
$$h: N \times N \to E_0/\Gamma_0$$

be the quotient forms associated with M_A (cf. [14], p. 178). The $(-1)^{q+1}$-hermitian form b is the Blanchfield form of K^{2q-1} (cf. [9], 14.3, p. 44). Clearly $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$. It is easy to check that if A and B are congruent Seifert matrices, then the quotient forms associated to M_A and M_B are isometric.

Conversely let $h: N \times N \to E_0/\Gamma_0$ be a unimodular $(-1)^{q+1}$-hermitian form such that $(N, h) \otimes_{\Gamma_0} \Lambda_0 = (L, b)$. Following Trotter (cf. [14], [15]) let us define a trace function $s: E_0 \to \mathbb{Q}$ by setting $s(f)$ equal to the coefficient of z^{-1} in the Laurent expansion of f at infinity. Set $[a_1, a_2] = s(h(a_1, a_2))$ for $a_1, a_2 \in N$. Then $[\ ]: N \times N \to \mathbb{Z}$ is a unimodular $(-1)^q$-symmetric $\mathbb{Z}$-bilinear form (cf. [14] pp. 292–294). We have $[za_1, a_2] = [a_1, (1-z)a_2]$, i.e. $(N, [\ ], z)$ is an isometric structure. It is easy to check that isometric $(-1)^{q+1}$-hermitian forms give rise to isomorphic isometric structures. Let S, Z be the matrices of [ ], z with respect to a $\mathbb{Z}$-basis of N. Set $A = ZS^{-1}$. Then A is a Seifert matrix, i.e. $A + (-1)^q A' = S^{-1}$ is unimodular. A is non-singular as $\det(A) = \det(Z)$. By [14], Proposition 2.11, (L, b) is isometric to the quotient form associated to M_A. Trotter's main theorem in [14] implies that A is in the S-equivalence class determined by K^{2q-1}. It is easy to check that if two isometric structures are isomorphic then the corresponding Seifert matrices are congruent so by Levine [8] the associated minimal Seifert surfaces are isotopic.

Remark. The existence of at least one minimal Seifert surface follows from Trotter, [13] and Levine, [8].

Let b and h be unimodular ε-hermitian forms as in Theorem 1. Let $\varphi \in \mathbb{Z}[X]$ be the minimal polynomial of $z: L \to L$ and let $\Gamma = \mathbb{Z}[X]/(\varphi)$. Set $\lambda(X) = (1-X)^{\deg \varphi}\varphi(1/1-X) \in \mathbb{Z}[X]$. We have $\lambda L = 0$ and $\lambda(1) = \pm 1$. Notice that λ is a generator of $\mathrm{Ann}_{\Lambda_0}(L)$, cf. Levine [11], proof of Theorem 7.1. Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda) = \Lambda_0/(\lambda)$. Then L is a Λ-module and b takes values in $(1/\lambda)\Lambda_0/\Lambda_0 \simeq \Lambda$. So we can consider b and h as unimodular ε-hermitian forms $b: L \times L \to \Lambda$. $h: N \times N \to \Gamma$.

We shall apply Theorem 1 to give a short proof of a theorem of Trotter, in a special case. Let $F \in \mathbb{Z}[X]$ be the characteristic polynomial of $z: L \to L$.

THEOREM 2 (Trotter, [14] Corollary 4.7). *Let $K^{2q-1} \subset S^{2q+1}$ be a simple knot, $q \geq 3$, such that $F(0) = \pm p$ where p is a prime. Then the knot K^{2q-1} has only one isotopy class of minimal Seifert surfaces.*

Let us assume that φ is *irreducible.* As φ and F have the same irreductible factors, F is then a power of φ. If the constant term of F is $\pm p$, where p is a prime number, then we must have $F = \varphi$.

Let $\Gamma = \mathbb{Z}[\alpha]$. Then $\Lambda = [\alpha^{-1}, \bar{\alpha}^{-1}]$ where $\bar{\alpha} = 1 - \alpha$. We have $\varphi(0) = \pm p$, therefore $\Gamma/(\alpha) \cong \mathbb{F}_p$, so (α) is a maximal ideal.

In the special case where φ is irreducible, Theorem 2 is a consequence of the following lemma:

LEMMA. *Let (I, h_1) and (J, h_2) be two unimodular ε-hermitian forms where I and J are Γ-ideals, such that $(I, h_1) \otimes \Lambda \cong (J, h_2) \otimes \Lambda$. Then $(I, h_1) \cong (J, h_2)$.*

Proof of Lemma. We want to show that if I and J are Γ-ideals such that $I\Lambda = J\Lambda$ then $\alpha^k \bar{\alpha}^m I = J$ for some integers k, m. As Γ is noetherian we can write $I = I_1 \cap I_2$, $J = J_1 \cap J_2$ where the I_i's, J_j's are the intersection of a finite number of primary ideals (cf. [17] Chap. IV §4 Theorem 4). We can assume that the radicals of I_1, J_1 are prime to $P=(\alpha)$ and to P and that the radical of the primary components of I_2 and J_2 is P or $\bar{P}$. By [17] Chap. IV §10 Theorem 17 the hypothesis $I\Lambda = J\Lambda$ implies that $I_1 = J_1$. Let Q be a P-primary component of I_2. Then there exists an integer n such that $\alpha^n \in Q$. Let us assume that n is minimal with this property. If $n=0$ then we have finished. We have $Q \subset P$ therefore $Q' = \alpha^{-1}Q \subset \Gamma$. Then either $Q' = \Gamma$ or Q' is P-primary so we can repeat the above procedure. We finally obtain $\alpha^{-n+1}Q = \Gamma$. Therefore $I_2 = (\alpha^k\bar{\alpha}^m)$, and a similar result holds for J_2.

Let $h_1 : I \times I \to \Gamma$, $h_2 : I \times I \to \Gamma$ be two unimodular ε-hermitian forms such that $(I, h_1) \otimes_\Gamma \Lambda = (I, h_2) \otimes_\Gamma \Lambda$. We have $h_i(x, y) = a_i x\bar{y}$, $i = 1, 2$. As h_1 and h_2 are unimodular, $a_1 a_2^{-1} = u$ is a unit of Γ. There exists $x \in \Lambda$ such that $x\bar{x} = u$. We have $x\Lambda = \Lambda$, therefore $x = v\alpha^k\bar{\alpha}^m$ where v is a unit of Γ. So $x\bar{x} = \alpha^{k+m}\bar{\alpha}^{k+m}v\bar{v} = u$. This implies that $k = -m$, so $x\bar{x} = v\bar{v} = u$, therefore h_1 and h_2 are isometric.

2. Dedekind knots

Let $K^{2q-1} \subset S^{2q+1}$ be a simple knot, $q \geq 2$, and let $b : L \times L \to \Lambda$ be the associated Blanchfield form, $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$ as above. We shall say that K^{2q-1} is a *Dedekind knot* if λ is irreducible and Λ is Dedekind. We shall now apply the results of [1] and [3] to the classification of Dedekind knots and of their minimal Seifert surfaces.

Let us denote E the field of quotients of Λ and F the fixed field of the involution. For every real embedding of F which extends to an imaginary embedding of E we have a *signature* invariant of $b : L \times L \to \Lambda$. We shall say that b is *definite* if F is totally real, E is totally imaginary and if every signature is maximal. Otherwise we say that b is *indefinite*. The determinant of (L, b) is the rank one form

$$\det(b) : \Lambda^n L \times \Lambda^n L \to \Lambda$$
$$(x_1 \Lambda \cdots \Lambda x_n, y_1 \Lambda \cdots \Lambda y_n) \to \det(b(x_i, y_j)_{ij})$$

where $n = \operatorname{rank}_\Lambda(L)$.

If $\varepsilon = -1$ and $\operatorname{rank}_\Lambda(L)$ is even, we also need a finite number of pfaffians. Let $\Lambda' = \Lambda \cap F$ and let p be a prime Λ'-ideal such that $p\Lambda = P^2$. The involution on Λ/P is trivial (cf. [6], §5), and the skew-hermitian form b induces a non-singular skew-symmetric form $\tilde{b}$ on $\tilde{L} = L/PL$. Let us denote by $\operatorname{Pf}_p(b)$ a *pfaffian* of this form. If (M, b) is another lattice such that $\varphi : (\tilde{L}, \tilde{b}) \to (\tilde{M}, \tilde{b})$ is an isometry, then $\operatorname{Pf}_p(L, b) \cdot \det(\varphi) = \operatorname{Pf}_p(M, b)$.

Let us recall the classification theorem of [3]. We have the following hypothesis:
(*) Either $\Lambda \neq \Gamma$ (or equivalently $\lambda(0) \neq \pm 1$) or the ε-hermitian forms $b_i : L_i \times L_i \to \Lambda$ are indefinite.

THEOREM 3. *Assume that the hypothesis* (*) *is satisfied. Then two unimodular ε-hermitian forms $b_1 : L_1 \times L_1 \to \Lambda$ and $b_2 : L_2 \times L_2 \to \Lambda$ are isometric if and only if they have the same rank, same signatures and isometric determinants, and if moreover $\varepsilon = -1$ and the forms have even rank, there exists an isometry f between* $\det(b_1)$ *and* $\det(b_2)$ *such that* $\det(f)\,\mathrm{Pf}_p(b_1) \equiv \mathrm{Pf}_p(b_2) \bmod P$ *if $p\Lambda = P^2$.*

Proof. This is a consequence of [3], Theorem 2 and Remark 1. Notice that if p is a prime of $\Gamma' = \Gamma \cap F$ such that $p\Lambda' = \Lambda'$ then $p\Gamma = P\bar{P}$ with $P \neq \bar{P}$. Indeed, $p\Lambda' = \Lambda'$ implies that p contains $\alpha\bar{\alpha}$ (see the proof of Theorem 2). The minimal polynomial of α over Γ' is $X^2 - X + \alpha\bar{\alpha}$. Therefore $\Gamma/p\Gamma = \Gamma'/p[X]/(X^2 - X) = \Gamma'/p \times \Gamma'/p$.

The isotopy classes of simple $(2q-1)$-knots, $q \geq 2$, are in bijection with the isometry classes of Blanchfield forms (cf. Kearton [7] or Levine [8] and Trotter [14].) Therefore the above theorem gives an isotopy classification of Dedekind knots satisfying (*). Notice that all non-fibered $(2q-1)$-knots, $q \geqslant 3$, satisfy (*). Indeed, an easy application of the h-cobordism theorem shows that a simple $(2q-1)$-knot, $q \geq 3$, is fibered if and only if $\lambda(0) = 1$.

COROLLARY 1. *Let K_1 and K_2 be Dedekind* $(2q-1)$*-knots such that the associated Blanchfield forms satisfy* (*), *and let K be any* $(2q-1)$*-knot. If the connected sum $K_1 + K$ is isotopic to $K_2 + K$ then K_1 and K_2 are isotopic.*
In particular, cancellation holds for non-fibered $(2q-1)$*-Dedekind knots if* $q \geq 3$.

Proof. Let b_1, b_2 and b be the Blanchfield forms of K_1, K_2 and K. We have an isometry between $b_1 \perp b$ and $b_2 \perp b$ where $\perp$ denotes orthogonal sum. The knot modules of K_1 and K_2 clearly have the same annihilator $\lambda \in \mathbb{Z}[X]$, $\lambda(1) = 1$. Let $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$. Taking tensor product over $\mathbb{Z}[X, X^{-1}]$ with Λ and then taking the $\mathbb{Z}$-torsion free part we may assume that $b : L \times L \to \Lambda$, where L is a projective Λ-module of finite rank. Now Theorem 3 implies that b_1 and b_2 are isometric.

In the fibered definite case there are counter-examples to cancellation (cf. [2]).

Minimal Seifert surfaces

The isotopy classes of the minimal Seifert surfaces of a given simple $(2q-1)$-knot K, $q \geq 3$, are classified by the isometry classes of the Trotter forms associated to K (cf. Theorem 1). Therefore Theorem 3 implies the following

COROLLARY 2. *Let K^{2q-1} be a Dedekind knot such that the associated Blanchfield form is indefinite and that Γ is Dedekind. Let S_1 and S_2 be two minimal Seifert surfaces of K and let (N_1, h_1) and (N_2, h_2) be the associated Trotter forms. Then S_1 and S_2 are isotopic if and only if there exists an isometry $f: \det(N_1, h_1) \to \det(N_2, h_2)$ such that*

$$\mathrm{Pf}_p\,(N_1, h_1) \det(f) \equiv \mathrm{Pf}_p\,(N_2, h_2) \bmod P \text{ if } p\Gamma = P^2.$$

Remark. We have $\Lambda = \Gamma[\alpha^{-1}, \bar{\alpha}^{-1}]$ so if Γ is Dedekind then Λ is Dedekind too. But the converse is not true. I thank Jonathan Hillman for the following example: let $\lambda(X) = 9X^4 - 3X^3 - 11X^2 - 3X + 9$, then $\varphi(X) = X^4 - 2X^3 + 34X^2 - 33X + 9$, $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$, $\Gamma = \mathbb{Z}[X]/(\varphi)$.

