Variations on the theme of the Trotter-Kato theorem for homogenization of periodic hyperbolic systems *

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Abstract

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a matrix elliptic second order differential operator $B_\varepsilon > 0$. Coefficients of the operator B_ε are periodic with respect to some lattice in $\mathbb{R}^d$ and depend on $\mathbf{x}/\varepsilon$. We study the quantitative homogenization for the solutions of the hyperbolic system $\partial_t^2\mathbf{u}_\varepsilon = -B_\varepsilon\mathbf{u}_\varepsilon$. In operator terms, we are interested in approximations of the operators $\cos(tB_\varepsilon^{1/2})$ and $B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})$ in suitable operator norms. Approximations for the resolvent B_ε^{-1} have been already obtained by T. A. Suslina. So, we rewrite hyperbolic equation as a system for the vector with components $\mathbf{u}_\varepsilon$ and $\partial_t\mathbf{u}_\varepsilon$, and consider the corresponding unitary group. For this group, we adapt the proof of the Trotter-Kato theorem by introduction of some correction term and derive hyperbolic results from elliptic ones.

Key words: homogenization, convergence rates, hyperbolic systems, Trotter-Kato theorem.

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Introduction

The paper is devoted to homogenization of periodic differential operators (DO's). More precisely, we are interested in the so-called operator error estimates, i. e., in quantitative homogenization results, admitting formulation in the uniform operator topology.

For elliptic and parabolic problems, estimates of such type are very well studied, see, e. g., books [CDaGr, Chapter 14] and [Sh], the survey [ZhPas2], the papers [BSu2, Su1] and references therein. For hyperbolic problems the situation is different. In the present paper, we deal with the hyperbolic systems in $\mathbb{R}^d$.

0.1 Problem setting

Let $\Gamma \subset \mathbb{R}^d$ be a lattice and let Ω be the elementary cell of the lattice Γ. For a Γ-periodic function ψ in $\mathbb{R}^d$, we denote $\psi^\varepsilon(\mathbf{x}) := \psi(\mathbf{x}/\varepsilon)$, where $\varepsilon > 0$, and $\overline{\psi} := |\Omega|^{-1} \int_\Omega \psi(\mathbf{x})\, d\mathbf{x}$.

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we study a selfadjoint matrix strongly elliptic second order DO B_ε, $0 < \varepsilon \leqslant 1$. The principal part of the operator B_ε is given in a factorized form $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, where $b(\mathbf{D})$ is a matrix homogeneous first order DO, and $g(\mathbf{x})$ is a Γ-periodic bounded and positive definite matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(\mathbf{D})$ and $g(\mathbf{x})$ are given below in Subsection 1.2.) The simplest example of the operator A_ε is the scalar elliptic operator $A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$ (acoustics operator). The matrix-valued example is the operator of elasticity theory. It can be written as $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, see [BSu2, Chapter 2, §5].

The operator B_ε is given by the differential expression

$$B_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^{d} \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j a_j^\varepsilon(\mathbf{x})^* \right) + Q^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x}). \quad (0.1)$$

Here $a_j(\mathbf{x})$, $j = 1, \dots, d$, and $Q(\mathbf{x})$ are Γ-periodic matrix-valued functions, in general, unbounded; a Γ-periodic matrix-valued function $Q_0(\mathbf{x})$ is such that $Q_0(\mathbf{x}) > 0$ and $Q_0, Q_0^{-1} \in L_\infty$. The constant λ is chosen so that the operator B_ε is positive definite. (The precise assumptions on the coefficients are given below in Subsection 1.3.) An example of operator (0.1) is the Schrödinger operator with the singular potential of the form $\varepsilon^{-1} v_\varepsilon$, see [Su1, §11].

The coefficients of the operator (0.1) oscillate rapidly for small ε. *Our goal* is to study the behaviour of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the hyperbolic system

$$\begin{cases} \partial_t^2 \mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t), & \mathbf{x}\in\mathbb{R}^d,\ t\in\mathbb{R}, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), & \partial_t \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \end{cases} \tag{0.2}$$

and to give an approximation for $\mathbf{u}_\varepsilon$ with the quantitative error estimate with explicit dependence of the norms of the initial data $\boldsymbol{\phi} \in H^2(\mathbb{R}^d;\mathbb{C}^n)$ and $\boldsymbol{\psi} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$. The solution $\mathbf{u}_\varepsilon$ is given by

$$\mathbf{u}_\varepsilon(\cdot,t) = \cos(tB_\varepsilon^{1/2})\boldsymbol{\phi} + B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi}. \tag{0.3}$$

So, we are interested in approximations for the cosine $\cos(tB_\varepsilon^{1/2})$ and sine functions $B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})$ in suitable operator norms. (The terms ,,cosine" and ,,sine functions" used here are common in semigroup theory, see [ABaHi, Definition 3.14.2 and (3.93)].)

0.2 Main results

It turns out that the solution $\mathbf{u}_\varepsilon$ of the problem (0.2) with rapidly oscillating coefficients behaves in the small period limit as the solution of the so-called *effective problem*:

$$\begin{cases} \partial_t^2 \mathbf{u}_0(\mathbf{x},t) = -B^0 \mathbf{u}_0(\mathbf{x},t), & \mathbf{x}\in\mathbb{R}^d,\ t\in\mathbb{R}, \\ \mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), & \partial_t \mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases}$$

Here B^0 is the *effective operator* with constant coefficients. The precise definition of B^0 can be found in Subsection 1.6.

Let us formulate the main results of the paper. The principal term of approximation for the solution $\mathbf{u}_\varepsilon$ is given by

$$\|\mathbf{u}_\varepsilon(\cdot,t) - \mathbf{u}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|)\|\boldsymbol{\phi}\|_{H^2(\mathbb{R}^d)} + C\varepsilon|t|\|\boldsymbol{\psi}\|_{H^1(\mathbb{R}^d)}.$$

It is also possible to approximate the time derivative $\partial_t \mathbf{u}_\varepsilon$ of the solution:

$$\|(\partial_t\mathbf{u}_\varepsilon)(\cdot,t) - (\partial_t\mathbf{u}_0)(\cdot,t)\|_{H^{-1}(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|)\|\boldsymbol{\phi}\|_{H^2(\mathbb{R}^d)} + C\varepsilon|t|\|\boldsymbol{\psi}\|_{H^1(\mathbb{R}^d)}.$$

According to (0.3) and the similar identity for the solution $\mathbf{u}_0$ of the effective problem, these estimates can be rewritten in operator terms:

$$\|\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|), \tag{0.4}$$

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon|t|, \tag{0.5}$$

$$\|B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|), \tag{0.6}$$

$$\|\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C\varepsilon|t|. \tag{0.7}$$

In accordance with the classical result [BrOtFMu], it is impossible to give H^1-approximation for the solution $\mathbf{u}_\varepsilon$ of the problem (0.2) with an arbitrary initial data $\boldsymbol{\phi}$. In [BrOtFMu], it was observed that such approximation can be constructed only for very special choices of the initial data. The argument is the following: convergence of the energy of the solution $\mathbf{u}_\varepsilon$ to the energy of $\mathbf{u}_0$ does not occur in the general situation. But the solution $\mathbf{u}_\varepsilon$ can be splitted into two parts: the first one is designed so that the corresponding energy converges to the energy for the effective equation and the second part tends to zero in some sense (but not in the uniform operator topology). In our considerations, we deal only with the first part $\mathbf{v}_\varepsilon$:

$$\begin{cases} \partial_t^2 \mathbf{v}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{v}_\varepsilon(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d,\ t\in\mathbb{R}, \\ \mathbf{v}_\varepsilon(\mathbf{x},0) = B_\varepsilon^{-1}\boldsymbol{\phi}(\mathbf{x}), \quad \partial_t \mathbf{v}_\varepsilon(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases} \tag{0.8}$$

Here $\boldsymbol{\phi}\in H^1(\mathbb{R}^d;\mathbb{C}^n)$ and $\boldsymbol{\psi}\in H^2(\mathbb{R}^d;\mathbb{C}^n)$. In operator terms, this case corresponds to consideration of the operator $\cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}$ instead of $\cos(tB_\varepsilon^{1/2})$ (see discussion in Subsection 4.3 below). We have

$$\begin{aligned} &\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon\mathcal{K}_1(\varepsilon;t)\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\leqslant C\varepsilon(1+|t|), \end{aligned} \tag{0.9}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon\mathcal{K}_2(\varepsilon;t)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\leqslant C\varepsilon(1+|t|). \end{aligned} \tag{0.10}$$

Here $\mathcal{K}_1(\varepsilon;t)$ and $\mathcal{K}_2(\varepsilon;t)$ are correctors. They contain rapidly oscillating factors and so depend on ε. In the general case, correctors contain some smoothing operator. We distinguish the cases, when the smoothing operator can be removed from the correctors. In particular, if $d\leqslant 4$, then the smoothing operator always can be replaced by the identity operator.

Let $\mathbf{v}_0$ be the solution of the effective problem for (0.8):

$$\begin{cases} \partial_t^2 \mathbf{v}_0(\mathbf{x},t) = -B^0 \mathbf{v}_0(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d,\ t\in\mathbb{R}, \\ \mathbf{v}_0(\mathbf{x},0) = (B^0)^{-1}\boldsymbol{\phi}(\mathbf{x}), \quad \partial_t \mathbf{v}_0(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}). \end{cases}$$

By $\mathbf{w}_\varepsilon$ we denote the first order approximation for the solution $\mathbf{v}_\varepsilon$: $\mathbf{w}_\varepsilon(\cdot,t) = \mathbf{v}_0(\cdot,t) + \varepsilon\mathcal{K}_1(\varepsilon;t)\boldsymbol{\phi} + \varepsilon\mathcal{K}_2(\varepsilon;t)\boldsymbol{\psi}$. Then

$$\begin{aligned} &\|(\partial_t\mathbf{v}_\varepsilon)(\cdot,t) - (\partial_t\mathbf{v}_0)(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant C\varepsilon|t|\|\boldsymbol{\phi}\|_{H^1(\mathbb{R}^d)} + C\varepsilon(1+|t|)\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)}, \\ &\|\mathbf{v}_\varepsilon(\cdot,t) - \mathbf{w}_\varepsilon(\cdot,t)\|_{H^1(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|)\|\boldsymbol{\phi}\|_{H^1(\mathbb{R}^d)} + C\varepsilon(1+|t|)\|\boldsymbol{\psi}\|_{H^2(\mathbb{R}^d)}. \end{aligned}$$

Note that our estimates grow with time as $O(|t|)$. Some growth seems natural because of the dispersion of waves in inhomogeneous media, see discussion in introduction to the paper [BenGl] and references therein.

0.3 Survey

The interest on homogenization results admitting a formulation in the uniform operator topology was stimulated by the work of M. Sh. Birman and T. A. Suslina [BSu1], where operator error estimates appeared in an explicit form at the first time. In [BSu1], the spectral theory approach to homogenization problems was applied. The method is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory. For further conceptual development of spectral method to the fibre homogenization, see [CooW]. Another approach to obtaining operator error estimates for the problems in $\mathbb{R}^d$ was suggested by V. V. Zhikov [Zh] and developed by him together with S. E. Pastukhova [ZhPas1]. Both methods were applied to elliptic and parabolic problems. For the class of operators B_ε under consideration, approximations for B_ε^{-1} and e^{-tB_ε} were obtained in [Su1, Su2, M1]. The effective operator B^0 was calculated in [Bo].

Hyperbolic systems were studied only via the spectral approach, see [BSu5, M4, M2, DSu1] and [CheW]. In [BSu5], the non-stationary Schrödinger equation also was considered. It was obtained that

$$\| \cos(tA_\varepsilon^{1/2}) - \cos(t(A^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|), \tag{0.11}$$

$$\|A_\varepsilon^{-1/2}\sin(tA_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(t(A^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|)^2, \tag{0.12}$$

$$\|e^{-itA_\varepsilon} - e^{-itA^0}\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|), \tag{0.13}$$

$t\in\mathbb{R}$. In [D], estimate (0.13) was generalized to the case of the operator B_ε. In [M4, M2], the estimate (0.12) was refined with respect to the type of the operator norm:

$$\|A_\varepsilon^{-1/2}\sin(tA_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(t(A^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C\varepsilon(1+|t|), \tag{0.14}$$

$t\in\mathbb{R}$, and $(H^2\to H^1)$-approximation for the operator $A_\varepsilon^{-1/2}\sin(tA_\varepsilon^{1/2})$ was obtained

$$\begin{aligned}&\big\|A_\varepsilon^{-1/2}\sin(tA_\varepsilon^{1/2}) - (A^0)^{-1/2}\sin(t(A^0)^{1/2}) - \varepsilon\mathrm{K}(\varepsilon;t)\big\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ &\leqslant C\varepsilon(1+|t|),\quad t\in\mathbb{R}.\end{aligned} \tag{0.15}$$

Here $\mathrm{K}(\varepsilon;t)$ is the corrector. In [DSu1], the sharpness of estimates (0.11) and (0.14) with respect to the type of the norm was proven in the general case. By ,,sharpness" we mean that it is impossible to obtain the error estimate of the order $O(\varepsilon)$ for approximation of $\cos(tA_\varepsilon^{1/2})$ in $(H^{2-\delta}\to L_2)$-norm with some $\delta > 0$ and $(H^{1-\delta} \to L_2)$-estimate for $A_\varepsilon^{-1/2}\sin(tA_\varepsilon^{1/2})$. M. Dorodnyi and T. Suslina [DSu3] showed that estimate (0.15) is also sharp with respect to the type of the norm. Moreover, estimates (0.11), (0.14), and (0.15) are also sharp with respect to time, i. e., the order $O(|t|)$, $|t| \to \infty$, cannot be refined in the general case. The proof available at [DSu2]. For completeness of the survey, we mention the paper [CheW], where only the case $d = 1$ was considered. For the problem of fractional elasticity, some order sharp estimate in L_2 was obtained in non-stationary setting as well as resolvent estimates (see [CheW, Theorems 3.2 and 7.1]).

Our estimates (0.4), (0.5), and (0.10) transfer results from [BSu5, M2] to the more general class of operators. So, we believe that estimates (0.4), (0.5), and (0.10) are sharp with respect to the type of the norm. Note that inequalities (0.5) and (0.14) have different behavior for small $|t|$. Estimates of the form (0.6), (0.7), and (0.9) are new even for the operator without lower order terms. It seems natural to expect that they are also sharp with respect to the type of the norm. Indeed, it is in accordance with usual difference in smoothness of the initial data for solution and its derivative in the setting of hyperbolic problems.

Recall that, according to the classical Trotter-Kato theorem (see, e. g., [Sa, Chapter X, Theorem 1.1]), the strong convergence of semigroups follows from the strong convergence of the corresponding resolvents. This result is fruitful for time-dependent homogenization problems. Let us note the recent work [ChEl], where the transfer of the Trotter-Kato theorem to weak and uniform operator topologies was studied and the results were applied to homogenization of parabolic equations (without operator error estimates). Let us also mention the paper [CooSav], where the homogenization of the attractors of the quasi-linear damped wave equation was derived from the $(L_2 \to L_2)$-approximation for the resolvent. But the results of [CooSav] can not be written in the uniform operator topology: the error estimates for homogenization are written in terms of the attractors, not the solutions[1], and the dependence of constants from the norms of the initial data was not traced. For operators, acting in a bounded domain, hyperbolic results were derived from elliptic ones via the Laplace transform, see [M3]. But the corresponding results do not look optimal with respect to the type of the operator norm. Let us also mention the work [W], where Laplace transform technique also was applied, but the rate of convergence was not traced explicitly. Finally, we highlight the paper [CheW], where the non-stationary homogenization result with the precise order error estimate was derived from elliptic one with the

[1] I. e., there are no operator-error analogues for Theorem 1.3 from [FiVish], where the quantitative homogenization of attaractors was studied in a more classical manner.

help of the Fourier–Laplace transform.

The method of the present paper also can be considered as a variation on the theme of the Trotter-Kato theorem.

0.4 Method

The proof relies on the rewriting of the hyperbolic equation as a system for the solution and its time derivative, the explicit formulas for the corresponding group of operators and the resolvent of the generator of the group, and the known results on homogenization of the resolvent B_ε^{-1} from [Su1]. Our key observation is that the quantitative homogenization results for the hyperbolic systems can be derived from the approximation of the resolvent by analogy with the proof of the Trotter-Kato theorem.

First, we rewrite hyperbolic problem (0.2) in the following form:

$$\partial_t \begin{pmatrix} \mathbf{u}_\varepsilon \\ \partial_t \mathbf{u}_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 & I \\ -B_\varepsilon & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\varepsilon \\ \partial_t \mathbf{u}_\varepsilon \end{pmatrix}, \quad \begin{pmatrix} \mathbf{u}_\varepsilon(\cdot,0) \\ \partial_t \mathbf{u}_\varepsilon(\cdot,0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{pmatrix}.$$

Denote $\mathfrak{A}_\varepsilon = \begin{pmatrix} 0 & I \\ -B_\varepsilon & 0 \end{pmatrix}$. Then, according to (0.3),

$$\begin{pmatrix} \mathbf{u}_\varepsilon \\ \partial_t \mathbf{u}_\varepsilon \end{pmatrix} = e^{t\mathfrak{A}_\varepsilon} \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \cos(tB_\varepsilon^{1/2}) & B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) \\ -B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) & \cos(tB_\varepsilon^{1/2}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{pmatrix}.$$

Our second step can be considered as technical modification of the proof of the classical Trotter-Kato theorem in the variant from [BSu5, Lemma 19.1]. The new observation is the following: we need to use approximation for the resolvent with corrector to obtain the principal term of approximation for the solution of hyperbolic problem. We use this trick to make estimates optimal with respect to the type of the operator norm.

Actually, estimates (0.4)–(0.7) are neither else then approximations for $e^{t\mathfrak{A}_\varepsilon}$ in $H^2 \times H^1 \to L^2 \times H^{-1}$-norm written entrywise. While in the context of works [BSu5], [DSu1], [DSu2] results in norms of operators acting to H^{-1} look new, the $L_2 \times H^{-1}$-norm appeared in a more classical homogenization context, see [FiVish, (1.13), (1.22)].

For the operator e^{-itB_ε}, the method simplifies significantly. It allows us to give a short proof of $(H^3 \to L_2)$-approximation of the unitary group e^{-itB_ε} (the original proof belongs to M. Dorodnyi [D]) and approximation with corrector for the operator $e^{-itB_\varepsilon}B_\varepsilon^{-1}$.

0.5 Plan of the paper

The paper consists of five sections and introduction. Section 1 contains definitions of the operators B_ε and B^0 and known results on homogenization of the resolvent B_ε^{-1}. In Section 2, the main results of the paper are formulated and proven. In Section 3, we discuss the possibility to remove the smoothing operator from the corrector. In Section 4, we apply results in operator terms to homogenization of the solutions of hyperbolic systems. Finally, in Section 5, we give alternative proof for approximation of the operator e^{-itB_ε} originally obtained in [D] and, for completeness of the presentation, approximation in the energy norm for $e^{-itB_\varepsilon}B_\varepsilon^{-1}$.

0.6 Notation

Let $\mathfrak{H}$ and $\mathfrak{H}_\bullet$ be complex separable Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ denote the inner product and the norm in $\mathfrak{H}$, respectively; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_\bullet}$ means the norm of the linear continuous operators from $\mathfrak{H}$ to $\mathfrak{H}_\bullet$. The algebra of all bounded linear operators acting in $\mathfrak{H}$ is denoted by $\mathcal{B}(\mathfrak{H})$.

The symbols $\langle\cdot,\cdot\rangle$ and $|\cdot|$ stand for the inner product and the norm in $\mathbb{C}^n$, respectively, $\mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(m\times n)$-matrix, then the symbol $|a|$ denotes the norm of the matrix a as the operator from $\mathbb{C}^n$ to $\mathbb{C}^m$.

We use the notation $\mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$, $iD_j=\partial_j=\partial/\partial x_j$, $j=1,\dots,d$, $\mathbf{D}=-i\nabla=(D_1,\dots,D_d)$. The classes L_p of vector-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ with values in $\mathbb{C}^n$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1\leqslant p\leqslant\infty$. The Sobolev spaces of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}\subset\mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$. For $n=1$, we simply write $L_p(\mathcal{O})$, $H^s(\mathcal{O})$ and so on, but, sometimes, if this does not lead to confusion, we use such simple notation for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0,T);\mathfrak{H})$, $1\leqslant p\leqslant\infty$, denotes the L_p-space of $\mathfrak{H}$-valued functions on the interval $(0,T)$.

Various constants in estimates are denoted by c, $\mathfrak{c}$, C, and $\mathfrak{C}$ (possibly, with indices and marks).

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1 Preliminaries. Known results

The main material of the present section contained in [Su1, Su2]. For the reader convenience, we repeat it in details.

1.1 Lattices in $\mathbb{R}^d$

Let $\Gamma \subset \mathbb{R}^d$ be a lattice generated by a basis $\mathbf{a}_1, \dots, \mathbf{a}_d \in \mathbb{R}^d$, i. e.,

$$\Gamma = \left\{ \mathbf{a} \in \mathbb{R}^d : \mathbf{a} = \sum_{j=1}^{d} \nu_j \mathbf{a}_j,\ \nu_j \in \mathbb{Z} \right\},$$

and let Ω be the elementary cell of Γ:

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{j=1}^{d} \tau_j \mathbf{a}_j,\ -\frac{1}{2} < \tau_j < \frac{1}{2} \right\}.$$

By $|\Omega|$ we denote the Lebesgue measure of Ω: $|\Omega| = \operatorname{meas}\Omega$.

