ToFu : Topologie eFfective et calcUl

Projet équipe-action
Mathematics : from Fundamentals to Applications

Research topics	:	Geometry and toplogy Combinatorial optimization
Institutions	:	Institut Fourier (IF), Universite Grenoble Alpes G-SCOP, Grenoble INP
Teams involved	:	Geométrie et topologie (IF) Optimisation Combinatoire (G-SCOP)
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Table des matières

1 Presentation of the team

The team consists of eight members in total (7 tenured faculty and one PhD student : Joanny Perret) du projet belonging to two laboraties on the Grenoble campus :
Institut Fourier (IF) and G-SCOP.

Laboratoire	Équipe	Nom
IF	Géométrie et Topologie	**Martin Deraux**
G-SCOP	Optimisation Combinatoire	**Louis Esperet**
G-SCOP	Optimisation Combinatoire	**Francis Lazarus**
IF	Géométrie et Topologie	**Greg McShane**
IF	Géométrie et Topologie	**Anne Parreau**
G-SCOP	Optimisation Combinatoire	**Joanny Perret**
IF	Géométrie et Topologie	**Andrea Seppi**
G-SCOP	Optimisation Combinatoire	**Matěj Stehlìk**

2 Synopsis

The purpose of the present project is to develop a synergy between colleagues with a backgrounds in fundamental mathematics and others coming from more an applied/computer science background. Such a synergy is prerequisite for progress on a certain important problems related to properties of topological spaces such as surfaces but more generally manifolds with particular geometric structures and the associated moduli spaces. The problems we intend to study are those which have both combinatorial and analytical aspects. Past experience shows that this kind of problem has often lead to rich exchanges between those who have a purely theoretical background on one side and a more applied/numerical one on the other. Fundamental to our approach is the investigation of topological constructions using computer software.

We believe that by combining the unique knowledge base of theoretical mathematicians and computer scientists present on the Grenoble campus we can make significant progress on a number of geometric problems which have an analytic, combinatorial and algorithmic nature. This is not merely a superficial sales pitch but something we have come to realise collectively over the past few years through participating together in seminars at IF and G-SCOP, in evaluation committees for our PhD students and in a number of workshops, summer schools and conferences.

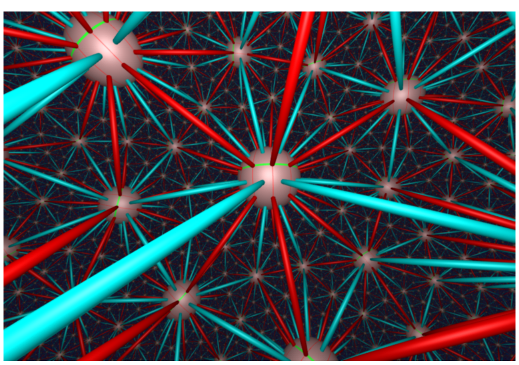
One of the things that makes our proposal particularly relevant at this time is that the theme of the Master of Mathematics (M2R) for 2020-21 will be geometric group theory, including its combinatorial aspects with applications to topology. In particular, Martin Deraux, Anne Parreau and Francis Lazarus will all be giving courses as part of this Master. So accepting our proposal would be an excellent opportunity to reinforce what is at the moment a nascent synergy between team members in quite distinct laboratories and academic programs. The possibility to recrute students from these Masters courses by offering fundings would be undoubtedly be good for the attractivity of Grenoble as a place to study both in the short and long term.

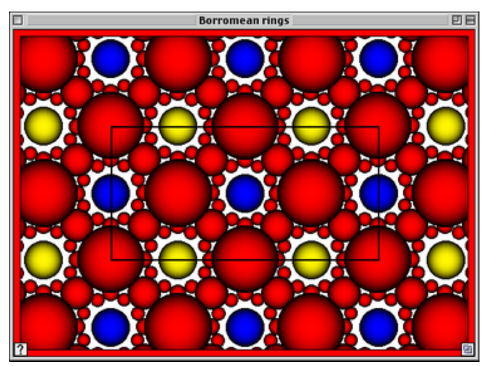
3 Research Context

We aim to study ways to describe the geometry of space from two points of view :
— combinatorial/algorithmic
— analytical
This double headed approach has been central to the field of low dimensional topology since its inception. It has proven spectacularly successful particularly now that computers allow us to make very accurate calculations rapidly as we shall now see.

