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A COMPLETE PROOF OF BEAL'S CONJECTURE

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I dedicate this work to my wife Wahida, my daughter Sinda, my son Mohamed Mazen and to the memory of my beloved parents

Abstract. — In 1997, Andrew Beal announced the following conjecture: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and C have a common factor.* We begin to construct the polynomial $P(x) = (x-A^m)(x-B^n)(x+C^l) = x^3-px+q$ with p, q integers depending on A^m, B^n and C^l. We resolve $x^3-px+q=0$ and we obtain the three roots x_1, x_2, x_3 as functions of p and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3-px+q=0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or do have not a common factor. Many cases are studied. Three numerical examples are given.

Résumé. — En 1997, Andrew Beal avait annoncé la conjecture suivante: *Soient $A, B, C, m, n,$ et l des entiers positifs avec $m, n, l > 2$. Si $A^m + B^n = C^l$ alors $A, B,$ et C ont un facteur commun.*
Nous commençons par construire le polynôme $P(x) = (x-A^m)(x-B^n)(x+C^l) = x^3-px+q$ avec p, q des entiers qui dépendent de A^m, B^n et C^l. Nous résolvons $x^3-px+q=0$ et nous obtenons les trois racines x_1, x_2, x_3 comme fonctions de p, q et d'un paramètre θ. Comme $A^m, B^n, -C^l$ sont les seules racines de $x^3-px+q=0$, nous discutons les conditions pourque x_1, x_2, x_3 soient des entiers. Plusieurs cas ont été étudiés. Trois exemples numériques sont présentés.

1. Introduction

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1.1. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1}$$

then $A, B,$ and C have a common factor.

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The purpose of this paper is to give a complete proof of Beal's conjecture. Our idea is to construct a polynomial $P(x)$ of order three having as roots A^m, B^n and $-C^l$ with the condition (1.1). We obtain $P(x) = x^3 - px + q$ where p, q are depending of A^m, B^n and C^l. Then we express $A^m, B^n, -C^l$ the roots of $P(x) = 0$ in function of p and a parameter θ that depends of the A, B, C. The calculations give that $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2\dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$ where a, b are two positive coprime integers. Beside the trivial cases, there are two main hypothesis to study:

- the first hypothesis is: $3 \mid a \quad and \quad b \mid 4p$,
- the second hypothesis is: $3 \mid p \quad and \quad b \mid 4p$.

We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending of each individual case, we obtain that A, B, C have or do have not a common factor. Our proof of the conjecture contains many cases to study. there are many cases where we use elementary number theory and some cases need more research to obtain finally the solution. I think that my new idea detailed above overcomes the apparent limitations of the methods I am using.

The paper is organized as follows. In section 1, It is an introduction of the paper. The trivial case, where $A^m = B^n$, is studied in section 2. The preliminaries needed for the proof are given in section 3 where we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. The section 4 is the preamble of the proof of the main theorem. Section 5 treats the cases of the first hypothesis $3 \mid a$ and $b \mid 4p$. We study the cases of the second hypothesis $3 \mid p$ and $b \mid 4p$ in section 6. Finally, we present three numerical examples and the conclusion in section 7.

In 1997, Andrew Beal [1] announced the following conjecture :

Conjecture 1.2. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.2}$$

then $A, B,$ and C have a common factor.

2. Trivial Case

We consider the trivial case when $A^m = B^n$. The equation (1.2) becomes:

$$2A^m = C^l \tag{2.1}$$

then $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^q.C_1$ with $q \geqslant 1$, $2 \nmid C_1$ and $2A^m = 2^{ql}C_1^l \Longrightarrow A^m = 2^{ql-1}C_1^l$. As $l > 2$, $q \geqslant 1$, then $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^r A_1$ with $r \geqslant 1$ and $2 \nmid A_1$. The equation (2.1),becomes:

$$2 \times 2^{rm} A_1^m = 2^{ql} C_1^l \tag{2.2}$$

As $2 \nmid A_1$ and $2 \nmid C_1$, we obtain the first condition :
(2.3)
there exists two positive integers r, q with $r.q \geqslant 1$ so that $\boxed{ql = mr + 1}$

Then from (2.2):

$$A_1^m = C_1^l \tag{2.4}$$

2.1. Case 1 $A_1 = 1 \Longrightarrow C_1 = 1$

Using the condition (2.3) above, we obtain $2.(2^r)^m = (2^q)^l$ and the Beal conjecture is verified.

2.2. Case 2 $A_1 > 1 \Longrightarrow C_1 > 1$

From the fundamental theorem of the arithmetic, we can write:

$$A_1 = a_1^{\alpha_1} \dots a_I^{\alpha_I}, \quad a_1 < a_2 < \dots < a_I \Longrightarrow A_1^m = a_1^{m\alpha_1} \dots a_I^{m\alpha_I} \tag{2.5}$$

$$C_1 = c_1^{\beta_1} \dots c_J^{\beta_J}, \quad c_1 < c_2 < \dots < c_J \Longrightarrow C_1^l = c_1^{l\beta_1} \dots c_J^{l\beta_J} \tag{2.6}$$

where a_i (respectively c_j) are distinct positive prime numbers and α_i (respectively β_j) are integers > 0.

From (2.4) and using the uniqueness of the factorization of A_1^m and C_1^l, we obtain necessary:

$$\begin{cases} I = J \\ a_i = c_i, \quad i = 1, 2, \dots, I \\ m\alpha_i = l\beta_i \end{cases} \tag{2.7}$$

As one $a_i \mid A^m \Longrightarrow a_i \mid B^m \Longrightarrow a_i \mid B$ and in this case, the Beal conjecture is verified.

We suppose in the following that $A^m > B^n$.

3. Preliminaries

Let $m, n, l \in \mathbb{N}^* \; > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{3.1}$$

We call:

$$\begin{aligned} P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l) \\ +x[A^m B^n - C^l(A^m + B^n)] + C^l A^m B^n \end{aligned} \tag{3.2}$$

Using the equation (3.1), $P(x)$ can be written as:

$$\boxed{P(x) = x^3 + x[A^m B^n - (A^m + B^n)^2] + A^m B^n(A^m + B^n)} \tag{3.3}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^m B^n = A^{2m} + A^m B^n + B^{2n}$$
$$q = A^m B^n (A^m + B^n)$$

As $A^m \neq B^n$, we have $p > (A^m - B^n)^2 > 0$. Equation (3.3) becomes:

$$P(x) = x^3 - px + q$$

Using the equation (3.2), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{3.4}$$

To resolve (3.4) let:

$$x = u + v$$

Then $P(x) = 0$ gives:

(3.5)

$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0$$

To determine u and v, we obtain the conditions:

$$u^3 + v^3 = -q$$
$$uv = p/3 > 0$$

Then u^3 and v^3 are solutions of the second order equation:

$$X^2 + qX + p^3/27 = 0 \tag{3.6}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27}$$

Let:

$$\begin{aligned}\bar{\Delta} = 27q^2 - 4p^3 &= 27(A^mB^n(A^m+B^n))^2 - 4[(A^m+B^n)^2 - A^mB^n]^3 \\ &= 27A^{2m}B^{2n}(A^m+B^n)^2 - 4[(A^m+B^n)^2 - A^mB^n]^3 \end{aligned} \tag{3.7}$$

Denoting :

$$\alpha = A^mB^n > 0$$
$$\beta = (A^m+B^n)^2$$

we can write (3.7) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta-\alpha)^3 \tag{3.8}$$

As $\alpha \neq 0$, we can also rewrite (3.8) as :

$$\bar{\Delta} = \alpha^3\left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3\right)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha}$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t-1)^3)$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t)$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:

The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m+B^n)^2 > 4\alpha = 4A^mB^n$$

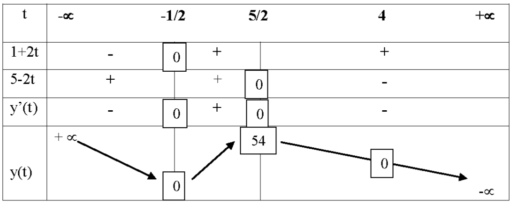

Figure 3.1. The table of variations

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (3.6) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

3.1. Expressions of the roots

Proof. — The solutions of (3.6) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}}$$

$$\text{and } \; sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0$$

$$cos\theta = -\frac{q}{2\rho} < 0$$

Then $\theta\,[2\pi] \in]+\dfrac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{3.9}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{3.10}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{3.11}$$

Let:

$$u = re^{i\psi} \tag{3.12}$$

$$\text{and } \ j = \frac{-1+i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{3.13}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1+i\sqrt{3}}{2} = \bar{j} \tag{3.14}$$

j is a complex cubic root of the unity $\Longleftrightarrow j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{3.15}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{3.16}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{3.17}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{3.18}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{3.19}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{3.20}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{3.21}$$

$$v_2 = \overline{u_2} \tag{3.22}$$

$$v_3 = \overline{u_3} \tag{3.23}$$

We obtain as real solutions of the equation (3.5):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\frac{\theta}{3} > 0 \tag{3.24}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{3.25}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{3.26}$$

We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p}cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$

$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{3.27}$$

As $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$, then $sin\dfrac{\theta}{3}$ and $cos\dfrac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{3.28}$$

which is true since $\dfrac{\theta}{3} \in]+\dfrac{\pi}{6}, +\dfrac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (3.4), we consider, as A^m is supposed great than B^n, the expressions:

(3.29)

$$\boxed{\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3} \\ \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho}cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho}cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}}$$

□

4. Preamble of the Proof of the Main Theorem

THEOREM 4.1. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{4.1}$$

then $A, B,$ and C have a common factor.

Proof. — $A^m = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\dfrac{\theta}{3}$ is also an integer. But :

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{4.2}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{4.3}$$

As A^{2m} is an integer and p is an integer, then $cos^2\dfrac{\theta}{3}$ must be written under the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{4.4}$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, ..., z$, $\alpha, \beta, ...$, $A, B, C, ...$ and $\Delta, \Phi, ...$ represent positive integers except the parameters θ, ρ, or others cited in the text, are reals.

4.1. Case $cos^2\dfrac{\theta}{3} = \dfrac{1}{b}$

We obtain:

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{4.5}$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} < \dfrac{3}{4} \Rightarrow \dfrac{1}{4} < \dfrac{1}{b} < \dfrac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

4.1.1. $b = 1$

$b = 1 \Rightarrow 4 < 3$ which is impossible.

4.1.2. $b = 2$

$b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$A^{2m} = (A^m)^2 = \frac{2p}{3} = 2.p' \Longrightarrow 2 \mid p' \Longrightarrow p' = 2^\alpha p_1^2$$
$$with \quad 2 \nmid p_1, \quad \alpha + 1 = 2\beta$$
$$A^m = 2^\beta p_1 \tag{4.6}$$
$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = p' = 2^\alpha p_1^2 \tag{4.7}$$

From the equation (4.6), it follows that $2 \mid A^m \Longrightarrow A = 2^i A_1$, $i \geqslant 1$ and $2 \nmid A_1$. Then, we have $\beta = i.m = im$. The equation (4.7) implies that $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

Case $2 \mid B^n$**:** - If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geqslant 1$, $2 \mid C^l \Longrightarrow 2 \mid C$ according to $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2im - 1 - jn \leqslant 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

Case $2 \mid C^l$**:** If $2 \mid C^l$: with the same method used above, we obtain the identical results.

4.1.3. $b = 3$

$b = 3 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{3} = \frac{4p}{9} \Rightarrow 9 \mid p \Rightarrow p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geqslant 3$, it follows that $2 \mid p'$, then $2 \mid A^m$. But $B^n C^l = 5p'$ and $2 \mid (B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

4.2. Case $a > 1,\ cos^2\dfrac{\theta}{3} = \dfrac{a}{b}$

We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b}\,; \quad A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{4.8}$$

where a, b verify one of the two conditions:

$$\boxed{\{3 \mid a \quad and \quad b \mid 4p\}} \text{ or } \boxed{\{3 \mid p \quad and \quad b \mid 4p\}} \tag{4.9}$$

and using the equation (3.10), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{4.10}$$

For these conditions, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\dfrac{p}{3}.cos^2\dfrac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (4.9) in the following two sections.

5. Hypothesis : $\{3 \mid a \quad and \quad b \mid 4p\}$

We obtain :

$$3 \mid a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{5.1}$$

5.1. Case $b = 2$ and $3 \mid a$

A^{2m} is written as:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{5.2}$$

Using the equation (5.1), A^{2m} becomes :

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{5.3}$$

but $cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

5.2. Case $b = 4$ and $3 \mid a$

A^{2m} is written :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{5.4}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{5.5}$$

which is impossible. Then the case $b = 4$ is impossible.

5.3. Case $b = p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{5.6}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{5.7}$$

$$\exists a" \ / \ a' = a"^2 \tag{5.8}$$

$$and \quad B^nC^l = p - A^{2m} = b - 4a' = b - 4a"^2 \tag{5.9}$$

The calculation of A^mB^n gives :

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{5.10}$$

The left member of (5.10) is an integer and p also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ is written under the form :

$$2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2} \tag{5.11}$$

where k_1, k_2 are two coprime integers and $k_2 \mid p \Longrightarrow p = b = k_2.k_3$, $k_3 \in \mathbb{N}^*$.