The basis $\mathbf{b}_1, \dots, \mathbf{b}_d$ in $\mathbb{R}^d$ dual to the basis $\mathbf{a}_1, \dots, \mathbf{a}_d$ is defined by the relations $\langle \mathbf{b}_i, \mathbf{a}_j \rangle = 2\pi\delta_{ij}$. The lattice Γ^* generated by the dual basis is called the lattice dual to Γ. We consider the first Brillouin zone

$$\Omega^* = \{ \mathbf{k} \in \mathbb{R}^d : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|,\ 0 \neq \mathbf{b} \in \Gamma^* \},$$

as a fundamental domain of the lattice Γ^*. By r_0 we denote the radius of the ball inscribed in $\operatorname{clos}\Omega^*$. Then $2r_0 = \min_{0 \neq \mathbf{b} \in \Gamma^*} |\mathbf{b}|$.

By $H^1_{\mathrm{per}}(\Omega)$ we denote the subspace of all functions in $H^1(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $f(\mathbf{x})$ is a Γ-periodic measurable matrix-valued function in $\mathbb{R}^d$, we put $f^\varepsilon(\mathbf{x}) := f(\mathbf{x}/\varepsilon)$, $\varepsilon > 0$; $\overline{f} := |\Omega|^{-1} \int_\Omega f(\mathbf{x})\, d\mathbf{x}$, and $\underline{f} := \left(|\Omega|^{-1} \int_\Omega f(\mathbf{x})^{-1}\, d\mathbf{x} \right)^{-1}$. Here, in the definition of $\overline{f}$ it is assumed that $f \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$, and in the definition of $\underline{f}$ it is assumed that the matrix $f(\mathbf{x})$ is square and nondegenerate, and

[2] To simplify the notation originally coming from the cited papers of M. Sh. Birman and T. A. Suslina, the replacement of $\widetilde{\Gamma}$ by Γ^*, $\widetilde{\Omega}$ by Ω^*, and $\widetilde{H}^1$ by H^1_{per} was carried out.

$f^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. By $[f^\varepsilon]$ we denote the operator of multiplication by the matrix-valued function $f^\varepsilon(\mathbf{x})$.

Let $\mathcal{U}$ be the Gelfand transformation associated with the lattice Γ. Initially, $\mathcal{U}$ is defined on the Schwartz class $\mathcal{S}(\mathbb{R}^d;\mathbb{C}^n)$ by the formula

$$(\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x}) = |\Omega^*|^{-1/2} \sum_{\mathbf{a}\in\Gamma} \exp(-i\langle \mathbf{k}, \mathbf{x}+\mathbf{a}\rangle)\mathbf{v}(\mathbf{x}+\mathbf{a}), \quad \mathbf{x}\in\Omega, \quad \mathbf{k}\in\Omega^*.$$

Herewith, $\int_{\Omega^*}\int_\Omega |(\mathcal{U}\mathbf{v})(\mathbf{k},\mathbf{x})|^2\, d\mathbf{x}\, d\mathbf{k} = \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{x})|^2\, d\mathbf{x}$, and $\mathcal{U}$ extends by continuity to a unitary operator

$$\mathcal{U} : L_2(\mathbb{R}^d;\mathbb{C}^n) \to \int_{\Omega^*} \oplus L_2(\Omega;\mathbb{C}^n)\, d\mathbf{k}$$

(for more details, see [BSu2, Chapter 2, Subsection 1.3]).

1.2 The class of operators A_ε

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the operator A_ε given by the differential expression $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$. Here $g(\mathbf{x})$ is a Γ-periodic $(m\times m)$-matrix-valued function (in general, with complex entries). We assume that $g(\mathbf{x}) > 0$ and $g, g^{-1} \in L_\infty(\mathbb{R}^d)$. Next, $b(\mathbf{D})$ is the differential operator given by

$$b(\mathbf{D}) = \sum_{j=1}^d b_j D_j, \tag{1.1}$$

where b_j, $j=1,\dots,d$, are constant $(m\times n)$-matrices (in general, with complex entries). It is assumed that $m \geqslant n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ of the operator $b(\mathbf{D})$ has maximal rank:

$$\operatorname{rank} b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

This condition is equivalent to the estimates

$$\alpha_0 \mathbf{1}_n \leqslant b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \leqslant \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta}\in\mathbb{S}^{d-1}, \quad 0 < \alpha_0 \leqslant \alpha_1 < \infty, \tag{1.2}$$

with some positive constants α_0 and α_1. From (1.2) it follows that

$$|b_j| \leqslant \alpha_1^{1/2}, \quad j=1,\dots,d. \tag{1.3}$$

The precise definition of the operator A_ε is given in terms of the quadratic form

$$\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle\, d\mathbf{x}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Under the above assumptions, this form is closed and nonnegative. Using the Fourier transformation and condition (1.2), it is easy to check that

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leqslant \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] \leqslant \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{1.4}$$

Let $c_1 := \alpha_0^{-1/2} \|g^{-1}\|^{1/2}_{L_\infty}$. Then the lower estimate (1.4) can be written as

$$\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leqslant c_1^2 \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}], \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{1.5}$$

1.3 Lower order terms

We study a selfadjoint operator B_ε whose principal part coincides with A_ε. To define the lower order terms, we introduce Γ-periodic $(n \times n)$-matrix-valued functions a_j, $j = 1, \dots, d$, (in general, with complex entries) such that

$$a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ if } d = 1, \quad \rho > d \text{ if } d \geqslant 2, \quad j = 1, \dots, d.$$

Let $\mathcal{Q}_0$ be the operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ that acts as multiplication by the Γ-periodic positive definite and bounded $(n\times n)$-matrix-valued function $\mathcal{Q}_0(\mathbf{x})$. We factorise this matrix as $\mathcal{Q}_0(\mathbf{x}) = f(\mathbf{x})^* f(\mathbf{x})$, where an $(n \times n)$-matrix-valued function $f(\mathbf{x})$ is assumed to be Γ-periodic.

Suppose that $d\mu(\mathbf{x})$ is a Γ-periodic Borel σ-finite measure in $\mathbb{R}^d$ with values in the class of Hermitian $(n\times n)$-matrices. Then $d\mu(\mathbf{x}) = \{d\mu_{jl}(\mathbf{x})\}$, $j, l = 1, \dots, n$. In other words, $d\mu_{jl}(\mathbf{x})$ is a complex-valued Γ-periodic measure in $\mathbb{R}^d$, and $d\mu_{jl} = d\mu^*_{lj}$. Suppose that the measure $d\mu$ is such that a function $|v(\mathbf{x})|^2$ is integrable with respect to each measure $d\mu_{jl}$ for any $v \in H^1(\mathbb{R}^d)$.

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider the sesquilinear form

$$\int_{\mathbb{R}^d} \langle d\mu(\mathbf{x})\mathbf{u}, \mathbf{v}\rangle = \sum_{j,l=1}^n \int_{\mathbb{R}^d} u_l v_j^* d\mu_{jl}(\mathbf{x}), \quad \mathbf{u}, \mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Assume that the measure $d\mu$ is subject to the following condition.

Condition 1.1. 1°. *There exist constants $\widetilde{c}_2 \geqslant 0$ and $c_3 \geqslant 0$ such that for any $\mathbf{u}, \mathbf{v} \in H^1(\Omega;\mathbb{C}^n)$ we have*

$$\left| \int_\Omega \langle d\mu(\mathbf{x})\mathbf{u}, \mathbf{v}\rangle \right| \leqslant (\widetilde{c}_2 \|\mathbf{D}\mathbf{u}\|^2_{L_2(\Omega)} + c_3 \|\mathbf{u}\|^2_{L_2(\Omega)})^{1/2} (\widetilde{c}_2 \|\mathbf{D}\mathbf{v}\|^2_{L_2(\Omega)} + c_3 \|\mathbf{v}\|^2_{L_2(\Omega)})^{1/2}. \tag{1.6}$$

2°. *We have*

$$\int_\Omega \langle d\mu(\mathbf{x})\mathbf{u}, \mathbf{u}\rangle \geqslant -\widetilde{c} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\Omega)} - c_0 \|\mathbf{u}\|^2_{L_2(\Omega)}, \quad \mathbf{u} \in H^1(\Omega;\mathbb{C}^n), \tag{1.7}$$

with constants $c_0 \in \mathbb{R}$ and $\widetilde{c}$ such that $0 \leqslant \widetilde{c} < \alpha_0 \|g^{-1}\|^{-1}_{L_\infty}$.

Examples of the forms satisfying Condition 1.1 can be found in [Su1, Subsection 5.5]. Here we provide only the main example.

Example 1.2. Suppose that the measure $d\mu$ is absolutely continuous with respect to Lebesgue measure, i. e., $d\mu(\mathbf{x}) = Q(\mathbf{x})\,d\mathbf{x}$, where $Q(\mathbf{x})$ is a Γ-periodic Hermitian $(n\times n)$-matrix-valued function in $\mathbb{R}^d$ such that

$$Q \in L_\varrho(\Omega), \quad \varrho = 1 \text{ for } d=1, \quad \varrho > \frac{d}{2} \text{ for } d \geqslant 2.$$

Then, by the Sobolev embedding theorem, for any $\nu > 0$ there exists a positive constant $C_Q(\nu)$ such that

$$\int_\Omega |Q(\mathbf{x})||\mathbf{u}|^2\,d\mathbf{x} \leqslant \nu \int_\Omega |\mathbf{D}\mathbf{u}|^2\,d\mathbf{x} + C_Q(\nu)\int_\Omega |\mathbf{u}|^2\,d\mathbf{x}, \quad \mathbf{u}\in H^1(\Omega;\mathbb{C}^n).$$

Then Condition 1.1 is satisfied with the constants $\widetilde{c}_2 = 1$, $c_3 = C_Q(1)$, $\widetilde{c} = \nu$, and $c_0 = C_Q(\nu)$, where $2\nu = \alpha_0\|g^{-1}\|^{-1}_{L_\infty}$.

1.4 The operator B_ε

Define the measure $d\mu^\varepsilon(\mathbf{x})$ as follows. For any Borel set $\Delta\subset\mathbb{R}^d$, we consider the set $\varepsilon^{-1}\Delta = \{\mathbf{y} = \varepsilon^{-1}\mathbf{x} : \mathbf{x}\in\Delta\}$ and put $\mu^\varepsilon(\Delta) = \varepsilon^d\mu(\varepsilon^{-1}\Delta)$. Consider the sesquilinear form $\mathfrak{q}_\varepsilon$ defined by

$$\mathfrak{q}_\varepsilon[\mathbf{u},\mathbf{v}] = \int_{\mathbb{R}^d}\langle d\mu^\varepsilon(\mathbf{x})\mathbf{u},\mathbf{v}\rangle, \quad \mathbf{u},\mathbf{v}\in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

Consider the following quadratic form

$$\begin{aligned}\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] = \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + 2\mathrm{Re}\sum_{j=1}^d (a_j^\varepsilon D_j\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)} \\ + \mathfrak{q}_\varepsilon[\mathbf{u},\mathbf{u}] + \lambda(Q_0^\varepsilon\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n).\end{aligned} \tag{1.8}$$

We fix a constant λ so that the form $\mathfrak{b}_\varepsilon$ is nonnegative (see (1.15) below). Let us check that the form $\mathfrak{b}_\varepsilon$ is closed. By the Hölder inequality and the Sobolev embedding theorem, it is easily seen (see [Su1, (5.11)–(5.14)]) that for any $\nu>0$ there exist constants $C_j(\nu)>0$ such that

$$\|a_j^*\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leqslant \nu\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C_j(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n),\ j=1,\dots,d.$$

Using the change of variable $\mathbf{y} := \varepsilon^{-1}\mathbf{x}$ and denoting $\mathbf{u}(\mathbf{x}) =: \mathbf{v}(\mathbf{y})$, we deduce

$$\begin{aligned}\|(a_j^\varepsilon)^*\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}|a_j(\varepsilon^{-1}\mathbf{x})^*\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} = \varepsilon^d\int_{\mathbb{R}^d}|a_j(\mathbf{y})^*\mathbf{v}(\mathbf{y})|^2\,d\mathbf{y} \\ &\leqslant \varepsilon^d\nu\int_{\mathbb{R}^d}|\mathbf{D}_{\mathbf{y}}\mathbf{v}(\mathbf{y})|^2\,d\mathbf{y} + \varepsilon^d C_j(\nu)\int_{\mathbb{R}^d}|\mathbf{v}(\mathbf{y})|^2\,d\mathbf{y} \\ &\leqslant \nu\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C_j(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0<\varepsilon\leqslant 1.\end{aligned} \tag{1.9}$$

Hence, by (1.4), for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\sum_{j=1}^{d} \|(a_j^\varepsilon)^* \mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \leqslant \nu \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + C(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \qquad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \leqslant 1. \tag{1.10}$$

If ν is fixed, then $C(\nu)$ depends only on d, ρ, α_0, the norms $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and the parameters of the lattice Γ.

For functions in $H^1(\mathbb{R}^d;\mathbb{C}^n)$, we write inequalities (1.6), (1.7) over shifted cells $\Omega + \mathbf{a}$, $\mathbf{a} \in \Gamma$, and sum up, obtaining similar inequalities with integration over $\mathbb{R}^d$. Using these arguments, changing variables $\mathbf{y} := \varepsilon^{-1}\mathbf{x}$ similarly to (1.9), and taking (1.4) into account, we can obtain that

$$|\mathfrak{q}_\varepsilon[\mathbf{u},\mathbf{v}]| \leqslant (c_2 \mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + c_3\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)})^{1/2} (c_2 \mathfrak{a}_\varepsilon[\mathbf{v},\mathbf{v}] + c_3\|\mathbf{v}\|^2_{L_2(\mathbb{R}^d)})^{1/2}, \tag{1.11}$$

$$\mathfrak{q}_\varepsilon[\mathbf{u},\mathbf{u}] \geqslant -(1-\kappa)\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] - c_0\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u},\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{1.12}$$

$0 < \varepsilon \leqslant 1$. Here

$$c_2 = \widetilde{c}_2 \alpha_0^{-1} \|g^{-1}\|_{L_\infty}, \quad \kappa = 1 - \widetilde{c}\alpha_0^{-1}\|g^{-1}\|_{L_\infty}, \quad 0 < \kappa \leqslant 1.$$

Combining (1.5), (1.10) with $\nu = 1$, and (1.11), we get

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] \leqslant (2 + c_1^2 + c_2)\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + (C(1) + c_3 + |\lambda|\|Q_0\|_{L_\infty})\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \qquad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \leqslant 1. \tag{1.13}$$

From (1.5) and (1.10) it follows that

$$2\left|\operatorname{Re}\sum_{j=1}^{d} (D_j \mathbf{u}, (a_j^\varepsilon)^*\mathbf{u})_{L_2(\mathbb{R}^d)}\right| \leqslant \frac{\kappa}{2}\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + c_4\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \qquad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n),\ 0 < \varepsilon \leqslant 1, \tag{1.14}$$

where $c_4 = 4\kappa^{-1}c_1^2 C(\nu_0)$ with $\nu_0 = 16^{-1}c_1^{-2}\kappa^2$.

Assume that the parameter λ is subject to the following restriction:

$$\begin{aligned} &\lambda > \|Q_0^{-1}\|_{L_\infty}(c_0 + c_4) \text{ if } \lambda \geqslant 0, \\ &\lambda > \|Q_0\|^{-1}_{L_\infty}(c_0 + c_4) \text{ if } \lambda < 0 \text{ (and } c_0 + c_4 < 0). \end{aligned} \tag{1.15}$$

This condition ensures that

$$\lambda(Q_0\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)} \geqslant (c_0 + c_4 + \beta)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n), \tag{1.16}$$

where $\beta > 0$ is defined in terms of λ as follows

$$\beta = \lambda \|Q_0^{-1}\|_{L_\infty}^{-1} - c_0 - c_4 \text{ if } \lambda \geqslant 0,$$
$$\beta = \lambda \|Q_0\|_{L_\infty} - c_0 - c_4 \text{ if } \lambda < 0 \text{ (and } c_0 + c_4 < 0).$$

Then from (1.8), (1.12), (1.14), and (1.16) it follows that

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] \geqslant \frac{\kappa}{2}\mathfrak{a}_\varepsilon[\mathbf{u},\mathbf{u}] + \beta\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \leqslant 1. \quad (1.17)$$

Thus, the form $\mathfrak{b}_\varepsilon$ is closed and positive definite. The selfadjoint operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ generated by this form is denoted by B_ε. Formally, we have

$$B_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j a_j^\varepsilon(\mathbf{x})^*\right) + Q^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x}). \quad (1.18)$$

Note that, by (1.4) and (1.13),

$$\mathfrak{b}_\varepsilon[\mathbf{u},\mathbf{u}] \leqslant c_5^2 \|\mathbf{u}\|^2_{H^1(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n); \quad (1.19)$$
$$c_5^2 := \max\{(2 + c_1^2 + c_2)\alpha_1 \|g\|_{L_\infty}; C(1) + c_3 + |\lambda| \|Q_0\|_{L_\infty}\}.$$

From (1.4) and (1.17) it follows that

$$\|\mathbf{u}\|_{H^1(\mathbb{R}^d)} \leqslant c_6 \|B_\varepsilon^{1/2}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n);$$
$$c_6 := (\min\{2^{-1}\kappa\alpha_0 \|g^{-1}\|^{-1}_{L_\infty}; \beta\})^{-1/2}. \quad (1.20)$$

Note that differential expression (1.18) also can be treated as a bounded operator $\mathfrak{B}_\varepsilon$ from $H^1(\mathbb{R}^d;\mathbb{C}^n)$ to $H^{-1}(\mathbb{R}^d;\mathbb{C}^n)$. It is easily seen that $\mathfrak{B}_\varepsilon|_{\operatorname{Dom} B_\varepsilon} = B_\varepsilon$. So, we write simply B_ε instead of $\mathfrak{B}_\varepsilon$. By (1.19) and the duality arguments, $\|B_\varepsilon\|_{H^1 \to H^{-1}} \leqslant c_5^2$.

By the duality arguments, the operator $B_\varepsilon^{1/2} : H^1 \to L_2$ can be extended to bounded operator from L_2 to H^{-1}. Indeed, by (1.19),

$$\|B_\varepsilon^{1/2}\mathbf{u}\|_{L_2(\mathbb{R}^d)} \leqslant c_5 \|\mathbf{u}\|_{H^1(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad (1.21)$$

and

$$c_5 \|\mathbf{u}\|_{H^1(\mathbb{R}^d)} \|\mathbf{v}\|_{L_2(\mathbb{R}^d)} \geqslant |(B_\varepsilon^{1/2}\mathbf{u},\mathbf{v})_{L_2(\mathbb{R}^d)}| = |(\mathbf{u}, B_\varepsilon^{1/2}\mathbf{v})_{L_2(\mathbb{R}^d)}|,$$
$$\mathbf{u},\mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n).$$

So,

$$\|B_\varepsilon^{1/2}\mathbf{v}\|_{H^{-1}(\mathbb{R}^d)} \leqslant c_5 \|\mathbf{v}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{v} \in H^1(\mathbb{R}^d;\mathbb{C}^n). \quad (1.22)$$

By continuity, we extend $B_\varepsilon^{1/2}$ to a bounded operator from L_2 to H^{-1}. By construction, the restriction of this extended operator onto H^1 coincides with the original operator $B_\varepsilon^{1/2}$. So, we denote this extended operator by $B_\varepsilon^{1/2}$. By (1.22),

$$\|B_\varepsilon^{1/2}\|_{L_2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant c_5. \tag{1.23}$$

For convenience of further references, in what follows by the ,,problem data" we mean the following set of parameters:

$$\begin{aligned}
&d, m, n, \rho; \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|a_j\|_{L_\rho(\Omega)}, j = 1, \dots, d;\\
&\widetilde{c}, c_0, \widetilde{c}_2, c_3 \text{ from Condition 1.1};\\
&\lambda, \|Q_0\|_{L_\infty}, \|Q_0^{-1}\|_{L_\infty}; \text{ the parameters of the lattice } \Gamma.
\end{aligned} \tag{1.24}$$

1.5 The effective matrix

The effective operator for $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ is given by $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$, where g^0 is a constant $(m\times m)$-matrix called the effective matrix. The matrix g^0 is defined in terms of the solution of an auxiliary cell problem. Suppose that a Γ-periodic $(n\times m)$-matrix-valued function $\Lambda(\mathbf{x})$ is the (weak) solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \qquad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.25}$$

Then the effective matrix is given by

$$g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \tag{1.26}$$

where

$$\widetilde{g}(\mathbf{x}) = g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{1.27}$$

It can be checked that g^0 is positive definite.

We also need the following estimates for the solution of the problem (1.25) proved in [BSu3, (6.28) and Subsection 7.3]:

$$\|\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} M_1, \quad M_1 = m^{1/2}(2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}, \tag{1.28}$$

$$\|\mathbf{D}\Lambda\|_{L_2(\Omega)} \leqslant |\Omega|^{1/2} M_2, \quad M_2 = m^{1/2}\alpha_0^{-1/2}\|g\|_{L_\infty}^{1/2}\|g^{-1}\|_{L_\infty}^{1/2}. \tag{1.29}$$

The effective matrix satisfies the estimates known as the Voigt–Reuss bracketing (see, e. g., [BSu2, Chapter 3, Theorem 1.5]).

Proposition 1.3. *Let g^0 be the effective matrix* (1.26)*. Then*

$$\underline{g} \leqslant g^0 \leqslant \overline{g}. \tag{1.30}$$

If $m = n$, then $g^0 = \underline{g}$.

Now we distinguish the cases where one of the inequalities in (1.30) becomes an identity, see [BSu2, Chapter 3, Propositions 1.6 and 1.7].

Proposition 1.4. *The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1, \dots, m, \tag{1.31}$$

where $\mathbf{g}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 1.5. *The identity $g^0 = \underline{g}$ is equivalent to the relations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in H^1_{\mathrm{per}}(\Omega; \mathbb{C}^m), \quad k = 1, \dots, m, \tag{1.32}$$

where $\mathbf{l}_k(\mathbf{x})$, $k = 1, \dots, m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

1.6 The effective operator

In order to define the effective operator for B_ε, consider the Γ-periodic $(n \times n)$-matrix-valued function $\widetilde{\Lambda}(\mathbf{x})$ which is the solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}) \widetilde{\Lambda}(\mathbf{x}) + \sum_{j=1}^{d} D_j a_j(\mathbf{x})^* = 0, \qquad \int_\Omega \widetilde{\Lambda}(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.33}$$

(Here equation is understood in the weak sense.)

