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Generalized Markoff equations and Chebyshev polynomials

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ABSTRACT

The Markoff equation is $x^2 + y^2 + z^2 = 3xyz$, and all of the positive integer solutions of this equation occur on one tree generated from $(1, 1, 1)$, called the Markoff tree. In this paper, we consider trees of solutions to $x^2+y^2+z^2 = xyz+A$. We say a tree satisfies the unicity condition if the maximum element of an ordered triple in the tree uniquely determines the other two. The unicity conjecture says that the Markoff tree satisfies the unicity condition. In this paper, we show that there exists a sequence of real numbers $\{c_n\}$ such that each tree generated from $(1, c_n, c_n)$ satisfies the unicity condition, and that these trees converge to the Markoff tree. We accomplish this by recasting polynomial solutions as linear combinations of Chebyshev polynomials, showing that these polynomials are distinct, and evaluating them at certain values.

0. Introduction

In 1879 and 1880 [Mar79,Mar80], A. Markoff used continued fractions to show that there is a one-to-one correspondence between the indefinite quadratic forms with minima

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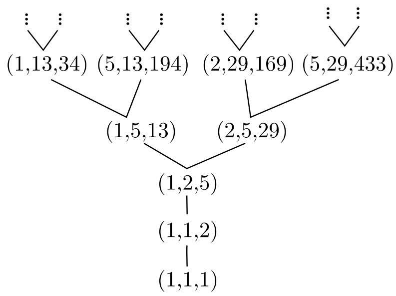

Fig. 1. The Markoff tree $\mathfrak{M}$.

greater than $\frac{1}{3}$ of the square root of its discriminant, and integral solutions to the following equation:

$$x^2 + y^2 + z^2 = 3xyz.$$

This equation is known as the Markoff equation. Many surprising connections have been made between integer solutions of the Markoff equation and various fields of mathematics, such as algebraic number theory, combinatorics, diophantine approximation, and hyperbolic geometry ([Aig13] is an excellent source for details of these connections). There is also a connection between integer solutions of the Markoff equation and exceptional representative sheaves on the complex projective plane $\mathbb{P}^2$ [Rud89]. We call $(x, y, z) \in \mathbb{R}^3$ an *ordered triple* if $x \leq y \leq z$, and we call (x, y, z) a *Markoff triple* if it is an ordered triple solution to the Markoff equation with x, y, and z all positive integers. If (x, y, z) is a Markoff triple then $(x, z, 3xz - y)$ and $(y, z, 3yz - x)$ are Markoff triples as well. Thus, any ordered triple solution creates additional solutions, forming a tree of solutions for the Markoff equation. In particular, we can generate a tree of solutions from $(1, 1, 1)$, which we refer to as the Markoff tree $\mathfrak{M}$, as shown in Fig. 1. Markoff used a method of descent to show that every Markoff triple descends all the way down to $(1, 1, 1)$ in a finite number of steps. Thus, all of the Markoff triples appear in $\mathfrak{M}$.

In 1913, Frobenius conjectured that the largest entry z of a Markoff triple uniquely determines the other two [Fro13]. It is common to say that the maximum element is *unique* when this is the case. This is known as the unicity conjecture and remains unsolved. Some partial results of the unicity conjecture have been settled. It is known that if z or $3z \pm 2$ is a prime, twice a prime, or four times a prime then it is unique [Bar96]; or if z is a prime power then it is unique [But01]. Currently, it is known that z is unique if $z = k \cdot p^\beta$, with $k \leq 10^{35}$, p prime and k relatively prime to p [But01]; and if $3z \pm 2 = k \cdot p^\beta$, with $k \leq 10^{10}$, p prime, and k relatively prime to p [CC13]. The upper bounds for k in these last two results are based on the empirical result that z is unique if $z < 10^{140}$ [Bar96].

There are several generalizations of the Markoff equation, including

$$x^2 + y^2 + z^2 = axyz + b,$$

which Mordell studied [Mor53];

$$\mathcal{M}_{z,n}: \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = zx_1x_2\ldots x_n,$$

which Hurwitz studied [Hur07]; and

$$ax^2 + by^2 + cz^2 = dxyz,$$

with a, b, and c all dividing d, which Rosenberger studied [Ros79]. In this paper, we consider generalized Markoff equations of the form

$$M_A: \qquad x^2 + y^2 + z^2 = xyz + A,$$

where A is any real number. There is a one-to-one correspondence between the equations Mordell looked at and M_{a^2b}. Specifically, (x, y, z) is a solution to $x^2+y^2+z^2 = axyz+b$ if and only if (ax, ay, az) is a solution to M_{a^2b}. In particular, there is a one-to-one correspondence between the Markoff equation and M_0. That is, the triple (x, y, z) is a solution to the Markoff equation if and only if $(3x, 3y, 3z)$ is a solution to M_0.

We define maps τ and σ that generate a tree of solutions corresponding to M_A as follows:

$$\tau(x, y, z) = (x, z, xz - y),$$
$$\sigma(x, y, z) = (y, z, yz - x).$$

Given a triple $\vec{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, we define $\mathfrak{T}(\vec{r})$ to be the tree rooted at $\vec{r}$ and generated by τ and σ. Later, when we consider $\vec{r}$ as a triple of polynomials instead of numbers, we refer to $\mathfrak{T}(\vec{r})$ as a *polynomial tree*. For ease of notation, we use $\mathfrak{T}(\vec{r})$ and $\mathfrak{T}(r_1, r_2, r_3)$ interchangeably (i.e., we use $\mathfrak{T}(r_1, r_2, r_3)$ instead of $\mathfrak{T}((r_1, r_2, r_3))$). We call $\vec{r}$ the *root* of $\mathfrak{T}(\vec{r})$ associated with M_A, and since the root satisfies M_A, we have

$$A = r_1^2 + r_2^2 + r_3^2 - r_1r_2r_3.$$

We emphasize here that we consider $\vec{r}$ and A to be real, not necessarily integral. For example, we consider the tree $\mathfrak{T}(3, \pi, \pi)$ associated with the equation $M_{9-\pi^2}$ in Section 4 after Theorem 4.4. G. McShane and H. Parlier also looked at trees of solutions with real numbers associated with M_0 [MP10]. Theorem 4.4 of this paper is similar to their Theorem 1.3, but our result holds for arbitrary $A \neq 0$, and we use different methods [MP10].

In cases where $\tau(\vec{x}) = \vec{x}, \sigma(\vec{x}) = \vec{x}$, or $\tau(\vec{x}) = \sigma(\vec{x})$ for any $\vec{x}$ in a tree of solutions with root $\vec{r}$, we use the notation $\mathfrak{T}(\vec{r})$ for the tree that collapses the branches that contain repeated triples, and we use the notation $\mathfrak{T}'(\vec{r})$ for the tree that does not (see Fig. 2 for an example).

In Section 1, we show that if $\vec{r}$ is an ordered triple with $2 \leq r_1$ then $\tau(\vec{x})$ and $\sigma(\vec{x})$ are ordered triples whenever $\vec{x}$ is an ordered triple, for every $\vec{x}$ in $\mathfrak{T}(\vec{r})$ (or in $\mathfrak{T}'(\vec{r})$). Hence,

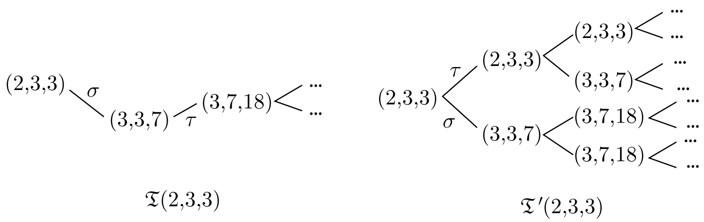

Fig. 2. The trees $\mathfrak{T}(2,3,3)$ and $\mathfrak{T}'(2,3,3)$.

