COUNTING DIHEDRAL SUBGROUPS

IN LATTICES OF $PSL(2, \mathbb{R})$

(joint work with Juan Souto)

RECIPROCAL GEODESICS

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HYPERBOLIC 2-ORBIFOLDS

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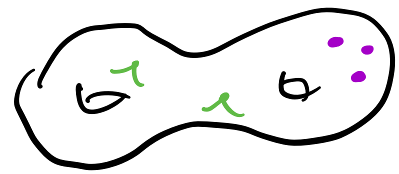
- $\Gamma < \underbrace{PSL(2,\mathbb{R})}_{= Isom^+(\mathbb{H}^2)}$ lattice (discrete, co-finite volume)

- $\mathbb{H}^2/\Gamma$ = hyperbolic surface or $\underbrace{\text{orbifold}}_{\text{if } \Gamma \text{ has torsion}}$



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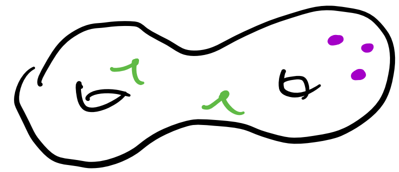
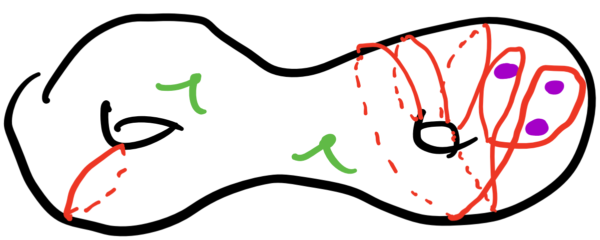


- γ closed geodesic on $\mathbb{H}^2/\Gamma$



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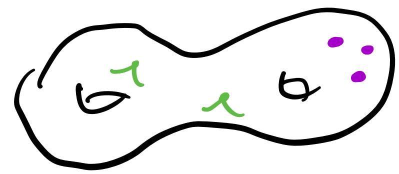

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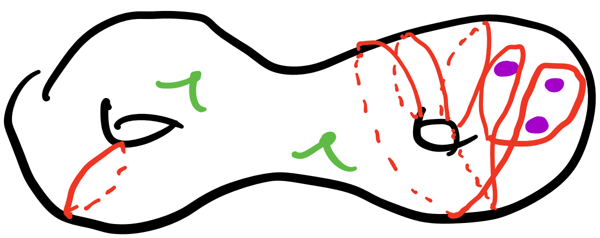

= homotopy class of a closed curve

= " " immersion $S^1 \hookrightarrow \mathbb{H}^2/\Gamma$

= periodic orbit of geodesic flow on $T^1(\mathbb{H}^2/\Gamma)$

COUNTING CLOSED GEODESICS (of bounded length)

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$$\#\{\gamma \subset \mathbb{H}^2/\Gamma \text{ closed geodesic} \mid \ell(\gamma) \leq L\} \sim \frac{e^L}{L}$$

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$$\mu_L = \sum_{\ell(\gamma)\leq L} \vec{\gamma}\ ,$$

$\vec{\gamma}$ measure given by integrating its geodesic flow orbit

$$\int f\, d\vec{\gamma} = \int_0^{\ell(\gamma)} f(\varphi_t(v))\,dt$$

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$$\frac{1}{\|\mu_L\|}\mu_L \to \mathrm{vol}, \quad \text{i.e.} \quad \frac{1}{\|\mu_L\|}\int f \, d\mu_L \longrightarrow \int f \, d\mathrm{vol} \quad \forall f \in C_c(T^1\mathbb{H}/\Gamma)$$

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at most
k-selfintersections

$\gamma \in Map(S) \cdot \gamma_0$
for some (any) fixed γ_0

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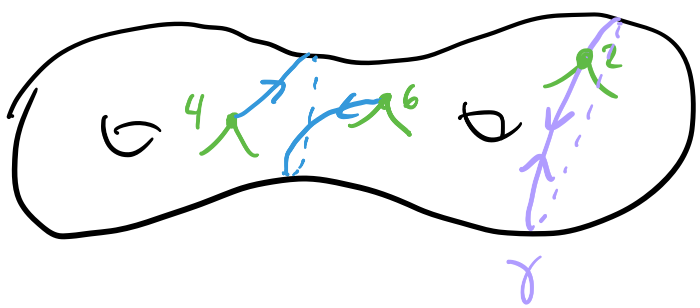