The following estimates for $\widetilde{\Lambda}$ were proven in [Su1, (7.51) and (7.52)]:

$$\|\widetilde{\Lambda}\|_{L_2(\Omega)} \leqslant (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} = |\Omega|^{1/2} \widetilde{M}_1, \tag{1.34}$$

$$\|\mathbf{D}\widetilde{\Lambda}\|_{L_2(\Omega)} \leqslant C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty} = |\Omega|^{1/2} \widetilde{M}_2, \tag{1.35}$$

where $C_a^2 := \sum_{j=1}^d \int_\Omega |a_j(\mathbf{x})|^2\, d\mathbf{x}$, $\widetilde{M}_1 := |\Omega|^{-1/2} (2r_0)^{-1} C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}$, and $\widetilde{M}_2 := |\Omega|^{-1/2} C_a n^{1/2} \alpha_0^{-1} \|g^{-1}\|_{L_\infty}$.

Define constant matrices V and W as follows:

$$V = |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\Lambda(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x},$$
$$W = |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x}.$$

The effective operator for the operator (1.18) is given by

$$B^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^{d} (\overline{a_j + a_j^*}) D_j - W + \overline{Q} + \lambda \overline{Q_0}.$$

Here $\overline{Q} = |\Omega|^{-1} \int_\Omega d\mu(\mathbf{x})$. The operator B^0 is the elliptic second order operator with constant coefficients, $\operatorname{Dom} B^0 = H^2(\mathbb{R}^d;\mathbb{C}^n)$. According to [Su2, (3.31), (4.9), Subsection 7.2], the operator B^0 is positive definite:

$$B^0 \geqslant \check{c} I, \quad \check{c} := \min\{2^{-1}\kappa\alpha_0 \|g^{-1}\|^{-1}_{L_\infty};\beta\}. \tag{1.36}$$

Moreover, its symbol $L(\boldsymbol{\xi})$ satisfies the estimate

$$L(\boldsymbol{\xi}) \geqslant \check{c}(|\boldsymbol{\xi}|^2+1)\mathbf{1}_n. \tag{1.37}$$

So,

$$\|\mathbf{u}\|_{H^1(\mathbb{R}^d)} \leqslant \check{c}^{-1/2}\|(B^0)^{1/2}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n), \tag{1.38}$$

and

$$\|(B^0)^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \check{c}^{-1/2}, \tag{1.39}$$

$$\|\mathbf{D}(B^0)^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} |\boldsymbol{\xi}||L(\boldsymbol{\xi})|^{-1/2} \leqslant \check{c}^{-1/2}, \tag{1.40}$$

$$\|(B^0)^{-1/2}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{1/2}||L(\boldsymbol{\xi})|^{-1/2} \leqslant \check{c}^{-1/2}, \tag{1.41}$$

$$\|(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^2(\mathbb{R}^d)} \leqslant \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)|L(\boldsymbol{\xi})|^{-1} \leqslant \check{c}^{-1}. \tag{1.42}$$

For the case, when the measure $d\mu$ is as in Example 1.2, in [MSu3, Lemma 1.6] it was proven that

$$L(\boldsymbol{\xi}) \leqslant C_L(|\boldsymbol{\xi}|^2+1)\mathbf{1}_n \tag{1.43}$$

for some positive constant C_L depending only on the problem data (1.24). To modify the proof from [MSu3] to our case, we need only to estimate the term $\overline{Q}$. It turns out that $\overline{Q} \leqslant c_3 \mathbf{1}_n$. Indeed, by Condition 1.1(1°),

$$\begin{aligned} |(\overline{Q}\mathbf{u},\mathbf{u})_{L_2(\mathbb{R}^d)}| &= |\Omega|^{-1}\Big|\int_{\mathbb{R}^d} \langle \int_\Omega d\mu(\mathbf{x})\mathbf{u}(\mathbf{y}),\mathbf{u}(\mathbf{y})\rangle\, d\mathbf{y}\Big| \\ &\leqslant |\Omega|^{-1}\sum_{\mathbf{a}\in\Gamma}\int_{\Omega_\mathbf{a}} \Big|\langle\int_\Omega d\mu(\mathbf{x})\mathbf{u}(\mathbf{y}),\mathbf{u}(\mathbf{y})\rangle\Big|\, d\mathbf{y} \leqslant c_3\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u}\in H^1(\mathbb{R}^d;\mathbb{C}^n). \end{aligned}$$

Now, from (1.43) it follows that

$$\|(B^0)^{1/2}\|_{L_2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{-1/2}|L(\boldsymbol{\xi})|^{1/2} \leqslant C_L^{1/2} \tag{1.44}$$

and

$$\|(B^0)^{1/2}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_L^{1/2}, \tag{1.45}$$

$$\|B^0\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_L, \tag{1.46}$$

$$\|(B^0)^{3/2}\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_L^{3/2}. \tag{1.47}$$

1.7 The operator Π_ε

Let Π_ε be the pseudodifferential operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ whose symbol is the characteristic function $\chi_{\Omega^*/\varepsilon}(\boldsymbol{\xi})$ of the set Ω^*/ε:

$$(\Pi_\varepsilon \mathbf{f})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\Omega^*/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle} \widehat{\mathbf{f}}(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \qquad (1.48)$$

Here $\widehat{\mathbf{f}}$ is the Fourier-image of the function $\mathbf{f}$. Obviously, $\Pi_\varepsilon \mathbf{D}^\alpha \mathbf{u} = \mathbf{D}^\alpha \Pi_\varepsilon \mathbf{u}$ for $\mathbf{u} \in H^l(\mathbb{R}^d;\mathbb{C}^n)$, if $|\alpha| \leqslant l$. Note, that $\Pi_\varepsilon^2 = \Pi_\varepsilon$ and

$$\|\Pi_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = 1. \qquad (1.49)$$

The following property of the operator Π_ε was proven in [PSu, Proposition 1.4].

Proposition 1.6. *We have*

$$\|\Pi_\varepsilon \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon r_0^{-1} \|\mathbf{D}\mathbf{u}\|_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad \varepsilon > 0.$$

For $\varkappa = 0$, the following result was obtained in [BSu4, Subsection 10.2]. For $0 < \varkappa < \infty$ the proof is quite similar.

Proposition 1.7. *Let f be a Γ-periodic function in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$. Let $0 \leqslant \varkappa < \infty$. Then*

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{H^{-\varkappa}(\mathbb{R}^d)\to H^{-\varkappa}(\mathbb{R}^d)} \leqslant |\Omega|^{-1/2} \|f\|_{L_2(\Omega)}, \quad \varepsilon > 0.$$

Proof. We have

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{H^{-\varkappa}(\mathbb{R}^d)\to H^{-\varkappa}(\mathbb{R}^d)} = \|(\mathbf{D}^2 + I)^{-\varkappa}[f^\varepsilon]\Pi_\varepsilon(\mathbf{D}^2 + I)^{\varkappa}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.$$

Let T_ε be the unitary rescaling operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ given by $(T_\varepsilon \mathbf{u})(\mathbf{x}) = \varepsilon^{d/2}\mathbf{u}(\mathbf{x})$. Then

$$(\mathbf{D}^2 + I)^{-\varkappa}[f^\varepsilon]\Pi_\varepsilon(\mathbf{D}^2 + I)^{\varkappa} = T_\varepsilon^*(\mathbf{D}^2 + \varepsilon^2 I)^{-\varkappa}[f]\Pi_1(\mathbf{D}^2 + \varepsilon^2 I)^{\varkappa} T_\varepsilon.$$

So,

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{H^{-\varkappa}(\mathbb{R}^d)\to H^{-\varkappa}(\mathbb{R}^d)} = \|(\mathbf{D}^2 + \varepsilon^2 I)^{-\varkappa}[f]\Pi_1(\mathbf{D}^2 + \varepsilon^2 I)^{\varkappa}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.$$

It turns out that under Gelfand transformation (see Subsection 1.1) the operator

$$(\mathbf{D}^2 + \varepsilon^2 I)^{-\varkappa}[f]\Pi_1(\mathbf{D}^2 + \varepsilon^2 I)^{\varkappa} = (\mathbf{D}^2 + \varepsilon^2 I)^{-\varkappa}[f](\mathbf{D}^2 + \varepsilon^2 I)^{\varkappa}\Pi_1$$

is decomposed into the direct integral of the operators

$$((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{-\varkappa}[f]((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{\varkappa}P,$$

depending on the quasi-momentum $\mathbf{k}\in\Omega^*$ and acting in $L_2(\Omega;\mathbb{C}^n)$ with the periodic boundary conditions. Here P is the operator of averaging over the cell: $P\mathbf{v}=|\Omega|^{-1}\int_\Omega \mathbf{v}(\mathbf{x})\,d\mathbf{x}$, $\mathbf{v}\in L_2(\Omega;\mathbb{C}^n)$, i. e., the projection of $L_2(\Omega;\mathbb{C}^n)$ onto the subspace of constant functions. Because of the presence of the projection P,

$$((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{-\varkappa}[f]((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{\varkappa}P=((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{-\varkappa}[f](\mathbf{k}^2+\varepsilon^2)^{\varkappa}P.$$

Thus,

$$\begin{aligned}&\|[f^\varepsilon]\Pi_\varepsilon\|_{H^{-\varkappa}(\mathbb{R}^d)\to H^{-\varkappa}(\mathbb{R}^d)}=\|(\mathbf{D}^2+\varepsilon^2 I)^{-\varkappa}[f]\Pi_1(\mathbf{D}^2+\varepsilon^2 I)^{\varkappa}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &\leqslant \sup_{\mathbf{k}\in\Omega^*}\|((\mathbf{D}+\mathbf{k})^2+\varepsilon^2 I)^{-\varkappa}[f]P(\mathbf{k}^2+\varepsilon^2 I)^{\varkappa}\|_{L_2(\Omega)\to L_2(\Omega)}\\ &\leqslant \|[f]P\|_{L_2(\Omega)\to L_2(\Omega)}\sup_{\mathbf{k}\in\Omega^*}(\mathbf{k}^2+\varepsilon^2)^{\varkappa}\|((\mathbf{D}+\mathbf{k})^2+\varepsilon^2)^{-\varkappa}\|_{L_2(\Omega)\to L_2(\Omega)}.\end{aligned}\tag{1.50}$$

Using the discrete Fourier transform and definition of the first Brillouin zone Ω^*, we obtain

$$\|((\mathbf{D}+\mathbf{k})^2+\varepsilon^2)^{-\varkappa}\|_{L_2(\Omega)\to L_2(\Omega)}=\sup_{\mathbf{b}\in\Gamma^*}(|\mathbf{b}+\mathbf{k}|^2+\varepsilon^2)^{-\varkappa}=(|\mathbf{k}|^2+\varepsilon^2)^{-\varkappa},\quad \mathbf{k}\in\Omega^*.$$

Together with (1.50), this implies

$$\|[f^\varepsilon]\Pi_\varepsilon\|_{H^{-\varkappa}(\mathbb{R}^d)\to H^{-\varkappa}(\mathbb{R}^d)}\leqslant\|[f]P\|_{L_2(\Omega)\to L_2(\Omega)}\leqslant|\Omega|^{-1/2}\|f\|_{L_2(\Omega)}.$$

$\square$

1.8 Approximation of the resolvent

In the present Subsection we formulate the homogenization results for B_ε^{-1}, obtained in [Su1, Theorems 9.2 and 9.7].

Let $K(\varepsilon)$ be the corrector

$$K(\varepsilon):=[\Lambda^\varepsilon]\Pi_\varepsilon b(\mathbf{D})(B^0)^{-1}+[\widetilde{\Lambda}^\varepsilon]\Pi_\varepsilon(B^0)^{-1}.\tag{1.51}$$

The continuity of the operator $K(\varepsilon)$ from $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$ follows from Proposition 1.7 with $\varkappa=0$ and inclusions $\Lambda,\widetilde{\Lambda}\in H^1_{\mathrm{per}}(\Omega)$. By Proposition 1.7 and (1.2), (1.28), (1.34), and (1.42),

$$\|K(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant C_K:=(M_1\alpha_1^{1/2}+\widetilde{M}_1)\check{c}^{-1}.\tag{1.52}$$

Theorem 1.8 ([Su1]). *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Let $K(\varepsilon)$ be the corrector* (1.51). *Denote*

$$\mathcal{R}_1(\varepsilon) := B_\varepsilon^{-1} - (B^0)^{-1}, \tag{1.53}$$

$$\mathcal{R}_2(\varepsilon) := B_\varepsilon^{-1} - (B^0)^{-1} - \varepsilon K(\varepsilon). \tag{1.54}$$

Then for $0 < \varepsilon \leqslant 1$ we have

$$\|\mathcal{R}_1(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_1\varepsilon,$$

$$\|\mathcal{R}_2(\varepsilon)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_2\varepsilon. \tag{1.55}$$

The constants C_1 and C_2 are controlled in terms of the problem data (1.24).

Note that inequalities (1.19) and (1.55) imply the estimate

$$\|B_\varepsilon^{1/2}\mathcal{R}_2(\varepsilon)\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_5 C_2\varepsilon. \tag{1.56}$$

2 Problem setting. Main results

2.1 Problem

Let $\mathbf{u}_\varepsilon$ be the generalized solution of the following Cauchy problem for hyperbolic system:

$$\begin{cases} \partial_t^2\mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon\mathbf{u}_\varepsilon(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d,\ t\in\mathbb{R}, \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t\mathbf{u}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{2.1}$$

Here $\boldsymbol{\phi}\in H^1(\mathbb{R}^d;\mathbb{C}^n)$ and $\boldsymbol{\psi}\in L_2(\mathbb{R}^d;\mathbb{C}^n)$.

The solution of problem (2.1) is given by

$$\mathbf{u}_\varepsilon(\cdot,t) = \cos(tB_\varepsilon^{1/2})\boldsymbol{\phi} + B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi}.$$

Thus,

$$\partial_t\mathbf{u}_\varepsilon(\cdot,t) = -B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\phi} + \cos(tB_\varepsilon^{1/2})\boldsymbol{\psi}.$$

Our goal is to study the behavior of the solution $\mathbf{u}_\varepsilon(\cdot,t)$ as $\varepsilon\to 0$. In other words, to approximate the operator functions $\cos(tB_\varepsilon^{1/2})$ and $B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})$ in suitable norms.

The problem (2.1) can be rewritten as follows

$$\partial_t\begin{pmatrix}\mathbf{u}_\varepsilon\\ \partial_t\mathbf{u}_\varepsilon\end{pmatrix} = \begin{pmatrix}0 & I\\ -B_\varepsilon & 0\end{pmatrix}\begin{pmatrix}\mathbf{u}_\varepsilon\\ \partial_t\mathbf{u}_\varepsilon\end{pmatrix}, \quad \begin{pmatrix}\mathbf{u}_\varepsilon(\cdot,0)\\ \partial_t\mathbf{u}_\varepsilon(\cdot,0)\end{pmatrix} = \begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\psi}\end{pmatrix}.$$

Denote

$$\mathfrak{A}_\varepsilon := \begin{pmatrix} 0 & I \\ -B_\varepsilon & 0 \end{pmatrix} : H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n) \to H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n),$$
$$\operatorname{Dom}\mathfrak{A}_\varepsilon := \operatorname{Dom} B_\varepsilon \times H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{2.2}$$

(Our choice of $\operatorname{Dom}\mathfrak{A}_\varepsilon$ guaranties that $\operatorname{Ran}\mathfrak{A}_\varepsilon \subset H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)$.) According to [Go, Chapter 2, Section 7], operator (2.2) generates a C_0-group

$$e^{t\mathfrak{A}_\varepsilon} = \begin{pmatrix} \cos(tB_\varepsilon^{1/2}) & B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) \\ -B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) & \cos(tB_\varepsilon^{1/2}) \end{pmatrix} \tag{2.3}$$

on the space $\operatorname{Dom}\mathfrak{b}_\varepsilon \times L_2(\mathbb{R}^d;\mathbb{C}^n) = H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)$ equipped with the graph norm of $B_\varepsilon^{1/2}$. By (1.19) and (1.20), this norm is equivalent to the standard norm in $H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)$. So,

$$e^{t\mathfrak{A}_\varepsilon} \in \mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)). \tag{2.4}$$

Then

$$\begin{pmatrix} \mathbf{u}_\varepsilon \\ \partial_t \mathbf{u}_\varepsilon \end{pmatrix} = e^{t\mathfrak{A}_\varepsilon} \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{pmatrix}.$$

It is easily seen that the resolvent of the operator $\mathfrak{A}_\varepsilon$ has the form

$$\mathfrak{A}_\varepsilon^{-1} = \begin{pmatrix} 0 & -B_\varepsilon^{-1} \\ I & 0 \end{pmatrix}, \quad \mathfrak{A}_\varepsilon^{-1} \in \mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)). \tag{2.5}$$

2.2 The operator $\mathfrak{A}_0$

Let $\mathfrak{A}_0$ be the effective operator for (2.2):

$$\mathfrak{A}_0 = \begin{pmatrix} 0 & I \\ -B^0 & 0 \end{pmatrix} : H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n) \to H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n),$$
$$\operatorname{Dom}\mathfrak{A}_0 = \operatorname{Dom} B^0 \times H^1(\mathbb{R}^d;\mathbb{C}^n) = H^2(\mathbb{R}^d;\mathbb{C}^n) \times H^1(\mathbb{R}^d;\mathbb{C}^n). \tag{2.6}$$

Then, similarly to (2.3)–(2.5),

$$e^{t\mathfrak{A}_0} = \begin{pmatrix} \cos(t(B^0)^{1/2}) & (B^0)^{-1/2}\sin(t(B^0)^{1/2}) \\ -(B^0)^{1/2}\sin(t(B^0)^{1/2}) & \cos(t(B^0)^{1/2}) \end{pmatrix}, \tag{2.7}$$

$$e^{t\mathfrak{A}_0} \in \mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)), \tag{2.8}$$

and

$$\mathfrak{A}_0^{-1} = \begin{pmatrix} 0 & -(B^0)^{-1} \\ I & 0 \end{pmatrix}, \quad \mathfrak{A}_0^{-1} \in \mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)). \tag{2.9}$$

Thus

$$\mathfrak{A}_0^{-2} = \begin{pmatrix} 0 & -(B^0)^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & -(B^0)^{-1} \\ I & 0 \end{pmatrix} = \begin{pmatrix} -(B^0)^{-1} & 0 \\ 0 & -(B^0)^{-1} \end{pmatrix}. \tag{2.10}$$

2.3 Principal term of approximation

The principal term of approximation is given by the following theorem. The proof can be found in Subsection 2.4 below.

Theorem 2.1. *Under the assumptions of Subsections* 1.1–1.6*, for* $t \in \mathbb{R}$ *and* $0 < \varepsilon \leqslant 1$ *we have*

$$\| \cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_3\varepsilon(1+|t|), \tag{2.11}$$

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_4\varepsilon(1+|t|), \tag{2.12}$$

$$\|B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C_5\varepsilon(1+|t|), \tag{2.13}$$

$$\| \cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C_6\varepsilon|t|. \tag{2.14}$$

The constants C_3*,* C_4*,* C_5*, and* C_6 *are controlled explicitly in terms of the problem data* (1.24)*.*

Remark 2.2. *The right-hand side in* (2.12) *can be replaced by* $C\varepsilon|t|$*, see* (2.60) *below.*

Corollary 2.3. *For* $t \in \mathbb{R}$ *and* $0 < \varepsilon \leqslant 1$ *we have*

$$\|e^{t\mathfrak{A}_\varepsilon} - e^{t\mathfrak{A}_0}\|_{H^2(\mathbb{R}^d)\times H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)\times H^{-1}(\mathbb{R}^d)} \leqslant (C_3+C_4+C_5+C_6)\varepsilon(1+|t|).$$

By the interpolation arguments, Theorem 2.1 implies the following result.

Theorem 2.4. *Let* $0 \leqslant r \leqslant 2$*. Then, under the assumptions of Theorem* 2.1*, for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$\| \cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^r(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_7\varepsilon^{r/2}(1+|t|)^{r/2}, \tag{2.15}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{r-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\quad \leqslant C_8\varepsilon^{r/2}(1+|t|)^{r/2}, \end{aligned} \tag{2.16}$$

$$\|B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2})\|_{H^r(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C_9\varepsilon^{r/2}(1+|t|)^{r/2}, \tag{2.17}$$

$$\| \cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{H^{r-1}(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant C_{10}\varepsilon^{r/2}|t|^{r/2}. \tag{2.18}$$

The constants C_7, C_8, C_9, and C_{10} depend only on r and the problem data (1.24).

Remark 2.5. *The right-hand side in* (2.16) *can be replaced by $C\varepsilon^{r/2}|t|^{r/2}$, see* (2.63) *below.*

Proof. Interpolating between the rough estimate

$$\|\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2})\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2$$

and (2.11), we obtain (2.15) with the constant $C_7 = 2^{1-r/2}C_3^{r/2}$.

Next, by the duality arguments and (1.20),

$$\begin{aligned} \|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &= \|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\leqslant c_6\|\sin(tB_\varepsilon^{1/2})\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = c_6. \end{aligned} \tag{2.19}$$

Similarly, according to (1.38),

$$\|(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \check{c}^{-1/2}.$$

Together with (2.19), this implies the inequality

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_6 + \check{c}^{-1/2}. \tag{2.20}$$

Interpolating between (2.12) and (2.20), we arrive at the estimate (2.16) with the constant $C_8 := (c_6 + \check{c}^{-1/2})^{1-r/2}C_4^{r/2}$.

From (1.23) and (1.44) it follows that

$$\|B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2})\|_{L_2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant c_5 + C_L^{1/2}. \tag{2.21}$$

Interpolating between (2.21) and (2.13), we obtain estimate (2.17), where $C_9 := (c_5 + C_L^{1/2})^{1-r/2}C_5^{r/2}$.