Then Λ is Dedekind by Levine's criterion (cf. [10], §28). On the other hand $\varphi(X) \in (3, X)^2$, so Γ is not Dedekind by Uchida's criterion (cf. [16]).

COROLLARY 3. *If K^{2q-1} is a Dedekind knot, $q \geq 3$, such that the associated Blanchfield form is indefinite and that Γ is Dedekind, the number of isotopy classes of minimal Seifert surfaces of K only depends on Λ.*

If moreover $\lambda(0) = \pm p$ where p is a prime number, then K has only one isotopy class of minimal Seifert surfaces. (This is a generalization of Theorem 1, *in the case of Dedekind knots.)*

Remark. The above corollary is no longer true if the Blanchfield form is definite. For instance let $\lambda(X) = aX^2 + (1-2a)X + a$, $\varphi(X) = X^2 - X + a$, where a is a positive integer, $a \neq 1$, and $1-4a$ is square free. Then $E = \mathbb{Q}[X]/(\lambda) = \mathbb{Q}(\sqrt{1-4a})$ is an imaginary quadratic field. Let $p(n)$ be the number of partitions of n into the sum of positive integers. There are at least $p(n)$ unimodular forms $h: N \times N \to \Gamma$, $\operatorname{rank}(N) = 4n$ such that $(N, h) \otimes_\Gamma \Lambda$ is isomorphic to $\langle 1 \rangle \perp \cdots \perp \langle 1 \rangle$ (cf. [2], Remark 2). On the other hand the number of unimodular forms $h: N \times N \to \Gamma$ such that $(N, h) \otimes_\Gamma \Lambda$ is isomorphic to $\langle 1 \rangle \perp \langle -1 \rangle \perp \cdots \perp \langle 1 \rangle$ does not depend on n.

3. Symmetries of knots

If X is an oriented manifold, let us denote X^- the same manifold with the opposite orientation. We shall say that a knot $K^{2q-1} \subset S^{2q+1}$ is *invertible* if it is isotopic to $(K^{2q-1})^- \subset S^{2q+1}$ *(+1)-amphicheiral* if it is isotopic to $K^{2q-1} \subset (S^{2q+1})^-$ and *(−1)-amphicheiral* if it is isotopic to $(K^{2q-1})^- \subset (S^{2q+1})^-$. F. Michel [11] has translated these conditions into algebraic conditions on the Blanchfield form (L, b) associated to K^{2q-1}, $q \geq 2$. Let us define $(\bar{L}, \bar{b})$ as follows: $\bar{L}$ is equal to L as $\mathbb{Z}$-modules, and the Λ-module structure of L is given by $\lambda^* x = \bar{\lambda} x$. Let $\bar{b}(x, y) = \overline{b(x, y)}$. Then K^{2q-1} is invertible if $(L, b) \cong (\bar{L}, \bar{b})$, (+1)-amphicheiral if $(L, b) \cong (\bar{L}, -\bar{b})$ and (−1)-amphicheiral if $(L, b) \cong (L, -b)$ (see [11], [5]).

In this section we shall apply Theorem 3 to determine the symmetries of Dedekind knots.

COROLLARY 4. *Let K^{2q-1} be a Dedekind knot, $q \geq 2$, and let (L, b) be the corresponding Blanchfield form. Then K^{2q-1} is (−1)-amphicheiral if and only if*

a) (F. Michel [11]) $\operatorname{rank}(L)$ *is odd and there exists a unit u of Λ such that* $u\bar{u} = -1$

b) $\operatorname{rank}_\Lambda(L)$ *is even and every signature of b is zero.*

Proof. It is easy to see that the conditions are necessary. Let us prove that they are also sufficient:

a) an isometry is given by multiplication with u

b) As $\operatorname{rank}(L)$ is even, $\det(-b) = \det(b)$, and we have $\operatorname{Pf}_p(-b) = (-1)^n \operatorname{Pf}_p(b)$ where $2n = \operatorname{rank}_\Lambda(L)$. Therefore $f(x) = (-1)^n x$ gives an isometry between $\det(b)$ and $\det(-b)$ such that $\det(f) \operatorname{Pf}_p(b) \equiv \operatorname{Pf}_p(-b) \bmod P$ if $p\Lambda = P^2$. As b and $-b$ are indefinite and have same signatures, they are isometric by Theorem 3.

The following is a consequence of Corollary 4:

COROLLARY 5. *Let K^{2q-1} be a Dedekind knot, $q \geq 2$, which has order two in the knot cobordism group (i.e. $K^{2q-1} + K^{2q-1}$ is nullcobordant where + denotes connected sum). Assume that the associated Blanchfield form has even rank. Then K^{2q-1} is (−1)-amphicheiral.*

In the case of odd rank, D. Coray and F. Michel have given counter-examples to the above statement in [4].

Let C_Λ be the group of isomorphism classes of Λ-ideals and let $C_\varepsilon = \{c \in C_\Lambda$ such that if $I \in c$ then $\bar{I} = xI$ with $x\bar{x} = \varepsilon\}$ (notice that if $c \in C_\Lambda$ contains an ideal I such that $\bar{I} = xI$, $x\bar{x} = \varepsilon$, then every $J \in c$ has this property. Indeed, let $J = aI$ then $\bar{J} = (\bar{a}/a)xJ$).

The following is a generalization of results of F. Michel, (cf. [11], Propositions 2 and 3):

COROLLARY 6. *Let K^{2q-1} be a Dedekind knot, $q \geq 2$, such that the associated Blanchfield form (L, b) satisfies $(*)$. Let c be the ideal class of the Λ-module L. Then K^{2q-1} is invertible (resp. $(+1)$-amphicheiral) if and only if the signatures of b and $\bar{b}$ (resp. $-\bar{b}$) are equal and $c \in C_\varepsilon$ with $\varepsilon = (-1)^{n(q+1)}$ (resp. $\varepsilon = (-1)^{nq}$), where $n = \mathrm{rank}_\Lambda (L)$.*

Proof. It is easy to check that there conditions are necessary, let us prove that they are also sufficient. Let us choose a basis $e_1, \ldots, e_n$ of V such that $L = Ie_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$ with $\bar{I} = xI$, $x\bar{x} = \varepsilon$. We can identify V and $\bar{V}$ using the isomorphism $f: V \to \bar{V}$, $f(\lambda e_i) = \lambda^* e_i$. We have $\bar{L} = \bar{I}e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$, and multiplication by $\pm x$ gives an isometry between $\det (L, b)$ and $\varepsilon(-1)^{n(q+1)} \det (\bar{L}, b)$. If n is even and b is skew-hermitian, we see that $\mathrm{Pf}_p (L, b) = \mathrm{Pf}_p (\bar{L}, b)$ for $p\Lambda = P^2$. Theorem 3 now gives the desired result.

In the fibered definite case there are counter-examples to the above corollary. Let for instance $\Lambda = \mathbb{Z}[\xi]$ where ξ is a 52th root of unity. Then there exists a non-trivial Λ-ideal I such that I^3 is principal and that I supports a rank one form b (cf. [12], [1] §1). Notice that I is not isomorphic to $\bar{I}$, therefore (I, b) cannot be isometric to $(\bar{I}, \bar{b})$. So by unique factorisation of definite forms (cf. [2]) $b \perp b \perp b$ cannot be isometric to $\bar{b} \perp \bar{b} \perp \bar{b}$.

EXAMPLE. Let I be a Λ-ideal which supports a rank one form b. Let $(L, b') = (I, b) \perp (\bar{I}, -\bar{b})$. The simple $(2q-1)$-knot, $q \geq 2$, which has Blanchfield pairing (L, b') is clearly $(+1)$-amphicheiral, but it also has the two other symmetries by Corollary 4 and Corollary 6. This answers a question of J. Hillman in [5], for the special case $\Lambda = \mathbb{Z}[w, \frac{1}{53}]$, $I = (5, w+1)$, with $w = 1 + \sqrt{-211}/2$.

4. Rank one forms

Theorem 3 essentially reduces the classification of non-fibered Dedekind $(4q+1)$-knots, $q \geqslant 1$, to the classification of rank one hermitian forms. These have been studied in [1], §1 and §2. Let C_Λ, $C_{\Lambda'}$, denote the ideal class groups (recall $\Lambda' = \{x \in \Lambda$ such that $\bar{x} = x\}$) and let $N: C_\Lambda \to C_{\Lambda'}$ be the norm homomorphism. Let U_Λ be the group of units of Λ, and $N(u) = u\bar{u}$. Let $I(\Lambda)$ be the set of isomorphism classes of rank one forms, which is a group under tensor product. The following diagram summarizes the relation between Γ-lattices and Λ-lattices. The rows and columns are exact.

$$\begin{tikzcd}
1 \arrow[r] & \operatorname{Ker}(f) \arrow[r] \arrow[d] & \operatorname{Ker}(g) \arrow[r] \arrow[d] & \operatorname{Ker}(h) \arrow[r] \arrow[d] & Y \arrow[r] & 1 \\
1 \arrow[r] & U_{\Gamma'}/N(U_\Gamma) \arrow[r] \arrow[d, "f"'] & I(\Gamma) \arrow[r] \arrow[d, "g"'] & \operatorname{Ker}(N_\Gamma) \arrow[r] \arrow[d, "h"'] & 1 & \\
1 \arrow[r] & U_{\Lambda'}/N(U_\Lambda) \arrow[r] \arrow[d] & I(\Lambda) \arrow[r] \arrow[d] & \operatorname{Ker}(N_\Lambda) \arrow[r] \arrow[d] & 1 & \\
& X \arrow[d] & 1 & 1 & & \\
& 1 & & & &
\end{tikzcd}$$

EXAMPLE. Let $\varphi(X) = X^2 - X + 122$, $\lambda(X) = 112X^2 - 223X + 112$ $\Lambda = \mathbb{Z}[X, X^{-1}]/(\lambda)$, $\Gamma = \mathbb{Z}[X]/(\varphi)$. Then we have the following diagram:

$$\begin{tikzcd}
& 1 \arrow[r] \arrow[d] & \mathbb{Z}/7 \arrow[r] \arrow[d] & \mathbb{Z}/14 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \times \mathbb{Z}/14 \arrow[r] \arrow[d] & \mathbb{Z}/14 \arrow[r] \arrow[d] & 1 & \\
1 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[d] & 1 & & \\
& \mathbb{Z}/2 \arrow[d] & 1 & & & \\
& 1 & & & &
\end{tikzcd}$$

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On the zeros of meromorphic solutions of second-order linear differential equations

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1. Introduction and main results

This paper is concerned with the differential equation,

$$f''+A(z)f=0, \tag{1.1}$$

where $A(z)$ is a meromorphic function on the plane. In an earlier paper [2] the authors investigated this equation in the case where $A(z)$ is an entire function, mainly from the point of view of determining the distribution of zeros of solutions. (Of course, in this case all solutions of (1.1) are entire.) The following theorem summarizes these results, and also includes some well-known facts (see [2] for references). As in [2], we will use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function f, and $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of f.

THEOREM A. *Let $A(z)$ be an entire function, and let f_1 and f_2 be any two linearly independent solutions of* (1.1). *Then:*

(A) *If $A(z)$ is a polynomial of degree $n\geq 1$, then the following hold:* (i) *Any solution $f\not\equiv 0$ of* (1.1) *is of order* $(n+2)/2$, *and* (ii) *At least one of the numbers* $\lambda(f_1), \lambda(f_2)$ *is* $(n+2)/2$.

(B) *If $A(z)$ is transcendental, any solution $f\not\equiv 0$ of* (1.1) *is of infinite order of growth.*

(C) *If $A(z)$ is transcendental, and $\sigma(A)$ is finite but not a positive integer, then* $\max\{\lambda(f_1), \lambda(f_2)\}\geq\sigma(A)$ *if* $\sigma(A)\geq\frac{1}{2}$, *while if* $\sigma(A)<\frac{1}{2}$, *then* $\max\{\lambda(f_1), \lambda(f_2)\}=\infty$.