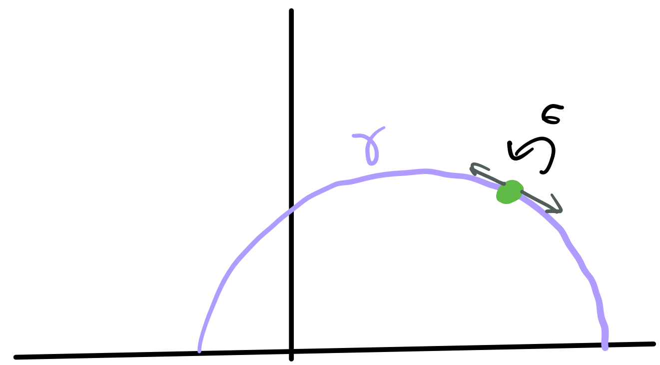

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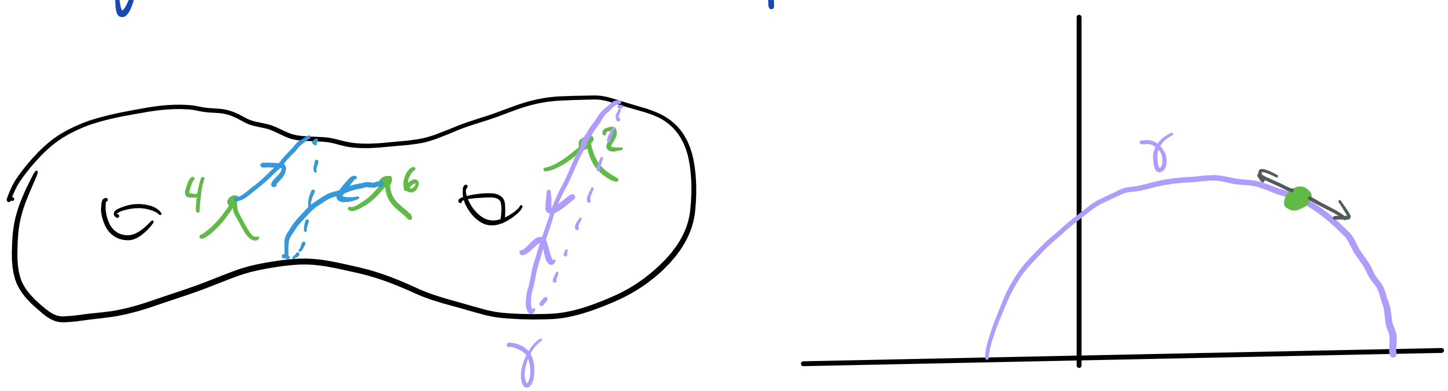
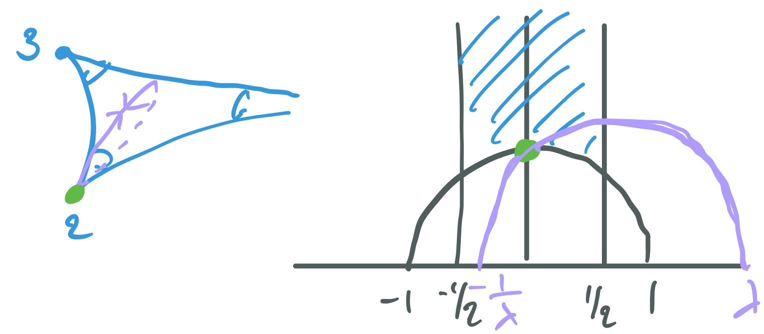


- $\gamma \rightsquigarrow$ hyperbolic element $\gamma \in \Gamma$ (conj. class)
- $[\gamma^{-1}] = [\gamma]$, i.e. $\exists\ \eta \in \Gamma$ s.t. $\eta \gamma^{-1} \eta^{-1} = \gamma$

E.g.: $\Gamma = PSL(2,\mathbb{Z})$

$\mathbb{H}^2/\Gamma$ = MODULAR SURFACE



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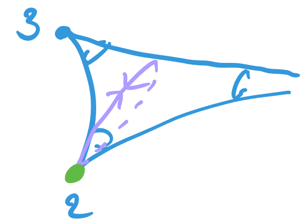

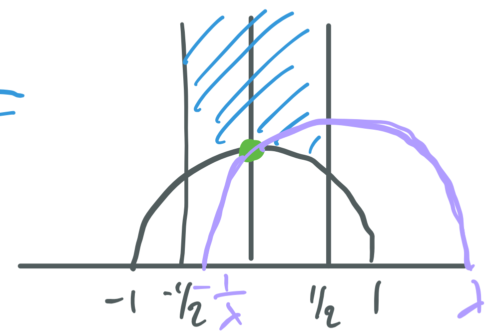

THEOREM (SARNAK '07)

$$\#\{\gamma \subset \mathbb{H}^2/PSL(2,\mathbb{Z}) \text{ reciprocal} \mid \ell(\gamma) \leq L\} \sim \frac{3}{8} e^{L/2}$$

