SMALL FOLIATIONS AT INFINITY

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1. Introduction

The central objects of study in this article are *quasi-fuchsian hyperbolic* 3*-manifolds* and pairs of *measured foliations at infinity* that appear naturally in their asymptotic boundary. In brief, the central goal of this article will be to prove that any pair of *small* measured foliations that fill up a hyperbolic surface can be realised on the boundary components at infinity of a quasi-fuchsian metric as the *horizontal foliation of the schwarzian differential*, provided that the quasi-fuchsian structure in consideration is in a neighbourhood of the Fuchsian locus. In particular, we want to prove:

Theorem 1.1. *$\exists V$ an open neighbourhood $\mathcal{F}(\Sigma) \subset \mathcal{QF}(\Sigma)$ such that the map $\mathfrak{F} : \mathcal{QF}(\Sigma) \to \mathcal{MF}(\Sigma) \times \mathcal{MF}(\Sigma)$ is homeomorphism between $V \setminus \mathcal{F}(\Sigma)$ and its image.*

In order to describe the meaning of the words in the statement above, we start of by fixing notations for the different objects of study in this article along with their role in formulating the theorem. Let Σ be a smooth orientable surface of genus $g \geq 2$. This equips it with a complete finite volume hyperbolic metric along with the structure of a *Riemann surface*, i.e 1-dimensional complex curves. This duality will be of importance through out the article. One way to realize this is that the group of isometries of $\mathbb{H}^2$ is $PSL_2(\mathbb{R})$ which in turn acts biholomorphicaly by möbius transformations on the upper half planem, while the other implication comes by the virtue of *Riemann Uniformisation Theorem.* Σ thus can be identified with the quotient of $\mathbb{H}^2$ under the action of a discrete subgroup $\Gamma < PSL_2(\mathbb{R})$.

The group of orientation preserving isometries of $\mathbb{H}^3$, being $PSL_2(\mathbb{C})$ thus contain the $PSL_2(\mathbb{R})$ and this in turn allows us to see the same Γ as a discrete subgroup of $PSL_2(\mathbb{C})$ acting properly discontinuously and freely on $\mathbb{H}^3$. The quotient in this case is a 3-manifold M, topologically homeomorphic to $\Sigma \times \mathbb{R}$, equipped with a finite volume complete hyperbolic metric, which we call *Fuchsian metric.* A fuchsian metric on M thus contain a totally geodesic copy of Σ seen as the image of $\mathbb{H}^2 \subset \mathbb{H}^3$ under the quotient map.

We call $\mathcal{F}(\Sigma)$ to be the set of all isotopy classes of fuchsian metrics on M. Thus $\mathcal{F}(\Sigma)$ is isomorphic to space isotopy classes of hyperbolic structures on Σ, which is the classical Teichmúller space, denoted henceforth as $\mathcal{T}(\Sigma)$. The duality between hyperbolic structures and riemann surface structure on Σ allows us to alternately define $\mathcal{T}(\Sigma)$ as the *equivalence class of conformal metrics* on Σ. A *conformal metric* on Σ is an assignment of charts into $\mathbb{C}^2$ where in every chart the metric takes the form $\lambda(z)|dz|$ for some function $\lambda : \Sigma \to (0, +\infty)$. Two conformal metrics c and c' are thus called equivalent if $\exists f : \Sigma \to (0, \infty)$ such that $c = e^{2f}c'$. The two interpretations of $\mathcal{T}(\Sigma)$ can be interchanged easiy by the fact that $\exists$ *a unique hyperbolic metric in each conformal class.* Thus choosing one is equal to choosing the other. For ease we will notify these two versions of $\mathcal{T}(\Sigma)$ by $\mathcal{T}_h(\Sigma)$ and $\mathcal{T}_c(\Sigma)$ respectively.

A way to classify the subgroup Γ is via it's *limit set* Λ_Γ which is the accumulation point of the orbit of a point under the group action and can be shown forms a proper circle on $\partial_\infty \mathbb{H}^3 \cong \mathbb{CP}^1$. M can be identified with $(\mathbb{H}^3 \setminus \Lambda_\Gamma)/\Gamma$. Note that the two boundary components of M at infinity, henceforth denoted $\partial^+_\infty M$ and $\partial^-_\infty M$, are essentially the quotient of the connected components of $\mathbb{CP}^1 \setminus \Lambda_\Gamma$ and Λ_Γ being a proper circle in the fuchsian case implies that the boundary components have the same hyperbolic structure as $\mathbb{H}^2/\Gamma$. Alternately, the induced metrics on $\partial^\pm_\infty M$ are in the same conformal class.

Quasi-fuchsian metrics are metrics obtained by *quasi-conformally* deforming a fuchsian metric and $\mathcal{QF}(\Sigma)$ will denote the set of all isotopy classes. In this case, the limit set Λ_Γ is a *jordan curve* in stead of a proper circle. This provides the connected components of $\partial^\pm_\infty M$ with two distinct hyperbolic metrics (h_+, h_-) and thus two distinct conformal classes $([c_+], [c_-])$. Moreover, given such a pair, $\exists$ a unique quasi-fuchsian metric realising it. To this end we have the classical *Ber's simultaneous uniformisation theorem* a version of which can be:

Theorem 1.2 (Bers). *The map $\mathfrak{U} : \mathcal{QF}(\Sigma) \to \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ sending a quasi-fuchsian metric $\mathfrak{M} : g \mapsto (h_+, h_-)$ (or equivalently $([c_+], [c_-]) \in \mathcal{T}_c(\Sigma) \times \mathcal{T}_c(\Sigma)$) is a homeomorphism. Moreover for every such pair in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$, the choice g is unique.*

Remark 1.3. *The space $\mathcal{QF}(\Sigma)$ has been show to carry an almost complex structure, namely at each point $g \in \mathcal{QF}(\Sigma)$ the tangent bundle splits as $T(\mathcal{QF}(\Sigma)) \cong \mathcal{T}(\Sigma) \bigoplus i_\chi \mathcal{T}(\Sigma)$ [Bon96], where i_χ is inherited from the character variety $\chi = \mathsf{Hom}(\pi_1(\Sigma), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$ whose connected component of the identity is in turn $\mathcal{QF}(\Sigma)$. This fact will be of significance in many ways for us.*

We now turn our attention to another classical object of study related to quasi-fuchsian structures g on M, called their *convex core*, denoted henceforth as $\mathcal{CC}(M)$. It can be defined to be the quotient of the convex hull of Λ_Γ under the action of Γ. Equivalently, it is also the smallest non-empty closed convex subset o M. Being topologically $\Sigma \times \mathbb{R}$ once more, the boundary components $\partial^\pm_\infty \mathcal{CC}(M)$ thus are two copies of Σ with distinct induced hyperbolic metric (m_+, m_-). Thurston conjectured the following:

Conjecture 1.4 (Thurston). *The map $\mathfrak{M} : \mathcal{QF}(\Sigma) \to \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ sending a quasi-fuchsian metric $\mathfrak{M} : g \mapsto (m_+, m_-)$ is a homeomorphism. Moreover for every such pair in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$, the choice of g is unique.*

So far, only existence has been shown [BO04] however uniqueness remains elusive in full generality.
$\partial^\pm_\infty \mathcal{CC}(M)$ further carry the structure of a *geodesic measured lamination* on them. We denote this pair as (λ_+, λ_-) and they belong to the set $\mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma)$, where $\mathcal{ML}(\Sigma)$ is the space of equivalence classes of measured laminations(check [Col79] for exact definitions). Essentially they can be thought as the closure of the set of simple closed geodesics on Σ with a *transverse measure*. The surfaces $\partial^\pm_\infty \mathcal{CC}(M)$ are in fact, *pleated along* along these laminations, meaning that the induced metric hyperbolic metric $m_\pm$ is singular along $\lambda_\pm$. Thurston thus again conjectured the following:

Conjecture 1.5 (Thurston). *The map $\mathfrak{B} : \mathcal{QF}(\Sigma) \to \mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma)$ sending a quasi-fuchsian metric $\mathfrak{B} : g \mapsto (\lambda_+, \lambda_-)$ is a homeomorphism. Moreover for every such pair in $\mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma)$,the choice of g is unique.*

[BO04] tackled this question and succesfully resolved in the case when the laminations are rational, i.e contain no dense leaves, and proved existence when the laminations are irrational. A crucial step in their proof is to classify the image of the map in terms of *filling up property of the pair* $\lambda_\pm$ which states that the two laminations realised on the same surface are transverse to each other at all points of intersection. Assuming this in the paper [Bon02] bonahon proved that the map $\mathfrak{B}$ is a *local homeomorphism* without subsuming the condition for rational/irrational laminations.

Theorem 1.6 (Bonahon). *There exists an open neighbourhood U of $\mathcal{F}(\Sigma)$ in $\mathcal{QF}(\Sigma)$, such that $\mathfrak{B}$ is a homeomorphism between $U \setminus \mathcal{F}(\Sigma)$ and its image.*

We want to replicate this exact statement for *measured foliations at infinity* of quasifuchsian manifold M. These are *measured foliations* $(\mathsf{f}_+, \mathsf{f}_-)$ that appear on $\partial^\pm_\infty M$ and are defined as *horizontal foliation of the schwarzian differential*. To define this notion, we need to consider the *complex projective structure* or $\mathbb{CP}^1$ structure associated to $\partial^\pm_\infty M$. This is equivalent to an assignment of charts into $\mathbb{CP}^1$ with transition maps being locally restrictions of action of $PSL_2(\mathbb{C})$. Let $\mathcal{CP}(\Sigma)$ denote the equivalence classes of such structures.
The components $\partial^\pm_\infty M$ inherit these structures naturally being quotient of open domains of $\mathbb{CP}^1$ under discrete subgroup of $PSL_2(\mathbb{C})$. Let us call them (ρ_+, ρ_-) and they lie in the same conformal class as $[c_\pm]$. Geometrically these are obtained by *grafting* an annulus along leaves of the pleating lamination $\lambda_\pm$ and the resulting $\rho_\pm$ is nothing but a metric which inherits the induced metric $m_\pm$ from $\partial^\pm \mathcal{CC}(M)$ with euclidean cylinders attached along the leaves of the pleating lamination.

The schwarzian derivative or differential is a quadratic differential associated to the unique locally univalent map $u : \Delta \to \mathbb{CP}^1$ uniformising $\rho_\pm$, where $\Delta \subset \mathbb{C}$ is an open domain. Locally in terms of u, it is expressed as $((\frac{u''}{u'})' - \frac{1}{2}(\frac{u''}{u'})^2)dz^2$, but we will abbreviate this data by denoting the schwarzians associated to $\partial^\pm_\infty M$ as (σ_+, σ_-). Intuitively the schwarzian measures the amount by which the a map of the type u as above differs from a mobiús transformation.

The space $\mathcal{CP}(\Sigma)$ described before forms a bundle $\mathcal{CP}(\Sigma) \to \mathcal{T}(\Sigma)$ with the fibers being parametrized by the schwarzian derivatives, while the map in the forward direction comes from the fact that any complex projective structure has a canonical underlying complex structure. This schwarzian thus descends to a *holomorphic quadratic differential* defined on this complex structure. The space of holomorphic quadratic differential $\mathcal{QD}(\Sigma, [c])$ on the other hand can be identified with the cotangent bundle $T^*_{[c]}\mathcal{T}(\Sigma)$ over $\mathcal{T}(\Sigma)$ at the point $[c] \in \mathcal{T}(\Sigma)$. $\mathcal{CP}(\Sigma)$ thus forms a sub-bundle of $\mathcal{QD}(\Sigma, [c])$ and the dimension of both is $6g-6$ by an application of the *Riemann-Roch theorem.* [Dum] Serves as an excellent survey for further details.

The *foliations at infinity* are defined as the *horizontal measured foliation* of the schwarzian at the two ends and were first termed by [KS08]. By horizontal foliation associated to a holomorphic quadratic differential q, we mean the integral curves of vector fields for which the real part of q, $\Re q = 0$. The transverse unit measure associated to such a foliation is given in local coordinates by the imaginary part, i.e, $\Im|\sqrt{q}||dz|$. We will call $\mathsf{f} := \mathrm{hor}_{[c]}q$, when f is the horizontal foliation of q on the surface $(\Sigma, [c])$ and we will call $q^{\mathsf{f}}_{[c]} \in T^*_{[c]}\mathcal{T}(\Sigma)$ to be the differential *realsing* f on $(\Sigma, [c])$.

A result of pivotal importance for this article is the following theorem of [HM79] that set up a bijective correspondence between $\mathsf{f} \in \mathcal{MF}(\Sigma) \leftrightarrow q^{\mathsf{f}}_{[c]} \in \mathcal{QD}(\Sigma, [c])$

Theorem 1.7 ([Gar84]). *Every $\mathsf{f} \in \mathcal{MF}(\Sigma)$ can be realised as the horizontal foliation of a holomorphic quadratic differential on a Riemann surface $(\Sigma, [c])$. Precisely, for a given $\mathsf{f} \in \mathcal{MF}(\Sigma)$, the map $q^{\mathsf{f}} : \mathcal{T}(\Sigma) \to \mathcal{QD}(\Sigma, [c])$, sending $[c] \to q^{\mathsf{f}}_{[c]}$ is a homeomorphism.*

Analogously, we can also define the notion of *vertical foliation* due to $q \in \mathcal{QD}(\Sigma, [c])$ as the integral curves of vector fields for which $\Im q = 0$ and the transverse unit measure associated to such a foliation is given in local coordinates by $\Re|\sqrt{q}||dz|$. If g is the vertical foliation of $q \in \mathcal{QF}(\Sigma, [c])$ then we will write $\mathsf{g} := \mathrm{ver}_{[c]}(q)$. The above theorem can be reinterpreted in terms of these in the exact same way.