3.1 Thurston and SnapPea

The dual analytic/combinatorial approach was pioneered by the likes of Felix Klein, Henri Poincaré and Max Dehn in the late 19th century and early 20th century. They studied surfaces and 3 manifolds using differential equations, combinatorial constructions and transformation groups. Of particular note was the proof of the classification of triangulated surfaces in 1907 by Dehn which uses a clever algorithm. Nearly 100 years later Bill Thurston used complex analytic techniques to prove existence theorems for hyperbolic structures on 3 manifolds [33] – that is to show that the fundamental group of the 3 manifold is isomorphic to a discrete group of isometries of hyperbolic 3 space. Thurston's results are is based on fixed point theorems on *Teichmuller space* and are non constructive. Though very elegant they were certainly not *effective*. With the advent of computers there were many new possibilities for constructing and exploring Thurston's manifolds. Jeffrey Weeks, as part of his 1985 doctoral thesis supervised by Thurston, created a remarkable piece of software called SnapPea which tries to find a hyperbolic structure on the complement of a knot. To do this it starts by finding a good decomposition of the space into tetrahedra analogous in some sense to Dehn's triangulation of a surface. It then associates to each tetrahedron a complex number and constructs a set of nonlinear equations of complex variables from the adjacencies between the tetrahedra. A solution of this system gives a complete hyperbolic metric on the space. SnapPea uses an iterative method - essentially Newton's method - to search for solutions. Once a solution is found many invariants of the manifold can be computed : volume, Chern-Simons invariant, etc. One can also understand relations to other 3-manifolds which are related to it by doing *Dehn surgery*.





Views from the inside of a hyperbolic manifold in SnapPy and SnapPea.

SnapPea has proven to be a very useful piece of software. It has been used to verify conjectures in base cases, investigate statistics of topological invariants and visualize 3 manifolds. Its architecture has allowed it to be extended to include additional functionality [25] and ported to a variety of operating systems. One important developpment is a version called Snap written by Oliver Goodman [11] which uses the number theory package Pari incorporating high precision arithmetic and number theoretic functions. This allows one to *certify* results which is not possible using ordinary floating point arithemetic.

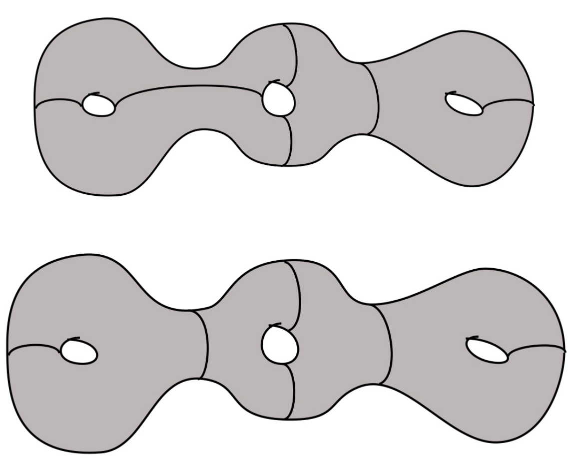
There is certainly a demand for quality free software packages like SnapPea which would allow researchers to study geometric objects in geometric topology and we will give several examples where our expertise could reply to this demand below.

3.2 Surface theory

It is perhaps surprising but due to work of Mostow [28] if there is a hyperbolic structure of finite volume on a 3 manifold like the ones that SnapPea finds then it is unique up to isometry. This is in contrast to the variety of such structures on a surface. In fact the key to proving Thurston's existence theorem is a profound study of these structures, the space of all Riemann surfaces of a fixed genus the so-called *Teichmueller space*. The Teichmueller space is naturally an analytic object. But the theory of how sufaces vary is extremely rich requiring many tools coming from different branches of mathematics - algebra, topology and differential geometry but also combinatorial and algorithmic analysis.

An important topic is understanding the group of automorphisms of Teichmueller space which turns out to be isomorphic to the diffeomorphisms of a surface up to isotopy. This is a group which is difficult to study using algebra as it is not known to be a linear group in general. Instead one studies it via its actions on spaces constructed from the combinatorics of curves on the surface. Of course it acts on Teichmüller space but also on Thurston's measured lamination space [10] and Harvey's curve complex. The former is a finite dimensional piecewise linear manifold which one should think of as the completion of the space of simple closed curves on the surface. The latter is an infinite graph associated to a surface : a vertex is a homotopy class of simple curves and vertices are joined by an edge if the corresponding curves can be chosen to be disjoint. The curve graph is known to be a negatively curved space in the sense of Gromov [27]. There are many interesting questions concerning the metric geometry/combinatorics of this graph [5] and several algorithms related to them : the Bestvina–Handel algorithm, Bell–Webb algorithm [4], Leasure and Shackleton algorithms.