(D) *For any σ, $0\leq\sigma\leq\infty$, there exists an entire transcendental function $A(z)$ of order σ such that* (1.1) *possesses a solution with no zeros. If σ is a positive integer or*

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∞, *there exists an entire function* $A(z)$ *of order* σ *such that* (1.1) *possesses two linearly independent solutions each having no zeros.*

(E) *If* $0<\sigma(A)\leq\infty$, *and if* $A(z)$ *has the property that* $\lambda(A)<\sigma(A)$, *then for any solution* $f\not\equiv 0$ *of* (1.1), *the inequality* $\lambda(f)\geq\sigma(A)$ *holds.*

The main technique used in the proofs of Parts (A) and (C) consisted of looking at the product $f_1 f_2$ of the solutions. The proof of Part (E) mainly used the Tumura–Clunie theory (see [6; §3.5]).

In the present paper, we consider the case of equation (1.1) where $A(z)$ is a meromorphic function on the plane, and we seek to determine to what extent results analogous to those in Theorem A hold. Of course, when $A(z)$ is meromorphic, there are some immediate difficulties. For example, it is well-known (see [4; p. 205]) that if $A(z)$ is entire, then the growth of any solution of (1.1) can be estimated in terms of the growth of $A(z)$ alone. However, this is not true if $A(z)$ is meromorphic (see [1] and [3]). but there are more basic difficulties in the case where $A(z)$ is meromorphic. For example, it is possible that no solution of (1.1) except the zero solution is single-valued on the plane. This obstacle can easily be handled since necessary and sufficient conditions on $A(z)$ can be found which guarantee that all solutions of (1.1) are meromorphic functions on the plane. Two such types of conditions are used in this paper. The primary one for our purposes is to represent $A(z)$ in terms of another meromorphic function $E(z)$ which will be the product of two meromorphic solutions of (1.1) (see Lemma B below). The second way is to represent $A(z)$ in terms of another meromorphic function $g(z)$ which will be the quotient of two solutions of (1.1). Of course, the latter is the classical technique of using the Schwarzian derivative of a meromorphic function $g(z)$, which we will denote by $\{g, z\}$. (See Fuchs [5; §2] or Hille [8; Chapter 10].) The necessary and sufficient conditions for single-valued meromorphic solutions are found in Lemmas A and B below.

It was shown in [2; §5(a)] that the obvious meromorphic analogue of Part A(ii) of Theorem A does not hold, since one can construct a rational function $A(z)$ having a pole of order $n\geq 1$ at ∞, for which equation (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on the plane such that $\max\{\lambda(f_1), \lambda(f_2)\}<(n+2)/2$. (We remark that Part A(i) of Theorem A is valid for any meromorphic solution $f\not\equiv 0$ of (1.1) when $A(z)$ has a pole of order n at ∞ (see [2; §5(a)] or [13]).) In the following theorem (which is proved in §4), we determine all rational functions $A(z)$ having a pole of some order n at ∞, for which (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane such that $\lambda(f_j)<(n+2)/2$ for $j=1,2$.

THEOREM 1. (a) *Let* $A(z)$ *be a rational function having a pole at* ∞ *of any* *order n, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in the plane such that* $\lambda(f_j)<(n+2)/2$ *for* $j=1,2$. *Then* $A(z)$ *must have the form,*

$$A=((E')^2-c^2-2EE'')/4E^2, \tag{1.2}$$

where c is a nonzero constant, and where $E(z)$ *is a rational function with the following properties:*

(i) $E(z)\not\equiv 0$ *and* $E(z)\to 0$ *as* $z\to\infty$;

(ii) *All zeros of* $E(z)$ *in the complex plane are simple*;

(iii) *All poles of* $E(z)$ *are of even order*;

(iv) *At any finite zero* z_0 *of* $E(z)$, *the number* $c/E'(z_0)$ *is an odd integer.*

In addition, it is also true that f_1 *and* f_2 *each have only finitely many zeros and finitely many poles in the plane, and any solution* f_3 *of* (1.1) *which is not a constant multiple of either* f_1 *or* f_2 *has the property that* $\lambda(f_3)=(n+2)/2$. *Finally, n must be even,* $n\geq 2$.

(b) *Conversely, let c be a nonzero constant, and let* $E(z)$ *be a rational function which possess properties* (i)–(iv). *Then, if* $A(z)$ *is the rational function defined by* (1.2), *then* $A(z)$ *has a pole at* ∞, *and the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in the plane with the following properties:*

(v) f_1 *and* f_2 *each have only finitely many zeros*;

(vi) $E=f_1f_2$ *and c is the Wronskian of* f_1 *and* f_2;

(vii) $f_1'/f_1=(\frac{1}{2})((E'/E)-(c/E))$, *and* $f_2'/f_2=(\frac{1}{2})((E'/E)+(c/E))$.

The example produced in [2; §5a] illustrating the phenomenon described in Theorem 1 corresponds to applying Part (b) to $E(z)=(z-2)(z-1)^{-2}$ and $c=1$. However, the general method of Part (b) allows us to obtain a simpler example, by choosing $E(z)=z^{-2}$, and taking $c\neq 0$ to be arbitrary. In this case, we find that the functions $z^{-1}\exp(\pm(c/6)z^3)$ are both solutions of (1.1) where $A(z)$ is given by $-2z^{-2}-(c^2/4)z^4$.

Turning to Part (B) of Theorem A, it is very easy to see that this can fail to hold if $A(z)$ is a transcendental meromorphic function. For example, it is easy to verify that when $A(z)=-2\sec^2 z$, equation (1.1) possesses the solutions $f_1(z)=\tan z$, and $f_2(z)=1+z(\tan z)$, which are linearly independent, and of finite order. In our next theorems (which are proved in §5), we determine all meromorphic functions $A(z)$ on the plane for which all solutions of (1.1) are meromorphic on the plane, and of finite order of growth. The constructions are stated in terms of both the quotient approach (Theorem 2A), and the product approach (Theorem 2B). As an application of these results (see the Remark in §5), it is shown that examples of the phenomenon can occur for any finite choice of $\sigma(A)$. We now state the results:

THEOREM 2A. (a) *Let* $A(z)$ *be meromorphic on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1 *and* f_2 *on the plane, each having finite order of growth. Then* $g = f_1/f_2$ *is a non-constant meromorphic function of finite order with the following properties:*

(i) *All poles of* g *are of odd order;*

(ii) *All zeros of* g' *are of even multiplicity;*

(iii) $A \equiv (\frac{1}{2})\{g, z\}$.

(b) *Conversely, suppose* $g(z)$ *is a nonconstant meromorphic function on the plane having finite order of growth and satisfying* (i) *and* (ii) *above. Then, with* A *defined by* (iii), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each having finite order of growth, and such that* $g = f_1/f_2$. *In addition, if either* g' *has infinitely many zeros, or if* g *has infinitely many multiple poles, then* $A(z)$ *has infinitely many poles (and so is not rational).*

THEOREM 2B. (a) *Let* c *be a nonzero constant, and let* $E(z) \not\equiv 0$ *be a meromorphic function on the plane having finite order of growth, and satisfying the following properties:*

(i) *All zeros of* E *are simple;*

(ii) *All poles of* E *are of even order;*

(iii) *If* $(z_1, z_2, \ldots)$ *is the zero-sequence of* $E(z)$, *then each number* $q_n = c/E'(z_n)$ *is an odd integer;*

(iv) *If* $s_n = (1 + |q_n|)/2$, *then the sequence obtained from* $(z_1, z_2, \ldots)$ *by letting* z_n *appear* s_n *times, has a finite exponent of convergence;*

(v) $m(r, 1/E) = 0(\log r)$ *n.e. as* $r \to \infty$.

Then, with $A(z)$ *defined by* (1.2), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each having finite order of growth. In addition, properties* (vi), (vii) *in Theorem* 1 *hold. Furthermore, if either* E *has infinitely many poles, or if for infinitely many* z_n *we have* $q_n \neq \pm 1$, *then* $A(z)$ *has infinitely many poles (and so is not rational).*

(b) *Conversely, let* $A(z)$ *be a meromorphic function on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *on the plane, each of finite order of growth. Then there exist a nonzero constant* c, *and a meromorphic function* $E(z) \not\equiv 0$ *of finite order on the plane such that* A *has the form* (1.2), *and* (i)–(v) *above hold.*

As a simple example of the construction given in Part (a) of Theorem 2B, we can take $E(z) = -\sin z$, and c to be any odd integer. Then the conditions (i)–(v) are fulfilled. From (1.2), we find $A(z)$ to be $(\frac{1}{4})+((1-c^2)/4\sin^2 z)$, and from the formulas for f_1 and f_2 in (vii) of Theorem 1, we find the meromorphic solutions $\sqrt{2}\sin(z/2)((1-\cos z)/\sin z)^{(c-1)/2}$, and $-\sqrt{2}\sin(z/2)((1-\cos z)/\sin z)^{-(c+1)/2}$ of (1.1). (Other examples are found in §5.)

The result in Part (C) of Theorem A shows that for entire transcendental functions $A(z)$, the only way for an equation (1.1) to possibly possess two linearly independent solutions each having no zeros, is in the case where $\sigma(A)$ is a positive integer or ∞. (Of course, such examples do exist from Part (D) of Theorem A.) However, as an application of our next result, we show that for transcendental meromorphic functions $A(z)$, there are examples of equations (1.1) for any choice of $\sigma(A)$ which possess two linearly independent meromorphic solutions on the plane each having no zeros. The following results give the general construction of all such equations, and the application mentioned above can be found in §6 along with the proofs. For completeness, we include both the product approach (Theorem 3A) and the quotient approach (Theorem 3B).

THEOREM 3A. (a) *Let $A(z)$ be meromorphic on the plane, and assume* (1.1) *possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane, each having no zeros. Then, there exist a nonzero constant c and an entire function $\psi\not\equiv 0$ with the following properties:*

(i) $A=(4\psi\psi''-8(\psi')^2-c^2\psi^6)/4\psi^2$;

(ii) *If $H(z)$ denotes a primitive of $-(c/2)\psi^2$ on the plane, then there are nonzero constants c_1 and c_2 such that,*

$$f_1=(c_1/\psi)e^{H}, \quad \textit{and} \quad f_2=(c_2/\psi)e^{-H}. \tag{1.3}$$

In addition, if $A(z)$ is transcendental, the following two properties hold:

(iii) *Every solution $f\not\equiv 0$ of* (1.1) *is of infinite order of growth on the plane*;

(iv) *Any solution $f\not\equiv 0$ of* (1.1) *which is linearly independent with each of f_1 and f_2, satisfies $\lambda(f)=\infty$.*

(b) *Conversely, let $\psi\not\equiv 0$ be an entire function, and let c be a nonzero constant. Define $A(z)$ by* (i), *and let H denote a primitive of $-(c/2)\psi^2$. Then for any nonzero constants c_1 and c_2, the meromorphic functions f_1 and f_2 defined by* (1.3) *are linearly independent solutions of* (1.1), *each having no zeros. In addition, any zero of ψ is a pole of $A(z)$, and so if ψ has infinitely many zeros, then A cannot be rational.*

THEOREM 3B. *Let $A(z)$ be meromorphic on a simply-connected region D. Then* (1.1) *possesses two linearly independent meromorphic solutions on D, each having no zeros on D, if and only if there exists a nonconstant analytic function*

$g(z)$ *on* D *such that,*

(i) g *has no zeros on* D;

(ii) *All zeros of* g' *on* D *are of even multiplicity*;

(iii) $A = (\frac{1}{2})\{g, z\}$.

The reason why the result in Part (C) of Theorem A can fail to hold for meromorphic coefficients $A(z)$ is explained by the next result which shows that what is actually occurring in the meromorphic case is a balance between zeros and poles of a solution. If poles as well as zeros are taken into consideration, then we have the following direct analogue of the first part of Part (C) of Theorem A, (to be proved in §7) where we use the notation $\bar{\lambda}(f)$ to denote the exponent of convergence of the sequence of zeros of f, each counted only once. (Of course, in this notation, $\lambda(1/f)$ is the exponent of convergence of the sequence of poles of f.)