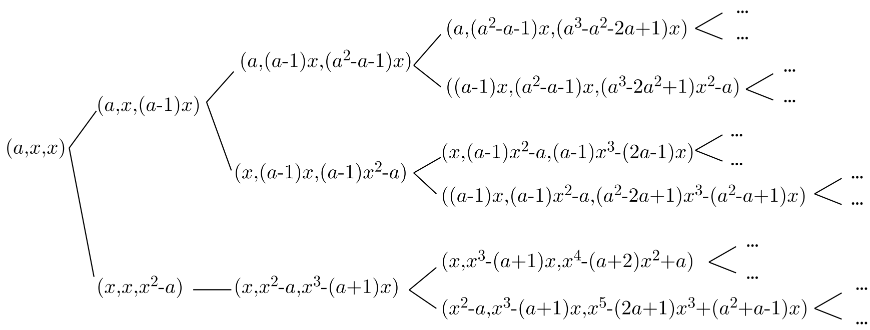

Fig. 3. The polynomial tree rooted at (a, x, x).

we assume from now on that $2 \leq r_1 \leq r_2 \leq r_3$. Of particular interest are polynomial trees rooted at the ordered triple (a, x, x) (see Fig. 3).

Definition 0.1. We say that $\mathfrak{T}(\vec{r})$ satisfies the *unicity condition* if the maximum element of any triple in $\mathfrak{T}(\vec{r})$ uniquely determines the other two.

Remark 1. The unicity conjecture states that the maximum element of a Markoff triple uniquely determines the other two. Since all Markoff triples appear in the Markoff tree, this is equivalent to saying that $\mathfrak{T}(3,3,3)$, satisfies the unicity condition. We emphasize here that it is possible that for some A, the maximal component of an integer solution may not uniquely determine the other two, yet trees of integer solutions for M_A could still satisfy the unicity condition.

One of our main results is the following theorem (especially when $a = c = 3$):

Theorem 4.5. *For any pair of rational numbers (a, c) with $2 < a \leq c$, there exists a sequence of real numbers $\{c_n\}$ such that the sequence of trees $\mathfrak{T}(a, c_n, c_n)$ converges to $\mathfrak{T}(a, c, c)$, and $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition for every n.*

The remainder of this paper will be organized as follows. In Section 1, we show that $\mathfrak{T}(\vec{r})$ is properly ordered. In Section 2, we use the Euclid tree to help describe $\mathfrak{T}(\vec{r})$ as a polynomial tree. In Section 3, we write these polynomials in terms of Chebyshev polynomials, and prove that all of the polynomials are distinct. In particular, we prove the main technical result of this paper, Theorem 3.2, which is used to prove our main result Theorem 4.5 and demonstrate the connection between Chebyshev polynomials and generalized Markoff equations. In Section 4, we mention what it means for a polynomial tree to satisfy the unicity condition (or satisfy the unicity condition up to level N). In this section, we also prove Theorem 4.5 using Theorem 3.2 and a countability argument based on the fact that a polynomial of degree n has at most n roots. Finally, we briefly mention how a sequence of trees converges in Section 5.

1. Proper ordering of trees

Throughout this section, we assume that $2 \le a \le x$, implying (a, x, x) is an ordered triple. Since (x, y, z) branches to $\tau(x, y, z)$ and $\sigma(x, y, z)$, if (a, x, x) is the root of a tree, then (a, x, x) branches to the two triples $(a, x, (a-1)x)$ and $(x, x, x^2 - a)$. It is straightforward to show that if $2 \le a \le x$ then $(a, x, (a-1)x)$ and $(x, x, x^2 - a)$ are both ordered triples. Next, we show that all triples generated from (a, x, x) are ordered triples.

Theorem 1.1. *If $2 \le a \le x$, then every triple in $\mathfrak{T}(a, x, x)$ and $\mathfrak{T}'(a, x, x)$ is an ordered triple.*

Proof. First, assume $2 < a$. Suppose that we have an ordered triple (b, c, d) in $\mathfrak{T}(a, x, x)$ with $b > 2$. If $bd - c \le d$ then $(b-1)d \le c$, which implies

$$\begin{aligned} d &< (b-1)d \qquad (\text{since } b > 2), \\ &\le c \qquad (\text{since } bd - c \le d), \\ &\le d \qquad (\text{since } b \le c \le d), \end{aligned}$$

a contradiction. Thus, $bd - c > d$. Similarly, $cd - b > d$. Hence, all nodes in the tree $\mathfrak{T}(a, x, x)$ are ordered triples.

Now assume $a = 2$. Then $\mathfrak{T}(2, 2, 2)$ only consists of the triple $(2, 2, 2)$ since $(2)(2) - (2) = 2$, and $\mathfrak{T}(2, 2, 2)$ collapses the branches with repeated triples. When $2 = a < x$, we do not branch in the τ direction since $(2)(x) - (x) = x$, and $(2, x, x)$ branches in σ direction to the triple $(x, x, x^2 - 2)$, where $x^2 - 2 > x$ since $x > 2$. Thereafter, our situation is as before and all nodes in the tree $\mathfrak{T}(2, x, x)$ are ordered triples. Therefore, all nodes in the polynomial tree $\mathfrak{T}(a, x, x)$ with $2 \le a \le x$ are ordered triples, and it follows that all nodes in the polynomial tree $\mathfrak{T}'(a, x, x)$ with $2 \le a \le x$ are ordered triples as well. $\square$

Theorem 1.1 shows that polynomials are properly ordered by the maps τ and σ when we evaluate x at values satisfying $2 \le a \le x$ (we emphasize here that x is a variable and a is a constant for these polynomials). It is mentioned after Lemmas 2.1, 2.2 in Section 2 that this is the same as ordering polynomials by their degrees.

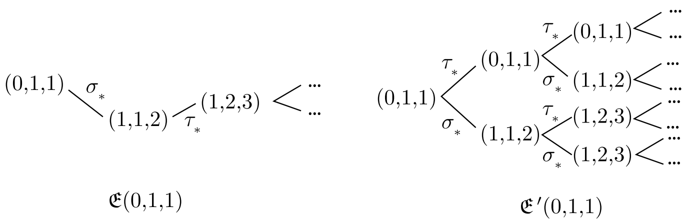

Fig. 4. The Euclid trees $\mathfrak{E}(0, 1, 1)$ and $\mathfrak{E}'(0, 1, 1)$.