Note that the operator $\cos(tB_\varepsilon^{1/2})$ can be extended to the bounded operator in $H^{-1}(\mathbb{R}^d;\mathbb{C}^n)$. Indeed, by (1.20) and (1.21), for $\mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ we have

$$\begin{aligned} \|\cos(tB_\varepsilon^{1/2})\mathbf{u}\|_{H^{-1}(\mathbb{R}^d)} &= \sup_{0\neq\mathbf{v}\in H^1(\mathbb{R}^d)} \frac{|(\cos(tB_\varepsilon^{1/2})\mathbf{u},\mathbf{v})_{L_2(\mathbb{R}^d)}|}{\|\mathbf{v}\|_{H^1(\mathbb{R}^d)}} \\ &= \sup_{0\neq\mathbf{v}\in H^1(\mathbb{R}^d)} \frac{|(\mathbf{u},\cos(tB_\varepsilon^{1/2})\mathbf{v})_{L_2(\mathbb{R}^d)}|}{\|\mathbf{v}\|_{H^1(\mathbb{R}^d)}} \\ &\leqslant \|\mathbf{u}\|_{H^{-1}(\mathbb{R}^d)} \sup_{0\neq\mathbf{v}\in H^1(\mathbb{R}^d)} \frac{\|\cos(tB_\varepsilon^{1/2})\mathbf{v}\|_{H^1(\mathbb{R}^d)}}{\|\mathbf{v}\|_{H^1(\mathbb{R}^d)}} \\ &\leqslant c_6\|\mathbf{u}\|_{H^{-1}(\mathbb{R}^d)} \sup_{0\neq\mathbf{v}\in H^1(\mathbb{R}^d)} \frac{\|B_\varepsilon^{1/2}\mathbf{v}\|_{L_2(\mathbb{R}^d)}}{\|\mathbf{v}\|_{H^1(\mathbb{R}^d)}} \\ &\leqslant c_5c_6\|\mathbf{u}\|_{H^{-1}(\mathbb{R}^d)}. \end{aligned}$$

So, by continuity we can extend the operator $\cos(tB_\varepsilon^{1/2})$ onto $H^{-1}(\mathbb{R}^d;\mathbb{C}^n)$, and
$$\|\cos(tB_\varepsilon^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant c_5c_6. \tag{2.22}$$
The operator $\cos(t(B^0)^{1/2})$ also can be extended onto $H^{-1}(\mathbb{R}^d;\mathbb{C}^n)$. Since the operator B^0 has constant coefficients, it commutes with differentiation, and
$$\begin{aligned}&\|\cos(t(B^0)^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)}\\ &=\|(\mathbf{D}^2+I)^{-1/2}\cos(t(B^0)^{1/2})(\mathbf{D}^2+I)^{1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 1.\end{aligned} \tag{2.23}$$
Combining (2.22) and (2.23), we obtain
$$\|\cos(tB_\varepsilon^{1/2})-\cos(t(B^0)^{1/2})\|_{H^{-1}(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \leqslant 1+c_5c_6. \tag{2.24}$$
Interpolating between (2.24) and (2.14), we arrive at estimate (2.18) with the constant $C_{10}:=C_6^{r/2}(1+c_5c_6)^{1-r/2}$. $\square$

2.4 Proof of Theorem 2.1

Proof of Theorem 2.1. Denote
$$\mathcal{E}(t):=e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{A}_0^{-1}e^{t\mathfrak{A}_0}.$$
It is bounded operator in $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$, because all factors are bounded in $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$. Then
$$\frac{d\mathcal{E}(t)}{dt}=e^{-t\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})e^{t\mathfrak{A}_0}.$$
(Derivative is taken in the strong operator topology in $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$, see [Pa, Chapter 1, Theorem 2.4(c)].) So,
$$\mathcal{E}(t)-\mathcal{E}(0)=e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{A}_0^{-1}e^{t\mathfrak{A}_0}-\mathfrak{A}_\varepsilon^{-1}\mathfrak{A}_0^{-1}=\int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})e^{s\mathfrak{A}_0}\,ds. \tag{2.25}$$
(The integral is understood as a strong Riemann integral in $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$.)

Let us multiply the identity (2.25) by $e^{-t\mathfrak{A}_0}$ from the right. Then
$$e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{A}_0^{-1}-\mathfrak{A}_\varepsilon^{-1}\mathfrak{A}_0^{-1}e^{-t\mathfrak{A}_0}=\int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds.$$

So,

$$(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2} = -e^{-t\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1})\mathfrak{A}_0^{-1} + (\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1})\mathfrak{A}_0^{-1}e^{-t\mathfrak{A}_0} + \int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds. \tag{2.26}$$

Denote

$$\mathfrak{G}(\varepsilon) := \begin{pmatrix} \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{Dom}\mathfrak{G}(\varepsilon) = H^2(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n). \tag{2.27}$$

Then, by (2.9),

$$\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1} = \begin{pmatrix} \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -(B^0)^{-1} \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & -K(\varepsilon) \\ 0 & 0 \end{pmatrix}. \tag{2.28}$$

Here $K(\varepsilon)$ is the operator (1.51). So,

$$\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1} \in \mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n) \times L_2(\mathbb{R}^d;\mathbb{C}^n)). \tag{2.29}$$

Note that

$$\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1} = \begin{pmatrix} 0 & -\mathcal{R}_1(\varepsilon) \\ 0 & 0 \end{pmatrix} \tag{2.30}$$

and

$$\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1} - \varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1} = \begin{pmatrix} 0 & -\mathcal{R}_2(\varepsilon) \\ 0 & 0 \end{pmatrix}, \tag{2.31}$$

where $\mathcal{R}_1(\varepsilon)$ and $\mathcal{R}_2(\varepsilon)$ are defined by (1.53) and (1.54), respectively.

By (2.26),

$$\begin{aligned}(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2} = &-e^{-t\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1})\mathfrak{A}_0^{-1} + (\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1})\mathfrak{A}_0^{-1}e^{-t\mathfrak{A}_0} \\ &+ \int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1} - \mathfrak{A}_0^{-1} - \varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds \\ &+ \int_0^t e^{-s\mathfrak{A}_\varepsilon}\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1}e^{(s-t)\mathfrak{A}_0}\,ds.\end{aligned} \tag{2.32}$$

Denote the consecutive summands in the right-hand side of (2.32) by $\mathcal{I}_j(\varepsilon;t)$, $j = 1,\dots,4$.

Then, by (2.3), (2.9), and (2.30),

$$\mathcal{I}_1(\varepsilon;t) = \begin{pmatrix} \cos(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon) & 0 \\ B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon) & 0 \end{pmatrix}. \tag{2.33}$$

Next, the operator $\mathcal{I}_2(\varepsilon;t)$ has the entries

$$\{\mathcal{I}_2(\varepsilon;t)\}_{11} = -\mathcal{R}_1(\varepsilon)\cos(t(B^0)^{1/2}), \tag{2.34}$$

$$\{\mathcal{I}_2(\varepsilon;t)\}_{12} = \mathcal{R}_1(\varepsilon)(B^0)^{-1/2}\sin(t(B^0)^{1/2}), \tag{2.35}$$

$$\{\mathcal{I}_2(\varepsilon;t)\}_{21} = \{\mathcal{I}_2(\varepsilon;t)\}_{22} = 0. \tag{2.36}$$

By using (2.3), (2.7), and (2.31), we see that the entries of the operator $\mathcal{I}_3(\varepsilon;t)$ have the form

$$\{\mathcal{I}_3(\varepsilon;t)\}_{11} = \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.37}$$

$$\{\mathcal{I}_3(\varepsilon;t)\}_{12} = -\int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds, \tag{2.38}$$

$$\{\mathcal{I}_3(\varepsilon;t)\}_{21} = \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.39}$$

$$\{\mathcal{I}_3(\varepsilon;t)\}_{22} = -\int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds. \tag{2.40}$$

Here integrals are understood in the strong sense (in different norms): in (2.37) in $(H^1 \to H^1)$-norm, in (2.38) in $(L_2 \to H^1)$-norm, etc. We can understand the integral from (2.37) in the strong $(H^1 \to L_2)$-sense and integrate by parts. Then

$$\begin{aligned}\{\mathcal{I}_3(\varepsilon;t)\}_{11} = &-\cos(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) + \mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2})\\ &-\int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds\end{aligned} \tag{2.41}$$

is bounded operator in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Next, the integral in (2.39) is understood in the strong $(H^1 \to L_2)$-sense. So, it makes sense in a weaker $(H^1 \to H^{-1})$-topology. Integrating by parts, we arrive at the expression

$$\begin{aligned}\{\mathcal{I}_3(\varepsilon;t)\}_{21} = &-B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\\ &+\int_0^t B_\varepsilon^{1/2}\cos(sB_\varepsilon^{1/2})B_\varepsilon^{1/2}\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.\end{aligned} \tag{2.42}$$

This operator is obviously $(L_2 \to H^{-1})$-bounded.

According to (2.3), (2.7), and (2.28), the operator $\mathcal{I}_4(\varepsilon;t)$ has the entries

$$\{\mathcal{I}_4(\varepsilon;t)\}_{11} = \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.43}$$

$$\{\mathcal{I}_4(\varepsilon;t)\}_{12} = -\int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds, \tag{2.44}$$

$$\{\mathcal{I}_4(\varepsilon;t)\}_{21} = \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.45}$$

$$\{\mathcal{I}_4(\varepsilon;t)\}_{22} = -\int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds. \tag{2.46}$$

Now, we compute the matrix entries of the left-hand side of (2.32). By (2.3), (2.7), and (2.10),

$$\{(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2}\}_{11} = -(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1}, \tag{2.47}$$

$$\{(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2}\}_{12} = (B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1}, \tag{2.48}$$

$$\{(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2}\}_{21} = -(B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1}, \tag{2.49}$$

$$\{(e^{-t\mathfrak{A}_\varepsilon} - e^{-t\mathfrak{A}_0})\mathfrak{A}_0^{-2}\}_{22} = -(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1}. \tag{2.50}$$

Combining (1.53), (1.54), (2.32)–(2.34), (2.41), (2.43), and (2.47), we obtain

$$\begin{aligned}
&-(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1} = \cos(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon) \\
&- \mathcal{R}_1(\varepsilon)\cos(t(B^0)^{1/2}) - \cos(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) + \mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2}) \\
&- \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds \\
&+ \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds \\
&= \cos(tB_\varepsilon^{1/2})\varepsilon K(\varepsilon) - \varepsilon K(\varepsilon)\cos(t(B^0)^{1/2}) \\
&- \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds \\
&+ \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds.
\end{aligned} \tag{2.51}$$

By construction, the matrix entries with indices ,,11" are bounded operators in $H^1(\mathbb{R}^d;\mathbb{C}^n)$. After integrating by parts in the term $\{\mathcal{I}_3(\varepsilon;t)\}_{11}$, we can understand (2.51) only as equality for the operators acting from $H^1(\mathbb{R}^d;\mathbb{C}^n)$

to $L_2(\mathbb{R}^d;\mathbb{C}^n)$. By continuity, we extend equality from $H^1(\mathbb{R}^d;\mathbb{C}^n)$ to the whole space $L_2(\mathbb{R}^d;\mathbb{C}^n)$.

By Proposition 1.7 with $\varkappa = 0$, (1.2), (1.28), and (1.34),

$$\|[\Lambda^\varepsilon]\Pi_\varepsilon b(\mathbf{D}) + [\widetilde{\Lambda}^\varepsilon]\Pi_\varepsilon\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant M_1\alpha_1^{1/2} + \widetilde{M}_1. \tag{2.52}$$

Together with (1.41) and (1.51), this implies

$$\|K(\varepsilon)(B^0)^{1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant (M_1\alpha_1^{1/2} + \widetilde{M}_1)\check{c}^{-1/2}. \tag{2.53}$$

Combining (1.52), (1.56), (2.51), and (2.53), we obtain

$$\begin{aligned} &\|(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\leqslant 2C_K\varepsilon + c_5C_2\varepsilon|t| + \check{c}^{-1/2}(\alpha_1^{1/2}M_1 + \widetilde{M}_1)\varepsilon|t|. \end{aligned} \tag{2.54}$$

Now from (1.46) and (2.54) we derive estimate (2.11) with the constant

$$C_3 := C_L\max\{2C_K; c_5C_2 + \check{c}^{-1/2}(\alpha_1^{1/2}M_1 + \widetilde{M}_1)\}.$$

We proceed to the proof of estimate (2.12). Combining (2.32), (2.33), (2.35), (2.38), (2.44), and (2.48), we have

$$\begin{aligned} &(B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1} \\ &= \mathcal{R}_1(\varepsilon)(B^0)^{-1/2}\sin(t(B^0)^{1/2}) \\ &- \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds \\ &- \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds. \end{aligned}$$

This equality can be understood as an identity for the operators acting from L_2 to L_2. So, it makes sense on the range of the operator $(B^0)^{1/2} : H^1 \to L_2$:

$$\begin{aligned} &(B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1/2} \\ &= \mathcal{R}_1(\varepsilon)\sin(t(B^0)^{1/2}) \\ &- \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\cos((s-t)(B^0)^{1/2})\,ds \\ &- \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\cos((s-t)(B^0)^{1/2})\,ds. \end{aligned}$$

The equality here is understood in the $(H^1 \to L_2)$-sense. By continuity, we extend it onto the whole space L_2. Integrating by parts in the first integral, we get

$$
\begin{aligned}
&(B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1/2}\\
&= \mathcal{R}_1(\varepsilon)\sin(t(B^0)^{1/2}) - \mathcal{R}_2(\varepsilon)\sin(t(B^0)^{1/2})\\
&- \int_0^t \sin(sB_\varepsilon^{1/2})B_\varepsilon^{1/2}\mathcal{R}_2(\varepsilon)\sin((s-t)(B^0)^{1/2})\,ds\\
&- \int_0^t \cos(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\cos((s-t)(B^0)^{1/2})\,ds.
\end{aligned}
$$

Combining this with (1.52)–(1.54), (1.56), and (2.53), we obtain

$$
\begin{aligned}
&\|(B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\
&\leqslant \varepsilon C_K + c_5C_2\varepsilon|t| + (M_1\alpha_1^{1/2} + \widetilde{M}_1)\check{c}^{-1/2}\varepsilon|t|.
\end{aligned}
$$

By (1.45), this implies (2.12) with the constant

$$
C_4 := C_L^{1/2}\max\{C_K; c_5C_2 + \check{c}^{-1/2}(M_1\alpha_1^{1/2} + \widetilde{M}_1)\}.
$$

Bringing together (2.32), (2.33), (2.36), (2.42), (2.45), and (2.49), we have

$$
\begin{aligned}
&-(B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{1/2}\sin(t(B^0)^{1/2}))(B^0)^{-1}\\
&= B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon) - B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\\
&+ \int_0^t B_\varepsilon^{1/2}\cos(sB_\varepsilon^{1/2})B_\varepsilon^{1/2}\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds\\
&+ \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds.
\end{aligned}
\tag{2.55}
$$

By construction, this equality should be understood in the $(H^1 \to H^{-1})$-sense. By continuity, it make sense in the $(L_2 \to H^{-1})$-topology. Together with (1.23), (1.46), (1.52)–(1.54), (1.56), and (2.53), equality (2.55) implies estimate (2.13) with the constant

$$
C_5 := C_L\max\{c_5C_K; c_5^2C_2 + c_5(M_1\alpha_1^{1/2} + \widetilde{M}_1)\check{c}^{-1/2}\}.
$$

Combining (2.32), (2.33), (2.36), (2.40), (2.46), and (2.50), we obtain

$$
\begin{aligned}
&(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1}\\
&= \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds\\
&+ \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\varepsilon K(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.
\end{aligned}
$$

This is the equality for the operators acting from L_2 to L_2. But L_2 is the subset of H^{-1} and the L_2-norm is stronger than the H^{-1}-norm, so this identity also can be understood in the $(L_2 \to H^{-1})$-sense. Applying it to functions from the space $(B^0)^{1/2} H^1(\mathbb{R}^d;\mathbb{C}^n) = L_2(\mathbb{R}^d;\mathbb{C}^n)$, we arrive at

$$
\begin{aligned}
&(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1/2} \\
&= \int_0^t B_\varepsilon^{1/2} \sin(sB_\varepsilon^{1/2}) \mathcal{R}_2(\varepsilon)(B^0)^{1/2} \cos((s-t)(B^0)^{1/2})\, ds \\
&+ \int_0^t B_\varepsilon^{1/2} \sin(sB_\varepsilon^{1/2}) \varepsilon K(\varepsilon)(B^0)^{1/2} \cos((s-t)(B^0)^{1/2})\, ds.
\end{aligned}
$$

Here the equality is understood in the $(H^1 \to H^{-1})$-topology. By continuity, we can understand it in the $(L_2 \to H^{-1})$-sense. Integrating by parts in the first integral, we obtain

$$
\begin{aligned}
&(\cos(tB_\varepsilon^{1/2}) - \cos(t(B^0)^{1/2}))(B^0)^{-1/2} \\
&= -\int_0^t B_\varepsilon^{1/2} \cos(sB_\varepsilon^{1/2}) B_\varepsilon^{1/2} \mathcal{R}_2(\varepsilon) \sin((st)(B^0)^{1/2})\, ds \\
&+ \int_0^t B_\varepsilon^{1/2} \sin(sB_\varepsilon^{1/2}) \varepsilon K(\varepsilon)(B^0)^{1/2} \cos((s-t)(B^0)^{1/2})\, ds.
\end{aligned}
$$

Combining this with (1.23), (1.45), (1.56), and (2.53), we arrive at estimate (2.14) with the constant $C_6 := C_L^{1/2}(c_5^2 C_2 + c_5(M_1\alpha_1^{1/2} + \widetilde{M}_1)\check{c}^{-1/2})$. $\square$

2.5 Approximation with corrector

Theorem 2.6. *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Denote*

$$
\mathcal{K}_1(\varepsilon;t) := (\Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon) \cos(t(B^0)^{1/2})(B^0)^{-1}, \tag{2.56}
$$

$$
\mathcal{K}_2(\varepsilon;t) := (\Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon)(B^0)^{-1/2} \sin(t(B^0)^{1/2}). \tag{2.57}
$$

Then for $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$
\begin{aligned}
&\| \cos(tB_\varepsilon^{1/2}) B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon \mathcal{K}_1(\varepsilon;t)\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
&\quad \leqslant C_{11}\varepsilon(1+|t|),
\end{aligned} \tag{2.58}
$$

$$
\begin{aligned}
&\|B_\varepsilon^{-1/2} \sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2} \sin(t(B^0)^{1/2}) - \varepsilon \mathcal{K}_2(\varepsilon;t)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
&\quad \leqslant C_{12}\varepsilon(1+|t|),
\end{aligned} \tag{2.59}
$$

$$
\|B_\varepsilon^{-1/2} \sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2} \sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_{13}\varepsilon|t|. \tag{2.60}
$$

The constants C_{11}*,* C_{12}*, and* C_{13} *are controlled in terms of the problem data* (1.24).

The proof of Theorem 2.6 can be found in Subsection 2.7 below. According to (2.3), (2.7), (2.11), and (2.27), Theorem 2.6 implies the following result.

Corollary 2.7. *Let $\mathfrak{A}_\varepsilon$, $\mathfrak{A}_0$, and $\mathfrak{G}(\varepsilon)$ be the operators* (2.2), (2.6), *and* (2.27), *respectively. Then for* $0<\varepsilon\leqslant 1$ *and* $t\in\mathbb{R}$ *we have*

$$\left\| e^{t\mathfrak{A}_\varepsilon}\begin{pmatrix} B_\varepsilon^{-1} & 0\\ 0 & I\end{pmatrix} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{t\mathfrak{A}_0}\begin{pmatrix} (B^0)^{-1} & 0\\ 0 & I\end{pmatrix}\right\|_{H^1(\mathbb{R}^d)\times H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)\times L_2(\mathbb{R}^d)}$$
$$\leqslant (C_3+C_{11}+C_{12}+C_{13})\varepsilon(1+|t|).$$

By interpolation arguments, from Theorem 2.6 we derive the following result.

Theorem 2.8. *Let* $0\leqslant q\leqslant 1$ *and let* $0\leqslant r\leqslant 2$. *Then for* $0<\varepsilon\leqslant 1$ *and* $t\in\mathbb{R}$ *we have*

$$\begin{aligned}&\|\cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}-\cos(t(B^0)^{1/2})(B^0)^{-1}-\varepsilon\mathcal{K}_1(\varepsilon;t)\|_{H^q(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ &\quad\leqslant C_{14}\varepsilon^q(1+|t|)^q,\end{aligned}\tag{2.61}$$

$$\begin{aligned}&\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})-(B^0)^{-1/2}\sin(t(B^0)^{1/2})-\varepsilon\mathcal{K}_2(\varepsilon;t)\|_{H^{1+q}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ &\quad\leqslant C_{15}\varepsilon^q(1+|t|)^q,\end{aligned}\tag{2.62}$$

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})-(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{-1+r}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant C_{16}\varepsilon^{r/2}|t|^{r/2}.\tag{2.63}$$

The constants C_{14} *and* C_{15} *depend only on* q *and the problem data* (1.24), *the constant* C_{16} *depends on* r *and the problem data* (1.24).