5.3.1. We suppose that $k_3 \neq 1$

We obtain :

$$A^m(A^m + 2B^n) = k_1.k_3 \tag{5.12}$$

Let μ be a prime integer with $\mu \mid k_3$, then $\mu \mid b$ and $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** A-1-1- If $\mu \mid A^m \Longrightarrow \mu \mid A$ and $\mu \mid A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu \mid 4a' \Longrightarrow (\mu = 2$, but $2 \mid a')$ or $(\mu \mid a')$. Then $\mu \mid a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu \mid (A^m + 2B^n)$ as:

$$A^m + 2B^n = \mu.t' \tag{5.13}$$

It follows :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m) \tag{5.14}$$

As $p = b = k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ and $b = \mu\mu'$, so we can write:

$$\mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m) \tag{5.15}$$

From the last equation, we obtain $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** A-1-2-1- If $\mu \mid B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we arrive to :

$$\mu \mid 3B^n \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases} \tag{5.16}$$

** A-1-2-2-1- If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

5.3.2. We assume now $k_3 = 1$

Then :

$$A^{2m} + 2A^m B^n = k_1 \tag{5.17}$$

$$b = k_2 \tag{5.18}$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{b} \tag{5.19}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p-a) = k_1^2 \tag{5.20}$$

but $a' = a"^2$, then $p - a$ is a square. Let:

$$\lambda^2 = p - a = b - a = b - 3a"^2 \Longrightarrow \lambda^2 + 3a"^2 = b \tag{5.21}$$

The equation (5.20) becomes:

$$4^2 a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{5.22}$$

taking the positive root, but $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, then :

$$A^m + 2B^n = 2\lambda \Longrightarrow \lambda = a" + B^n \tag{5.23}$$

** A-2-1- As $A^m = 2a" \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, with $i \geqslant 1$ and $2 \nmid A_1$, then $A^m = 2a" = 2^{im} A_1^m \Longrightarrow a" = 2^{im-1} A_1^m$, but $im \geqslant 3 \Longrightarrow 4 \mid a"$. As $p = b = A^{2m} + A^m B^n + B^{2n} = \lambda = 2^{im-1} A_1^m + B^n$. Taking its square, then :

$$\lambda^2 = 2^{2im-2} A_1^{2m} + 2^{im} A_1^m B^n + B^{2n}$$

As $im \geqslant 3$, we can write $\lambda^2 = 4\lambda_1 + B^{2n} \Longrightarrow \lambda^2 \equiv B^{2n} (\mathrm{mod} 4) \Longrightarrow \lambda^2 \equiv B^{2n} \equiv 0 (\mathrm{mod}\ 4)$ or $\lambda^2 \equiv B^{2n} \equiv 1 (\mathrm{mod}\ 4)$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0 (\mathrm{mod}\ 4) \Longrightarrow 4 \mid \lambda^2 \Longrightarrow 2 \mid (b - a)$. But $2 \mid a$ because $a = 3a' = 3a"^2 = 3 \times 2^{2(im-1)} A_1^{2m}$ and $im \geqslant 3$. Then $2 \mid b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1 (\bmod 4)$. As $A^m = 2^{im-1} A_1^m$ and $im-1 \geqslant 2$, then $A^m \equiv 0 (\bmod 4)$. As $B^{2n} \equiv 1 (\bmod 4)$, then B^n verifies $B^n \equiv 1 (\bmod 4)$ or $B^n \equiv 3 (\bmod 4)$ which gives for the two cases $B^n C^l \equiv 1 (\bmod 4)$.

We have also $p = b = A^{2m} + A^m B^n + B^{2n} = 4a' + B^n.C^l = 4a"^2 + B^n C^l \Longrightarrow B^n C^l = \lambda^2 - a"^2 = B^n.C^l$, then $\lambda, a" \in \mathbb{N}^*$ are solutions of the Diophantine equation :

$$x^2 - y^2 = N \tag{5.24}$$

with $N = B^n C^l > 0$. Let $Q(N)$ be the number of the solutions of (5.24) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the equation (5.24) (see theorem 27.3 in [2]):

- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 (\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leqslant x < [x]+1$.

As $\lambda, a"$ is a couple of solutions of the Diophantine equation (5.24), then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 2\lambda \tag{5.25}$$

$$d - d' = 2a" \tag{5.26}$$

** A-2-1-2-1- As $C^l > B^n$, we take $d = C^l$ and $d' = B^n$. It follows:

$$2\lambda = C^l + B^n = A^m + 2B^n \tag{5.27}$$

$$2a" = C^l - B^n = A^m \tag{5.28}$$

From the paragraph A-2-1 above, we have $\lambda = p = A^{2m} + A^m B^n + B^{2n} > (A^m + 2B^n)$, then the case $d = C^l$ and $d' = B^n$ gives a contradiction.

** A-2-1-2-2- Now, we consider the case $d = c_1^{lr-1} C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^r C_1$, $r \geqslant 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (5.25-5.26):

$$c_1^{lr-1} C_1^l + c_1.B^n = 2\lambda \tag{5.29}$$

$$c_1^{lr-1} C_1^l - c_1.B^n = 2a" \tag{5.30}$$

As $l \geqslant 3$, from the last two equations above, it follows that $c_1 \mid (2\lambda)$ and $c_1 \mid (2a")$. Then $c_1 = 2$, or $c_1 \mid \lambda$ and $c_1 \mid a"$.

** A-2-1-2-2-1- We suppose $c_1 = 2$. As $2 \mid A^m$ and $2 \mid C^l$ because $l \geqslant 3$, it follows $2 \mid B^n$, then $2 \mid (p = b)$. Then the contradiction with a, b coprime.

** A-2-1-2-2-2- We suppose $c_1 \neq 2$ and $c_1 \mid a"$ and $c_1 \mid \lambda$. $c_1 \mid a" \Longrightarrow c_1 \mid a$ and $c_1 \mid (A^m = 2a")$. $B^n = C^l - A^m \Longrightarrow c_1 \mid B^n$. It follows that $c_1 \mid (p = b)$. Then the contradiction with a, b coprime.

The others cases of the expressions of d and d' with d, d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (4.10) given by $b < 4a < 3b$. In our case, the condition becomes :

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{5.31}$$

and $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} \overbrace{>}^{?} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

5.4. Case $b \mid p \Rightarrow\ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ and $3 \mid a$

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{5.32}$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{5.33}$$

but $\sqrt[3]{\rho^2} = \frac{p}{3}$, using $cos^2\frac{\theta}{3} = \frac{3.a'}{b}$, we obtain:

(5.34)

$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a')$$

As $p = b.p'$, and $p' > 1$, so we have :

$$B^n C^l = p'(b - 4a') \tag{5.35}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{5.36}$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4a'p' = (A^m)^2 \Longrightarrow p' \mid a'$. But $B^n C^l = p'(b - 4a') \Longrightarrow p' \mid B^n$ or $p' \mid C^l$.

** B-1-1- If $p' \mid B^n \Longrightarrow p' \mid B \Longrightarrow B = p' B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1} B_1^n C^l = b - 4a'$. But $n > 2 \Rightarrow (n-1) > 1$ and $p' \mid a'$, then $p' \mid b \Longrightarrow a$ and b are not coprime, then the contradiction.

** B-1-2- If $p' \mid C^l \Longrightarrow p' \mid C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4a'p' \Longrightarrow A^m = 2a".p_1$ with $a' = a"^2$ and $p' = p_1^2$, then $a", p_1$ are also coprime. As $A^m = 2a".p_1$ then $2 \mid a"$ or $2 \mid p_1$.

** B-2-1-1- $2 \mid a"$, then $2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a".p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \Longrightarrow p' = \omega^{2m}$, then: $B^n C^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = b - 4a'$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a, then $\omega \nmid b$. The conjecture (1.2) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geqslant 1$, in this case with the same method, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (b-4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (1.2) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m} B_1^n.C^l = b - 4a'$. As $\omega \mid C$ using $C^l = A^m + B^n$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l} B_1^n.C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^n C_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega \mid (b - 4a')$ with ω, a, b coprime and the conjecture (1.2) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega \mid C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$ and $f \geqslant 1$ an integer, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f.m}\Omega^{2m}(b - 4a') \Longrightarrow \omega_1 \mid (B^n C^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** B-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2mf-nj}\Omega^{2m}(b - 4a')$:

** B-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(b - 4a')$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b - 4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a', then coprime with a, we deduce $\omega_1 \nmid b$. Then the conjecture (1.2) is verified.

** B-2-1-1-2-2-1-2- If $2f.m - n.j \geqslant 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C \Longrightarrow \omega_1 \mid (b - 4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (1.2) is verified.

** B-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(b - 4a')$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(b - 4a')$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^n C_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (b - 4a')$ with ω_1, a, b coprime and the conjecture (1.2) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1 \mid C^l$.

** B-2-1-2- If $2 \mid p_1$, then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a" \Longrightarrow 2 \mid a"$ as $m \geqslant 3$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$, as $A^m = 2a"p_1$, p_1 is written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p' = 2^{2m-2}\omega^{2m}$. It follows $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a') \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** B-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a')$, it follows if $2 \mid (b-4a') \Longrightarrow 2 \mid b$ but as $2 \nmid a'$, there is no contradiction with a, b coprime and the conjecture (1.2) is verified.

** B-2-1-2-2-2- If $2 \mid C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a' are supposed not coprime. Let ω be a prime integer so that $\omega \mid a'$ and $\omega \mid p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4a'p' \Longrightarrow 3 \mid A$, but $3 \mid p' \Longrightarrow 3 \mid p$, as $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p'$ that gives $a = 3^\alpha a_1 = 3a' \Longrightarrow a' = 3^{\alpha-1}a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a'p' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha-1+\mu}.a_1.p_1 \Longrightarrow \alpha+\mu-1 = 2im$. As $p = bp' = b.3^\mu p_1 = 3^\mu.b.p_1$. The exponent of the term 3 of p is k, the exponent of the term 3 of the left member of the last equation is μ. If $3 \mid b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $min(2im, 2jn, im+jn) = \mu$, recall that $\alpha+\mu-1 = 2im$. But $B^nC^l = p'(b-4a')$ that gives $3^{(nj+hl)}B_1^nC_1^l = 3^\mu p_1(b - 4 \times 3^{(\alpha-1)}a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. Let $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. Then, we obtain the conditions:

$$k = min(2im, 2jn, im+jn) = \mu \tag{5.37}$$

$$\alpha + \mu - 1 = 2im \tag{5.38}$$

$$\epsilon = hl = min(im, jn) \tag{5.39}$$

$$3^{(nj+hl)}B_1^nC_1^l = 3^\mu p_1(b - 4 \times 3^{(\alpha-1)}a_1) \tag{5.40}$$

** B-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (5.38) becomes:

$$\mu = 2im$$

and the first equation (5.37) is written as:

$$k = min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leqslant 2jn \Longrightarrow im \leqslant jn \Longrightarrow hl = im$, and (5.40) gives $\mu = 2im = nj + hl = im + nj \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (1.2) is verified.
- If $k = 2jn \Longrightarrow 2jn = 2im \Longrightarrow im = jn = hl$. Hence $3 \mid A, 3 \mid B$ and $3 \mid C$ and the conjecture (1.2) is verified.
- If $k = im + jn = 2im \Longrightarrow im = jn \Longrightarrow \epsilon = hl = im = jn$ case that is seen above and we deduce that $3 \mid A, 3 \mid B$ and $3 \mid C$, and the conjecture (1.2) is verified.

** B-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geqslant 2$ and $a' = 3^{\alpha-1}a_1$.

- If $k = 2im \Longrightarrow 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = 2jn = \mu \Longrightarrow 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \Longrightarrow jn < im \Longrightarrow 2jn < im+jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im+jn)$. We obtain $jn = hl < im$ and the equation (5.40) becomes:

$$B_1^n C_1^l = p_1(b - 4 \times 3^{(\alpha-1)} a_1)$$

The conjecture (1.2) is verified.

- If $k = im + jn \leqslant 2im \Longrightarrow jn \leqslant im$ and $k = im + jn \leqslant 2jn \Longrightarrow im \leqslant jn \Longrightarrow im = jn \Longrightarrow k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.

- If $k = im + jn < 2im \Longrightarrow jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4a'p' = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p'(b - 4a') = \omega^\mu p_1(b - 4a') \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** B-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = \omega^\mu b p_1 = \omega^k(\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then :

- If $\mu = k$, then $\omega \nmid b$ and the conjecture (1.2) is verified.

- If $k > \mu$, then $\omega \mid b$, but $\omega \mid a$ we deduce the contradiction with a, b coprime.

- If $k < \mu$, it follows from :

$$\omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, we obtain the same results as B-2-2-2-1- above.

5.5. Case $b = 2p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' = (A^m)^2 \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$$

Then $2 \mid a$ and $2 \mid b$ that is a contradiction with a, b coprime.

5.6. Case $b = 4p$ and $3 \mid a$

We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a' = (A^m)^2 = a"^2$$
$$\text{with} \quad A^m = a"$$

Let us calculate $A^m B^n$, we obtain:

$$A^m B^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^m B^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3}$$

Let:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{5.41}$$

The left member of (5.41) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2 \mid p \Longrightarrow p = k_2.k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^m B^n = k_3.k_1$$

Let μ be a prime integer and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** C-1-1- If $\mu \mid (A^m = a") \Longrightarrow \mu \mid (a"^2 = a') \Longrightarrow \mu \mid (3a' = a)$. As $\mu \mid k_3 \Longrightarrow \mu \mid p \Longrightarrow \mu \mid (4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{5.42}$$

$\mu \mid (A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t'$$

Then:

$$\begin{aligned} A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n} \\ \Longrightarrow p = t'^2\mu^2 - 2t' B^n \mu + B^n(B^n - A^m) \end{aligned}$$

As $b = 4p = 4k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid b \Longrightarrow \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t' B^n) + 4B^n(B^n - A^m)$$

The last equation implies $\mu \mid 4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** C-1-1-1- If $\mu \mid B^n \Longrightarrow$ then the contradiction with (5.42).

** C-1-1-2- If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we have :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu \mid B^n$ then the contradiction with (5.42).

** C-1-1-2-2- If $\mu = 3$, then $3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$A^{2m} + 2A^m B^n = k_1 \tag{5.43}$$

$$p = k_2$$

$$\frac{2\sqrt{3}}{3} sin \frac{2\theta}{3} = \frac{k_1}{p}$$

We take the square of the last equation, we obtain :

$$\frac{4}{3} sin^2 \frac{2\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} cos^2 \frac{\theta}{3} = \frac{k_1^2}{p^2}$$

$$\frac{16}{3} sin^2 \frac{\theta}{3} . \frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{5.44}$$

but $a' = a"^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (5.44) becomes :

$$a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = a"\lambda \tag{5.45}$$

taking the positive root. Using (5.43), we have:

$$k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \Longrightarrow \lambda^2 + 3a"^2 = b$, then the couple $(\lambda, a")$ is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{5.46}$$

with $X = \lambda$ and $Y = a"$. But using one theorem on the solutions of the equation given by (5.46), b is written under the form (see theorem 37.4 in [3]):

$$b = 2^{2s} \times 3^t . p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1 (\mathrm{mod}\, 6)$, the q_j are also prime integers so that $q_j \equiv 5 (\mathrm{mod}\, 6)$. Then, as $b = 4p$:

- If $t \geqslant 1 \Longrightarrow 3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 \pmod 6$ and $q_j \equiv 5 \pmod 6$. Finally, we obtain that $p \equiv 1 \pmod 6$. We will verify if this condition does not give contradictions.