THEOREM 4. *Let* $A(z)$ *be a transcendental meromorphic function on the plane of finite order* σ, *where* σ *is not a positive integer, and assume that* f_1 *and* f_2 *are two linearly independent meromorphic solutions on the plane of* (1.1). *Then, if* $\sigma > 0$, *we have*

$$\max\{\bar{\lambda}(f_1), \bar{\lambda}(f_2), \lambda(1/f_1)\} \geq \sigma. \tag{1.4}$$

If $\sigma = 0$, *then at least one of the following three sets must be infinite: the set of zeros of* f_1; *the set of zeros of* f_2; *the set of poles of* f_1.

We remark here that in contrast to the strong result in the second part of Part (C) of Theorem A when $A(z)$ is an entire function of order less than $\frac{1}{2}$, no such result is possible in the meromorphic case as evidenced by examples constructed in §5 (following the proof of Theorem 2B).

As in the case of Part (C) of Theorem A we next show that if the poles of a solution are taken into consideration in the case when $A(z)$ is meromorphic, then a direct analogue of Part (E) of Theorem A holds for meromorphic $A(z)$. This result follows very easily from a theorem of W. Hayman [9; Theorem 4], and the theorem of Hayman permits us to obtain the conclusion under the weaker condition $\bar{\lambda}(A) < \sigma(A)$, thus answering a question raised in [2; p. 352]. The theorem (which will be proved in §7) is as follows:

THEOREM 5. *Let* $A(z)$ *be a transcendental meromorphic function on the plane of order* σ, *where* $0 < \sigma \leq \pm\infty$, *and assume that* $\bar{\lambda}(A) < \sigma$. *Then, if* $f(z) \not\equiv 0$ *is a meromorphic solution on the plane of* (1.1), *we have*

$$\max\{\bar{\lambda}(f), \bar{\lambda}(1/f)\} \geq \sigma. \tag{1.5}$$

In the next result, we consider the situation of an equation (1.1) where $A(z)$ is meromorphic, and where (1.1) possesses two linearly independent meromorphic solutions each of whose zero-sequences has a finite exponent of convergence. We address the question of what can be said about the distribution of zeros of other solutions. The answer is very simple, and is given by the following theorem which is proved in §7, and is followed by a simple corollary for the case when $A(z)$ is entire.

THEOREM 6. *Let $A(z)$ be a transcendental meromorphic function on the plane, and assume that* (1.1) *possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane, satisfying $\bar{\lambda}(f_1)<\infty$ and $\bar{\lambda}(f_2)<\infty$. Then, any solution $f\not\equiv 0$ of* (1.1) *which is not a constant multiple of either f_1 or f_2 satisfies,*

$$\max\{\bar{\lambda}(f), \bar{\lambda}(1/f)\}=\infty, \tag{1.6}$$

unless all solutions of (1.1) *are of finite order. In the special case where $\bar{\lambda}(1/A)<\infty$ (e.g. A is of finite order), we can conclude that $\bar{\lambda}(f)=\infty$ unless all solutions of* (1.1) *are of finite order.*

COROLLARY 7. *Let $A(z)$ be a transcendental entire function, and assume that* (1.1) *possesses two linearly independent solutions f_1 and f_2 such that $\lambda(f_1)<\infty$ and $\lambda(f_2)<\infty$. Then, any solution $f\not\equiv 0$ of* (1.1) *which is not a constant multiple of either f_1 or f_2 satisfies $\lambda(f)=\infty$.*

For our final results, we return to the case where $A(z)$ in (1.1) is an entire transcendental function, and to the methods developed in [2] for dealing with this case. As seen from Theorem A, when the order of $A(z)$ is a positive integer or ∞, there seem to be no general results concerning the zeros of solutions of (1.1) except in the special case $\lambda(A)<\sigma(A)$. In our final theorem, we develop a positive result which will be proved in §8, and as a corollary, we apply this result to a special class of equations. Other applications are given in §8. We prove:

THEOREM 8. *Let $A(z)$ be an entire trancendental function of finite order σ, and let $\delta(r)=\min\{|A(z)|:|z|=r\}$ for $r>0$. Assume there is a subset U of $[1,\infty)$ having infinite logarithmic measure, and two constants c_1 and α such that*

$$c_1>0, \qquad \alpha>2(\sigma-1), \quad \textit{and} \quad \delta(r)\geq c_1 r^{\alpha} \textit{ for } r \textit{ in } U. \tag{1.7}$$

Then for any two linearly independent solutions f_1, f_2 of (1.1), *we have,*

$$\max\{\lambda(f_1), \lambda(f_2)\}\geq 1+(\alpha/2). \tag{1.8}$$

Remark. The original proof given in [2] of Theorem A(C) for the case $\sigma(A)<\frac{1}{2}$, follows from Theorem 8 and the well-known minimum modulus theorem of P. Barry.

COROLLARY 9. *Let $P(\rho)$ be a nonconstant polynomial such that $P(0)\neq 0$. Let β be a nonzero complex number, and let m be a positive integer. Then if f_1 and f_2 are any two linearly independent solutions of*

$$f''+z^m P(e^{\beta z})f=0, \tag{1.9}$$

we have

$$\max\{\lambda(f_1),\lambda(f_2)\}\geq 1+(m/2). \tag{1.10}$$

We remark that in light of an example constructed in [2; p. 356], the conclusion of Corollary 9 can fail to hold if $m=0$.

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2. Preliminaries

(a) For a meromorphic function $f(z)$ on the plane, we will use the standard notation of the Nevanlinna theory (see [6] or [9]) including the notation $\bar{N}(r,f)$ for the counting function for the distinct poles of f, as well as the notations $\sigma(f)$, $\lambda(f)$, and $\bar{\lambda}(f)$, which were introduced in §1. Following Hayman [7], we use the abbreviation "n.e." (nearly everywhere) to mean "everywhere in $(0,\infty)$ except in a set of finite measure."

(b) For a nonconstant meromorphic function $g(z)$ in a region D, we will use the standard notation $\{g,z\}$ for the Schwarzian derivative of $g(z)$,

$$\{g,z\}=(g'''/g')-(\tfrac{3}{2})(g''/g')^2. \tag{2.1}$$

(c) If $E(z)\not\equiv 0$ is meromorphic on a region D, and c is a nonzero constant, we will use the notation,

$$\langle E,c\rangle=((E')^2-c^2-2EE'')/4E^2. \tag{2.2}$$

It is very easy to verify that for any nonconstant meromorphic function $g(z)$, we have,

$$\langle\pm(cg/g'),c\rangle=(\tfrac{1}{2})\{g,z\}. \tag{2.3}$$

(d) For any two meromorphic functions, f and g, we will denote their Wronskian by $W(f, g)$.

(e) We will require the following elementary fact: If $A(z)$ is meromorphic on the plane, and if $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$ are meromorphic functions on the plane which satisfy (1.1), then $\sigma(f_1) = \sigma(f_2)$. (The proof is very simple: It is obvious if f_1 and f_2 are linearly dependent.) In the case of linear independence, we have,

$$d((f_2/f_1))/dz = c/f_1^2, \tag{2.4}$$

where $c = W(f_1, f_2)$. This relation immediately shows that $\sigma(f_1) \geq \sigma(f_2)$ in the light of Whittaker's result (see [6; p. 104]) that $\sigma(g) = \sigma(g')$ for meromorphic functions g. Reversing the roles of f_1 and f_2 now proves the statement.

3. Single-valued solutions

In this section we give necessary and sufficient conditions for all solutions of equation (1.1) to be meromorphic (and hence single-valued) in a simply-connected region D. (We remark that all regions considered are subsets of the finite plane, and hence do not contain the point at infinity.)

LEMMA A. (a) *Let $A(z)$ be meromorphic in a region D, and assume that* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D. Then $g = f_1/f_2$ possesses the following properties:*

(i) *All poles of $g(z)$ in D are of odd order*;

(ii) *All zeros of $g'(z)$ in D are of even multiplicity.*

(iii) $A \equiv (\frac{1}{2})\{g, z\}$.

(b) *Conversely, let $g(z)$ be a nonconstant meromorphic function in a simply-connected region D, which possesses properties* (i) *and* (ii), *and define $A(z)$ by* (iii). *Then the equation* (1.1) *possesses two linearly independent meromorphic solutions f_1, f_2 in D such that $g = f_1/f_2$.*

Proof. Part (a). It is well-known [5; p. 6] that (iii) holds. Denoting $c = W(f_1, f_2)$, we have $g' = -c/f_2^2$ from which (i) and (ii) follow.

Part (b). Since the zeros (resp. poles) of g' in D are of even multiplicity (resp. even order), and since D is simply-connected, clearly there exists in D a meromorphic branch $\phi(z)$ of $(g'(z))^{-1/2}$. *With* $A(z)$ defined by (iii), it is well-known [5; p. 6] that ϕ and $g\phi$ are linearly independent solutions of (1.1) which proves Part (b).

LEMMA B. (a) *Let $A(z)$ be meromorphic in a region D, and assume that* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in D. Set* $E = f_1 f_2$ *and* $c = W(f_1, f_2)$. *Then,*

(i) *All zeros of* $E(z)$ *in D are simple;*

(ii) *All poles of* $E(z)$ *in D are of even order;*

(iii) *At any zero* z_0 *of E in D, the number* $c/E'(z_0)$ *is an odd integer;*

(iv) $A \equiv \langle E, c \rangle$.

(b) *Conversely, let* $E(z) \not\equiv 0$ *be a meromorphic function in a simply-connected region D, and let c be a nonzero constant such that* (i), (ii), *and* (iii) *above hold. Then, if* $A(z)$ *is defined by* (iv), *the equation* (1.1) *possesses two linearly independent meromorphic solutions* f_1, f_2 *in D such that*

(v) $E = f_1 f_2$ *and* $c = W(f_1, f_2)$,

and

(vi) $f_1'/f_1 = (\frac{1}{2})((E'/E) - (c/E))$; $f_2'/f_2 = (\frac{1}{2})((E'/E) + (c/E))$.

Proof. Part (a). Set $g = f_1/f_2$, and so g satisfies (i)–(iii) of Lemma A, as well as,

$$g'/g = -c/E. \tag{3.1}$$

Conclusion (i) now follows immediately. Furthermore, any pole of E of order m must be a zero of g' of order m, and hence m is even by Lemma A(ii) proving conclusion (ii). From (3.1), it follows that at any zero z_0 of $E(z)$ in D, the function $g(z)$ has either a zero, say of multiplicity m, or a pole, say of order n, and in addition,

$$-c/E'(z_0) = \text{Residue of } g'/g \text{ at } z = z_0. \tag{3.2}$$

If g has a zero at z_0, then either $m = 1$ or $m - 1$ is even by Lemma A(ii). In any case, m is odd, and since the right-hand side of (3.2) is m in this case, we obtain conclusion (iii). If g has a pole at z_0, then n is odd by Lemma A(i), and again $c/E'(z_0)$ is odd by (3.2). This proves conclusion (iii). Finally (iv) follows immediately from (2.3), (3.1), and Lemma A(iii), since

$$A = (\tfrac{1}{2})\{g, z\} = \langle -E, c \rangle = \langle E, c \rangle. \tag{3.3}$$

Conversely, if $E(z) \not\equiv 0$ is meromorphic in a simply-connected region D and satisfies (i)–(iii), define $H_1 = (\frac{1}{2})((E'/E) - (c/E))$, and $H_2 = (\frac{1}{2})((E'/E) + (c/E))$. Any pole z_0 of H_1 or H_2 must clearly be a zero or pole of E. If E has a zero at z_0, then the zero is simple, and we have,

$$H_1(z) = (\tfrac{1}{2})(1 - (c/E'(z_0)))(z - z_0)^{-1} + \phi_1(z), \tag{3.4}$$

where $\phi_1(z)$ is analytic in a neighborhood of z_0. In view of condition (iii), we see that $H_1(z)$ is either analytic at z_0 or it has a simple pole with integer residue. The same statment holds for $H_2(z)$. Now assume that $E(z)$ has a pole at z_0. Then c/E is analytic at z_0 and so by condition (ii) we see that H_1 and H_2 have simple poles at z_0 with integer residue. Hence all poles of H_1 and H_2 in D are simple with integer residues. Since D is simply-connected it follows from standard techniques that H_1 and H_2 are the logarithmic derivatives of certain meromorphic functions f_1 and f_2 in D so that (vi) holds. By simple calculation from (vi), we see that f_1 and f_2 are solutions of (1.1) when A is defined by (iv). Adding the two relations (vi), it easily follows that for some constant $K\neq 0$, we have $E=Kf_1f_2$. Subtracting the two relations in (vi), we see that $W(f_1, f_2)=c/K$. It thus follows that the two solutions Kf_1 and f_2 satisfy all the conditions in (v) and (vi) proving Part (b).