Proof. According to (1.17), $B_\varepsilon\geqslant\beta I$. So,

$$\|B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant\beta^{-1/2}.\tag{2.64}$$

Using this argument and (1.20), we obtain

$$\|\cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant c_6\|B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant c_6\beta^{-1/2}.\tag{2.65}$$

By (1.42),

$$\|\cos(t(B^0)^{1/2})(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant\|(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant\check{c}^{-1}.\tag{2.66}$$

Next, by Proposition 1.7 with $\varkappa=0$ and (1.2), (1.28), (1.29),

$$\begin{aligned}&\|\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})\cos(t(B^0)^{1/2})(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\ &\leqslant(\varepsilon M_1+M_2)\alpha_1^{1/2}\|\mathbf{D}(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\ &+\varepsilon M_1\alpha_1^{1/2}\|\mathbf{D}^2(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.\end{aligned}\tag{2.67}$$

Similarly, by Proposition 1.7 and (1.34), (1.35),

$$\begin{aligned}&\|\varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon\cos(t(B^0)^{1/2})(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\&\leqslant(\varepsilon\widetilde{M}_1+\widetilde{M}_2)\|(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}+\varepsilon\widetilde{M}_1\|\mathbf{D}(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}.\end{aligned}\tag{2.68}$$

Bringing together (1.42), (2.56), (2.67), and (2.68), we obtain

$$\|\mathcal{K}_1(\varepsilon;t)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant C_{17},\quad C_{17}:=(2M_1+M_2)\alpha_1^{1/2}\check{c}^{-1}+(2\widetilde{M}_1+\widetilde{M}_2)\check{c}^{-1}.\tag{2.69}$$

Now from (2.65), (2.66), and (2.69) it follows that

$$\begin{aligned}&\|\cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}-\cos(t(B^0)^{1/2})(B^0)^{-1}-\varepsilon\mathcal{K}_1(\varepsilon;t)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\&\leqslant c_6\beta^{-1/2}+\check{c}^{-1}+C_{17}.\end{aligned}\tag{2.70}$$

Interpolating between (2.70) and (2.58), we arrive at estimate (2.61) with the constant $C_{14}:=(c_6\beta^{-1/2}+\check{c}^{-1}+C_{17})^{1-q}C_{11}^q$.

We proceed to the proof of estimate (2.62). By (1.20),

$$\|\sin(tB_\varepsilon^{1/2})B_\varepsilon^{-1/2}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant c_6.\tag{2.71}$$

Next, by (1.38),

$$\|\sin(t(B^0)^{1/2})(B^0)^{-1/2}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant\check{c}^{-1/2}.\tag{2.72}$$

By Proposition 1.7, (1.2), (1.28), (1.29), (1.39), and (1.40),

$$\begin{aligned}&\|\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})\sin(t(B^0)^{1/2})(B^0)^{-1/2}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\\&\leqslant(\varepsilon M_1+M_2)\alpha_1^{1/2}\|\mathbf{D}(B^0)^{-1/2}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\\&+\varepsilon M_1\alpha_1^{1/2}\|\mathbf{D}^2(B^0)^{-1/2}\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\leqslant(2M_1+M_2)\alpha_1^{1/2}\check{c}^{-1/2}.\end{aligned}\tag{2.73}$$

Similarly, by using Proposition 1.7, (1.34), (1.35), (1.39), and (1.40), we obtain

$$\|\varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon\sin(t(B^0)^{1/2})(B^0)^{-1/2}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant(2\widetilde{M}_1+\widetilde{M}_2)\check{c}^{-1/2}.\tag{2.74}$$

Now from (2.57) and (2.71)–(2.74) it follows that

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})-(B^0)^{-1/2}\sin(t(B^0)^{1/2})-\varepsilon\mathcal{K}_2(\varepsilon;t)\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}\leqslant C_{18},\tag{2.75}$$

where $C_{18}:=c_6+\check{c}^{-1/2}+(2M_1+M_2)\alpha_1^{1/2}\check{c}^{-1/2}+(2\widetilde{M}_1+\widetilde{M}_2)\check{c}^{-1/2}$. Interpolating between (2.75) and (2.59), we arrive at estimate (2.62) with the constant $C_{15}:=C_{18}^{1-q}C_{12}^q$.

Finally, interpolating between (2.20) and (2.60), we obtain estimate (2.63), where $C_{16}:=(c_6+\check{c})^{1-r/2}C_{13}^{r/2}$. □

2.6 The case where the corrector is equal to zero

Assume that $g^0 = \overline{g}$, i. e., relations (1.31) are satisfied. In addition, suppose that

$$\sum_{j=1}^{d} D_j a_j(\mathbf{x})^* = 0. \tag{2.76}$$

Then the Γ-periodic solutions of problems (1.25) and (1.33) are equal to zero: $\Lambda(\mathbf{x}) = 0$ and $\widetilde{\Lambda}(\mathbf{x}) = 0$.

Proposition 2.9. *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Suppose that relations* (1.31) *and* (2.76) *hold. Let* $0 \leqslant r \leqslant 2$. *Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$\begin{aligned} &\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^{-1+r}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad\leqslant (c_5 c_6^3 + \check{c}^{-1})^{1-r/2} C_{11}^{r/2} \varepsilon^{r/2}(1+|t|)^{r/2}, \end{aligned} \tag{2.77}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^r(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad\leqslant (c_6 + \check{c}^{-1/2})^{1-r/2} C_{12}^{r/2} \varepsilon^{r/2}(1+|t|)^{r/2}. \end{aligned} \tag{2.78}$$

Proof. By Theorem 2.6, for $t \in \mathbb{R}$ and $0 < \varepsilon \leqslant 1$ we have

$$\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_{11}\varepsilon(1+|t|), \tag{2.79}$$

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_{12}\varepsilon(1+|t|). \tag{2.80}$$

By (1.20), (2.22), and the duality arguments,

$$\begin{aligned} &\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}\|_{H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c_5 c_6 \|B_\varepsilon^{-1}\|_{H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad\leqslant c_5 c_6^2 \|B_\varepsilon^{-1/2}\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} = c_5 c_6^2 \|B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c_5 c_6^3. \end{aligned} \tag{2.81}$$

Similarly, according to (1.38) and (2.23),

$$\| \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \check{c}^{-1}.$$

Together with (2.81), this implies

$$\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^{-1}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c_5 c_6^3 + \check{c}^{-1}. \tag{2.82}$$

Interpolating between (2.82) and (2.79), we arrive at estimate (2.77).

By (1.20) and (1.38),

$$\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c_6 + \check{c}^{-1/2}. \tag{2.83}$$

Interpolating between (2.83) and (2.80), we get (2.78). $\square$

2.7 Proof of Theorem 2.6

Proof of Theorem 2.6. Denote

$$\Sigma(t) := e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}e^{t\mathfrak{A}_0}.$$

Since all the factors are bounded operators (see (2.4), (2.5), (2.8), (2.9), and (2.29)), $\Sigma(t)\in\mathcal{B}(H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n))$. Then, according to semigroup theory (see [Pa, Chapter 1, Theorem 2.4(c)]), there exists derivative in the strong topology in $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$:

$$\frac{d\Sigma(t)}{dt} = e^{-t\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}-\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{t\mathfrak{A}_0}+\varepsilon e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{G}(\varepsilon)e^{t\mathfrak{A}_0}.$$

Thus

$$\begin{aligned}\Sigma(t)-\Sigma(0) &= e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}e^{t\mathfrak{A}_0}-\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\\ &= \int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}-\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{s\mathfrak{A}_0}\,ds\\ &+\int_0^t \varepsilon e^{-s\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{G}(\varepsilon)e^{s\mathfrak{A}_0}\,ds.\end{aligned}$$

(The integrals are understood as strong Riemann integrals in the $H^1(\mathbb{R}^d;\mathbb{C}^n)\times L_2(\mathbb{R}^d;\mathbb{C}^n)$-topology.) Let us multiply this identity by $e^{-t\mathfrak{A}_0}$ from the right:

$$\begin{aligned}&e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}-\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}e^{-t\mathfrak{A}_0}\\ &=\int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}-\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds\\ &+\int_0^t \varepsilon e^{-s\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{G}(\varepsilon)e^{(s-t)\mathfrak{A}_0}\,ds.\end{aligned}\tag{2.84}$$

We have

$$\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}=(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1}+\left[\mathfrak{A}_\varepsilon^{-1},(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right].$$

(Here $[\cdot,\cdot]$ denotes the commutator of operators.) Combining this with (2.84), we obtain

$$\begin{aligned}&e^{-t\mathfrak{A}_\varepsilon}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1}-(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1}e^{-t\mathfrak{A}_0}\\ &=-e^{-t\mathfrak{A}_\varepsilon}\left[\mathfrak{A}_\varepsilon^{-1},(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right]+\left[\mathfrak{A}_\varepsilon^{-1},(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right]e^{-t\mathfrak{A}_0}\\ &+\int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}-\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds\\ &+\int_0^t \varepsilon e^{-s\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{G}(\varepsilon)e^{(s-t)\mathfrak{A}_0}\,ds.\end{aligned}$$

So,

$$
\begin{aligned}
&e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-2} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{-t\mathfrak{A}_0}\mathfrak{A}_0^{-2} = -e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-1}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})\\
&- e^{-t\mathfrak{A}_\varepsilon}\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1} + (I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})e^{-t\mathfrak{A}_0}\\
&- e^{-t\mathfrak{A}_\varepsilon}\left[\mathfrak{A}_\varepsilon^{-1},(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right] + \left[\mathfrak{A}_\varepsilon^{-1},(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right]e^{-t\mathfrak{A}_0}\\
&+\int_0^t e^{-s\mathfrak{A}_\varepsilon}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}-\varepsilon\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1})e^{(s-t)\mathfrak{A}_0}\,ds\\
&+\int_0^t \varepsilon e^{-s\mathfrak{A}_\varepsilon}\mathfrak{A}_\varepsilon^{-1}\mathfrak{G}(\varepsilon)e^{(s-t)\mathfrak{A}_0}\,ds =: \sum_{j=1}^{7}\mathcal{J}_j(\varepsilon;t).
\end{aligned}
\tag{2.85}
$$

According to (2.3), (2.7), (2.10), and (2.27), the left-hand side of identity (2.85) has the entries

$$
\begin{aligned}
&\{e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-2} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{-t\mathfrak{A}_0}\mathfrak{A}_0^{-2}\}_{11} = -\cos(tB_\varepsilon^{1/2})(B^0)^{-1}\\
&+(I+\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})+\varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos(t(B^0)^{1/2})(B^0)^{-1},
\end{aligned}
\tag{2.86}
$$

$$
\begin{aligned}
&\{e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-2} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{-t\mathfrak{A}_0}\mathfrak{A}_0^{-2}\}_{12} = B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})(B^0)^{-1}\\
&-(I+\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})+\varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)(B^0)^{-1/2}\sin(t(B^0)^{1/2})(B^0)^{-1},
\end{aligned}
\tag{2.87}
$$

$$
\begin{aligned}
&\{e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-2} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{-t\mathfrak{A}_0}\mathfrak{A}_0^{-2}\}_{21}\\
&= -B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})(B^0)^{-1} + (B^0)^{1/2}\sin(t(B^0)^{1/2})(B^0)^{-1},
\end{aligned}
\tag{2.88}
$$

$$
\{e^{-t\mathfrak{A}_\varepsilon}\mathfrak{A}_0^{-2} - (I+\varepsilon\mathfrak{G}(\varepsilon))e^{-t\mathfrak{A}_0}\mathfrak{A}_0^{-2}\}_{22} = -(\cos(tB_\varepsilon^{1/2})-\cos(t(B^0)^{1/2}))(B^0)^{-1}.
\tag{2.89}
$$

We proceed to consideration of the terms $\mathcal{J}_j(\varepsilon;t)$, $j=1,\dots,7$. By (2.3), (2.9), and (2.30),

$$
\mathcal{J}_1(\varepsilon;t) = \begin{pmatrix} 0 & -B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon) \\ 0 & \cos(tB_\varepsilon^{1/2})\mathcal{R}_1(\varepsilon)\end{pmatrix}.
\tag{2.90}
$$

Next, using (2.3), (2.5), and (2.28), we obtain

$$
\mathcal{J}_2(\varepsilon;t) = \begin{pmatrix} \cos(tB_\varepsilon^{1/2})\varepsilon K(\varepsilon) & 0 \\ B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\varepsilon K(\varepsilon) & 0\end{pmatrix}.
\tag{2.91}
$$

Now, let us consider the term $\mathcal{J}_3(\varepsilon;t)$. According to (2.28) and (2.30),

$$
\mathfrak{G}(\varepsilon)\mathfrak{A}_0^{-1}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1}) = 0.
$$

So,

$$
\mathcal{J}_3(\varepsilon;t) = \mathfrak{A}_0^{-1}(\mathfrak{A}_\varepsilon^{-1}-\mathfrak{A}_0^{-1})e^{-t\mathfrak{A}_0}
$$

and (by (2.7), (2.9), and (2.30))

$$\{\mathcal{J}_3(\varepsilon;t)\}_{11} = \{\mathcal{J}_3(\varepsilon;t)\}_{12} = 0, \tag{2.92}$$

$$\{\mathcal{J}_3(\varepsilon;t)\}_{21} = -\mathcal{R}_1(\varepsilon)(B^0)^{1/2}\sin(t(B^0)^{1/2}), \tag{2.93}$$

$$\{\mathcal{J}_3(\varepsilon;t)\}_{22} = -\mathcal{R}_1(\varepsilon)\cos(t(B^0)^{1/2}).$$

To evaluate the matrix entries of the terms $\mathcal{J}_4(\varepsilon;t)$ and $\mathcal{J}_5(\varepsilon;t)$, we need to calculate the commutator $[\mathfrak{A}_\varepsilon^{-1}, (I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}]$. By (2.9) and (2.28),

$$(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1} = \begin{pmatrix} 0 & -(B^0)^{-1} - \varepsilon K(\varepsilon) \\ I & 0 \end{pmatrix}.$$

Together with (2.5), this implies

$$\mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1} = \begin{pmatrix} -B_\varepsilon^{-1} & 0 \\ 0 & -(B^0)^{-1} - \varepsilon K(\varepsilon) \end{pmatrix}, \tag{2.94}$$

$$(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1} = \begin{pmatrix} -(B^0)^{-1} - \varepsilon K(\varepsilon) & 0 \\ 0 & -B_\varepsilon^{-1} \end{pmatrix}. \tag{2.95}$$

Recall notation (1.54). Combining (2.94) and (2.95), we obtain

$$\begin{aligned} \left[\mathfrak{A}_\varepsilon^{-1}, (I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\right] &= \mathfrak{A}_\varepsilon^{-1}(I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1} - (I+\varepsilon\mathfrak{G}(\varepsilon))\mathfrak{A}_0^{-1}\mathfrak{A}_\varepsilon^{-1} \\ &= \begin{pmatrix} -\mathcal{R}_2(\varepsilon) & 0 \\ 0 & \mathcal{R}_2(\varepsilon) \end{pmatrix}. \end{aligned} \tag{2.96}$$

Now from (2.3) and (2.96) it follows that

$$\mathcal{J}_4(\varepsilon;t) = \begin{pmatrix} \cos(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) & B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) \\ B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) & -\cos(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) \end{pmatrix}. \tag{2.97}$$

By (2.7) and (2.96),

$$\mathcal{J}_5(\varepsilon;t) = \begin{pmatrix} -\mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2}) & \mathcal{R}_2(\varepsilon)(B^0)^{-1/2}\sin(t(B^0)^{1/2}) \\ \mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin(t(B^0)^{1/2}) & \mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2}) \end{pmatrix}. \tag{2.98}$$

Next, by using (2.3), (2.7), and (2.31), we calculate $\mathcal{J}_6(\varepsilon;t)$. It turns out that

$$\{\mathcal{J}_6(\varepsilon;t)\}_{11} = \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.99}$$

$$\{\mathcal{J}_6(\varepsilon;t)\}_{12} = -\int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds, \tag{2.100}$$

$$\{\mathcal{J}_6(\varepsilon;t)\}_{21} = \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds, \tag{2.101}$$

$$\{\mathcal{J}_6(\varepsilon;t)\}_{22} = -\int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.$$

Finally, by (2.3), (2.5), (2.7), and (2.27),

$$\begin{aligned}&\{\mathcal{J}_7(\varepsilon;t)\}_{11}\\ &= -\varepsilon\int_0^t B_\varepsilon^{-1/2}\sin(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds,\end{aligned} \tag{2.102}$$

$$\begin{aligned}&\{\mathcal{J}_7(\varepsilon;t)\}_{12}\\ &= -\varepsilon\int_0^t B_\varepsilon^{-1/2}\sin(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)(B^0)^{-1/2}\sin((s-t)(B^0)^{1/2})\,ds,\end{aligned} \tag{2.103}$$

$$\{\mathcal{J}_7(\varepsilon;t)\}_{21} = \varepsilon\int_0^t \cos(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds, \tag{2.104}$$

$$\begin{aligned}&\{\mathcal{J}_7(\varepsilon;t)\}_{22}\\ &= \varepsilon\int_0^t \cos(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)(B^0)^{-1/2}\sin((s-t)(B^0)^{1/2})\,ds.\end{aligned}$$

Combining (2.85), (2.86), (2.90)–(2.92), (2.97)–(2.99), and (2.102), we obtain

$$\begin{aligned}&-\cos(tB_\varepsilon^{1/2})(B^0)^{-1} + (I + \varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos(t(B^0)^{1/2})(B^0)^{-1}\\ &\quad = \cos(tB_\varepsilon^{1/2})\varepsilon K(\varepsilon) + \cos(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) - \mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2})\\ &\quad + \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds\\ &\quad - \varepsilon\int_0^t B_\varepsilon^{-1/2}\sin(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.\end{aligned}$$

It is the equality for the operators, acting from H^1 to H^1. Taking (1.54) and (2.56) into account, we arrive at

$$
\begin{aligned}
&\cos(tB_\varepsilon)^{1/2}B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon\mathcal{K}_1(\varepsilon;t) \\
&= \mathcal{R}_2(\varepsilon)\cos(t(B^0)^{1/2}) - \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds \\
&+ \varepsilon\int_0^t B_\varepsilon^{-1/2}\sin(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.
\end{aligned}
$$

Together with (1.20), (1.45), (1.55), (1.56), and (2.52), this implies (2.58) with the constant $C_{11} := \max\{C_2; c_5c_6C_L^{1/2}C_2 + c_6(M_1\alpha_1^{1/2} + \widetilde{M}_1)\}$.

Recall the notation (2.57). Combining (2.85), (2.87), (2.90)–(2.92), (2.97), (2.98), (2.100), and (2.103), we arrive at

$$
\begin{aligned}
&B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})(B^0)^{-1} - (B^0)^{-1/2}\sin(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon\mathcal{K}_2(\varepsilon;t)(B^0)^{-1} \\
&= -\varepsilon B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})K(\varepsilon) + \mathcal{R}_2(\varepsilon)(B^0)^{-1/2}\sin(t(B^0)^{1/2}) \\
&- \int_0^t \cos(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds \\
&- \varepsilon\int_0^t B_\varepsilon^{-1/2}\sin(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)(B^0)^{-1/2}\sin((s-t)(B^0)^{1/2})\,ds.
\end{aligned}
\tag{2.105}
$$

Here the equality $\mathcal{R}_2(\varepsilon) - \mathcal{R}_1(\varepsilon) = -\varepsilon K(\varepsilon)$ was taken into account. By construction, identity (2.105) should be understood in the $(H^1 \to L_2)$-sense. By continuity, we extend domains of the operators from the left- and right-hand sides of identity (2.105) onto the space L_2. Moreover, the ranges of all operators in (2.105) lie in H^1.

By Theorem 1.8, the elementary inequality $|\sin x|/|x| \leqslant 1$, $x \in \mathbb{R}$, and (1.20), (1.51), (1.52), (1.56), (2.53), and (2.105),

$$
\begin{aligned}
&\left\| \left(B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon\mathcal{K}_2(\varepsilon;t)\right)(B^0)^{-1}\right\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
&\leqslant c_6C_K\varepsilon + (1+c_5c_6)C_2\varepsilon|t| + c_6\check{c}^{-1/2}(M_1\alpha_1^{1/2} + \widetilde{M}_1)\varepsilon|t|.
\end{aligned}
$$

Together with (1.46), this implies estimate (2.59) with the constant $C_{12} := C_L\max\{c_6C_K; (1+c_5c_6)C_2 + c_6\check{c}^{-1/2}(M_1\alpha_1^{1/2} + \widetilde{M}_1)\}$.

Combining (2.85), (2.88), (2.90), (2.91), (2.93), (2.97), (2.98), (2.101), and (2.104), we arrive at the $(H^1 \to L_2)$-equality

$$\begin{aligned}
&-B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})(B^0)^{-1} + (B^0)^{1/2}\sin(t(B^0)^{1/2})(B^0)^{-1} \\
&\quad = B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\varepsilon K(\varepsilon) - \mathcal{R}_1(\varepsilon)(B^0)^{1/2}\sin(t(B^0)^{1/2}) \\
&\quad + B_\varepsilon^{1/2}\sin(tB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon) + \mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin(t(B^0)^{1/2}) \\
&\quad + \int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds \\
&\quad + \varepsilon\int_0^t \cos(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.
\end{aligned}$$

Taking (1.53), (1.54), and (2.57) into account, we get

$$\begin{aligned}
&B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon\mathcal{K}_2(\varepsilon;t) \\
&\quad = -\int_0^t B_\varepsilon^{1/2}\sin(sB_\varepsilon^{1/2})\mathcal{R}_2(\varepsilon)(B^0)^{1/2}\sin((s-t)(B^0)^{1/2})\,ds \\
&\quad - \varepsilon\int_0^t \cos(sB_\varepsilon^{1/2})(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\cos((s-t)(B^0)^{1/2})\,ds.
\end{aligned}$$

(The equality here is understood in the $(H^1 \to L_2)$-sense.) Together with (1.45), (1.56), and (2.52), this implies

$$\begin{aligned}
&\| B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon\mathcal{K}_2(\varepsilon;t)\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\
&\quad \leqslant (c_5 C_2 C_L^{1/2} + M_1\alpha_1^{1/2} + \widetilde{M}_1)\varepsilon|t|.
\end{aligned} \tag{2.106}$$

Next, since the operator B^0 commutes with differentiation, by (2.52) and the elementary inequality $|\sin x|/|x| \leqslant 1$, we have

$$\begin{aligned}
\|\mathcal{K}_2(\varepsilon;t)\|_{H^1(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} &\leqslant (M_1\alpha_1^{1/2} + \widetilde{M}_1)\|(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\
&\leqslant (M_1\alpha_1^{1/2} + \widetilde{M}_1)|t|.
\end{aligned}$$

Together with (2.106), this implies estimate (2.60) with the constant $C_{13} := c_5 C_2 C_L^{1/2} + 2M_1\alpha_1^{1/2} + 2\widetilde{M}_1$. $\square$

Remark 2.10. *Consideration of element* (2.89) *does not give us any new information compared with estimate* (2.11).