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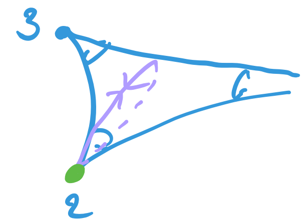

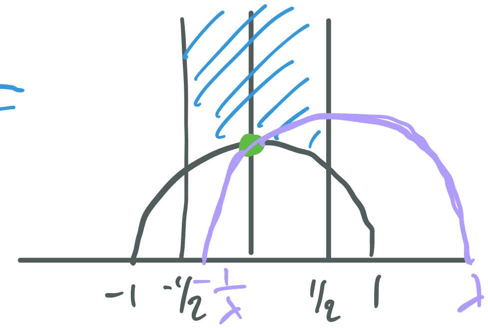

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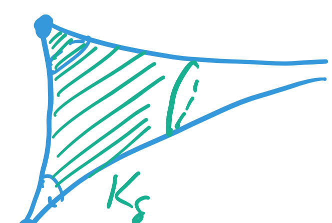
THEOREM (BOURGAIN-KONTOROVICH '19) $\forall\ \delta > 0$

$\exists\ K_\delta \subset \mathbb{H}^2/PSL(2,\mathbb{Z})$ compact such that

$$\#\{\underbrace{\gamma \subset K_\delta}_{\hookrightarrow\ \gamma \text{ "low-lying"}} \text{ RECIPROCAL} \mid \ell(\gamma) \le L\} > e^{(1-\delta)L/2}$$



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[(3) Reciprocal geodesics equidistribute in $T^1(\mathbb{H}^2/\Gamma)$]

SARNAK: $\liminf\limits_{L\to\infty} \frac{1}{\|\mu_L\|}\mu_L(K) \geq c\cdot \mathrm{vol}(K)$ $\forall$ compact $K \subset T^1(\mathbb{H}^2/\Gamma)$

$$\mu_L = \sum_{\substack{\gamma \text{ recip.} \\ \ell(\gamma)\leq L}} \vec{\gamma}$$

Theorem (E.-Souto '22) Let $\Gamma < PSL(2,\mathbb{R})$ be any lattice containing order 2 elements (involutions)

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$$c(\Gamma) = \frac{1}{4}\left(\sum_{\sigma} \frac{1}{|N_\Gamma(\sigma)|}\right)^2$$

sum over $\sigma \in \mathcal{I}_\Gamma/\Gamma$, conj. classes of involutions

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  - T_D is hyperbolic $\rightsquigarrow$ acts on a geodesic $A_D \subset \mathbb{H}^2$
  - $\gamma_D := A_D/T_D$ is a reciprocal geodesic
  - $\ell(D) := \ell(A_D/D) = \frac{1}{2}\ell(\gamma_D)$

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- If $D = \langle \sigma, \sigma' \rangle = \langle \eta, \eta' \rangle$ then (η, η') is D-CONJUGATE to either (σ, σ') or (σ', σ)

- Let $I \subset \Gamma$ be set of all involutions

$$\left.\begin{aligned} I \times I / \Delta &\longrightarrow D_\Gamma \\ (\sigma, \sigma') &\longmapsto \langle \sigma, \sigma' \rangle \end{aligned}\right\}$$

- WELL-DEFINED
- SURJECTIVE

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- $\Gamma \circlearrowright I$ & $\Gamma \circlearrowright D_\Gamma$ by conjugation

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$$\coprod_{(\sigma, \bar{\sigma}) \in J \times J} \{\sigma\} \times \Gamma \cdot \bar{\sigma} \setminus \{\sigma\} \longrightarrow I_\Gamma \times I_\Gamma \setminus \Delta /_{\Gamma}$$
$$(\sigma, \gamma \bar{\sigma} \gamma^{-1}) \longmapsto [(\sigma, \gamma \bar{\sigma} \gamma^{-1})]$$

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- Let J be a set of reps. of the elements in I/Γ

$$\coprod_{(\sigma,\bar\sigma)\in J\times J} \{\sigma\} \times \Gamma \cdot \bar\sigma \setminus \{\sigma\} \longrightarrow I_\Gamma \times I_\Gamma \setminus \Delta / \Gamma \xrightarrow{\text{above}} D_\Gamma / \Gamma$$

$$\underbrace{(\sigma, \gamma\bar\sigma\gamma^{-1}) \longmapsto [(\sigma, \gamma\bar\sigma\gamma^{-1})]}_{\text{For each } (\sigma,\bar\sigma),\ |N_\Gamma(\sigma)|\text{-to-}1} \xmapsto{2-1} [\langle \sigma, \gamma\bar\sigma\gamma^{-1} \rangle]$$