Remark. Lemma B can be interpreted as giving a complete answer to the question of determining when a meromorphic function $E(z)\not\equiv 0$ in a simply connected region D is the product of two linearly independent meromorphic solutions of an equation (1.1) where A is meromorphic in D. The corresponding question when we replace "meromorphic" by "analytic" throughout, is answered by the following result:

LEMMA C. *Let $E(z)\not\equiv 0$ be analytic in a simply-connected region D. Then $E(z)$ is the product of two linearly independent analytic solutions in D of an equation* (1.1) *where $A(z)$ is analytic in D, if and only if there is a nonzero constant c such that at every zero of $E(z)$ in D, the value of $E'(z)$ is either c or $-c$.*

Proof. If $E(z)$ is the product f_1f_2 of two analytic solutions in D of (1.1) where A is analytic on D, then from Lemma B, we know that $A\equiv\langle E, c\rangle$ for some nonzero constant c. Since A is analytic on D, it follows from (2.2) that if $E(z_0)=0$ then $E'(z_0)=\pm c$.

Conversely, if $E(z)$ has the property that at every zero, the value of E' is c or $-c$ for some fixed $c\neq 0$, then E satisfies (i)–(iii) of Lemma B so $E(z)$ is the product of two linearly independent meromorphic solutions f_1, f_2 of (1.1) where A is given by $\langle E, c\rangle$. To show A is analytic on D, we can write,

$$A(z)=h(z)/4(E(z))^2, \quad \text{where} \quad h=(E')^2-c^2-2EE''. \tag{3.5}$$

Since $h'=-2EE'''$, it follows that at any zero of E, the analytic function h has at least a double zero, and so $A(z)$ is analytic on D by (3.5). Of course, then f_1 and f_2 are also analytic on D by standard results. This proves Lemma C.

LEMMA D. *Let $A(z)$ be meromorphic on a simply-connected region D, and assume that equation (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 in D. Set $E = f_1 f_2$ and $c = W(f_1, f_2)$. Then:*

(a) *If f_1 has a zero z_0 in D of multiplicity n, then f_2 is analytic and nonzero at z_0 if $n = 1$, while if $n > 1$, then f_2 has a pole at z_0 of order $n-1$.*

(b) *If f_1 has a pole at a point z_0 in D of order n, then either f_2 has a zero at z_0 of multiplicity $n+1$, or f_2 has a pole at z_0 of order n.*

(c) *$E(z)$ has a zero at a point z_0 in D if and only if exactly one of the functions f_1, f_2 has a zero at z_0.*

(d) *For any constant c_1, the equation $A = \langle F, c_1 \rangle$ possesses a meromorphic solution $F \not\equiv 0$ in D. Any function $F(z) \not\equiv 0$ which is meromorphic in a subregion of D and satisfies $A = \langle F, c_1 \rangle$ for some constant c_1 is a product of two solutions of (1.1) whose Wronskian is c_1.*

(e) *If D is the whole complex plane, then the Nevanlinna characteristic of E satisfies the following estimate n.e. as $r \to \infty$:*

$$T(r, E) = 0(\bar{N}(r, 1/E) + T(r, A) + \log r). \tag{3.6}$$

Proof. Part (a). This follows immediately from the relation,

$$d((f_2/f_1))/dz = c/f_1^2, \tag{3.7}$$

since f_2/f_1 must have a pole at z_0 of order $2n-1$.

Part (b). We have $f_1'/f_1 = -n(z-z_0)^{-1} + \phi_1(z)$, where ϕ_1 is analytic at z_0. From this we obtain

$$-A = f_1''/f_1 = (n^2+n)(z-z_0)^{-2} + (z-z_0)^{-1}\psi_1(z), \tag{3.8}$$

where ψ_1 is analytic at z_0. From (3.8) we see that the indicial equation for (1.1) at z_0 has roots $n+1$ and $-n$, and so from standard results [8; pp. 155–161] the equation (1.1) possesses an analytic solution $f_3(z)$ in a neighborhood of z_0 which has a zero at z_0 of multiplicity $n+1$. Then $f_3 = c_1 f_1 + c_2 f_2$ for some constants c_1 and c_2, and we must have $c_2 \neq 0$ since f_1 has a pole at z_0. Writing $f_2 = c_2^{-1}(f_3 - c_1 f_1)$, we see that f_2 has a zero of multiplicity $n+1$ at z_0 if $c_1 = 0$, while if $c_1 \neq 0$, f_2 has a pole of order n at z_0. This proves Part (b). (An alternate proof of Part (b) which is analogous to the proof of Part (a), can be given using (3.7) with f_1 and f_2 reversed.)

Part (c). This follows easily from Part (a) noting that any zero of E is a zero of one of the functions f_1 or f_2.

Part (d). If c_1 is given and is nonzero, then $F = (c_1/c)E$ will satisfy $A = \langle F, c_1 \rangle$ by Lemma B(iv). If $c_1 = 0$, it is easily verified that $F = f_1^2$ satisfies $A = \langle F, c_1 \rangle$. Now assume that $F\not\equiv 0$ satisfies $A=\langle F, c_1\rangle$ for some constant c_1. It is easily verified that each of F, f_1^2, f_1f_2, f_2^2 satisfies the linear differential equation,

$$w'''+4Aw'+2Aw=0. \tag{3.9}$$

Since f_1^2, f_1f_2, and f_2^2, are linearly independent, the function F is a linear combination of them, and so is a product of two linear combinations of f_1 and f_2. If c_2 is the Wronskian of these two linear combinations, it is easy to see (see Lemma B(iv)) that $A=\langle F, c_2\rangle$ and so $c_1=\pm c_2$. This proves Part (d).

Part (e). By Lemma B(iv), we have $A=\langle E, c\rangle$. We rewrite this equation in the form,

$$E^2=c^2/((E'/E)^2-2(E''/E)-4A). \tag{3.10}$$

We now apply the Nevanlinna theory (including the lemma on the logarithmic derivative) to (3.10), and we obtain,

$$T(r, E)=0(\bar{N}(r, E)+\bar{N}(r, 1/E)+T(r, A)+\log r) \tag{3.11}$$

holding n.e. as $r\to\infty$. However, any pole of E must be a pole of f_1 or f_2 and hence (see (3.8)) at most a double pole of A. It now follows from (3.11) that (3.6) holds n.e. as $r\to\infty$.

4. Proof of Theorem 1

Part (a). Let $A(z)$ be a rational function having a pole of order n at ∞, and assume that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane such that $\lambda(f_j)<(n+2)/2$ for $j=1,2$. Set $E=f_1f_2$ and $c=W(f_1, f_2)$, so that $E(z)\not\equiv 0$ and c is a nonzero constant. Then, from Lemma B, $A(z)$ has the form (1.2), and $E(z)$ possesses properties (ii), (iii), and (iv) listed in Theorem 1. From (3.8), clearly the poles of f_1 and f_2 in the plane can only occur at the poles of A, and so $E(z)$ is analytic in a neighborhood of ∞, say $|z|>K$. If $E(z)$ has an essential singularity at ∞, then the Wiman–Valiron Theory ([11: Chapter 4], [12: Chapters 9 and 10], or [14: Chapter 1]) is applicable to (1.2), and since $A(z)$ has a pole of order n at ∞, it would follows that $\sigma(E)=(n+2)/2$. But in view of (3.6), the rationality of $A(z)$, and the fact that $E(z)$ is not rational, we then obtain $\bar{\lambda}(E)=(n+2)/2$. But then at least one of the two solutions f_j would satisfy $\bar{\lambda}(f_j)=(n+2)/2$ contradicting the hypothesis. Hence the meromorphic function $E(z)$ has at most a pole at ∞, and so is rational. Since E'/E and E''/E both tend to zero as $z \to \infty$, it follows from (1.2) that $E \to 0$ as $z \to \infty$ (and, in fact, has a zero of multiplicity $n/2$ at ∞.) From Lemma D, Part (a), we see that both f_1 and f_2 have only finitely many zeros in the plane since E has only finitely many zeros. Finally, let $f_3 = c_1 f_1 + c_2 f_2$ where c_1 and c_2 are nonzero constants, and set $E_1 = f_1 f_3$. If we assume $\lambda(f_3) < (n+2)/2$, the same argument as above would show that E_1 is rational. But $E_1 = c_1 f_1^2 + c_2 E$, and so f_1^2 would be rational. This implies f_1 is rational, and so $A = -f_1''/f_1$ tends to zero as $z \to \infty$ contradicting the hypothesis. Hence $\lambda(f_3) = (n+2)/2$ proving Part (a) completely.

Part (b). Let $A(z)$ be the rational function defined by (1.2), where $c \neq 0$ and the rational function $E(z)$ satisfy (i)–(iv). Clearly $A(z)$ has a pole at ∞, and by Lemma B, the equation (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane, such that conclusions (vi) and (vii) in Theorem 1 hold. Since $E = f_1 f_2$ and E is rational, it follows from Part (a) of Lemma D that each of f_1, f_2 has only finitely many zeros in the plane. This proves Part (b) completely.

5. Solutions of finite order

In this section, we prove Theorems 2A and 2B, and give examples.

Proof of Theorem 2A. Part (a) follows immediately from Lemma A(a).

Part (b). Assume now that g is a nonconstant meromorphic function on the plane having finite order of growth, and possessing properties (i) and (ii). Then with A defined by (iii), it follows from Lemma A(b), that (1.1) possesses two independent meromorphic solutions f_1, f_2 on the plane with $g = f_1/f_2$. Then $g' = -c/f_2^2$ where $c = W(f_1, f_2)$, and so f_2 is of finite order. Since $f_1 = g f_2$, we also have that f_1 is of finite order. To prove the last statement, we observe first that it is easy to verify (e.g. see [5; p. 5]) that if $g(z)$ has a Laurent expansion around a point z_0 of the form,

$$g(z) = c_0 + \sum_{\substack{k=p \\ k \neq 0}}^{\infty} c_k (z - z_0)^k, \tag{5.1}$$

where $p \neq 0$ is an integer, and $c_p \neq 0$, then

$$\{g(z), z\} = ((1-p^2)/2)(z-z_0)^{-2} + (z-z_0)^{-1}\phi(z), \tag{5.2}$$

where $\phi(z)$ is analytic at z_0. Since the left-hand side of (5.2) is $2A(z)$, it easily follows that any zero of g' or any multiple pole of g is a pole of A, and this proves Theorem 2A completely.

Proof of Theorem 2*B*. Part (a). Under the stated conditions (i)–(iii), on E and c, and with $A(z)$ defined by (1.2), it follows from Lemma B that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 in the plane such that $E=f_1f_2$, $c=W(f_1,f_2)$, and

$$f_1'/f_1=(\tfrac{1}{2})((E'/E)-(c/E)); \qquad f_2'/f_2=(\tfrac{1}{2})((E'/E)+(c/E)). \tag{5.3}$$

We now analyze the sequence of zeros and sequence of poles of f_1. Suppose ρ is a zero of f_1 of multiplicity q. By Lemma D(a), the point ρ is a zero of E, so $\rho=z_n$ for some n. From (5.3), the residue of f_1'/f_1 is $(1-q_n)/2$ at z_n, so $q=(1-q_n)/2$ (where $q_n=c/E'(z_n)$). Since $q\geq 1$, we see that $q_n\leq -1$, and thus $q=s_n=(1+|q_n|)/2$. Thus the zero-sequence of f_1 is contained in the sequence described in (iv) of Theorem 2B, and thus has a finite exponent of convergence. Now suppose w is a pole of f_1 of order t. Then by Lemma D(b), either E has a pole at w of order $2t$, or E has a zero at w, in which case $w=z_n$. In the latter case, it follows from (5.3) that $-t=(\tfrac{1}{2})(1-q_n)$. Hence $q_n\geq 3$ and $t=s_n-1$. It follows that the sequence of poles of f_1 is contained in the union of two sequences R_1 and R_2, where R_1 is the sequence obtained from the pole sequence of E by eliminating one-half of the occurrences of each pole, and where R_2 is the sequence obtained from $(z_1, z_2, \ldots)$ by repeating z_n only s_n-1 times. Since E is of finite order, clearly R_1 has a finite exponent of convergence. In view of condition (iv) of the theorem, R_2 has a finite exponent of convergence. Thus the sequence of poles of f_1, (like the sequence of zeros of f_1), has a finite exponent of convergence. Hence we may write, $f_1=(Q_1/Q_2)e^Q$, where Q_1 and Q_2 are canonical products of finite order, and where Q is entire. Now by the Nevanlinna theory, each of $m(r, E'/E)$, $m(r, Q_1'/Q_1)$ and $m(r, Q_2'/Q_2)$ is $0(\log r)$ as $r\to\infty$. In view of condition (v) and (5.3), it now follows that $m(r, Q')=0(\log r)$ as $r\to\infty$, and thus Q is a polynomial. Hence f_1 is of finite order. It now follows from §2(e) that f_2 is also of finite order.

