

# Inner Product Spaces

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## 1 Inner Products

A semi-inner-product  $\langle u, v \rangle$  on a *real vector space* is a real-valued function of two vectors  $u, v$  that is *symmetric*, *linear* in its arguments, and *positive*. That is, for all vectors  $u, v, w$  and scalars  $\alpha \in \mathbb{R}$ ,

$$\langle u, v \rangle = \langle v, u \rangle, \quad \langle u + \alpha v, w \rangle = \langle u, w \rangle + \alpha \langle v, w \rangle, \quad \text{and} \quad \langle u, u \rangle \geq 0.$$

An inner product is a semi-inner-product that is *positive definite*, i.e., that satisfies  $\langle u, u \rangle = 0$  if and only if  $u = 0$ . We tend to talk more about inner products than semi-inner products, but there are a few semi-inner products we use often that aren't positive-definite.

Here are some examples of semi-inner products.

- For real scalars, we have the product  $\langle u, v \rangle = uv$ .
- On finite dimensional vectors  $v \in \mathbb{R}^n$ , we have the dot product,  $\langle u, v \rangle_2 := \sum_{i=1}^n u_i v_i = u^T v$ .
- On functions  $v(x)$ , in terms of a random variable  $X$  with distribution  $P$ , we have the population inner product  $\langle u, v \rangle_{L_2(P)} = E[u(X)v(X)]$  and the covariance  $\text{Cov}_P(u, v) = E[\{u(X) - E[u(X)]\}\{v(X) - E[v(X)]\}]$ .

**Exercise 1** *Prove that these examples are semi-inner-products.*

For each of these, there is an associated seminorm  $\rho(v) = \sqrt{\langle v, v \rangle}$ . In fact, they're all included in the list of examples in the Vector Spaces Homework. To work out which it is, you can write out  $\langle v, v \rangle$  for the specific semi-inner-product you're thinking about, then compare to the example seminorms' definitions.

**Exercise 2** *For each of these examples of semi-inner-products, what is the corresponding seminorm?*

**Sample Inner Products** Just like with seminorms, sometimes when we're working with a sample  $X_1 \dots X_n$ , we think of functions as vectors: for functions  $u$  and  $v$ ,  $\langle u, v \rangle_2 = \sum_{i=1}^n u(X_i)v(X_i)$ . And, as with the seminorms, this is just a scaled version of the population inner product for the *empirical distribution*  $P_n$ , which is the distribution that puts probability  $1/n$  on each of our  $n$  observations.

$$\begin{aligned}\langle u, v \rangle_{L_2(P_n)} &= \sum_{x \in X_1 \dots X_n} P(X = x) u(x)v(x) \\ &= \sum_{i=1}^n \frac{1}{n} u(X_i)v(X_i) = \frac{1}{n} \langle u, v \rangle_2.\end{aligned}$$

## 1.1 Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality is the first tool we reach for when bounding a semi-inner-product. For any semi-inner-product  $\langle \cdot, \cdot \rangle$ ,  $|\langle u, v \rangle| \leq \rho(u)\rho(v)$  where  $\rho(v) = \sqrt{\langle v, v \rangle}$ ; furthermore, given any  $u$ , there is always a vector  $v$  of a given 'length'  $\rho(v)$  for which this bound is attained.

**Exercise 3** *Think about the Cauchy-Schwarz inequality in context of the inner product  $\langle u, v \rangle = uv$  on scalars, the dot product  $\langle u, v \rangle_2 = u^T v$ , and the covariance inner product  $\text{Cov}_P(u, v)$ . In each context, what does it say? Be as context-specific as you can; repeating the definition three times is not an instructive exercise. A sentence or two will do for each.*

I will not ask you to prove the Cauchy-Schwarz inequality, but if you're interested, take a look at one of the proofs on Wikipedia.

## 1.2 Hölder's Inequality

To bound the dot product on vectors in  $\mathbb{R}^n$ , Hölder's inequality is the second tool we reach for. While this is a fairly general tool, we often use a simple special case that's easy to prove: the one for the dot product,  $|\langle u, v \rangle_2| \leq \|u\|_1 \|v\|_\infty$ .<sup>1</sup>

**Exercise 4** *Prove it! If it takes you more than one line, you're doing it wrong.*

There are also versions for some inner products on functions. We'll want one for sample inner products analogous to the one we have for the dot product on vectors above:  $\langle u, v \rangle_{L_2(P_n)} \leq \|u\|_{L_1(P_n)} \|v\|_{L_\infty(P_n)}$ .

