Solutions to Markov Chain Exercises: From Die Rolls to Hypercubes

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Exercise 1: Identification of Markov Chains

Let $(Y_i)_{i\geq 1}$ be i.i.d. variables uniform on $\{1, \dots, 6\}$.

1. Running Maximum: $X_n = \max\{Y_1, \dots, Y_n\}$

Markov Chain: Yes.
Reasoning: $X_{n+1} = \max(X_n, Y_{n+1})$. The future state depends only on the current maximum X_n and the new independent roll Y_{n+1}.
Transition Matrix P:

$$P_{ij} = \begin{cases} i/6 & \text{if } j = i \\ 1/6 & \text{if } j > i \\ 0 & \text{if } j < i \end{cases}$$

2. Sliding Maximum: $X_n = \max\{Y_{n-1}, Y_n\}$

Markov Chain: No.
Reasoning: The state X_n does not capture enough information about the last roll Y_n. *Counter-example:* If $X_n = 5$, it could be that $Y_n = 1$ (making $X_{n+1} = \max(1, Y_{n+1})$) or $Y_n = 5$ (making $X_{n+1} = \max(5, Y_{n+1})$). These lead to different transition probabilities.

3. Counting Occurrences: $X_n = \sum_{i=1}^{n} \mathbf{1}_{Y_i=6}$

Markov Chain: Yes.
Reasoning: $X_{n+1} = X_n + \mathbf{1}_{Y_{n+1}=6}$. The next value is determined solely by the current count and the independent result of the next roll.

4. Index of Last Six: $X_n = \max\{k \leq n : Y_k = 6\}$

Markov Chain: Yes.
Reasoning: $X_{n+1} = n + 1$ if $Y_{n+1} = 6$, and $X_{n+1} = X_n$ otherwise. The transition depends only on the current index X_n and Y_{n+1}.

Exercise 2: Random Walks on Graphs

1. The Segment (2 Vertices)

Invariant Measure: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The unique invariant measure is $\pi = (1/2, 1/2)$.
Convergence: The chain is periodic (period 2). The law μ_n oscillates between δ_1 and δ_2 and **does not converge**.

2. The Triangle (3 Vertices)

Invariant Measure: The graph is regular; $\pi = (1/3, 1/3, 1/3)$.
Convergence: $p_n = P(X_n = 1) = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n$. As $n \to \infty$, $p_n \to 1/3$. The law **converges** because the graph is aperiodic (contains an odd cycle).

3. The Cube (8 Vertices, Hypercube Q_3)

Matrix Structure: Although the cube is vertex-transitive, its transition matrix is **not circulant**. A circulant matrix requires a cyclic ordering that the 3D connectivity of a cube does not support.
Invariant Measure: Since the graph is 3-regular, $\pi = (1/8, \ldots, 1/8)$.
Convergence: The cube is a **bipartite graph**. This implies the chain is periodic with period 2. Consequently, the law of X_n **does not converge** to the stationary distribution; it oscillates between the two partitions of the bipartite set.

1 Exercise 1: Sequence Analysis

Let $(Y_i)_{i\geq 1}$ be i.i.d. random variables with law $\mathcal{U}(\{1,\dots,6\})$.

1.1 Running Maximum: $X_n = \max\{Y_1,\dots,Y_n\}$

Markov Property: $X_{n+1} = \Phi(X_n, Y_{n+1})$ where $\Phi(x,y) = \max(x,y)$. Since Y_{n+1} is independent of $\sigma(Y_1,\dots,Y_n)$, it is independent of $\sigma(X_1,\dots,X_n)$. Thus, (X_n) is a Markov Chain.

Transition Matrix P: For $i,j \in \{1,\dots,6\}$:

$$P_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} \mathbb{P}(Y_{n+1} \leq i) = \frac{i}{6} & \text{if } j = i \\ \mathbb{P}(Y_{n+1} = j) = \frac{1}{6} & \text{if } j > i \\ 0 & \text{if } j < i \end{cases}$$

1.2 Sliding Maximum: $X_n = \max\{Y_{n-1}, Y_n\}$

Proof of Non-Markovian Nature: Consider $\mathbb{P}(X_3 = 1 | X_2 = 5)$.

- If $(Y_1, Y_2) = (5,1)$, then $X_2 = 5$. Then $X_3 = \max(1, Y_3)$, so $\mathbb{P}(X_3 = 1) = \mathbb{P}(Y_3 = 1) = 1/6$.
- If $(Y_1, Y_2) = (1,5)$, then $X_2 = 5$. Then $X_3 = \max(5, Y_3)$, so $\mathbb{P}(X_3 = 1) = 0$.

Since $\mathbb{P}(X_3 = 1 | X_2 = 5, X_1 = x)$ depends on the past (the value of Y_1 hidden in X_1), the Markov property fails.

1.3 Counting Sixes: $X_n = \sum_{i=1}^n \mathbf{1}_{Y_i=6}$

Analysis: This is a random walk on $\mathbb{N}$ with $X_{n+1} = X_n + \xi_{n+1}$ where $\xi \sim \text{Bernoulli}(1/6)$. **Transition Probabilities:** $P_{i,i} = 5/6$ and $P_{i,i+1} = 1/6$.