2. Euclid trees and polynomial trees

The *Euclid tree* $\mathfrak{E}(\vec{r}_*)$ is the tree rooted at $\vec{r}_* = (r_{*1}, r_{*2}, r_{*3})$ with each r_{*i} a nonnegative integer satisfying $r_{*1} + r_{*2} = r_{*3}$, and defined by the following branching operations:

$$\tau_*(x_*, y_*, z_*) = (x_*, z_*, x_* + z_*),$$
$$\sigma_*(x_*, y_*, z_*) = (y_*, z_*, y_* + z_*).$$

As before, we define $\mathfrak{E}'(\vec{r}_*)$ to be the Euclid tree that does not collapse the branches when $\sigma_*(\vec{x}_*) = \vec{x}_*, \tau_*(\vec{x}_*) = \vec{x}_*$, or $\tau_*(\vec{x}_*) = \sigma_*(\vec{x}_*)$ (see Fig. 4 for an example). The Euclid trees $\mathfrak{E}(1, 2, 3)$ and $\mathfrak{E}'(0, 1, 1)$ are used in Definition 2.3. Note that $\mathfrak{E}(1, 2, 3)$ is a subtree of $\mathfrak{E}'(0, 1, 1)$ (see Fig. 4). It follows from the maps σ_* and τ_* that $x_* + y_* = z_*$ for all $(x_*, y_*, z_*) \in \mathfrak{E}(\vec{r}_*)$. The tree $\mathfrak{E}(\vec{r}_*)$ is called a Euclid tree since descending the tree is equivalent to performing the Euclidean algorithm. Note that the descent in these trees is unique, since the Euclidean algorithm is unique. Of particular interest are Euclid trees with $\gcd(r_{*1}, r_{*2}) = 1$, which guarantees that the Euclidean algorithm only involves relatively prime numbers.

For any composite map $\mu = \sigma^{e_1} \circ \tau^{e_2} \circ \cdots \circ \sigma^{e_k}$ defined on $\mathfrak{T}(\vec{r})$, we define μ_* on $\mathfrak{E}'(\vec{r}_*)$ by $\sigma_*^{e_1} \circ \tau_*^{e_2} \circ \cdots \circ \sigma_*^{e_k}$. Then, for any $\vec{r}$ and $\vec{r}_*$, we define:

$$\Psi : \qquad \mathfrak{T}(\vec{r}) \to \mathfrak{E}'(\vec{r}_*)$$
$$\mu(\vec{r}) \mapsto \mu_*(\vec{r}_*).$$

Note that Ψ depends on $\vec{r}$ and $\vec{r}_*$. Let us consider $\vec{r}$ and $\vec{r}_*$ with $r_{*1} < r_{*2} < r_{*3}$ and $\deg(r_k) = r_{*k}$ for $k = 1, 2, 3$. Under these conditions, it is easy to see that $\mathfrak{E}'(\vec{r}_*) = \mathfrak{E}(\vec{r}_*)$ and Ψ is invertible.

Before proving the next two lemmas, we mention their significance. Lemma 2.1 shows that the triples in $\mathfrak{E}(\vec{r}_*)$ uniquely determine the polynomial triples in $\mathfrak{T}(\vec{r})$. Lemma 2.2 shows that identities (1) and (2) (which appear on page 9) rely on *unique* m and k in their recursions, a fact that we use in Theorem 3.2 in Section 3.

Lemma 2.1. *Let* $r_{*1} < r_{*2} < r_{*3}$ *and* $\deg(r_k) = r_{*k}$ *for* $k = 1, 2, 3$. *Then for every* $\vec{x}$ *in* $\mathfrak{T}(\vec{r})$, *we have* $\Psi(\vec{x}) = (\deg(x_1), \deg(x_2), \deg(x_3))$.

Proof. The result clearly holds for $\vec{x} = \vec{r}$. If we pick any arbitrary nonzero polynomials P_1, P_2, and P_3 such that $\deg(P_1) < \deg(P_2) < \deg(P_3)$, then it is clear that $\deg(P_1P_3 - P_2) = \deg(P_1) + \deg(P_3)$ and $\deg(P_2P_3 - P_1) = \deg(P_2) + \deg(P_3)$. Therefore, it follows inductively from the maps σ, σ_*, τ, and τ_* that $\Psi(\vec{x}) = (\deg(x_1), \deg(x_2), \deg(x_3))$ holds for all $\vec{x}$ in $\mathfrak{T}(\vec{r})$. $\square$

Therefore, Ψ maps ordered triples of polynomials to their degrees, and since Ψ is invertible, the degree triples uniquely determine the polynomial triples.

Lemma 2.2. *For any relatively prime integers n and j with $0 < j < \frac{n}{2}$, there exist exactly two solution pairs of integers (n_1, j_1) and $(n_2, j_2) = (n - n_1, j - j_1)$ with each $n_k < n$ such that $|nj_k - n_kj| = 1$ and $0 < j_k < \frac{n_k}{2}$ holds for $k = 1, 2$.*

Proof. It follows from the Euclidean algorithm and the fact that j has a unique inverse in $\mathbb{Z}_n^\times$ that there exist only two positive integer solutions (n_1, j_1) and (n_2, j_2) satisfying $|nj_k - n_kj| = 1$, one for $nj_k - n_kj = 1$ and one for $nj_k - n_kj = -1$. Then $n_2 = n - n_1$ and $j_2 = j - j_1$ hold because

$$\begin{aligned} |n(j - j_1) - j(n - n_1)| &= |nj - nj_1 - nj + jn_1| \\ &= |nj_1 - jn_1| \\ &= 1. \end{aligned}$$

For each k, if $nj_k - b_kj = \pm 1$ then $nj_k = b_kj \pm 1 < \frac{b_kn}{2} \pm 1$. Thus, we can conclude that $j_k < \frac{b_k}{2}$ for each k. $\square$

Let us fix $\vec{r} = (x, x^2 - a, x^3 - (a + 1)x)$ and $\vec{r_*} = (1, 2, 3)$. From before, we know that the tree $\mathfrak{C}(1, 2, 3)$ (which appears in [Coh79] and [Zag82]) consists of all triples $\vec{x_*} = (x_{*1}, x_{*2}, x_{*3})$ with $\gcd(x_{*1}, x_{*2}) = 1$, $x_{*1} + x_{*2} = x_{*3}$, and $x_{*1} \leq x_{*2} \leq x_{*3}$. Each $\vec{x_*}$ descends (uniquely) to $(1, 2, 3)$. Thus, for any integer $n > 2$, there are $\frac{1}{2}\varphi(n)$ triples $\vec{x_*}$ in $\mathfrak{C}(1, 2, 3)$ so that $x_{*3} = n$, where φ is the Euler totient function (see [Zag82] for details).

We have a bijection Ψ from $\mathfrak{T}(x, x^2 - a, x^3 - (a + 1)x)$ to $\mathfrak{C}(1, 2, 3)$ (see Fig. 5), from which it follows that each $\vec{x}$ in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ with $\deg(x_1) \leq \deg(x_2) \leq \deg(x_3)$ appears exactly once in the tree. Our main goal is to prove Theorem 3.2, which shows that each polynomial appears exactly once in the tree as a maximum element.

Before we continue, we should make a remark about the ordering of polynomial triples. It is natural to assume that polynomials are ordered by their degrees, but by “ordered triple” we mean ordered by magnitude when we plug in a value for x satisfying $2 \leq a \leq x$. By Theorem 1.1 and the previous two lemmas, these orderings are the same.

Definition 2.3. For each polynomial x_3 that appears as a maximum element in $\mathfrak{T}(\vec{r})$ where $\vec{r} = (x, x^2 - a, x^3 - (a + 1)x)$, we assign two parameters n and j in the following

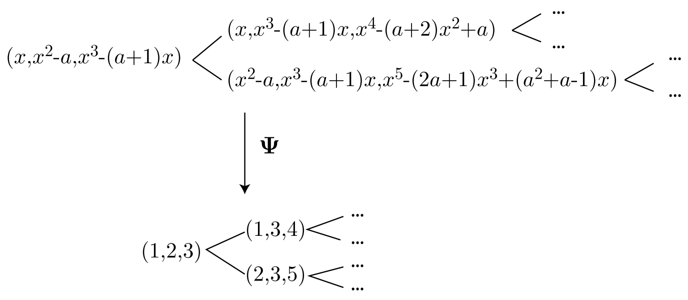

Fig. 5. The map $\Psi : \mathfrak{T}(x, x^2 - a, x^3 - (a+1)x) \to \mathfrak{E}(1, 2, 3)$.