**Exercise 5** *Prove it! If you want, you can write a new proof, but it may be more instructive to show that it's implied by the case for vectors in  $\mathbb{R}^n$ .*

## 1.3 Triangle Inequality

When we showed that a few of our examples of seminorms are in fact seminorms in last week's homework, we didn't deal with any examples of seminorms associated with semi-inner-products. Let's do that now. Or the hard part, anyway.

**Exercise 6** Prove that, for any semi-inner product  $\langle u, v \rangle$ , the seminorm  $\rho(v) = \sqrt{\langle v, v \rangle}$  satisfies the triangle inequality.

**Hint.** You want to show that  $\rho(u + v)^2 \leq \{\rho(u) + \rho(v)\}^2$ . You know that  $\rho(u + v)^2 = \langle u + v, u + v \rangle$ . Expand this as the sum of four terms using *linearity*, then see what you can work out using the Cauchy-Schwarz inequality.

**Hint.** If you are not entirely comfortable with notation  $\langle u, v \rangle$  for inner products, use the more familiar notation  $u^T v$ .

## 2 Complex Vector Spaces

This semester, we'll mostly be working with real numbers, real vectors, and real-valued functions. But just like in high school algebra, it's occasionally useful to work with complex ones. Notationally,  $x \in \mathbb{C}$  is a complex number,  $v \in \mathbb{C}^n$  is a complex vector, and  $v : \mathcal{X} \rightarrow \mathbb{C}$  is a complex-valued function. If you've forgotten how to work with complex numbers, here's what you'll need to know.

1. A complex number is  $x + iy$  where  $x$  and  $y$  are real numbers and  $i = \sqrt{-1}$ .
2. These add and multiply as you'd expect: if  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ , then  $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$  and  $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + i^2 y_1 y_2 + i(y_1 x_2 + x_1 y_2) = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2)$ . Think about this: what happened to the  $i^2$  in the last equality?
3. We tend to think of a complex number  $z = x + iy$  as a vector in the plane with magnitude  $|x + iy| = \sqrt{x^2 + y^2}$  and angle  $\tan^{-1}(y/x)$ . We call  $\bar{z} = x - iy$  the complex conjugate of  $z = x + iy$ , which allows us a convenient expression for the magnitude of  $z$ :  $|z| = \sqrt{z \bar{z}}$ . Note that  $\overline{a + b} = \bar{a} + \bar{b}$ , and  $\overline{ab} = \bar{a} \bar{b}$ , and  $\bar{\bar{a}} = a$  for  $a, b \in \mathbb{C}$ . We will also refer to the complex conjugate of a vector or a function, which is interpreted *elementwise*. For a vector  $v \in \mathbb{C}^n$  with elements  $v_i$ ,  $\bar{v}$  is the vector with elements  $\bar{v}_i$ ; for a complex-valued function  $v$ ,  $\bar{v}$  is the function with  $\bar{v}(x) = \overline{v(x)}$  for all  $x$ .

When we're thinking about spaces of vectors  $v \in \mathbb{C}^n$  or functions  $v : \mathcal{X} \rightarrow \mathbb{C}$ , we'll want to think of them as elements of a *complex vector space*, i.e., as elements in a vector space where the *scalars* are complex numbers. For example, ...

- for a vector  $v \in \mathbb{C}^n$  and a scalar  $\alpha \in \mathbb{C}$ ,  $u = \alpha v \in \mathbb{C}^n$  will be the vector with elements  $u_i = \alpha v_i$ .
- for a function  $v : \mathcal{X} \rightarrow \mathbb{C}$  and a scalar  $\alpha \in \mathbb{C}$ ,  $u = \alpha v : \mathcal{X} \rightarrow \mathbb{C}$  is the function with  $u(x) = \alpha v(x)$  for all  $x \in \mathcal{X}$ .