We will present the table of the value modulo 6 of $p = A^{2m} + A^m B^n + B^{2n}$ in function of the values of $A^m, B^n \pmod 6$. We obtain the table below:

Table 5.1. Table of $p \pmod 6$

A^m, B^n	0	1	2	3	4	5
0	0	**1**	4	3	4	**1**
1	**1**	3	**1**	**1**	3	**1**
2	4	**1**	0	**1**	4	3
3	3	**1**	**1**	3	**1**	**1**
4	4	3	4	**1**	0	**1**
5	**1**	**1**	3	**1**	**1**	3

** C-2-2-1-1- Case $A^m \equiv 0 (\mathrm{mod} 6) \Longrightarrow 2 \mid (A^m = a") \Longrightarrow 2 \mid (a' = a"^2) \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime. All the cases of the first line of the table 5.1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1 (\mathrm{mod} 6)$ and $B^n \equiv 0 (\mathrm{mod} 6)$, then $2 \mid B^n \Longrightarrow B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3 \mid p$, then $3 \mid b$, but $3 \mid a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ is a solution of the Diophantine equation:

$$x^2 + 3y^2 = p \tag{5.47}$$

The solution $x = A^m + B', y = B'$ is unique because $x - y$ verifies $x - y = A^m$. If (u, v) is another pair solution of (5.47), with $u, v \in \mathbb{N}^*$, then we obtain:

$$u^2 + 3v^2 = p$$
$$u - v = A^m$$

Then $u = v + A^m$ and we obtain the equation of second degree $4v^2 + 2vA^m - 2B'(A^m + 2B') = 0$ that gives as positive root $v_1 = B' = y$, then $u = A^m + B' = x$. It follows that p in (5.47) has an unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As p is an odd integer number, we applique one of Euler's theorems on convenient numbers "numerus idoneus" (see [4],[5]) : *Let m be an odd number relatively prime to n which is properly represented by x^2+ny^2. If the equation $m=x^2+ny^2$ has only one solution with $x,y>0$, then m is a prime number.* Then p is prime and $4p$ has an unique representation (we put $U=2u$, $V=2v$, with $U^2+3V^2=4p$ and $U-V=2A^m$). But $b=4p \Longrightarrow \lambda^2+3a"^2=(2(A^m+B'))^2+3(2B')^2$, the representation of $4p$ is unique gives:

$$\lambda = 2(A^m+B') = 2a" + B^n$$
$$and \quad a" = 2B' = B^n = A^m$$

But $A^m > B^n$, then the contradiction.

** C-2-2-1-3- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$, then $3 \mid B^n \Longrightarrow B^n = 3B'$. We can write $b=4p=(2A^m+3B')^2+3(3B')^2=\lambda^2+3a"^2$. The unique representation of b as $x^2+3y^2=\lambda^2+3a"^2 \Longrightarrow a"=A^m=3B'=B^n$, then the contradiction with $A^m>B^n$.

** C-2-2-1-5- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$, then $C^l \equiv 0(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2(\bmod 6) \Longrightarrow 2 \mid a" \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a,b coprime.

** C-2-2-1-7- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$, then $C^l \equiv 4(\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, we can write that $p=(C'-B^n)^2+3C'^2$, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4(\bmod 6) \Longrightarrow 2 \mid a" \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5(\mod 6)$ and $B^n \equiv 0(\mod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$, then $C^l \equiv 0(\bmod 6) \Longrightarrow 2 \mid C^l$, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2 \mid C^l \Longrightarrow C^l = 2C'$, p is written as $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5(\mod 6)$ and $B^n \equiv 4(\mod 6)$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 5.1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

5.7. Case $3 \mid a$ and $b = 2p',\ b \neq 2$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$B^nC^l = k(p' - 2a') \tag{5.48}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{5.49}$$

** D-1- We suppose that k is prime.

** D-1-1- If $k=2$, then we have $p=2p'=b \Longrightarrow 2 \mid b$, but $A^{2m}=4a'=(A^m)^2 \Longrightarrow A^m=2a$" with $a'=a"^2$, then $2 \mid a" \Longrightarrow 2 \mid (a=3a"^2)$, it follows the contradiction with a,b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m}=2k.a'=(A^m)^2 \Longrightarrow k \mid a'$ and $2 \mid a' \Longrightarrow a'=2.k.a"^2 \Longrightarrow A^m=2.k.a"$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A=k^i.A_1$ with $i \geqslant 1$ and $k \nmid A_1$. $k^{im}A_1^m=2ka" \Longrightarrow 2a"=k^{im-1}A_1^m$. From $B^nC^l=k(p'-2a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** D-1-2-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B=k^j.B_1$ with $j \geqslant 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^nC^l=p'-2a'=p'-4ka"^2$. As $n \geqslant 3 \Longrightarrow nj-1 \geqslant 2$, then $k \mid p'$ but $k \neq 2 \Longrightarrow k \mid (2p'=b)$, but $k \mid a' \Longrightarrow k \mid (3a'=a)$. It follows the contradiction with a,b coprime.

** D-1-2-2- If $k \mid C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k=\omega^s.k_1$, with $s \geqslant 1$, $\omega \nmid k_1$. The equations (5.48-5.49) become:

$$B^nC^l=\omega^s.k_1(p'-2a')$$
$$\text{and} \quad A^{2m}=2\omega^s.k_1.a'$$

** D-2-1- We suppose that $\omega=2$, then we have the equations:

$$A^{2m}=2^{s+1}.k_1.a' \tag{5.50}$$

$$B^nC^l=2^s.k_1(p'-2a') \tag{5.51}$$

** D-2-1-1- Case: $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a,b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (5.50) gives $2 \mid A^{2m} \Longrightarrow A=2^iA_1$, with $i \geqslant 1$ and $2 \nmid A_1$. It follows that $2im=s+1$.

** D-2-1-2-1- We suppose that $2 \nmid (p'-2a') \Longrightarrow 2 \nmid p'$. From the equation (5.51), we obtain that $2 \mid B^nC^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B=2^jB_1$ with $2 \nmid B_1$ and $j \geqslant 1$, then $B_1^nC^l=2^{s-jn}k_1(p'-2a')$:

- If $s-jn \geqslant 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, and no contradiction with $C^l=2^{im}A_1^m+2^{jn}B_1^n$, and the conjecture (1.2) is verified.

- If $s - jn \leqslant 0$, from $B_1^n C^l = 2^{s-jn} k_1(p' - 2a') \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2 \mid C^l$.

** D-2-1-2-2- We suppose now that $2 \mid (p' - 2a') \Longrightarrow p' - 2a' = 2^\mu.\Omega$, with $\mu \geqslant 1$ and $2 \nmid \Omega$. We recall that $2 \nmid a'$. The equation (5.51) is written as:

$$B^n C^l = 2^{s+\mu}.k_1.\Omega$$

This last equation implies that $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geqslant 1$ and $2 \nmid B_1$. Then $B_1^n C^l = 2^{s+\mu-jn}.k_1.\Omega$:

- If $s + \mu - jn \geqslant 1$, then $2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$, and the conjecture (1.2) is verified.

- If $s + \mu - jn \leqslant 0$, from $B_1^n C^l = 2^{s+\mu-jn} k_1.\Omega \Longrightarrow 2 \nmid C^l$, then contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \Longrightarrow 2 \mid C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2 \mid C^l$.

** D-2-2- We suppose that $\omega \neq 2$. We have then the equations:

$$A^{2m} = 2\omega^s.k_1.a' \tag{5.52}$$

$$B^n C^l = \omega^s.k_1.(p' - 2a') \tag{5.53}$$

As $\omega \neq 2$, from the equation (5.52), we have $2 \mid (k_1.a')$. If $2 \mid a' \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime.

** D-2-2-1- Case: $2 \nmid a'$ and $2 \mid k_1 \Longrightarrow k_1 = 2^\mu.\Omega$ with $\mu \geqslant 1$ and $2 \nmid \Omega$. From the equation (5.52), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ with $i \geqslant 1$ and $2 \nmid A_1$, then $2im = 1 + \mu$. The equation (5.53) becomes:

$$B^n C^l = \omega^s.2^\mu.\Omega.(p' - 2a') \tag{5.54}$$

From the equation (5.54), we obtain $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** D-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$, with $j \in \mathbb{N}^*$ and $2 \nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p' - 2a')$, then we have $B_1^n C^l = \omega^s 2^{\mu-jn}\Omega(p' - 2a')$:

- If $\mu - jn \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.

- If $\mu - jn \leqslant 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-1-2- We suppose that $2 \mid (p' - 2a') \Longrightarrow p' - 2a' = 2^\alpha.P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^n C^l = \omega^s 2^{\mu+\alpha-jn}\Omega.P$:

- If $\mu + \alpha - jn \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.

- If $\mu + \alpha - jn \leqslant 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-2- We suppose now that $2 \mid C^n \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

5.8. Case $3 \mid a$ and $b = 4p',\ b \neq 4$ with $p' \mid p$

$3 \mid a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 5.6), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p'-a')$$

As $p = b.p'$, and $p' > 1$, we have :

$$B^nC^l = k(p' - a') \tag{5.55}$$

$$\text{and} \quad A^{2m} = k.a' \tag{5.56}$$

** E-1- We suppose that k is prime. From $A^{2m} = k.a' = (A^m)^2 \Longrightarrow k \mid a'$ and $a' = k.a"^2 \Longrightarrow A^m = k.a"$. Then $k \mid A^m \Longrightarrow k \mid A \Longrightarrow A = k^i.A_1$ with $i \geqslant 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka" \Longrightarrow a" = k^{mi-1}A_1^m$. From $B^nC^l = k(p' - a') \Longrightarrow k \mid (B^nC^l) \Longrightarrow k \mid B^n$ or $k \mid C^l$.

** E-1-1- We suppose that $k \mid B^n \Longrightarrow k \mid B \Longrightarrow B = k^j.B_1$ with $j \geqslant 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^nC^l = p' - a'$. As $n.j-1 \geqslant 2 \Longrightarrow k \mid (p'-a')$. But $k \mid a' \Longrightarrow k \mid a$, then $k \mid p' \Longrightarrow k \mid (4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k \mid C^l$, using the same method with the above hypothesis $k \mid B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \Longrightarrow p = 4p' = b$, it is the case 5.3 studied above.

** E-2-2- We suppose that $k \geqslant 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geqslant 1$, $\omega \nmid k_1$. The equations (5.55-5.56) become:

$$B^nC^l = \omega^s.k_1(p'-a') \tag{5.57}$$

$$\text{and} \quad A^{2m} = \omega^s.k_1.a' \tag{5.58}$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2 \mid a' \Longrightarrow 2 \mid (3a' = a)$, but $2 \mid (4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (5.58), it follows that $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$ with $2 \nmid A_1$ and:

$$B^nC^l = 2^sk_1(p'-a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p'-a')$, from the above expression, we have $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-1-1- If $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$. Then $B_1^nC^l = 2^{2im-jn}k_1(p'-a')$:

- If $2im - jn \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2im - jn \leqslant 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-1-2- If $2 \mid C^l \Longrightarrow 2 \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2 \mid (p'-a')$. As $2 \nmid a' \Longrightarrow 2 \nmid p'$, $2 \mid (p'-a') \Longrightarrow p'-a' = 2^{\alpha}.P$ with $\alpha \geqslant 1$ and $2 \nmid P$. The equation (5.57) is written as :

$$B^n C^l = 2^{s+\alpha} k_1.P = 2^{2im+\alpha} k_1.P \tag{5.59}$$

then $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** E-2-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$, with $2 \nmid B_1$. The equation (5.59) becomes $B_1^n C^l = 2^{2im+\alpha-jn} k_1 P$:

- If $2im+\alpha-jn \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (1.2) is verified.
- If $2im+\alpha-jn \leqslant 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n \Longrightarrow 2 \mid C^l$.

** E-2-2-1-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$A^{2m} = \omega^s.k_1.a' \tag{5.60}$$

$$B^n C^l = \omega^s.k_1(p'-a') \tag{5.61}$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (5.60), we have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p'-a')$. From the equation (5.61) above, we have $\omega \mid (B^n C^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{2im-jn} k_1 (p'-a')$:

- If $2im-jn \geqslant 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (1.2) is verified.
- If $2im-jn \leqslant 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega \mid (p'-a') \Longrightarrow \omega \nmid p'$ as ω and a' are coprime. $\omega \mid (p'-a') \Longrightarrow p'-a' = \omega^{\alpha}.P$ with $\alpha \geqslant 1$ and $\omega \nmid P$. The equation (5.61) becomes :

$$B^nC^l = \omega^{s+\alpha}k_1.P = \omega^{2im+\alpha}k_1.P \tag{5.62}$$

then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$, with $\omega \nmid B_1$. The equation (5.62) is written as $B_1^n C^l = 2^{2im+\alpha-jn}k_1 P$:

- If $2im+\alpha-jn \geqslant 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2im+\alpha-jn \leqslant 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta.a"$ with $\omega \nmid a"$. The equation (5.60) becomes:

$$A^{2m} = \omega^s k_1 a' = \omega^{s+\beta}k_1.a"$$

We have $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$ with $\omega \nmid A_1$ and $s+\beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p'-a') \Longrightarrow \omega \nmid p' \Longrightarrow \omega \nmid (b = 4p')$. From the equation (5.61), we obtain $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** E-2-2-2-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$. Then $B_1^n C^l = 2^{s-jn}k_1(p'-a')$:

- If $s-jn \geqslant 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $s-jn \leqslant 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \Longrightarrow \omega \mid C^l$.

** E-2-2-2-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega \mid (p'-a' = p'-\omega^\beta.a") \Longrightarrow \omega \mid p' \Longrightarrow \omega \mid (4p' = b)$, but $\omega \mid a' \Longrightarrow \omega \mid a$. Then the contradiction with a, b coprime.

The study of the cases of 5.8 is achieved.