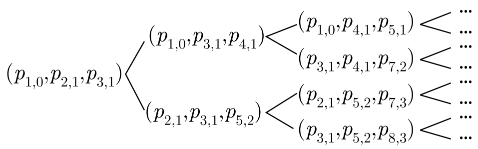

Fig. 6. The tree $\mathfrak{T}(p_{1,0}, p_{2,1}, p_{3,1}) = \mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$.

way. For each $\vec{x}$, there exists a composite map μ (depending on $\vec{x}$) such that $\vec{x} = \mu(\vec{r})$. We let $\Psi(\vec{x}) = \mu_*(1, 2, 3) = (x_{1_*}, x_{2_*}, x_{3_*})$ in $\mathfrak{E}(1, 2, 3)$ and $\mu_*(0, 1, 1) = (y_{1_*}, y_{2_*}, y_{3_*})$ in $\mathfrak{E}'(0, 1, 1)$. Then we let $n = x_{3_*}$ and $j = y_{3_*}$. We now define x_3 as $p_{n,j}$. We also define $p_{1,0} = x$ and $p_{2,1} = x^2 - a$.

The following corollary shows that this definition of $p_{n,j}$ is well-defined. In Section 3, Theorem 3.2 shows that $p_{n,j}$ is unique.

Corollary 2.4. *The pair (n, j) uniquely determines $p_{n,j}$.*

Proof. First, we want to show that j satisfies the properties in Lemma 2.2. Clearly, it does for the triple $(x, x^2 - a, x^3 - (a+1)x)$. Suppose we have a triple (x_1, x_2, x_3) in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ where for $k = 1, 2, 3$, each x_k is associated with parameters (n_k, j_k), and that $n_1 + n_2 = n_3$, $j_1 + j_2 = j_3$, and $n_2 j_1 - n_1 j_2 = \pm 1$. We just need to show that $n_3 j_2 - n_2 j_3 = \pm 1$ holds (with $n_3 j_1 - n_1 j_3 = \pm 1$ being shown in a similar way). So,

$$\begin{aligned} n_3 j_2 - n_2 j_3 &= (n_1 + n_2) j_2 - n_2 (j_1 + j_2) \\ &= n_1 j_2 - n_2 j_1 = \mp 1. \end{aligned}$$

Also, note that if $j_1 < \frac{n_1}{2}$ and $j_2 < \frac{n_2}{2}$ then $j_3 = j_1 + j_2 < \frac{n_1 + n_2}{2} = \frac{n_3}{2}$.

It now follows from previous paragraph and from Lemmas 2.1 and 2.2 that $p_{n,j}(x)$ is well-defined, and when it is a maximum element (with respect to degrees), it uniquely determines $\vec{x}$. □

Thus, $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x) = \mathfrak{T}(p_{1,0}, p_{2,1}, p_{3,1})$, as in Fig. 6. As a result of Lemma 2.2 and Corollary 2.4, and the definitions of the maps τ and σ, we get the following recursion, which holds throughout all of $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$, where m and k are unique and dependent on n and j:

$$p_{n,j} = (p_{m,k})(p_{n-m,j-k}) - (p_{n-2m,j-2k}). \tag{1}$$

Now we take a look at $\mathfrak{T}(a, x, (a-1)x)$. We use $q_{n,j} = q_{n,j}(x)$ to denote the polynomials in this tree. We define the $q_{n,j}$'s in a similar way as the $p_{n,j}$'s. That is to say, if the composite map μ maps $(x, x^2 - a, x^3 - (a+1)x)$ to $(p_{n_1,j_1}, p_{n_2,j_2}, p_{n_3,j_3})$ then μ maps $(a, x, (a-1)x)$ to $(q_{n_1,j_1}, q_{n_2,j_2}, q_{n_3,j_3})$. For example, $q_{1,0} = a$, $q_{2,1} = x$, $q_{5,2} = (a-1)x^2 - a$ and $q_{7,3} = (a-1)x^3 - (2a-1)x$. Then each $q_{n,j}$ occurs exactly once in $\mathfrak{T}(a, x, (a-1)x)$, and the following recursion

$$q_{n,j} = (q_{m,k})(q_{n-m,j-k}) - (q_{n-2m,j-2k}) \tag{2}$$

holds throughout this tree where m and k are unique and dependent on n and j.

Since the degrees of a, x, and $(a-1)x$ are 0, 1, and 1, respectively, $q_{n,j}$ is a polynomial of degree j, not degree n. Also, in $\mathfrak{T}(x, x^2-a, x^3-(a+1)x)$, there are only a finite number of polynomials of degree n for each n, which are all monic, but in $\mathfrak{T}(a, x, (a-1)x)$, there are infinitely many polynomials of degree n for each n which, besides $q_{2,1} = x$, are all not monic when $a > 2$ (this will be shown later in the proof of Theorem 3.2). Since $q_{2,1} = x$ does not represent the maximum element of any triple in $\mathfrak{T}(a, x, (a-1)x)$, all of the polynomials that represent the maximum element of a triple in $\mathfrak{T}(a, x, (a-1)x)$ are not monic. We use the fact that the polynomials in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ are all monic and the maximal polynomials in $\mathfrak{T}(a, x, (a-1)x)$ are all not monic in Theorem 3.2.

Remark 2. There is an alternate way to define the parameters n and j for the Markoff tree $\mathfrak{T}(3, 3, 3)$ using Farey fractions, worth mentioning because Farey fractions were shown to give an ordering of some of the Markoff numbers (see [Fro13] or [Aig13]). For those who are familiar with the Farey fraction indexing of the Markoff numbers, we use the notation that $(1, 2, 5)$ corresponds to $(m_{\frac{0}{1}}, m_{\frac{1}{1}}, m_{\frac{1}{2}})$ [Aig13]. Similarly, in the tree $\mathfrak{T}(3, 3, 3)$, the triple $(3, 6, 15)$ corresponds to $(3m_{\frac{0}{1}}, 3m_{\frac{1}{1}}, 3m_{\frac{1}{2}})$. The polynomials of $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$, when evaluated at $x = a = 3$, satisfy

$$p_{n,j} = 3m_{\frac{j}{n-j}}.$$

This can be easily verified by using induction and identity (1).

3. Chebyshev polynomials

Let us consider triples in $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ of the form $\tau^{k-2}(x, x^2 - a, x^3 - (a+1)x) = (p_{1,0}, p_{k,1}, p_{k+1,1})$. The τ map implies that $p_{k+1,1}(x) = x(p_{k,1}(x)) - p_{k-1,1}(x)$ holds for all $k > 2$. This recursion is similar to the recursion for the Chebyshev polynomials.

Chebyshev polynomials are a sequence of orthogonal polynomials that were first studied by Chebyshev. They have many interesting properties and applications in various branches of mathematics, and Chebyshev polynomials are important in approximation theory because their roots are used as nodes in polynomial interpolation [Riv90]. In the next two paragraphs, we list some important identities and facts about the Chebyshev polynomials that are used to establish identities (3)–(8) (see [Riv90] for details).