π

• Let $I \subset \Gamma$ be set of all involutions   • $\Gamma \circlearrowright I$ & $\Gamma \circlearrowright D_\Gamma$ by conjugation

$$\begin{aligned} \left(I \times I /_{\Delta}\right) /_{\Gamma} &\longrightarrow D_\Gamma /_{\Gamma} \\ [(\sigma, \sigma')] &\longmapsto [\langle \sigma, \sigma' \rangle] \end{aligned}$$

• WELL-DEFINED
• SURJECTIVE
• 2-to-1

• Let J be a set of reps. of the elements in I/Γ

$$\begin{tikzcd} \displaystyle\coprod_{(\sigma,\bar\sigma)\in J\times J} \{\sigma\} \times \Gamma\cdot\bar\sigma \setminus \{\sigma\} \arrow[r] \arrow[rr, bend left, "\pi"] & I_\Gamma \times I_\Gamma \setminus \Delta /_{\Gamma} \arrow[r, "\text{above}"] & D_\Gamma /_{\Gamma} \end{tikzcd}$$

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• π is surjective & $(N_\Gamma(\sigma) + N_\Gamma(\sigma'))$-to-1.

$$\pi : \bigsqcup_{(\sigma,\bar{\sigma}) \in J \times J} \{\sigma\} \times \Gamma \cdot \bar{\sigma} \setminus \{\sigma\} \longrightarrow D_\Gamma / \Gamma$$

$$(\sigma, \gamma\bar{\sigma}\gamma^{-1}) \longmapsto [(\sigma, \gamma\bar{\sigma}\gamma^{-1})]$$

Rewrite in terms of fixed points

$\rightsquigarrow$ LATTICE POINT COUNTING

- $\Pi : \coprod_{(\sigma,\bar\sigma)\in J\times J} \{\sigma\} \times \Gamma\cdot\bar\sigma \setminus \{\sigma\} \longrightarrow D_\Gamma / \Gamma$

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- $\sigma \in I \longleftrightarrow p_\sigma \in \mathbb{H}$ its unique fixed point

- $\Pi : \coprod_{(\sigma,\bar\sigma)\in J\times J} \{p_\sigma\} \times \Gamma(p_{\bar\sigma}) \longrightarrow D_\Gamma / \Gamma$

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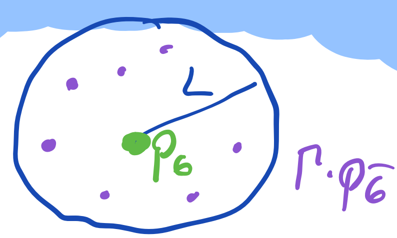
- Restrict to those $p' \in \Gamma p_{\bar\sigma}$ at distance $\leq L$ from p_σ:

- $\Pi_L : \bigsqcup_{(\sigma,\bar\sigma)\in J\times J} B^*(p_\sigma, L) \cap \Gamma\cdot p_{\bar\sigma} \longrightarrow D_\Gamma(L)/\Gamma$



- **Recap:**

$$\pi_L : \bigsqcup_{(\sigma,\bar\sigma)} B^*(p_\sigma, L) \cap \Gamma\cdot p_{\bar\sigma} \longrightarrow \mathcal{D}_\Gamma(L)/\Gamma$$
$$(p_\sigma, \gamma\cdot p_{\bar\sigma}) \longmapsto [\langle \sigma, \gamma\bar\sigma\gamma^{-1}\rangle]$$

is surjective & $|N_\Gamma(\sigma)| + |N_\Gamma(\bar\sigma)|$ - to - 1 when restricted to each $(\sigma,\bar\sigma) \in J\times J$.