In fact there is another graph which models the geometry of Teichmueller space much better – the pants graph. This is a graph whose vertices are homotopy classes of $3g-3$ disjoint simple closed curves and two verrtices are adjacent if they differ by a single curve. The complement of the curves is a union of(triply connected domains which are called by the more familiar term *pants*. A Jeff Brock in his thesis proved that the pants graph with the obvious distance function and Teichmueller space with the Weil-Petersson metric are quasi isometric. There are many deep relation between the Weil-Petersson geometry of Teichmueller space, lengths of simple closed curves on surfaces [8], harmonic maps between surfaces [35] and even hyperbolic volumes of 3 manifolds obtained from suspensions of surface diffeomorphisms [21].



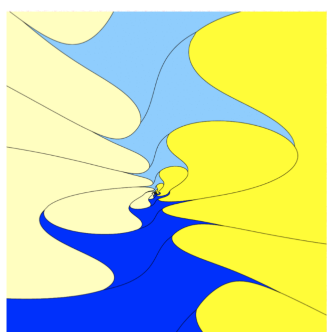
A pair ofadjacent vertices in the pants graph.

So studying the mapping class group of a surface requires an understanding of the lengths of closed simple geodesics which in turn requires both analytic tools coming from hyperbolic geometry and Teichmüller theory but has also a fundamentally algorithmic approach. This fits well with the goal of our project – combining the knowledge of theoretical mathematicians and computer scientists in order to study questions in geometry which have both a combinatorial and algorithmic nature.

3.3 Geometric structures in general

There are many variations on Thurston's theory of surfaces and his hyperbolization for 3-manifolds. These are often formulated in terms of finding a good (discrete, faithful) representations of the fundamental group into the automorphism group of some space and one then tries to study their properties by analogy with what is known for Teichmueller space [24]. One such variation is the construction of spherical CR structures, i.e. geometric structures modeled on the boundary at infinity of the complex hyperbolic plane. There is no invariant metric for the corresponding group action, and this is closer in spirit to the theory of (flat) conformal structures.

One of the amazing things about Thurston's theorem is that there is a simple combinatorial/algebraic condition which allows one to decide whether a 3-manifold admits a hyperbolic structure. In contrast, a characterization of 3-manifolds (compact or not) that admit such a spherical CR structure remains an important open problem see for example [13]. Many non-hyperbolic manifolds are known to admit such structures ; one basic but important class of examples is given by quasi-Fuchsian groups, obtained by deformation from the obvious inclusion of the group $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SU}(2,1)$. It is also known that many hyperbolic 3-manifolds also admit a spherical CR structure. This is much more surprising, since there is no embedding of $\mathrm{SL}(2,\mathbb{C})$ in $\mathrm{SU}(2,1)$! So another important open question is the characterization of (closed) 3-manifolds that admit a spherical CR structure. Partial progress on these questions has been recently by Schwartz [31], Deraux and Falbel [14], Parker and Will [29]. One hopes that their work can be applied more generally by, for example, applying similar techniques in a more systematic fashion. This will involve using an interesting mix of combinatorial, topological and geometrical methods.



View from the inside of a complex hyperbolic manifold

Another important class of discrete subgroups of $\mathrm{SU}(2,1)$ (or more generally $\mathrm{SU}(n,1)$, or even more general Lie groups) is given by holonomy groups of suitable geometric structures on moduli spaces In particular, there is an important family of groups, studied by Deligne-Mostow [12] and Thurston [32], which arise from the moduli spaces of flat metrics on the sphere with fixed cone angle singularities. This family is of special interest since it contains many of the known non-arithmetic lattices of $\mathrm{SU}(n,1)$. Some parts of the Deligne-Mostow-Thurston construction were extended by Veech [34] to cover flat metrics on surfaces of positive genus. Recently Ghazouani and Pirio [19] have pushed Veech's work a little further by but the situation is still far from being completely understood – for example it is not known which holonomy groups are discrete, and if discrete, which are arithmetic.

3.4 Structural graph theory and combinatorial maps

Between 1983 and 2004, Robertson and Seymour have written a series of 20 papers adding up to more than 500 pages to prove the Wagner conjecture saying that every minor closed class of graphs is characterized by a finite family of forbidden minors [1]. An important step in the proof is to describe the structure of the graphs excluding a fixed minor. This structure theorem [30] states that such graphs can be obtained from cellular embeddings of graphs in surfaces of bounded genus, to which one applies some standard operations such as clique-sums [23]. A cellular embedding of a graph is called a combinatorial map (or combinatorial surface). Combinatorial maps are thus central objects in graph theory not only as discrete models of surfaces but also as the main components of the graph structures. Understanding the structure of combinatorial maps is thus a fundamental question from both the topological and combinatorial viewpoints. Spectral graph theory, based on graph Laplacians, has been successful to obtain important structural properties such as the Cheeger inequality [9]. The Laplacian is also a main tool to analyse the geometry of hyperbolic surfaces. In particular, there is a striking connection to the length spectrum through the Selberg trace formula. Developing an analogous theory on combinatorial surfaces is a challenging objective.

