Machine Learning Theory

Least Squares and the Efron-Stein Inequality

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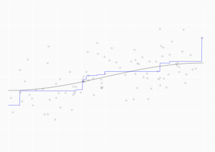
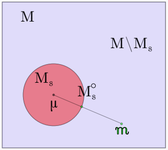
Where We Left Things

$$\hat{\mu} = \underset{m \in \mathcal{M}}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} \{ Y_i - m(X_i) \}^2 \quad \text{for a convex set } \mathcal{M}$$





Claim. When $Y_i = \mu(X_i) + \varepsilon_i$ for $\mu \in \mathcal{M}$ and $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$,

$$\|\hat{\mu} - \mu\| < s \quad \text{w.p. } 1 - \delta \text{ if } \quad \frac{s^2}{2} \overset{(a)}{\geq} \mathrm{E} \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu \rangle + s\sigma \sqrt{\frac{2c\log(n)}{\delta n}}.$$

What We Actually Proved.

$$\|\hat{\mu} - \mu\| < s \quad \text{whenever} \quad \frac{s^2}{2} \overset{(b)}{\geq} \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu \rangle$$

Loose End. w.p. $1 - \delta$, $(a) \implies (b)$. That is, ...

$$\max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu \rangle \leq \mathrm{E} \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu \rangle + s\sigma \sqrt{\frac{2c\log(n)}{\delta n}} \quad \text{w.p.} \quad 1 - \delta.$$

Our Maximum is Approximately Constant

What we want to show.

$$Z = \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu \rangle \quad \text{satisfies} \quad Z \le \operatorname{E} Z + s\sigma\sqrt{\frac{2c\log(n)}{\delta n}} \quad \text{w.p.} \quad 1-\delta.$$

We'll show something a bit stronger.

$$|Z - \operatorname{E} Z| \le s\sigma\sqrt{\frac{2c\log(n)}{\delta n}} \quad \text{w.p.} \quad 1-\delta.$$

This is implied by Chebyshev's inequality. A special case of Markov's inequality.

$$\begin{aligned} &P\left\{|Z - \operatorname{E} Z| \le \frac{\operatorname{sd}(Z)}{\sqrt{\delta}}\right\} \\ &= P\left\{|Z - \operatorname{E} Z|^2 \le \frac{\operatorname{Var}(Z)}{\delta}\right\} \\ &\le \frac{\operatorname{E}|Z - \operatorname{E} Z|^2}{\frac{\operatorname{Var}(Z)}{\delta}} = \frac{\operatorname{Var}(Z)}{\frac{\operatorname{Var}(Z)}{\delta}} = \delta. \end{aligned}$$

All we need to do is bound the variance. We need to show that ...

$$\frac{\operatorname{sd}(Z)}{\sqrt{\delta}} \le s\sigma\sqrt{\frac{2}{\delta n}} \quad \text{i.e.} \quad \operatorname{Var}(Z) \le s^2\sigma^2\,\frac{2c\log(n)}{n}.$$

Variance and Independent Copies

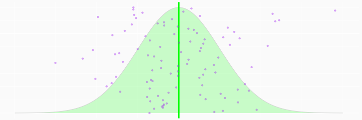
$$\mathrm{Var}[Z] = \mathrm{Var}[f(\varepsilon)] \quad \text{for} \quad f(u) = \max_{m \in \mathcal{M}_s^\circ} \sum_{i=1}^{n} u_i \{m(X_i) - \mu(X_i)\}.$$

- Z is a pretty complicated function of our noise vector ε. To bound its variance, ...
- ...we'll need to think about it a bit differently than you're probably used to.

$$\begin{aligned} \mathrm{Var}[Z] &= \mathrm{E}\left[\{Z - \mathrm{E}[Z]\}^2\right] \\ &= \frac{1}{2}\,\mathrm{E}\left[\left\{Z - \tilde{Z}\right\}^2\right] \end{aligned}$$

where Z and $\tilde{Z}$ are independent and identically distributed.