The Chebyshev polynomials of the first and second kind, denoted $T_n = T_n(x)$ and $U_n = U_n(x)$, respectively, are defined recursively as follows:

$$\begin{aligned} T_{n+1}(x) &= (2x)T_n(x) - T_{n-1}(x) \qquad (n \geq 1), \qquad \text{and} \\ T_0 &= 1, \qquad T_1 = x, \end{aligned}$$

for the first kind, and

$$\begin{aligned} U_{n+1}(x) &= (2x)U_n(x) - U_{n-1}(x) \qquad (n \geq 1), \qquad \text{and} \\ U_0 &= 1, \qquad U_1 = 2x, \end{aligned}$$

for the second kind. The first few such polynomials with nonnegative indices are:

$$\begin{aligned} T_0 &= 1, & U_0 &= 1, \\ T_1 &= x, & U_1 &= 2x, \\ T_2 &= 2x^2 - 1, & U_2 &= 4x^2 - 1, \\ T_3 &= 4x^3 - 3x, & U_3 &= 8x^3 - 4x, \\ T_4 &= 8x^4 - 8x^2 + 1, & U_4 &= 16x^4 - 12x^2 + 1. \qquad \square \end{aligned}$$

Remark 3. The following characterization shows a connection between Chebyshev polynomials of the second kind and twin primes (see [Yam13] for details). The following statements are equivalent:

(i) n and $n + 2$ are primes,
(ii) $U_n(\frac{x}{2}) + 1$ has exactly two irreducible factors, and
(iii) $U_n(\frac{x}{2}) - 1$ has exactly two irreducible factors.

The Chebyshev polynomials of the first kind satisfy the following for all x:

$$T_n(\cos x) = \cos(nx).$$

The Chebyshev polynomials of the second kind satisfy the following:

$$U_n(\cos x) = \frac{\sin((n + 1)x)}{\sin(x)}.$$

The following identities are derived from these last two identities using the product to sum formulas for cosine and sine:

$$T_jT_k = \frac{1}{2}(T_{j+k} + T_{j-k}),$$

$$T_jU_k = \frac{1}{2}(U_{k+j} + U_{k-j}),$$

$$U_jU_k = \frac{T_{j+k+2} - T_{j-k}}{2(T_1^2 - 1)}.$$

Now, notice that when we fix $a = 1$, we get the following relationship between $p_{n,j}$ and the Chebyshev polynomials of the second kind:

$$p_{1,0}(2x) = 2x = U_1(x), \qquad \text{and}$$

$$p_{2,1}(2x) = 4x^2 - 1 = U_2(x).$$

We can prove $p_{k,1}(2x) = U_k(x)$, for all $k \geq 2$ by induction using identity (1). Also, notice that when we fix $a = 2$, we get the following relationship between $p_{n,j}$ and the Chebyshev polynomials of the first kind:

$$p_{1,0}(2x) = 2x = 2T_1(x), \qquad \text{and}$$

$$p_{2,1}(2x) = 4x^2 - 2 = 2T_2(x).$$

We can prove $p_{k,1}(2x) = 2T_k(x)$, for all $k \geq 2$ by induction using identity (1) as well.

Next, we show that the $p_{n,j}$'s with an arbitrary value of a can always be written in terms of the $p_{n,j}$'s that use the specific values of $a = 1$ and $a = 2$. To help avoid confusion with the parameter a, when $a = 1$ we define

$$u_1 = p_{1,0}, \qquad \text{and}$$

$$u_k = p_{k,1} \qquad (k \geq 2),$$

and when $a = 2$, we define

$$t_1 = p_{1,0}, \qquad \text{and}$$

$$t_k = p_{k,1} \qquad (k \geq 2).$$

We use the notation t_n and u_n for the polynomials at the fixed values of $a = 2$ and $a = 1$, respectively, because of their direct connection to the Chebyshev polynomials T_n and U_n. Hence, we have the following identities (which hold for all j, k):

$$t_j t_k = t_{j+k} + t_{j-k}, \tag{3}$$

$$t_j u_k = u_{k+j} + u_{k-j}, \tag{4}$$

$$u_j u_k = \frac{t_{j+k+2} - t_{j-k}}{t_1^2 - 4}, \tag{5}$$

$$t_{k+1} = t_1 t_k - t_{k-1}, \tag{6}$$

and
$$u_{k+1} = u_1 u_k - u_{k-1}. \tag{7}$$

Notice that the first three identities (3), (4), (5) come from the Chebyshev polynomial identities derived from the product to sum formulas of sine and cosine, and the last two identities come from identities (1) and (2). Let us consider $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ for any real a. We now show that $p_{n,1}$ with the general a can be represented in terms of $p_{n,1}$ with a fixed as 1 or 2. Observe that

$$\begin{aligned}
p_{1,0} &= x \\
&= t_1 - (a-2)u_{-1} \qquad (\text{since } u_{-1} = 0), \\
p_{2,1} &= x^2 - a \\
&= t_2 - (a-2)u_0, \\
p_{3,1} &= x^3 - (a+1)x = t_3 - (a-2)u_1.
\end{aligned}$$

We show that $p_{n,1} = t_n - (a-2)u_{n-2}$, for all n. Assume by induction that we have shown that $p_{n,1} = t_n - (a-2)u_{n-2}$ holds for all n up to k. Then

$$\begin{aligned}
p_{k+1,1} &= (p_{k,1})(p_{1,0}) - p_{k-1,1} \qquad (\text{by (1)}) \\
&= (t_k - (a-2)u_{k-2})(t_1) - (t_{k-1} - (a-2)u_{k-3}) \\
&\qquad\qquad (\text{by the induction hypothesis}) \\
&= (t_1 t_k - t_{k-1}) - (a-2)(u_1 u_{k-2} - u_{k-3}) \qquad (\text{since } t_1 = u_1) \\
&= t_{k+1} - (a-2)u_{k-1} \qquad (\text{by (6) and (7)}).
\end{aligned}$$

Therefore, we have obtained the following result for $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$:

$$p_{n,1} = t_n - (a-2)u_{n-2}, \qquad n \geq 2. \tag{8}$$

Hence, for every real number a, we can represent all the polynomials as linear combinations of Chebyshev polynomials. In a similar way, we can show by induction that the polynomials in $\mathfrak{T}(a, x, (a-1)x)$ satisfy

$$q_{n,1} = (u_{n-2}(a) - u_{n-3}(a))x = (u_{n-2}(a) - u_{n-3}(a))t_1(x).$$

We emphasize here that the t_n's and u_n's are polynomials of the variable x, and that the expression $u_{n-2}(a) - u_{n-3}(a)$ is just the coefficient of $t_1 = x$. For example,

$$\begin{aligned}q_{5,1}(x) &= (a^3 - a^2 - 2a + 1)x \\ &= (u_3(a) - u_2(a))x \\ &= (u_3(a) - u_2)(a))t_1(x),\end{aligned}$$

and it is still a polynomial of degree 1. Therefore, we can represent all the polynomials of $\mathfrak{T}(a, x, (a-1)x)$ as linear combinations of Chebyshev polynomials as well.

Our next result looks at M_4, which Zagier showed fails the unicity condition as part of his analysis of calculating the number of Markoff numbers below a certain bound (M_4 appears as Eq. 13 of [Zag82]).