5.9. Case $3 \mid a$ and $b \mid 4p$

$a = 3a'$ and $4p = k_1 b$. As $A^{2m} = \dfrac{4p}{3}cos^2\dfrac{\theta}{3} = \dfrac{4p}{3}\dfrac{3a'}{b} = k_1 a'$ and B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3a'}{b}\right) = \frac{k_1}{4}(b - 4a')$$

As B^nC^l is an integer, we must obtain $4 \mid k_1$, or $4 \mid (b - 4a')$ or ($2 \mid k_1$ and $2 \mid (b - 4a')$).
** F-1- If $k_1 = 1 \Rightarrow b = 4p$: it is the case 5.6.

** F-2- If $k_1 = 4 \Rightarrow p = b$: it is the case 5.3.

** F-3- If $k_1 = 2$ and $2 \mid (b - 4a')$: in this case, we have $A^{2m} = 2a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$. $2 \mid (b - 4a') \Longrightarrow 2 \mid b$ then the contradiction with a, b coprime.

** F-4- If $2 \mid k_1$ and $2 \mid (b - 4a')$: $2 \mid (b - 4a') \Longrightarrow b - 4a' = 2^\alpha\lambda$, α and $\lambda \in \mathbb{N}^* \geqslant 1$ with $2 \nmid \lambda$; $2 \mid k_1 \Longrightarrow k_1 = 2^t k_1'$ with $t \geqslant 1 \in \mathbb{N}^*$ with $2 \nmid k_1'$ and we have:

$$A^{2m} = 2^t k_1' a' \tag{5.63}$$

$$B^nC^l = 2^{t+\alpha-2} k_1' \lambda \tag{5.64}$$

From the equation (5.63), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, $i \geqslant 1$ and $2 \nmid A_1$.
** F-4-1- We suppose that $t = \alpha = 1$, then the equations (5.63-5.64) become :

$$A^{2m} = 2k_1' a' \tag{5.65}$$

$$B^nC^l = k_1' \lambda \tag{5.66}$$

From the equation (5.65) it follows that $2 \mid a' \Longrightarrow 2 \mid (a = 3a')$. But $b = 4a' + 2\lambda \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2- We suppose that $t + \alpha - 2 \geqslant 1$ and we have the expressions:

$$A^{2m} = 2^t k_1' a' \tag{5.67}$$

$$B^nC^l = 2^{t+\alpha-2} k_1' \lambda \tag{5.68}$$

** F-4-2-1- We suppose that $2 \mid a' \Longrightarrow 2 \mid a$, but $b = 2^\alpha\lambda + 4a' \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (5.67), we have $2 \mid A^{2m} \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$ and $B^n C^l = 2^{t+\alpha-2} k_1' \lambda \Longrightarrow 2 \mid B^n C^l \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** F-4-2-2-1- We suppose that $2 \mid B^n$. We have $2 \mid B \Longrightarrow B = 2^j B_1$, $j \geqslant 1$ and $2 \nmid B_1$. The equation (5.68) becomes $B_1^n C^l = 2^{t+\alpha-2-jn} k_1' \lambda$:

- If $t+\alpha-2-jn > 0 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (1.2) is verified.

- If $t+\alpha-2-jn < 0 \Longrightarrow 2 \mid k_1' \lambda$, but $2 \nmid k_1'$ and $2 \nmid \lambda$. Then this case is impossible.

- If $t+\alpha-2-jn = 0 \Longrightarrow B_1^n C^l = k_1' \lambda \Longrightarrow 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** F-4-2-2-2- We suppose that $2 \mid C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4 \mid k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k_2'$, we have :

$$A^{2m} = 4k_2' a' \tag{5.69}$$

$$B^n C^l = k_2'(b - 4a') \tag{5.70}$$

** F-5-1- We suppose that k_2' is prime, from (5.69), we have $k_2' \mid a'$. From (5.70), $k_2' \mid (B^n C^l) \Longrightarrow k_2' \mid B^n$ or $k_2' \mid C^l$.

** F-5-1-1- We suppose that $k_2' \mid B^n \Longrightarrow k_2' \mid B \Longrightarrow B = k_2'^{\beta}.B_1$ with $\beta \geqslant 1$ and $k_2' \nmid B_1$. It follows that we have $k_2'^{n\beta-1} B_1^n C^l = b - 4a' \Longrightarrow k_2' \mid b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k_2' \mid C^l$.

** F-5-2- We suppose that k_2' is not prime.

** F-5-2-1- We suppose that k_2' and a' are coprime. From (5.69), k_2' can be written under the form $k_2' = q_1^{2j}.q_2^2$ and $q_1 \nmid q_2$ and q_1 prime. We have $A^{2m} = 4q_1^{2j}.q_2^2 a' \Longrightarrow q_1 \mid A$ and $B^n C^l = q_1^{2j}.q_2^2(b - 4a') \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-1-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{2j-fn} q_2^2(b - 4a')$:
- If $2j - f.n \geqslant 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$ but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (1.2) is verified.

- If $2j - f.n = 0$, we have $B_1^n C^l = q_2^2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b-4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (1.2) is verified.
- If $2j - f.n < 0 \Longrightarrow q_1 \mid (b-4a') \Longrightarrow q_1 \nmid b$ because a' is coprime with q_1, and $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (1.2) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-5-2-2- We suppose that k_2', a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_2'$ and $q_1 \mid a'$. We write k_2' under the form $q_1^j.q_2$ with $j \geqslant 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k_2'a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. Then from $B^n C^l = q_1^j q_2(b-4a')$, it follows that $q_1 \mid (B^n C^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-5-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geqslant 1$ and $q_1 \nmid B_1$. Then, we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2(b-4a') \Longrightarrow B_1^n C^l = q_1^{j-n\beta} q_2(b-4a')$.
- If $j - n\beta \geqslant 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then the conjecture (1.2) is verified.
- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b-4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b-4a') \Longrightarrow q_1 \mid b$, because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6- If $4 \nmid (b-4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4 \mid (b-4a') \Rightarrow 4 \mid b$, and $b - 4a' = 4^t.g$, $t \geqslant 1$ with $4 \nmid g$, then we have :

$$A^{2m} = k_1 a'$$
$$B^n C^l = k_1.4^{t-1}.g$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1 a'$ we deduce easily that $k_1 \mid a'$. From $B^n C^l = k_1.4^{t-1}.g$ we obtain that $k_1 \mid (B^n C^l) \Longrightarrow k_1 \mid B^n$ or $k_1 \mid C^l$.

** F-6-1-1- We suppose that $k_1 \mid B^n \Longrightarrow k_1 \mid B \Longrightarrow B = k_1^j.B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k_1^{n.j} B_1^n C^l = k_1.4^{t-1}.g \Longrightarrow k_1^{n.j-1} B_1^n C^l = 4^{t-1}.g$. But $n \geqslant 3$ and $j \geqslant 1$, then $n.j - 1 \geqslant 2$. We deduce as $k_1 \neq 2$ that $k_1 \mid g \Longrightarrow k_1 \mid (b-4a')$, but $k_1 \mid a' \Longrightarrow k_1 \mid b$, then the contradiction with

a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1 \mid C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1. Then $A^{2m} = 2k'a' \Longrightarrow 2 \mid a' \Longrightarrow 2 \mid a$, as $4 \mid b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j.q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geqslant 1$. $B^nC^l = q_1^j.q_2 4^{t-1} g \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-2-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^n C^l = q_1^{j-f.n} q_2 4^{t-1} g$.
- If $j - f.n \geqslant 1 \Longrightarrow q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives also $q_1 \mid C$ and the conjecture (1.2) is verified.
- If $j - f.n = 0$, we have $B_1^n C^l = q_2 4^{t-1} g$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, then $q_1 \mid (b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (1.2) is verified.
- If $j - f.n < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1 \mid C$ and the conjecture (1.2) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1 \mid C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1 \mid k_1$ and $q_1 \mid a'$. We write k_1 under the form $q_1^j.q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1 a' \Longrightarrow q_1 \mid A^{2m} \Longrightarrow q_1 \mid A$. From $B^nC^l = q_1^j q_2 (b - 4a')$, it follows that $q_1 \mid (B^nC^l) \Longrightarrow q_1 \mid B^n$ or $q_1 \mid C^l$.

** F-6-2-3-1- We suppose that $q_1 \mid B^n \Longrightarrow q_1 \mid B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geqslant 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta} B_1^n C^l = q_1^j q_2 (b - 4a') \Longrightarrow B_1^n C^l = q_1^{j-n\beta} q_2 (b - 4a')$:
  - If $j - n\beta \geqslant 1$, then $q_1 \mid C^l \Longrightarrow q_1 \mid C$, but $C^l = A^m + B^n$ gives $q_1 \mid C$, and the conjecture (1.2) is verified.
  - If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2(b - 4a')$, but $q_1 \mid A$ and $q_1 \mid B$ then $q_1 \mid C$ and we obtain $q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$ because $q_1 \mid a' \Longrightarrow q_1 \mid a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \Longrightarrow q_1 \mid (b - 4a') \Longrightarrow q_1 \mid b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1 \mid C^l$.

6. Hypothèse: $\{3 \mid p \quad and \quad b \mid 4p\}$

6.1. Case $b = 2$ and $3 \mid p$

$3 \mid p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow 1 < 2a < 3 \Rightarrow a = 1 \Longrightarrow cos^2\frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 4.1.2).

6.2. Case $b = 4$ and $3 \mid p$

we have $3 \mid p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2$$

as a, b are coprime, then the case $b = 4$ and $3 \mid p$ is impossible.

6.3. Case: $b \neq 2, b \neq 4, b \neq 3$, $b \mid p$ and $3 \mid p$

As $3 \mid p$, then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b}$$

We consider the case: $b \mid p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (If $p" = 1$, then $p = 3b$, see paragraph 6.8 Case $k' = 1$). Finally, we obtain:

$$A^{2m} = \frac{4bp"a}{b} = 4ap"; \quad B^nC^l = p".(3b - 4a)$$

** G-1- We suppose that p" is prime, then $A^{2m} = 4ap" = (A^m)^2 \Longrightarrow p" \mid a$. But $B^nC^l = p"(3b - 4a) \Longrightarrow p" \mid B^n$ or $p" \mid C^l$.

** G-1-1- If $p" \mid B^n \Longrightarrow p" \mid B \Longrightarrow B = p"B_1$ with $B_1 \in \mathbb{N}^*$. Then $p"^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $p" \mid a$, then $p" \mid 3b \Longrightarrow p" = 3$ or $p" \mid b$.

** G-1-1-1- If $p" = 3 \Longrightarrow 3 \mid a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$, then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a"$. As $p"^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n > 2 \Longrightarrow n - 2 \geqslant 1$, then $3 \mid b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p" \mid b$, as $p" \mid a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p" \mid C^l$, we obtain identical results (contradictions).

** G-2- We consider now that p" is not prime.

** G-2-1- $p", a$ coprime: $A^{2m} = 4ap" \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p" = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2 \mid a'$ or $2 \mid p_1$.

** G-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $p" = p_1^2$.

** G-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \Longrightarrow p" = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b - 4a)$.

** G-2-1-1-2-1- If ω is prime, $\omega \neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$

and $\omega \nmid b$, the conjecture (1.2) is verified.

** G-2-1-1-2-1-1-2- If $2m - nj \geqslant 1$, using the method as above, we obtain $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (1.2) is verified.

** G-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m} B_1^n.C^l = 3b - 4a$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, then $C = \omega^h.C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l} B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^n C_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega \mid (3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega \mid C^l$.

** G-2-1-1-2-2- We suppose that $p" = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$, $f \geqslant 1$, and $\omega_1 \mid A$. Then $B^n C^l = \omega_1^{2f.m} \Omega^{2m}(3b - 4a) \Longrightarrow \omega_1 \mid (B^n C^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** G-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.m-nj} \Omega^{2m}(3b - 4a)$:

** G-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (1.2) is verified.

** G-2-1-1-2-2-1-2- If $2f.m - n.j \geqslant 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b - 4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture (1.2) is verified.

** G-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $\omega_1 \mid C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^n C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b - 4a)$ with ω_1, a, b coprime and the conjecture (1.2) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1 \mid C^l$.

** G-2-1-2- We suppose that $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $p" = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2 \mid a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 can written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p" = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** G-2-1-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B$. As $2 \mid A$, then $2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (1.2) is verified.

** G-2-1-2-2-2- We suppose that $2 \mid C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that $p", a$ are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p"$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap" \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid p"$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p" = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap" = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.p" = 3b.3^\mu p_1 = 3^{\mu+1}.b.p_1$, the exponent of the factor 3 of p is k, the exponent of the factor 3 of the left member of the last equation is $\mu+1$ added of the exponent β of 3 of the term b, with $\beta \geqslant 0$, let $min(2im, 2jn, im+jn) = \mu+1+\beta$ and we recall that $\alpha+\mu = 2im$. But $B^nC^l = p"(3b-4a)$, we obtain $3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_1(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \Longrightarrow 3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call

$\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. We obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{6.1}$$

$$\alpha + \mu = 2im \tag{6.2}$$

$$\epsilon = hl = min(im, jn)$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_1 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1)$$

** G-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (6.2) becomes:

$$1 + \mu = 2im$$

and the first equation (6.1) is written as:

$$k = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $k = 2im \Longrightarrow \beta = 0$ then $3 \nmid b$. We obtain $2im \leqslant 2jn \Longrightarrow im \leqslant jn$, and $2im \leqslant im + jn \Longrightarrow im \leqslant jn$. The third equation gives $hl = im$ and the last equation gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = nj$, then $im = nj = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (1.2) is verified.

- If $k = 2jn$ or $k = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_1(b - 4a_1)$. As a, b are coprime, the conjecture (1.2) is verified.

** G-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geqslant 2$.

- If $k = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ that gives $\alpha = 1 + \beta \geqslant 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime.

- If $k = 2jn = \mu + 1 + \beta \leqslant 2im \Longrightarrow \mu + 1 + \beta \leqslant \mu + \alpha \Longrightarrow 1 + \beta \leqslant \alpha \Longrightarrow \beta \geqslant 1$. If $\beta \geqslant 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime.