This rich interplay between foliations and differentials is further strengthened by the following result of importance where every pair of foliations (f, g) filling up a surface are shown to be realizable as the horizontal and vertical foliations of a holomorphic quadratic differential. This is inspired by the obvious converse where the horizontal and vertical foliations of a quadratic differential fill up a surface by definition.

Theorem 1.8 (Gardiner-Masur). *A pair of foliation (f, g) fill up Σ iff $\exists! [c] \in \mathcal{T}(\Sigma)$ and $q \in \mathcal{QD}(\Sigma, [c])$ such that (f, g) can be realized as $(\mathrm{hor}_{[c]}(q), \mathrm{ver}_{[c]}(q))$. Moreover, for every every $c \in [c]$ the choice of q is also unique.*

An important corollary that comes out of the proof of the above theorem is that, $[c]$ is again the *unique critical point* for the function $ext(\mathsf{f}) + ext(\mathsf{g}) : \Sigma \to \mathbb{R}$, where $ext_{[c]}(\mathsf{f})$ denotes the extremal length of f over the class $[c]$, which can be defined as the infrimum of the length of f measured in the class $[c]$. When f is $\mathrm{hor}_{[c]}(q^{\mathsf{f}})$, we have the expression:

$$ext_{[c]}(q) = \int_{(\Sigma, [c])} ||q^{\mathsf{f}}||$$

and the corollary is an interpretation of *Gardiner's formula*, to be elaborated upon in the next section.

The map $\mathfrak{F} : \mathcal{QF}(\Sigma) \to \mathcal{MF}(\Sigma) \times \mathcal{MF}(\Sigma)$ introduced in the beginning can thus be defined to send $g \in \mathcal{QF}(\Sigma) \mapsto (\mathsf{f}_+, \mathsf{f}_-) \in \mathcal{MF}(\Sigma) \times \mathcal{MF}(\Sigma)$, the pair of it's measured foliations at infinity.

One motivation to replicate [BO04] result in terms of measured foliation at infinities is because of the properties that the boundary of $\mathcal{CC}(M)$ and $\partial_\infty M$ seem to share along with the geometric objects that appear on them. For a tabulated summary of this consult [KS08]. Moreover, conjecture 1.3 above due to Thurston can be seen to be in parallel to Ber's theorem. The question thus that can be further asked about these foliations $(\mathsf{f}_+, \mathsf{f}_-)$ can be thought of as mirror of conjecture 1.4 above. In particular, we can ask if the map $\mathfrak{F}$ is also a homeomorphism and produces a bijection between the two spaces it bridges. Our aim thus is to answer this question locally á la [BO04].

1.1. **Sketch of proof and article outline.** Essentially we want to apply inverse function theorem to $\mathfrak{F}$ in a neighbourhood of $\mathcal{F}(\Sigma)$ to construct U. First, restricted to the neighbourhood of $\mathcal{F}(\Sigma)$, and replicating the condition from [BO04], we classify the image of $\mathfrak{F}$ to be the pair of foliations $(\mathsf{f}_+, \mathsf{f}_-)$ that fill up. Section 3 is then devoted to identifying the point g_0 for which a path of quasi-fuchsian metrics g_t starting from $\mathcal{F}(\Sigma)$ at $t = 0$ has the image $(t\mathsf{f}_+, t\mathsf{f}_-)$ for a pre-specified pair of filling foliations $(\mathsf{f}_+, \mathsf{f}_-)$, at first order and for t small enough. This identification will be done in terms solely in terms of $(\mathsf{f}_+, \mathsf{f}_-)$ so chosen.

Section 4 then applies inverse function theorem at this point to construct such paths which realize a prescribed pair in the sense above. The way this will be done is to construct certain submanifold of $\mathcal{QF}(\Sigma)$ termed $\mathcal{W}_+^{\mathsf{f}_+}$ and $\mathcal{W}_-^{\mathsf{f}_-}$, which are defined to carry the foliations $t\mathsf{f}_+$ at $\partial_\infty^+ M$ and $t\mathsf{f}_+$ at $\partial_\infty^- M$ respectively, for $t \geq 0$. We will then study the intersection of $\mathcal{W}_+^{\mathsf{f}_+} \cap \mathcal{W}_-^{\mathsf{f}_-}$ and show that their boundaries intersect transversely when one considers the ambient space to be $\widetilde{\mathcal{QF}(\Sigma)}$, the blow-up of $\mathcal{QF}(\Sigma)$ at $\mathcal{F}(\Sigma)$. Transversality will then allow us to claim $\widetilde{\mathcal{W}_+^{\mathsf{f}_+}} \cap \widetilde{\mathcal{W}_-^{\mathsf{f}_-}}$ is locally diffeomorphic near $\partial\widetilde{\mathcal{W}_+^{\mathsf{f}_+}} \cap \partial\widetilde{\mathcal{W}_-^{\mathsf{f}_-}}$ to $\widetilde{\mathbb{R}^2}$, the blow-up of $\mathbb{R}^2$ at the origin. near This will allow us to construct the paths realising a prescribe pair of filling foliations at first-order by pulling-bacl the diagonal $(t, t) \in \widetilde{\mathbb{R}^2}$.

Section 5 then deals with constructing this U explicitly and showing $\mathfrak{F}|_U$ is a homeomorphism onto it's image, which we will see to be essentially the union of rays in $\mathcal{MF}(\Sigma) \times \mathcal{MF}(\Sigma)$ formed by scaling a pair $t(\mathsf{f}_+, \mathsf{f}_-), t \geq 0$ that fill up.

Section 2 develops some basic properties concerning the sections q^{f} which will be used later and serves as a primer on facts regarding measured foliations on riemann surfaces required for our purpose. In particular, for a pair (f, g) which fill up we introduce a line $\mathsf{P}(\mathsf{f}, \mathsf{g})$ defined as the set of path $t \mapsto [c_t] \in \mathcal{T}(\Sigma)$ which are the unique critical points for $ext(\sqrt{t}\mathsf{f}) + ext(\mathsf{g})$ for each $t \geq 0$. This will be used to identify the intersection of $\partial\widetilde{\mathcal{W}_+^{\mathsf{f}_+}} \cap \partial\widetilde{\mathcal{W}_-^{\mathsf{f}_-}}$ in the blown up space. An additional remark of independent interest will be to see that $\mathsf{P}(\mathsf{f}, \mathsf{g})$ is an alternate description for geodesics in $\mathcal{T}(\Sigma)$ with respect to the *Teichmüller metric.*

2. Measured foliations on a Riemann surface

Let Σ be a surface as before with genus $g \geq 2$ and is equipped with a complete hyperbolic metric. Σ also has a natural structure of a Riemann surface, which corresponds to the hyperbolic one through the Riemann uniformisation map. So here after when we say "Σ is a riemann surface" we mean the couple $(\Sigma, [c])$, where c is the conformal class of the Riemann surface structure. This implies the existence of local charts on Σ to domains in $\mathbb{C}$ and transition map which are biholomorphic maps. A *holomorphic quadratic differential* q on a riemann surface $(\Sigma, [c])$ is a symmetric covariant 2-tensor which in local coordinates can be seen as $\phi(z)dz^2$. The space of holomorphic quadratic differential denoted as $Q(\Sigma, [c])$ thus forms a bundle over $\mathcal{T}(\Sigma)$ of dimension $(6g - 6)$. $Q(\Sigma, [c])$ can also be identified with $T^*_{[c]}\mathcal{T}(\Sigma)$.

Quadratic differentials naturally enrich Σ with the geometry of it's trajectories. Precisely, given $q = \phi(z)dz^2$ on (Σ, c) we can define the *horizontal foliation* (resp. *vertical foliation*)of q, which we denote by $\mathrm{hor}(q)$ (resp. $\mathrm{ver}(q)$) as the set of curves $\gamma : I \to (\Sigma, c)$, such that $\phi(\gamma(t))(\gamma(t)')^2 > 0$(resp. < 0). Observe that on $\mathbb{C}$ with the standard metric and the quadratic differential dz^2, this condition gives me all the horizontal lines (or vertical lines) on $\mathbb{C}$, thus inspiring the nomenclature. This is independent of the choice of local coordinates as well. Also we have the simple but important observation:

Lemma 2.1. $\mathrm{hor}(q)$ *is equivalent to* $\mathrm{ver}(-q)$ *over the same riemann surface* (Σ, c).

That is to say that changing the sign of the quadratic differential exchanges the vertical and horizontal foliations associated to it. This will be of significance as the pair $(\mathrm{hor}(q), \mathrm{ver}(q))$ satisfy the topological property of *filling up* Σ.

Definition 2.2. *A pair of measured foliations* (f, g) *are said to fill up* Σ *if every non zero measured foliation on* Σ *has non-zero geometric intersection number with at least one of* f *or* g*. Equivalently the condition can be that every component of* $\Sigma - (\mathsf{f} \cup \mathsf{g})$ *is either a topological disc bounded by the union of finitely many geodesic arcs, or a topological annulus bounded on one side by union of finitely many geodesic arcs.*

Observe that, the pair $(\mathrm{hor}(q), \mathrm{ver}(q))$ automatically satisfies the condition of filling up Σ.

A series of author studied these objects, namely Hubbard-Masur, Gardiner Masur. One primary question they answer is whether under a fixed riemann surface structure, any measured foliation can be uniquely realized as a horizontal trajectory of some quadratic differential. Furthermore, we also ask whether a pair (f, g) can be realized as the horizontal and vertical trajectory of the same q under the same assumption. The answers are affirmative and can be summarized as follows:

Theorem 2.3. *The map* $\mathrm{hor} : Q(\Sigma, [c]) \to \mathcal{MF}(\Sigma)$*, sending a quadratic differential* q *to its horizontal measured foliation* $\mathrm{hor}(q)$ *is a homeomorphism.*

So, any measured foliation f can be uniquely realised as the horizontal foliation of a quadratic differential q^{f} on a given riemann surface. Thus associated to each element of $\mathcal{MF}(\Sigma)$ is a section $q^{\mathsf{f}} : \mathcal{T}(\Sigma) \to T^*\mathcal{T}(\Sigma)$, sending $[c] \in \mathcal{T}(\Sigma)$ to $q^{f}_{[c]} \in T^*\mathcal{T}(\Sigma)$. Same can be said if we pick the unique hyperbolic metric m in the conformal class $[c]$ and through the dual interpretation of $\mathcal{T}(\Sigma)$ we get a section of $T^*\mathcal{T}(\Sigma)$ associating m to q^{f}_{m}.

Likewise, we can also take the section $q^{-\mathsf{f}} : \mathcal{T}(\Sigma) \to T^*\mathcal{T}(\Sigma)$, which associates to f the quadratic differential whose vertical foliation is the same. Observe that, $-q^{\mathsf{f}} = q^{-\mathsf{f}}$.

Theorem 2.4. *A pair* (f, g) *of measured foliations on* Σ *is filling if and only there is a conformal structure* c *and a holomorphic quadratic differential* $q \in (\Sigma, c)$ *such that* (f, g) *are respectively measure equivalent to the vertical and horizontal foliations of* q*. Moreover, the conformal class* $[c]$ *is determined uniquely and for each* $c \in [c]$ *the quadratic differential* q *is also unique.*

An observation of importance for this article is a key step in the proof of the above theorem, where we show that $[c]$ as above is also a critical point for the function $ext(\mathsf{f}) + ext(\mathsf{g}) : \mathcal{T}(\Sigma) \to \mathbb{R}_{\geq 0}$, i.e, $[c]$ is the point of minima for the above function.

We can now combine the statements above to get an equivalent formulation of the filling condition (f, g) in terms of the sections $(q^{\mathsf{f}}, q^{\mathsf{g}})$.

Proposition 2.5. *Let* (f, g) *be a pair of measured foliations that fill up* Σ *and* $(q^{\mathsf{f}}, q^{-\mathsf{g}}) : \mathcal{T}(\Sigma) \to T^*\mathcal{T}(\Sigma)$ *be the associated sections defined above. Then the sections intersect transversely exactly at one point in* $T^*\mathcal{T}(\Sigma)$ *whose projection onto* $\mathcal{T}(\Sigma)$ *is the unique minima of the function* $ext(\mathsf{f}) + ext(\mathsf{g})$*.*

Proof. First we will show that if the sections intersect then they do so uniquely at one point and as a result (f, g) fill up Σ.

