Let

$$P = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that

$$Q = SPS^{-1} = \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}.$$

Let $G_m < \mathrm{SL}(2,\mathbb{Z})$ denote the group generated by P, Q. One has

$$QP = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix} = \begin{pmatrix} 1-m^2 & m \\ -m & 1 \end{pmatrix}$$

which admits a square root in $P\mathrm{SL}(2,\mathbb{Z})$ that is:

$$C^2 = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} m^2-1 & -m \\ m & 1 \end{pmatrix}.$$

The matrices

$$C = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}, C' = \begin{pmatrix} 0 & 1 \\ -1 & -m \end{pmatrix}$$

generate a free group G_m^* and

$$CC' = \begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix}.$$

Lemma 4.1. *The groups G_m and G_m^* have a common fundamental domain*

- *The quotient $\mathbb{H}/G_m$ is a surface with two cusps and a single geodesic boundary component of length $2\ell(\alpha)$.*
- *The quotient $\mathbb{H}/G_m*$ is a (degenerate) pair of pants with a single cusp and boundary consisting of a pair of closed geodesics each of lengths $\ell(\alpha)$.*
- *The orbit of i under G_m and G_m^* is the same.*

Lemma 4.2. *For every integer k there is a bicuspidal geodesic on $\mathbb{H}/G_m$ which has λ-length k^2m^2+1, and is invariant under the involution induced by S.*

Proof. Consider the image of i under the sequence of elements of G_m given by:

$$Q^kP = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -km & 1 \end{pmatrix} = \begin{pmatrix} 1-km^2 & m \\ -km & 1 \end{pmatrix}$$

As before the imaginary part of $Q^kP.i$ is given by

$$\frac{1}{k^2m^4+1}$$

So there is a half infinite geodesic in $\mathbb{H}$ joining the ideal point ∞ to $Q^kP.i$. The projection of this segment to $\mathbb{H}/G_m$ together with its image under the involution form a bicuspidal geodesic of the required λ-length.