Derivation of the weak form from Maxwell's equations

The four maxwells equations can be written as,

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This equation says that for a changing magnetic field, an electric field is generated perpendicular to it, with the amount of rotation/twisting (read more on curl of a vector) of it proportional to the magnitude.

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

For this application, the first term on the right-hand side can be neglected as there are no displacement charges. Then the equation says that the curl of magnetic field intensity is equal to the current density vector. This is expected, when there is a current flowing in a wire, there is a magnetic field circling (curl of H) around it.

$$\nabla . B = 0$$

This equation says that the divergence of a magnetic field is equal to zero. This means that the flux of the magnetic field entering any surface in a volume is equal to the amount of flux exiting the volume. This is also why there can be no magnetic monopoles.

$$\nabla . D = \rho$$

This equation is similar to the previous equation, but now, the amount of flux coming out of any volume is equal to the amount of charge in this volume. This is because electric charges exist.!

Let us now prepare to derive the final equations. Before that lets do some preparations.

The current can be split into two parts,

$$J = J_e + J_s$$

$$J = \sigma\left(-\frac{\partial A}{\partial t} - \nabla V\right) + J_s$$

Where J_e is used to represent eddy currents and J_s is used to represent current from wires. In ht case of the present simulation, the latter term is not significant and therefore it can be dropped.

Now, we can write the magnetic field as,

$$B = \mu_0 \mu_r H + B_r$$

Where $\mu_0$is the permeability of free space and $\mu_r$is relative permeability. H is the magnetic field intensity due to the electric current. Watch this video for a wonderful practical explanation on H and B by Ben Krasnow. B_r is the remnant magnetic field from the magnet which is,

$$B_r = \mu_0 M$$

From these equations, H can be written as,

$$H = \frac{(B - B_r)}{(\mu_0 \mu_r)}$$

$$H = \frac{B}{(\mu_0 \mu_r)} - \frac{M}{\mu_r}$$

Writing in terms of μ, which is equal to $\mu_0 \mu_r$,

$$H = \frac{B}{\mu} - \frac{\mu_0}{\mu} M$$

Where M is the magnetisation vector for the magnet. This vector dictates the direction and magnitude of the magnetic field from the magnet.

Now, from the second Maxwells equations,

$$\nabla \times H = J$$

$$\nabla \times \left(\frac{B}{\mu} - \frac{\mu_0}{\mu} M \right) = \sigma \left(-\frac{\partial A}{\partial t} - \nabla V \right) + J_s$$

Taking the *M* term to the RHS,

$$\nabla \times \left(\frac{B}{\mu} \right) = \sigma \left(-\frac{\partial A}{\partial t} - \nabla V \right) + \nabla \times \frac{\mu_0}{\mu} M + J_s$$

Now, from the third Maxwells equation, we know that $\nabla . B = 0$. When the divergence is zero, there exists a term, a potential, let's call it *A,* which can be defined as,

$$B = \nabla \times A$$

This term, *A,* is called the magnetic potential. This is convenient as seen in the following steps. Substituting this in the equation,

$$\nabla \times \left(\frac{1}{\mu} \nabla \times A \right) = -\sigma \frac{\partial A}{\partial t} - \sigma \nabla V + \nabla \times \frac{\mu_0}{\mu} M + J_s$$

Now, we have an equation for *A,* which is what we need to solve. However, this is not convenient for deriving the weak form. Therefore, lets use the Curl-Curl identity from vector calculus,

$$\nabla \times (\nabla \times A) = \nabla(\nabla . A) - \nabla^2 A$$

Since $\nabla . A = 0$ (Coulomb gauge),

$$\nabla \times (\nabla \times A) = -\nabla^2 A$$

Using this equation, we can write the final equation to derive the magnetic field,

$$\boldsymbol{\frac{1}{\mu} \nabla^2 A = \sigma \frac{\partial A}{\partial t} + \sigma \nabla V - \nabla \times \frac{\mu_0}{\mu} M - J_s}$$

However, we have to simultaneously solve for the electric field as well. This is done as the following.