- It's the mean squared deviation of Z from its expectation.
- And *half* of the mean squared deviation of Z from *an independent copy* of Z.



Let's use all this to tackle a simplified version of our problem. We'll lose the $\max$.

Calculate $\operatorname{Var}[f(\varepsilon)]$ for $f(u) = \sum_{i=1}^{n} u_i$.

We can do this calculation the ‘usual way’ with these independent copies.

$$
\begin{aligned}
\operatorname{Var}\left[f(\varepsilon)\right] &= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}\varepsilon_i - \sum_{i=1}^{n}\tilde{\varepsilon}_i\right\}^2\right] \\
&= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}(\varepsilon_i - \tilde{\varepsilon}_i)\right\}^2\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)(\varepsilon_j - \tilde{\varepsilon}_j)\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)^2\right]
\end{aligned}
$$

We can use our independent copies to write this more abstractly,
keeping everything ‘inside’ our summing function f.

$$
\begin{aligned}
\varepsilon_i - \tilde{\varepsilon}_i &= (\varepsilon_1 + \ldots + \varepsilon_i + \tilde{\varepsilon}_{i+1} + \ldots + \tilde{\varepsilon}_n) - (\varepsilon_1 + \ldots + \varepsilon_{i-1} + \tilde{\varepsilon}_i + \ldots + \tilde{\varepsilon}_n) \\
&= f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right) \quad \text{where} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}
\end{aligned}
$$

$$\text{Calculate} \quad \mathrm{Var}[f(\varepsilon)] \quad \text{for} \quad f(u) = \sum_{i=1}^{n} u_i.$$

We can do this calculation the ‘usual way’ with these independent copies.

$$\mathrm{Var}\left[f(\varepsilon)\right] = \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}\varepsilon_i - \sum_{i=1}^{n}\tilde{\varepsilon}_i\right\}^2\right] \qquad = \frac{1}{2}\,\mathrm{E}\left[\{f(\varepsilon) - f(\tilde{\varepsilon})\}^2\right]$$

$$= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}(\varepsilon_i - \tilde{\varepsilon}_i)\right\}^2\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)(\varepsilon_j - \tilde{\varepsilon}_j)\right]$$

$$= \frac{1}{2}\sum_{i=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)^2\right]$$

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$$\varepsilon_i - \tilde{\varepsilon}_i = (\varepsilon_1 + \ldots + \varepsilon_i + \tilde{\varepsilon}_{i+1} + \ldots + \tilde{\varepsilon}_n) - (\varepsilon_1 + \ldots + \varepsilon_{i-1} + \tilde{\varepsilon}_i + \ldots + \tilde{\varepsilon}_n)$$

$$= f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right) \quad \text{where} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}$$

$$\text{Calculate} \quad \mathrm{Var}[f(\varepsilon)] \quad \text{for} \quad f(u) = \sum_{i=1}^{n} u_i.$$

We can do this calculation the ‘usual way’ with these independent copies.

$$\begin{aligned}
\mathrm{Var}\left[f(\varepsilon)\right] &= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}\varepsilon_i - \sum_{i=1}^{n}\tilde{\varepsilon}_i\right\}^2\right] \\
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\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\,\mathrm{E}\left[\{f(\varepsilon) - f(\tilde{\varepsilon})\}^2\right] \\
&= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n} f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}^2\right]
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&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}\left\{f\left(\varepsilon^{[j]}\right)\right.\right.
\end{aligned}$$

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&= f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right) \quad \text{where} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}
\end{aligned}$$

$$\text{Calculate} \quad \mathrm{Var}[f(\varepsilon)] \quad \text{for} \quad f(u) = \sum_{i=1}^{n} u_i.$$

We can do this calculation the ‘usual way’ with these independent copies.

