Topic D: Functions and Sequences

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Key points

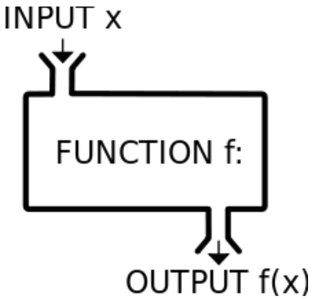
- the meaning of a function
- the notation used for functions
- a composite function, evaluation of a composite function
- the meaning of sequences, the term of sequence
- evaluation of the *n*-term of the sequence
- recurrent sequences

Where will I ever use this?!?

- MATHFUN/Discrete Maths (CS)
- THEOCS (CS)
- Recurrence relations: fundamental to understand complexity of divide & conquer algorithms

Definition of a function

A function is a 'rule' that receives an input and produces an output.



A function produces a *single* output for any given input.

Example

For example, the rule may be

"add 3 to the input".

- If 5 is the input, then $5 + 3 = 8$ will be the output.
- If -2 is the input, then $-2 + 3 = 1$ will be the output.
- If a variable x is the input then $x + 3$ will be the output.

The values of input/output variables need to be specified, e.g. numbers, strings, ...

Our focus: the functions in which input/output variables are real/whole numbers

Notation used for functions

- In mathematical notation we write $f : x \to x + 3$, or more commonly

$$f(x) = x + 3.$$

- We could represent the same function using different letters, e.g. $h(t) = t + 3$.

Example

Example. A function multiplies the input by 4. Write down the function in mathematical notation.
Let us call the function f and the input x.Then:

$$f(x) = 4x.$$

Example. A function divides the input by 6 and then adds 3 to the result. Write the function in mathematical notation.
Let us call the function z and the input t.Then:

$$z(t) = \frac{t}{6} + 3$$

.

Calculating the output from a given input

Often we are given a function and need to calculate the output from a given input.

Example. A function f is defined by $f(x) = 3x + 1$. Calculate the output when the input is 4.

$$f(4) = 3 \times 4 + 1 = 13.$$

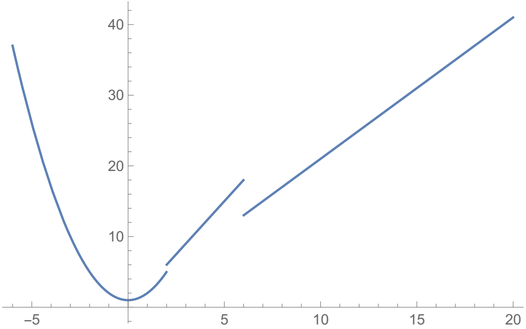
Example. A function g is defined by $g(t) = 2t^2 - 1$. Find $g(3)$:

$$g(3) = 2 \times (3)^2 - 1 = 2 \times 9 - 1 = 17.$$

Calculation output from a given input

Example. A piecewise function on real numbers:

$$g(x) = \begin{cases} x^2 + 1 & \text{when } x \leq 2 \\ 3x & \text{when } 2 < x \leq 6 \\ 2x + 1 & \text{when } x > 6 \end{cases}$$



Calculation output from a given input

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Evaluate $g(0)$, $g(4)$, $g(7)$.

- Since 0 lies between -1 and 2 we use the first part of the definition, that is $g(x) = x^2 + 1$, hence $g(0) = 1$.
- 4 lies between 2 and 6, therefore $g(4) = 3 \times 4 = 12$
- Finally, $g(7) = 2 \times 7 + 1 = 15$

Composite functions

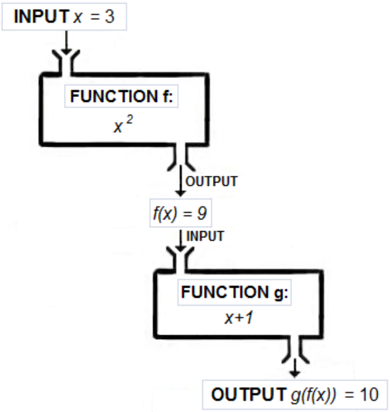
Sometimes we need to apply two or more functions one after the other: the output of one function becomes the input of the next function.

Suppose $f(x)$ and $g(x)$ are given functions, let the output of $f(x)$ become the input to $g(x)$.

We call $g(f(x))$ a composite function (or $(g \circ f)(x)$).

Example

Example. Given $f(x) = x^2$ and $g(x) = x + 1$. Find a value of the composite function $g(f(x))$ for $x = 3$.



Example

Example. Given $f(x) = x^2$ and $g(x) = x + 1$. Find a value of the composite functions $f(g(x))$ and $g(f(x))$ for $x = 3$.

Solution.

$$f(g(3)) = f(3+1) = f(4) = 4^2 = 16$$

$$g(f(3)) = g(3^2) = g(9) = 9 + 1 = 10$$

So, the order is important!

Have you noticed the style of maths writing ?

Example

Example. Given $f(x) = x^2$ and $g(x) = x + 1$. Find the composite functions $f(g(x))$ and $g(f(x))$.
For a variable x:

$$f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$$

$$g(f(x)) = g(x^2) = x^2 + 1$$

In general $f(g(x))$ and $g(f(x))$ are different functions.

Example

Example. Given $f(x) = 3x^2 + 1$. Find the composite functions $f(f(x))$ and $f(f(f(x)))$.

For a variable x:

$$f(f(x)) = f(3x^2 + 1) = 3(3x^2 + 1)^2 + 1 =$$

$$= 3(3x^2 + 1)(3x^2 + 1) + 1 = 27x^4 + 18x^2 + 4$$

And $f(f(f(x)))$?

$$f(f(f(x))) = f(27x^4 + 18x^2 + 4) =$$

$$= 3(27x^4 + 18x^2 + 4)^2 + 1 = \dots$$

Sequences

A sequence is an enumerated collection of numbers where order matters and repetitions are allowed.
For example,

$$1, 3, 5, 7, 9, 11$$
$$-1, -2, -3, \ldots$$

are both sequences.
Each number in the sequence is called a term of the sequence.
The number of terms in the first sequence is six – finite sequences.
The second sequence goes on for ever, – inifinite sequences.

Notation used for sequences

We use a subscript notation to refer to different terms in a sequence.

Example. In the sequence 1, 3, 5, 7, 9, 11, $\ldots$ the first term can be labeled by x_1, the second term x_2 and so on.

The terms of a sequence can often be written by using a formula, e.g.

$$x_k = 2k - 1.$$

Example. The terms of a sequence x are given by $x_k = 4k^2$. Write down the terms x_1, x_2 and x_3.

$x_1 = 4 \times 1^2 = 4, \qquad x_2 = 4 \times 2^2 = 16,$

$x_3 = 4 \times 3^2 = 36.$

Recurrence relations

A recurrence relation is an equation that recursively defines a sequence:

- once one or more initial terms are given,
- each further term of the sequence is defined as a function of the preceding terms.

Example. Write down the first 12 terms of the following sequence:

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2 \qquad F_0 = 0, \; F_1 = 1$$

Example

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \qquad F_0 = 0,\ F_1 = 1$$

$$F_2 = F_1 + F_0 = 1 + 0 = 1,$$

$$F_3 = F_2 + F_1 = 1 + 1 = 2,$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3,$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5, \ldots$$

The sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Think about a program – quite tricky!