3 Removal of the smoothing operator from the corrector

3.1 Removal of Π_ε under additional assumptions on Λ and $\widetilde{\Lambda}$

It turns out that the smoothing operator can be removed from the corrector if the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ are subject to some additional assumptions.

Condition 3.1. *Assume that the* Γ*-periodic solution* $\Lambda(\mathbf{x})$ *of problem* (1.25) *is bounded, i. e.,* $\Lambda \in L_\infty(\mathbb{R}^d)$.

The cases when the Condition 3.1 is fulfilled automatically were distinguished in [BSu4, Lemma 8.7].

Proposition 3.2. *Suppose that at least one of the following assumptions is satisfied*:
1°) $d \leqslant 2$;
2°) *the dimension* $d \geqslant 1$ *is arbitrary, and the differential expression* A_ε *is given by* $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is a symmetric matrix with real entries*;
3°) *the dimension* d *is arbitrary, and* $g^0 = \underline{g}$, *i. e., relations* (1.32) *are satisfied.*
Then Condition 3.1 *holds.*

In order to remove Π_ε from the term involving $\widetilde{\Lambda}^\varepsilon$, it suffices to impose the following condition.

Condition 3.3. *Assume that the* Γ*-periodic solution* $\widetilde{\Lambda}(\mathbf{x})$ *of problem* (1.33) *is such that*

$$\widetilde{\Lambda} \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \geqslant 3.$$

The following result was obtained in [Su1, Proposition 8.11].

Proposition 3.4. *Condition* 3.3 *is fulfilled, if at least one of the following assumptions is satisfied*:
1°) $d \leqslant 4$;
2°) *the dimension* d *is arbitrary, and the differential expression* A_ε *has the form* $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, *where* $g(\mathbf{x})$ *is a symmetric matrix-valued function with real entries.*

Remark 3.5. *If $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a symmetric matrix-valued function with real entries, from* [LaU, Chapter III, Theorem 13.1] *it follows that $\Lambda, \widetilde{\Lambda} \in L_\infty$ and the norm $\|\Lambda\|_{L_\infty}$ is controlled in terms of d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and Ω; the norm $\|\widetilde{\Lambda}\|_{L_\infty}$ does not exceed a constant depending on d, ρ, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and Ω. In this case, Conditions* 3.1 *and* 3.3 *are fulfilled simultaneously.*

Theorem 3.6. *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Assume that Conditions* 3.1 *and* 3.3 *hold. Denote*

$$\mathcal{K}_1^0(\varepsilon; t) := (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon) \cos(t(B^0)^{1/2})(B^0)^{-1}, \tag{3.1}$$

$$\mathcal{K}_2^0(\varepsilon; t) := (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)(B^0)^{-1/2} \sin(t(B^0)^{1/2}). \tag{3.2}$$

Let $0 \leqslant q \leqslant 1$. Then for $0 < \varepsilon \leqslant 1$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} &\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon \mathcal{K}_1^0(\varepsilon; t)\|_{H^q(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{19} \varepsilon^q (1 + |t|)^q, \\ &\|B_\varepsilon^{-1/2} \sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2} \sin(t(B^0)^{1/2}) - \varepsilon \mathcal{K}_2^0(\varepsilon; t)\|_{H^{1+q}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{20} \varepsilon^q (1 + |t|)^q. \end{aligned}$$

The constants C_{19} and C_{20} depend only on q, p, $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$, and on the problem data (1.24).

Proof of Theorem 3.6 is given below in Subsection 3.3. In the proof we use some properties of the matrix-valued functions Λ and $\widetilde{\Lambda}$ from the next subsection.

3.2 Properties of the matrix-valued functions Λ and $\widetilde{\Lambda}$

The following results were obtained in [PSu, Corollary 2.4] and [MSu1, Lemma 3.5 and Corollary 3.6].

Lemma 3.7. *Suppose that Λ is the Γ-periodic solution of problem* (1.25). *Assume also that $\Lambda \in L_\infty$. Then for any $u \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$ we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2 |u(\mathbf{x})|^2 \, d\mathbf{x} \leqslant \mathfrak{c}_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \mathfrak{c}_2 \varepsilon^2 \|\Lambda\|^2_{L_\infty} \int_{\mathbb{R}^d} |\mathbf{D}u(\mathbf{x})|^2 \, d\mathbf{x}.$$

The constants $\mathfrak{c}_1$ and $\mathfrak{c}_2$ depend only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

Lemma 3.8. *Suppose that the Γ-periodic solution $\widetilde{\Lambda}$ of problem* (1.33) *satisfies Condition* 3.3. *Then for $0 < \varepsilon \leqslant 1$ the operator $[\widetilde{\Lambda}^\varepsilon]$ is a continuous mapping from $H^1(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$, and $\|[\widetilde{\Lambda}^\varepsilon]\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)}$, where $C_\Omega^{(p)}$ is the norm of the embedding $H^1(\Omega) \hookrightarrow L_{2(p/2)'}(\Omega)$. Here $(p/2)' = \infty$ for $d = 1$, and $(p/2)' = p(p-2)^{-1}$ for $d \geqslant 2$.*

Lemma 3.9. *Suppose that $\widetilde{\Lambda}$ is the* Γ*-periodic solution of problem* (1.33)*. Suppose also that $\widetilde{\Lambda}$ satisfies Condition* 3.3*. Then for any $u \in H^2(\mathbb{R}^d)$ and $0 < \varepsilon \leqslant 1$ we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\widetilde{\Lambda})^\varepsilon(\mathbf{x})|^2 |u(\mathbf{x})|^2\, d\mathbf{x} \leqslant \widetilde{\mathfrak{c}}_1 \|u\|^2_{H^1(\mathbb{R}^d)} + \widetilde{\mathfrak{c}}_2 \varepsilon^2 \|\widetilde{\Lambda}\|^2_{L_p(\Omega)} (C_\Omega^{(p)})^2 \|\mathbf{D}u\|^2_{H^1(\mathbb{R}^d)}.$$

The constants $\widetilde{\mathfrak{c}}_1$ and $\widetilde{\mathfrak{c}}_2$ depend only on n, d, α_0, α_1, ρ, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the norms $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and the parameters of the lattice Γ.

To prove Theorem 3.6, we need the following lemmas. With the another smoothing operator (the Steklov smoothing), these lemmas were proven in [MSu3, Lemmas 7.7 and 7.8]. For our smoothing operator Π_ε, the proof is quite similar.

Lemma 3.10. *Suppose that Condition* 3.1 *is satisfied. Let Π_ε be the operator* (1.48)*. Then for $0 < \varepsilon \leqslant 1$ we have*

$$\|[\Lambda^\varepsilon] b(\mathbf{D})(\Pi_\varepsilon - I)\|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_\Lambda. \tag{3.3}$$

The constant $\mathfrak{C}_\Lambda$ depends only on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, the parameters of the lattice Γ, and the norm $\|\Lambda\|_{L_\infty}$.

Proof. Let $\mathbf{\Phi} \in H^2(\mathbb{R}^d; \mathbb{C}^n)$. From (1.2), (1.49), and Condition 3.1, it follows that

$$\|\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\|_{L_2(\mathbb{R}^d)} \leqslant 2\alpha_1^{1/2} \|\Lambda\|_{L_\infty} \|\mathbf{D}\mathbf{\Phi}\|_{L_2(\mathbb{R}^d)}. \tag{3.4}$$

Consider the derivatives:

$$\partial_j \left(\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\right) = \varepsilon^{-1} (\partial_j \Lambda)^\varepsilon (\Pi_\varepsilon - I) b(\mathbf{D})\mathbf{\Phi} + \Lambda^\varepsilon (\Pi_\varepsilon - I) b(\mathbf{D}) \partial_j \mathbf{\Phi}.$$

Hence,

$$\begin{aligned} \|\mathbf{D}\left(\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\right)\|^2_{L_2(\mathbb{R}^d)} &\leqslant 2\varepsilon^{-2} \|(\mathbf{D}\Lambda)^\varepsilon (\Pi_\varepsilon - I) b(\mathbf{D})\mathbf{\Phi}\|^2_{L_2(\mathbb{R}^d)} \\ &\quad + 2\|\Lambda\|^2_{L_\infty} \|(\Pi_\varepsilon - I) b(\mathbf{D})\mathbf{D}\mathbf{\Phi}\|^2_{L_2(\mathbb{R}^d)}. \end{aligned}$$

By Lemma 3.7, this yields

$$\begin{aligned} \|\mathbf{D}\left(\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\right)\|^2_{L_2(\mathbb{R}^d)} &\leqslant 2\mathfrak{c}_1 \varepsilon^{-2} \|(\Pi_\varepsilon - I) b(\mathbf{D})\mathbf{\Phi}\|^2_{L_2(\mathbb{R}^d)} \\ &\quad + 2\|\Lambda\|^2_{L_\infty} (\mathfrak{c}_2 + 1) \|(\Pi_\varepsilon - I) b(\mathbf{D})\mathbf{D}\mathbf{\Phi}\|^2_{L_2(\mathbb{R}^d)}. \end{aligned}$$

So, combining (1.2), (1.49), and Proposition 1.6, we get

$$\|\mathbf{D}\left(\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\right)\|^2_{L_2(\mathbb{R}^d)} \leqslant \alpha_1 \left(2\mathfrak{c}_1 r_0^{-2} + 8\|\Lambda\|^2_{L_\infty} (\mathfrak{c}_2 + 1)\right) \|\mathbf{D}^2\mathbf{\Phi}\|^2_{L_2(\mathbb{R}^d)}. \tag{3.5}$$

Finally, relations (3.4) and (3.5) imply the inequality

$$\begin{aligned}&\|\Lambda^\varepsilon b(\mathbf{D})(\Pi_\varepsilon - I)\mathbf{\Phi}\|_{H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_\Lambda \|\mathbf{D}\mathbf{\Phi}\|_{H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_\Lambda \|\mathbf{\Phi}\|_{H^2(\mathbb{R}^d)},\\ &\mathbf{\Phi} \in H^2(\mathbb{R}^d;\mathbb{C}^n), \quad \mathfrak{C}_\Lambda^2 := \alpha_1 \left(2\mathfrak{c}_1 r_0^{-2} + (8\mathfrak{c}_2 + 12)\|\Lambda\|^2_{L_\infty}\right).\end{aligned}$$

This is equivalent to estimate (3.3). $\square$

Lemma 3.11. *Suppose that Condition* 3.3 *is satisfied. Let* Π_ε *be the smoothing operator* (1.48). *Then for* $0 < \varepsilon \leqslant 1$ *we have*

$$\|[\widetilde{\Lambda}^\varepsilon](\Pi_\varepsilon - I)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_{\widetilde{\Lambda}}.$$

The constant $\mathfrak{C}_{\widetilde{\Lambda}}$ *depends only on* n, d, α_0, α_1, ρ, p, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the norms* $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, *and* $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$, *and the parameters of the lattice* Γ.

Proof. Let $\mathbf{\Phi} \in H^2(\mathbb{R}^d;\mathbb{C}^n)$. From (1.49) and Lemma 3.8 it follows that

$$\|\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi}\|_{L_2(\mathbb{R}^d)} \leqslant 2C_\Omega^{(p)}\|\widetilde{\Lambda}\|_{L_p(\Omega)}\|\mathbf{\Phi}\|_{H^1(\mathbb{R}^d)}. \tag{3.6}$$

Consider the derivatives:

$$\partial_j\big(\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi}\big) = \varepsilon^{-1}(\partial_j\widetilde{\Lambda})^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi} + \widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\partial_j\mathbf{\Phi}.$$

Together with Lemmas 3.8 and 3.9, this yields

$$\begin{aligned}\|\mathbf{D}\big(\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi}\big)\|^2_{L_2(\mathbb{R}^d)} &\leqslant 2\widetilde{\mathfrak{c}}_1\varepsilon^{-2}\|(\Pi_\varepsilon - I)\mathbf{\Phi}\|^2_{H^1(\mathbb{R}^d)}\\ &+ 2(\widetilde{\mathfrak{c}}_2 + 1)\|\widetilde{\Lambda}\|^2_{L_p(\Omega)}(C_\Omega^{(p)})^2\|\mathbf{D}(\Pi_\varepsilon - I)\mathbf{\Phi}\|^2_{H^1(\mathbb{R}^d)}.\end{aligned}$$

Combining this with (1.49) and Proposition 1.6, we obtain

$$\|\mathbf{D}\big(\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi}\big)\|^2_{L_2(\mathbb{R}^d)} \leqslant \big(2\widetilde{\mathfrak{c}}_1 r_0^{-2} + 8(\widetilde{\mathfrak{c}}_2 + 1)\|\widetilde{\Lambda}\|^2_{L_p(\Omega)}(C_\Omega^{(p)})^2\big)\|\mathbf{D}\mathbf{\Phi}\|^2_{H^1(\mathbb{R}^d)}. \tag{3.7}$$

Now, (3.6) and (3.7) imply that

$$\|\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\mathbf{\Phi}\|_{H^1(\mathbb{R}^d)} \leqslant \mathfrak{C}_{\widetilde{\Lambda}}\|\mathbf{\Phi}\|_{H^2(\mathbb{R}^d)}, \quad \mathbf{\Phi} \in H^2(\mathbb{R}^d;\mathbb{C}^n).$$

Here $\mathfrak{C}^2_{\widetilde{\Lambda}} := 2\widetilde{\mathfrak{c}}_1 r_0^{-2} + (8\widetilde{\mathfrak{c}}_2 + 12)(C_\Omega^{(p)})^2\|\widetilde{\Lambda}\|^2_{L_p(\Omega)}$. $\square$

3.3 Proof of Theorem 3.6

Proof of Theorem 3.6. By (1.42),

$$\begin{aligned} &\| \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^q(\mathbb{R}^d)\to H^2(\mathbb{R}^d)} \\ &\quad \leqslant \| \cos(t(B^0)^{1/2})(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^2(\mathbb{R}^d)} \leqslant \check{c}^{-1}. \end{aligned} \tag{3.8}$$

According to estimate (1.37) for the symbol of B^0,

$$\begin{aligned} &\|(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{1+q}(\mathbb{R}^d)\to H^2(\mathbb{R}^d)} \\ &\quad \leqslant \|(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d)\to H^2(\mathbb{R}^d)} \leqslant \check{c}^{-1/2}. \end{aligned} \tag{3.9}$$

Together with Theorem 2.8, Lemmas 3.10 and 3.11, estimates (3.8) and (3.9) imply the results of Theorem 3.6 with the constants $C_{19} := C_{14}+(\mathfrak{C}_\Lambda+\mathfrak{C}_{\widetilde{\Lambda}})\check{c}^{-1}$ and $C_{20} := C_{15} + (\mathfrak{C}_\Lambda + \mathfrak{C}_{\widetilde{\Lambda}})\check{c}^{-1/2}$. $\square$

3.4 Removal of the smoothing operator from the corrector for $3 \leqslant d \leqslant 4$

If $d \leqslant 2$, then, according to Propositions 3.2 and 3.4, Theorem 3.6 is applicable. So, let $d \geqslant 3$. Now we are interested in the possibility to remove the smoothing operator from the corrector without any additional assumptions on the matrix-valued functions Λ and $\widetilde{\Lambda}$.

If $3 \leqslant d \leqslant 4$, it turns out that the smoothing operator Π_ε can be eliminated from both terms of the corrector. But now restrictions on q are stronger than ones from Theorem 2.8. These new restrictions are caused by the multiplier properties of the matrix-valued function Λ^ε. The following result was obtained in [MSu2, Lemma 6.3].

Lemma 3.12. *Let $\Lambda(\mathbf{x})$ be the Γ-periodic matrix-valued solution of problem* (1.25). *Assume that $d \geqslant 3$ and put $l = d/2$.*
1°. *For $0 < \varepsilon \leqslant 1$, the operator $[\Lambda^\varepsilon]$ is a continuous mapping from $H^{l-1}(\mathbb{R}^d;\mathbb{C}^m)$ to $L_2(\mathbb{R}^d;\mathbb{C}^n)$ and*

$$\|[\Lambda^\varepsilon]\|_{H^{l-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C^{(0)}.$$

2°. *Let $0 < \varepsilon \leqslant 1$. Then for the function $\mathbf{u} \in H^l(\mathbb{R}^d;\mathbb{C}^m)$ we have the inclusion $\Lambda^\varepsilon\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ and the estimate*

$$\|\Lambda^\varepsilon\mathbf{u}\|_{H^1(\mathbb{R}^d)} \leqslant C^{(1)}\varepsilon^{-1}\|\mathbf{u}\|_{L_2(\mathbb{R}^d)} + C^{(2)}\|\mathbf{u}\|_{H^l(\mathbb{R}^d)}.$$

The constants $C^{(0)}$, $C^{(1)}$, and $C^{(2)}$ depend on m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

Theorem 3.13. *Suppose that the assumptions of Subsections* 1.1–1.6 *are satisfied. Let* $3 \leqslant d \leqslant 4$. *Let* $1/2 \leqslant q \leqslant 1$ *for* $d = 3$ *and* $q = 1$ *for* $d = 4$. *Let* $\mathcal{K}_1^0(\varepsilon;t)$ *and* $\mathcal{K}_2^0(\varepsilon;t)$ *be the operators* (3.1), (3.2), *respectively. Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$\begin{aligned} &\| \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1} - \cos(t(B^0)^{1/2})(B^0)^{-1} - \varepsilon\mathcal{K}_1^0(\varepsilon;t)\|_{H^q(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{21}\varepsilon^q(1+|t|)^q, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon\mathcal{K}_2^0(\varepsilon;t)\|_{H^{1+q}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\quad \leqslant C_{22}\varepsilon^q(1+|t|)^q. \end{aligned} \tag{3.11}$$

The constants C_{21} *and* C_{22} *depend only on the problem data* (1.24) *and* q.

Proof. By Proposition 1.6, Lemma 3.12(2°), and (1.2), (1.42), (1.49),

$$\begin{aligned} &\|\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - I)b(\mathbf{D})\cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^q(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\leqslant C^{(1)}\|(\Pi_\varepsilon - I)b(\mathbf{D})(B^0)^{-1}\|_{H^q(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &+ \varepsilon C^{(2)}\|(\Pi_\varepsilon - I)b(\mathbf{D})(B^0)^{-1}\|_{H^q(\mathbb{R}^d)\to H^l(\mathbb{R}^d)} \\ &\leqslant \varepsilon\alpha_1^{1/2}(r_0^{-1}C^{(1)} + 2C^{(2)})\check{c}^{-1}, \quad l = d/2. \end{aligned} \tag{3.12}$$

Similarly, by Proposition 3.4, Lemma 3.11, and (1.42),

$$\|\varepsilon\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)\cos(t(B^0)^{1/2})(B^0)^{-1}\|_{H^q(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \varepsilon\mathfrak{C}_{\widetilde{\Lambda}}\check{c}^{-1}. \tag{3.13}$$

Combining (2.56), (2.61), (3.1), (3.12), and (3.13), we arrive at estimate (3.10) with the constant $C_{21} := C_{14} + \alpha_1^{1/2}(r_0^{-1}C^{(1)} + 2C^{(2)})\check{c}^{-1} + \mathfrak{C}_{\widetilde{\Lambda}}\check{c}^{-1}$.

Now we proceed to the proof of estimate (3.11). Combining Proposition 1.6, Lemma 3.12(2°), estimate (1.37) for the symbol of the operator B^0, and (1.2), (1.49), we get

$$\begin{aligned} &\|\varepsilon\Lambda^\varepsilon(\Pi_\varepsilon - I)b(\mathbf{D})(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{1+q}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \\ &\leqslant \varepsilon\alpha_1^{1/2}(r_0^{-1}C^{(1)} + 2C^{(2)})\check{c}^{-1/2}. \end{aligned} \tag{3.14}$$

Next, by Proposition 3.4, Lemma 3.11, and estimate (1.37) for the symbol of the operator B^0,

$$\|\varepsilon\widetilde{\Lambda}^\varepsilon(\Pi_\varepsilon - I)(B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^{1+q}(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \varepsilon\mathfrak{C}_{\widetilde{\Lambda}}\check{c}^{-1/2}. \tag{3.15}$$

Bringing together (2.57), (2.62), (3.2), (3.14), and (3.15), we arrive at estimate (3.11) with the constant $C_{22} := C_{15} + \alpha_1^{1/2}(r_0^{-1}C^{(1)} + 2C^{(2)})\check{c}^{-1/2} + (r_0^{-1}\widetilde{C}^{(1)} + 2\widetilde{C}^{(2)})\check{c}^{-1/2}$. □

4 Homogenization of solutions of hyperbolic systems

It the present section, we apply the results in operator terms to homogenization of solutions of hyperbolic systems.