We will make use of a formula of Gardiner relating the differential of the extremal length function associated to a measured foliation and the quadratic differential realising it as a horizontal foliation. Let $\mathsf{f} \in \mathcal{MF}(\Sigma)$, then for a choice of conformal class $[c] \in \mathcal{T}(\Sigma)$, we get a unique quadratic differential $q^{\mathsf{f}}_{[c]}$ such that f is measure equivalent to the horizontal foliation of $q^{\mathsf{f}}_{[c]}$. We define the extremal length $ext(\mathsf{f})$ associated to f as the L^1 norm of $q^{\mathsf{f}}_{[c]}$ on the riemann surface $(\Sigma, [c])$. That is

$$ext_{[c]}(\mathsf{f}) = \int_{(\Sigma,[c])} ||q^{\mathsf{f}}_{[c]}||$$

Gardiner's formula [Gar84] then states that

$$\mathsf{d}ext_{[c]}(\mathsf{f})(\tau) = 2Re(\langle q^{\mathsf{f}}_{[c]}, \tau\rangle), \forall \tau \in T_{[c]}\mathcal{T}(\Sigma).$$

Essentially, Gardiner's formula sends the section $\mathsf{d}ext_{[c]}(\mathsf{f})$ to the section $q^{\mathsf{f}}_{[c]}$. So given, $(\mathsf{f},\mathsf{g}) \in \mathcal{MF}(\Sigma)$, the sections $q^{\mathsf{f}}_{[c]}, q^{-\mathsf{g}}_{[c]}$ intersect iff the sections $\mathsf{d}ext_{[c]}(\mathsf{f})$ and $\mathsf{d}ext_{[c]}(-\mathsf{g})$ do.
Notice that $\mathsf{d}ext_{[c]}(-\mathsf{g})(\tau) = 2Re(\langle q^{-\mathsf{g}}_{[c]}, \tau\rangle) = 2Re(\langle -q^{\mathsf{g}}_{[c]}, \tau\rangle) = -\mathsf{d}ext_{[c]}(\mathsf{g})(\tau)$.
So the condition can be interpreted as to finding $[c] \in \mathcal{T}(\Sigma)$ such that $\mathsf{d}ext_{[c]}(\mathsf{f})(\tau) = -\mathsf{d}ext_{[c]}(\mathsf{g})(\tau)$ $\iff$ $\mathsf{d}ext_{[c]}(\mathsf{f})(\tau) + \mathsf{d}ext_{[c]}(\mathsf{g})(\tau) = 0$ i.e, $[c]$ is a critical point of the function $ext(\mathsf{f}) + ext(\mathsf{g})$ as stated above. This critical point exists (miyachi log-plurisub) and is unique. We call this point of minima $[c_0]$.
Moreover, at the point $q^{\mathsf{f}}_{[c_0]} = -q^{\mathsf{g}}_{[c_0]}$ from Gardiner's formula, which is equivalent to the fact that (f,g) are $(\mathrm{hor}(q^{\mathsf{f}}_{[c_0]}), \mathrm{ver}(q^{q^{\mathsf{f}}_{[c_0]}}))$, and thus they fill from the theorem of Hubbard-Masur. This also guarantees the uniqueness of $q^{\mathsf{f}}_{[c_0]}$.
Now we show that the sections intersect transversely. Consider the map $\phi : \mathcal{T}(\Sigma) \to T^*\mathcal{T}(\Sigma)$, given by $\phi([c]) := q^{f}_{[c]} - q^{-g}_{[c]}$. At the point $[c_0]$ we thus $\phi([c_{[0]}]) = 0$. Let $d\phi : T\mathcal{T}(\Sigma) \to T(T^*\mathcal{T}(\Sigma))$. We need to show that $\forall \tau \in T_{[c_0]}\mathcal{T}(\Sigma)$, $d\phi(\tau) = T_{[c_0]}(q^{\mathsf{f}})(\tau) - T_{[c_0]}(q^{-\mathsf{g}})(\tau) = 0 \implies \tau = 0$.
Let us consider the equivalence classes of curves $\gamma : \mathbb{R} \to \mathcal{T}(\Sigma)$ passing through $[c_0]$ to define an element in $T_{[c_0]}\mathcal{T}(\Sigma)$. This generates equivalence classes of curves $(\gamma(t), q^{\mathsf{f}}_{\gamma(t)})$ in $T^*\mathcal{T}(\Sigma)$ passing through $([c_0], q^{\mathsf{f}}_{[c_0]})$, and thus define an element of the tangent space $T_{([c_0], q^{\mathsf{f}}_{[c_0]})}(q^{\mathsf{f}}_{[c]}) \subset T(T^*\mathcal{T}(\Sigma))$.
Observe that $\forall [c] \in \mathcal{T}(\Sigma)$, the sections $([c], q^{\mathsf{f}}_{[c]})$ gives us the transverse measure associated to f on $(\Sigma, [c])$, which in local coordinates are expressed by integrating the measure $|\Im\sqrt{q^{\mathsf{f}}_{[c]}(z)}dz|$. Thus the points $([c], q^{\mathsf{f}}_{[c]})$ are essentially a choice of transverse measure that you can associate to a given foliation f on Σ. The kernel of the measure on the other hand gives us the vertical foliation associated to $q^{f}_{[c]}$ or equivalently the horizontal foliation of $-q^{\mathsf{f}}_{[c]}$. A path $(c_{\gamma(t)}, q^{\mathsf{f}}_{\gamma(t)})$ that preserves the foliation f topologically on Σ nut changes the transverse measure associated to it.
If $\tau \in T_{[c_0]}\mathcal{T}(\Sigma)$ be such that $T_{[c_0]}(q^{\mathsf{f}})(\tau) = T_{[c_0]}(q^{-\mathsf{g}})(\tau)$,then $\exists \gamma(t) \in \mathcal{T}^*T(\Sigma)$ associated to τ along which both the foliations (f,g) and their transverse measures are preserved simultaneously. But since they fill up Σ, $(c_{[0]}, q^{\mathsf{f}}_{c[0]})$ is the unique such point in $T^*\mathcal{T}(\Sigma)$ which realizes this pair. So, such a path is not possible and we have that $\tau = 0$. Thus the intersections are transverse.

□

Remark 2.6. *The case where the foliations don't fill up is left out in the proposition as then the sections do not intersect and this follows from HM again.*

2.1. **A line in $\mathcal{T}(\Sigma)$.** We just noted that a pair of filling foliations (f,g) identifies a point in the cotangent space $(c_0, q^{f}_{[c_0]})$ in $T^*\mathcal{T}(\Sigma)$. In fact, just the point $[c_0]$ is enough to uniquely determine $q^{f}_{[c_o]} = -q^{g}_{[c_0]}$. We will denote the unique hyperbolic metric in $[c_0]$ by $\mathsf{p}(\mathsf{f},\mathsf{g}) \in \mathcal{T}(\Sigma)$.
We have the trivial observation that this point is uniquely determined by the first coordinate. I.e, if $\mathsf{p}(\mathsf{f},\mathsf{g}) = \mathsf{p}(\mathsf{f}',\mathsf{g})$, then $\mathsf{f} = \mathsf{f}'$.

$\mathbb{R}_{>0}$ has a natural action on $Q(\Sigma, [c])$ which sends every non-zero $q \in Q(\Sigma, [c]) \mapsto t^2 q, \forall t \in (o, \infty)$. We can thus define $Q^1(\Sigma, [c])$ to be quotient $(Q(\Sigma, [c]) - 0)/\mathbb{R}_{\geq 0}$ under this action. Clearly $Q^1(\Sigma, [c]) \cong UT^*_{[c]}\mathcal{T}(\Sigma)$. The next proposition is a similar result for the sections $\bar{q}^{\mathsf{f}}$, which are the image of q^{f} under the quotient map.
A simple computation in local coordinates shows that the horizontal trajectories of $t^2 q^{\mathsf{f}}$ on $(\Sigma, [c])$ are given by $t\mathsf{f}$, which is the topologically the same foliation as $\mathsf{f} = \mathrm{hor}(q^{\mathsf{f}}_{[c_0]})$ but with the transverse measure scaled by a factor of t. The section $\bar{q}^{\mathsf{f}}_{[c]}$ is thus an equivalence class of quadratic differential under the action of $\mathbb{R}_{\geq 0}$ which realizes $t\mathsf{f}$ as it's horizontal trajectories on $(\Sigma, [c])$. Note that this also implies $\mathsf{p}(\mathsf{f},\mathsf{g}) = \mathsf{p}(t\mathsf{f}, t\mathsf{g})$.
However, we will observe that the projection of the intersections of the two sections $\bar{q}^{\mathsf{f}}, \bar{q}^{\mathsf{g}}$ onto $\mathcal{T}(\Sigma)$ gives me a properly embedded 1-dimensional submanifold of $\mathcal{T}(\Sigma)$ given by the image of the map $t \mapsto \mathsf{p}(t\mathsf{f}, \mathsf{g})$. We will call this line $\mathsf{P}(\mathsf{f},\mathsf{g})$.

Proposition 2.7. *Let* (f, g) *be a pair of measured foliations that fill up* Σ*, then the projection of the intersection of the sections* $\bar{q}^{\mathsf{f}}$*,*$\bar{q}^{-\mathsf{g}} \subset UT^*\mathcal{T}(\Sigma)$ *onto* $\mathcal{T}(\Sigma)$ *is* $\mathsf{P}(\mathsf{f}, \mathsf{g})$*, which is a properly embedded* 1*-dimensional submanifold of* $\mathcal{T}(\Sigma)$*.*

Proof. We basically mimic (bonahon prop) with appropriate tweaks.
For $[c] \in \mathcal{T}(\Sigma)$ if, $\bar{q}^{\mathsf{f}}_{[c]} = \bar{q}^{-\mathsf{g}}_{[c]} \implies t^2 q^{\mathsf{f}}_{[c]} = q^{-\mathsf{g}}_{[c]} \implies q^{t\mathsf{f}}_{[c]} = q^{-\mathsf{g}}_{[c]}$ for some $t > 0$. From the preceding discussions this is equivalent to the fact that $[c]$ is the unique point of minima of the function $ext(t\mathsf{f}) + ext(\mathsf{g})$ subsuming that $(t\mathsf{f}, \mathsf{g})$ fill up Σ given (f, g) do as well. As $[c]$ isidentified with $\mathsf{p}(t\mathsf{f}, \mathsf{g})$ the projection of the intersections is along the line $\mathsf{P}(\mathsf{f}, \mathsf{g})$ as defined.

The fact that $\mathsf{P}(\mathsf{f}, \mathsf{g})$ is a submanifold will follow from the fact that the intersection of the sections $q^{t\mathsf{f}}, q^{-\mathsf{g}}$ is transverse at the point $(p(t\mathsf{f}, \mathsf{g}), q^{t\mathsf{f}}_{\mathsf{p}(t\mathsf{f},\mathsf{g})}) \in T^*\mathcal{T}(\Sigma)$ from the previous proposition. We thus have that the sections $\bar{q}^{\mathsf{f}}$ and $\bar{q}^{-\mathsf{g}}$ intersect transversely along the fiber of the line $\mathsf{P}(\mathsf{f}, \mathsf{g})$. The intersection is thus a submanifold of $UT^*\mathcal{T}(\Sigma)$ and it's projection onto $\mathcal{T}(\Sigma)$, being a local diffeomorphism, equips $\mathsf{P}(\mathsf{f}, \mathsf{g})$ with the structure of a submanifold.

To get the last bit about this line being properly embedded in $\mathcal{T}(\Sigma)$ it suffices to show that the map $t \to p(t\mathsf{f}, \mathsf{g})$ is proper. This is where we digress from Bonahon's proof. The quadratic differential realizing $(t\mathsf{f}, \mathsf{g})$ as it's horizontal and vertical foliations at the point $\mathsf{p}(t\mathsf{f}, \mathsf{g})$ also equips Σ with a flat metric in the same conformal class which call $\rho(t\mathsf{f}, \mathsf{g})$. Thus again it is enough to show that $t \to \rho(t\mathsf{f}, \mathsf{g})$ is a proper map. As local charts for this metric are given by the leaves of $(t\mathsf{f}, \mathsf{g})$, if $t \to \infty$ then $\rho(t\mathsf{f}, \mathsf{g})$ corresponds to a flat metric on Σ with charts that are infinitely stretched along f, thus it remains outside a bounded set. Thus the maps are proper and we prove our claim. □

3. Fuchsian Neighbourhood via Minimal Surfaces

Recall the map $\Phi : \mathcal{QF}(\Sigma) \to \mathcal{MF} \times \mathcal{MF}$, which we intend to show is a local homeomorphism around the fuchsian locus. Moreover recall that in the introduction we have already hinted at what the open neighbourhood around the fuchsian locus we are considering might be. We will approach this issue by considering a subset of quasi-fuchsian manifolds called "almost fuchsian" ,$\mathcal{AF}(\Sigma)$, which contain a unique immersed minimal surface with principal curvatures between $(-1, 1)$. Moreover,

Proposition 3.1 ([Sep16])**.** $\exists$ *an uniform neighbourhood* $\mathcal{N}(\mathcal{F}(\Sigma))$ *of the fuchsian locus* $\mathcal{F}(\Sigma)$ *in* $\mathcal{QF}(\Sigma)$ *such that* $\mathcal{N}(\mathcal{F}(\Sigma)) \subset \mathcal{AF}(\Sigma)$

Restricting ourselves thus to this very neighbourhood we will see during the course of this section that under this assumption, we have a filling up property associated to the two foliations at infinity, mimicking that of the bending laminations as observed in [BO04].

First we recall some background differential geometry adapter for the cause. If Σ is minimally immersion in a hyperbolic 3-manifold then it is equipped with the pair $(I, I\!I)$ where I is a smooth riemannian metric that Σ inherits from the ambient space and $I\!I$ is the second fundamental form of a minimal isometric immersion, which is a symmetric bilinear form on $T\Sigma$. Given this pair, we also have a unique self-adjoint operator $B : T\Sigma \to T\Sigma$, satisfying the relation $I\!I(x, y) = I(Bx, y) = I(x, By)$, where $x, y \in T_p\Sigma, \forall p \in \Sigma$. In a local orthonormal coordinate for I, B can be represented as the matrix $(I)^{-1}I\!I$. $tr_I(I\!I)$ and $\det_I(I\!I)$ are given by the trace and determinant of B respectively. Moreover, the eigenvalues of B being the direction of principal curvature associated to the immersion, $tr(B)$ is the mean curvature and $\det(B)$ is the extrinsic curvature of the immersion Σ.

Furthermore, these three quantities produce another map $I\!I\!I : T\Sigma \to T\Sigma$ such that $I\!I\!I(x, y) = I(Bx, By), \forall x, y \in T_p(\Sigma)$.
Associated to an immersion of Σ the datum $I, I\!I, I\!I\!I$ are called the *first,second and third* fundamental forms, and B is called the shape operator.
Fundamental theorem of surface theory [KS08] then provide us with the following condition of the pair $(I, I\!I)$.

(1) Trace of $I\!I$ with respect to I vanishes iff $I\!I$ is the real part of a holomorphic quadratic differential q over Σ associated to the complex structure it inherits from I.
(2) If (1) holds, then q is holomorphic iff $I\!I$ satisfies the Codazzi equation, $d^{\nabla} I\!I = 0$.
(3) If (1) and (2) both hold, then $(I, I\!I)$ is a minmal immersion of Σ iff the pair $(I, I\!I)$ satisfies the Gauss-Codazzi equations: $K_I = -1 + \det_I(I\!I)$, where K_I is the intrinsic curvature.

