Reconnaissance de caractères manuscrits par factorisation de matrices

Le point de vue quasi-bayésien

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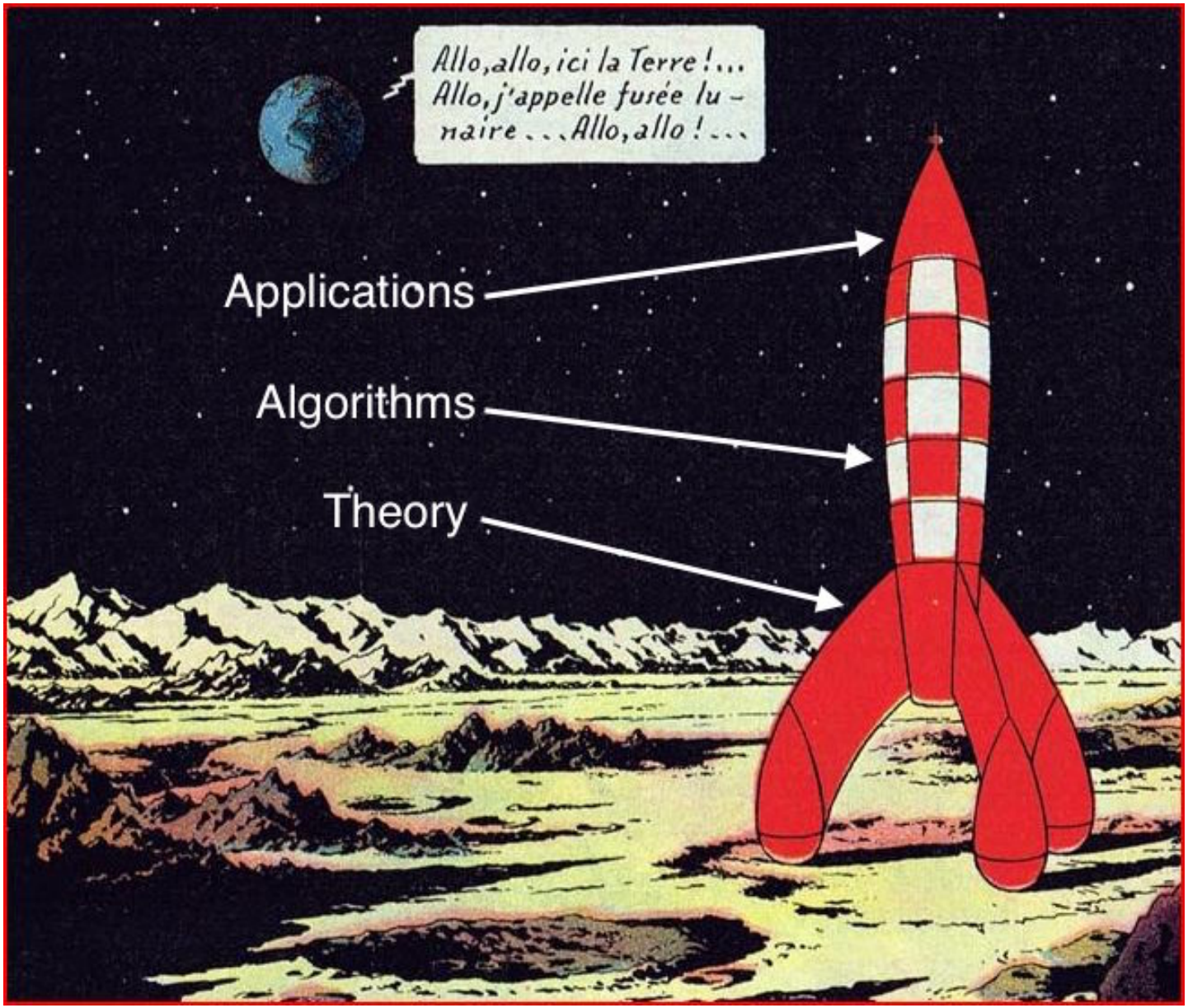
Allo, allo, ici la Terre !...
Allo, j'appelle fusée lu-
naire ...Allo, allo !...
Applications
Algorithms
Theory

{Statistical,Machine} Learning: building automatic procedures to infer general rules from examples.