To talk about inner products on complex vector spaces, we need to make one small change to our definition of a semi-inner product from Section 1. We have to talk about *conjugate-symmetry* instead of *symmetry*. A semi-inner-product

$\langle u, v \rangle$  on a complex vector space is a complex-valued function of two vectors  $u, v$  that is *conjugate-symmetric*, *linear* in its arguments, and *positive*. That is, for all vectors  $u, v, w$  and scalars  $\alpha \in \mathbb{C}$ ,

$$\langle u, v \rangle = \overline{\langle v, u \rangle}, \quad \langle u + \alpha v, w \rangle = \langle u, w \rangle + \alpha \langle v, w \rangle, \quad \text{and} \quad \langle u, u \rangle \geq 0.$$

Why conjugate-symmetry? Think about the simplest complex vector space: the complex numbers  $\mathbb{C}$ . By using the inner product  $\langle u, v \rangle = u\bar{v}$  for  $u, v \in \mathbb{C}$ , we get the magnitude  $|v| = \sqrt{v\bar{v}}$  as the norm  $\|v\| = \sqrt{\langle v, v \rangle}$ . More generally, ...

- For vectors  $u, v \in \mathbb{C}^n$ , we typically use the inner product  $\langle u, v \rangle = u^T \bar{v}$ , and we get the norm  $\|v\| = \sqrt{\sum_i |v_i|^2}$ .
- For functions  $u, v : [0, 1] \rightarrow \mathbb{C}$ , we typically use the inner product  $\langle u, v \rangle = \int_0^1 u(x) \overline{v(x)} dx$ , and we get the norm  $\|v\| = \sqrt{\int_0^1 |v(x)|^2 dx}$ .

If you look back at what's written above and in last week's homework, everything still works. Whenever  $\langle \cdot, \cdot \rangle$  is a semi-inner product on a complex vector space and  $\|v\| = \sqrt{\langle v, v \rangle}$  is the corresponding seminorm, the Cauchy-Schwarz inequality  $|\langle u, v \rangle| \leq \rho(u)\rho(v)$  and the triangle inequality  $\rho(u+v) \leq \rho(u) + \rho(v)$  hold. And when  $\langle \cdot, \cdot \rangle$  is the inner product  $\langle u, v \rangle = u^T \bar{v}$  on  $\mathbb{C}^n$ , Hölder's inequality  $|\langle u, v \rangle| \leq \|u\|_\infty \|v\|_1$  holds where  $\|u\|_\infty = \max_i |u_i|$  is the maximum of the magnitudes of the elements of  $u$  and  $\|v\|_1 = \sum_i |v_i|$  is the sum of the magnitudes of the elements of  $v$ .

If you want some practice working with complex numbers, try the exercises in Appendix A, where you'll prove a few of these.

### 3 Self-adjoint Operators

In this problem, we'll generalize of the idea of a symmetric matrix.

You can think of an  $n \times n$  matrix  $A \in \mathbb{R}^{n \times n}$  as a linear operator on the vector space  $\mathbb{R}^n$ , i.e. a function from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  that's linear in the sense that that  $A(\alpha u + \beta v) = \alpha Au + \beta Av$  for any  $\alpha, \beta \in \mathbb{R}$  and  $u, v \in \mathbb{R}^n$ . And we can talk about linear operators on other vectors spaces. For example,  $\frac{d}{dx}$  is a linear operator on the space of infinitely-differentiable functions, as  $\frac{d}{dx}\{\alpha u(x) + \beta v(x)\} = \alpha \frac{d}{dx}u(x) + \beta \frac{d}{dx}v(x)$ .

When we're working with an inner product  $\langle u, v \rangle$  on our vector space  $\mathcal{V}$ , we can define the *adjoint*  $A^*$  of a linear operator  $A$  to be another linear operator satisfying  $\langle A^*u, v \rangle = \langle u, Av \rangle$  for all vectors  $u$  and  $v$ . Here are some examples.

#### 3.1 Operators on finite dimensional spaces

When we're working with the dot product  $\langle u, v \rangle_2 = u^T v$  on  $\mathbb{R}^n$ , the adjoint of a matrix  $A \in \mathbb{R}^{n \times n}$  is its *transpose*  $A^T$ .

$$\langle A^T u, v \rangle_2 = (A^T u)^T v = u^T A v = \langle u, A v \rangle_2.$$

When we're working with the dot product  $\langle u, v \rangle_2 = u^T \bar{v}$  on  $\mathbb{C}^n$ , the adjoint of a matrix  $A \in \mathbb{C}^{n \times n}$  is its *conjugate transpose*  $\bar{A}^T$ . That is, it's the matrix whose elements are the complex conjugates of the elements in  $A^T$ .

$$\langle \bar{A}^T u, v \rangle_2 = (\bar{A}^T u)^T \bar{v} = u^T \bar{A} \bar{v} = u^T \overline{Av} = \langle u, Av \rangle.$$