4 Objectives

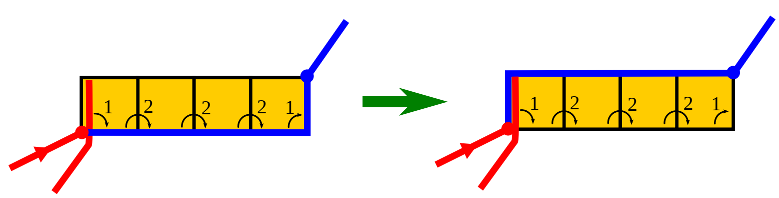
We now describe the main research challenges that we intend to tackle.

4.1 Combinatorial length and rigidity

Calculate the length spectrum of a combinatorial surface, that is a graph which is cellularly embedded in a surface. For such a graph one associates a length to each homotopy class of loops which is just the minimal number of edges traversed by a circuit in the graph representing it. The combinatorial length spectrum is the collection of all these lengths. One can study the inverse spectral problem : to what extent does this spectrum determine the surface ? Exploring the connection of the length spectrum with graph Laplacian spectrum could be a promising angle of attack. Another intriguing question relates to the action of the mapping class group on this combinatorial length spectrum. Indeed, an automorphism of the surface permutes the homotopy class of curves, hence the length spectrum. Does the discrete nature of the graph metric induce specific properties of this action ? In other words, can we learn the graph structure from this action ?

4.2 Algorithmic study of lengths

Lazarus and Damiand have added a package to the C++ library CGAL which tests if two curves are in the same free homotopy class. The algorithm test has linear complexity : in about 30 seconds one can test two paths of combinatorial length of 30 million on a surface of genus 100. The next step is to implement a test to see if a path is homotopic to a simple loop. There is already an algorithm, of “quasi”-linear complexity [15], which just needs to be implemented.



A switch operation in the algorithm [15] that tests if a curve is homotopic to a simple curve.

With this one can count the number of simple curves of length less than a given threshold and hopefully check the numerical values of certain important constants which appear in the work of Mirzakhani.

4.3 Curve complex and pants graph

Curver is a program for performing calculations in the curve complex which implements the Bell–Webb algorithm to determine the Nielsen–Thurston type of a mapping class [2, 3]. The algorithm runs in polynomial time but the constants involved currently make this implementation impractical. There is a sister program called Flipper [1] which does calculations for surfaces with boundary. There seems to be room to improve this situation and implement this situation. Namely, finding a way of making the algorithm practical by solving the problem of large constants and extending the work to cover closed surfaces without boundary [26]. This is closely related to the questions concerning the geometry of the pants graph and the Weil-Petersson geometry of Teichmueller space as defined above.

On the analytic side we hope to extend the inequalities between entropy of diffeomorphisms and hyperbolic invariants [21] to other problems and relate this to combinatorial problems of curve graphs etc. On the effective side we will develop and use programs to explore the combinatorics of the various graphs that one can associate to a surface.

4.4 General geometric structures

Here we have two very concrete goals

4.4.1 CR SnapPy

As mentionned above would like to applying the techniques developed by Schwartz, Deraux, Falbel, Parker, Will, in a more systematic fashion so as to understand which 3-manifolds admit a spherical CR structure. Obviously since Week's SnapPea software has proved so useful in understanding hyperbolic structures on 3-manifolds it would useful to have a similar tool to explore CR structure. So we hope to develop a spherical CR version of SnapPy, which would allow the user to study, given a fundamental domain in complex hyperbolic space for a give discrete subgroup of $\mathrm{SU}(2,1)$, properties of the manifold at infinity.

4.4.2 Cellulations of moduli space

To better understand the variations on the Deligne-Mostow-Thurston construction of Veech and Ghazouani-Pirio it is important to understand the geometry of tha associated moduli space. We expect that a concrete understanding of their construction in terms of triangulations (or more general cell-decompositions) should given insight into these open questions. If this produces complex hyperbolic (or more general geometric) structures on some well-chosen moduli spaces, we expect that combinatorial methods will be crucial to understand the fine topological/geometric/dynamical structure of these moduli spaces.

