# Billiards

If $\Delta$ is a hyperbolic triangle with angles $\pi/p, \pi/q, \pi/r$, and $1/p + 1/q + 1/r < 1$, then the reflexions in the sides of $\Delta$ generate a group of isometries of the hyperbolic plane isomorphic to the triangle group $\Delta(p, q, r)$.

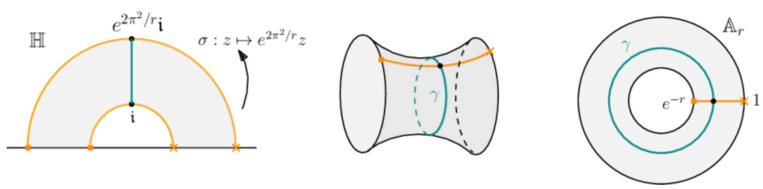
The reflexions are a canonical set of generators and provided $p, q, r < \infty$ the Cayley graph of the group with respect to these generators is quasi-isometric to the hyperbolic plane (this is Svarc-Milnor lemma).

This means that the translation length $\ell(\gamma)$ of an element of the triangle group and the distance in the Cayley graph $\|\gamma\|_W$ are comparable:

$$c_1\|\gamma\|_W \leq \ell(\gamma) \leq c_2\|\gamma\|_W$$

where $\|\gamma\|_W$ is the word length of $\gamma$ with respect to the canonical generators. This is basically just compacity of the fundamental domain of the triangle group.

Let $A$ be the covering space of $\mathbb{H}^2$ corresponding to the subgroup triangle group generated by $\gamma$. It is easy to see that $A$ is an annulus, with a single closed geodesic which is the lift of $\gamma$.




(This is stolen from Yiling Yang's paper).

Now the word length can be computed by counting the number of translates $g(\Delta) \subset A$ of the fundamental domain that the lift of $\gamma$ intersects. Let $D$ denote the diameter of $\Delta$ then each $g(\Delta)$ is contained in a regular neighborhood of radius $D$ of the lift of $\gamma$ to $A$.

Thus

$$\|\gamma\|_W \cdot \mathrm{Area}(\Delta) = \mathrm{Area}(\cup g(\Delta)) \leq \mathrm{Area}(N(D))$$

where $N(D)$ is the regular neighborhood of radius $D$ of the lift of $\gamma$ to $A$.

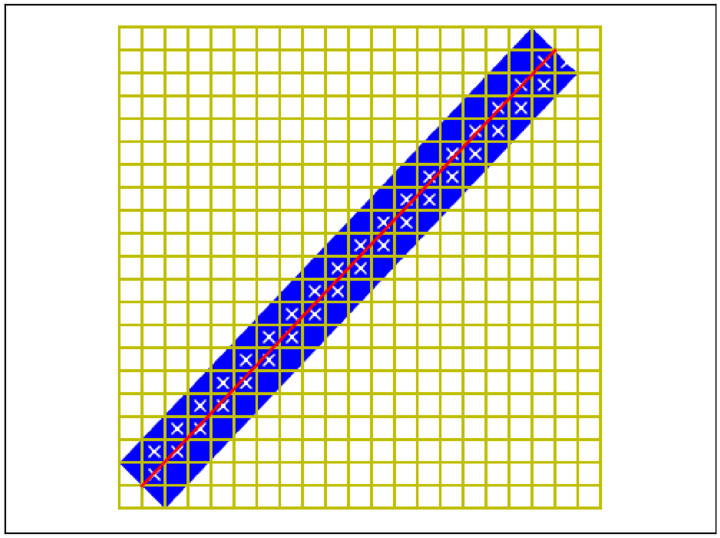
The area of $N(D)$ is $\ell(\gamma) \cdot 2\sinh(D)$ (I think).

This is just an adaptation of the same argument in Euclidean geometry. Below is a picture of a billiard path (red curve) in the standard square torus. The fundamental domains are the squares and the ones with a cross meet the billiard path. They are contained in the blue rectangle which is a lift of the regular neighborhood in the annulus (cylinder) covering space.

The sides of the rectangle have lengths

- $\ell(\gamma)$

- and $2D = 2\sqrt{2}$, ie twice the diameter of the square and



(I made this myself)

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## References

As for references about statistics of lengths:

**Mark Policott** - old policott sharp - recent cantrell policott

**Moira Chas** - experimental - also interesting