The relations (vi) and (vii) in Theorem 1 also hold (see (5.3)).

To prove the last statements, we observe first that any pole of E is a pole of at least one of f_1, f_2, and hence clearly is a pole of A (see (3.8)). Now set $g=f_1/f_2$ so that (3.1) holds. Hence at any z_n, we have relation (3.2), and thus at z_n, g has either a zero of multiplicity $-q_n$ if $q_n<0$, or a pole of order q_n if $q_n>0$. Since $A=(\tfrac{1}{2})\{g, z\}$ by Lemma A, we see from (5.1) and (5.2) that,

$$A(z)=((1-q_n^2)/4)(z-z_n)^{-2}+(z-z_n)^{-1}\psi(z), \tag{5.4}$$

where ψ is analytic at z_n. Hence, if $q_n\neq\pm 1$, then A has a pole at z_n. This proves Part (a).

Part (b). Let $E=f_1f_2$, and $c=W(f_1,f_2)$. Then E is of finite order, and by Lemma B, properties (i)–(iii) hold, and A is given by (1.2). Set $g=f_1/f_2$ so that $g'/g=-c/E$. Since g is of finite order, we obtain (v) from the Nevanlinna theory. As in the proof of Part (a), we see that at each z_n, g has either a zero of multiplicity $-q_n$ if $q_n<0$, or a pole of order q_n if $q_n>0$. Since both the sequence of zeros of g, and the sequence of poles of g each have a finite exponent of convergence, it now follows easily that the sequence described in (iv) also has a finite exponent of convergence, and thus Part (b) is proved.

Remark. In this remark, we show that for any nonnegative real number α, there exists a transcendental meromorphic function $A(z)$ on the plane of order α, such that every solution $f(z)\not\equiv 0$ of (1.1) is a transcendental meromorphic function on the plane of order α. The construction is quite easy. Let ψ be a transcendental entire function of order α having only simple zeros such that $\lambda(\psi)=\alpha$. Let g denote a primitive of ψ^2. Then if $A=(\frac{1}{2})\{g,z\}$, we see by Lemma A that g is the quotient f_1/f_2 of two linearly independent meromorphic solutions on the plane of (1.1). Since $g'=-c/f_2^2$, where $c=W(f_1,f_2)$, we see that ψ^{-1} and $g\psi^{-1}$ are meromorphic solutions on the plane of (1.1). From (5.1) and (5.2), every zero of g' is a double pole of A. Hence every zero of ψ is a double pole of A, and so A is a transcendental meromorphic function of order at least α. However, since g is of order α, we also have $\sigma(A)\leq\alpha$ and so $\sigma(A)=\alpha$. The solution ψ^{-1} is of order α, ans so by §2(e), every solution (except the zero solution) is of order α. Of course, all solutions (except zero) are transcendental since A is transcendental. The examples constructed here have the property $\sigma(f)=\sigma(A)$ for all solutions $f\not\equiv 0$ of (1.1). In the following example, we construct an equation (1.1) where A is a transcendental meromorphic function of finite order on the plane, all of whose solutions $f\not\equiv 0$ are meromorphic functions of finite order on the plane satisfying $\sigma(f)>\sigma(A)$.

EXAMPLE. Set $E(z)=\cos(z^{1/2})$. Then $E(z)$ is an entire function having simple zeros at the points $z_n=((2n+1)\pi/2)^2$ for $n=0,1,\ldots$, and no other zeros. It is easy to verify that if we choose $c=1/\pi$, then for each n we have

$$q_n=c/E'(z_n)=(-1)^{n+1}(2n+1), \tag{5.5}$$

and so q_n is odd integer. Defining s_n as in Theorem 2B, Part (iv), we have $s_n=n+1$, and it is easy to see that the sequence obtained from $(z_0,z_1,\ldots)$ by repeating z_n s_n-times has exponent of convergence equal to 1. The function $E(z)$ satisfies the differential equation,

$$1/E(z)^2=1+4z(E'(z)/E(z))^2, \tag{5.6}$$

and since E is of order $\frac{1}{2}$, we have from (5.6) that $m(r, 1/E) = 0(\log r)$ as $r \to \infty$. Hence from Theorem 2B, Part (a), if we set $A = \langle E, c\rangle$, then (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on the plane, each having finite order of growth, satisfying $E = f_1 f_2$, $c = W(f_1, f_2)$, and such that (5.3) holds. From (5.3) and (5.5), we see that the residue at z_n of f_2'/f_2 is $(1+q_n)/2$. Hence if n is odd, then f_2 has a zero at z_n of multiplicity $n+1$. Since the exponent of convergence of the sequence obtained from $(z_1, z_3, \ldots)$ by repeating z_{2k+1} $(2k+2)$-times is obviously equal to 1, we can conclude that $\sigma(f_2) \geq 1$. Using §2(e), we can now conclude that $\sigma(f) \geq 1$ for every solution $f \not\equiv 0$ of (1.1). Of course, since E is of order $\frac{1}{2}$, we see that $A = \langle E, c\rangle$ is of order at most $\frac{1}{2}$. In fact, A is of order precisely $\frac{1}{2}$, since for $n \geq 1$ we have $q_n \neq \pm 1$, and so from the proof (see (5.4)) of Theorem 2B, the function $A(z)$ has a double pole at $z_1, z_2, \ldots$. This shows that $\sigma(A) \geq \frac{1}{2}$ and thus $\sigma(A) = \frac{1}{2}$. Hence $\sigma(A) < \sigma(f) < \infty$ for all solutions $f \not\equiv 0$.

6. Zero-free solutions

Proof of Theorem 3A. Part (a). Set $E = f_1 f_2$, and $c = W(f_1, f_2)$. By assumption, E has no zeros, and by Lemma B, all poles of E are of even order. Hence E has the form $1/\psi^2$ where ψ is an entire function, and by Lemma B the representation (i) holds since the right side of (i) is $\langle 1/\psi^2, c\rangle$. Now let H denote a primitive of $-(c/2)\psi^2$. From the relations $E = f_1 f_2$ and $c = W(f_1, f_2)$, we see that (5.3) holds, and hence

$$f_1'/f_1 = -((\psi'/\psi) + (c/2)\psi^2); \qquad f_2'/f_2 = -((\psi'/\psi) - (c/2)\psi^2). \tag{6.1}$$

Since $H' \equiv -(c/2)\psi^2$, the representations (1.3) follow immediately.

Now assume that $A(z)$ is transcendental. Then H must be transcendental, for in the contrary case, ψ would be a polynomial, and A would be rational by (i). Now, in view of (1.3), and the definition of H, we have as $r \to \infty$,

$$T(r, e^H) \leq T(r, f_1) + (\tfrac{1}{2})T(r, H') + 0(1). \tag{6.2}$$

Since $T(r, H') = 0(T(r, H))$ n.e. as $r \to \infty$, and $T(r, e^H)/T(r, H) \to +\infty$ as $r \to \infty$ (see [6; pp. 54, 55]) we see from (6.2) that,

$$T(r, e^H) \leq 2T(r, f_1) + 0(1) \text{ n.e. as } r \to \infty. \tag{6.3}$$

Since e^H is of infinite order, the same is true for f_1, and also for all solutions $f \not\equiv 0$ by §2(e).

Now let $f=\alpha f_1+\beta f_2$ where α and β are nonzero constants, and set $E_1=ff_1$. Since f_1 has no zeros, clearly any zero of E_1 must be a zero of f. We now apply Lemma D, Part (e), to both E and E_1. From the relation $f_1^2=(1/\alpha)(E_1-\beta E)$, we thus obtain,

$$T(r, f_1)=0(T(r, A)+\bar{N}(r, 1/f)+\log r), \tag{6.4}$$

n.e. as $r\to\infty$. Since $A=-f_1''/f_1$, we have

$$m(r, A)=0(\log T(r, f_1)+\log r), \quad \text{n.e. as } r\to\infty. \tag{6.5}$$

Since f_1 has no zeros, any pole of A must be a pole of f_1 and hence a zero of ψ by (1.3). Since A can have at most double poles (see (5.2) and Lemma A), we see $N(r, A)=0(N(r, 1/\psi))$ as $r\to\infty$, and so from (6.4) and (6.5) we have,

$$T(r, f_1)=0(N(r, 1/\psi)+\bar{N}(r, 1/f)+\log r), \tag{6.6}$$

n.e. as $r\to\infty$. Since $\psi^2=-(2/c)H'$, it now follows from (6.3) and (6.6) that,

$$T(r, e^H)=0(\bar{N}(r, 1/f)+\log r) \quad \text{n.e. as } r\to\infty. \tag{6.7}$$

Since $\sigma(e^H)=\infty$, we thus obtain $\bar{\lambda}(f)=\infty$. This proves Part (a).

Part (b). Set $E=1/\psi^2$, and $A=\langle E, c\rangle$. Then, by Lemma B, Part (b), equation (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on the plane such that $E=f_1f_2$, $c=W(f_1, f_2)$, and (5.3) holds. Since $E=1/\psi^2$, we see that (6.1) holds, and since $H'=-(c/2)\psi^2$, we now see that the functions defined by (1.3) are solutions of (1.1). Any zero of ψ is a pole of E, and by Lemma D(b), a pole of f_1. Thus (see (3.8)), any zero of ψ is a pole of A. This proves Theorem 3A.

Remark. In this remark, we show that for any α, $0\leq\alpha\leq+\infty$, there exists a transcendental mermorphic function $A(z)$ on the plane of order α such that (1.1) possesses two linearly independent meromorphic solutions on the plane, each having no zeros. The construction is very simple. Let ψ be an entire function of order α with only simple zeros, and satisfying $\lambda(\psi)=\alpha$. Let c be a nonzero constant, and set $A=\langle(1/\psi^2), c\rangle$. Then by Theorem 3A, the equation (1.1) possesses two linearly independent meromorphic solutions, each having no zeros, and $\lambda(1/A)=\alpha$. Thus, $\sigma(A)\geq\alpha$. But obviously, $\sigma(A)\leq\sigma(\psi)=\alpha$, so $\sigma(A)=\alpha$.

Proof of Theorem 3B. If (1.1) possesses two linearly independent meromorphic solutions f_1 and f_2 on D, then setting $g=f_1/f_2$, we see that the conclusions (ii) and (iii) hold by Lemma A. Since $g'=-c/f_2^2$, we see that if f_2 has no zeros on D, then g must be analytic on D. Since $(1/g)' = c/f_1^2$, we see that if f_1 has no zeros on D, then $1/g$ is analytic on D, so (i) holds.

Conversely, under conditions (i)–(iii) and the analyticity of g, it follows from Lemma A that (1.1) possesses two linearly independent meromorphic solutions f_1, f_2 on D such that $g = f_1/f_2$. Since $g' = -c/f_2^2$, and $(1/g)' = c/f_1^2$, it follows that if g has no zeros or poles on D, then f_1 and f_2 have no zeros on D.

7. Distribution of zeros or poles of solutions

Proof of Theorem 4. We are given that $\sigma = \sigma(A)$ is finite, but not a positive integer. Set $E = f_1 f_2$, and consider first the case $\sigma > 0$. Assume that (1.4) fails to hold. Since the zeros of E are all simple, we then obtain $\lambda(E) < \sigma$. In view of Lemma D(b), any pole of E, say of order k, must be a pole of f_1 of order $k/2$. Hence by our assumption, we also obtain $\lambda(1/E) < \sigma$. Since $\lambda(E) < \sigma$, it follows from Lemma D(e), that $\sigma(E) \leq \sigma(A) = \sigma$. However, from Lemma B we also have $A = \langle E, c\rangle$, where $c = W(f_1, f_2)$, and so $\sigma(A) \leq \sigma(E)$. Thus $\sigma(E) = \sigma$. Now we may write $E = (G_1/G_2)e^G$, where G_1 and G_2 are entire canonical products of order less than σ, and G is a polynomial. Hence we obtain $\sigma(e^G) = \sigma$ which is absurd since σ is not an integer. This contradiction proves (1.4) if $\sigma > 0$.