- If $k = im + jn \Longrightarrow im + jn \leqslant 2im \Longrightarrow jn \leqslant im$, and $im + jn \leqslant 2jn \Longrightarrow im \leqslant jn$, then $im = jn$. As $k = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$, we obtain $\alpha = 1 + \beta \geqslant 2 \Longrightarrow \beta \geqslant 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** G-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p" = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap" = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p"(3b - 4a) = \omega^\mu p_1(3b - 4a) \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** G-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bp" = 3\omega^\mu b p_1 = \omega^k(\omega^{2im-k} A_1^{2m} + \omega^{2jn-k} B_1^{2n} + \omega^{im+jn-k} A_1^m B_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then:

- If $k=\mu$, then $\omega \nmid b$ and the conjecture (1.2) is verified.
- If $k>\mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime.
- If $k<\mu$, it follows from:

$$3\omega^{\mu} b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ then the contradiction with $\omega \nmid A_1$ or $\omega \nmid B_1$.

** G-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

6.4. Case $b=3$ and $3 \mid p$

As $3 \mid p \Longrightarrow p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4\times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3}$$

As A^{2m} is an integer and a, b are coprime and $cos^2\dfrac{\theta}{3} < 1$ (see equation (3.9)), then we have necessary $3 \mid p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3\times 3p" = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^m B^n)$, the hypothesis $p" = 1$ is impossible, then $p" > 1$, and we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4\times 3p"a}{3} = 4p"a\,; \quad B^n C^l = p".(9-4a)$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{a}{3} < \dfrac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow$ as $a>1, a=2$ and we obtain:

$$A^{2m} = 4p"a = 8p"\,; \quad B^n C^l = \frac{3p"(9-4a)}{3} = p" \tag{6.3}$$

The two last equations above imply that $p"$ is not a prime. We can write $p"$ as : $p" = \prod_{i\in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of $\mathbb{N}^*$ and $i \in I$ a finite set of indexes. We can write also $p" = p_1^{\alpha_1}.q_1$ with $p_1 \nmid q_1$. From (6.3), we have $p_1 \mid A$ and $p_1 \mid B^n C^l \Longrightarrow p_1 \mid B^n$ or $p_1 \mid C^l$.

** H-1- We suppose that $p_1 \mid B^n \Longrightarrow B = p_1^{\beta_1}.B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geqslant 1$. Then, we obtain $B_1^n C^l = p_1^{\alpha_1 - n\beta_1}.q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geqslant 1 \Longrightarrow p_1 \mid C^l \Longrightarrow p_1 \mid C$, in accord with $p_1 \mid (C^l = A^m + B^n)$, it follows that the conjecture (1.2) is verified.
- If $\alpha_1 - n\beta_1 = 0 \Longrightarrow B_1^n C^l = q_1 \Longrightarrow p_1 \nmid C^l$, it is a contradiction with $p_1 \mid (A^m - B^n) \Longrightarrow p_1 \mid C^l$. Then this case is impossible.

- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1-\alpha_1}B_1^nC^l = q_1 \Longrightarrow p_1 \mid q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1 \mid C^l$, using the same method as for the case $p_1 \mid B^n$, we obtain identical results.

6.5. Case $3 \mid p$ and $b = p$

We have $cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{a}{p}$ and:

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3 \mid a$, but $3 \mid p \Longrightarrow 3 \mid b$. As a and b are coprime, then the contradiction and the case $3 \mid p$ and $b = p$ is impossible.

6.6. Case $3 \mid p$ and $b = 4p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow 3 \mid [(4p) = b]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

6.7. Case $3 \mid p$ and $b = 2p$

$3 \mid p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3 \mid a$$

as A^{2m} is an integer. But $3 \mid p \Longrightarrow 3 \mid (2p) \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

6.8. Case $3 \mid p$ and $b \neq 3$ a divisor of p

We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3 \mid k \Longrightarrow k = 3k'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = 4ak'$$

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' is prime, then $A^{2m} = 4ak' = (A^m)^2 \Longrightarrow k' \mid a$. But $B^nC^l = k'(3b - 4a) \Longrightarrow k' \mid B^n$ or $k' \mid C^l$.

** I-1-1-1- If $k' \mid B^n \Longrightarrow k' \mid B \Longrightarrow B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $k' \mid a$, then $k' \mid 3b \Longrightarrow k' = 3$ or $k' \mid b$.

** I-1-1-1-1- If $k' = 3 \Longrightarrow 3 \mid a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A \Longrightarrow A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1}A_1^m = 2a' \Longrightarrow 3 \mid a' \Longrightarrow a' = 3a"$. But $k'^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n \geqslant 3 \Longrightarrow n - 2 \geqslant 1$, then $3 \mid b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k' \mid b$, but $k' \mid a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k' \mid C^l$, using the same method as for the case $k' \mid B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2 \mid a'$ or $2 \mid p_1$.

** I-1-2-1-1- We suppose that $2 \mid a'$, then $2 \mid a' \Longrightarrow 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \Longrightarrow k' = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b-4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-1-1-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b-4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$, as $C^l = A^m + B^n \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$, and $\omega \mid (3b-4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \Longrightarrow \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (1.2) is verified.

- If $2m - nj \geqslant 1$, using the same method, we have $\omega \mid C^l \Longrightarrow \omega \mid C$ and $\omega \mid (3b-4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then the conjecture (1.2) is verified.

- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m} B_1^n.C^l = 3b-4a$. As $C^l = A^m + B^n \Longrightarrow \omega \mid C$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l} B_1^n.C_1^l = 3b-4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega \mid B_1^n C_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If $n.j - 2m + h.l > 0 \Longrightarrow \omega \mid (3b-4a)$ with ω, a, b coprime, it implies that the conjecture (1.2) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega \mid C^l$, using the same method as for the case $\omega \mid B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k' = \omega^{2m}$ and ω not a prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geqslant 1$ an integer, and $\omega_1 \mid A$, then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b-4a) \Longrightarrow \omega_1 \mid (B^nC^l) \Longrightarrow \omega_1 \mid B^n$ or $\omega_1 \mid C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1 \mid B^n \Longrightarrow \omega_1 \mid B \Longrightarrow B = \omega_1^j B_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.fm-nj}\Omega^{2m}(3b-4a)$.

- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$, and $\omega_1 \mid (3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \Longrightarrow \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (1.2) is verified.

- If $2f.m - n.j \geqslant 1$, we have $\omega_1 \mid C^l \Longrightarrow \omega_1 \mid C$ and $\omega_1 \mid (3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (1.2) is verified.

- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f} B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1 \mid C$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l} B_1^n.C_1^l = \Omega^{2m}(3b-4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1 \mid B_1^n C_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1 \mid (3b-4a)$ with ω_1, a, b

coprime, then the conjecture (1.2) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1 \mid B^n$, we obtain identical results if $\omega_1 \mid C^l$.

** I-1-2-1-2- If $2 \mid p_1$: then $2 \mid p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $k' = p_1^2$.

** I-1-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2 \mid a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2 \mid p_1$. As $A^m = 2a'p_1$, p_1 is written under the form $p_1 = 2^{m-1}\omega^m \Longrightarrow p_1^2 = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = k'(3b-4a) = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** I-1-2-1-2-2-1- If $2 \mid B^n \Longrightarrow 2 \mid B$, as $2 \mid A \Longrightarrow 2 \mid C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2 \mid (3b-4a) \Longrightarrow 2 \mid b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (1.2) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2 \mid C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega \mid a$ and $\omega \mid p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \Longrightarrow 3 \mid A$, but $3 \mid p$. As $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3 \mid B^{2n} \Longrightarrow 3 \mid B$, then $3 \mid C^l \Longrightarrow 3 \mid C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^s.g$ with $s = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3) \mid a$ and $(\omega = 3) \mid k'$ that give $a = 3^\alpha a_1$, $3 \nmid a_1$ and $k' = 3^\mu p_2$, $3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_2 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.k' = 3b.3^\mu p_2 = 3^{\mu+1}.b.p_2$. The exponent of the factor 3 of p is s, the exponent of the factor 3 of the left member of the last equation is $\mu+1$ added of the exponent β of 3 of the factor b, with $\beta \geqslant 0$, let $min(2im, 2jn, im+jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^nC^l = k'(4b-3a)$ that gives $3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_2(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_2(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call $\epsilon = min(im, jn)$, we obtain $\epsilon = hl = min(im, jn)$. We have then the conditions:

$$s = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{6.4}$$

$$\alpha + \mu = 2im \tag{6.5}$$

$$\epsilon = hl = min(im, jn) \tag{6.6}$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^{\mu+1} p_2 (3^\beta b_1 - 4 \times 3^{(\alpha-1)} a_1) \tag{6.7}$$

** I-1-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (6.5) becomes:

$$1 + \mu = 2im$$

and the first equation (6.4) is written as :

$$s = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $s = 2im \Longrightarrow \beta = 0 \Longrightarrow 3 \nmid b$. We obtain $2im \leqslant 2jn \Longrightarrow im \leqslant jn$, and $2im \leqslant im + jn \Longrightarrow im \leqslant jn$. The third equation (6.6) gives $hl = im$. The last equation (6.7) gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (1.2) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (1.2) is verified.

** I-1-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geqslant 2$.

- If $s = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geqslant 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3 \mid b$, but $3 \mid a$ then the contradiction with a, b coprime and the conjecture (1.2) is not verified.

- If $s = 2jn = \mu + 1 + \beta \leqslant 2im \Longrightarrow \mu + 1 + \beta \leqslant \mu + \alpha \Longrightarrow 1 + \beta \leqslant \alpha \Longrightarrow \beta = 1$. If $\beta = 1 \Longrightarrow 3 \mid b$ but $3 \mid a$, then the contradiction with a, b coprime and the conjecture (1.2) is not verified.

- If $s = im + jn \Longrightarrow im + jn \leqslant 2im \Longrightarrow jn \leqslant im$, and $im + jn \leqslant 2jn \Longrightarrow im \leqslant jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1 + \beta \geqslant 2 \Longrightarrow \beta \geqslant 1 \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime and the conjecture (1.2) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu}.a_1.p_2 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = k'(3b - 4a) = \omega^\mu p_2 (3b - 4a) \Longrightarrow \omega \mid B^n C^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** I-1-2-2-2-1- $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B^n = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$. As $p = bp' = 3bk' = 3\omega^\mu b p_2 =$

$\omega^s(\omega^{2im-s}A_1^{2m}+\omega^{2jn-s}B_1^{2n}+\omega^{im+jn-s}A_1^mB_1^n)$ with $s = min(2im, 2jn, im+jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (1.2) is verified.
- If $s > \mu$, then $\omega \mid b$, but $\omega \mid a$ then the contradiction with a, b coprime and the conjecture (1.2) is not verified.
- If $s < \mu$, it follows from:

$$3\omega^{\mu}bp_1 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^mB_1^n)$$

that $\omega \mid A_1$ or $\omega \mid B_1$ that is the contradiction with the hypothesis and the conjecture (1.2) is not verified.

** I-1-2-2-2-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^hC_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega \mid (C^l - A^m) \Longrightarrow \omega \mid B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k' = 1$: then $k' = 1 \Longrightarrow p = 3b$, then we have $A^{2m} = 4a = (2a')^2 \Longrightarrow A^m = 2a'$, then $a = a'^2$ is even and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3} - cos\frac{\theta}{3}\right) = \frac{p\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a$$

and we have also:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2b\sqrt{3}sin\frac{2\theta}{3} \tag{6.8}$$

The left member of the equation (6.8) is a naturel number and also b, then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2 \mid b \Longrightarrow b = k_2.k_3$.

** I-2-1- $k' = 1$ and $k_3 \neq 1$: then $A^{2m} + 2A^mB^n = k_3.k_1$. Let μ be a prime integer so that $\mu \mid k_3$. If $\mu = 2 \Rightarrow 2 \mid b$, but $2 \mid a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu \mid k_3$, then $\mu \mid A^m(A^m + 2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** I-2-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid 4a \Longrightarrow \mu \mid a$. As $\mu \mid k_3 \Longrightarrow \mu \mid b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain:

$$p = t'^2 \mu^2 - 2t' B^n \mu + B^n(B^n - A^m)$$

As $p = 3b = 3k_2.k_3$ and $\mu \mid k_3$ then $\mu \mid p \Longrightarrow\ p = \mu.\mu'$, then we obtain:

$$\mu'.\mu = \mu(\mu t'^2 - 2t' B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** I-2-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using that $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \Longrightarrow \mu \mid B \\ or \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2 k_3 = 3k_2 k'_3$, it follows $p = 3b = 9k_2 k'_3$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^m B^n$ then:

$$9k_2 k'_3 - 3A^m B^n = (A^m - B^n)^2$$

that we write as:

$$3(3k_2 k'_3 - A^m B^n) = (A^m - B^n)^2 \tag{6.9}$$

then:

$$3 \mid (3k_2 k'_3 - A^m B^n) \Longrightarrow\ 3 \mid A^m B^n \Longrightarrow 3 \mid A^m \ or\ 3 \mid B^n$$

** I-2-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A$ and we have also $3 \mid A^{2m}$, but $A^{2m} = 4a \Longrightarrow\ 3 \mid 4a \Longrightarrow 3 \mid a$. As $b = 3k_2 k'_3$ then $3 \mid b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- $3 \mid B^m$: If $3 \mid B^n \Longrightarrow 3 \mid B$, but the equation (6.9) implies $3 \mid (A^m - B^n)^2 \Longrightarrow 3 \mid (A^m - B^n) \Longrightarrow 3 \mid A^m \Longrightarrow 3 \mid A$. The last case above has given that $3 \nmid A$. Then the case $3 \mid B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$, then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{6.10}$$

with k_1, b coprime. We write (6.10) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$, we obtain:

$$3 \times 4^2.a(b-a) = k_1^2 \Longrightarrow k_1^2 = 3 \times 4^2.a'^2(b-a)$$

it implies that :

$$b - a = 3\alpha^2 \Longrightarrow b = a'^2 + 3\alpha^2 \Longrightarrow k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \Longrightarrow 3\alpha = a' + B^n$$

We consider now that $3 \mid (b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{6.11}$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (6.11), b is written as (see theorem in [2]):

$$b = 2^{2s} \times 3^t.p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1 (\bmod 6)$, the q_j are also prime numbers so that $q_j \equiv 5 (\bmod 6)$, then :

- If $s \geqslant 1 \Longrightarrow 2 \mid b$, as $2 \mid a$, then the contradiction with a, b coprime.
- If $t \geqslant 1 \Longrightarrow 3 \mid b$, but $3 \mid (b-a) \Longrightarrow 3 \mid a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1 (\bmod 6)$ and $q_j \equiv 5 (\bmod 6)$. Finally, we obtain that $b \equiv 1 (\bmod 6)$. We will verify then this condition.