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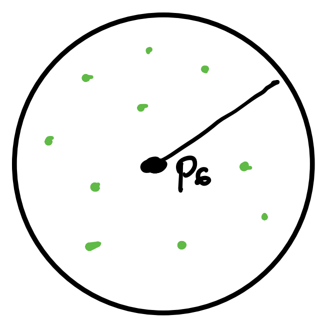
$$\Longrightarrow \quad |\mathcal{D}_\Gamma(L)/_\Gamma| = \sum_{(\sigma,\bar\sigma)\in \mathcal{J}\times\mathcal{J}} \frac{|\Gamma p_{\bar\sigma} \cap B^*(p_\sigma, L)|}{|N_\Gamma(\sigma)| + |N_\Gamma(\bar\sigma)|}$$



Delsarte: $\left| \Gamma p_{\bar{\sigma}} \cap B(p_\sigma, L) \right| \sim \dfrac{\mathrm{vol}(B(p_\sigma, L))}{\left| \mathrm{Stab}_\Gamma(p_{\bar{\sigma}}) \cdot \mathrm{vol}(\mathbb{H}^2/\Gamma) \right|}$

Delsarte: $$|\Gamma p_{\bar{\epsilon}} \cap B(p_\epsilon, L)| \sim \frac{\mathrm{vol}(B(p_\epsilon, L))}{|\mathrm{Stab}_\Gamma(p_{\bar{\epsilon}}) \cdot \mathrm{vol}(\mathbb{H}^2/\Gamma)|}$$

$$\sim \frac{e^L}{2\,|N_\Gamma(\bar{\epsilon})| \cdot |\chi(\mathbb{H}^2/\Gamma)|}$$

- $\mathrm{vol}(B) = 2\pi(\cosh(R) - 1)$
  $\sim \pi e^R$
- $\mathrm{vol}(\mathbb{H}^2/\Gamma) = 2\pi|\chi|$
- $\mathrm{Stab}_\Gamma(p_{\bar{\epsilon}}) = N_\Gamma(\bar{\epsilon})$

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$$\Longrightarrow \; |P_\Gamma(L)/\Gamma| \sim \frac{e^L}{|\chi(\mathbb{H}^2/\Gamma)|} \underbrace{\sum_{\substack{(\sigma,\bar{\sigma}) \\ \in J \times J}} \frac{1}{2\,|N_\Gamma(\bar{\sigma})|\,(|N_\Gamma(\sigma)| + |N_\Gamma(\bar{\sigma})|)}}_{C(\Gamma)}$$

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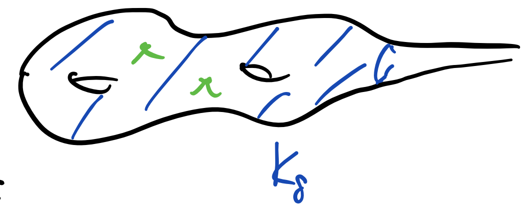
$$\Longrightarrow \left|\mathcal{D}_\Gamma(L)/\Gamma\right| \sim \frac{e^L}{|\chi(\mathbb{H}^2/\Gamma)|} \underbrace{\sum_{\substack{(\sigma,\bar{\sigma}) \\ \in \mathcal{J}\times\mathcal{J}}} \frac{1}{2|N_\Gamma(\bar{\sigma})|(|N_\Gamma(\sigma)|+|N_\Gamma(\bar{\sigma})|}}_{C(\Gamma)}$$

$$\Longrightarrow \#\{\text{RECIPROCAL GEODESICS } \gamma \mid \ell(\gamma) \leq L\} \sim \frac{e^{L/2} \cdot C(\Gamma)}{|\chi(\mathbb{H}^2/\Gamma)|}$$

\#

IDEA FOR LOW-LYING



$\forall\ \delta > 0\ \ \exists$ compact $K_\delta \subset \mathbb{H}^2/\Gamma$ such that

$$\#\{\text{reciprocal geodesics } \gamma \subset K_\delta \mid \ell(\gamma) \leq L\} > e^{(1-\delta)L/2}$$

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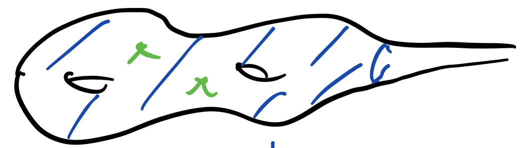

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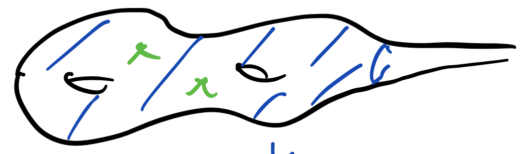

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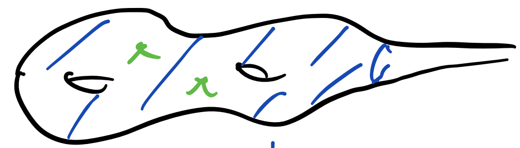

