Topic TB1/CS, SE, COMP: Logarithms

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Key points

- understand logarithm, base of a logarithm
- laws of logarithms
- simplifying expressions with logarithms
- calculating logarithms to any base

Introducing logarithms

A friendly person: Mike starts with one friend on Facebook and every day he doubles the number of friends. After how many days will he have 1024 friends?

1st day: 1 friend
2nd day: 2 friends
...

... after n days: 2^n, hence we have to find n such that $2^n = 1024$

???

$[n = 10]$

Introducing logarithms

Or we can write the answer like $\log_2 1024 = 10$

Why?

Powers: $2^{\boxed{10}} = 1024$, where 2 is the base and 10 is the power/index

Logarithms: $\log_2 1024 = \boxed{10}$, logarithm to the base 2 of 1024 is 10.

In general

$$y = a^x \text{ and } \log_a y = x \text{ are equivalent}$$

when $a > 1$.

Worked examples

Examples. Evaluate the following logarithms:

1. $\log_2 16 = 4$
2. $\log_3 9 = 2$
3. $\log_{16} 2 = \frac{1}{4}$
4. $\log_4 4 = 1$
5. $\log_4 1 = 0$
6. $\log_2 2^n = n$

Base of a logarithm

The most commonly used bases are:

- 10: denoted by log or $\log_{10}$, $\log 1000 = 3$
- 2: denoted by $\log_2$,$\log_2 32 = 5$
- natural logarithm e: denoted by $\log_e$ or ln (most scientific calculators have a 'ln' button)

e is Euler's constant, $e = 2.71828\ldots$

First law of logs

Adding logarithms:

$$\log_a x + \log_a y = \log_a xy$$

All bases must be the same.

1. $\log_{10} 50 + \log_{10} 2 = \log_{10}(50 \times 2) = \log_{10} 100 = 2$
2. $\log_2 4 + \log_2 8 = \log_2(4 \times 8) = \log_2 32 = 5$
3. $\log_4 16 + \log_4 8 + \log_4 2 = \log_4(16 \times 8 \times 2) = \log_4 256 = 4$
4. $\log_5 t + \log_5 4t + \log_5 \frac{1}{4t^2} = \log_5 \frac{t \times 4t}{4t^2} = \log_5 1 = 0$

Second law of logarithms

Subtracting logarithms:

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

All bases must be the same.

1. $\log_5 50 - \log_5 2 = \log_5(\frac{50}{2}) = \log_5 25 = 2$
2. $\log_2 16 - \log_2 8 = \log_2(\frac{16}{8}) = \log_2 2 = 1$
3. $\log_4 16 + \log_4 8 - \log_4 2 = \log_4(\frac{16\times 8}{2}) = \log_4 64 = 3$
4. $\log_5 t + \log_5 4t - \log_5(4t^2) = \log_5 \frac{t\times 4t}{4t^2} = \log_5 1 = 0$

Third law of logarithms

$$n \log_a x = \log_a x^n$$

This law applies if n is an integer, fractional, positive or negative.

1. $2\log_2 8 = \log_2(8^2) = \log_2 64 = 6$
2. $2\log_{16} 4 = \log_{16}(4^2) = \log_{16} 16 = 1$
3. $\frac{1}{2}\log 16 = \log(16^{\frac{1}{2}}) = \log\sqrt{16} = \log 4$

Example.

Example.
Simplify

$$\begin{aligned}
\log_2 Y - 3\log_2 2Y + 2\log_2 4Y \\
&= \log_2 Y - \log_2 (2Y)^3 + \log_2 (4Y)^2 \\
&= \log_2 Y - \log_2 8Y^3 + \log_2 16Y^2 \\
&= \log_2 \left(\frac{Y \times 16Y^2}{8Y^3} \right) \\
&= \log_2 2 \\
&= 1
\end{aligned}$$

Calculating logarithms to any base

Sometimes it is useful to know how to change the base of a logarithm.

Let a, b, and x be positive real numbers such that $a \neq 1$, $b \neq 1$:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example.

$$\log_8 2^{15} = \frac{\log_2 2^{15}}{\log_2 8} = \frac{15}{3} = 5$$