An upshort of the above discussion is that if we have a minimal immersion of a surface Σ into a hyperbolic 3-manifold, then the second fundamental form $I\!I$ can be realized as the real part of a holomorphic quadratic differential. Moreover the fact that $tr_I(I\!I) = 0$ is preserved under a conformal change to I. This allows us to consider the set of minimal immersions with prescribed conformal class.

As in [Uhl84] consider the ray $([m], t^2\Re(q)), t \geq 0$, where m is the hyperbolic metric representing the fixed conformal class of our choice and q is a holomorphic quadratic differential on $(\Sigma, [m])$ seen as an element of the $T^*\mathcal{T}(\Sigma)$.
For a fixed t let $u_t : \Sigma \to \mathbb{R}_{\geq 0}$ be such that the induced metric on Σ be given by $e^{2u_t}m$. Imposing the condition (3) from above by assuming these immersion are minimal gives us the following 1-parameter family of equations:

$$e^{-2u_t}(\Delta u_t - 1) = -1 + e^{-4u_t} t^2 \det_m(\Re(q))$$

Remark 3.2. *We have assumed here the following fact that if $g' = e^{2u}g$ be two conformal metric and $K_{g'}$ and K_g be the associated Gaussian curvatures, then $K_{g'} = e^{-2u}(\Delta u + K_g)$, where Δ is the geometric laplacian.*

An application of Implicit function theorem led [Uhl84] to conclude the following:

Theorem 3.3 (Uhlenbeck)**.** *There exists constant $\tau_0 > 0$, depending only on $([m], q)$ such that for $t \in [0, \tau_o]$ there is a minimal immersion of Σ with data $([m], t\Re(q))$ into a hyperbolic 3-manifold*

Recall that almost-fuchsian manifolds contain only one closed embedded minimal surface with principal curvature in $(-1, 1)$ ([KS08]). Uhlenbeck'e solution curve thus provide us in turn with a family of almost-fuchsian metrics parametrized with the data of it's minimal surface given by $([m], t^2\Re(q))$ for $t < \tau$. This can been made precise in [KS08] as follows:

Theorem 3.4. *For $0 < t < \tau$, $\exists$ an unique almost-fuchsian metric with an unique minimal surface whose $(I, I\!I)$ is prescribed by the pair $([m], t^2\Re(q)) \in T^*\mathcal{T}(\Sigma)$.*

Upshot of the above discussion is that the path $\beta_q : [0, \tau) \to T^*\mathcal{T}(\Sigma)$, where $\beta_q(t) \mapsto ([m], t^2\Re(q))$, provides us with a unique 1-parameter family of metrics in $\mathcal{AF}(\Sigma)$ with the data of the minimal surface being realized by the same pair.

[Uhl84] also notes that an almost-fuchsian manifold further admits a foliation by surfaces equidistant from the minimal surface. Owing to previous machinery developed for instance in [KS08], we have thus the notion of *first and second fundamental forms at infinity* at the boundary at infinity associated a minimal immersion.
To define them first we need the following lemma:

Lemma 3.5 ([KS08])**.** *Let Σ be a complete, oriented, smooth surface with principal curvatures in $(-1, 1)$ immersed minimally into an almost fuchsian manifold homeomorphic to $\Sigma \times (-\infty, \infty)$ and let $(I, I\!I, I\!I\!I, B)$ be the associated data of the immersion. Then $\forall r \in \mathbb{R}$ the set of point Σ_r from Σ is a smooth embedded surface with data $(I_r, I\!I_r, I\!I\!I_r, B_r)$ where :*

(1) $I_r(x, y) = I((cosh(r)E + sinh(r)B)x, (cosh(r)E + sinh(r)B)y)$
(2) $I\!I_r = \frac{1}{2}\frac{dI_r}{dr}$
(3) $B_r = (cosh(r)E + sinh(r)B)^{-1}(sinh(r)E + cosh(r)B)$

where E is the identity operator and Σ_r is identified to Σ through the closest point projection.

Remark 3.6. *Note that we do not provide an expression for $I\!I\!I_r$ as it is intrinsically obtained from the other three.*
The *fundamental forms at infinity* quantify the asymptotic behaviour of the quantities described above

as $r \to \infty$. In particular, it estimates the data at the conformal class at infinity of an almost fuchsian manifold M with respect to the unique embedded surface Σ it contains.

Formally, $I^* = \lim_{r\to\infty} 2e^{-2r} I_r$ and $I\!I^* = \lim_{r\to\infty}(I_r - I\!I\!I_r)$. However the lemma above gives us explicit formulae to express the same in terms of $(I, I\!I, I\!I\!I)$ and we use that to define:

Definition 3.7. *Adhering to the notations introduced above, the first fundamental form at infinity is given by the expression $I^* = \frac{1}{2}(I + 2I\!I + I\!I\!I)$ and the second fundamental form at infinity is given by $I\!I^* = \frac{1}{2}(I - I\!I\!I)$.*

The pair $(I^*, I\!I^*)$ satisfy a certain modified version of gauss codazzi equations at infinity which we will skip for now. The thing for importance to us is expression for curvature associated to I^* which we call K^*. [KS07] further provide us with an expression for it using the data of the immersed minimal surface:

Lemma 3.8. *With the notation as above,*

$$K^* = \frac{2K}{\det(E+B)} = \frac{-1+\det(B)}{1+\det(B)}$$

where K is the gaussian curvature of the minimal immersion of Σ.

Remark 3.9. *the second equality follows from the fact that $Tr(B) = 0$ the immersion being minimal and $(I, I\!I)$ satisfy the Gauss-Codazzi equations.*

Beware, nowhere does it state in the set-up above that I^* computed for an immersion is hyperbolic. In fact, [KS08] note that when multiplied by the correct conformal factor to take I^* to the unique hyperbolic metric in it's conformal class, the corresponding change in $I\!I^*$ is closely related to the schwarzian derivative σ associated to the conformal class. Precisely,

Theorem 3.10. *If I^* is hyperbolic, then $(I\!I^*)_0 = -\Re(\sigma)$, where $(I\!I^*)_0$ denotes the traceless part of the second fundamental form at infinity.*

Let us now focus on the path $\beta_q(t)$, $t \in (0, \epsilon)$ seen as a 1-parameter family of almost fuchsian matrics prescribed by the data $t \to ([m], t^2\Re(q))$ of the unique minimal surface it contains. Note that ϵ is so chosen such that it is $< \tau$ where τ is the constant depending on $([m], q)$ from Uhlenbeck's solution curve. Let us fix some notations: For a fixed $t > 0$, the data of the minimal surface Σ embedded in M almost-fuchsian will be expressed as $I_t = e^{2u_t} m$, $I\!I_t = t^2\Re(q)$. Simple computation shows that that $I\!I\!I_t = e^{-2u_t} t^2 m \det_m(\Re(q))$. Let the associated fundamental forms at infinity be $I^*_t, I\!I^*_t$ and K^*_t be the curvature. Further, let the schwarzian differentials associated to the two ends of $\beta_q(t)$ be called σ^t.

Remark 3.11. *Observe that since we keep the conformal class fixed, this path seen in $\mathcal{QF}(\Sigma)$ is orthogonal to $\mathcal{F}(\Sigma)$, i.e, it is along the direction normal to the fuchsian locus and the derivative at $t = 0$ of $\beta_q(t)$ gives us an element in the normal bundle $\mathcal{NF}(\Sigma)$. This will be of consequence later.*

Lemma 3.12. *I^*_t is hyperbolic at first order at $t = 0$.*

Proof. Recall that each u_t defined above satisfies the gauss equation,

$$e^{-2u_t}(\Delta u_t - 1) = -1 + e^{-4u_t} t^2 \det_m(\Re(q))$$

Consider the non-linear map $F : W^{(2,2)}(\Sigma) \times [0, \infty)] \mapsto L^2(\Sigma)$ given as :

$$F(u,t) = \Delta u - 1 + e^{2u} - e^{-2u} t^2 det_m(\Re(q))$$

For fixed t, straightforward computation gives the linearized operator and the Fréchet derivative of $F(u,t)$ as

$$L(u,t) = -\Delta - 2(e^{2u} + e^{-2u} t^2 det_m(\Re(q))$$
$$dF(u,t)(\dot{u}, \dot{t}) = \Delta \dot{u} + 2(e^{2\dot{u}} + t^2 e^{-2\dot{u}} det_m(\Re(q)) - 2t\dot{t}e^{-2u} det_m(\Re(q))$$

[Uhl84] observed that dF is onto when the eigenvalues of L are > 0 and this we can apply inverse function theorem. Also $L(0,0) > 0$ is implied by $F(0,0) = 0$. Thus we apply implicit function theorem to get the solution curve $\gamma : [0,\tau] \to W^{(2,2)}(\Sigma) \times [0,+\infty)$ where $\gamma(t) := (u(t),t)$ satisfies $F(u(t),t) = 0, \forall t \in [0,\tau]$, in particular $\gamma(0,0) = 0$. Implicit differentiation of the curve $u(t)$ with respect to t at $t = 0$, thus gives

$$u'(0) = -\frac{F_t(0,0)}{F_u(0,0)} = -\frac{2t\dot{t}e^{-2u}\det_m(\Re(q))}{L(u,t)}\Big|_{t=0,u(0)=0} = 0$$

Note that the denominator doesn't vanish, so the fraction is defined.
Now recall that $I_t^* = \frac{1}{2}(I_t + 2I\!I_t + I\!I\!I_t) = \frac{1}{2}(e^{2u_t}m + 2t\Re(q) + t^2e^{-2u_t}m\det_m(\Re(q))$. So taking derivative at $t = 0$ gives us:

$$\frac{d}{dt}\Big|_{t=0} I_t^* = \frac{1}{2}(2u'(0)e^{2u_0}m + 2\Re(q) + 0) = \Re(q).$$

Morover recall that

$$K_t^* = \frac{-1+\det(B_t)}{1+\det(B_t)} = \frac{-1+t^2e^{-4u_t}\det(B)}{1+t^2e^{-4u_t}\det(B)} = -1 + 2t^2e^{-4u_t}\det(B) + O(t^2)$$

We see thus that $\frac{d}{dt}\Big|_{t=0} K_t^* = 2(2te^{-4u_t} + t^2u_t'e^{-4u_t})\Big|_{t=0} = 0$.
So at first order, the curvature associates to the I_t^* doesn't change and is equal to -1 at $t = 0$.

Lemma 3.13. *For the path $\beta_q(t)$ as above, $\frac{d}{dt}\Big|_{t=0}(I\!I_t^*)_0 = \Re(q)$*

Proof. This follows from straightforward computation. First observe that $\frac{d}{dt}\Big|_{t=0} I\!I_t^* = \frac{d}{dt}\Big|_{t=0}(I_t - I\!I\!I_t)$ from the computation above.
[KS08] further show that that the mean curvature $H_t^* = -K_t^*$. Writing $I\!I_t^* = (I\!I_t^*)_0 + H_t^*I_t^*$ and taking derivative at $t = 0$ gives, $\frac{d}{dt}\Big|_{t=0}(I\!I_t^*)_0 = \frac{d}{dt}\Big|_{t=0} I_t^*$. We thus prove the lemma coupling this with the proposition above.

The condition for I^* being hyperbolic at first order at $t = 0$ enable us to use [KS08] to remark that $(I\!I_t^*)_0$ is equal to $-\Re(\sigma_t)$ at first order at $t = 0$. Thus we have the following:

Lemma 3.14. *For the path $\beta_q(t)$, $\frac{d}{dt}\Big|_{t=0}\sigma^t = q$*

Note that we have done all the computation at one boundary component at infinity of M, which is almost-fuchsian. However, recall that M admits a foliation by surfaces "parallel" to the minimal surface, and the corresponding computation for the other component will differ by a sign. To be precise, $I_t^* = \frac{1}{2}(I_t - 2I\!I_t + I\!I\!I_t)$ when we consider the component at the boundary at the other end. The rest of the computation follow as it is. Keeping this in mind we rename the two schwarzian derivatives at infinity by $\sigma_t^{\pm}$ and claim:

Corollary 3.15. *For the path $\beta_q(t)$, $\frac{d}{dt}\Big|_{t=0}\sigma_{\pm}^t = \pm q$*

Note that at $t = 0$, $\sigma_{\pm}^t = 0$ and the horizontal foliations do not exist. Thus the image $\Phi(\beta_q(t))$ can be thought of as a path in $\mathcal{MF} \times \mathcal{MF}$ originating from $(0,0)$.

We will now show that the first order behaviour of the schwarzian gives us a first order behaviour of the foliations associated to it with respect to the horizontal foliation of $-q$. Essentially we want to say that the foliations are "close" in some sense. Recall that this can be done with the help of intersection number, which in turn can be computed using the transverse measure.