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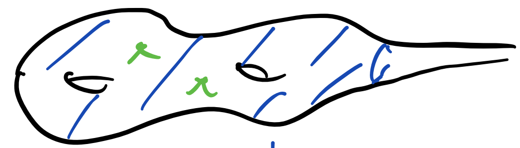
$$\delta(\Gamma_n) = \limsup \frac{1}{L} \log |\{y \in \Gamma_n \cdot x_0 \text{ w/ } d(x_0, y) \le L\}|$$

IDEA FOR LOW-LYING



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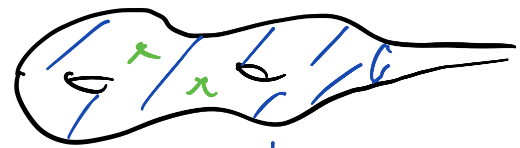

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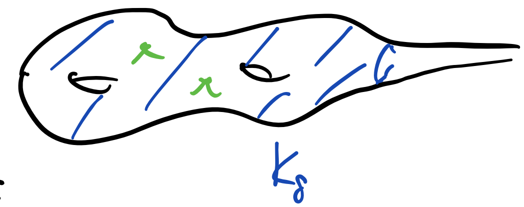
IDEA FOR LOW-LYING



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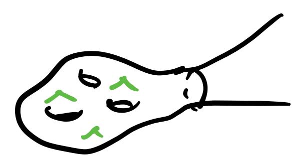
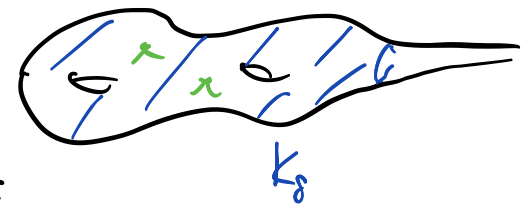


IDEA FOR LOW-LYING



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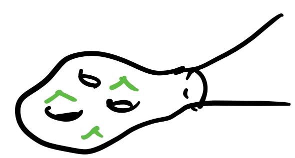
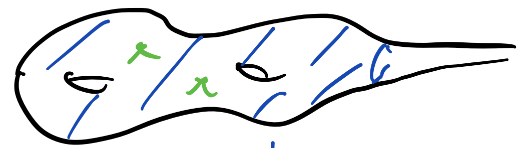
  $\Longrightarrow$ $\exists$ $K \subset \mathbb{H}^2/\Gamma$ containing all (reciprocal) geodesics



IDEA FOR LOW-LYING



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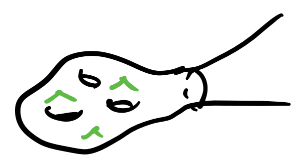

$\Longrightarrow \exists \ K \subset \mathbb{H}^2/\Gamma$ containing all (reciprocal) geodesics
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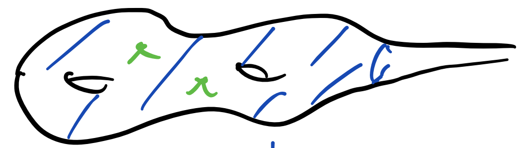
$\Longrightarrow$ there are $\geq \dfrac{1}{2\,|N_\Gamma(\sigma)|} \left| \Gamma' . p_\sigma \cap B^*(p_\sigma, L/2) \right|$ recip. geod

IDEA FOR LOW-LYING



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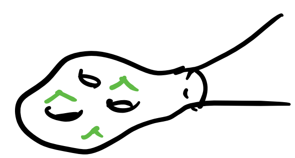

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Ex: $\Gamma = PSL(2,\mathbb{Z})$

Let Γ_k = subgp. generated by $\{\tau, \eta^i \tau \eta^{-i} \mid i = -k, \ldots, k\}$

$\tau(z) = -\frac{1}{z}$, $\eta(z) = z+2$

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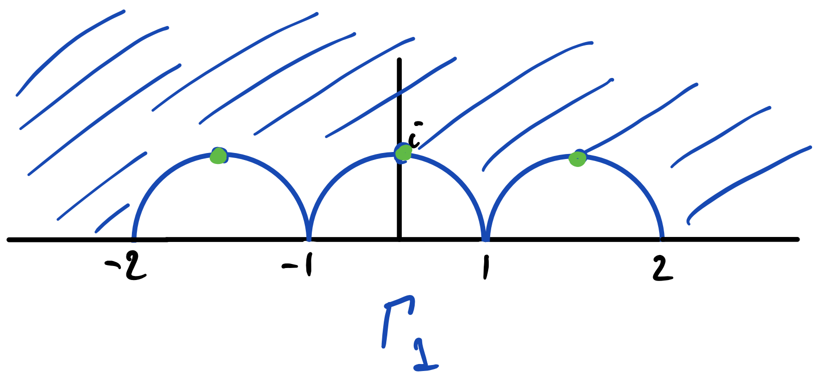

