APPROXIMATION OF FUNCTIONS

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Weierstrass first demonstrated (') that any continuous function could be represented, with a given approximation, by a polynomial. I will point out some basic considerations to demonstrate this theorem and some of its consequences.

Let f(x) be a finite and continuous function in an interval (a, b). We can divide the interval (a, b) by the points $x_0 = a, x_1, x_2, ..., x_{n-1}, x_n = b$ such that in each interval (x_i, x_{i+1}) the oscillation of the function is less than a given positive number ϵ

Insert in the curve $y = f(x)$ the polygonal line $A_0A_1, ...A_n$ whose summits have as abscissa $x_0x_1...x_n$; it represents in the interval (a, b) a continuous function $y = \phi(x)$ which differs from $f(x)$ by less than ϵ. $\phi(x)$ is equal to the continuous function ψ_1 represented in the interval (a, b) by the line which carries the side A_0A_1, plus a function a represented by a polygonal line $A_0^{'}A_1^{'}...A_n^{'}$, whose first side $A_0^{'}A_1^{'}$, is on the x axis. ϕ_1 is the sum of two continuous functions ψ_2 and ϕ_2; ψ_2 is zero between x_0 and x_1 and is represented by the line carrying $A_1^{'}A_2^{'}$, between x_1, and x_n, ϕ_2 is represented by a polygonal line $A_0^{"}A_1^{"}...A_n^{"}$; where $A_0^{"}, A_1^{"}, A_2^{"}$ are on the x-axis. Finally, we arrive at

$$\phi = \psi_1 + \psi_2 + ... + \psi_n$$

(') Journal of Liouville, in the year 1886.

ψ_i being a continuous null function between a and x_{i-1}, and represented by a line segment between x_{i-1} and b. If the change of variable is made

$$X = mx + n$$

by appropriately choosing m and n, ψ_i will be defined in a portion (α, β) of the interval (-1, +1) by the relation

$$\psi_i = \kappa(X + |X|)$$

which can be written as

$$\psi_i = \kappa[X + \sqrt{1 + (X^2 - 1)}]$$

If the radical is developed by the binomial formula and $X^2 - 1$ as a variable, we obtain a series of polynomials in X, and consequently converges uniformly in x. The sum of n analogous developments is a uniformly convergent series of polynomials representing $\phi(x)$. Taking a sufficient number of terms in this series we obtain a polynomial $P(x)$ which differs from $\phi(x)$ by less than η, with η being chosen in advance. $P(x)$ differs from $f(x)$ by less than $\epsilon + \eta$, therefore:

I. Given a finite and continuous function in an interval (a, b) we may find a polynomial which, in the whole interval, differs from it by less than any positive quantity given in advance.

II. As a first application, Weierstrass develops a continuous function in a series of polynomials as follows: Let ϵ_1, ϵ_2, ... be positive quantities such that the series $\sum \epsilon_n$ is convergent, the series whose sum of the first n terms is a polynomial P_n, which differs from the given continuous function f(x) by less than ϵ_n converges uniformly to f(x). In addition, this series is absolutely convergent because one has

$$|u_n| = |P_n - P_{n-1}| \leq |P_n - f| + |P_{n-1} - f| \leq \epsilon_n + \epsilon_{n-1}$$

III. Another consequence of Weierstrass's theorem is that any series of continuous functions in an interval (a, b) can be replaced by a series of polynomials. Indeed, ϵ_1, ϵ_2, ... being positive numbers tending towards zero, the series of polynomials whose sum of the first n terms differs from the sum of the first n terms of the proposed series of less than ϵ_n answer to the question.

Functions with points of discontinuities. - By taking the derivative, term by term, of the series of polynomials representing

$$x + |x|$$

we obtain a series of convergent polynomials in the interval (-1, +1) at point O, representing 0 for negative x, 1 for positive x. By the addition of such series to a series of polynomials representing a continuous function, we develop a series of polynomials of any function having a finite number of discontinuities. The series obtained is divergent at the points of discontinuities. By a different process we will arrive at a more general conclusion.