Now suppose $\sigma = 0$ but the conclusion fails. Then as above, E has only finitely many zeros, and finitely may poles. By Lemma D(e), E is of order zero, and so E is rational. However, this implies $A = \langle E, c\rangle$ is rational contrary to hypothesis. This proves Theorem 4 completely.

Proof of Theorem 5. Since $f(z)$ is a solution of (1.1) where $\sigma(A) > 0$, it is obvious that $f(z)$ cannot be rational, nor be of the form e^{az+b} for constants a and b. Hence we can invoke [7; Theorem 4], and we obtain n.e. as $r \to \infty$,

$$T(r, f/f') = 0(\bar{N}(r, f) + \bar{N}(r, 1/f) + \bar{N}(r, 1/f'')). \tag{7.1}$$

In addition, since f satisfies (1.1), we have,

$$\bar{N}(r, 1/f'') \leq \bar{N}(r, 1/f) + \bar{N}(r, 1/A). \tag{7.2}$$

By assumption, $\bar{\lambda}(A) < \sigma$. Hence, if we assume that (1.5) fails to hold, then it follows from (7.1) and (7.2), that f/f' is of order less than σ. By Jensen's formula, we then see that if $\psi = f'/f$, then $\sigma(\psi) < \sigma$. However, from (1.1) it easily follows that $-A = \psi' + \psi^2$, and so we would obtain $\sigma(A) \leq \sigma((\psi) < \sigma = \sigma(A)$ which is absurd. This contradiction proves Theorem 5.

Proof of Theorem 6. Assume (1.1) possesses linearly independent meromorphic solutions f_1 and f_2 such that $\bar{\lambda}(f_1)<\infty$ and $\bar{\lambda}(f_2)<\infty$. Set $E_1=f_1f_2$, and let $f=\alpha f_1+\beta f_2$ where α and β are nonzero constants, and set $E_2=ff_1$. Assume that (1.6) fails to hold, so that $\bar{\lambda}(f)<\infty$ and $\bar{\lambda}(1/f)<\infty$. From these relations we easily see that $\bar{\lambda}(E_1)<\infty$, and $\bar{\lambda}(E_2)<\infty$. By Lemma D(e), there is a constant $b>0$ such that n.e. as $r\to\infty$,

$$T(r, E_j)=0(r^b+T(r, A)) \quad \text{for } j=1, 2. \tag{7.3}$$

Since $E_2=\alpha f_1^2+\beta E_1$, we thus obtain,

$$T(r, f_1)=0(r^b+T(r, A)) \quad \text{n.e. as } r\to\infty. \tag{7.4}$$

Since $A=-f''/f$, we see that (6.5) holds, and since any pole of A is at most double (see (3.8)) and is either a zero or pole of f, we also have,

$$N(r, A)\leq 2(\bar{N}(r, 1/f)+\bar{N}(r, f)). \tag{7.5}$$

Hence by assumption, $N(r, A)=0(r^a)$ as $r\to\infty$ for some $a>0$. Together with (7.4) and (6.5), we obtain $T(r, f_1)=0(r^{a+b})$ n.e. as $r\to\infty$, from which it follows (see [2; §2(A), p. 353)] that f_1 is of finite order. Hence by §2(e), all solutions are of finite order if (1.6) fails to hold.

Now assume that $\bar{\lambda}(1/A)<\infty$. Then since any pole of f is a pole of A (see (3.8)), we have $\bar{\lambda}(1/f)\leq\bar{\lambda}(1/A)<\infty$, and so (1.6) takes the form $\bar{\lambda}(f)=\infty$. This proves Theorem 6.

Proof of Corollary 7. This result follows immediately from the last statement in Theorem 6 together with the fact (see Part (B) of Theorem A) that when A is a transcendental entire function, all solutions $f\not\equiv 0$ of (1.1) have infinite order.

8. New results when A is entire

Proof of Theorem 8. Let f_1 and f_2 be linearly independent solutions of (1.1), and set $E=f_1f_2$. Then by Lemma B, we have

$$4A=(E'/E)^2-2(E''/E)-(c/E)^2, \tag{8.1}$$

where $c=W(f_1, f_2)$. Of course E cannot be a polynomial since A is transcendental. Hence we can apply the Wiman–Valiron theory to (8.1), and we obtain the existence of a set D in $[1, \infty)$ of finite logarithmic measure such that if r does not belong to D, and z is a point on $|z|=r$ at which $|E(z)|=M(r,E)$, then

$$2\,|A(z)|\leq (v(r)/r)^2 \tag{8.2}$$

where $v(r)$ denotes the central index of E. Since U is of infinite logarithmic measure, we can find a sequence $\{r_n\}\to+\infty$ such that r_n belongs to U but not to D. From (8.2) and (1.7), we then obtain,

$$2c_1r_n^\alpha\leq(v(r_n)/r_n)^2 \quad \text{for all } n, \tag{8.3}$$

and it now follows (see [11; p. 34]) that $\sigma(E)\geq 1+(\alpha/2)$. Of course by (1.7), we also have $\sigma(A)<1+(\alpha/2)$. In view of (3.6), we then obtain $\lambda(E)\geq 1+(\alpha/2)$ from which (1.8) immediately follows. This proves Theorem 8.

Application of Theorem 8. We consider the differential equations

$$f''+z^m\sin^p(z^q)f=0, \qquad f''+z^m\cos^p(z^q)f=0, \tag{8.4}$$

where m, p, and q are positive integers. Then from Theorem 8 we can conclude that if $m>2(q-1)$, and f_1 and f_2 are two linearly independent solutions of either the first equation in (8.4) or the second equation, then

$$\max\{\lambda(f_1),\lambda(f_2)\}\geq 1+(m/2). \tag{8.5}$$

We will indicate the proof for the first equation, the second being similar. For any $\varepsilon>0$, there is a constant $K_\varepsilon>0$ such that

$$|\sin(z^q)|\geq K_\varepsilon, \tag{8.6}$$

if $|z^q-n\pi|\geq\varepsilon$ for all integers n (see [10; p. 71]). If V denotes the union of all intervals $((n\pi-\varepsilon)^{1/q},(n\pi+\varepsilon)^{1/q})$ for $n=0,1,\ldots$, and if U denotes the complement of V with respect to $[1,\infty)$, then it is easy to see that U has infinite logarithmic measure, and for $A(z)=z^m\sin^p(z^q)$, the minimum modulus $\delta(r)$ of A satisfies $\delta(r)\geq r^mK_\varepsilon^p$ for r in U. Since $\sigma(A)=q$, the conclusion (8.5) now follows from Theorem 8 if $m>2(q-1)$.

Proof of Corollary 9. This is similar to the proof above, and we omit it.

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On Grauert's conjecture and the characterization of Moishezon spaces

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0. Introduction

In this paper, we pursue the investigation of the global structure of 1-convex spaces which we begun in [1] [10a.b.c]. In this direction, we provide here an affirmative answer to the following:

Grauert's conjecture: Let S be an exceptional set of some 1*-convex space X. Then there exists a coherent ideal sheaf* $\tilde{J} \subset 0_X$ *such that* $\tilde{J} \mid S$ *is weakly positive.*

In the same token, Moishezon spaces can be characterized as follows:

A compact, irreducible $\mathbb{C}$-analytic space is Moishezon iff it carries a torsion free positive analytic coherent sheaf.

This result answers affirmatively a problem posed in [1], [9].

The organization of this paper goes as follows:

In Section 1, the equivalence of various notions of positivity for analytic coherent sheaves over compact $\mathbb{C}$-analytic spaces is established. The crucial vanishing theorem for arbitrary 1-convex spaces is proved in Section 2. In Section 3, we shall tackle Grauert's conjecture as well as the characterization of Moishezon spaces.

In the following, we shall use freely the basic definitions and notations employed in [1], [10a.b.c].

1. The positivity of analytic coherent sheaves

DEFINITION 1. Let S be a compact, irreducible $\mathbb{C}$-analytic space and let $\theta \in \mathrm{Coh}(S)$. Then θ is said to be

(i) *weakly positive* if the zero section of $L(\theta)$ admits a 1-convex neighbor-

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hood, where $L(\theta)$ is the linear fibre space associated to θ in the sense of Grauert [2].

(ii) *cohomologically positive* if $H^i(S, S^k(\theta)\otimes\mathscr{F})=0$ for $\forall i\geq 1$, $k\gg 0$ and $\mathscr{F}\in\mathrm{Coh}\,(S)$, where $S^k(\theta)$ denotes the k-th fold symmetric tensor product of θ.

(iii) *ample* if $S^k(\theta)\otimes\mathscr{F}$ is generated by its global sections for $k\gg 0$ and any $\mathscr{F}\in\mathrm{Coh}\,(S)$.

We are now in a position to prove the main result of this section.

THEOREM 1. *Let S be a compact, irreducible $\mathbb{C}$-analytic space and let $\theta\in\mathrm{Coh}\,(S)$. Then the following conditions are equivalent:*

(i) *θ is weakly positive*

(ii) *θ is cohomologically positive*

(iii) *θ is ample.*

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are standard (see e.g. [2], [3], [4], [7]). It remains for us to show that (iii)$\Rightarrow$(i):

So let E be the linear fibre space associated to θ, let $\mathbb{P}(E)$ be the associated projective fibre space and let $L(E):=0_{\mathbb{P}(E)}(-1)$ be the tautological line bundle on $\mathbb{P}(E)$. Notice that there exists a canonical biholomorphism

$$\phi: E\setminus S \simeq L(E)\setminus\mathbb{P}(E) \tag{*}$$

Let $H:=L^*(E)$ and let H be the locally free sheaf associated to H. Then one has the following diagram

$$\begin{array}{ccc} H & & \theta \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & \xrightarrow{p} & S \end{array}$$

and the following isomorphism

$$S^k(\theta)\otimes\mathscr{F}\simeq p_*(H^k\otimes p^*(\mathscr{F})) \tag{\#}$$

for $k\gg 0$ and $\forall\mathscr{F}\in\mathrm{Coh}\,(S)$ (see [9]).

Claim. H is ample.

In fact, for any $\hat{\mathscr{F}}\in\mathrm{Coh}\,(\mathbb{P}(E))$ there exists an integer $k\gg 0$, such that the following morphism

$$p^*p_*(\hat{\mathscr{F}}\otimes H^k)\to\hat{\mathscr{F}}\otimes H^k \tag{†}$$

is surjective (see e.g. [7]). Now let us replace $\hat{\mathscr{F}}$ by $p^*(\mathscr{F})$ in (†) where $\mathscr{F}:=p_*(\hat{\mathscr{F}}\otimes H^n)\in \operatorname{Coh}(S)$. Thus we obtain the following surjective morphism of analytic coherent sheaves:

$$p^*p_*(p^*\mathscr{F}\otimes H^n)\to p^*\mathscr{F}\otimes H^n \quad \text{for} \quad n\gg 0 \tag{††}$$

Now (††) together with (#) and (†) give us the following surjective composite morphism:

$$p^*(S^N(\theta)\otimes\mathscr{F})\to\hat{\mathscr{F}}\otimes H^N \quad \text{for} \quad N\gg 0$$

Since θ is ample, this implies the ampleness of the line bundle H. Hence our claim is proved.

In view of result in [3], [4] $L(E)$ is a weakly negative line bundle in the sense of [2]. Consequently (*) tells us that θ is weakly positive. *Q.E.D.*

Remarks. (i) Notice that Theorem 1 is well known in the special case where θ is locally free [2], [3], [4]. Furthermore, Theorem 1 does not hold in general if S is not assumed to be compact.

(ii) From now on, any analytic coherent sheaf satisfying one of the equivalent conditions in Theorem 1, will be called simply positive coherent sheaf.

2. Vanishing theorem for 1-convex spaces

In [10a] we established a vanishing theorem for "embeddable" 1-convex spaces. We would like to present here a crucial vanishing theorem for arbitrary 1-convex spaces.