In the (~~rather~~ not so?) long term: mimic the inductive functioning of the humain brain to develop an artificial intelligence.

In the Big Data Era, very dynamic field at the crossroads of Computer Science and Statistics. Strategic focus at Inria!

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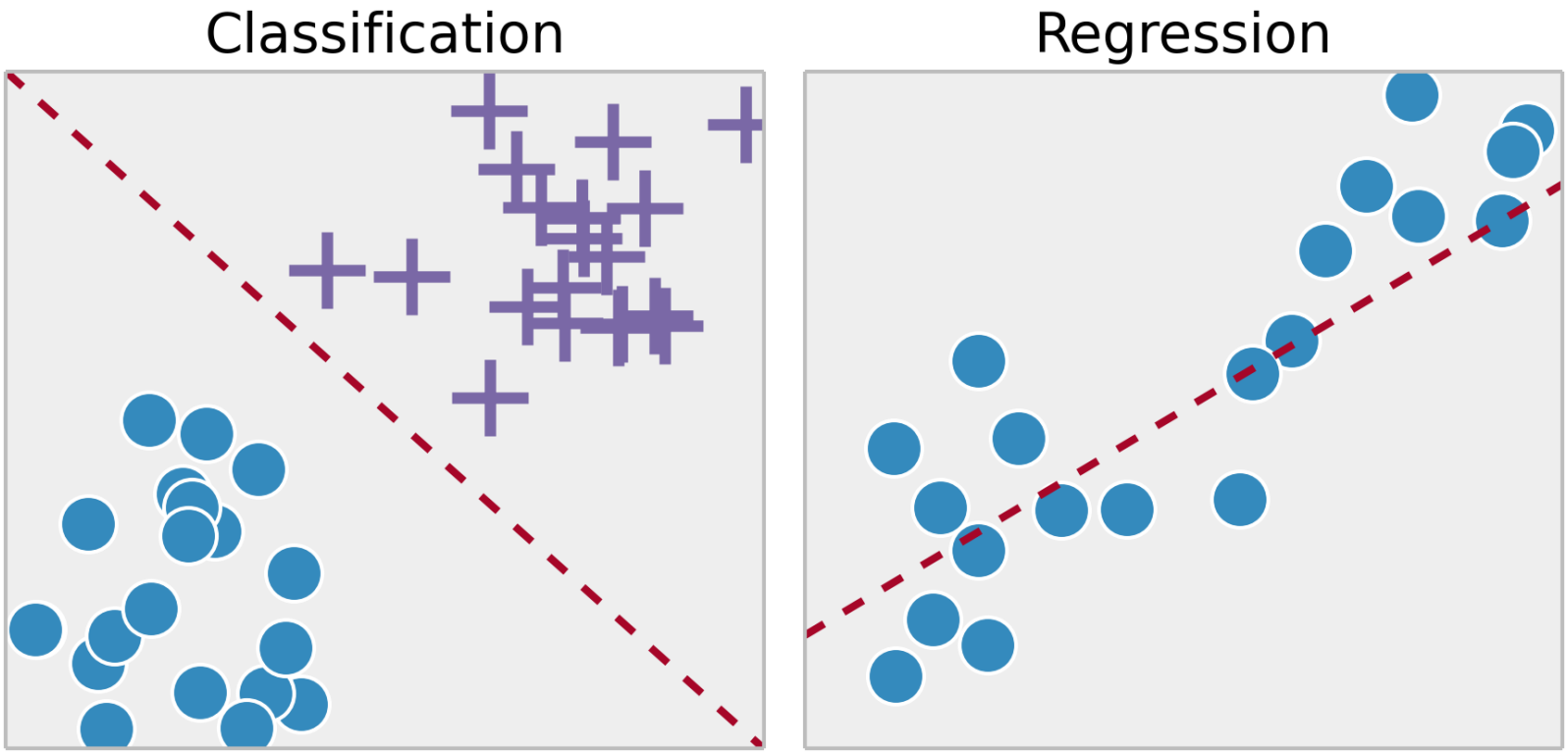
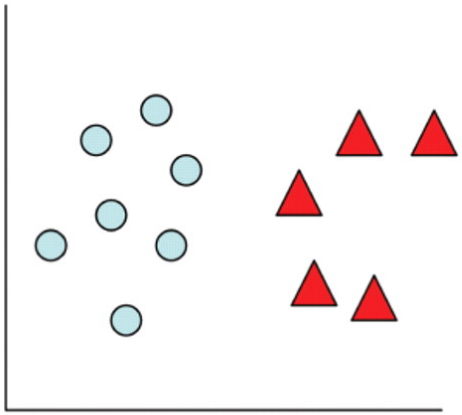
- Classification: $\mathcal{Y}$ is discrete.
- Regression: $\mathcal{Y}$ is a continuum.
- Online learning: observations are revealed over time.
- Batch learning: all observations are revealed at once.