$$\begin{aligned}
\mathrm{Var}\left[f(\varepsilon)\right] &= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}\varepsilon_i - \sum_{i=1}^{n}\tilde{\varepsilon}_i\right\}^2\right] \\
&= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n}(\varepsilon_i - \tilde{\varepsilon}_i)\right\}^2\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)(\varepsilon_j - \tilde{\varepsilon}_j)\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\mathrm{E}\left[(\varepsilon_i - \tilde{\varepsilon}_i)^2\right]
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$$\begin{aligned}
&= \frac{1}{2}\,\mathrm{E}\left[\{f(\varepsilon) - f(\tilde{\varepsilon})\}^2\right] \\
&= \frac{1}{2}\,\mathrm{E}\left[\left\{\sum_{i=1}^{n} f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}^2\right] \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\Bigl[\Bigl\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\Bigr\}\Bigl\{f\left(\varepsilon^{[j]}\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\mathrm{E}\left[\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}^2\right]
\end{aligned}$$

We can use our independent copies to write this more abstractly,
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$$\begin{aligned}
\varepsilon_i - \tilde{\varepsilon}_i &= (\varepsilon_1 + \ldots + \varepsilon_i + \tilde{\varepsilon}_{i+1} + \ldots + \tilde{\varepsilon}_n) - (\varepsilon_1 + \ldots + \varepsilon_{i-1} + \tilde{\varepsilon}_i + \ldots + \tilde{\varepsilon}_n) \\
&= f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right) \quad \text{where} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}
\end{aligned}$$

The Variance of Sums: $\mathrm{Var}[f(\varepsilon)]$ for $f(u) = \sum_{i=1}^n u_i$

$$\mathrm{Var}[f(\varepsilon)] = \frac{1}{2}\sum_{i=1}^n \mathrm{E}\left[\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}^2\right] \qquad \text{for} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}$$

$$= \frac{1}{2}\sum_{i=1}^n \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}^2\right] \qquad \text{for} \quad \varepsilon_j^{(i)} = \begin{cases} \tilde{\varepsilon}_i & j = i \\ \varepsilon_j & j \ne i \end{cases}$$

We can derive the (simpler) second formula from the one we've just worked out. Here's the argument.

- The pair of vectors $\varepsilon^{[i]}, \varepsilon^{[i+1]}$ have the same joint distribution as $\varepsilon, \varepsilon^{(i)}$.
- It follows that any functions of those pairs,

  $$\text{e.g.} \quad f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right) \quad \text{and} \quad f(\varepsilon) - f(\varepsilon^{(i)}),$$

  have the same distribution. And therefore the same expectation.

How do we know our pairs have the same distribution?

- The first vectors, $\varepsilon^{[i]}$ and ε, have the same distribution.
- To get the second vector from the first, we do the same thing.
  We replace the ith component with an independent copy.

The Efron-Stein inequality: $\mathrm{Var}[f(\varepsilon)]$ for *arbitrary* f

$$\mathrm{Var}\left[f(\varepsilon)\right] \le \frac{1}{2}\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}^2\right] \quad \text{for} \quad \varepsilon_j^{(i)} = \begin{cases} \tilde{\varepsilon}_i & j = i \\ \varepsilon_j & j \ne i \end{cases}$$

- Something very cool happens when we write things this way.
  - What we've derived isn't just a new formula for the variance of a sum.
  - It's a variance bound for *any function* of a vector of independent random variables.
- We call this the *Efron-Stein inequality*.
- There's an equivalent 'positive part' version that's sometimes easier to use.

$$\mathrm{Var}\left[f(\varepsilon)\right] \le \sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}_+^2\right] \quad \text{for} \quad \{z\}_+ = \max\{z, 0\}.$$

- This is nice because $f(x) = \{x\}+^2$ is increasing (whereas $f(x) = x^2$ is not).
- And that means we can substitute an upper bound for what's inside it.