** I-2-2-1-1- We present the table below giving the value of $A^m+B^n=C^l$ modulo 6 in function of the value of $A^m, B^n \pmod 6$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0(\bmod\ 6)$ and $A^m \equiv 3 \pmod 6$ (respectively of $B^n \equiv 0 \pmod 6$ and $B^n \equiv 3 \pmod 6$), they present cases with contradictions:

Table 6.1. Table of $C^l \pmod 6$

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

** I-2-2-1-1-1- For the case $C^l \equiv 0 \pmod 6$ and $C^l \equiv 3 \pmod 6$, we deduce that $3 \mid C^l \Longrightarrow 3 \mid C \Longrightarrow C = 3^h C_1$, with $h \geqslant 1$ and $3 \nmid C_1$. It follows that $p - B^n C^l = 3b - 3^{lh} C_1^l B^n = A^{2m} \Longrightarrow 3 \mid (A^{2m} = 4a) \Longrightarrow 3 \mid a \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0 \pmod 6$, $C^l \equiv 2 \pmod 6$ and $C^l \equiv 4 \pmod 6$, we deduce that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h C_1$, with $h \geqslant 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^n C^l = 4a + 2^{lh} C_1^l B^n \Longrightarrow 2 \mid 3b \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1 (\bmod\ 6)$ and $B^n \equiv 4 (\bmod\ 6)$ (respectively $B^n \equiv 2 \pmod 6$): then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geqslant 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$ that $2 \mid b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5 (\bmod\ 6)$ and $B^n \equiv 2 (\bmod\ 6)$: then $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j B_1$ with $j \geqslant 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^n C^l = 4a + 2^{jn} B_1^n C^l$, then $2 \mid b$ and we obtain the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2 \pmod 6$ and $B^n \equiv 5 \pmod 6$: as $A^m \equiv 2 \pmod 6 \Longrightarrow A^m \equiv 2 \pmod 3$, then A^m is not a square and also for B^n. Hence, we can write A^m and B^n as:

$$A^m = a_0 . \mathcal{A}^2$$
$$B^n = b_0 \mathcal{B}^2$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mathcal{A}) = 1$ and $(b_0, \mathcal{B}) = 1$. We have also $p = 3b = A^{2m} + A^m B^n + B^{2n} = (A^m - B^n)^2 + 3A^m B^n \Longrightarrow 3 \mid (b - A^m B^n) \Longrightarrow A^m B^n \equiv b (\bmod 3)$ but $b = a + 3\alpha^2 \Longrightarrow b \equiv a \equiv a'^2 (\bmod 3)$, then $A^m B^n \equiv a'^2 (\bmod 3)$. But $A^m \equiv 2 (\bmod 6) \Longrightarrow 2a' \equiv 2 (\bmod 6) \Longrightarrow 4a'^2 \equiv 4 (\bmod 6) \Longrightarrow a'^2 \equiv 1 (\bmod 3)$. It follows that $A^m B^n$ is a square, let $A^m B^n = \mathcal{N}^2 = \mathcal{A}^2.\mathcal{B}^2.a_0.b_0$. We call $\mathcal{N}_1^2 = a_0.b_0$. Let p_1 be a prime number so that $p_1 \mid a_0 \Longrightarrow a_0 = p_1.a_1$ with $p_1 \nmid a_1$. $p_1 \mid \mathcal{N}_1^2 \Longrightarrow p_1 \mid \mathcal{N}_1 \Longrightarrow \mathcal{N}_1 = p_1^t \mathcal{N}_1'$ with $t \geqslant 1$ and $p_1 \nmid \mathcal{N}_1'$, then $p_1^{2t-1}\mathcal{N}_1'^2 = a_1.b_0$. As $2t \geqslant 2 \Longrightarrow 2t - 1 \geqslant 1 \Longrightarrow p_1 \mid a_1.b_0$ but $(p_1, a_1) = 1$, then $p_1 \mid b_0 \Longrightarrow p_1 \mid B^n \Longrightarrow p_1 \mid B$. But $p_1 \mid (A^m = 2a')$, and $p_1 \neq 2$ because $p_1 \mid B^n$ and B^n is odd, then the contradiction. Hence, $p_1 \mid a' \Longrightarrow p_1 \mid a$. If $p_1 = 3$, from $3 \mid (b - a) \Longrightarrow 3 \mid b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime that divides A^m and B^n, then $p_1 \mid (p = 3b) \Longrightarrow p_1 \mid b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3 (\bmod 6)$ and we choose the case $b \equiv 1 (\bmod 6)$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4 (\bmod 6)$ and $B^n \equiv 1 (\bmod 6)$. We return to the equation (6.11) that b verifies :

$$b = X^2 + 3Y^2 \tag{6.12}$$

$$\textit{with} \quad X = a'; \quad Y = \alpha$$

$$\textit{and} \quad 3\alpha = a' + B^n$$

Suppose that it exists another solution of (6.12):

$$b = X^2 + 3Y^3 = u^2 + 3v^2 \Longrightarrow 2u \neq A^m, 3v \neq a' + B^n$$

But $B^n = \dfrac{6\alpha - A^m}{2} = 3\alpha - a'$ and b verifies also :$3b = p = A^{2m} + A^m B^n + B^{2n}$, it is impossible that u, v verify:

$$6v = 2u + 2B^n$$

$$3b = 4u^2 + 2uB^n + B^{2n}$$

If we consider that : $6v - 2u = 6\alpha - 2a' \Longrightarrow u = 3v - 3\alpha + a'$, then $b = u^2 + 3v^2 = (3v - 3\alpha + a')^2 + 3v^2$, it gives:

$$2v^2 - B^n v + \alpha^2 - a'\alpha = 0$$

$$2v^2 - B^n v - \frac{(a' + B^n)(A^m - B^n)}{9} = 0$$

The resolution of the last equation gives with taking the positive root (because $A^m > B^n$), $v_1 = \alpha$, then $u = a'$. It follows that b in (6.12) has an unique representation under the form X^2+3Y^2 with $X, 3Y$ coprime. As b is odd, we applique one of Euler's theorems on the convenient numbers "numerus idoneus" as cited above (Case C-2-2-1-2). It follows that b is prime.

We have also $p = 3b = A^{2m}+A^mB^n+B^{2n} = 4a'^2+B^n.C^l \Longrightarrow 9\alpha^2-a'^2 = B^n.C^l$, then $3\alpha, a' \in \mathbb{N}^*$ are solutions of the Diophantine equation:

$$x^2 - y^2 = N \tag{6.13}$$

with $N = B^nC^l > 0$. Let $Q(N)$ be the number of the solutions of (6.13) and $\tau(N)$ the number of ways to write the factors of N, then we announce the following result concerning the number of the solutions of (6.13) (see theorem 27.3 in [2]):

- If $N \equiv 2 \pmod 4$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 \pmod 4$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0 \pmod 4$, then $Q(N) = [\tau(N/4)/2]$.

We recall that $A^m \equiv 0(\mod 4)$. Concerning B^n, for $B^n \equiv 0(\mod 4)$ or $B^n \equiv 2(\mod 4)$, we find that $2 \mid B^n \Longrightarrow 2 \mid \alpha \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

For the last case $B^n \equiv 3 \pmod 4 \Longrightarrow C^l \equiv 3 \pmod 4 \Longrightarrow N = B^nC^l \equiv 1 (\mod 4) \Longrightarrow Q(N) = [\tau(N)/2] > 1$.

As $(3\alpha, a')$ is a couple of solutions of the Diophantine equation (6.13) and $3\alpha > a'$, then $\exists\ d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$d + d' = 6\alpha \tag{6.14}$$

$$d - d' = 2a' \tag{6.15}$$

** I-2-2-1-1-6-1 Now, we consider the case $d = c_1^{lr-1}C_1^l$ where c_1 is a prime integer with $c_1 \nmid C_1$ and $C = c_1^rC_1$, $r \geqslant 1$. It follows that $d' = c_1.B^n$. We rewrite the equations (6.14-6.15):

$$c_1^{lr-1}C_1^l + c_1.B^n = 6\alpha \tag{6.16}$$

$$c_1^{lr-1}C_1^l - c_1.B^n = 2a' \tag{6.17}$$

As $l \geqslant 3$, from the last two equations above, it follows that $c_1 \mid (6\alpha)$ and $c_1 \mid (2a')$. Then $c_1 = 2$, or $c_1 = 3$ and $3 \mid a'$ or $c_1 \neq 3 \mid \alpha$ and $c_1 \mid a'$.

** I-2-2-1-1-6-1-1 We suppose $c_1 = 2$. As $2 \mid (A^m = 2a') \Rightarrow 2 \mid a$ and $2 \mid C^l$ because $l \geqslant 3$, it follows $2 \mid B^n$, then $2 \mid (p = 3b)$. Then the contradiction with a, b coprime.

** I-2-2-1-1-6-1-2 We suppose $c_1 = 3 \Rightarrow c_1 \mid (a = 3a')$ and $c_1 = 3 \mid a'$. It follows that $(c_1 = 3) \mid (b = a'^2 + 3\alpha^2)$, then the contradiction with a, b coprime.
** I-2-2-1-1-6-1-3 We suppose $c_1 \neq 3$ and $c_1 \mid 3\alpha$ and $c_1 \mid a'$. It follows that $c_1 \mid a$ and $c_1 \mid b$, then the contradiction with a, b coprime.

The others cases of the expressions of d and d' not coprime so that $N = B^n C^l = d.d'$ give also contradictions.

** I-2-2-1-1-6-2 The last case is to consider $d = C^l$ and $d' = B^n$, so we obtain the only solution $(3\alpha, a')$ of the Diophantine equation (6.13). It follows that $Q(N) = 1$, then the contradiction with $Q(N) = [\tau(N)/2] > 1$ the number of the solution of (6.13).

It follows that the condition $3 \mid (b - a)$ is a contradiction.

The study of the case 6.8 is achieved.

6.9. Case $3 \mid p$ and $b \mid 4p$

The following cases have been soon studied:
* $3 \mid p$, $b = 2 \Longrightarrow b \mid 4p$: case 6.1,
* $3 \mid p$, $b = 4 \Longrightarrow b \mid 4p$: case 6.2,
* $3 \mid p \Longrightarrow p = 3p'$, $b \mid p' \Longrightarrow p' = bp"$, $p" \neq 1$: case 6.3,
* $3 \mid p$, $b = 3 \Longrightarrow b \mid 4p$: case 6.4,
* $3 \mid p \Longrightarrow p = 3p'$, $b = p' \Longrightarrow b \mid 4p$: case 6.8.

** J-1- Particular case: $b = 12$. In fact $3 \mid p \Longrightarrow p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b \mid 4p$. But $b < 4a < 3b$, that gives $12 < 4a < 36 \Longrightarrow 3 < a < 9$. As $2 \mid b$ and $3 \mid b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \dfrac{4p}{3}.\dfrac{a}{b} = \dfrac{5bp'}{3b} = \dfrac{5p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p' = 3p"$ with $p" \in \mathbb{N}^*$, then $p = 9p"$, we obtain the expressions:

$$A^{2m} = 5p" \tag{6.18}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 4p" \tag{6.19}$$

As $n, l \geqslant 3$, we deduce from the equation (6.19) that $2 \mid p" \Longrightarrow p" = 2^\alpha p_1$ with $\alpha \geqslant 1$ and $2 \nmid p_1$. Then (6.18) becomes: $A^{2m} = 5p" = 5 \times 2^\alpha p_1 \Longrightarrow 2 \mid A \Longrightarrow A = 2^i A_1$, $i \geqslant 1$ and $2 \nmid A_1$. We have also $B^nC^l = 2^{\alpha+2}p_1 \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-1-1-1- We suppose that $2 \mid B^n \Longrightarrow B = 2^j B_1$, $j \geqslant 1$ and $2 \nmid B_1$. We obtain $B_1^n C^l = 2^{\alpha+2-jn}p_1$:

- If $\alpha + 2 - jn > 0 \Longrightarrow 2 \mid C^l$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ and the conjecture (1.2) is verified.
- If $\alpha + 2 - jn = 0 \Longrightarrow B_1^n C^l = p_1$. From $C^{=}2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2 \mid C^l$ that implies that $2 \mid p_1$, then the contradiction with $2 \nmid p_1$.
- If $\alpha + 2 - jn < 0 \Longrightarrow 2^{jn-\alpha-2}B_1^n C^l = p_1$, it implies that $2 \mid p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2 \mid C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3}.\frac{a}{b} = \frac{12p'}{3}.\frac{7}{12} = \frac{7p'}{3} \Longrightarrow 3 \mid p' \Longrightarrow p = 9p"$, we obtain:

$$A^{2m} = 7p"$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 2p"$$

The last equation implies that $2 \mid B^nC^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3 \mid p \Rightarrow p = 3p'$ and $b \mid 4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1 b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^mB^n$). But $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{12p'}{3}\frac{a}{b} = \frac{4p'.a}{12p'} = \frac{a}{3} \Rightarrow 3 \mid a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} = \dfrac{k_1.a}{3} = a = (A^m)^2 = a'^2 \Longrightarrow A^m = a'$. The term A^mB^n gives $A^mB^n = \dfrac{p\sqrt{3}}{3}sin\dfrac{2\theta}{3} - \dfrac{a}{2}$, then:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} = 2p'\sqrt{3}sin\frac{2\theta}{3} \tag{6.20}$$

The left member of (6.20) is an integer number and also p', then $2\sqrt{3}sin\dfrac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two integer numbers and are coprime and $k_3 \mid p' \Longrightarrow p' = k_3.k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^mB^n = k_2.k_4 \tag{6.21}$$

Let μ be a prime number so that $\mu \mid k_4$, then $\mu \mid A^m(A^m+2B^n) \Longrightarrow \mu \mid A^m$ or $\mu \mid (A^m + 2B^n)$.

** J-3-1-1- $\mu \mid A^m$: If $\mu \mid A^m \Longrightarrow \mu \mid A^{2m} \Longrightarrow \mu \mid a$. As $\mu \mid k_4 \Longrightarrow \mu \mid p' \Rightarrow \mu \mid (4p' = b)$. But a, b are coprime, then the contradiction.