Lemma 3.1. *The equation M_4 fails the unicity conjecture infinitely many times.*

Proof. By observation, we see that $(2, x, x)$ is a solution for any real x. When $x = 2$, the entire tree collapses to the only solution of $(2, 2, 2)$. For $x > 2$, the root $(2, x, x)$ moves in σ direction to $(x, x, x^2 - 2)$ which moves to $\mathfrak{T}(t_1, t_2, t_3)$. From before when $a = 2$, the polynomials of $\mathfrak{T}(t_1, t_2, t_3)$ satisfy $p_{n,1} = t_n$, for all $n \geq 1$. Then

$$\begin{aligned}p_{n,2} &= p_{2k+1,2} \\ &= p_{k,1}p_{k+1,1} - p_{1,0} \qquad \text{(by (1))} \\ &= t_k t_{k+1} - t_1 \\ &= t_{2k+1} \qquad \text{(by (6))}.\end{aligned}$$

Assume $p_{n,j} = t_n$ holds for all n and for all $j < J$. Then, for any n, there exists a unique pair of integers (m, j) with $j < J$ such that

$$\begin{aligned}p_{n,J} &= p_{m,j}p_{n-m,J-j} - p_{n-2m,J-2j} \qquad \text{(by (1))} \\ &= t_m t_{n-m} - t_{n-2m} \qquad \text{(by the induction hypothesis)} \\ &= t_n \qquad \text{(by (6))}.\end{aligned}$$

Since n was arbitrary, $p_{n,j} = t_n$, for all n and j. Hence, the $\frac{1}{2}\varphi(n)$ polynomials of degree n that appear (in each tree) are the same, and the result follows. □

Remark 4. Since M_4 has (t_j, t_k, t_{j+k}) as a solution for all real x and any relatively prime integers j, k, we obtain the following identity:

$$T_j(x)^2 + T_k(x)^2 + T_{j+k}(x)^2 = 2T_j(x)T_k(x)T_{j+k}(x) + 1.$$

Since each $t_n(2\cos\theta) = 2\cos(n\theta)$, we also obtain the following trigonometric identity:

$$\cos(j\theta)^2 + \cos(k\theta)^2 + \cos((j+k)\theta)^2 = 2\cos(j\theta)\cos(k\theta)\cos((j+k)\theta) + 1,$$

where j, k are relatively prime positive integers. This result seems to be known, and it is easy to verify that this trigonometric identity holds for all real j, k by using the sum addition formula for cosine and the Pythagorean identity. Also, Riedel mentions that M_4 has (t_j, t_k, t_{j+k}) as a solution for all real x on page 9 of [Rie].

It is already known that $U_n = 2(T_n + T_{n-2} + \cdots + T_1)$ when n is odd, and $U_n = 2(T_n + T_{n-2} + \cdots + T_2) + 1$ when n is even (see page 9 of [Riv90]). Hence, $u_n = t_n + t_{n-2} + \cdots + t_1$ when n is odd, and $u_n = t_n + t_{n-2} + \cdots + 1$ when n is even. Thus, identity (8) can be written as

$$p_{n,1} = t_n - (a-2)(t_{n-2} + t_{n-4} + \cdots). \tag{9}$$

Identity (9) is used in the proof of the next theorem, which is the main technical result of this paper.

Theorem 3.2. *All entries in the polynomial tree $\mathfrak{T}(a, x, x)$ are distinct when $a > 2$. More specifically, any two polynomials of degree n from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ differ by a polynomial whose degree is exactly $n - 2$; any two polynomials of degree n from $\mathfrak{T}(a, x, (a-1)x)$ differ by a polynomial whose degree is exactly n; and any two polynomials of degree n with one from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ and one from $\mathfrak{T}(a, x, (a-1)x)$ differ by a polynomial whose degree is exactly n.*

Proof. When $a > 2$, we do not get trees associated with $A = 4$, which does not have distinct polynomials by Lemma 3.1. For simplicity, let $\alpha = a - 2$. It is clear that polynomials of different degrees are different, so we only need to show that polynomials of the same degree are different.

First, we look at $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$. Notice that

$$\begin{aligned} p_{2k+1,2} &= p_{k,1}p_{k+1,1} - p_{1,0} \qquad \text{(by (1))} \\ &= (t_k - \alpha t_{k-2} - \langle * \rangle)(t_{k+1} - \alpha t_{k-1} - \langle * \rangle) - t_1 \\ &\qquad \text{(by identity (9))} \\ &= t_k t_{k+1} - \alpha(t_k t_{k-1} + t_{k+1} t_{k-2}) - \langle * \rangle \\ &= t_{2k+1} - 2\alpha t_{2k-1} - \langle * \rangle, \end{aligned}$$

where $\langle * \rangle$ denotes lower degree terms. Hence, $p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle$ holds for all n, and for $j = 1, 2$. Assume $p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle$, holds for all n, and for all $j < J$. Then, for any n, there exists a unique pair of integers (m, j) (uniqueness guaranteed by Lemma 2.2) with $j < J$ such that

$$\begin{aligned} p_{n,J} &= p_{m,j}p_{n-m,J-j} - p_{n-2m,J-2j} \qquad \text{(by (1))} \\ &= (t_m - j\alpha t_{m-2} - \langle * \rangle)(t_{n-m} - (J-j)\alpha t_{n-m-2} - \langle * \rangle) \end{aligned}$$

$$
\begin{aligned}
&- (t_{n-2m} - (J-2j)\alpha t_{n-2m-2} - \langle * \rangle) \qquad \text{(by the induction hypothesis)} \\
&= t_m t_{n-m} - \alpha(j t_{m-2} t_{n-m} + (J-j) t_{n-m-2} t_m) - \langle * \rangle \\
&= t_n - J\alpha t_{n-2} - \langle * \rangle.
\end{aligned}
$$

Since n was arbitrary, we obtain the following result:

$$p_{n,j} = t_n - j\alpha t_{n-2} - \langle * \rangle,$$

which holds for all n and j. Therefore, for any $j_1 \neq j_2$, $p_{n,j_1} - p_{n,j_2} = (j_1 - j_2)\alpha t_{n-2} - \langle * \rangle$, which is a polynomial of degree $n-2$. In particular, they are not equal.

Next, we look at $\mathfrak{T}(a, x, (a-1)x)$. Recall that $q_{n,j}$ is a polynomial of degree j not n. We know that $q_{n,1} = (u_{n-2}(a) - u_{n-3}(a))x$. Theorem 1.1 guarantees that $n_1 > n_2$ implies $q_{n_1,1} > q_{n_2,1}$, as long as $a > 2$. Let us refer to the leading coefficient of $q_{n,1}$ as C_n ($C_2 = 1, C_3 = a-1$, etc.). Since $q_{2k+1,2} = q_{k,1}q_{k+1,1} - q_{1,0}$ by identity (2), it follows that the leading term of $q_{2k+1,2}$ is $(C_k C_{k+1})x^2$. More generally, it can be established that the leading term of $q_{n,j}$ is $(C_k^{j-r} C_{k+1}^r)x^j$, where $n = kj + r$, with $0 < r < j$.

We already have mentioned that $n_1 > n_2$ implies that the leading coefficient of $q_{n_1,1}$ is bigger than the leading coefficient of $q_{n_2,1}$. Pick any $j > 1$. Let $n_1 > n_2$. We show that the leading coefficient of $q_{n_1,j}$ is bigger than the leading coefficient of $q_{n_2,j}$. By applying Euclidean division, there exist integers k_1, k_2, r_1, and r_2 such that $n_1 = jk_1 + r_1$ and $n_2 = jk_2 + r_2$.