$$\mathrm{Var}\left[f(\varepsilon)\right] \le \sum_{i=1}^{n} \mathrm{E}\{F_i\}_+^2 \le \sum_{i=1}^{n} \mathrm{E}\, F_i^2 \quad \text{for} \quad F_i \ge f(\varepsilon) - f\left(\varepsilon^{(i)}\right).$$

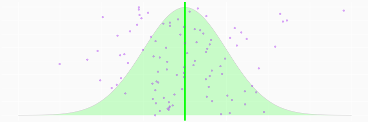
The ‘Positive Part’ Efron-Stein inequality

$$\begin{aligned}
\operatorname{Var}\left[f(\varepsilon)\right] &\leq \frac{1}{2}\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}^{2}\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}_{+}^{2}\right] \quad \text{for} \quad \{z\}_{+} = \max\{z, 0\}.
\end{aligned}$$

- What’s changed from the first formula to the second?
  - The differences on the right have been replaced with their *positive parts*.
  - We’ve lost the $\frac{1}{2}$ to compensate.
- Why is this equivalent? Symmetry.
- For any random variable S with a symmetric distribution[1], $\mathrm{E}\, S^2 = 2\,\mathrm{E}\{S\}_{+}^{2}$.

Proof.

$$\begin{aligned}
S^2 &= \{S\}_{+}^{2} + \{-S\}_{+}^{2} \\
&= \mathrm{E}\{S\}_{+}^{2} + \mathrm{E}\{-S\}_{+}^{2} \\
&= 2\,\mathrm{E}\{S\}_{+}^{2}.
\end{aligned}$$



$\square$

[1] A random variable S has a symmetric distribution if S and $-S$ have the same distribution.

The Variance of our Maximum

$$\mathrm{Var}[f(\varepsilon)] \le \sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f\left(\varepsilon^{(i)}\right)\right\}_+^2\right] \quad \text{for} \quad f(x) = \max_{m \in \mathcal{M}_s^\circ} \langle x, m - \mu\rangle$$

What do the terms on the right look like?

$$\begin{aligned}
f(\varepsilon) - f\left(\varepsilon^{(i)}\right) &= \max_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu\rangle - \max_{m \in \mathcal{M}_s^\circ} \left\langle \varepsilon^{(i)},\ m - \mu\right\rangle \\
&\le \langle \varepsilon,\ \hat{m} - \mu\rangle - \left\langle \varepsilon^{(i)},\ \hat{m} - \mu\right\rangle \quad \text{for} \quad \hat{m} = \operatorname*{argmax}_{m \in \mathcal{M}_s^\circ} \langle \varepsilon,\ m - \mu\rangle \\
&= \left\langle \varepsilon - \varepsilon^{(i)},\ \hat{m} - \mu\right\rangle = \frac{1}{n}\{\hat{m}(X_i) - \mu(X_i)\}(\varepsilon_i - \tilde{\varepsilon}_i).
\end{aligned}$$

Plugging in these bounds, we get …

$$\begin{aligned}
\mathrm{Var}[f(\varepsilon)] &\le \frac{1}{n} \times \mathrm{E}\,\frac{1}{n}\sum_{i=1}^{n} \underbrace{\{\hat{m}(X_i) - \mu(X_i)\}^2}_{U_i} \underbrace{(\varepsilon_i - \tilde{\varepsilon}_i)^2}_{V_i} && = \frac{1}{n} \times \mathrm{E}\langle U,\ V\rangle_{L_2(\mathrm{P_n})} \\
&= \frac{1}{n} \times \frac{1}{n}\sum_{i=1}^{n}\{\hat{m}(X_i) - \mu(X_i)\}^2\, \mathrm{E}\max_{i \in 1 \ldots n} (\varepsilon_i - \tilde{\varepsilon}_i)^2 && = \frac{1}{n} \times \mathrm{E}\|U\|_{L_1(\mathrm{P_n})}\|V\|_{L_\infty(\mathrm{P}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n} \times s^2 \times \mathrm{E}\max_{i \in 1 \ldots n} (\varepsilon_i - \tilde{\varepsilon}_i)^2 \\
&\le \frac{1}{n} \times s^2 \times \sigma^2 c \log(n).
\end{aligned}$$