Let f(x) be a function which in the interval (a, b) has discontinuities for a denumerable set of values

$$x_0 = a, \quad x_1 = b, \quad x_2, \quad x_3, ...$$

Let us denote the points $A_0, A_1, ..., A_n$ representing the function y = f(x) for $x = x_0, x_1, ..., x_n$. Some of these points may be infinite. Let x_k be one of the n values considered for x, and x_l is one of these n values which is immediately superior to it.

1. Suppose that A_k and A_l have a finite distance.

A. Between x_k and x_l there is no interval where the function is continuous, or there are several; Let us trace the segment $A_k A_l$.

B. Between x_k and x_l there exists an interval (x', x") and only one where f(x) is continuous. In this interval the function is represented by a arc of the curve $\alpha\beta$

a. f(x) is continuous on the right for x = x' and on the left for x = x"; draw the segment $A_k\alpha$, the arc $\alpha\beta$, the segment βA_l; the segments $A_k\alpha$, βA_l may be void.

b. f(x) is not continuous on the right for x = x' nor on the left for x = x", draw the Segment $A_k P_n$, the arc $P_n Q_n$, the segment $Q_n A_l$; P_n and Q_n, being two points of the arc $\alpha\beta$ which tend respectively to α and β when n grows indefinitely.

c. f(x) is continuous on the right for x = x' and is not continuous at left for x = x", start the plot as in (a), let us terminate it as in case (b).

d. f(x) is not right-handed for x = x 'and is continuous at left for x = x", let's start the plot as in (b), let's finish it as in (a).

2. If one of the two points A_k, A_l, or both, are at infinity, we will replace that of those two points which are at infinity by points of the same abscissa and whose ordinate will increase with n.

In all cases, we draw in the interval (a, b) a curve representing a function $\phi_n(x)$, we can even assume that the function f is infinite in a certain number, finite or not, of intervals, provided that the ends of these intervals as points of discontinuities and to make the role of the curve $\alpha\beta$ at a parallel to Ox whose ordinate will increase indefinitely with n.

The function $\phi_n(x)$ has as limit for n infinity f(x), by $\phi_n(X)$ tends to $f_n(X)$ when X is taken arbitrarily in (a, b). This is obvious if X is a discontinuity value, or belongs to an interval where the function is continuous, or is end of such an interval. For another value, X is the limit of a sequence $x^{(1)}, x^{(2)}$... of discontinuity values, f(X) is continuous for x = X, f(X) is the limit of the sequence $f(x^{(1)})f(x^{(2)})$,... On the other hand, for n sufficiently large $\phi_n(x)$ is between $f(\alpha)$ and $f(\beta)$, α and β being the ends of that of the intervals, obtained with the help of the subdivision points $x_0, x_1, ..., x_n$, which contains X, whence

$$\lim_{n\to\infty} \phi_n(x) = \lim_{p\to\infty} f(x^{(p)}) = f(X)$$

So

$$f(x) = \phi_1(x) + \sum [\phi_n(x) - \phi_{n-n}(x)]$$

f(x) is representable by a series of continuous functions and, consequently, by a series of polynomials.

IV. *Any function continuous in an interval (a, b), except for one set of values of the variable, can be developed in this interval in series of polynomials, absolutely* *and uniformly convergent in any interval where there are no points of discontinuities.*

It is easy to give examples of functions having an infinity of discontinuities.

First example. - Between $\frac{1}{n}$ and $\frac{1}{n+1}$, the function $f(x)$ is equal to any continuous function A_n. At the points of discontinuities, it has any finite or infinite value. A_n will be limited or not; this will be, for example, $A_n = \frac{1}{x - \frac{1}{n}}$

Second example. - $x_1, x_2, ...$ given, we consider a function zero if x is not one of the given values, and equal to $\frac{1}{n}$ for $x = x_n$. The points $x_1, x_2, ...$ are the only points of discontinuities.

There are therefore, in particular, series of null polynomials for all commensurable (or algebraic) values of an interval and for those only.

Third example. - functions of limited variation of Mr. Jordan.

In the above, we did not get the most general representable by series of polynomials; Mr. Baire has (*Comptes rendus*, 21 March 1898) a necessary and sufficient condition for a function of a variable to be representable by a series of polynomials.

Functions of several variables. - Let f(x, y) be a function of two finite and continuous variables with respect to the set (x, y) for $a \leq x \leq b$; $c \leq y \leq d$ .. Let us try to represent it approximately by a polynomial.