Case 1: $k_1 = k_2$, $r_1 > r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1} C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_1}^{j-r_2} C_{k_1+1}^{r_2}$. Since

$$C_{k_1}^{j-r_1} C_{k_1+1}^{r_1} = \left(\frac{C_{k_1+1}}{C_{k_1}}\right)^{r_1-r_2} C_{k_1}^{j-r_2} C_{k_1+1}^{r_2}$$

and

$$\left(\frac{C_{k_1+1}}{C_{k_1}}\right)^{r_1-r_2} > 1,$$

the result follows for this case.

Case 2: $k_1 > k_2$, $r_1 \geq r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1} C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_2}^{j-r_2} C_{k_2+1}^{r_2}$. Since $k_1 > k_2$ implies $C_{k_1} > C_{k_2}$, we get:

$$
\begin{aligned}
C_{k_1}^{j-r_1} C_{k_1+1}^{r_1} &> C_{k_2}^{j-r_1} C_{k_2+1}^{r_1} \\
&\geq C_{k_2}^{j-r_2} C_{k_2+1}^{r_2} \qquad \text{(by case 1).}
\end{aligned}
$$

Thus, the result follows in this case.

Case 3: $k_1 - k_2 = 1$, $r_1 = 1$, $r_2 = j-1$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-1} C_{k_1+1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_1-1} C_{k_1}^{j-1}$. Since $C_{k_1+1} > C_{k_1-1}$, the result follows in this case.

Case 4: $k_1 > k_2$, $r_1 < r_2$. The leading coefficient of $q_{n_1,j}$ is $C_{k_1}^{j-r_1}C_{k_1+1}^{r_1}$, and the leading coefficient of $q_{n_2,j}$ is $C_{k_2}^{j-r_2}C_{k_2+1}^{r_2}$. Then

$$\begin{aligned} C_{k_1}^{j-r_1}C_{k_1+1}^{r_1} &\geq C_{k_2+1}^{j-r_1}C_{k_2+2}^{r_1} && \text{(by case 2, since } k_1-k_2 \geq 1) \\ &\geq C_{k_2+1}^{j-1}C_{k_2+2} && \text{(by case 1, since } r_1 \geq 1) \\ &> C_{k_2}C_{k_2+1}^{j-1} && \text{(by case 3)} \\ &\geq C_{k_2}^{j-r_2}C_{k_2+1}^{r_2} && \text{(by case 1, since } j-1 \geq r_2). \end{aligned}$$

Thus, the result follows in this case. These four cases represent all possibilities. Hence, all polynomials are distinct in $\mathfrak{T}(a, x, (a-1)x)$. Furthermore, when two polynomials have the same degree n, their leading coefficients are different, and therefore, their difference is a polynomial of degree n.

We now know that the maximal polynomials are monic polynomials in $\mathfrak{T}(x, x^2 - a, x^3 - (a + 1)x)$ but not in $\mathfrak{T}(a, x, (a - 1)x)$ (as long as $a > 2$), so it is clear that any two polynomials of degree n with one from $\mathfrak{T}(x, x^2 - a, x^3 - (a+1)x)$ and the other from $\mathfrak{T}(a, x, (a - 1)x)$ are distinct and have a difference that is exactly degree n. □

The previous two results show that when $a = 2$, the polynomials of the same degree are all the same, but when $a > 2$, all the polynomials are different (we also know the exact degree when we subtract any two polynomials of the same degree). However, this does not imply that in the latter case, the polynomials are all distinct at a specific value, which is what the unicity condition implies. However, it suggests that there is something special about M_4 and why it should fail the unicity condition more significantly than any other parameters.

4. Satisfying the unicity condition for infinitely many parameters (up to level N)

Suppose for fixed a that there exist two different polynomials in $\mathfrak{T}(a, c, c)$, say p_{n_1,j_1} and p_{n_2,j_2}, such that $p_{n_1,j_1}(c) = p_{n_2,j_2}(c) = m$, for some real numbers m and c with $m \geq c \geq a$. This is equivalent to the tree $\mathfrak{T}(a, c, c)$ failing the unicity condition, because m is no longer unique as a maximum element in $\mathfrak{T}(a, c, c)$ (note that we are using the same a, c, and m as in previous sentence). More generally, the statement "(a_i, b_i, m) is an ordered triple solution of $\mathfrak{T}(a, c, c)$ for $i = 1, 2$ implies $a_1 = a_2, b_1 = b_2$" is equivalent to the statement "$(p_{n_1,j_1} - p_{n_2,j_2})(c) \neq 0$, $(p_{n_1,j_1} - q_{n_1,j_1})(c) \neq 0$, $(p_{n_1,j_1} - q_{n_2,j_2})(c) \neq 0$, $(p_{n_2,j_2} - q_{n_1,j_1})(c) \neq 0$, $(p_{n_2,j_2} - q_{n_2,j_2})(c) \neq 0$, and $(q_{n_1,j_1} - q_{n_2,j_2})(c) \neq 0$ in $\mathfrak{T}(a, c, c)$, for all n_1, n_2, j_1, j_2". For the sake of brevity, this last statement can be phrased as "$(\lambda_1 - \lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all n_1, j_1, n_2, j_2. Also, when we say "for all n_1, n_2, j_1, j_2", we mean for all n_1, n_2, j_1, j_2 such that p_{n_1,j_1}, q_{n_1,j_1}, p_{n_2,j_2} and q_{n_2,j_2} represent distinct spots in $\mathfrak{T}(a, c, c)$. Thus, we have the following definition.

Definition 4.1. When $(\lambda_1-\lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all n_1, j_1, n_2, j_2, we say $\mathfrak{T}(a,c,c)$ satisfies the unicity condition, and when $(\lambda_1-\lambda_2)(c) \neq 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all n_1, j_1, n_2, $j_2 \leq N$, we say $\mathfrak{T}(a,c,c)$ satisfies the unicity condition up to level N.

In the following proof, we start with (a,x,x) with a being constant but x being a variable to create the polynomials $p_{n,j}$ and $q_{n,j}$, which are functions of x instead of numbers associated with a specific M_A. Then we find rational numbers q that give us the $\mathfrak{T}(a,q,q)$'s satisfying the unicity condition up to level N, and then we find the parameters associated with $\mathfrak{T}(a,q,q)$; i.e., we want $A = a^2+2q^2-aq^2$. Notice that if a is rational, then the maps τ and σ always produce polynomials with rational coefficients in $\mathfrak{T}(a,x,x)$, because the coefficients are just products and differences of rational numbers. Hence, if a is rational, then $\mathfrak{T}(a,c,c)$ contains polynomials (of c now) with rational coefficients, even if A and c are irrational. This fact is used after Theorem 4.4.