** J-3-1-2- $\mu \mid (A^m + 2B^n)$: If $\mu \mid (A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu \mid (A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain $p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$. As $p = 3p'$ and $\mu \mid p' \Rightarrow \mu \mid (3p') \Rightarrow \mu \mid p$, we can write : $\exists \mu'$ and $p = \mu\mu'$, then we arrive to:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu \mid B^n(B^n - A^m) \Longrightarrow \mu \mid B^n$ or $\mu \mid (B^n - A^m)$.

** J-3-1-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu \mid (B^n - A^m)$: If $\mu \mid (B^n - A^m)$ and using $\mu \mid (A^m + 2B^n)$, we obtain :

$$\mu \mid 3B^n \Longrightarrow \begin{cases} \mu \mid B^n \\ or \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu \mid B^n$: If $\mu \mid B^n \Longrightarrow \mu \mid B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3 \mid k_4 \Longrightarrow k_4 = 3k_4'$, and we have $p' = k_3k_4 = 3k_3k_4'$, it follows that $p = 3p' = 9k_3k_4'$, then $9 \mid p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then we obtain:

$$9k_3k_4' - 3A^mB^n = (A^m - B^n)^2$$

that we write : $3(3k_3k_4' - A^mB^n) = (A^m - B^n)^2$, then : $3 \mid (3k_3k_4' - A^mB^n) \Longrightarrow\ 3 \mid A^mB^n \Longrightarrow 3 \mid A^m$ or $3 \mid B^n$.

** J-3-1-2-2-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Rightarrow 3 \mid a$, but $3 \mid p' \Rightarrow 3 \mid (4p') \Rightarrow 3 \mid b$, then the contradiction with a, b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3 \mid B^n$: If $3 \mid B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3 \mid A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \Longrightarrow p' = k_3k_4 = k_3$. Then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{6.22}$$

with k_2, p' coprime, we write (6.22) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b-a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3 \mid (b-a), \quad and \quad b - a = b - a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (6.21) and that $3 \mid k_2 \Longrightarrow 3 \mid A^m(A^m + 2B^n) \Longrightarrow 3 \mid A^m$ or $3 \mid (A^m + 2B^n)$.

** J-3-2-1- $3 \mid A^m$: If $3 \mid A^m \Longrightarrow 3 \mid A^{2m} \Longrightarrow 3 \mid a$, but $3 \mid (b-a) \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime.

** J-3-2-2- $3 \mid (A^m + 2B^n) \Longrightarrow 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \Longrightarrow k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$3\alpha = A^m + 2B^n \tag{6.23}$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$x^2 + 3y^2 = b \tag{6.24}$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2 \mid a' \Longrightarrow 2 \mid a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1 (\bmod 4)$ or $A^m \equiv 3 (\bmod 4)$. If u, v verify (6.24), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (6.23): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \Longrightarrow b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (6.24).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 0 (\bmod 4)$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$p = a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \Longrightarrow 3 \mid (a' + B') \Longrightarrow a' + B' = 3B"$$
$$p' = B'^2 + 3B"^2 \Longrightarrow b = 4p' = (2B')^2 + 3(2B")^2 = a'^2 + 3\alpha^2$$

that gives $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 1 (\bmod 4)$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$p = C^{2l} - C^l B^n + B^{2n} = 4C'^2 - 2C'B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p'$$
$$\Longrightarrow 3 \mid (C' - B^n) \Longrightarrow C' - B^n = 3C"$$
$$p' = C'^2 + 3C"^2 \Longrightarrow b = 4p' = (2C')^2 + 3(2C")^2 = a'^2 + 3\alpha^2$$

We obtain $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1 (\mod 4)$ and $B^n \equiv 2 (\mod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1 (\mod 4)$ and $B^n \equiv 3 (\mod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3 (\mod 4)$ and $B^n \equiv 0 (\mod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3 (\mod 4)$ and $B^n \equiv 1 (\mod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3 (\mod 4)$ and $B^n \equiv 2 (\mod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3 (\mod 4)$ and $B^n \equiv 3 (\mod 4)$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- . It gives contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3 \mid k_1 \Longrightarrow k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3} = \dfrac{3k'_1 b}{3} \dfrac{a}{b} = k'_1 a$ and $B^n C^l = \dfrac{p}{3} \left(3 - 4cos^2 \dfrac{\theta}{3} \right) = \dfrac{k'_1}{4}(3b - 4a)$. As $B^n C^l$ is an integer number, we must have $4 \mid (3b - 4a)$ or $4 \mid k'_1$ or $[2 \mid k'_1$ and $2 \mid (3b - 4a)]$.

** J-4-1- We suppose that $4 \mid (3b - 4a)$.

** J-4-1-1- We suppose that $3b - 4a = 4 \Longrightarrow 4 \mid b \Longrightarrow 2 \mid b$. Then, we have:

$$A^{2m} = k'_1 a$$
$$B^n C^l = k'_1$$

** J-4-1-1-1- If k'_1 is prime, from $B^n C^l = k'_1$, it is impossible.

** J-4-1-1-2- We suppose that $k'_1 > 1$ is not prime. Let ω be a prime number so that $\omega \mid k'_1$.

** J-4-1-1-2-1- We suppose that $k'_1 = \omega^s$, with $s \geqslant 6$. Then we have :

$$A^{2m} = \omega^s.a \tag{6.25}$$
$$B^n C^l = \omega^s \tag{6.26}$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k'_1 are not coprime , then $2 \mid a$, as $2 \mid b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k'_1 are coprime, then $2 \nmid a$. From (6.26), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s.a$, but $A^m = 2^l - 2^n \Longrightarrow A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s.a \Longrightarrow 2^{2l} + 2^{2n} = 2^s(a+2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \Longrightarrow n < l \Longrightarrow 2n < s$, then $2^{2n}(1 + 2^{2l-2n} - 2^{s+1-2n}) = 2^n 2^l.a$. We call $l = n + n_1 \Longrightarrow 1 + 2^{2l-2n} - 2^{s+1-2n} = 2^{n_1}.a$, but the left member is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k'_1 = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k'_1 are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $t \nmid a_1$. Then, we have:

$$A^{2m} = \omega^{s+t}.a_1 \tag{6.27}$$
$$B^n C^l = \omega^s \tag{6.28}$$

From (6.28), we deduce that $B^n = \omega^n$, $C^n = \omega^l$, $s = n + l$ and $A^m = \omega^l - \omega^n > 0 \Longrightarrow l > n$. We have also $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \Longrightarrow \omega$ is odd, then $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2$ is even, then $2 \mid a_1 \Longrightarrow 2 \mid a$, it is in contradiction with a, b coprime, then this case is impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$A^{2m} = \omega^s.a \tag{6.29}$$
$$B^n C^l = \omega^s \tag{6.30}$$

From (6.30), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \Longrightarrow \omega$ is odd and $A^{2m} = \omega^s.a = (\omega^l - \omega^n)^2$ is even, then $2 \mid a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geqslant 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s.k_2.a$$
$$B^nC^l = \omega^s.k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geqslant 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s.k_2.a$$
$$B^nC^l = \omega^s.k_2 \tag{6.31}$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s.k_2.a \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $i \geqslant 1$ and $\omega \nmid A_1$, then $s = 2i.m$. From (6.31), we have $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $j \geqslant 1$ and $\omega \nmid B_1$. then :

$$B_1^nC^l = \omega^{2im-jn}k_2$$

- If $2im - jn > 0$, $\omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (1.2) is verified.

- If $2im - jn = 0 \Longrightarrow B_1^nC^l = k_2$, as $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - jn < 0 \Longrightarrow \omega^{jn-2im}B_1^nC^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega \mid a \Longrightarrow a = \omega^t.a_1$ and $\omega \nmid a_1$. So we have :

$$A^{2m} = \omega^{s+t}.k_2.a_1 \tag{6.32}$$
$$B^nC^l = \omega^s.k_2 \tag{6.33}$$

As $A^{2m} = \omega^{s+t}.k_2.a_1 \Longrightarrow \omega \mid A \Longrightarrow A = \omega^iA_1$ with $i \geqslant 1$ and $\omega \nmid A_1$, then $s + t = 2im$. From (6.33), we have $\omega \mid (B^nC^l) \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $j \geqslant 1$ and $\omega \nmid B_1$. then:

$$B_1^n C^l = \omega^{2im-t-jn} k_2$$

- If $2im - t - jn > 0$, $\omega \mid C^l \Longrightarrow \omega \mid C$, no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (1.2) is verified.

- If $2im - t - jn = 0 \Longrightarrow B_1^n C^l = k_2$, As $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega \mid (C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \Longrightarrow \omega^{jn+t-2im} B_1^n C^l = k_2 \Longrightarrow \omega \mid k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega \mid C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- $3b - 4a \neq 4$ and $4 \mid (3b - 4a) \Longrightarrow 3b - 4a = 4^s \Omega$ with $s \geqslant 1$ and $4 \nmid \Omega$. We obtain:

$$A^{2m} = k'_1 a \tag{6.34}$$

$$B^n C^l = 4^{s-1} k'_1 \Omega \tag{6.35}$$

** J-4-1-2-1- We suppose that $k'_1 = 2$. From (6.34), we deduce that $2 \mid a$. As $4 \mid (3b - 4a) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k'_1 = 3$. From (6.34) we deduce that $3^3 \mid A^{2m}$. From (6.35), it follows that $3^3 \mid B^n$ or $3^3 \mid C^l$. In the last two cases, we obtain $3^3 \mid p$. But $4p = 3k'_1 b = 9b \Longrightarrow 3 \mid b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k'_1 is prime $\geqslant 5$:

** J-4-1-2-3-1- Suppose that k'_1 and a are coprime. The equation (6.34) gives $(A^m)^2 = k'_1.a$, that is impossible with $k'_1 \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k'_1 and a are not coprime. Let $k'_1 \mid a \Longrightarrow a = k'^{\alpha}_1 a_1$ with $\alpha \geqslant 1$ and $k'_1 \nmid a_1$. The equation (6.34) is written as :

$$A^{2m} = k'_1 a = k'^{\alpha+1}_1 a_1$$

The last equation gives $k'_1 \mid A^{2m} \Longrightarrow k'_1 \mid A \Longrightarrow A = k'^{i}_1.A_1$, with $k'_1 \nmid A_1$. If $2i.m \neq (\alpha + 1)$, it is impossible. We suppose that $2i.m = \alpha + 1$, then

$k'_1 \mid A^m$. We return to the equation (6.35). If k'_1 and Ω are coprime, it is impossible. We suppose that k'_1 and Ω are not coprime, then $k'_1 \mid \Omega$ and the exponent of k'_1 in Ω is so the equation (6.35) is satisfying. We deduce easily that $k'_1 \mid B^n$. Then $k'^2_1 \mid (p = A^{2m}+B^{2n}+A^mB^n)$, but $4p = 3k'_1b \Longrightarrow k'_1 \mid b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k'_1 \geqslant 4$ is not a prime.

** J-4-1-2-4-1- We suppose that $k'_1 = 4$, we obtain then $A^{2m} = 4a$ and $B^nC^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 6.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k'_1 > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k'_1 are coprime. From the expression $A^{2m} = k'_1.a$, we deduce that $a = a_1^2$ and $k'_1 = k"^2_1$. It gives :

$$A^m = a_1.k"_1$$
$$B^nC^l = 4^{s-1}k"^2_1.\Omega$$

Let ω be a prime so that $\omega \mid k"_1$ and $k"_1 = \omega^t.k"_2$ with $\omega \nmid k"_2$. The last two equations become :

$$A^m = a_1.\omega^t.k"_2 \tag{6.36}$$
$$B^nC^l = 4^{s-1}\omega^{2t}.k"^2_2.\Omega \tag{6.37}$$

From (6.36), $\omega \mid A^m \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$ and $im = t$. From (6.37), we obtain $\omega \mid B^nC^l \Longrightarrow \omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-1-1- If $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j.B_1$ with $\omega \nmid B_1$. From (6.36), we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k"^2_2.\Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s-j.n-2}k"^2_2.\Omega$:

- If $2t + 2s - jn - 2 \leqslant 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t + 2s - jn - 2 \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (1.2) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2 \mid \Omega \Longrightarrow \Omega = 2.\Omega_1$ because $4 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s+1-j.n-2}k"^2_2\Omega_1$:

- If $2t + 2s - jn - 3 \leqslant 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.

- If $2t+2s-jn-3 \geqslant 1 \Longrightarrow 2 \mid C^l \Longrightarrow 2 \mid C$ and the conjecture (1.2) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^n C^l = \omega^{2t-j.n} 4^{s-1}.k"_2^2.\Omega$:
-If $2t-jn \leqslant 0 \Longrightarrow \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$.
-If $2t-jn \geqslant 1 \Longrightarrow \omega \mid C^l \Longrightarrow \omega \mid C$ and the conjecture (1.2) is verified.

** J-4-1-2-4-2-1-2- If $\omega \mid C^l \Longrightarrow \omega \mid C \Longrightarrow C = \omega^h.C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k'_1 are not coprime. Let ω be a prime so that $\omega \mid a$ and $\omega \mid k'_1$. We write:

$$a = \omega^\alpha.a_1$$
$$k'_1 = \omega^\mu.k"_1$$

with $a_1, k"_1$ coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu}.a_1.k"_1$. The term $B^n C^l$ becomes:

$$B^n C^l = 4^{s-1}.\omega^\mu.k"_1.\Omega \tag{6.38}$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \Longrightarrow 2 \mid a$, but $2 \mid b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geqslant 3$, we have $\omega \mid a$. If $\omega \mid b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m}, we obtain $\omega \mid A^{2m} \Longrightarrow \omega \mid A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$, $i \geqslant 1$ and $2i.m = \alpha + \mu$. From (6.38), we deduce that $\omega \mid B^n$ or $\omega \mid C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega \mid B^n \Longrightarrow \omega \mid B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$ and $j \geqslant 1$. Then, $B_1^n C^l = 4^{s-1} \omega^{\mu-jn}.k"_1.\Omega$:

* $\omega \nmid \Omega$:
- If $\mu - jn \geqslant 1$, we have $\omega \mid C^l \Longrightarrow \omega \mid C$, there is no contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$ and the conjecture (1.2) is verified.
- If $\mu - jn \leqslant 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im} A_1^m + \omega^{jn} B_1^n$. Then this case is impossible.