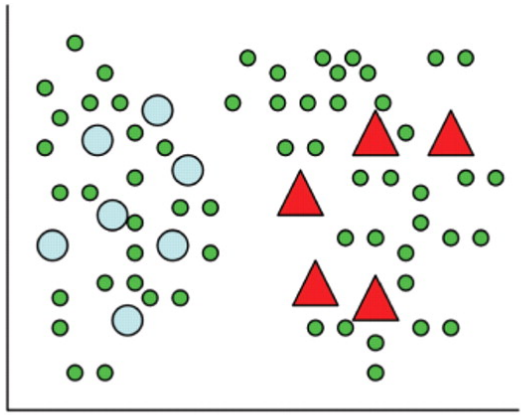
- Supervised learning: all of the Y_is are observed.



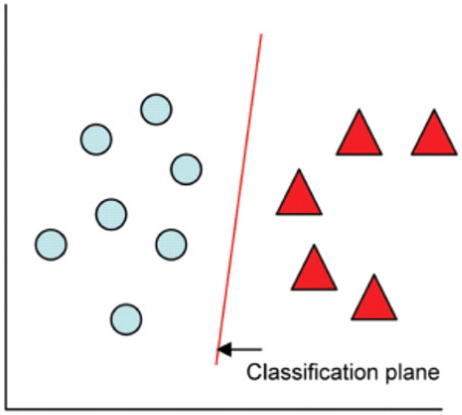
- Semi-supervised learning: some of the Y_is are observed (labeling is expensive or difficult).



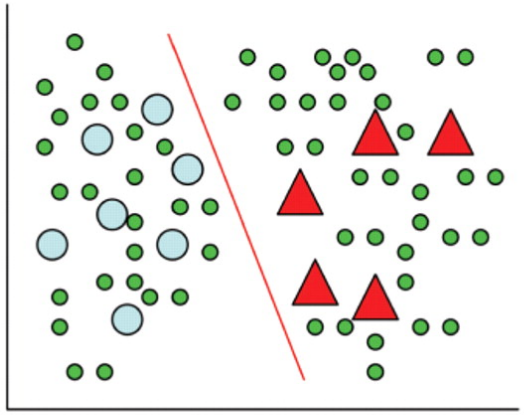
Labeled Data
(a)



Labeled and Unlabeled Data
(b)