**Why we use complex spaces.** Even when we really intend to work with real-valued vectors, it's useful to think about matrices as operators on  $\mathbb{C}^n$  and think of the dot product  $\langle u, v \rangle_2$  as  $u^T \bar{v}$ . We have to deal with complex numbers in any case, as matrices  $A \in \mathbb{R}^{n \times n}$  can have complex eigenvalues and eigenvectors. And the inner product  $u^T v$  we use on  $\mathbb{R}^n$  isn't an inner product on complex vectors at all, as the norm  $\|v\|^2 = \langle v, v \rangle$  associated with an inner product must be positive and  $u^T u$  will be negative for imaginary vectors.

### 3.2 Operators on spaces of functions

When we're working with the inner product  $\langle u, v \rangle = \int_{-\infty}^{\infty} u(x)v(x)dx$  on the vector space of infinitely-differentiable functions  $v : \mathbb{R} \rightarrow \mathbb{R}$  with  $v(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$ , the adjoint of the linear operator  $\frac{d}{dx}$  is  $-\frac{d}{dx}$ . To see this, we integrate by parts.

$$\begin{aligned} \left\langle u, \frac{d}{dx} v \right\rangle &= \int_{-\infty}^{\infty} u(x)v'(x)dx && \text{by definition} \\ &= u(x)v(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u'(x)v(x)dx && \text{because } (uv)' = u'v + uv' \\ &= 0 - \int_{-\infty}^{\infty} u'(x)v(x)dx && \text{because } u(x)v(x) \xrightarrow{x \rightarrow \pm\infty} 0 \\ &= \left\langle -\frac{d}{dx} u, v \right\rangle. \end{aligned}$$

Note that it's important that our vector space includes only functions that go to zero as  $x \rightarrow \pm\infty$ ; otherwise our 'boundary term'  $u(x)v(x)|_{-\infty}^{\infty}$  would be nonzero and we could not say that  $-\frac{d}{dx}$  was the adjoint of  $\frac{d}{dx}$ .

Specifying the vector space and inner product we're using is more important when talking about operators on spaces of functions than operators on finite-dimensional vectors. We can essentially get away with assuming we're talking about  $\mathbb{C}^n$  and  $\langle u, v \rangle = u^T \bar{v}$  in the latter case because that's what everyone always does; we don't have unspoken defaults like this for operators on functions.

**The Complex Case.** The adjoint is still  $-\frac{d}{dx}$  if we're thinking about  $\frac{d}{dx}$  as a linear operator on the space of *complex-valued* infinitely-differentiable functions with  $v(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  with the inner product  $\langle u, v \rangle = \int_{-\infty}^{\infty} u(x)\overline{v(x)}dx$ . It's useful to think this way for the same reason it's useful to think about  $\mathbb{C}^n$  instead of  $\mathbb{R}^n$ .

You may not be familiar with derivatives and integrals involving complex-valued functions. That's no big deal.<sup>2</sup> For a complex-valued function  $u : \mathbb{R} \rightarrow \mathbb{C}$ ,

$u(x) = u_r(x) + iu_i(x)$  where  $u_r$  and  $u_i$  are real-valued functions, differentiation and integration are done component-wise. That is,  $\frac{d}{dx}u(x) = \frac{d}{dx}u_r(x) + i\frac{d}{dx}u_i(x)$  and  $\int u(x)dx = \int u_r(x)dx + i\int u_i(x)dx$ . We can show that  $-\frac{d}{dx}$  is the adjoint of  $\frac{d}{dx}$  by using integration by parts as above on the real and imaginary components separately.

### 3.3 Self-adjointness

A self-adjoint operator on a vector space  $\mathcal{V}$  with an inner product  $\langle u, v \rangle$  is, as you would expect, an operator that is its own adjoint. That is, we say an operator  $A$  is self-adjoint if  $\langle Au, v \rangle = \langle u, Av \rangle$ . Symmetric matrices, i.e. matrices  $A$  with  $A^T = A$ , are self-adjoint on  $\mathbb{R}^n$  with the usual inner product  $\langle u, v \rangle = u^T v$ . *Conjugate-symmetric* matrices, i.e. matrices  $A$  with  $A^T = \bar{A}$ , are self-adjoint on  $\mathbb{C}^n$  with the usual inner product  $\langle u, v \rangle = u^T \bar{v}$ .