Lets take the divergence of equation no.

$$\nabla.\left(\nabla\times\left(\frac{B}{\mu}\right)\right) = -\nabla.\left(\sigma\frac{\partial A}{\partial t}\right) - \nabla.(\sigma\nabla V) + \nabla.\left(\nabla\times\frac{\mu_0}{\mu}M\right) + \nabla.(J_s)$$

Now, we know that the divergence of J is equal to zero (derived from the last Maxwells equation), also, the divergence of curl as in the LHS term is also equal to zero (vector calculus identity), therefore,

$$0 = -\nabla\cdot\left(\sigma\frac{\partial A}{\partial t}\right) - \nabla\cdot(\sigma\nabla V) + \nabla.\left(\nabla\times\frac{\mu_0}{\mu}M\right)$$

Here, the last term is also a divergence of curl, so that is equal to zero too. Taking the electric potential to the LHS, the final equation for the electric scalar potential is,

$$\boldsymbol{\sigma\nabla^2 V = -\nabla\cdot\left(\sigma\frac{\partial A}{\partial t}\right)}$$

Did you note that we used all the four Maxwells equations to derive these two equations?! We now have the system of two equations which is therefore compliant to all the four Maxwells equations. We can now derive the weak form for these two equations and start solving this in FEniCS. Let's write down these two equations one more time for clarity.

$$Magnetic\ vector\ potential, \frac{1}{\mu}\nabla^2 A = \sigma\frac{\partial A}{\partial t} + \sigma\nabla V - \nabla\times\frac{\mu_0}{\mu}M - J_s$$

$$Electric\ scalar\ potential, \sigma\nabla^2 V = -\nabla\cdot\left(\sigma\frac{\partial A}{\partial t}\right)$$

Now, to write the weak form, let us multiply both sides by the test function, *v* and integrating these equations,

$$\int_\Omega \left(\frac{1}{\mu}\nabla^2 A\right).v\,dx = \int_\Omega \left(\sigma\frac{\partial A}{\partial t} + \sigma\nabla V - \nabla\times\frac{\mu_0}{\mu}M\right).v\,dx$$

We know that the LHS can be written using integration by parts as,

$$\int_\Omega \left(\frac{1}{\mu}\nabla^2 A\right).v\,dx = \nabla A.v|_{bcs} - \frac{1}{\mu}\int_\Omega \nabla A.\nabla v\,dx$$

Where, *bcs* are the boundary conditions, which are either Neuman or Dirichlet conditions that vanish those terms on the boundaries.

Substituting this, we get the final equation in its weak form to be solved,

$$f(A) = \frac{1}{\mu}\int_\Omega \nabla A.\nabla v\,dx + \int_\Omega \left(\sigma\frac{\partial A}{\partial t} + \sigma\nabla V - \nabla\times\frac{\mu_0}{\mu}M\right).v\,dx$$

Similarly,

$$f(V) = \sigma\int_\Omega \nabla V.\nabla q\,dx - \int_\Omega \left(\sigma\frac{\partial A}{\partial t}\right).\nabla q\,dx$$

Now, this is written in FEniCs as,

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f_A = (inner(nu*grad(A),grad(v))*dx
    + inner(Constant((1/(1.3e-6)))*Constant((4*pi*1e-7))*(-M),curl(v))*dx
    + inner(sigma*((A-A0)/dt),v)*dx
    + inner(sigma*grad(V),v)*dx
    )

f_V = (inner(sigma*grad(V),grad(q))*dx
    - inner(sigma*((A-A0)/dt),grad(q))*dx
    )

a1 = lhs(f_A+f_V)
l1 = rhs(f_A+f_V)
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a1 and l1 are then assembled into matrices and solved with the help of a GPU. The full code is available here.

It is highly recommended to go through FEniCS documentations [2] as they have extensive content on getting started with FEniCS and on how to use the software. It may be slightly difficult initially, but once you get going, it is truly a remarkable piece of kit for simulations.

References

[1]. James McDonagh, Nunzio Palumbo, Neeraj Cherukunnath, Nikolay Dimov, Nada Yousif, "Modelling a permanent magnet synchronous motor in FEniCSx for parallel high-performance simulations", *Finite Elements in Analysis and Design*, Volume 204, 2022, 103755, ISSN 0168-874X, https://doi.org/10.1016/j.finel.2022.103755. (https://www.sciencedirect.com/science/article/pii/S0168874X22000312).

[2]. Logg, Anders, Kent-Andre Mardal, and Garth Wells, eds. *Automated solution of differential equations by the finite element method: The FEniCS book*. Vol. 84. Springer Science & Business Media, 2012.