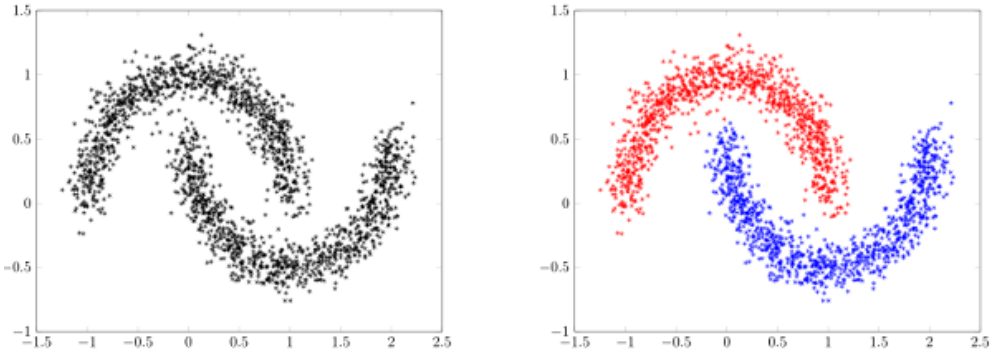


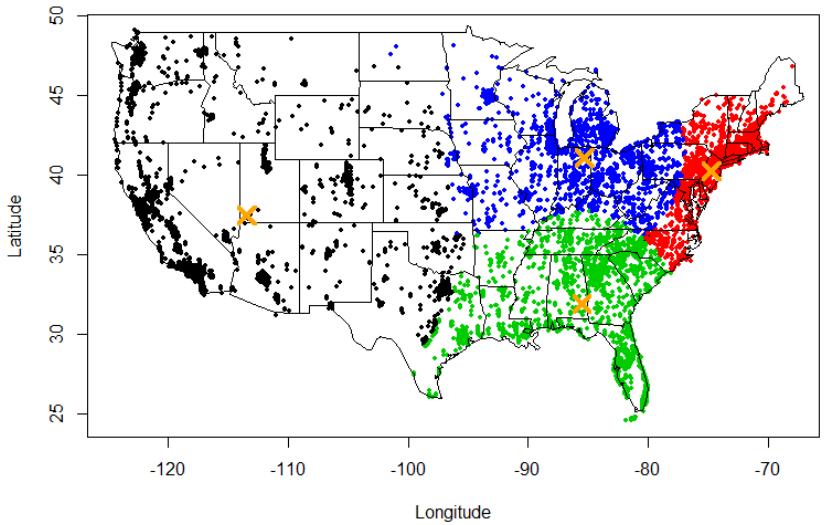
Supervised Learning
(c)



Semi-Supervised Learning
(d)

- Unsupervised learning: none of the Y_is are observed (detect patterns).





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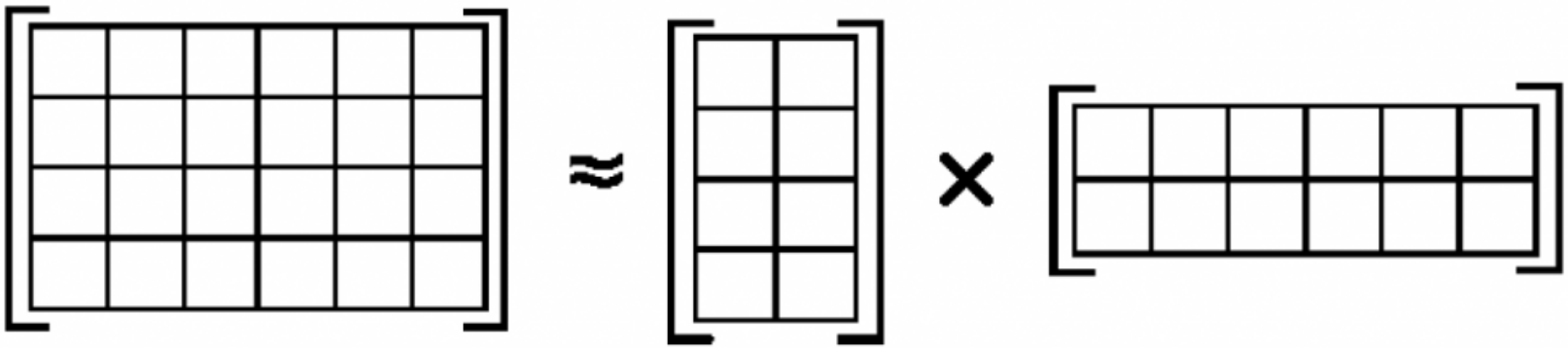
Non-negative Matrix Factorization (NMF)