THEOREM 2. *Let (X, S) be a 1-convex space. Then there exists a coherent ideal sheaf $J\subset 0_X$ supporting on S such that*

$$H^i(X, J^k\otimes\mathscr{F})=0 \quad \text{for } \forall i\geq 1,\ k\gg 0 \text{ and } \mathscr{F}\in\operatorname{Coh}(X)$$

Proof. Step 1. Since X is 1-convex, there exist a Stein space Y and a proper, surjective and holomorphic morphism $\pi: X\to Y$ inducing a biholomorphism $X\setminus S\simeq Y\setminus T$ where $T:=\pi(S)$ (see e.g. [2]). Now the main result in [5] tells us that there exist:

(i) a coherent ideal sheaf $m_T\subset 0_Y$ supporting on T

(ii) a monoidal transformation $\chi:\tilde{X}\to Y$ with respect to m_T inducing a biholomorphism $\tilde{X}\setminus\tilde{S}\simeq Y\setminus T$ where $\tilde{S}:=\chi^{-1}(T)$

(iii) a proper surjective and holomorphic morphism $p:\tilde{X}\to X$ such that the following diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "p"] \arrow[dr, dashed, "\chi"'] & X \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

Now let $I:=m_T\cdot 0_{\tilde{X}}$ (resp. $J:=m_T\cdot 0_X$) be the inverse image ideal sheaf of m_T by χ (resp. by π). Since T consists of finitely may points, it follows from [6] that

(a) p is a monoidal transformation with respect to J

(b) $I\mid\tilde{S}$ is an ample line bundle

Step 2. In view of (b), by using a standard spectral sequence argument, exactly as in the proof of Theorem 1 in [10b] one can show that

$$R^ip_*(I^k\otimes p^*\mathscr{F})=0 \quad \text{for} \quad \forall i\geq 1,\ k\gg 0 \text{ and } \mathscr{F}\in \mathrm{Coh}\,(X) \tag{†}$$

Claim. The natural sheaf morphism

$$J^k\otimes\mathscr{F}\to p_*(I^k\otimes p^*\mathscr{F}) \tag{§}$$

is an isomorphism for $k\gg 0$, $\mathscr{F}\in\mathrm{Coh}\,(X)$ along S.

In fact in view of the following exact sequence

$$0\to \mathrm{Ker}\cdot f=:\kappa\to 0_X \xrightarrow{f} p_*0_{\tilde{X}}$$

and the compactness of S, one can find an integer $k\gg 0$, such that $J^k\cdot\kappa=0$; hence in view of Artin–Rees lemma, one has $J^k\cap\kappa=0$. Consequently, in view of (a), a result in [3] (Chap. III Théorème 2.3.1) tells us that the sheaf morphism

$$J^k\to p_*(I^k) \tag{$*$}$$

is an isomorphism along S, for $k\gg 0$.

Now for any $x\in S$, let us consider the following local resolution of coherent sheaves for any $\mathscr{F}\in\mathrm{Coh}\,(X)$

$$0_X^m\to 0_X^q\to\mathscr{F}\to 0 \tag{††}$$

From (†) and (††), one obtains the following commutative diagram of analytic coherent sheaves with exact rows and with k arbitrary large:

$$\begin{tikzcd}
\arrow[r] & p_*(p^*0_X^m\otimes I^k) \arrow[r] & p_*(p^*0_X^q\otimes I^k) \arrow[r] & p_*(p^*\mathscr{F}\otimes I^k) \arrow[r] & 0 \\
\arrow[r] & 0_X^m\otimes J^k \arrow[r] \arrow[u, dashed, "\alpha"] & 0_X^q\otimes J^k \arrow[r] \arrow[u, dashed, "\beta"] & \mathscr{F}\otimes J^k \arrow[r] \arrow[u, dashed, "\gamma"] & 0
\end{tikzcd}$$

In view of (*), it follows readily that α and β are isomorphism; consequently so does γ and our claim is proved.

Since X is 1-convex, (†) and (§) imply that

$$H^i(X, J^k\otimes\mathscr{F})=0 \quad \text{for} \quad \forall i\geq 1, k\gg 0 \text{ and } \mathscr{F}\in \operatorname{Coh}(X) \quad Q.E.D.$$

COROLLARY 3. *Let X be a 1-convex manifold. Let us assume that its exceptional set is non singular and of pure codimension one. Then there exists a torsion free $J\in \operatorname{Coh}(X)$ such that $J \mid S$ is positive.*

Proof. Since S is non singular and of pure codimension 1, it follows from Theorem 2 that, for $\forall i\geq 1$, $k\gg 0$ and $\mathscr{F}\in \operatorname{Coh}(S)$,

$$H^i(X, J^k\otimes\mathscr{F})\simeq H^i(S, J^k/J^{k+1}\otimes\mathscr{F})\simeq H^i(S, S^k(J/J^2\otimes\mathscr{F})\simeq 0$$

(Notice that since both X and S are non singular with $\dim X=\dim S+1$, the ideal J is locally principal hence $S^k(J/J^2)\simeq J^k/J^{k+1}$ for $k\gg 0$). Consequently in view of Theorem 1, it follows readily that $J \mid S$ is positive. *Q.E.D.*

Remarks. As far as Corollary 3 is concerned, we would like to point out 2 facts:

(i) It seems likely that the exceptional set S is projective algebraic; however at this writing, we are not able to prove it yet, so we would like to come back in the future.

(ii) In constrast to the case where $\dim\cdot X=2$, [10c] $J \mid S$ in general, is not the normal bundle of S in X (see [2], [8]).

3. The existence of positive analytic coherent sheaves

Our main goal here is to strengthen Corollary 3. First of all the following result will be needed.

LEMMA 4 [2]. *Let S be a compact $\mathbb{C}$-analytic space and let us assume that, for every positive dimensional subspace T of S, there exist an n and a non zero section of $L^n \otimes 0_T$ which vanishes at some point of T. Then S is projective algebraic and L is ample.*

We are now in a position to establish the main result of this section.

Theorem 5. *Let (X, S) be a 1-convex space. Then there exists a coherent ideal sheaf $\mathcal{F} \subset 0_X$ supporting on S such that $\mathcal{F} \mid S$ is positive.*

Proof. Let $x, y \in S$ with $x \neq y$ and let $I_{x,y}$ be the coherent ideal sheaf of germs of holomorphic functions vanishing at x and y. Now Theorem 2 and the compactness of S tell us that, for any points $x \neq y \in S$, there exists $K \gg 0$, such that the restriction map

$$H^0(S, \theta) \to \theta_x \oplus \theta_y$$

is surjective where $\theta := J^K \otimes 0_S$. In view of the surjectivity of $\theta \otimes S^{n-1}(\theta) \to S^n(\theta)$, one obtains the epimorphism

$$H^0(S, S^n(\theta)) \to S^n(\theta)_x \oplus S^n(\theta)_y \tag{*}$$

for any pair of points $x \neq y$ in S.

Let us consider the following diagram and let us use the same notations as in the proof of Theorem 1 above:

$$\begin{tikzcd} H \arrow[d, dashed] & \theta \arrow[d, dashed] \\ \mathbb{P}(E) \arrow[r, "p"] & S \end{tikzcd}$$

where E is the linear fibre space associated to

Claim. H is an ample line bundle.

In fact, in view of Lemma 4, it suffices to show that, for any closed analytic subvariety $T \subset \mathbb{P}(E)$ which is not a point, there exists a section $\sigma \in \Gamma(T, H^n \mid T)$ which vanishes at some point of T, but does not vanish identically on T.

However this is obvious if T is contained in the fibre of p; so let $t_1, t_2 \in T$ be such that $p(t_1) =: x \neq y := p(t_2)$. Since H is relatively ample [7], one can find a section $\tau \in \Gamma(p^{-1}(x), H^n) = p_*(H^n)_x$ such that $\tau(t_1) \neq 0$ for some $n \gg 0$. In view of

(*) and the isomorphism $S^n(\theta)\simeq p_*(H^n)$ for $n \gg 0$, the pull back section map

$$H^0(S, p_*(H^n)) \to H^0(\mathbb{P}(E), H^n)$$

will provide us a global section $s \in H^0(\mathbb{P}(E), H^n)$ with $s(t_1)=\tau(t_1)\neq 0$ and $s(t_2)=0$. Now $\sigma := s \mid T$ will be our desired section and our claim is proved.

Consequently, as in the proof of Theorem 1, the ampleness of H implies the weakly positivity (hence the positivity) for θ. *Q.E.D.*

Remark. This result provides an affirmative answer to a conjecture posed by Grauert [2] (see also [1]).

COROLLARY 6. *Let S be a compact, irreducible $\mathbb{C}$-analytic space. Then S is Moishezon iff S carries a positive torsion free analytic coherent sheaf θ.*

Proof. If S carries a positive torsion free sheaf θ, then θ is weakly positive in view of Theorem 1. Therefore S is Moishezon (see e.g. [1]).

Now if S is Moishezon, a main result in [1] tells us that S can be realized as an exceptional set of some 1-convex space X. Hence Theorem 5 will imply the existence of a positive coherent and torsion free sheaf θ on S. *Q.E.D.*

COROLLARY 7. (*Blowing down problem*).
Let S be a compact $\mathbb{C}$-analytic subvariety of some $\mathbb{C}$-analytic space X. Then S is exceptional iff there exists an ideal sheaf $J \subset 0_X$ supporting on S such that the analytic coherent sheaf J/J^2 is positive.

Proof. Let us assume that there exists an ideal sheaf $J \subset 0_X$ supporting on S such that J/J^2 is positive. Hence, following Theorem 1, J/J^2 is weakly positive. A result in [2] tells us that S is exceptional.

Now if S is exceptional, it follows from an analytical version of Chow's Lemma [5] and Theorem 5 above that there exists an analytic coherent ideal sheaf $J \subset 0_X$ such that J/J^2 is positive. *Q.E.D.*

Comments. (i) An algebraic version of Corollary 7 is well known in Algebraic Geometry (see e.g. J. Mazur; Conditions for the existence of contractions in the category of algebraic spaces. Trans. AMS 209 (1975) p. 259–265).

(ii) Let X, S and J be as in Corollary 7. Let $\pi : \tilde{X} \to X$ be the blowing up of X with respect to J and let $\tilde{S}$ be an effective Cartier divisor on $\tilde{X}$ determined by $I := J \cdot 0_{\tilde{X}}$. Now let E be the linear fibre space determined by $\theta := J/J^2$, let $\mathbb{P}(E)$ be the projectivization of E and let $L(E)$ be the tautological line bundle on $\mathbb{P}(E)$.

Hence in view of (*) in Theorem 1, it follows readily that

θ is positive iff $L(E)$ is weakly negative (†)

Now let $\tilde{L}$ be the line bundle on $\tilde{X}$, determined by $\tilde{S}$, then one check easily that:

$$\tilde{S} \subset \mathbb{P}(E) \cap \tilde{X} \quad \text{and} \quad \tilde{N} := \tilde{L} \mid \tilde{S} \simeq L(E) \mid \tilde{S} \tag{††}$$

Recently, T. Peternell gave an another proof for Corollary 6 (Über exzeptionelle Mengen; Manus. Math., 37 (1982) p. 19–26). However, his proof contains a serious gap. He claimed that (Satz 3) θ is weakly positive iff $\tilde{N}$ is weakly negative. But this is simply not true; in fact, it follows from (††) that the weak negativity of $\tilde{N}$ only implies the weak negativity of $L(E)$ when restricted to $\tilde{S}$ which is, in general, merely a subspace (or a primary component, in the sense of [9]) of $\mathbb{P}(E)$. Consequently, in view of (†), this does not imply the positivity for θ!

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(i) Since this paper was submitted, results in Sections 1 and 3 were obtained independently by V. Ancona, by using purely algebro geometric techniques. See V. Ancona: Faisceaux amples sur les espaces analytiques. Trans. A.M.S. 274 (1982) p. 89–100; also see Ancona/Tomassini: Modifications analytiques. Lec. notes in Math. Vol. 943, Springer-Verlag.

(ii) The author also would like to express his sincere thanks to the referee for his thoughtful comments and suggestions.

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Note added in proof: By using the same notations as in comments ii) above, let $\tau : \tilde{N} \to C$ be the blowing down of $\tilde{N}$ along $\pi|_{\tilde{S}} : \tilde{S} \to S$ and let I be the ideal sheaf in 0_C determined by S. In order to patch up his previous gap (loc. cit.) Peternell (Erratum et Addendum zu der Arbeit: Über exzeptionelle Mengen; Manus. Math., 42 (1983) p. 259–263) proposed another proof which is based on the following erroneous claim, among others:

$$0_C \simeq I^{k+1} \oplus (\oplus I^{\nu}/I^{\nu+1}) \quad \text{for} \quad \forall k \in \mathbb{N} \tag{$*$}$$

He referred to Grauert's paper [2] for a proof of $(*)$ which does not exist!