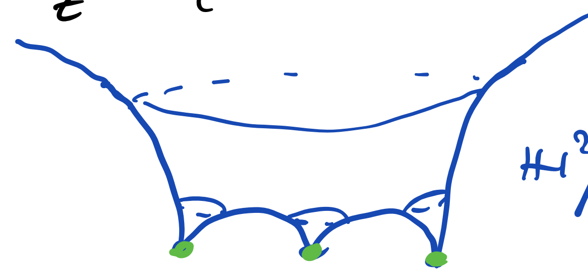

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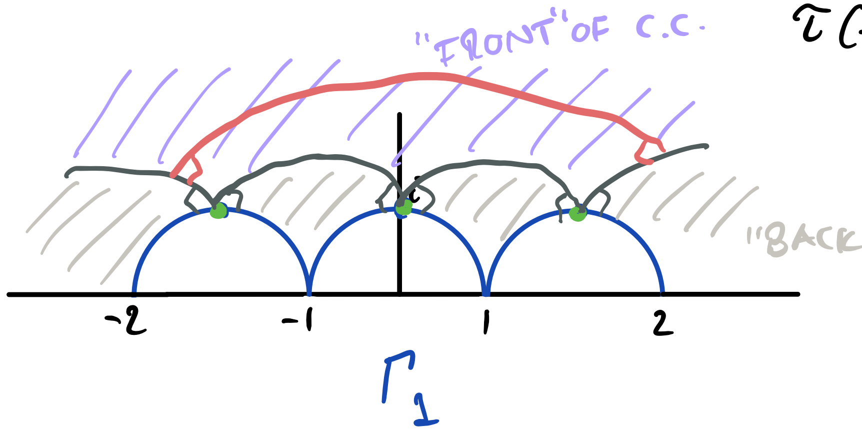

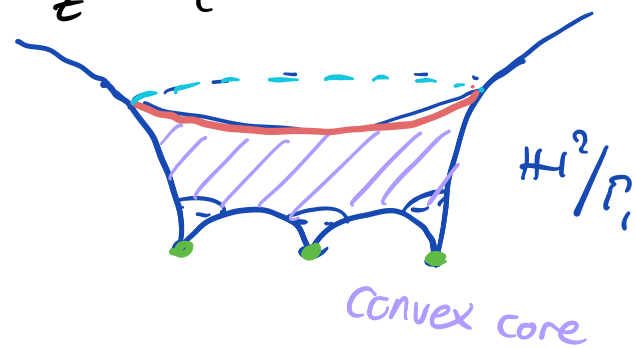

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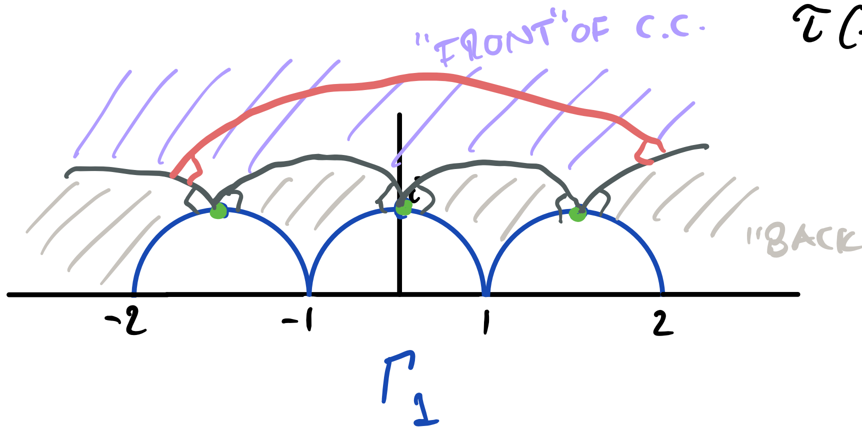

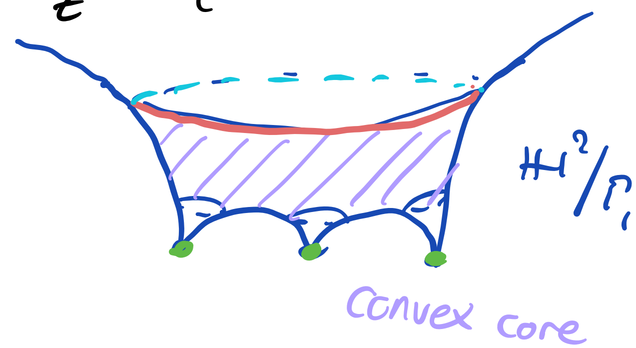

FACT: $\delta(\Gamma_k) \geq \frac{1}{2} + \sqrt{\frac{1}{4} - C \cdot \frac{\ell(\partial CC)}{vol(C.C)}}$