Theorem 4.2. *Given any rational number $a > 2$, any positive integer N, and any subset X of $[a,\infty)$ that contains infinitely many rational numbers, there exist infinitely many rational numbers $q \in X$ such that $\mathfrak{T}(a,q,q)$ satisfies the unicity condition up to level N.*

Proof. For any $a > 2$, the polynomials are all distinct by Theorem 3.2. Let $F_{n_1,n_2,j_1,j_2} = \{x \in X\colon\ (\lambda_1-\lambda_2)(c) = 0$ for each distinct pair λ_1, λ_2 in $\{p_{n_1,j_1}, p_{n_2,j_2}, q_{n_1,j_1}, q_{n_2,j_2}\}$, for all $n_1, j_1, n_2, j_2\}$. Since p_{n_1,j_1} and p_{n_2,j_2} are distinct polynomials, their difference is a nonzero polynomial of degree at most the maximum of n_1 and n_2. Hence, $p_{n_1,j_1} = p_{n_2,j_2}$ for at most $\max\{n_1,n_2\}$ real numbers. Using a similar argument for the other five polynomial differences, the set F_{n_1,n_2,j_1,j_2} contains only a finite number of points in X. Since $j_1 < \frac{n_1}{2} \leq \frac{N}{2}$ and $j_2 < \frac{n_2}{2} \leq \frac{N}{2}$, the collection of F_{n_1,n_2,j_1,j_2} for all positive integers n_1, n_2, j_1, j_2 with $n_1, n_2 \leq N$ is finite. Therefore, the union of all F_{n_1,n_2,j_1,j_2} must be a finite subset in X. Thus, $\mathfrak{T}(a,q,q)$ satisfies the unicity condition up to level N, for infinitely many rational $q \in X$, as desired. $\square$

Remark 5. Let us fix any pair of rational numbers (a,c) with $2 < a \leq c$. As a consequence of Theorem 4.2 (using $X = [c, c+\epsilon]$), given any $\epsilon > 0$, we now know that there are infinitely many rational numbers q in the interval $(c, c+\epsilon)$ such that $\mathfrak{T}(a,q,q)$ satisfies the unicity condition up to level N, for any large N. Specifically, we obtain the following corollary:

Corollary 4.3. *Given any arbitrarily large N and any small $\epsilon > 0$ (each independent of each other), there exist infinitely many rational numbers $q \in [3, 3+\epsilon]$ such that $\mathfrak{T}(3,q,q)$ satisfies the unicity condition up to level N.*

Remark 6. Since all of the zeros of a finite collection of polynomials are clearly bounded, for every N there exists x_N such that for every rational $a > 2$, $\mathfrak{T}(a,c,c)$ satisfies the unicity condition up to level N for every rational number $c > x_N$. If for some a, all of the zeros of all the differences of any two polynomials are uniformly bounded, say by M, then $\mathfrak{T}(a, c, c)$ satisfies the unicity condition for all $c > M$.

Remark 7. The argument in Theorem 4.2 cannot work for $a = 2$ because all of the polynomials of the same degree are the same in M_4. Hence, every real number satisfies $(p_{n,j_1} - p_{n,j_2})(x) = 0$, so $F_{n,n,j_1,j_2} = X$, instead of a finite number of points in X. This once again shows that M_4 is special.

Theorem 4.2 can be expanded to trees $\mathfrak{T}(a, c, c)$ for any real a and any real c.

Theorem 4.4. *Given any real number $a > 2$, the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a, x, x)$ satisfies the unicity condition is the complement of a countable set. In particular, the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a, x, x)$ satisfies the unicity condition is uncountable and dense in $[a, \infty)$.*

Proof. Just like in Theorem 4.2, when $a \neq 2$, the polynomials are all distinct. Therefore, the sets F_{n_1,n_2,j_1,j_2} (which were defined in Theorem 4.2) only contain a finite number of points. When we consider the union of all the F_{n_1,n_2,j_1,j_2}'s, we get a countable union of finite sets. Therefore, the complement of this union, which is the set of all real numbers $x \geq a$ such that $\mathfrak{T}(a, x, x)$ satisfies the unicity condition, is the complement of a countable set, and in particular, is uncountable and dense in $[a, \infty)$, as desired. □

Theorem 4.4 shows that for the ordered pairs (a, c) with $2 < a \leq c$, $\mathfrak{T}(a, c, c)$ satisfies the unicity condition for almost every c, i.e., for all but a set of measure zero (in Theorem 1.3 of [MP10], they use the phrase "Baire dense"). However, the existence of a tree $\mathfrak{T}(a, c, c)$ with a and c both rational or algebraic is still an open problem. When a is rational, all of the polynomials in $\mathfrak{T}(a, c, c)$ have rational coefficients. Hence, as a consequence of Theorem 4.4, $\mathfrak{T}(3, \omega, \omega)$ satisfies the unicity condition for every transcendental number ω (note the similarity between this last statement and the last sentence of [MP10]). For example, in $M_{9-\pi^2}$, $\mathfrak{T}(3, \pi, \pi)$ satisfies the unicity condition since π is transcendental. Since transcendental numbers are dense among the real numbers, we now know that there exists a sequence of real numbers $\{c_n\}$ such that $c_n \to 3$ as $n \to \infty$ and $\mathfrak{T}(3, c_n, c_n)$ satisfies the unicity condition for all n. The concept of $c_n \to 3$ as $n \to \infty$ is related to the concept of the sequence of trees $\mathfrak{T}(3, c_n, c_n)$ converging to $\mathfrak{T}(3, 3, 3)$, which we define in Section 5. More generally, we obtain the following theorem with our main result as a consequence:

Theorem 4.5. *For any pair of rational numbers (a, c) with $2 < a \leq c$, there exists a sequence of real numbers $\{c_n\}$ such that the sequence of trees $\mathfrak{T}(a, c_n, c_n)$ converges to $\mathfrak{T}(a, c, c)$, and $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition for every n.*

Note that Theorem 4.5 does not mention whether $\mathfrak{T}(a, c, c)$ satisfies the unicity condition or not. Hence, Theorem 4.5 even holds for pairs of rational numbers (a, c) such that $\mathfrak{T}(a, c, c)$ fails the unicity condition.

5. Convergence of trees

In this section, we explain what we mean by convergence of a sequence of trees. Let $\{c_n\}$ be a sequence of real numbers such that $\mathfrak{T}(a, c_n, c_n)$ satisfies the unicity condition (or satisfies the unicity condition up to level N) for all n. Let $c_n \to c$ as $n \to \infty$ in the usual sense, i.e., for all $\epsilon > 0$, there exists $N(\epsilon) = N > 0$ such that if $n > N$ then $|c_n - c| < \epsilon$.

All polynomial functions are continuous, so each $p_{m,j}$ and $q_{m,j}$ that appear in $\mathfrak{T}(a, c, c)$ is continuous. Notice that each polynomial occurs in the same spot of every tree, regardless of parameters. For example, $p_{5,2} = x^5 - (2a + 1)x^3 + (a^2 + a - 1)x$ occurs as the maximum element of a triple at $\tau(x, x^2 - a, x^3 - (a + 1)x)$, for each $\mathfrak{T}(a, c_n, c_n)$ and for $\mathfrak{T}(a, c, c)$. Hence, we have the following definition.

Definition 5.1. We say that the sequence of the trees $\mathfrak{T}(a, c_n, c_n)$ converges to the tree $\mathfrak{T}(a, c, c)$, if for each $p_{m,j}$ and $q_{m,j}$, $p_{m,j}(c_n)$ converges to $p_{m,j}(c)$ and $q_{m,j}(c_n)$ converges to $q_{m,j}(c)$. We let $p_{m,j}(c_n)$ converge to $p_{m,j}(c)$ in the usual sense, i.e., for every $\epsilon > 0$, there exists $M > 0$ (where $M = M(p_{m,j})$ depends on the polynomial) such that if $n > M$ then $|p_{m,j}(c_n) - p_{m,j}(c)| < \epsilon$.

Since the M's may vary from polynomial to polynomial, we do not necessarily have uniform convergence.

Since all the trees of generalized Markoff equations consist of ordered triples of polynomials, we say $\mathfrak{T}(a, c_n, c_n)$ converges to $\mathfrak{T}(a, c, c)$, when for each $p_{m,j}$, $p_{m,j}(c_n)$ converges to $p_{m,j}(c)$ (and for each $q_{n,j}$ as well).

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