Definition 3.16. *The intersection number of simple closed curve γ with a measured foliation f, realised as the horizontal foliation of q^{f} is given by*

$$i(\gamma,\mathsf{f}) = \int_\gamma |\Im(\sqrt{q^f(z)}dz)|$$

A similar expression with $\Re$ in stead of $\Im$ in the formula above gives us the intersection number in the case the given foliation is viewed as the vertical foliation of some quadratic differential. We will now show:

Lemma 3.17. *For any simple closed curve γ on Σ we have:*

$$i(\gamma, \mathsf{f}^t) - i(\gamma, \mathrm{hor}_{[c]}(t^2 q)) = o(\sqrt{t})$$

Proof. Reinterpreting this statement of the lemma we proceed with the computation :

$$\begin{aligned} i(\gamma, \mathsf{f}^t) - i(\gamma, \mathrm{hor}_{[c]}(-tq)) &= \int_\gamma |\Im(\sqrt{\sigma_t(z)}dz)| - \int_\gamma |\Im(\sqrt{tq(z)}dz)| \\ &= \int_\gamma |\Im(\sqrt{t^2 q(z) + O(t)}dz)| - \int_\gamma |\Im(\sqrt{t^2 q(z)}dz)| \\ &= \int_\gamma |\Im(\sqrt{tq(z)}dz)| - \int_\gamma |\Im(\sqrt{tq(z)}dz)| + O(t) \int_\gamma |(\Im dz)| \\ &= O(t) \int_\gamma |(\Im dz)| \\ &= O(t) \end{aligned}$$

The term which is the difference of the integrals refers to the difference of vertical and horizontal length of the same curve in the flat-metric induced by the quadratic differential and thus is bounded, the flat metric being finite.

Remark 3.18. *The topology in the space of measured foliation is the weak–$*$ topology induced from the function $\mathsf{f} \to i(\gamma, \mathsf{f})$ for any simple closed curve γ on Σ. Thus the above lemma tells us that the two foliations compared are contained in an open neighbourhood of each other. Moreover, two measured foliation are said to be equivalent if their intersection number with any simple closed curve is equal. Thus, another interpretation of the lemma can be that the foliation at infinity $\mathsf{f}^t_\pm$ is equivalent to the horizontal and vertical foliations of q realised on the riemann surface $(\Sigma, [m])$ at first order at $t = 0$.*

We can now propose a condition that the two foliations need to satisfy being the image of $\beta_q(t)$ under the map Φ. Un-surprisingly it is the "filling up" property that has been shown to be satisfied by the bending laminations in full generality.

Theorem 3.19. *The horizontal foliations of the schwarzian derivatives at the two ends at infinity fill up Σ near the fuchsian locus.*

Proof. First let's restrict ourselves to the path $\beta_q(t)$. Notice for this path, the two foliations $\mathsf{f}^t_\pm$ are in an open neighbourhood of the horizontal and vertical foliation of the quadratic differential q prescribing the data of the minimal surface. Recall that filling up is an open property, i.e, the set of filling measured foliations which fill up a surface form an open ball in $\mathcal{MF} \times \mathcal{MF}$. One way to see this is again through the theorem of Hubbard-Masur, which essentially state that any pair of filling foliations are in one-to-one correspondence with $T^*\mathcal{T}(\Sigma)$, which is an open ball of dimension $12g - 12$. The foliation $\mathsf{f}^t_\pm$ thus fill.
Now given any arbitrary point in $([m'], q')$ $T^*\mathcal{T}(\Sigma)$ one finds a minimal embedding of Σ with the prescribed data by considering the ray $([m'], t\Re(q'))$ where we keep the conformal class of $[m']$ fixed as before and look for solutions of the gauss equation. We claim again that a solution exists for t smaller than some fixed constant depending on the initial point on $T^*\mathcal{T}(\Sigma)$ chosen. We again see by mimicking the calculation above that the variation of the schwarzians at the two end is given by $\pm q'$ and we conclude the filling up property for the horizontal foliations by the repeating argument above.

The goal of this article is essentially proving that the map Φ is a local local homeomorphism or in other words, we want to study the invertibility of $d\Phi$ near the fuchsian locus so that we can apply inverse function theorem. Note that having established the corollary above, we can restrict the map to $\Phi : \mathcal{AF}(\Sigma) \to \mathcal{FMF}(\Sigma)$. Given $(\mathsf{f}, \mathsf{g}) \in \mathcal{FMF}$ which fill up, we have a ray $t(\mathsf{f}, \mathsf{g})$ for $t > 0$. This gives us a substitute for "tangent directions" in $\mathcal{FMF}$. We want to reformulate all that we have proved in

this section to identify the point on $\mathcal{F}(\Sigma)$ where a path in $\mathcal{AF}(\Sigma)$ with prescribed tangent direction in $\mathcal{FMF}(\Sigma)$ hits the fuchsian locus. This will facilitate the application of the inverse function theorem in the following section.
Compiling the sequence of results above gives us:

Proposition 3.20. *Consider, for $t \in [0, \epsilon)$, a one parameter family of almost-fuchsian metrics $\beta_q(t)$ prescribed by the pair $([m], t^2\Re(q))$ of it's unique embedded minimal surface, and let the horizontal foliations of the schwarzian derivatives at the two ends be given as $\mathsf{f}^t_{\pm}$. The f^t_+ (respectively f^t_-) is measure equivalent to the vertical foliation (respectively horizontal foliation) of tq on the riemann surface $(\Sigma, [m])$ at first order at $t = 0$. Moreover $[m]$ is the unique point of minima for the function $ext(\mathrm{hor}(q)) + ext(\mathrm{ver}(q))$.*

We now want to generalise this statement for an arbitrary path with given first order behavior of the foliations at infinity.

Proposition 3.21. *Suppose $(\mathsf{f}_+, \mathsf{f}_-)$ be a pair of foliations that fill up Σ. Let g_t be 1-parameter family of almost-fuchsian manifolds for $t \geq 0$ starting from the Fuchsian locus, such that the foliations at infinity $(\mathsf{f}^t_+, \mathsf{f}^t_-)$ are measure equivalent to $(t\mathsf{f}_+, t\mathsf{f}_-)$ at first order at $t = 0$. Then g_0 is the unique point where $ext(\mathsf{f}_+) + ext(\mathsf{f}_-)$ is minimized. Moreover, the component of $\frac{d}{dt}\Big|_{t=0} g_t$ normal to $\mathcal{F}(\Sigma)$ is given by $q \in T^*_{g_0}(\mathcal{T}(\Sigma))$ which realizes the pair $(\mathsf{f}_+, \mathsf{f}_-)$ on $(\Sigma, [g_0])$.*

Proof. We start off by parametrising the path g_t of almost fuchsian metrics near the fuchsian locus with the data of its unique embedded minimal surface once more. First let us recollect that the usual recipe to construct an almost-fuchsian manifold via the unique minimal surface begins by choosing a point on the co-tangent space of $\mathcal{T}(\Sigma)$, and then look for a solution curve to an equation of the type of UhL by keeping the conformal class fixed and scaling the real part of the quadratic differential. For the hypothesis above, we thus pick a family of minimal surface immersion data by the pair $(e^{\phi_t}|dz|^2, \Re(q_t))$. Here we write down the metric on Σ directly without specifying the conformal class it belongs to explicitly like in the case before, i.e, we take the fuchsian uniformisation of the induced metric of the minimal surface. Owing to the conditions for the solutions to the gauss equation, $q_t \in T^*_{e^{\phi_t}|dz|^2}\mathcal{T}(\Sigma)$ is again small enough in norm. Moreover the map ϕ_t can also be thought of as a louiville field having certain transformational properties which in turn can also be used to derive the metric at the two components in terms of the data of the minimal surface. See [KS08] section 2 for more details in this regard.
We can however proceed with the strategy as before to compute the variation of the schwarzians at the two ends. As in notation before, $I_t = e^{\phi_t}|dz^2|$ and $I\!I_t = \Re(q_t)$. Let $q_t = (\Re(q_t) + i\Im(q_t))dz^2$. Computing in orthonormal basis gives us expression for $I\!I\!I_t$. Also, $q_0 = 0$ since the path originates from the fuchsian locus and $e^{\phi_0}|dz|^2$ is point g_0 mentioned in the statement. I^*_t thus takes the form:

$$I^*_t = \frac{1}{2}\begin{pmatrix} e^{\phi_t} & 0 \\ 0 & e^{\phi_t} \end{pmatrix} + \begin{pmatrix} \Re(q_t) & -\Im(q_t) \\ -\Im(q_t) & -\Re(q_t) \end{pmatrix} + \frac{e^{-\phi_t}}{2}\begin{pmatrix} \Re(q_t)^2 & \Re(q_t)\Im(q_t) \\ -\Re(q_t)\Im(q_t) & \Im(q_t)^2 \end{pmatrix}$$

$$\implies \frac{d}{dt}\Big|_{t=0} I^*_t = \frac{1}{2}\begin{pmatrix} \phi'(0)e^{\phi_0} & 0 \\ 0 & \phi'(0)e^{\phi_0} \end{pmatrix} + \begin{pmatrix} \Re(q'(0)) & -\Im(q'(0)) \\ -\Im(q'(0)) & -\Re(q'(0)) \end{pmatrix} = \frac{1}{2}\phi'(0)e^{\phi_0}|dz|^2 + \Re(q'(0))$$

Unlike in the previous case $I\!I^*$ doesn't remain unchanged at first order and this can be seen as:

$$I\!I^*_t = \frac{1}{2}(I_t - I\!I\!I_t) \implies \frac{d}{dt}\Big|_{t=0} I\!I^*_t = \frac{d}{dt}\Big|_{t=0} I_t - \frac{d}{dt}\Big|_{t=0} I\!I\!I_t = \frac{1}{2}\phi'(0)e^{\phi_0}|dz|^2$$

It remains to compute the variation of the curvature K^*_t at first order at $t = 0$. Recall in the coordinates chosen, $B_t = (I_t)^{-1}(I\!I_t)$ and thus we have

$$K^*_t = \frac{-1 + \det(B_t)}{1 + \det(B_t)} = -1 + 2e^{-2\phi_t}||q_t||^2$$

$$\implies \frac{d}{dt}\Big|_{t=0} K^*_t = -4\phi'(0)e^{-2\phi_0}||q_0||^2 + 4e^{-2\phi_0}||q'(0)q_0|| = 0$$

Thus again, I^* remains hyperbolic at first order at $t = 0$ enabling us to say that $\frac{d}{dt}\Big|_{t=0} I\!I_0^* = -\Re(\frac{d}{dt}\Big|_{t=0}\sigma_t)$ using [KS08]. But once more, $I\!I_t^* = (I\!I_t^*)_0 + H_t^* I_t^* \implies \frac{d}{dt}\Big|_{t=0} I\!I_t^* = \frac{d}{dt}\Big|_{t=0}(I\!I_t^*)_0 + \frac{d}{dt}\Big|_{t=0} I_t^* \implies \frac{d}{dt}\Big|_{t=0}(I\!I_t^*)_0 = -\Re(q'(0))$

Thus we see that at $t = 0$, $\frac{d}{dt}\Big|_{t=0}\sigma_\pm^t = q'(0)$.
Now recall that the first order behavior of the horizontal foliations from the schwarzian is given by $(t\mathsf{f}_+, t\mathsf{f}_-)$ at first order at $t = 0$. Let q be the pair that realizes this foliation at first order at the point $e^{\phi_0}|dz|^2$. The computation thus implies that the first-order behavior of the schwarzian is given by q at $t = 0$, and g_0 is indeed the point of minima for the sum of the extremal lengths of $(\mathsf{f}_+, \mathsf{f}_-)$ from [HM79]. The assertion that the normal component of $\frac{d}{dt}\Big|_{t=0} g_t$ is given by q comes from the almost-complex structure of $\mathcal{QF}(\Sigma)$ realized as the connected component of the character variety $\chi := \mathsf{Hom}(\pi_1(\Sigma), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C})$, which is basically the conjugacy classes of representation of $\pi_1(\Sigma)$ into the isometry group of $\mathbb{H}^3$ which is $PSL_2(\mathbb{C})$. Owing to this, the tangent space at any point but in particular at a point $g \in \mathcal{F}(\Sigma) \cong \mathcal{T}(\Sigma)$ decomposes as $T_g\mathcal{QF}(\Sigma) \cong T_g\mathcal{T}(\Sigma) \bigoplus i_\chi T_g\mathcal{T}(\Sigma)$, where i_χ is derived from the complex structure of χ. Moreover the normal bundle $N\mathcal{F}(\Sigma)$ is equal to $i_\chi T\mathcal{T}(\Sigma)$. We see thus that $\frac{d}{dt}\Big|_{t=0} g_t = \frac{d}{dt}\Big|_{t=0} e^{\phi_t}|dz|^2 + \frac{d}{dt}\Big|_{t=0} q_t$, where the first term in the sum gives me the tangent direction of the path g_t along $\mathcal{F}(\Sigma)$ and thus the normal direction is given by $\frac{d}{dt}\Big|_{t=0} q_t = q'(0) = q$ as claimed.

Remark 3.22. *Essentially what this proposition claims is that prescribing the foliations at infinity at first order for a path of quasi-fuchsian metrics starting at the fuchsian locus, is sufficient to determine the point at $\mathcal{F}(\Sigma)$ from which this path starts and also the tangent direction normal to $\mathcal{F}(\Sigma)$ at the point. Note that the normal direction is thus along the path $\beta_q(t)$ constructed before with base point g_0 at $t = 0$.*

4. Existence of paths realising prescribed small foliations

As the section title suggests we aim to construct paths near the fuchsian locus with prescribed first-order behaviour of the foliations at infinity. To be precise, contrary to the previous section, we want to existence of a path g_t such that the prescribed foliation at first order $t = 0$ is given by a pair $(t\mathsf{f}_+ . t\mathsf{f}_-)$ for t small enough, where $(\mathsf{f}_+, \mathsf{f}_-)$ are assumed to fill up Σ. Note that, although the path $\beta_q(t)$ from previous section does the job, it is chosen to be orthogonal to $\mathcal{F}(\Sigma)$ exclusively so it alone doesn't suffice.
We will again follow Bonahon's construction of similar paths realising bending lamination of convex core boundaries, and try to translate it into our setting.
The outline for this section is as follows assuming definitions:

Given a measured foliation $\mathsf{f} \in \mathcal{MF}(\Sigma)$, we will consider certain submanifolds of $\mathcal{QF}(\Sigma)$, which we call $\mathcal{W}_\mathsf{f}^+$, such that the horizontal foliation at the boundary component at $+\infty$ is given by $t\mathsf{f}$ $\forall t \geq 0$. Likewise, we also consider $\mathcal{W}_\mathsf{f}^-$, with the foliation being $t\mathsf{f}$ in the boundary at $-\infty, \forall t \geq 0$. So given a pair of foliations $(\mathsf{f}_+, \mathsf{f}_-)$ that fill up we will study $\mathcal{W}_{\mathsf{f}_+}^+ \bigcap \mathcal{W}_{\mathsf{f}_-}^-$ and claim they have a transverse intersection if the ambient space is blown up. Let $\widetilde{\mathcal{QF}(\Sigma)}$ be the blow-up. Since transverse intersection of two submanifolds is again a submanifold, we will then show that this intersection in $\widetilde{\mathcal{QF}(\Sigma)}$ is locally diffeomorphic to $\widetilde{\mathbb{R}^2}$, where the latter is the blow up of $\mathbb{R}^2$ at the origin.