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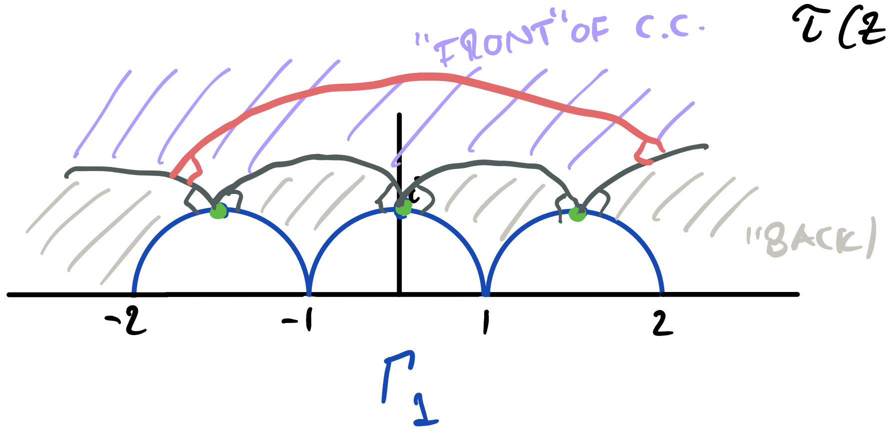

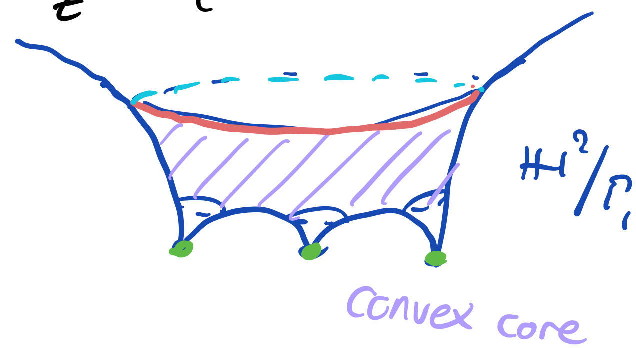

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- So if $\frac{\ell(\text{boundary convex core})}{\text{vol (convex core)}} \to 0$ then $\delta(\Gamma_k) \to 1$.

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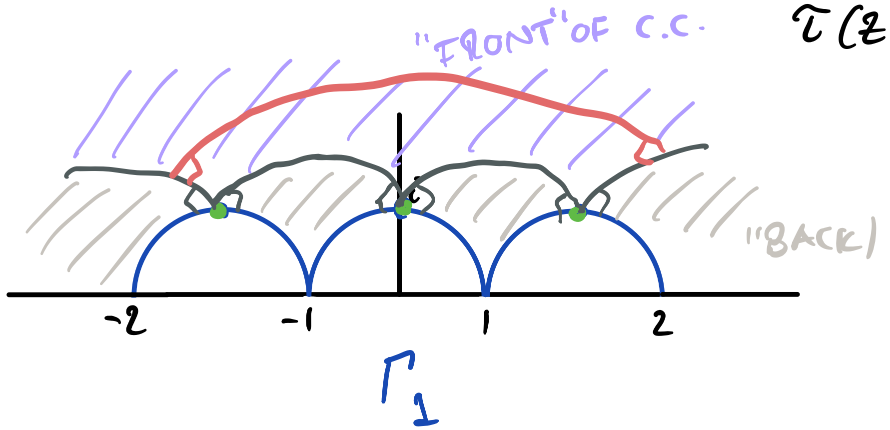

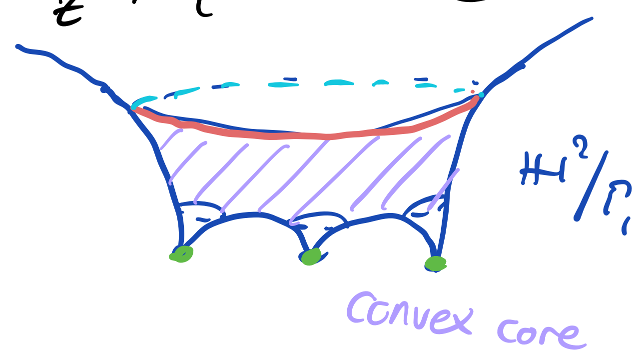

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- Here, vol(convex core) grow linearly, while length grow log as $k \to \infty$.

THANK YOU!