4.1 Principal term of approximation

Let $\mathbf{u}_\varepsilon$ be the generalized solution of the problem

$$\begin{cases} \partial_t^2 \mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^d,\ t \in (0,T), \\ \mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{u}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

Here $\boldsymbol{\phi} \in H^r(\mathbb{R}^d;\mathbb{C}^n)$, $\boldsymbol{\psi} \in H^{r-1}(\mathbb{R}^d;\mathbb{C}^n)$, and $\mathbf{F} \in L_1((0,T); H^{r-1}(\mathbb{R}^d;\mathbb{C}^n))$ for some $0 < T \leqslant \infty$ and $0 \leqslant r \leqslant 2$. We have

$$\begin{aligned} \mathbf{u}_\varepsilon(\cdot,t) &= \cos(tB_\varepsilon^{1/2})\boldsymbol{\phi} + B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t B_\varepsilon^{-1/2}\sin((t-s)B_\varepsilon^{1/2})\mathbf{F}(\cdot,s)\,ds, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \partial_t \mathbf{u}_\varepsilon(\cdot,t) &= -\sin(tB_\varepsilon^{1/2})B_\varepsilon^{1/2}\boldsymbol{\phi} + \cos(tB_\varepsilon^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t \cos((t-s)B_\varepsilon^{1/2})\mathbf{F}(\cdot,s)\,ds. \end{aligned} \tag{4.3}$$

Let $\mathbf{u}_0$ be the solution of the effective problem

$$\begin{cases} \partial_t^2 \mathbf{u}_0(\mathbf{x},t) = -B^0 \mathbf{u}_0(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^d,\ t \in (0,T), \\ \mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{u}_0)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{cases}$$

Combining estimates (2.15), (2.17), (2.18), and (2.63) with identities (4.2), (4.3) and similar representations for $\mathbf{u}_0$ and $\partial_t \mathbf{u}_0$, we arrive at the following result.

Theorem 4.1. *Under the assumptions of Subsections* 1.1–1.6 *and* 4.1*, for* $0 < \varepsilon \leqslant 1$ *and* $t \in (0,T)$*, we have*

$$\begin{aligned} &\|\mathbf{u}_\varepsilon(\cdot,t) - \mathbf{u}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant C_7 \varepsilon^{r/2}(1+|t|)^{r/2}\|\boldsymbol{\phi}\|_{H^r(\mathbb{R}^d)} \\ &\quad + C_{16}\varepsilon^{r/2}|t|^{r/2}\big(\|\boldsymbol{\psi}\|_{H^{r-1}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{r-1}(\mathbb{R}^d))}\big), \\ &\|(\partial_t \mathbf{u}_\varepsilon)(\cdot,t) - (\partial_t \mathbf{u}_0)(\cdot,t)\|_{H^{-1}(\mathbb{R}^d)} \leqslant C_9 \varepsilon^{r/2}(1+|t|)^{r/2}\|\boldsymbol{\phi}\|_{H^r(\mathbb{R}^d)} \\ &\quad + C_{10}\varepsilon^{r/2}|t|^{r/2}\big(\|\boldsymbol{\psi}\|_{H^{r-1}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{r-1}(\mathbb{R}^d))}\big). \end{aligned}$$

The constants C_7*,* C_9*,* C_{10}*, and* C_{16} *are controlled explicitly in terms of* r *and the problem data* (1.24).

4.2 Approximation with corrector

Now, consider the following problem:

$$\begin{cases} \partial_t^2 \mathbf{v}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{v}_\varepsilon(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d,\ t\in(0,T), \\ \mathbf{v}_\varepsilon(\mathbf{x},0) = B_\varepsilon^{-1}\boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{v}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{4.4}$$

Here $\boldsymbol{\phi}\in H^q(\mathbb{R}^d;\mathbb{C}^n)$, $\boldsymbol{\psi}\in H^{1+q}(\mathbb{R}^d;\mathbb{C}^n)$, and $\mathbf{F}\in L_1((0,T);H^{1+q}(\mathbb{R}^d;\mathbb{C}^n))$ for some $0<T\leqslant\infty$ and $0\leqslant q\leqslant 1$. We have

$$\begin{aligned} \mathbf{v}_\varepsilon(\cdot,t) &= \cos(tB_\varepsilon^{1/2})B_\varepsilon^{-1}\boldsymbol{\phi} + B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t B_\varepsilon^{-1/2}\sin((t-s)B_\varepsilon^{1/2})\mathbf{F}(\cdot,s)\,ds, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \partial_t\mathbf{v}_\varepsilon(\cdot,t) &= -\sin(tB_\varepsilon^{1/2})B_\varepsilon^{-1/2}\boldsymbol{\phi} + \cos(tB_\varepsilon^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t \cos((t-s)B_\varepsilon^{1/2})\mathbf{F}(\cdot,s)\,ds. \end{aligned} \tag{4.6}$$

Let $\mathbf{v}_0$ be the solution of the corresponding effective problem:

$$\begin{cases} \partial_t^2 \mathbf{v}_0(\mathbf{x},t) = -B^0 \mathbf{v}_0(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^d,\ t\in(0,T), \\ \mathbf{v}_0(\mathbf{x},0) = (B^0)^{-1}\boldsymbol{\phi}(\mathbf{x}), \quad (\partial_t \mathbf{v}_0)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{4.7}$$

Then

$$\begin{aligned} \mathbf{v}_0(\cdot,t) &= \cos(t(B^0)^{1/2})(B^0)^{-1}\boldsymbol{\phi} + (B^0)^{-1/2}\sin(t(B^0)^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t (B^0)^{-1/2}\sin((t-s)(B^0)^{1/2})\mathbf{F}(\cdot,s)\,ds, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \partial_t\mathbf{v}_0(\cdot,t) &= -\sin(t(B^0)^{1/2})(B^0)^{-1/2}\boldsymbol{\phi} + \cos(t(B^0)^{1/2})\boldsymbol{\psi} \\ &\quad + \int_0^t \cos((t-s)(B^0)^{1/2})\mathbf{F}(\cdot,s)\,ds. \end{aligned} \tag{4.9}$$

Let $\mathbf{w}_\varepsilon$ be the first order approximation for the solution $\mathbf{v}_\varepsilon$:

$$\mathbf{w}_\varepsilon(\cdot,t) = \mathbf{v}_0(\cdot,t) + \varepsilon(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)\mathbf{v}_0(\cdot,t). \tag{4.10}$$

From Theorem 2.8, (2.15) with $r=q+1$, and (4.5), (4.6), (4.8)–(4.10) we derive the following result.

Theorem 4.2. *Under the assumptions of Subsections* 1.1–1.7 *and* 4.2*, let* $0\leqslant q\leqslant 1$*. Then for* $0<\varepsilon\leqslant 1$ *and* $t\in(0,T)$ *we have*

$$\begin{aligned} &\|(\partial_t\mathbf{v}_\varepsilon)(\cdot,t) - (\partial_t\mathbf{v}_0)(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant C_{16}\varepsilon^{(q+1)/2}|t|^{(q+1)/2}\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &\quad + C_7\varepsilon^{(q+1)/2}(1+|t|)^{(q+1)/2}\big(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}\big), \\ &\|\mathbf{v}_\varepsilon(\cdot,t) - \mathbf{w}_\varepsilon(\cdot,t)\|_{H^1(\mathbb{R}^d)} \leqslant C_{14}\varepsilon^q(1+|t|)^q\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &\quad + C_{15}\varepsilon^q(1+|t|)^q\big(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}\big). \end{aligned} \tag{4.11}$$

The constants C_7, C_{14}, C_{15}, and C_{16} are controlled explicitly in terms of q and the problem data (1.24).

According to Propositions 3.2 and 3.4 and Theorems 3.6 and 3.13, for $d \leqslant 2$ we can always replace function (4.10) by

$$\check{\mathbf{w}}_\varepsilon(\cdot, t) = \mathbf{v}_0(\cdot, t) + \varepsilon(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)\mathbf{v}_0(\cdot, t) \tag{4.12}$$

in approximation (4.11), i. e., to remove the smoothing operator from the corrector. This changes only the constants in estimate. For $d = 3, 4$, we also can remove the smoothing operator, but not for all $q \in (0, 1]$.

Theorem 4.3. *Under the assumptions of Subsections* 1.1–1.7 *and* 4.2*, let $d \leqslant 4$. Let $0 \leqslant q \leqslant 1$ for $d = 1, 2$; $1/2 \leqslant q \leqslant 1$ for $d = 3$; and let $q = 1$ for $d = 4$. Let $\check{\mathbf{w}}_\varepsilon(\cdot, t)$ be the function* (4.12)*. Then for $0 < \varepsilon \leqslant 1$ and $t \in \mathbb{R}$ we have*

$$\begin{aligned} &\|\mathbf{v}_\varepsilon(\cdot, t) - \check{\mathbf{w}}_\varepsilon(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leqslant C_{23}\varepsilon^q(1 + |t|)^q\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &\quad + C_{24}\varepsilon^q(1 + |t|)^q(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}). \end{aligned} \tag{4.13}$$

The constants C_{23} and C_{24} depend only on q and the problem data (1.24).

Under the additional assumption that Conditions 3.1 and 3.3 hold, we also can remove smoothing operator from the corrector due to Theorem 3.6.

Theorem 4.4. *Let the assumptions of Subsections* 1.1–1.7 *and* 4.2 *be satisfied. Assume that Conditions* 3.1 *and* 3.3 *hold. Let $\check{\mathbf{w}}_\varepsilon(\cdot, t)$ be function* (4.12)*. Then for $0 < \varepsilon \leqslant 1$ and $t \in \mathbb{R}$*

$$\begin{aligned} &\|\mathbf{v}_\varepsilon(\cdot, t) - \check{\mathbf{w}}_\varepsilon(\cdot, t)\|_{H^1(\mathbb{R}^d)} \leqslant C_{19}\varepsilon^q(1 + |t|)^q\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &\quad + C_{20}\varepsilon^q(1 + |t|)^q(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}). \end{aligned}$$

The constants C_{19} and C_{20} depend only on q, p, $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$, and on the problem data (1.24).

4.3 Discussion of the results

According to the abstract theory of hyperbolic equations (see, e. g., [BSol, §8.2.4]), the following law of conservation of energy holds for the solution $\mathbf{v}_\varepsilon$ of problem (4.4) with $\mathbf{F} = 0$:

$$\begin{aligned} &\|\partial_t \mathbf{v}_\varepsilon(\cdot, t)\|^2_{L_2(\mathbb{R}^d)} + \|B_\varepsilon^{1/2}\mathbf{v}_\varepsilon(\cdot, t)\|^2_{L_2(\mathbb{R}^d)} \\ &\quad = \|\partial_t \mathbf{v}_\varepsilon(\cdot, 0)\|^2_{L_2(\mathbb{R}^d)} + \|B_\varepsilon^{1/2}\mathbf{v}_\varepsilon(\cdot, 0)\|^2_{L_2(\mathbb{R}^d)} \\ &\quad = \|\boldsymbol{\psi}\|^2_{L_2(\mathbb{R}^d)} + \|B_\varepsilon^{-1/2}\boldsymbol{\phi}\|^2_{L_2(\mathbb{R}^d)}. \end{aligned} \tag{4.14}$$

For the solution $\mathbf{v}_0$ of problem (4.7) with $\mathbf{F}=0$ we also have the conservation of energy:

$$\|\partial_t \mathbf{v}_0(\cdot,t)\|^2_{L_2(\mathbb{R}^d)} + \|(B^0)^{1/2}\mathbf{v}_0(\cdot,t)\|^2_{L_2(\mathbb{R}^d)} = \|\boldsymbol{\psi}\|^2_{L_2(\mathbb{R}^d)} + \|(B^0)^{-1/2}\boldsymbol{\phi}\|^2_{L_2(\mathbb{R}^d)}. \tag{4.15}$$

As was mentioned in Introduction, we consider problem of the form (4.4) instead of (4.1) to have convergence of the corresponding energy to the energy of the solution of the effective problem. This convergence allows us to give energy norm approximation for the solution. For $\boldsymbol{\phi} \in H^2(\mathbb{R}^d;\mathbb{C}^n)$, convergence of the energy (4.14) to the energy (4.15) is a consequence of the following lemma.

Lemma 4.5. *Under the assumptions of Subsections* 1.1–1.6, *for* $0<\varepsilon\leqslant 1$ *we have*

$$\|B_\varepsilon^{-1/2} - (B^0)^{-1/2}\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \mathfrak{C}\varepsilon. \tag{4.16}$$

The constant $\mathfrak{C}$ *depends only on the problem data* (1.24).

Proof. According to [VilKr, Chapter III, §3, Subsection 4], we have

$$B_\varepsilon^{-1/2} = \frac{1}{\pi}\int_0^\infty \nu^{-1/2}(B_\varepsilon+\nu I)^{-1}\,d\nu.$$

The similar representation holds for the operator $(B^0)^{-1/2}$. So,

$$\begin{aligned}
&(B_\varepsilon^{-1/2} - (B^0)^{-1/2})(B^0)^{-1}\\
&= \frac{1}{\pi}\int_0^\infty \nu^{-1/2}\left((B_\varepsilon+\nu I)^{-1} - (B^0+\nu I)^{-1}\right)(B^0)^{-1}\,d\nu.
\end{aligned}$$

By using the resolvent identity, we obtain

$$\begin{aligned}
&(B_\varepsilon^{-1/2} - (B^0)^{-1/2})(B^0)^{-1}\\
&= \frac{1}{\pi}\int_0^\infty \nu^{-1/2}B_\varepsilon(B_\varepsilon+\nu I)^{-1}(B_\varepsilon^{-1} - (B^0)^{-1})(B^0+\nu I)^{-1}\,d\nu.
\end{aligned} \tag{4.17}$$

We have

$$\|B_\varepsilon(B_\varepsilon+\nu I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \sup_{x\geqslant 0}\frac{x}{x+\nu}\leqslant 1, \quad \nu\geqslant 0. \tag{4.18}$$

Next, by (1.36),

$$\|(B^0+\nu I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \sup_{x\geqslant \check{c}}\frac{1}{x+\nu}\leqslant \frac{1}{\check{c}+\nu}, \quad \nu\geqslant 0. \tag{4.19}$$

Now from Theorem 1.8 and (4.17)–(4.19) it follows that

$$\|(B_\varepsilon^{-1/2} - (B^0)^{-1/2})(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \frac{1}{\pi} C_1 \varepsilon \int_0^\infty \nu^{-1/2} (\check{c} + \nu)^{-1}\, d\nu$$
$$\leqslant \frac{C_1\varepsilon}{\pi}\Big(\check{c}^{-1}\int_0^1 \nu^{-1/2}\, d\nu + \int_1^\infty \nu^{-3/2}\, d\nu\Big) = \frac{2C_1\varepsilon}{\pi}(\check{c}^{-1}+1). \tag{4.20}$$

Together with (1.46), this implies estimate (4.16) with the constant $\mathfrak{C} := 2\pi^{-1}(\check{c}^{-1}+1)C_1 C_L$. $\square$

Remark 4.6. *Estimate* (4.16) *does not look optimal with respect to the type of the norm. It is natural to expect that the correct type of the norm is* $(H^1 \to L_2)$*- one. But to prove such estimate we need to have approximation of the operator* $(B_\varepsilon + \nu I)^{-1}$, $\nu \in \mathbb{R}_+$, *in* $(H^1 \to L_2)$*-norm with the error estimate of the form* $C\varepsilon(1+\nu)^{-1}$. *It is one of the results of the author's work in progress* [M5].

4.4 The case where the corrector is equal to zero

Assume that relations (1.31) and (2.76) hold. Then the corrector is equal to zero (see Subsection 2.6), i. e., $\mathbf{v}_\varepsilon(\cdot,t) = \mathbf{v}_0(\cdot,t)$. Proposition 2.9 implies the following result.

Proposition 4.7. *Let the assumptions of Subsections* 1.1–1.7 *be satisfied. Suppose that relations* (1.31) *and* (2.76) *hold. Let* $\mathbf{v}_\varepsilon(\cdot,t)$ *and* $\mathbf{v}_0(\cdot,t)$ *be the solutions of problems* (4.4) *and* (4.7), *respectively, where* $\boldsymbol{\phi} \in H^{-1+r}(\mathbb{R}^d;\mathbb{C}^n)$, $\boldsymbol{\psi} \in H^r(\mathbb{R}^d;\mathbb{C}^n)$, *and* $\mathbf{F} \in L_1((0,T);H^r(\mathbb{R}^d;\mathbb{C}^n))$ *for some* $0 < T \leqslant \infty$ *and* $0 \leqslant r \leqslant 2$. *Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$\|\mathbf{v}_\varepsilon(\cdot,t) - \mathbf{v}_0(\cdot,t)\|_{H^1(\mathbb{R}^d)} \leqslant (c_6^2 + \check{c}^{-1})^{1-r/2} C_{11}^{r/2} \varepsilon^{r/2} (1+|t|)^{r/2} \|\boldsymbol{\phi}\|_{H^{-1+r}(\mathbb{R}^d)}$$
$$+ (c_6 + \check{c}^{-1/2})^{1-r/2} C_{12}^{r/2} \varepsilon^{r/2}(1+|t|)^{r/2} (\|\boldsymbol{\psi}\|_{H^r(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^r(\mathbb{R}^d))}).$$

4.5 Approximation of the flux

Theorem 4.8. *Let the assumptions of Subsections* 1.1–1.7 *and* 4.2 *be satisfied. Let* $0 \leqslant q \leqslant 1$. *Let* $\widetilde{g}$ *be the matrix-valued function* (1.27). *Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in (0,T)$, *for the flux* $\mathbf{p}_\varepsilon(\cdot,t) := g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon(\cdot,t)$ *we have an approximation*

$$\begin{aligned}&\|\mathbf{p}_\varepsilon(\cdot,t) - \widetilde{g}^\varepsilon \Pi_\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,t) - g^\varepsilon (b(\mathbf{D})\widetilde{\Lambda})^\varepsilon \Pi_\varepsilon \mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \\ &\leqslant C_{25}\varepsilon^q (1+|t|)^q \|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &+ C_{26}\varepsilon^q(1+|t|)^q(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}).\end{aligned} \tag{4.21}$$

The constants C_{25} and C_{26} are controlled explicitly in terms of q and the problem data (1.24).

Proof. By (1.2) and (4.11),

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon(\cdot,t) - g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant \alpha_1^{1/2}\|g\|_{L_\infty} C_{14}\varepsilon^q(1+|t|)^q\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} + \alpha_1^{1/2}\|g\|_{L_\infty} C_{15}\varepsilon^q(1+|t|)^q\big(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}\big). \tag{4.22}$$

Combining (1.1) and (4.10), we obtain

$$g^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon = g^\varepsilon b(\mathbf{D})\mathbf{v}_0 + g^\varepsilon(b(\mathbf{D})\Lambda)^\varepsilon\Pi_\varepsilon b(\mathbf{D})\mathbf{v}_0 + g^\varepsilon(b(\mathbf{D})\widetilde{\Lambda})^\varepsilon\Pi_\varepsilon\mathbf{v}_0 + \varepsilon\sum_{j=1}^d g^\varepsilon b_j(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)D_j\mathbf{v}_0. \tag{4.23}$$

The fourth term in the right-hand side of (4.23) can be estimated with the help of (1.3) and (2.52):

$$\|\varepsilon\sum_{j=1}^d g^\varepsilon b_j(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)D_j\mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon C_{27}\|\mathbf{v}_0(\cdot,t)\|_{H^2(\mathbb{R}^d)}, \tag{4.24}$$

where $C_{27} := \|g\|_{L_\infty}(d\alpha_1)^{1/2}(M_1\alpha_1^{1/2} + \widetilde{M}_1)$. From (4.8) and the estimate (1.37) for the symbol of the operator B^0 it follows that

$$\begin{aligned}\|\mathbf{v}_0(\cdot,t)\|_{H^2(\mathbb{R}^d)} &\leqslant \|\mathbf{v}_0(\cdot,t)\|_{H^{2+q}(\mathbb{R}^d)} \leqslant \|(B^0)^{-1}\boldsymbol{\phi}\|_{H^{2+q}(\mathbb{R}^d)} \\ &\quad + \|(B^0)^{-1/2}\boldsymbol{\psi}\|_{H^{2+q}(\mathbb{R}^d)} + \int_0^t \|(B^0)^{-1/2}\mathbf{F}(\cdot,s)\|_{H^{2+q}(\mathbb{R}^d)}\,ds \\ &\leqslant \check{c}^{-1}\|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} + \check{c}^{-1/2}\big(\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}\big),\end{aligned} \tag{4.25}$$

for any $0 \leqslant q \leqslant 1$. Next, by Proposition 1.6 and (1.2),

$$\|g^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,t) - g^\varepsilon\Pi_\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant \varepsilon r_0^{-1}\alpha_1^{1/2}\|g\|_{L_\infty}\|\mathbf{D}^2\mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)}. \tag{4.26}$$

Now from (1.27), and (4.22)–(4.26) we derive required estimate (4.21) with the constants

$$\begin{aligned}C_{25} &:= \alpha_1^{1/2}\|g\|_{L_\infty}C_{14} + C_{27}\check{c}^{-1} + r_0^{-1}\alpha_1^{1/2}\|g\|_{L_\infty}\check{c}^{-1}, \\ C_{26} &:= \alpha_1^{1/2}\|g\|_{L_\infty}C_{15} + C_{27}\check{c}^{-1/2} + r_0^{-1}\alpha_1^{1/2}\|g\|_{L_\infty}\check{c}^{-1/2}.\end{aligned}$$

$\square$

4.6 On the possibility to remove Π_ε from approximation of the flux

If $d \leqslant 4$, we can derive approximation of the flux $\mathbf{p}_\varepsilon$ from Theorem 4.3. The proof repeats the proof of Theorem 4.8 with some simplifications.