Thus, given $\mathsf{f} \in \mathcal{MF}(\Sigma)$

Definition 4.1. $\mathcal{W}_\mathsf{f}^+$ *(respectively $\mathcal{W}_\mathsf{f}^-$) is the set of quasi-fuchsian metrics such that the horizontal foliation due to the schwarzian at the boundary component at $+\infty$ (respectively $-\infty$) is given by* $t\mathsf{f}, \forall t \geq 0$.

We will proceed to show that these are indeed submanifolds with boundaries of dimension $(6g-6)+1$, where $(6g-6)$ is the dimension of $\mathcal{T}(\Sigma)$. With that goal in mind we recall the notion of periodic coordinates for quadratic differentials on a Riemann surface, i.e, essentially $T^*\mathcal{T}(\Sigma)$. Indirectly these provide a coordinate for $\mathsf{f} \in \mathcal{MF}(\Sigma)$ as through the Gardiner-Masur map each such f corresponds to a point in $T^*\mathcal{T}(\Sigma)$. If one has that a quasi-fuchsian metric is parametrized by $\mathbb{CP}^1$-structure in just one component, then we can use these coordinates to describe quasi-fuchsian metrics with prescribed foliation data at that component, as a $\mathbb{CP}^1$ structure at the boundary components at $\pm\infty$ is nothing but the pair $(c_\pm, \sigma_\pm)$ with the usual notation.

Lemma 4.2. *A quasi-fuchsian structure is uniquely determined by the conformal class and schwarzian derivative at one component.*

Proof. This can be seen as successive applications of two well-known results in the area. First recall [Bon96] that $g \in \mathcal{QF}(\Sigma)$ can be uniquely determined by the pair of (m_+, λ_+) where m_+ is the induced metric on the boundary of the convex core and λ_+ is the corresponding bending lamination. The grafting map $\phi : \mathcal{T}(\Sigma) \times \mathcal{ML}(\Sigma) \to \mathbb{CP}^1$ being a homeomorphism thurston in turn implies the statement as it sends the data (m_+, λ_+) to (c_+, σ_+).

4.1. **A small digression on classification of measured foliations via normal forms.** Having parametrized a quasi-fuchsian metric by the conformal class c_+ and schwarzian derivative σ_+ at one component, we intent to in turn parametrize them using $(12g-12)$ coordinates such that these together express the data of the foliation f_+ along with it's induced measure, which is realised as the horizontal foliation of the schwarzian on this conformal class. For this we recall the normal form on measured foliations adhering to exposé 6 of [Col79]. The following will be a condensed version of such.

Given a foliation f on a surface Σ of genus $g \geq 2$, we can choose a *normal form* to the foliation, which we denote as $\mathcal{NF}(\Sigma)$.

Remark 4.3. *Before recalling the definition, we will like to focus on the fact that each equivalence class of measured foliation has an unique equivalence class of associated normal form and vice-versa. Thus there is bijective correspondence between the two, and since the foliation f is considered as a representative in it's equivalence class, it is enough to assert that we can choose a normal form to this foliation.*

A normal form of a measured foliation f, with transverse measure μ, on Σ is essentially a special selection of $(3g-3)$ simple closed curves that decompose the surface into $(2g-2)$ *pair of pants* and $3g-3$ annuli such that the data of the foliation gets parametrised by the induced measures on the length of each of these l_i, which we will call $m_i \in \mathbb{R}_{\geq 0}$; and two other parameters $(s_i, b_i) \in \mathbb{R}^2_{\geq 0}$ to be described below. Together these triples (m_i, s_i, b_i) further satisfy the property that the satisfy degeneracy condition of triangle inequality, denoted as $\partial(\nabla \leq)$. Thus the normal form allows us to embed a measured foliation as:

$$\mathcal{B} = \Big\{(m_i s_i, b_i) \in \mathbb{R}^{9g-9}_{\geq 0} | (m_i, s_i, t_i) \in \partial(\nabla \leq)\Big\}$$

It has been shown in [Col79] that $\mathcal{B} \setminus \{0\}$ is in bijection with $\mathcal{NF}(\Sigma)$ and forms a cone in $\mathbb{R}^{9g-9}_{\geq 0}$ homeomorphic to $\mathbb{R}^{6g-6}$. Moreover, these coordinates provide $\mathcal{MF}(\Sigma)$ with the structure of a topological manifold.

The decomposition can be described as following: We first pick $(3g-3)$ simple closed curves labelled $\{l_1, l_2, \ldots l_{3g-3}\}$ that decompose Σ into $(2g-2)$ pair of pants, labelled $\{P_1, P_2, \ldots P_{2g-2}\}$. We then replace each curve l_i with an annulus labelled $A_i = l_i \times [-1, 1]$ and consider the connected components of the pants in the complement of these annuli, which we relabel P'_i. Thus we obtain a decomposition of the surface as the union of $(3g-3)$ annuli $\{A_1, A_2, \ldots A_{3g-3}\}$ and $(2g-2)$ pair of pants $\big\{P'_1, P'_2, \ldots P'_{2g-2}\big\}$. This procedure reduces the problem of classifying the measured foliation f on Σ into that of classifying it on annuli and pair of pants.

For a pair of pants invidually, upto assuming that the boundary components are not leaves, the foliation can be classified by encoding the induced measure of each boundary component.

For an annuli, on the other hand, classifying a measure foliation depends on 3-parameters (m, s, t). Given an annuli A we join the two boundary components by first an arc which we cal α and then by an arc differing from α by a counter-clock twist which we call $\overline{\alpha}$. We then define $m = \mu(\partial A)$, s to be $\inf(\mu(\alpha'))$ for all paths homotopic to α with fixed end points and t_i to be $\inf(\mu(\overline{\alpha'}))$ for all paths homotopic to $\overline{\alpha}$ with fixed end points. It has been shown in [Col79] that these triplets (m, s, t) uniquely determine equivalence classed of measured foliation on the annuli and further belong to the set $\partial(\nabla \leq)$, thus providing coordinates for measured foliations into a cone in $\mathbb{R}^3_{\geq 0}$, which is homeomorphic to $\mathbb{R}^2$. Going back to the decomposition of Σ induced by $\{l_1, l_2, \dots l_{3g-3}\}$ into $(3g-3)$ annuli A_i and $(2g-2)$ pair of pants P'_j, the entire data of the measured foliation thus can be encaptured bu the data (m_i, s_i, b_i) associated to each A_i. This account for the pants P'_j as well since the boundaries of the pants and the annuli coincide and thus record the same induced measure.

In our context, where f is realised as the horizontal foliation of a holomorphic quadratic differential q, recall that in local charts the transverse measure is given by $\Im|\sqrt{f(z)}dz|$, where $f(z)$ is an analytic map. Restricting to the case of the annuli, $m_i = \int_{l_i} \Im|\sqrt{f(z)}dz|$. s_i is given by the width of the annulus which is given by $\int_{l_i} \Re|\sqrt{f(z)}dz|$. The terms b_i corresponds to the twist and is determined by the degeneracy condition.

Suppose our given measured foliation f has coordinates (m_i^0, s_i^0, b_i^0) for $i = 1, 2, \dots, 3g-3$. Scaling the transverse measure of our given measured foliation as tf, which accounts to it being realised by the quadratic differential $t^2 q$ on $(\Sigma, [c])$, thus accounts to scaling the first two factors uniformly as $(t(m_i^0, s_i^0), b'_i)$ for $t \in \mathbb{R}_{\geq 0}$ where each b'_i will be uniquely determined having known the other two. This embeds the space of riemann surfaces with foliations tf for $t \geq 0$

Proposition 4.4. *$\mathcal{W}^+_{\mathsf{f}}$ is a submanifold of $\mathcal{QF}(\Sigma)$ of dimension $6g-5$ with boundary $\mathcal{F}(\Sigma)$.*

Proof. Let the set of curves $\{l_i\}_{i=1}^{i=3g-3}$ induce the normal form for the equivalence class of f realised as the horizontal foliation of the schwarzian σ_+ on $(\Sigma, [c])$. Along with the fact the quasifuchsian metric g can be uniquely determined by the pair (c_+, σ_+) at one end, these $3g-3$ curves in turn enable us to provide coordinates for the complex projective structure at the boundary at infinity of g into $\mathbb{R}^{12g-12}$. The first $6g-6$ coordinates denote an embedding of $\mathcal{T}(\Sigma)$ with respect to the fenchel-nielsen coordinates with respect to this pant decomposition, thus encoding the data of the unique hyperbolic metric in the conformal class $[c]$, the next $(6g-6)$ coordinates can be considered as above by considering the normal form induced by l_i for f. This provides as embedding of f into $\mathbb{R}^{6g-6}$ by considering the couples $(h_{\sigma_+}(l_i), v_{\sigma_+}(l_i))$ for each i. We thus have the map η

$$\eta(g) \to (c_+, \sigma_+) \to \mathcal{T}(\Sigma) \times (\int_{l_i} \Im|\sqrt{\sigma_+(z)}dz|, \int_{l_i} \Re|\sqrt{\sigma_+(z)}dz|)$$

η provides us with a smooth embedding of $\mathcal{QF}(\Sigma)$ into $\mathcal{T}(\Sigma) \times \mathcal{B}_{\mathsf{f}}$, where $\mathcal{B}_{\mathsf{f}}$ denote the set of triples associated to each annuli A_i in a neighbourhood of l_i and from what we have seen these can be parametrized by the first two components when l_i are assumed to non-parallel to the leaves of f in terms of the horizontal and vertical measure induced by σ_+. That is $\mathcal{B}_f$ is completely determined by the $3g-3$ pairs $(h_{l_i}(\sigma_+), v_{l_i}(\sigma_+))$.

Note that under this embedding, $\eta(\mathcal{F}(\Sigma)) \cong \mathcal{T}(\Sigma) \times \{0\}$ by just the fenchel-nielsen coordinates induced by the curves l_i. If the foliation f is parametrised by the set B_f, then the foliation $t\mathsf{f}$ has coordinates tB_f for $t \geq 0$. One intuitive way to realize is again through the encoding of the horizontal length and vertical length of the curves l_i with respect to the flat metric generated by the quadratic differential $t^2 q$ which generates $t\mathsf{f}$. Although here, the twist parameter also changes continuously depending on the first two, to satisfy the degeneracy condition, and this in turn provides us with new coordinates for $\mathcal{T}(\Sigma)$ as well, varying smoothly as we scale f.

It is clear that the map η sends an $g \in \mathcal{W}^+_{\mathsf{f}}$ to $\mathcal{T}(\Sigma) \times t\mathcal{B}_{\mathsf{f}}$, for $t \geq 0$. Conversely, the map parameterising a quasi-fuchsian metric g in terms of the conformal class and schwarzian derivative in one components is a homeomorphism. Thus given a $g_0 \in \mathcal{W}^+_{\mathsf{f}}$ and g in a suitable neighbourhood of g_0 with $\eta(g) \in \mathcal{T}(\Sigma) \times \mathcal{B}_{\mathsf{f}}$, then horizontal foliation at the component at $+\infty$ for the quasi-fuchsian structure g is given by $t\mathsf{f}$. This is essentially because for two metrics g and g' close enough, the schwarzians and conformal class do not differ by much and thus the foliation realised by them remains the same.

Thus $\mathcal{W}^{\mathsf{f}}_+$ is inherits the structure of a smooth submanifold of $\mathcal{QF}(\Sigma)$ owing to the smooth coordinates induced by the normal form, and is a submanifold of dimension $(6g-6)+1 = 6g-5$.

We want to realise quasifuchsian metrics with foliations at infinity realised by a prescribed filling pair $(t\mathsf{f}_+, t\mathsf{f}_-)$ for t small enough, and thus it makes sense to consider the set $\mathcal{W}^{\mathsf{f}_+}_+ \cap \mathcal{W}^{\mathsf{f}_-}_-$, where $\mathcal{W}^{\mathsf{f}}_-$ denotes the length of the quasi-fuchsian metrics with foliation f_- appearing on the boundary component at $-\infty$. This can also be shown to be a submanifold by mirroring the argument above.

Notice that these two submanifolds are both of dimension $6g-5$ with boundary of dimension $6g-6$ in an ambient manifold $\mathcal{QF}(\Sigma)$ of dimension $12g-12$. Thus their intersection can not be transverse by standard dimension arguments. For this purpose, we tweak the ambient space a bit following [BO04] so that the intersection is transverse in this new space.

Blow up $\widetilde{\mathcal{QF}(\Sigma)}$ of $\mathcal{QF}(\Sigma)$: Recall the standard blow up procedure utilized to resolve singularities of curves at a point in affine spac $\mathbb{A}^{n+1}$. Blowing up at some point; suppose $0 \in \mathbb{A}^{n+1}$; boils down to replacing 0 with the projectivised space $\mathbb{PA}^{n+1}$ such that the resulting space $\widetilde{\mathbb{A}^{n+1}}$ is topologically homeomorphic to $\mathbb{A}^{n+1} \setminus \{0\} \cup \mathbb{PA}^{n+1}$. Note that we are essentially replacing a point we want to blow up with it's unit projectivised tangent space. This provides us with a map $b : \widetilde{\mathbb{A}^{n+1}} \to \mathbb{A}^{n+1}$, such that for any open neighbourhood V of 0, $b^{-1}(V) \subset \mathbb{PA}^{n+1}$.