There are natural cellulations of the moduli space hyperbolic (complex) structures. There are various constructions for these due to amongst others Penner, Epstein-Penner [16], Bowditch-Epstein [7] Kontsevich [22]. Underlying these constructions are the lengths of certain curves on the surfaces and embedded trivalent graphs. The tri valent graph is dual to "the best possible" triangulation of the surface in the Epstein-Penner construction and this triangulation is obtained via the calculation of a convex hull (which is computationally very cheap). One hopes that interaction between team members could lead to an extension of these constructions to the space of flat moduli space and the development of a computationally efficient implementation.

4.5 Harmonic maps

A very important tool in the study of Teichmüller space, namely the space of deformation of Riemann surface structures, is represented by harmonic maps [35] [20]. Harmony is software developed in C++ by Loustau and Gaster [18], which computes harmonic maps between two

1. `https://markcbell.github.io/build/html/software.html`

hyperbolic surfaces, described in terms of their Fenchel-Nielsen coordinates, by means of a discrete heat flow method. In other words, a hyperbolic surface is approximated by a finite graph which encodes its geometry, then one searches a discrete harmonic map by a recursive procedure stopping when a prescribed tolerance is reached. Very interesting developments would be constituted by developing programs to compute harmonic maps with target a metric tree, or a quasi-Fuchsian three-manifold, or to compute minimal Lagrangian maps between closed hyperbolic surfaces [6].

5 Méthodologie envisagée

Grenoble has a very strong tradition both as a center for in low dimensional topology as well as for discrete mathematics and computer science (including graph theory and some aspects of algorithms). We believe that the group of researchers from the Institut Fourier and G-SCOP involved in this project posess a blend of complementary skills perfectly adapted to working together on the objectives above. The G-SCOP team *Optimisation Combinatoire* is certainly one of the leading groups in graph theory with an emphasis on topological aspects of graphs on surfaces [17]. Similarly the topology team at the Institut Fourier is a leading group in low dimensional topology especially the theory of surfaces with its many facets - dynamical, variational and spectral, interactions with classical and quantuum algebra.

Each of the challenges in the above scientific description of the project could certainly yield at least one subject for a PhD in the context of this project. Students would not only have the chance to work with members of our team but would have interactions with the many seminar speakers in the geometry and topology seminars at the Institut Fourier and the possibility of short visits to other teams with whom the team members have scientific collaborations (ICJ Lyon, IMJ Paris, University of Luxembourg).

6 Résultats attendus

The first challenge concerning the length spectrum of a combinatorial surface sits at the interface of graph theory and surface topology. Progress on this should provide a new point of view that fits perfectly in the structural approach of Robertson and Seymour. We are quite confident when we say that there are few research groups with the same level of competence in this domain.

The other challenges are related to implementations and experimental mathematics, which are again strengths of the mathematical/computer science community Grenoble. It is our intention to develop and distribute open source softwares and packages.

7 Principales étapes envisagées

8 Request budget and scientific justification

We plan to hire one PhD students and one post-doctoral student. Both the PhD student and the postdoctoral student will be supervised jointly by members of the IF and the G-SCOP. Ideally the PhD student would come from the M2R at the IF in the academic year 2020-21. Depending on the background of the student, the PhD thesis will focus on the questions raised in sections ??. The post-doctoral student is expected to have skills in both theoretical mathematics and algorithms to be involved in the research direction raised in section (??).

In addition to this, we plan to organize a workshop gathering all members of the team and which could be entitled "???" and several meetings or one-day workshops with related ANR projects (???). We also intend to organize a bimonthly all day meeting at which all members would participate to reinforce the cohesion of the TOFU team and facilitate the exchange of ideas.

A joint meeting of the SMF and AMS will take place in June 2021 and we would like to organise a special session. This would be the perfect occasion to establish new contacts and identify promising questions where geometry and computer science may fruitfully interact. In the same time it would dramatically increase the visibility of our initiative and be an efficient way to attract very good post-doctoral students.

— Financement de thèse : 100.000 €
  Le sujet portera sur ...
— Post-doc (1 an) 50.000 €
  ...
— Gratifications de stage : 6.000 €
  correspondant à 3 stages M2 de 5 mois.
— Invitations de chercheurs extérieurs : 6.000 €
  ...
— Missions : 20.000 €
  (1200€par personne et par an)
— Matériel : 4.000 €
— Fonctionnement : 4.000 €
— Congrès-colloques : 10.000 €
— TOTAL demandé : 200.000 €

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