One can find values $x_1 \leq x_2 \leq ... \leq x_n$; $y_1 \leq y_2 \leq ... \leq y_n$ such that $x_i \leq x \leq x_{i+1}$, $y_j \leq y \leq y_{j+1}$, the oscillation of the function is less than a given number ϵ.

Let A_{ij} be the point of coordinates $x = x_i, y = y_j, z = f(x_i, y_j)$. The paraboloid of directional planes: zOx, zOy passing through $A_{i,j}$; $A_{i+1,j}$; $A_{i,j+1}$; $A_{i+1,j+1}$ represents for $x_i \leq x \leq x_{i+1}$; $y_j \leq y \leq y_{j+1}$

a function which differs from f(x) by less than ϵ. However, the function continuously represented by the set of these paraboloid fragments is a sum of continuous functions such as the following:

$$k[XY + |XY| + X|Y| + Y|X|]$$

which can be represented by a uniformly convergent series of polynomials. We finish as before.

The same reasoning succeeds with any number of variables; the geometric image alone is rapidly lacking. It will be necessary to speak, for example, of linear functions with respect to each of the variables where one spoke of right or paraboloid.

The propositions I, II, III are therefore still true when replaces *function of a continuous variable in an interval (a, b)* by *function of several continuous variables with respect to the set in a finite domain.*

Artifices analogous to those which led to the theorem will lead to proposals such as the following:

V. If a function of two variables, defined on a finite domain Ω, connected or not, everywhere continuous in relation to all variables, except at points forming a countable set and on of the curves C (') forming a countable set is such that, on each of the curves C, with respect to a parameter fixing in a manner continues the position of a point on this curve, the function is, except in a countable set of points, it is representable by a series of polynomials, absolutely and uniformly convergent in any domain containing no discontinuities.

We can also assume that the function is infinite on some arcs of curves, C or in some finite or non-finite number of domains The boundary curves of these areas as forming part of the curves C.

(') The word curve is taken in a restricted sense; None of the curves C must pass in the vicinity of all the points of an area.

Functions of several continuous variables with respect to each of them. - M. Baire (*Comptes rendus*, 1897) has shown how could construct functions of several continuous variables by in relation to each of them without being so in relation to the whole. The method used above allows for the development of these functions.

Let f(x, y) be a function defined for $a \leq x \leq b$, $c \leq y \leq d$ continuous separately from x and y. Divide the interval (a, b) into n equal parts; let $x_0 = a, x_1, x_2, ..., x_n - 1, x_n = b$ be the points of division.

Consider the function $\phi_n(x, y)$ continuous with respect to the set xy and defined between x_p and x_{p+1} by

$$\phi_n(x, y) = \frac{f(x_{p+1}, y)(x - x_p) - f(x_p, y)(x - x_{p+1})}{x_{p+1} - x_p}$$

The functions $\phi_n(x, y)$ have f(x, y) for a limit when n grows indefinitely. Indeed, let X, Y be a system of values for x, y. Choose n large enough that in each interval x_i, x_{i+1} the oscillation of the function f(x, Y) is less than ϵ, ϵ being given in advance. If X is in the interval x_p, x_{p+1} the expression

$$|F(X, Y) - \phi(X, Y)|$$

is smaller than the greater of the next two

$$|F(X, Y) - f(x_p, Y)| \qquad |F(X, Y) - f(x_{p+1}, Y)|$$

which are smaller than ϵ.

One can therefore write

$$f(x, y) = \phi_1(x, y) + \sum_{n=1}^{n \to \infty} (\phi_{n+1} - \phi_n)$$

f(x, y) is representable by a series of continuous functions, and, consequently, by a series of polynomials; it may even be observed that this series is uniformly convergent for a given y.

The same reasoning proves that:

VI. *A function of the variables $x_1, x_2, ..., x_n; y_1, y_2, ..., y_p$, continuous with respect to the set of x and with respect to the set of y, is representable in any finite domain by a series of polynomials.*

In the case of more than two variables, one leads to proposals such as the following:

VII. *A function of four variables x, y, z, t, certain domain in relation to each of these variables, is representative in this field:*

1. *By a series of continuous functions with respect to the sets (x, y) (x, z) (x, t)*;

(2) *Or by a series of series of continuous functions with respect to the sets (x, y, z) (x, y, t)*;

(3) *Or by a series of series of polynomial series.*

This proposition subsists if one plays the role of x, y, z, t, has in sets $(x_1, x_2, ..., x_p)$, $(y_1, y_2, ..., y_q)$, $(z_1, z_2, ..., z_r)$, $(t_1, t_2, ..., t_s)$.