1.4 Last Occurrence Index: $X_n = \max\{k \in \{1,\dots,n\} : Y_k = 6\}$

Recurrence: $X_{n+1} = (n+1)\mathbf{1}_{Y_{n+1}=6} + X_n \mathbf{1}_{Y_{n+1}\neq 6}$. **Transition:** $P(X_{n+1} = n+1 | X_n = i) = 1/6$ and $P(X_{n+1} = i | X_n = i) = 5/6$.

2 Exercise 2: Simple Random Walks on Graphs

2.1 The Two-Vertex Graph (K_2)

Matrix: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **Invariant Measure:** $\pi = (1/2, 1/2)$ is the unique solution to $\pi P = \pi$. **Law of** X_n: $\mu_n = \mu_0 P^n$. If $\mu_0 = (1,0)$, then $\mu_n = (1,0)$ for n even and $(0,1)$ for n odd. **Non-convergence:** The eigenvalue $\lambda = -1$ (since $\det(P - \lambda I) = \lambda^2 - 1$) implies a period of 2.

2.2 The Triangle Graph (K_3)

Matrix: $P = \frac{1}{2}(J - I)$ where J is the all-ones matrix. **General Law:** $p_n = \mathbb{P}(X_n = 1)$. By $p_{n+1} = \frac{1}{2}(1 - p_n)$, we solve the characteristic equation to find:

$$p_n = \frac{1}{3} + (p_0 - \frac{1}{3})(-\frac{1}{2})^n$$

Convergence: Since $|-1/2| < 1$, $p_n \to 1/3$. The chain is aperiodic (odd cycle).

2.3 The Cube Graph (Q_3)

Adjacency: Vertices $V = \{0,1\}^3$. $x \sim y$ if Hamming distance $d_H(x,y) = 1$. **Transition Matrix:** $P = \frac{1}{3}A$, where A is the adjacency matrix. **Bipartiteness:** Let V_0 be vertices with even bit-sum and V_1 with odd bit-sum. P only allows transitions $V_0 \to V_1$ and $V_1 \to V_0$. **Spectrum:** The eigenvalues of the Cube random walk are $\{1, 1/3, -1/3, -1\}$.

- The eigenvalue -1 confirms the chain is bipartite (period 2).
- Consequently, μ_n does not converge to $\pi = (1/8, \dots, 1/8)$ but oscillates between V_0 and V_1.

3 The Markov Property: A Formal Definition via Conditional Independence

4 Introduction

The Markov property is the mathematical formalization of the idea that the *future* is independent of the *past*, provided that the *present* state is known. While it is often expressed using conditional probabilities, its most rigorous restatement is through the lens of conditional independence.

5 Formal Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ be a stochastic process taking values in a countable state space S.

5.1 The Conditional Probability View

Traditionally, we state that for all $n \geq 0$ and all states $i_0, i_1, \dots, i_n, j \in S$:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i_n)$$

5.2 The Conditional Independence View

In terms of independence, (X_n) is a Markov chain if for every n, the **future** is conditionally independent of the **past** given the **present**.

Formally, let:

- **Past:** $\mathcal{P}_n = \sigma(X_0, \dots, X_{n-1})$ (the σ-algebra generated by the history).
- **Present:** $\sigma(X_n)$ (the information at the current time).
- **Future:** $\mathcal{F}_n = \sigma(X_{n+1}, X_{n+2}, \dots)$ (the σ-algebra of all future events).

The Markov property holds if for any future event $B \in \mathcal{F}_n$ and any past event $A \in \mathcal{P}_n$:

$$\mathbb{P}(B \cap A \mid X_n) = \mathbb{P}(B \mid X_n)\mathbb{P}(A \mid X_n)$$

This implies that once X_n is fixed, no event in the history A can provide any additional predictive power regarding the occurrence of the future event B.

6 Application and Violation

6.1 Markovian Systems (Independence of Increments)

Most Markov chains are constructed as $X_{n+1} = f(X_n, \epsilon_{n+1})$, where ϵ_{n+1} is an "innovation" or "noise" term. If ϵ_{n+1} is independent of $(X_0, \dots, X_n)$, the sequence is Markovian.

- **Example:** In a random walk on a cube or K_4, the choice of the next neighbor is a random variable ϵ_{n+1} that is totally independent of the path taken to reach the current vertex.

6.2 Non-Markovian Systems (Dependency Link)

The "Sliding Maximum" $X_n = \max(Y_{n-1}, Y_n)$ fails because the variable Y_n is shared between the present X_n and the future $X_{n+1} = \max(Y_n, Y_{n+1})$. Because Y_n is also partially constrained by the past state $X_{n-1} = \max(Y_{n-2}, Y_{n-1})$, the past and future remain "linked" by Y_n. Knowing only the maximum X_n does not fully decouple this link, thus violating the requirement for conditional independence.