Now let's talk about self-adjoint operators on spaces of functions. A classic example is the differential operator  $-\frac{d^2}{dx^2}$  on the space of **2-periodic** complex-valued twice-differentiable functions,  $\{v : [-1, 1] \rightarrow \mathbb{C} : v(-1) = v(1)\}$ , with inner product  $\langle u, v \rangle = (1/2) \int_{-1}^1 u(x) \overline{v(x)} dx$ .<sup>3</sup>

**Exercise 7** Prove that the operator  $-\frac{d^2}{dx^2}$  on this space is self-adjoint. That is, prove that  $\langle -\frac{d^2}{dx^2}u, v \rangle = \langle u, -\frac{d^2}{dx^2}v \rangle$  for periodic functions  $u$  and  $v$ .

**Hint.** Integrate by parts twice. Why is it important that  $u$  and  $v$  be periodic?

### 3.4 Diagonalizing self-adjoint operators

Just like a matrix, a linear operator  $L$  has eigenvalues and eigenvectors: scalars  $\lambda$  and vectors  $v$  for which  $Lv = \lambda v$ .<sup>4</sup> In our example, they are defined by the differential equation  $-\frac{d^2}{dx^2}v = \lambda v$ . And like a symmetric matrix, a self-adjoint linear operator's eigenvalues are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal.

**Exercise 8** Prove that if  $L$  is a self-adjoint operator on a complex vector space with an inner product  $\langle u, v \rangle$ , then its eigenvalues are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal. That is, prove that

1. If  $Lv = \lambda v$  for some vector  $v$  and scalar  $\lambda \in \mathbb{C}$ , then  $\lambda \in \mathbb{R}$ .
2. If  $Lv = \lambda v$  and  $Lu = \eta u$  for vectors  $v$  and  $u$  and  $\lambda \neq \eta \in \mathbb{C}$ , then  $\langle u, v \rangle = 0$ .

Having done this, explain why this implies that, for integers  $j$  and  $k$  with  $j \neq k$ ,

$$\int_{-1}^1 \sin(\pi kx) \sin(\pi jx) dx = \int_{-1}^1 \cos(\pi kx) \cos(\pi jx) dx = \int_{-1}^1 \cos(\pi kx) \sin(\pi jx) dx = 0.$$

**Hint.** Recall from Section 2 that if  $\langle u, v \rangle$  is an inner product on a complex vector space, then for any vectors  $u$  and  $v$ ,

$$\langle u, v \rangle = \overline{\langle v, u \rangle}, \quad \langle u + \alpha v, w \rangle = \langle u, w \rangle + \alpha \langle v, w \rangle, \quad \text{and} \quad \langle u, u \rangle \geq 0.$$

**Hint.** What are  $\frac{d^2}{dx^2} \sin(\pi kx)$  and  $\frac{d^2}{dx^2} \cos(\pi kx)$ ?

Later on, we'll use the results we've proven to talk about the models defined using the *Sobolev seminorm*  $\rho(v) = \sqrt{\int_0^1 |v'(x)|^2 dx}$  and its generalizations.

## A Inner Products on Complex Vector Spaces: Exercises

These exercises are optional.

**Exercise 9 (*Optional*).** *Prove that, for any semi-inner product  $\langle u, v \rangle$  on a complex vector space, the seminorm  $\rho(v) = \sqrt{\langle v, v \rangle}$  satisfies the triangle inequality.*

*You may assume that the Cauchy-Schwarz inequality  $\langle u, v \rangle \leq \rho(u)\rho(v)$  holds.*

**Tip.** Do you need to change your solution to Exercise 6? If so, how?

**Exercise 10 (*Optional*).** *Prove Hölder's inequality for  $\mathbb{C}^n$ .*



## Notes

<sup>1</sup>If you'd like to get a sense of Hölder's inequality in full generality, take a look at this wikipedia article.

<sup>2</sup>What makes calculus involving complex numbers different is not really dealing with *complex-valued functions*, but rather with *functions of a complex variable*.

<sup>3</sup>If you prefer to think of these as periodic functions from  $\mathbb{R} \rightarrow \mathbb{C}$  you can, since all a 2-periodic function does outside the interval  $[-1, 1]$  is repeat what it does on  $[-1, 1]$ . Some people like to think of these as functions *on the circle*, too.

<sup>4</sup>The eigenvectors of operators on vector spaces of functions are sometimes called eigenfunctions.