* $\omega \mid \Omega$: we write $\Omega = \omega^\beta.\Omega_1$ with $\beta \geqslant 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s.\Omega = 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4a + 4^s.\omega^\beta.\Omega_1 = 4\omega^\alpha.a_1 + 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b =$

$4\omega(\omega^{\alpha-1}.a_1+4^{s-1}.\omega^{\beta-1}.\Omega_1)$. If $\omega=3$ and $\beta=1$, we obtain $b=4(3^{\alpha-1}a_1+4^{s-1}\Omega_1)$ and $B_1^nC^l=4^{s-1}3^{\mu+1-jn}.k"_1\Omega_1$.

- If $\mu-jn+1\geqslant 1$, then $3\mid C^l$ and the conjecture (1.2) is verified.
- If $\mu-jn+1\leqslant 0$, then $3\nmid C^l$ and it is the contradiction with $C^l=3^{im}A_1^m+3^{jn}B_1^n$.

Now, if $\beta\geqslant 2$ and $\alpha=im\geqslant 3$, we obtain $3b=4\omega^2(\omega^{\alpha-2}a_1+4^{s-1}\omega^{\beta-2}\Omega_1)$. If $\omega=3$ or not, then $\omega\mid b$, but $\omega\mid a$, then the contradiction with a,b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega\mid C^l\Longrightarrow\omega\mid C\Longrightarrow C=\omega^hC_1$ with $\omega\nmid C_1$ and $h\geqslant 1$. Then, $B^nC_1^l=4^{s-1}\omega^{\mu-hl}.k"_1.\Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4\mid k_1'$.

** J-4-2-1- $k_1'=4\Longrightarrow 4p=3k_1'b=12b\Longrightarrow p=3b=3p'$, this case has been studied (see case I-2- paragraph 6.8).

** J-4-2-2- $k_1'>4$ with $4\mid k_1'\Longrightarrow k_1'=4^sk"_1$ and $s\geqslant 1$, $4\nmid k"_1$. Then, we obtain:

$$A^{2m}=4^sk"_1a=2^{2s}k"_1a$$
$$B^nC^l=4^{s-1}k"_1(3b-4a)=2^{2s-2}k"_1(3b-4a)$$

** J-4-2-2-1- We suppose that $s=1$ and $k_1'=4k"_1$ with $k"_1>1$, so $p=3p'$ and $p'=k"_1b$, this is the case 6.3 already studied.

** J-4-2-2-2- We suppose that $s>1$, then $k_1'=4^sk"_1\Longrightarrow 4p=3\times 4^sk"_1b$ and we obtain:

$$A^{2m}=4^sk"_1a \tag{6.39}$$
$$B^nC^l=4^{s-1}k"_1(3b-4a) \tag{6.40}$$

** J-4-2-2-2-1- We suppose that $2\nmid(k"_1.a)\Longrightarrow 2\nmid k"_1$ and $2\nmid a$. As $(A^m)^2=(2^s)^2.(k"_1.a)$, we call $d^2=k"_1.a$, then $A^m=2^s.d\Longrightarrow 2\mid A^m\Longrightarrow 2\mid A\Longrightarrow A=2^iA_1$ with $2\nmid A_1$ and $i\geqslant 1$, then: $2^{im}A_1^m=2^s.d\Longrightarrow s=im$. From the equation (6.40), we have $2\mid(B^nC^l)\Longrightarrow 2\mid B^n$ or $2\mid C^l$.

** J-4-2-2-2-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, with $j \geqslant 1$ and $2 \nmid B_1$. The equation (6.40) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k"_1(3b-4a) = 2^{2im-jn-2} k"_1(3b-4a)$$

* We suppose that $2 \nmid (3b-4a)$:

- If $2im - jn - 2 \geqslant 1$, then $2 \mid C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (1.2) is verified.

- If $2im - jn - 2 \leqslant 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\mu \mid (3b-4a)$, $\mu \geqslant 1$:

- If $2im + \mu - jn - 2 \geqslant 1$, then $2 \mid C^l$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (1.2) is verified.

- If $2im + \mu - jn - 2 \leqslant 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geqslant 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2 \mid (k"_1.a)$:

** J-4-2-2-2-2-1- We suppose that $k"_1$ and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2 \mid k"_1 \Longrightarrow k"_1 = 2^{2\mu}.k"^2_2$ and $a = a_1^2$, then the equations (6.39-6.40) become:

$$A^{2m} = 4^s.2^{2\mu} k"^2_2 a_1^2 \Longrightarrow A^m = 2^{s+\mu}.k"_2.a_1 \tag{6.41}$$

$$B^n C^l = 4^{s-1} 2^{2\mu} k"^2_2 (3b-4a) = 2^{2s+2\mu-2} k"^2_2 (3b-4a) \tag{6.42}$$

The equation (6.41) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geqslant 1$ and $im = s + \mu$. From the equation (6.42), we have $2 \mid (B^n C^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geqslant 1$, then $B_1^n C^l = 2^{2s+2\mu-jn-2} k"^2_2 (3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2im + 2\mu - jn - 2 \geqslant 1 \Rightarrow 2 \mid C^l$, then there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (1.2) is verified.

- If $2im + 2\mu - jn - 2 \leqslant 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b - 4a)$, $\alpha \geqslant 1$ so that a, b remain coprime:

- If $2im + 2\mu + \alpha - jn - 2 \geqslant 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2im + 2\mu + \alpha - jn - 2 \leqslant 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geqslant 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k"_1$ and $2 \mid a \Longrightarrow a = 2^{2\mu}.a_1^2$ and $k"_1 = k"^2_2$, then the equations (6.39-6.40) become:

$$A^{2m} = 4^s.2^{2\mu}a_1^2k"^2_2 \Longrightarrow A^m = 2^{s+\mu}.a_1.k"_2. \tag{6.43}$$

$$B^nC^l = 4^{s-1}k"^2_2(3b - 4a) = 2^{2s-2}k"^2_2(3b - 4a) \tag{6.44}$$

The equation (6.43) gives $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geqslant 1$ and $im = s + \mu$. From the equation (6.44), we have $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geqslant 1$. Then we obtain $B_1^nC^l = 2^{2s-jn-2}k"^2_2(3b - 4a)$:

* We suppose that $2 \nmid (3b - 4a) \Longrightarrow 2 \nmid b$:

- If $2im - jn - 2 \geqslant 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2im - jn - 2 \leqslant 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b - 4a)$, $\alpha \geqslant 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geqslant 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that $k"_1$ and a are not coprime $2 \mid a$ and $2 \mid k"_1$. Let $a = 2^t.a_1$ and $k"_1 = 2^\mu k"_2$ and $2 \nmid a_1$ and $2 \nmid k"_2$. From (6.39), we have $\mu + t = 2\lambda$ and $a_1.k"_2 = \omega^2$. The equations (6.39-6.40) become:

$$A^{2m} = 4^s k"_1 a = 2^{2s}.2^\mu k"_2.2^t.a_1 = 2^{2s+2\lambda}.\omega^2 \Longrightarrow A^m = 2^{s+\lambda}.\omega \tag{6.45}$$

$$B^nC^l = 4^{s-1}2^\mu k"_2(3b-4a) = 2^{2s+\mu-2}k"_2(3b-4a) \tag{6.46}$$

From (6.45) we have $2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^iA_1$,$i \geqslant 1$ and $2 \nmid A_1$. From(6.46), $2s+\mu-2 \geqslant 1$, we deduce that $2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geqslant 1$. Then we obtain $B_1^nC^l = 2^{2s+\mu-jn-2}k"^2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2s+\mu-jn-2 \geqslant 1 \Rightarrow 2 \mid C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.
- If $2s+\mu-jn-2 \leqslant 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha \mid (3b-4a)$, for one value $\alpha \geqslant 1$. As $2 \mid a$, then $2^\alpha \mid (3b-4a) \Longrightarrow 2 \mid (3b-4a) \Longrightarrow 2 \mid (3b) \Longrightarrow 2 \mid b$, then the contradiction with a, b coprime.

** J-4-2-2-2-2-2-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C \Longrightarrow C = 2^h.C_1$, with $h \geqslant 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2 \mid k'_1$ and $2 \mid (3b-4a)$: then we obtain $2 \mid k'_1 \Longrightarrow k'_1 = 2^t.k"_1$ with $t \geqslant 1$ and $2 \nmid k"_1$, $2 \mid (3b-4a) \Longrightarrow 3b-4a = 2^\mu.d$ with $\mu \geqslant 1$ and $2 \nmid d$. We have also $2 \mid b$. If $2 \mid a$, it is a contradition with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (6.39-6.40) become:

$$A^{2m} = 2^t.k"_1.a = (A^m)^2 \tag{6.47}$$

$$B^nC^l = 2^{t-1}k"_1.2^{\mu-1}d = 2^{t+\mu-2}k"_1.d \tag{6.48}$$

From (6.47), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k"_1.a$, it gives $A^m = 2^\lambda.\omega \Longrightarrow 2 \mid A^m \Longrightarrow 2 \mid A \Longrightarrow A = 2^i.A_1$ with $i \geqslant 1$ and $2 \nmid A_1$. From (6.48), we have $2\lambda+\mu-2 \geqslant 1$, then

$2 \mid (B^nC^l) \Longrightarrow 2 \mid B^n$ or $2 \mid C^l$:

** J-4-3-1- We suppose that $2 \mid B^n \Longrightarrow 2 \mid B \Longrightarrow B = 2^jB_1$, with $j \geqslant 1$ and $2 \nmid B_1$. Then we obtain $B_1^nC^l = 2^{2\lambda+\mu-jn-2}.k"_1.d$.

- If $2\lambda + \mu - jn - 2 \geqslant 1 \Rightarrow 2 \mid C^l \Longrightarrow 2 \mid C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (1.2) is verified.

- If $2s + t + \mu - jn - 2 \leqslant 0 \Rightarrow 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2 \mid C^l \Longrightarrow 2 \mid C$. With the same method used above, we obtain identical results.

□

The Main Theorem is proved.

7. Examples and Conclusion

7.1. Numerical Examples

7.1.1. Example 1:

We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \quad q = 8 \times 3^{11}, \quad \bar{\Delta} = 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0$$

$$\rho = \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, \quad cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{7.1}$$

As $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} \Longrightarrow cos^2\frac{\theta}{3} = \frac{3A^{2m}}{4p} = \frac{3 \times 2^4}{73} = \frac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73$; then we obtain:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6.b \tag{7.2}$$

We verify easily the equation (7.1) to calculate $cos\theta$ using (7.2). For this example, we can use the two conditions from (4.9) as $3 \mid a$,$b \mid 4p$ and $3 \mid p$. The cases 5.4 and 6.3 are respectively used. For the case 5.4, it is the case B-2-2-1- that was used and the conjecture (1.2) is verified. Concerning the case 6.3, it is the case G-2-2-1- that was used and the conjecture (1.2) is verified.

7.1.2. Example 2:

The second example is: $7^4+7^3=14^3$. We take $A^m=7^4\,,B^n=7^3$ and $C^l=14^3$. We obtain $p=57\times 7^6=3\times 19\times 7^6\,,\quad q=8\times 7^{10}\,,\quad \overline{\Delta}=27q^2-4p^3=27\times 4\times 7^{18}(16\times 49-19^3)=-27\times 4\times 7^{18}\times 6075<0\,,\quad \rho=19\times 7^9\times\sqrt{19}\,,\quad cos\theta=-\dfrac{4\times 7}{19\sqrt{19}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{7^2}{4\times 19}=\dfrac{a}{b}\Longrightarrow a=7^2,\ b=4\times 19$, then $cos\dfrac{\theta}{3}=\dfrac{7}{2\sqrt{19}}$ and we have the two principal conditions $3\mid p$ and $b\mid(4p)$. The calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{7}{2\sqrt{19}}\right)^3-3\frac{7}{2\sqrt{19}}=-\frac{4\times 7}{19\sqrt{19}}$$

Then, we obtain $3\mid p\Rightarrow p=3p'$, $b\mid(4p)$ with $b\neq 2,4$ then $12p'=k_1b=3\times 7^6b$. It concerns the paragraph 6.9 of the second hypothesis. As $k_1=3\times 7^6=3k'_1$ with $k'_1=7^6\neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4\mid(3b-4a)$. So we verify :

$$3b-4a=3\times 4\times 19-4\times 7^2=32\Longrightarrow 4\mid(3b-4a)$$

with $A^{2m}=7^8=7^6\times 7^2=k'_1.a$ and k'_1 not a prime, with a and k'_1 not coprime with $\omega=7\nmid\Omega(=2)$. We find that the conjecture (1.2) is verified with a common factor equal to 7 (prime and divisor of $k'_1=7^6$).

7.1.3. Example 3:

The third example is: $19^4+38^3=57^3$ with $A^m=19^4$, $B^n=38^3$ and $C^l=57^3$. We obtain $p=19^6\times 577\,,\quad q=8\times 27\times 19^{10}\,,\quad \overline{\Delta}=27q^2-4p^3=4\times 19^{18}(27^3\times 16\times 19^2-577^3)<0\,,\quad \rho=\dfrac{19^9\times 577\sqrt{577}}{3\sqrt{3}}\,,\quad cos\theta=-\dfrac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{3\times 19^2}{4\times 577}=\dfrac{a}{b}\Longrightarrow a=3\times 19^2,\ b=4\times 577$, then $cos\dfrac{\theta}{3}=\dfrac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3\mid a$ and $b\mid(4p)$. Here again, the calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{19\sqrt{3}}{2\sqrt{577}}\right)^3-3\frac{19\sqrt{3}}{2\sqrt{577}}=-\frac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3 \mid a \Rightarrow a = 3a' = 3 \times 19^2$, $b \mid (4p)$ with $b \neq 2, 4$ and $b = 4p'$ with $p = kp'$ soit $p' = 577$ and $k = 19^6$. This concerns the paragraph 5.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega = 19$, a', ω not coprime and $\omega = 19 \nmid (p' - a') = (577 - 19^2)$ with $s - jn = 6 - 1 \times 3 = 3 \geqslant 1$, and the conjecture (1.2) is verified.

7.2. Conclusion

The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

THEOREM 7.1. — *Let A, B, C, m, n, and l be positive natural numbers with $m, n, l > 2$. If :*

$$A^m + B^n = C^l \tag{7.3}$$

then A, B, and C have a common factor.

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