The same artifices as before allow us to assume that these functions have discontinuities of the kind encountered in the Theorems IV and V.

Approximations using Fourier sequences. - Weierstrass demonstrated that any continuous function having the period 2π can be represented with such approximation that one wants, by a finite sequence of Fourier or, which is the same thing, by a polynomial in cos(x) and sin(x) (').

M. Picard deduces this proposition from the properties of the integral of Poisson, and concludes the possibility of representing approximately a continuous function by a polynomial.

In many ways, the inversion of the approximation by a polynomial with the approximation by a Fourier sequence.

Let f(x) be a continuous function having the period 2π. We can find a continuous function $\phi(x)$ having the period 2π which differs

(') Mr Volterra (*Sul Principio di Dirichlet (Rendiconti del Circolo Matematico di Palermo*, Volume XI)] demonstrates this proposal by the way: one can approach as much as one wants from a curve to using a polygonal line. Such a line represents a function which, having only a finite number of maxima and minima, can, according to Dirichlet, be developed in Fourier series uniformly convergent.

from f(x) by less than ϵ and which is such that, for α small enough, we have

$$\phi(\alpha) = \phi(2\pi - \alpha), \qquad \phi(\frac{\pi}{2} - \alpha) = \phi(\frac{\pi}{2} + \alpha)$$

$$\phi(\pi - \alpha) = \phi(\pi + \alpha), \qquad \phi(\frac{3\pi}{2} - \alpha) = \phi(\frac{3\pi}{2} + \alpha)$$

Let

$$\phi(x) = A(cos(x)) + sin(x)B(cos^2(x)) + sin(x)cos(x)C(cos^2(x)).$$

By exchanging in this expression x at π - x, π + x, 2π - x we shoot (we see??)

$$A(cos(x)) = \frac{\phi(x) + \phi(2\pi - x)}{2},$$

$$B(cos^2(x)) = \frac{\phi(x) - \phi(2\pi - x) + \phi(\pi - x) - \phi(\pi + x)}{4sin(x)},$$

$$C(cos^2(x)) = \frac{\phi(x) - \phi(2\pi - x) - \phi(\pi - x) + \phi(\pi + x)}{4sin(x)cos(x)},$$

Therefore A, B, C are finite and continuous functions of cos(x), and by consequence, $\phi(x)$ or f(x) can be represented with such approximation, that is, by a polynomial in sin(x) and cos(x).

With the proviso that the intervals in question are smaller than 2π, theorems I, II, III are exact if we replace in their polynomial forms a finite Fourier sequence

This result can be deduced even more easily from the theorem on functions of two variables.

A continuous function having the period 2π can, in fact, be considered as attached to the points of the circumference

$$X = cos(x), \qquad Y = sin(x).$$

Let F(X, Y) be a continuous function with respect to the set (X, Y) and equal on the circumference to the proposed function. One can find a polynomial P(X, Y) which differs by less than ϵ in a domain comprising the circumference; therefore P(cos(x), sin(x)) differs from the function proposed by less than ϵ.

In this form, the demonstration becomes generalized immediately.

Let a continuous function with respect to the set of n variables x_1, x_2, ..., x_n; having the period π for each of the n first variables and the period 2π for x_n, such that for $x_i = 0$ (i = 1, 2, ..., n) the function is independent of the variables $x_{i+1}, ..., x_n$. It may be regarded as attached to the points of the variety

$$X_1 = cos(x_1), X_2 = sin(x_1)cos(x_2), ...X_n = sin(x_1)sin(x_2)...sin(x_{n-1})cos(x_n)$$

$$X_{n+1} = sin(x_1)sin(x_2)...sin(x_{n-1})sin(x_n)$$

where

$$X_1^2 + X_2^2 + ... + X_{n+1}^2 = 0$$

It is therefore representable with such approximation that one wants by a polynomial in $X_1, X_2, ..., X_{n+1}$.