A Proof of the Efron-Stein inequality

$$
\begin{aligned}
\mathrm{Var}\left[f(\varepsilon)\right] &= \mathrm{E}\, f(\varepsilon)^2 - \left\{\mathrm{E}\, f(\varepsilon)\right\}^2 \\
&= \mathrm{E}\, f(\varepsilon)^2 - \mathrm{E}\, f(\varepsilon)\, \mathrm{E}\, f(\tilde{\varepsilon}) \\
&= \mathrm{E}\, f(\varepsilon)\left\{f(\varepsilon) - \mathrm{E}\, f(\varepsilon)\right\} \\
&= \mathrm{E}\, f(\varepsilon)\left\{\sum_{i=1}^{n} f(\varepsilon^{[i]}) - f(\varepsilon^{[i+1]})\right\} \\
&= \sum_{i=1}^{n} \mathrm{E}\, f(\varepsilon)\left\{f(\varepsilon^{[i]}) - f(\varepsilon^{[i+1]})\right\} \quad \text{where} \quad \varepsilon_j^{[i]} = \begin{cases} \varepsilon_j & j \le i \\ \tilde{\varepsilon}_j & j > i \end{cases}
\end{aligned}
$$

The Swapping Trick.

$$
\begin{aligned}
f(\varepsilon)\overbrace{\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}}^{A_i = g_i(\varepsilon)} &\to f(\varepsilon^{(i)})\overbrace{\left\{f\left(\varepsilon^{[i+1]}\right) - f\left(\tilde{\varepsilon}^{[i]}\right)\right\}}^{B_i = g_i(\varepsilon^{(i)})} \quad \text{for} \quad \tilde{\varepsilon}_j^{(i)} = \begin{cases} \tilde{\varepsilon}_i & j = i \\ \varepsilon_j & j \ne i \end{cases} \\
&= -f(\varepsilon^{(i)})\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}.
\end{aligned}
$$

- Think of the ith term as a function of ε: $g_i(\varepsilon) = f(\varepsilon)\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}$.
- Swapping $\varepsilon_i \longrightarrow \tilde{\varepsilon}_i$ doesn't change the distribution of ε.
- So it doesn't change the distribution — or expectation — of $g_i(\varepsilon)$.

That means we can replace $\mathrm{E}\, A_i$ in our variance formula with $\mathrm{E}(A_i + B_i)/2$.

$$
\frac{A_i + B_i}{2} = \frac{1}{2}\left[\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}\right]
$$

The rest. Once we've made the substitution, the rest boils down to Cauchy-Schwarz and our observation, from a few slides back, that $\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}^2$ and $\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}^2$ have the same distribution.

$$
\begin{aligned}
\mathrm{Var}\left[f(\varepsilon)\right] &= \frac{1}{2}\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}\left\{f\left(\varepsilon^{[i]}\right) - f\left(\varepsilon^{[i+1]}\right)\right\}\right] \\
&\le \frac{1}{2}\sqrt{\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}^2\right] \sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon^{[i]}) - f(\varepsilon^{[i+1]})\right\}^2\right]} \\
&= \frac{1}{2}\sqrt{\left\{\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}^2\right]\right\}^2} \\
&= \frac{1}{2}\sum_{i=1}^{n} \mathrm{E}\left[\left\{f(\varepsilon) - f(\varepsilon^{(i)})\right\}^2\right].
\end{aligned}
$$

References

- The proof of the Efron-Stein inequality is based on lecture 10 in Sourav Chatterjee's class Stein's method and applications.
- The bound on the variance of the maximum is based on Example 3.6 in Chapter 3 of Boucheron, Lugosi, and Massart's *Concentration inequalities: A nonasymptotic theory of independence*.