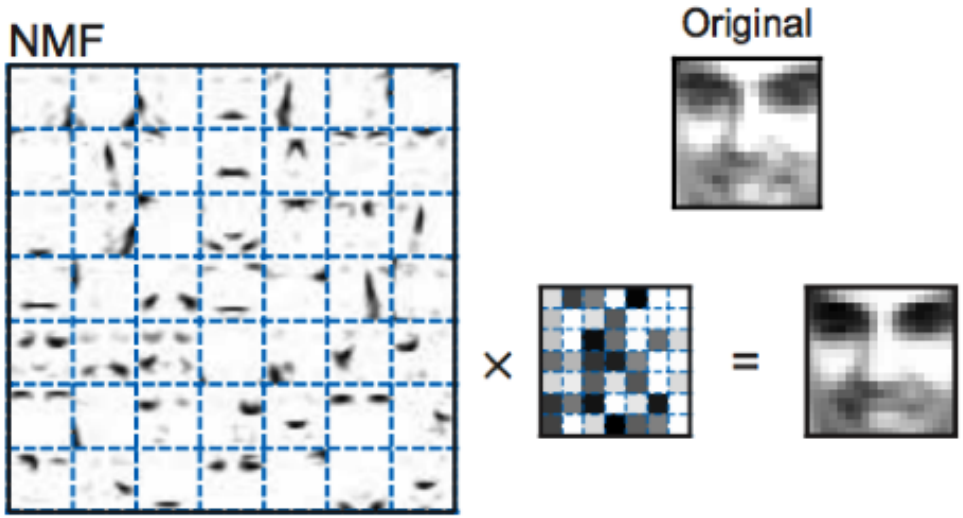
NMF amounts to decompose an $m_1 \times m_2$ matrix M as a product of two low rank matrices with non-negative entries.

$$M \approx UV^\top,$$

where U is $m_1 \times K$ and V is $m_2 \times K$, and $K \ll m_1 \wedge m_2$.



- Audio/video source separation $\longrightarrow$ [Video] [Demo]
- Denoising/restoration
- Speaker recognition
- Music classification
- Image processing: object recognition
- Topics extraction in texts
- Recommender systems
- ...



We observe an $m_1 \times m_2$ matrix Y and we assume

$$Y = M + \mathcal{E}$$

with $\mathbb{E}(\mathcal{E}) = 0$ and $\mathbb{V}(\mathcal{E}) = \sigma^2 \mathrm{Id}$.

The goal is to find a "good" factorization of M, in the sense of the Frobenius norm

$$\|A\|_F = \sqrt{\langle A, A \rangle_F},$$

$$\langle A, B \rangle_F = \mathrm{Tr}(AB^\top) = \sum_{i=1}^{p} \sum_{j=1}^{q} A_{i,j} B_{i,j}.$$

The quasi-Bayesian approach in one slide!

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Loss:

$$R_n(\phi) = \frac{1}{n}\sum_{i=1}^{n} \ell(Y_i, \phi(X_i)).$$

(examples: $\ell(a,b) = (a-b)^2$, $\ell(a,b) = \|a-b\|_F$, etc.)

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Quasi-posterior

$$\widehat{\rho}_\lambda(\cdot) \propto \exp\left(-\lambda R_n(\cdot)\right)\pi(\cdot),$$

for some inverse temperature $\lambda > 0$.

In general, $\exp\left(-\lambda R_n(\cdot)\right)$ is not a likelihood (hence the term quasi-Bayesian).

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Realization

$$\widehat{\phi}_\lambda \sim \widehat{\rho}_\lambda.$$

And so on.

Probably Approximately Correct (PAC) oracle inequalities

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For any $\epsilon > 0$,

$$\mathbb{P}\left(R\left(\widehat{\phi}_\lambda\right) - R^\star \leq \spadesuit \inf_{\phi \in \mathcal{F}} \left\{ R(\phi) - R^\star + \frac{\Delta(\phi, \epsilon)}{n^\alpha} \right\} \right) \geq 1 - \epsilon,$$

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Let $d = \dim(\mathcal{X})$

- $\Delta(\phi, \epsilon) \propto d + \log \frac{1}{\epsilon}$
- $\Delta(\phi, \epsilon) \propto \log d + \log \frac{1}{\epsilon}$

The PAC-Bayesian theory

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PAC-Bayesian Theory

$$= \text{Quasi-Bayesian learning} + \text{PAC oracle inequalities}$$

- Shawe-Taylor and Williamson (1997). A PAC analysis of a Bayes estimator, *COLT*
- McAllester (1998). Some PAC-Bayesian theorems, *COLT*
- McAllester (1999). PAC-Bayesian model averaging, *COLT*
- Catoni (2004). Statistical Learning Theory and Stochastic Optimization, Springer
- Catoni (2007). PAC-Bayesian Supervised Classification: The Thermodynamics of Statistical Learning, IMS