$\widetilde{\mathcal{QF}(\Sigma)}$ is obtained in an analogous way where, $\forall [c] \in \mathcal{F}(\Sigma) \cong \mathcal{T}(\Sigma)$, we replace the point in $\mathcal{QF}(\Sigma)$ with the unit normal bundle $UN_{[c]}\mathcal{F}(\Sigma)$ at that point. Owing to the almost complex structure of $\mathcal{QF}(\Sigma)$, this is topologically the same as replacing the point $[c]$ with it's unit tangent space $UT\mathcal{F}(\Sigma)$, as $UN\mathcal{F}(\Sigma) \cong i_\chi UT\mathcal{F}(\Sigma) \cong UT\mathcal{F}(\Sigma) \cong \mathcal{QD}^1(\Sigma)$. $\widetilde{QF}$ is topologically $\{\mathcal{QF}(\Sigma) \setminus \mathcal{F}(\Sigma)\} \cup UN\mathcal{F}(\Sigma)$. Moreover, $\widetilde{\mathcal{QF}(\Sigma)}$ naturally inherits the smooth structure of $\mathcal{QF}(\Sigma)$ with $\partial\widetilde{QF} \cong UN\mathcal{F}(\Sigma)$. Thus:

Lemma 4.5. *$\widetilde{QF}$ is a smooth manifold of dimension $(6g-6)+(6g-5)$ with boundary $\partial\widetilde{QF} \cong UN\mathcal{F}(\Sigma)$.*

Remark 4.6. *Intuitively one can find sense to do this, as given a curve $\gamma_t \in \mathcal{QF}(\Sigma)$ for $t \in [0, +\epsilon)$, the image $\mathfrak{F}(\gamma_t)$ is singular at $t = 0$, fuchsian manifolds not having any foliations at infinity. In particular, $d\mathfrak{F}|_{\mathcal{F}(\Sigma)}$ is non-invertible. Replacing $\mathcal{F}(\Sigma)$ with a copy of it's projectived tangent bundle, helps us to resolve this by mapping the curve γ_t into $\gamma_t \cup \gamma'(0)$, $t \in (0, +\epsilon)$. What we did in section 3 was to specify $\gamma(0)$ and $\gamma'(0)$ for a path having prescribed image. This will allow us to see that "$d\mathfrak{F}$ is invertible" in the charts provided by the blow up in a neighbourhood of $\mathcal{F}(\Sigma)$, allowing us to construct the paths we desire.*

We turn our attention to $\widetilde{\mathcal{W}_{\mathsf{f}}^{\pm}}$, the image of $\mathcal{W}_{\mathsf{f}_\pm}^{\pm}$ under the blow-up and claim:

Lemma 4.7. *$\widetilde{\mathcal{W}_{\mathsf{f}}^{\pm}}$ is a submanifold of $\widetilde{\mathcal{QF}(\Sigma)}$ of dimension $(6g-6)+1$, with $\partial\widetilde{\mathcal{W}_{\mathsf{f}}^{\pm}} \subset \partial\widetilde{QF} \cong UN\mathcal{F}(\Sigma)$.*

Proof. We again use the coordinates introduced before, namely if f is realized as the horizontal foliation for some $q \in \mathcal{QD}(\Sigma, [c])$, then the coordinates we get in $\mathbb{R}^{12g-12}$ are given by choosing a normal form and then accounting for the foliation restricted to each annuli formed in the procedure. We see that, if m is the unique hyperbolic metric in the class $[c]$, the coordinates for f will be given by $\{m_{l_i}, h_q(l_i), v_q(l_i), ext_q(l_i)\}$, where $\{l_1, l_2, \dots l_{3g-3}\}$ are the curves inducing the pant decomposition for the normal form. Note that, outside the neighbourhood of $\mathcal{F}(\Sigma)$ introduced in the blow-up, the coordinates do not change for $\widetilde{W_{\mathsf{f}}^{\pm}}$. In particular, we only need to parametrize $\partial\widetilde{W_{\mathsf{f}}^{\pm}}$ which is now clearly given by some element in $\partial\widetilde{\mathcal{QF}(\Sigma)}$. On the other hand, the elements of $\widetilde{\mathcal{QF}(\Sigma)}$ are given by $\bar{q}^{\mathsf{f}}$ which correspond to the tangent direction for the path βq^{f} from the previous section. This path on the other hand shows the property of realising the foliation tf in it's positive component in a small enough neighbourhood of $\mathcal{F}(\Sigma)$. Thus locally, η identifies $\mathcal{W}_+^{\mathsf{f}}$ with the element $\beta_{q^{\mathsf{f}}}$ at each point of $\mathcal{F}(\Sigma) \cong \mathcal{T}(\Sigma)$. This again being a smooth assignment of coordinates completes the argument regarding transversality of the intersections. The fact that their dimension are $(6g-6)+1$ follows from a similar scaling argument as before as essentially we are replacing the boundary of $\mathcal{W}_{\mathsf{f}}^{+}$ with $UN\mathcal{F}(\Sigma)$, which are the same dimension, while the complement remains the same.

Having identified the $\partial\widetilde{\mathcal{W}_{\mathsf{f}}^{\pm}}$ with the tangent direction along which it hits the fuchsian locus via the method above, we can now thus aim to replicate the statement regarding transverse intersection of $\partial\widetilde{\mathcal{W}_{\mathsf{f}_+}^{+}}$ and $\partial\widetilde{\mathcal{W}_{\mathsf{f}_-}^{-}}$ for a pair of foliations $(\mathsf{f}_+, \mathsf{f}_-)$ which has been assumed to fill. f_- being the horizontal foliaition of $-q$ on $(\Sigma, [c])$, helps us do that.

Proposition 4.8. *If the pair $(\mathsf{f}_+,\mathsf{f}_-)$ fill up Σ then the boundaries $\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}$ and $\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ intersect transversely in $\widetilde{\mathcal{QF}(\Sigma)}$ and moreover the projection of the intersection onto $\mathcal{T}(\Sigma)$ is given by the line $\mathsf{P}(\mathsf{f}_+,\mathsf{f}_-)$.*

Proof. The proof is quite easy given the assumption that $(\mathsf{f}_+,\mathsf{f}_-)$ fill up and in that case the sections $q^{\mathsf{f}_+}=-q^{\mathsf{f}_-}$ in $\mathcal{QD}(\Sigma,[c])$ intersect transversely and so the section $\bar{q}^{\mathsf{f}_+}=-\bar{q}^{\mathsf{f}_-}$ for all $[c]\in\mathcal{T}(\Sigma)$. Again, $i_\chi\bar{q}^{\mathsf{f}}\in UN\ F$ as we have seen through the almost complex structure of $\mathcal{QF}(\Sigma)$. Thus, the intersection of $\partial\widetilde{W^+_{\mathsf{f}_+}}$ and $\partial\widetilde{W^+_{\mathsf{f}_-}}$ in $\widetilde{\mathcal{QF}(\Sigma)}$ is at the point $(\mathsf{p}(t\mathsf{f},\mathsf{g}),i_\chi q^{tf}_{(t\mathsf{f},\mathsf{g})})$ using the notation from section 2, the projection onto which is clearly the line $\mathsf{P}(\mathsf{f},\mathsf{g})$ by definition.

To combine and summarise, we essentially showed that $\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ is properly embedded 2-dimensional submanifold of $\widetilde{\mathcal{QF}(\Sigma)}$ in a neighbourhood of $\partial\widetilde{\mathcal{QF}(\Sigma)}$ and that it's boundary $\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ is in $\partial\widetilde{\mathcal{QF}(\Sigma)}$. This is reminiscent of the situation when you have a curve in $\mathbb{R}^2$ with singularity at the origin and then to resolve that situation you consider $\widetilde{\mathbb{R}^2}$, the blow up of $\mathbb{R}^2$ at the origin; which is obtained by removing the origin and replacing it with it's projectivised tangent bundle; and then study the curve in the blow up space with the singularity removed with the projective charts around 0 tracking the tangent direction to the curve at the origin.

To formalize the notion we introduce the map $\widetilde{\pi}:\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}\to\widetilde{\mathbb{R}^2}$, which is a lift of $\pi:\mathcal{W}^+_{\mathsf{f}_+}\cap\mathcal{W}^-_{\mathsf{f}_-}\to\mathbb{R}^2$. sending $g\in\mathcal{W}^+_{\mathsf{f}_+}\cap\mathcal{W}^-_{\mathsf{f}_-}\mapsto(s,t)\in\mathbb{R}^2$, where $\mathfrak{B}(g)=(s\mathsf{f}_+,t\mathsf{f}_-)$. We can now claim the following using [Bon02]:

Theorem 4.9. *The map $\widetilde{\pi}$ is a local homeomorphism near $\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ onto it's image.*

Proof. What we essentially want to show is that $\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ is transverse near it's boundary. We can then apply the fact that transverse intersection of two submanifolds is itself a submanifold.

We thus need to study the derivative of the map considered above at a point $p\in\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ and show that the tangent map $d_p\widetilde{\pi}:T_p\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap T_p\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\to T_{\widetilde{\pi}(p)}\widetilde{\mathbb{R}^2}$; sending individual tangent directions of the intersection of $\widetilde{\mathcal{W}^\pm_{\mathsf{f}_\pm}}\cap\partial\widetilde{\mathcal{QF}(\Sigma)}$ to the corresponding ray in $\widetilde{\mathbb{R}^2}$; is injective.

Notice that $T_p\widetilde{\pi}(v)=0$ for some $v\in T_p\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap T_p\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$, implies that $v\in T_p\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap T_p\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}\subset T_p\partial\mathcal{QF}(\Sigma)$. One way to see this will be notice that $T_p\widetilde{\pi}(v)=0$, implies that v can be represented by a path which preserves both the foliations and their measure upto scaling by t, and thus must be along the direction tangent to the intersection of $\partial\mathcal{W}^+_{\mathsf{f}_+}\cap\partial\mathcal{W}^-_{\mathsf{f}_-}$, where the direction is determined by the $q\in T^*\mathcal{T}(\Sigma)$ realising $(\mathsf{f}_+,\mathsf{f}_-)$.

Introduce the chart $c:\widetilde{\mathbb{R}}^2\to\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$ which sends $(x,y)\to(\frac{x}{y},y)$. Note that this serves as a "blow up" chart for $\widetilde{\mathbb{R}^2}$ at the point $\widetilde{\pi}(p)$. Recall that $\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ has been shown to be the lift of the line $\mathsf{P}(\mathsf{f}_+,\mathsf{f}_-)$ into $T^*\mathcal{T}(\Sigma)$. Thus in order to study the image of the map $c\circ\widetilde{\pi}$ for a point p in the intersection, we need to study the image of the vectors, $i_\chi\bar{q}^{\mathsf{f}_+}_m$, where m is hyperbolic metric in the conformal class chosen and serves as a parameter for $\mathcal{T}(\Sigma)$. We want to show, that the image of this vector is given by $(0,t)$, which will mean that the point on the line seen as the intersection of the boundaries, gets mapped to the ray by which the foliations f_+ gets scaled. Pick a curve in $t\mapsto\widetilde{g}_t\in\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}\cap\partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ such that $\widetilde{g}_0$ is the point $(m,i_\chi\bar{q}^{\mathsf{f}_+}_m)$ and such that $\frac{d}{dt}\Big|_{t=0}\widetilde{g}_t$ has non zero orthogonal component. $\widetilde{g}_t$ in turn descends to a curve $g_t\in\mathcal{QF}(\Sigma)$ with $g_0=m$. Let $\mathfrak{B}(g_t)=(a(t)\mathsf{f}_+,b(t)\mathsf{f}_-)$, where $a(t),b(t)$ are smooth curves such that $a(0)=b(0)=0$. By the way, the curves $a(t),b(t)$ exist by virtue of the definition of $\mathcal{W}^+_{\mathsf{f}_+}\cap\mathcal{W}^-_{\mathsf{f}_-}\ni g_t$. Moreover, the component of $\frac{d}{dt}\Big|_{t=0}g_t$ orthogonal to $\mathcal{F}(\Sigma)$ is also non-zero.

This brings us back to the case of proposition cite where we have a curve emanating from $\mathcal{F}(\Sigma)$ with specified first order behavior of the foliations at $\pm\infty$. Thus we know a description of the point in $\mathcal{F}(\Sigma)$ from which it emanates by virtue of proposition cite. That is, $m=\mathsf{p}(a'(0)\mathsf{f}_+,b'(0)\mathsf{f}_-)$, the critical point for the function $ext(a'(0)\mathsf{f}_+)+ext(b'(0)\mathsf{f}_-)$. But by assumption $m=\mathsf{p}(t\mathsf{f}_+,\mathsf{f}_-)$; so $\frac{a'(0)}{b'(0)}=t$, as $\mathsf{p}(\mathsf{f}_+,\mathsf{f}_-)$

is unique.
Thus $c \circ \widetilde{\pi}(\bar{q}^{\mathsf{f}}_m) = \lim_{t\to 0} c \circ \widetilde{\pi}(\tilde{g}_t) = \lim_{t\to 0} c \circ \pi(g_t) = \lim_{t\to 0} c \circ (a(t), b(t)) = \lim_{t\to 0}(\frac{a(t)}{b(t)}, b(t)) = (t, 0)$.
This shows that if $v \in T_p\widetilde{\mathcal{W}^+_{\mathsf{f}_+}} \cap T_p\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ with $d_p\widetilde{\pi}(v) = 0$ then v is zero.
Thus the map $\widetilde{\pi}$ is an immersion into $\widetilde{\mathbb{R}^2}$ at the points $p \in \partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}} \cap \partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$. The fact that $\widetilde{\pi}|_{\partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}} \cap \partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}}$ is surjective follows from dimension arguments.