Theorem 4.9. *Under the assumptions of Theorem 4.8, let $d \leqslant 4$. If $d = 3$, we additionally assume that $1/2 \leqslant q \leqslant 1$; and, if $d = 4$, we suppose that $q = 1$. Then for $0 < \varepsilon \leqslant 1$ and $t \in \mathbb{R}$, we have*

$$\begin{aligned} &\|\mathbf{p}_\varepsilon(\cdot,t) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot,t) - g^\varepsilon (b(\mathbf{D})\widetilde{\Lambda})^\varepsilon \mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \\ &\leqslant C_{28}\varepsilon^q (1+|t|)^q \|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &+ C_{29}\varepsilon^q (1+|t|)^q (\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}). \end{aligned} \tag{4.27}$$

The constants C_{28} and C_{29} depend on q and the problem data (1.24)*.*

Proof. By (1.2) and (4.13),

$$\begin{aligned} &\|g^\varepsilon b(\mathbf{D})\mathbf{v}_\varepsilon(\cdot,t) - g^\varepsilon b(\mathbf{D})\check{\mathbf{w}}_\varepsilon(\cdot,t)\|_{L_2(\mathbb{R}^d)} \leqslant \alpha_1^{1/2}\|g\|_{L_\infty} C_{23}\varepsilon^q (1+|t|)^q \|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\ &+ \alpha_1^{1/2}\|g\|_{L_\infty} C_{24}\varepsilon^q(1+|t|)^q (\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}). \end{aligned} \tag{4.28}$$

From (1.1) and (4.12) it follows that

$$\begin{aligned} g^\varepsilon b(\mathbf{D})\check{\mathbf{w}}_\varepsilon &= g^\varepsilon b(\mathbf{D})\mathbf{v}_0 + g^\varepsilon (b(\mathbf{D})\Lambda)^\varepsilon b(\mathbf{D})\mathbf{v}_0 \\ &+ g^\varepsilon (b(\mathbf{D})\widetilde{\Lambda})^\varepsilon \mathbf{v}_0 + \varepsilon \sum_{j=1}^d g^\varepsilon b_j (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon) D_j \mathbf{v}_0. \end{aligned} \tag{4.29}$$

Let us estimate the last summand in the right-hand side of (4.29). By (1.2), (1.3), Propositions 3.2 and 3.4, Lemmas 3.8 and 3.12(1°),

$$\begin{aligned} &\|\varepsilon \sum_{j=1}^d g^\varepsilon b_j (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon) D_j \mathbf{v}_0(\cdot,t)\|_{L_2(\mathbb{R}^d)} \\ &\leqslant (d\alpha_1)^{1/2}\|g\|_{L_\infty} (C^{(0)}\alpha_1^{1/2}\|\mathbf{v}_0(\cdot,t)\|_{H^{l+1}(\mathbb{R}^d)} + \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)} \|\mathbf{v}_0(\cdot,t)\|_{H^2(\mathbb{R}^d)}). \end{aligned} \tag{4.30}$$

(Here $C^{(0)} := \|\Lambda\|_{L_\infty}$ and $l = 1$ for $d = 1, 2$.) Note that $\|\mathbf{v}_0(\cdot,t)\|_{H^{l+1}(\mathbb{R}^d)} \leqslant \|\mathbf{v}_0(\cdot,t)\|_{H^{2+q}(\mathbb{R}^d)}$ for $d = 3, 4$. So, together with (1.27), (4.25), and (4.28), estimate (4.30) implies inequality (4.27) with the constants

$$\begin{aligned} C_{28} &:= (d\alpha_1)^{1/2}\|g\|_{L_\infty}(C_{23} + C^{(0)}\alpha_1^{1/2}\check{c}^{-1} + \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)} \check{c}^{-1}), \\ C_{29} &:= (d\alpha_1)^{1/2}\|g\|_{L_\infty}(C_{24} + C^{(0)}\alpha_1^{1/2}\check{c}^{-1/2} + \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)} \check{c}^{-1/2}). \end{aligned}$$

□

If $d \geqslant 5$, under Conditions 3.1 and 3.3, we can remove the smoothing operator Π_ε from approximation of the flux. The proof of the following result is based on Theorem 4.4 and is quite similar to that of Theorem 4.9. We omit the details.

Theorem 4.10. *Under the assumptions of Theorem* 4.8*, let Conditions* 3.1 *and* 3.3 *hold. Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$*, we have*

$$\begin{aligned}
&\|\mathbf{p}_\varepsilon(\cdot, t) - \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot, t) - g^\varepsilon (b(\mathbf{D})\widetilde{\Lambda})^\varepsilon \mathbf{v}_0(\cdot, t)\|_{L_2(\mathbb{R}^d)} \\
&\quad \leqslant C_{30}\varepsilon^q (1 + |t|)^q \|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\
&\quad + C_{31}\varepsilon^q (1 + |t|)^q (\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}).
\end{aligned}$$

The constants

$$\begin{aligned}
C_{30} &:= (d\alpha_1)^{1/2} \|g\|_{L_\infty} (C_{19} + \|\Lambda\|_{L_\infty} \alpha_1^{1/2} \check{c}^{-1} + \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)} \check{c}^{-1}), \\
C_{31} &:= (d\alpha_1)^{1/2} \|g\|_{L_\infty} (C_{20} + \|\Lambda\|_{L_\infty} \alpha_1^{1/2} \check{c}^{-1/2} + \|\widetilde{\Lambda}\|_{L_p(\Omega)} C_\Omega^{(p)} \check{c}^{-1/2}),
\end{aligned}$$

depend on q*,* $\|\Lambda\|_{L_\infty}$*,* $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$*,* p*, and the problem data* (1.24)*.*

4.7 The special case

Assume that $g^0 = \underline{g}$, i. e., relations (1.32) are satisfied. Then, by Proposition 3.2(3°), Condition 3.1 holds. Herewith, according to [BSu3, Remark 3.5], the matrix-valued function (1.27) is constant and coincides with g^0, i. e., $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. Thus, $\widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{v}_0(\cdot, t) = g^0 b(\mathbf{D})\mathbf{v}_0(\cdot, t)$.

Assume that condition (2.76) holds. Then the Γ-periodic solution of problem (1.33) also equals to zero: $\widetilde{\Lambda}(\mathbf{x}) = 0$. So, Theorem 4.10 implies the following result.

Proposition 4.11. *Under the assumptions of Theorem* 4.8*, suppose that relations* (1.32) *and* (2.76) *hold. Then for* $t \in \mathbb{R}$ *and* $0 < \varepsilon \leqslant 1$ *we have*

$$\begin{aligned}
&\|\mathbf{p}_\varepsilon(\cdot, t) - g^0 b(\mathbf{D})\mathbf{v}_0(\cdot, t)\|_{L_2(\mathbb{R}^d)} \leqslant C_{30}\varepsilon^q (1 + |t|)^q \|\boldsymbol{\phi}\|_{H^q(\mathbb{R}^d)} \\
&\quad + C_{31}\varepsilon^q (1 + |t|)^q (\|\boldsymbol{\psi}\|_{H^{1+q}(\mathbb{R}^d)} + \|\mathbf{F}\|_{L_1((0,t);H^{1+q}(\mathbb{R}^d))}).
\end{aligned}$$

5 On approximation of the operator e^{-itB_ε}

5.1 Principal term of approximation

In the present subsection we give an alternative proof of Theorem 3.1.1 from [D].

Theorem 5.1 ([D]). *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Let* $0 \leqslant r \leqslant 3$. *Then for* $0 < \varepsilon \leqslant 1$ *and* $t \in \mathbb{R}$ *we have*

$$\| e^{-itB_\varepsilon} - e^{-itB^0} \|_{H^r(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leqslant C_{32} \varepsilon^{r/3} (1 + |t|)^{r/3}. \tag{5.1}$$

The constant C_{32} *depends only on* r *and the problem data* (1.24).

Proof. Denote $\Upsilon(t) := e^{-itB_\varepsilon} B_\varepsilon^{-1} (B^0)^{-1} e^{itB^0}$. Then, according to (1.53),

$$\frac{d\Upsilon(t)}{dt} = i e^{-itB_\varepsilon} \mathcal{R}_1(\varepsilon) e^{itB^0}.$$

Thus,

$$\begin{aligned} \Upsilon(t) - \Upsilon(0) &= e^{-itB_\varepsilon} B_\varepsilon^{-1} (B^0)^{-1} e^{itB^0} - B_\varepsilon^{-1} (B^0)^{-1} \\ &= i \int_0^t e^{-isB_\varepsilon} \mathcal{R}_1(\varepsilon) e^{isB^0} \, ds. \end{aligned}$$

Multiplying this equality by e^{-itB^0} from the right, we obtain

$$\begin{aligned} & e^{-itB_\varepsilon} B_\varepsilon^{-1} (B^0)^{-1} - B_\varepsilon^{-1} (B^0)^{-1} e^{-itB^0} \\ &= i \int_0^t e^{-isB_\varepsilon} \mathcal{R}_1(\varepsilon) e^{i(s-t)B^0} \, ds. \end{aligned}$$

The ranges of all operators in this identity lie in $H^1(\mathbb{R}^d; \mathbb{C}^n)$, so we can multiply it by $B_\varepsilon^{1/2}$ from the left:

$$\begin{aligned} & e^{-itB_\varepsilon} B_\varepsilon^{-1/2} (B^0)^{-1} - B_\varepsilon^{-1/2} (B^0)^{-1} e^{-itB^0} \\ &= i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \mathcal{R}_1(\varepsilon) e^{i(s-t)B^0} \, ds. \end{aligned}$$

Thus,

$$\begin{aligned} (e^{-itB_\varepsilon} - e^{-itB^0})(B^0)^{-3/2} &= -e^{-itB_\varepsilon} (B_\varepsilon^{-1/2} - (B^0)^{-1/2})(B^0)^{-1} \\ &+ (B_\varepsilon^{-1/2} - (B^0)^{-1/2})(B^0)^{-1} e^{-itB^0} + i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \mathcal{R}_1(\varepsilon) e^{i(s-t)B^0} \, ds. \end{aligned} \tag{5.2}$$

Let us consider the last summand in the right-hand side of (5.2). According to (1.53) and (1.54),

$$\begin{aligned} i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \mathcal{R}_1(\varepsilon) e^{i(s-t)B^0} \, ds &= i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \mathcal{R}_2(\varepsilon) e^{i(s-t)B^0} \, ds \\ &+ i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \varepsilon K(\varepsilon) e^{i(s-t)B^0} \, ds. \end{aligned} \tag{5.3}$$

By (1.56),

$$\Big\| \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \mathcal{R}_2(\varepsilon) e^{i(s-t)B^0}\, ds \Big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_5 C_2 \varepsilon |t|. \tag{5.4}$$

Integrating by parts and taking (1.51) into account, we rewrite the last summand in the right-hand side of (5.3) as follows:

$$\begin{aligned} & i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{1/2} \varepsilon K(\varepsilon) e^{i(s-t)B^0}\, ds = -\int_0^t \frac{de^{-isB_\varepsilon}}{ds} B_\varepsilon^{-1/2} \varepsilon K(\varepsilon) e^{i(s-t)B^0}\, ds \\ & = -e^{-itB_\varepsilon} B_\varepsilon^{-1/2} \varepsilon K(\varepsilon) + B_\varepsilon^{-1/2} \varepsilon K(\varepsilon) e^{-itB^0} \\ & + i \int_0^t e^{-isB_\varepsilon} B_\varepsilon^{-1/2} \varepsilon (\Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \Pi_\varepsilon) e^{i(s-t)B^0}\, ds. \end{aligned} \tag{5.5}$$

By (1.52) and (2.64),

$$\| - e^{-itB_\varepsilon} B_\varepsilon^{-1/2} \varepsilon K(\varepsilon) + B_\varepsilon^{-1/2} \varepsilon K(\varepsilon) e^{-itB^0} \|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2\varepsilon \beta^{-1/2} C_K. \tag{5.6}$$

We proceed to estimation of the third summand in the right-hand side of (5.5). By (1.2),

$$\begin{aligned} \|b(\mathbf{D})\|_{L_2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} &\leqslant \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{-1/2} |b(\boldsymbol{\xi})| \\ &\leqslant \alpha_1^{1/2} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^{-1/2} |\boldsymbol{\xi}| \leqslant \alpha_1^{1/2}. \end{aligned} \tag{5.7}$$

By (1.20) for a function $\mathbf{u} = B_\varepsilon^{-1/2}\mathbf{v}$, $\mathbf{v} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$,

$$\|(\mathbf{D}^2+I)^{1/2} B_\varepsilon^{-1/2} \mathbf{v}\|_{L_2(\mathbb{R}^d)} = \|B_\varepsilon^{-1/2}\mathbf{v}\|_{H^1(\mathbb{R}^d)} \leqslant c_6 \|\mathbf{v}\|_{L_2(\mathbb{R}^d)}.$$

So,

$$\|B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} = \|(\mathbf{D}^2+I)^{1/2} B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_6.$$

By the duality arguments,

$$\|B_\varepsilon^{-1/2}\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_6. \tag{5.8}$$

Now from Proposition 1.7 and (1.28), (5.7), and (5.8) it follows that

$$\begin{aligned} & \|B_\varepsilon^{-1/2} \varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D})\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ & \leqslant \varepsilon \|B_\varepsilon^{-1/2}\|_{H^{-1}(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \|[\Lambda^\varepsilon]\Pi_\varepsilon\|_{H^{-1}(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \|b(\mathbf{D})\|_{L_2(\mathbb{R}^d)\to H^{-1}(\mathbb{R}^d)} \\ & \leqslant \varepsilon c_6 \alpha_1^{1/2} M_1. \end{aligned} \tag{5.9}$$

By Proposition 1.7 and (1.34), (2.64),

$$\|B_\varepsilon^{-1/2}\varepsilon[\widetilde{\Lambda}^\varepsilon]\Pi_\varepsilon\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant \varepsilon\beta^{-1/2}\widetilde{M}_1. \tag{5.10}$$

From (5.9) and (5.10) it follows that

$$\begin{aligned}\Big\| i\int_0^t e^{-isB_\varepsilon}B_\varepsilon^{-1/2}\varepsilon(\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})+\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{i(s-t)B^0}\,ds\Big\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ \leqslant (c_6\alpha_1^{1/2}M_1+\beta^{-1/2}\widetilde{M}_1)\varepsilon|t|.\end{aligned} \tag{5.11}$$

It remains to estimate the first two summands in the right-hand side of (5.2). Using (4.20), (5.2)–(5.6), and (5.11), we arrive at

$$\|(e^{-itB_\varepsilon}-e^{-itB^0})(B^0)^{-3/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_{33}\varepsilon(1+|t|),$$

where

$$C_{33} := \max\{4\pi^{-1}(\check{c}^{-1}+1)C_1+2\beta^{-1/2}C_K; c_5C_2+c_6\alpha_1^{1/2}M_1+\beta^{-1/2}\widetilde{M}_1\}.$$

Together with (1.47) this implies that

$$\|e^{-itB_\varepsilon}-e^{-itB^0}\|_{H^3(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant C_{33}C_L^{3/2}\varepsilon(1+|t|). \tag{5.12}$$

Interpolating between the rough estimate

$$\|e^{-itB_\varepsilon}-e^{-itB^0}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant 2$$

and (5.12), we derive the required estimate (5.1) with the constant $C_{32} := 2^{1-r/3}C_{33}^{r/3}C_L^{r/2}$. $\square$

5.2 Approximation of the operator $e^{-itB_\varepsilon}B_\varepsilon^{-1}$ in the energy norm

As for hyperbolic problems, we can give approximation for the solution of the non-stationary Schrödinger equation in the energy norm only for the very specific choice of the initial data. In operator terms, we deal with the operator $e^{-itB_\varepsilon}B_\varepsilon^{-1}$. We give the corresponding approximation for completeness of the presentation.

Theorem 5.2. *Let the assumptions of Subsections* 1.1–1.6 *be satisfied. Denote*

$$\mathcal{K}_3(\varepsilon;t) := (\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D})+\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{-itB^0}(B^0)^{-1}. \tag{5.13}$$

Let $0\leqslant r\leqslant 2$. *Then for* $0<\varepsilon\leqslant 1$ *and* $t\in\mathbb{R}$ *we have*

$$\|e^{-itB_\varepsilon}B_\varepsilon^{-1}-e^{-itB^0}(B^0)^{-1}-\varepsilon\mathcal{K}_3(\varepsilon;t)\|_{H^r(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_{34}\varepsilon^{r/2}(1+|t|)^{r/2}. \tag{5.14}$$

Here the constant C_{34} *depends only on* r *and the problem data* (1.24).

Remark 5.3. *Lemma* 4.5 *together with Theorem* 5.2 *allows us to obtain approximation for the operator* $e^{-itB_\varepsilon}B_\varepsilon^{-1/2}$ *in the* $(H^3 \to H^1)$*-norm with the error estimate of the form* $C\varepsilon(1+|t|)$.

Remark 5.4. *Using properties of the matrix-valued functions* Λ *and* $\widetilde{\Lambda}$ (*see Subsection* 3.2*, Lemma* 3.12*, and Lemma* 6.5 *from* [MSu2]), *it is always possible to remove* Π_ε *from the corrector* (5.13) *for* $d = 1, 2$. *If* $3 \leqslant d \leqslant 6$, *it is possible under the additional restriction on* r: $d/2 - 1 \leqslant r \leqslant 2$. *If Conditions* 3.1 *and* 3.3 *hold simultaneously, it is also possible to remove* Π_ε *from the corrector.*

Proof of Theorem 5.2. Denote

$$\Psi(t) := e^{-itB_\varepsilon}B_\varepsilon^{-1}((B^0)^{-1} + \varepsilon K(\varepsilon))e^{itB^0}.$$

Then, according to (1.51) and (1.54),

$$\frac{d\Psi(t)}{dt} = ie^{-itB_\varepsilon}\mathcal{R}_2(\varepsilon)e^{itB^0} + ie^{-itB_\varepsilon}B_\varepsilon^{-1}(\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{itB^0}.$$

So,

$$\begin{aligned}
&\Psi(t) - \Psi(0) = e^{-itB_\varepsilon}B_\varepsilon^{-1}((B^0)^{-1} + \varepsilon K(\varepsilon))e^{itB^0} - B_\varepsilon^{-1}((B^0)^{-1} + \varepsilon K(\varepsilon)) \\
&= i\int_0^t e^{-isB_\varepsilon}\mathcal{R}_2(\varepsilon)e^{isB^0}\,ds + i\int_0^t e^{-isB_\varepsilon}B_\varepsilon^{-1}(\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{isB^0}\,ds.
\end{aligned}$$

Multiplying this identity by e^{-itB^0} from the right, we obtain

$$\begin{aligned}
&e^{-itB_\varepsilon}B_\varepsilon^{-1}(B^0)^{-1} - B_\varepsilon^{-1}((B^0)^{-1} + \varepsilon K(\varepsilon))e^{-itB^0} \\
&= -e^{-itB_\varepsilon}B_\varepsilon^{-1}\varepsilon K(\varepsilon) + i\int_0^t e^{-isB_\varepsilon}\mathcal{R}_2(\varepsilon)e^{i(s-t)B^0}\,ds \\
&+ i\int_0^t e^{-isB_\varepsilon}B_\varepsilon^{-1}(\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{i(s-t)B^0}\,ds.
\end{aligned}$$

Together with (1.54) and (5.13), this implies

$$\begin{aligned}
&(e^{-itB_\varepsilon}B_\varepsilon^{-1} - e^{-itB^0}(B^0)^{-1} - \varepsilon\mathcal{K}_3(\varepsilon;t))(B^0)^{-1} \\
&= \mathcal{R}_2(\varepsilon)(B^0)^{-1}e^{-itB^0} + B_\varepsilon^{-1}\varepsilon K(\varepsilon)e^{-itB^0} - e^{-itB_\varepsilon}B_\varepsilon^{-1}\varepsilon K(\varepsilon) \\
&+ i\int_0^t e^{-isB_\varepsilon}\mathcal{R}_2(\varepsilon)e^{i(s-t)B^0}\,ds \\
&+ i\int_0^t e^{-isB_\varepsilon}B_\varepsilon^{-1}(\varepsilon\Lambda^\varepsilon\Pi_\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon\Pi_\varepsilon)e^{i(s-t)B^0}\,ds.
\end{aligned} \tag{5.15}$$

Combining (1.42), (1.52), (1.56), (2.64), (5.6), (5.11), and (5.15), we arrive at the estimate

$$\begin{aligned}&\left\| B_\varepsilon^{1/2}\left(e^{-itB_\varepsilon}B_\varepsilon^{-1} - e^{-itB^0}(B^0)^{-1} - \varepsilon\mathcal{K}_3(\varepsilon;t)\right)(B^0)^{-1}\right\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \\ &\leqslant C_{35}\varepsilon(1+|t|), \\ &C_{35} := \max\{c_5C_2\check{c}^{-1} + 2\beta^{-1/2}C_K; c_5C_2 + c_6\alpha_1^{1/2}M_1 + \beta^{-1/2}\widetilde{M}_1\}.\end{aligned} \tag{5.16}$$

Bringing together (1.20), (1.46), and (5.16), we get

$$\|e^{-itB_\varepsilon}B_\varepsilon^{-1} - e^{-itB^0}(B^0)^{-1} - \varepsilon\mathcal{K}_3(\varepsilon;t)\|_{H^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_{36}\varepsilon(1+|t|), \tag{5.17}$$

where $C_{36} := c_6C_{35}C_L$.

By (1.20) and (2.64),

$$\|e^{-itB_\varepsilon}B_\varepsilon^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant c_6\|B_\varepsilon^{-1/2}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \leqslant c_6\beta^{-1/2}. \tag{5.18}$$

Next, according to (1.42),

$$\|e^{-itB^0}(B^0)^{-1}\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant \check{c}^{-1}. \tag{5.19}$$

Finally, by analogy with (2.67) and (2.68),

$$\|\mathcal{K}_3(\varepsilon;t)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant (2\varepsilon M_1 + M_2)\alpha_1^{1/2}\check{c}^{-1} + (2\varepsilon\widetilde{M}_1 + \widetilde{M}_2)\check{c}^{-1/2}. \tag{5.20}$$

Bringing together (5.18)–(5.20), we obtain

$$\|e^{-itB_\varepsilon}B_\varepsilon^{-1} - e^{-itB^0}(B^0)^{-1} - \varepsilon\mathcal{K}_3(\varepsilon;t)\|_{L_2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \leqslant C_{37}, \tag{5.21}$$

where $C_{37} := c_6\beta^{-1/2} + \check{c}^{-1} + (2M_1 + M_2)\alpha_1^{1/2}\check{c}^{-1} + (2\widetilde{M}_1 + \widetilde{M}_2)\check{c}^{-1/2}$. Interpolating between (5.21) and (5.17), we arrive at estimate (5.14) with the constant $C_{34} := C_{37}^{1-r/2}C_{36}^{r/2}$. $\square$

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