Gradient Descent

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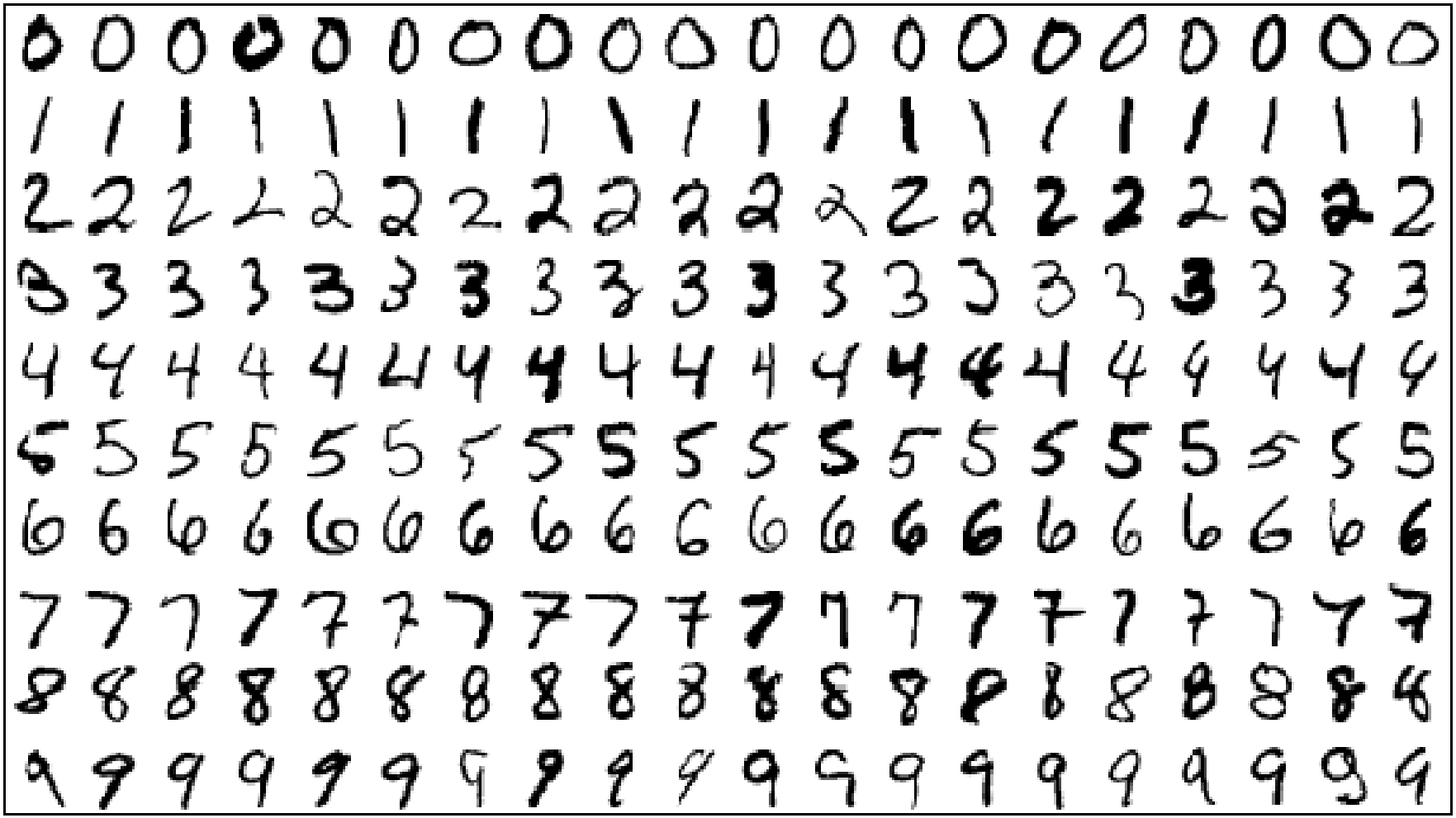
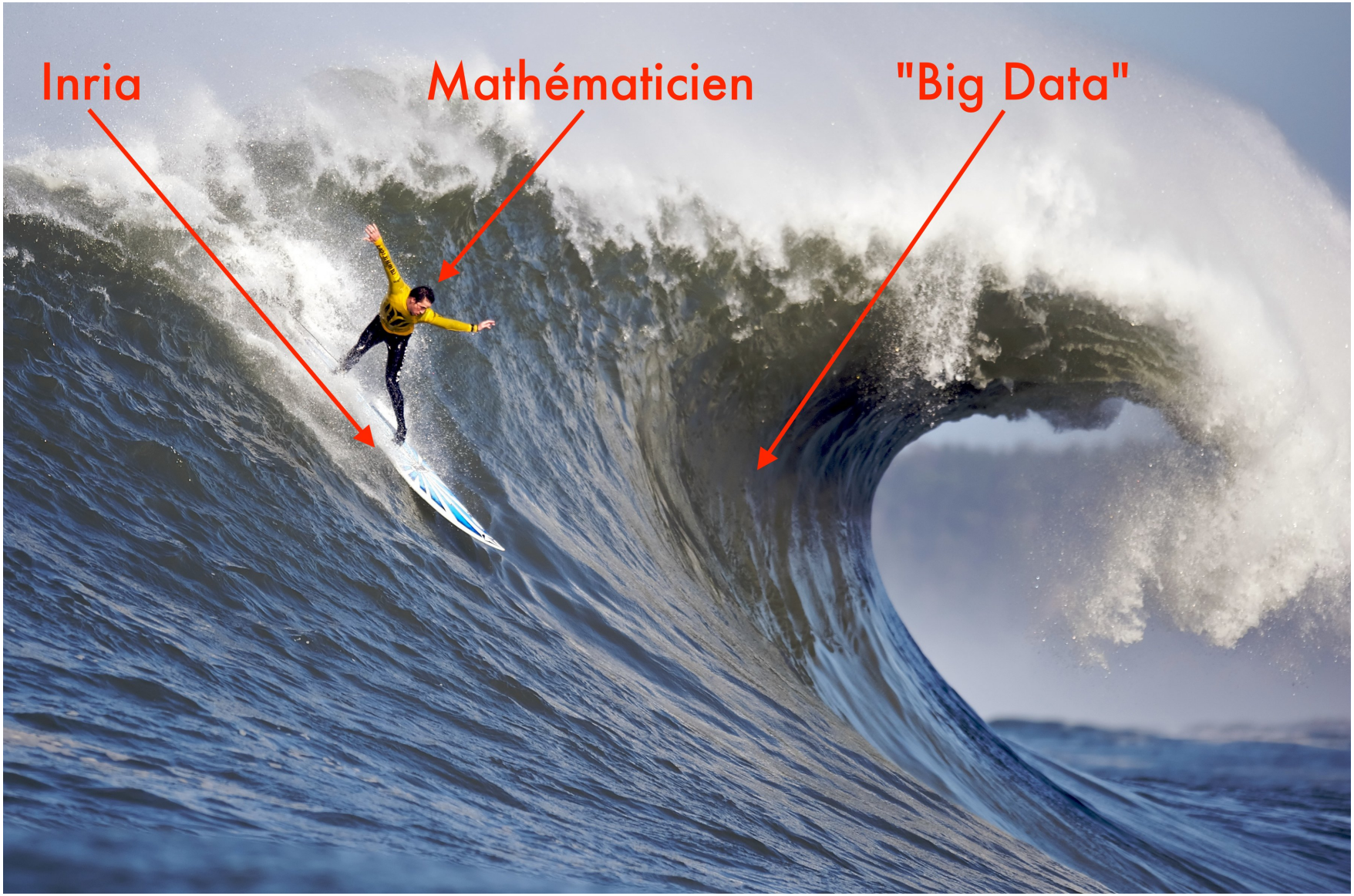
Output: $(x_k)_{k=0}^K$ and $x_K \simeq \arg\min_x f(x)$.

Digits recognition

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