We can now address the goal of the section:

Proposition 4.10. *Let $(\mathsf{f}_+, \mathsf{f}_-)$ be a pair of measured foliations that fill up Σ and let $\mathsf{p}(\mathsf{f}_+, \mathsf{f}_-)$ be the critical point of the function $ext(\mathsf{f}_+) + ext(\mathsf{f}_-)$. Then ,$\forall t \in [0, \epsilon)\exists t \mapsto g_t$ a curve, with $g_0 = \mathsf{p}(\mathsf{f}_+, \mathsf{f}_-)$, such that the $\mathfrak{B}(g_t) = (t\mathsf{f}_+, t\mathsf{f}_-)$ at first order at $t = 0$.*

Proof. We essentially use the last lemma to claim that there exists $\widetilde{g_t} \in \partial\widetilde{\mathcal{W}^+_{\mathsf{f}_+}} \cap \partial\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$, a curve with $b \circ \widetilde{\pi} \circ \widetilde{g_t} = (t, t)$ with b being the blow-up map $\widetilde{\mathbb{R}^2} \to \mathbb{R}^2$. The proposition then follows by definition as $\widetilde{g_t}$ descends to $g_t \in \mathcal{QF}(\Sigma)$ with $\mathfrak{B}(g_t) = (t\mathsf{f}_+, t\mathsf{f}_-)$ at first order at $t = 0$.

5. Constructing the local inverse

In this section we can finally look into constructing a map which will serve as a local inverse for $\mathfrak{F}$ in a neighbourhood of $\mathcal{F}(\Sigma)$. Note that we have already ascertained that the measured foliations at infinity fill in a neighbourhood of the fuchsian locus. On the other hand, from proposition 4.10, for every pair $(\mathsf{f}_+, \mathsf{f}_-) \in \mathcal{FMF}(\Sigma)$ there exists an $\epsilon_{\mathsf{f}_\pm}$ and a map $\Psi_{\mathsf{f}_\pm} : \mathcal{R}_{\mathsf{f}_\pm} \to \mathcal{QF}(\Sigma)$, where $\mathcal{R}_{\mathsf{f}_\pm}$ denotes the set $(0, \epsilon_{\mathsf{f}_\pm}](\mathsf{f}_+, \mathsf{f}_-)$ such that $\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}$ and $\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ intersect transversely along $\Psi(\mathcal{R}_{\mathsf{f}_\pm})$ and the image of a point in $\mathcal{QF}(\Sigma)$ is a quasifuchsian metric with measured foliations given by $c(\mathsf{f}_+, \mathsf{f}_-)$ with $c \in (0, \epsilon_{\mathsf{f}_\pm}]$.
To proceed further, we need to consider the blow up of $\widetilde{\mathcal{FMF}(\Sigma)}$ which is obtained by associating the "unit tangent directions" given by elements of the set $\mathcal{FMF}(\Sigma)/\mathbb{R}_{>0}$. This is done, as $\{(0, 0)\} \notin \mathcal{FMF}(\Sigma)$ and thus doesn't belong to the image of $\mathfrak{B}$.
We can thus consider $\widetilde{\Psi_{\mathsf{f}_\pm}} : \widetilde{\mathcal{R}_{\mathsf{f}_\pm}} \to \widetilde{\mathcal{QF}(\Sigma)}$, where as the notation suggests we lift everything to the blow-up spaces. Namely, $\widetilde{\Psi_{\mathsf{f}_\pm}}$ sends the initial direction of $\widetilde{\mathcal{R}_{\mathsf{f}_\pm}}$ to the point $([c], i_\chi q^{\mathsf{f}_+}_{[c]})$ with $[c]$ being the point of the unique minimum of the sum of the extremal lengths of f_+ and f_- and $q^{\mathsf{f}_+} = -q^{\mathsf{f}_-}$ being the quadratic differential realising $(\mathsf{f}_+, \mathsf{f}_-)$.
We essentially want to construct an open set $\widetilde{U}$ and a continuous map $\widetilde{\Psi} : \widetilde{U} \to \mathcal{QF}(\Sigma)$ by patching up individual rays $\widetilde{\mathcal{R}_{\mathsf{f}_\pm}}$ such that the map $\widetilde{\Psi}$ restricts to $\widetilde{\Psi_{\mathsf{f}_\pm}}$ on each $\widetilde{\mathcal{R}_{\mathsf{f}_\pm}}$.
Since $\widetilde{\Psi}|_{\widetilde{\mathcal{R}_{\mathsf{f}_\pm}}}$ was constructed by considering the transverse intersection of $\widetilde{\mathcal{W}^-_{\mathsf{f}_-}}$ and $\widetilde{\mathcal{W}^+_{\mathsf{f}_+}}$ near $\partial\mathcal{QF}(\Sigma)$, it is essential to show the following to proceed further:

Lemma 5.1. *$\mathcal{W}^\pm_{\mathsf{f}_n}$ converges uniformly to $\mathcal{W}^\pm_{\mathsf{f}}$ in compact subsets when f_n converges to f for the Weak-$*$ topology of $\mathcal{MF}(\Sigma)$*

Proof. We will first recall the f_n converging to f means that $\exists$ sequence of positive real numbers $\{t_n\}$ such that $t_n i(\mathsf{f}_n, \gamma)$ converges to $i(\mathsf{f}, \gamma)$ for any simple closed curve γ.
Consider the normal form introduced before. Note that, given $3g - 3$ simple closed curves labelled $\{l_1, l_2, \ldots l_{3g-3}\}$ that decompose the surface into pants and annuli, the coordinates for the measured foliation is given by just the restriction of the measured foliation in the annuli. Thus for each annuli A_i described as before, we have a set of tuples tracking the intersection of the curve with the measured foliation and the width of the annuli. When the measured foliation is induced by the quadratic differential q in the conformal class $[c]$, these quantities on the other hand are given by intergrating $\Re|\sqrt{q}|$ and $\Re|\sqrt{q}|$ and will be termed as the $h_q(l_i)$ and $v_q(l_i)$.
The extremal length of an annulus is realized by the curve which joins the two boundary components of the annulus with a twist in the positive direction. Thus in our case the extremal length of each annuli will be given by $\sqrt{h_q(l_i)^2 + v_q(l_i)^2} = \int_{l_i} |\sqrt{q}|$, and this will account for the twist parameter of the coordinates

for the hyperbolic in the conformal class considered.
We thus have $12g-12$ parameters for a point in $T^*\mathcal{T}(\Sigma)$ given by the 4-tuple $\{m(l_i), \tau(l_i), h_q(l_i), v_q(l_i))\}$, where $([c], q) \in T^*\mathcal{T}(\Sigma)$ is seen as (Σ, m) equipped with the horizontal measured foliation f of a quadratic differential q in that conformal class and m is the unique hyperbolic metric in this class. Here $h_q(l_i)$ is recording the intersection number of l_i with f.
This provides a smooth embedding of $([c], q)$ into $\mathbb{R}^{12g-12}$ in terms of a foliation on a surface. When this point in $T^*\mathcal{T}(\Sigma)$ is given by $([c_+], \sigma_+)$ as in our case, denoting the conformal class and schwarzian derivative at one end at infinity of a quasi-fuchsian manifold, we in turn get a smooth embedding of $\mathcal{I} : \mathcal{QF}(\Sigma) \cong ([c_+], \sigma_+) \hookrightarrow \mathbb{R}^{12g-12}$.
Moreover this embedding is proper. One way to see this is via the flat metric induced by the quadratic differential q realising f which is in the same conformal class as m and $\mathsf{f} \to \infty$ implies that every component of the coordinate goes to ∞. It is easy to see thus as $\mathsf{f}_n \to \mathsf{f}$, then $\mathcal{W}_{\mathsf{f}_n} \to \mathcal{W}_{\mathsf{f}}$ in compact subsets as $\mathcal{W}_{\mathsf{f}}$ corresponds to $\mathcal{I}^{-1} S_{\mathsf{f}}$ where S_{f} is the image of all possible $([c_+], \sigma_+)$ under the map $\mathcal{I}$.

Having shown this, we can now proceed to the main goal of the section. Define $\widetilde{U}$ to be the union of all the rays $\widetilde{\mathcal{R}_{\mathsf{f}_\pm}}$. This implies that $\widetilde{\Psi}$ to be the map from $\widetilde{U} \to \mathcal{QF}(\Sigma)$ which restricts to $\Psi_{\mathsf{f}_\pm}$ on each $widetilde\mathcal{R}_{\mathsf{f}_\pm}$.

Theorem 5.2. $\widetilde{\Psi}$ *is a homeormorphism from* $\widetilde{U}$ *onto it's image*

Proof. The last proposition implies that $\mathcal{W}^+_{\mathsf{f}_+}$ and $\mathcal{W}^-_{\mathsf{f}_-}$ depend continuously on $(\mathsf{f}_+, \mathsf{f}_-)$. Now, since $\widetilde{\Psi}|_{\mathcal{R}_{\mathsf{f}_\pm}}$ is given by $\widetilde{\Psi_{\mathsf{f}_\pm}}$ and $\mathcal{R}_{\mathsf{f}_\pm} = (0, \epsilon_{\mathsf{f}_\pm}](\mathsf{f}_+, \mathsf{f}_-)$, the function $\mathcal{FMF}(\Sigma) \to \mathbb{R}_{>0}$ given by $(\mathsf{f}_+, \mathsf{f}_-) \to \epsilon_{\mathsf{f}_\pm}$ is continuous as well. Morever, we also have that $\Psi_{\mathsf{f}_\pm}$ varies continuously with $(\mathsf{f}_+, \mathsf{f}_-)$. Thus $\widetilde{\Psi}$ is continuous by definition, since it's restriction to each ray is continuous and the rays varying continuously with $(\mathsf{f}_+, \mathsf{f}_-)$ in turn gives us that $\widetilde{U}$ is open.
If $\pi : \widetilde{\mathcal{QF}(\Sigma)} \to \mathcal{QF}(\Sigma)$ be the projection then $\mathfrak{B} \circ \pi \circ \widetilde{\Psi}$ restricts to the identity on $\widetilde{U} \setminus \widetilde{FMF}$ and since a pair of small filling foliation being realised by a path in $\mathcal{QF}(\Sigma)$ is uniquely determined the point in $\mathcal{F}(\Sigma)$ and the orthogonal direction to $\mathcal{F}(\Sigma)$, the map $\widetilde{QF}$ is injective.
Notice further, that the map $\widetilde{\Psi}$ sends points on $\partial\widetilde{\mathcal{FMF}}(\Sigma)$ to $\partial\widetilde{\mathcal{QF}}(\Sigma)$. By invariance of domain, we thus have that the image $\widetilde{\Psi}(\widetilde{U}) \subset \widetilde{\mathcal{QF}(\Sigma)}$ is also open, and we have that $\widetilde{\Psi} : \widetilde{U} \to \widetilde{V}$ is a homeomorphism.

Theorem 5.3. $\exists$ *an open neighbourhood* V *of* $\mathcal{F}(\Sigma) \in \mathcal{QF}(\Sigma)$ *such that* $\mathfrak{B}$ *restricted to* $V \setminus \mathcal{F}(\Sigma)$ *and it's image*

Proof. We have essentially proved the result above when every object is replaced by it's corresponding object in the blow up spaces. That is, $\widetilde{\Psi}^{-1}(\widetilde{U} \setminus \partial\mathcal{QF}(\Sigma))$ coincides with $\mathfrak{B}$. The theorem thus follows from that fact that $\pi(\widetilde{V}) = \pi(\widetilde{\Psi}(\widetilde{U})) = V$ is open in $\mathcal{QF}(\Sigma)$ since π is an open map.

6. Remarks

We will now give a proof for the fact that the line $\mathsf{P}(\mathsf{f}, \mathsf{g})$ is a geodesic for the teichmuller metric.
To see this, recall that the teichmuller metric between two pints $c_1, c_2 \in \mathcal{T}(\Sigma)$ can be given by the formula of Kerkhoff

$$d_T(c_1, c_2) = \frac{1}{2} log(\frac{ext_{c_1(l)}}{ext_{c_2}(l)})$$

for some curve l on Σ. On the other hand for the extremal length to be a continuous function we require $ext(tl) = t^2 ext(l)$.
Moreover $SL_2(\mathbb{R})$ acts on $T^*\mathcal{T}(\Sigma)$ and the diagonal flow generates the geodesic for the teichmuller metric.

Corollary 6.1. $\mathsf{P}(\mathsf{f}, \mathsf{g})$ *is a teichmuller geodesic.*

Proof. In our case, the points on $\mathsf{P}(\mathsf{f},\mathsf{g}) \subset UT\mathcal{T}(\Sigma)$ are essentially sequence of metrics on Σ which is induced by the flat metric for a quadratic differential having horizontal foliaiton given by $t\mathsf{f}$ where as the vertical one remains the same. Equipped with the teichmuller metric, $UT\mathcal{T}(\Sigma)$ is the double cover for the projectivisation of $T^*\mathcal{T}(\Sigma)$, which means that the path is given by the action of the matrix $diag(t^2, 1)$. Upon passing onto the $T\mathcal{T}(\Sigma)$ this gives us that the path $\mathsf{P}(\mathsf{f},\mathsf{g})$ so claimed, is given by the action of $\frac{1}{t}diag(t^2, 1) = diag(t, t^{-1})$ which is preciesly the diagonal flow for the $SL_2(\mathbb{R})$ action on $\mathcal{T}(\Sigma)$ generating the geodesics.

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