Algorithm Design - HW2

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1 Exercise 1

a)

ILP
minimize $\sum_{i=0}^{|O|}\sum_{j=0}^{|F|} x_{ij} \cdot w(j,i)$
subject to
$\sum_{i=0}^{|O|} x_{ij} = 1$ ∀ j ∈ F
$\sum_{j=0}^{|F|} x_{ij} = 1$ ∀ i ∈ O
$x_{ij} \in \{0,1\}$ [Integer values]

LP
minimize $\sum_{i=0}^{|O|}\sum_{j=0}^{|F|} x_{ij} \cdot w(j,i)$
subject to
$\sum_{i=0}^{|O|} x_{ij} = 1$ ∀ j ∈ F
$\sum_{j=0}^{|F|} x_{ij} = 1$ ∀ i ∈ O
$0 \leq x_{ij} \leq 1$ [Fractional values]

In addition we have to add $|O| - |F|$ elements in F so that we have $|O| = |F|$ and then link every new node with every "outfit node" with edges of very high cost (larger the every other): in this was we have a balanced graph and can solve the problem with linear programming.

b) Suppose we have a LP solution with fractional values. For sure we have a fractional value $x_{i,j}$. Since the sum of variables adjacent to j is 1 there must be one other variable adjacent to j with fractional value $x_{k,j}$. For the same reason there must be an other fractional value adjacent to k $w_{k,z}$ and so on until we find a cycle to vertex i (without loosing generality). The cycle is even since the graph is bipartite. We now pick a constant e sufficiently small so that, adding it to all even variables and removing it from all odd variable, both remain between 0 and 1 and the sum of variables near each vertex remains 1 satisfying the constraints. The largest e is the smallest difference between an odd variable and 0 or the smallest difference between an even variable and 1. Now we have one less fractional variable and repeating this process after n steps we reach an integer solution.

2 Exercise 2

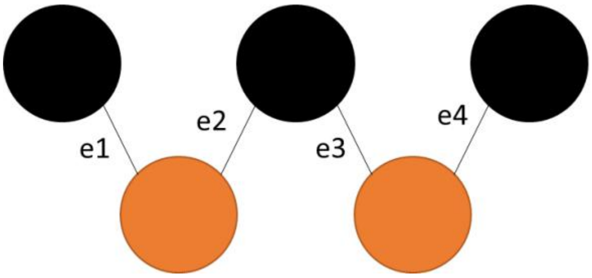
a)Suppose OPT is the optimal vertex cover while S is the vertex cover produced by Chris' algorithm. A_e indicates the event of choosing a node in OPT while B_e indicates the event of choosing a node not in OPT (both considering edge e). $A = \sum_e A_e$, $B = \sum_e B_e$, $S = A + B$ and $A \leq OPT$. Each time we pick an edge there are 2 case:

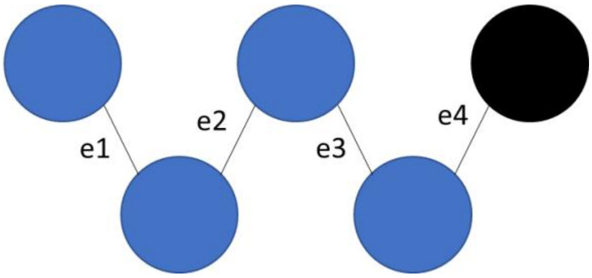
- Both endpoints belong to OPT: $A_e = 1$ and $B_e = 0$
- One endpoint belongs to OPT while the other not: $A_e = \frac{1}{2}$ and $B_e = \frac{1}{2}$

So we can claim that $Pr[B_e] \leq Pr[A_e]$ from which follows $E[B] \leq E[A]$ and again

$$E[S] = E[A + B] \leq 2E[A] \leq 2OPT$$

The following example show the OPT vertex cover (orange) and a possible vertex cover from Chris' algorithm (blue).





b) We claim that ∀ c ≥ 1 the set of building chosen by Chris' algorithm $|B'|$ might be ≥ c$|OPT|$. Indeed the worst case is the "star configuration" of the network in which there is a single node connected to all others. If Chris is very unlucky, his algorithm might choose N-1 times outer nodes having a N-1 times worse vertex cover. For this reason given c and a graph in star configuration with c+1 nodes we may have $|B'| \geq c|OPT|$.

3 Exercise 3

Payoff table for Philip (rows) and Bill (column):

	Rock	Paper	Scissor
Rock	α,0	$\alpha - 1$,1	α+1,-1
Paper	+1,-1	0,0	-1,+1
Scissor	-1,+1	+1,-1	0,0

a) Given Bill's mixed strategy (σ_r,σ_p,σ_s)=(σ_r,σ_p,1-σ_r-σ_p) we find the strategy for Philip to make Bill's choices indifferent setting the expected payoff for each strategy equal to each other:
$E_r = \alpha\sigma_r + (\alpha\text{-}1)\sigma_p + (\alpha\text{+}1)(1\text{-}\sigma_r\text{-}\sigma_p) =$
$E_p = 1\sigma_r + 0\sigma_p + (\text{-}1)(1\text{-}\sigma_r\text{-}\sigma_p) =$
$E_s = -1\sigma_r + 1\sigma_p + 0(1\text{-}\sigma_r\text{-}\sigma_p)$
The solution to this system is Bill's mixed strategy ($\frac{1}{3}$,$\frac{(\alpha+1)}{3}$,$\frac{(1-\alpha)}{3}$). We do the same for Philip obtaining his mixed strategy ($\frac{1}{3}$,$\frac{1}{3}$,$\frac{1}{3}$).
Now we can calculate expected payoff for Philip adding together the expected payoff for each combination of strategies times the probability of that:
$$Payoff = \frac{1}{3} \cdot [\frac{1}{3} \cdot \alpha + \frac{(\alpha+1)}{3} \cdot (\alpha - 1) + \frac{(1-\alpha)}{3} \cdot (\alpha + 1)] +$$
$$\frac{1}{3} \cdot [\frac{1}{3} \cdot 1 + \frac{(\alpha+1)}{3} \cdot 0 + \frac{(1-\alpha)}{3} \cdot (-1)] +$$
$$\frac{1}{3} \cdot [\frac{1}{3} \cdot (-1) + \frac{(\alpha+1)}{3} \cdot 1 + \frac{(1-\alpha)}{3} \cdot 0] = \frac{\alpha}{3}$$
b) For $\alpha \geq 1$ for every Bill's choice, Philip's rock payoff is always greater than paper's one so we can remove paper row. In the same way Bill's rock choice is dominant on his scissor choice so we remove scissor column. In this new matrix we can repeat the same procedure of point a and we get Philip's payoff $\frac{2\alpha - 1}{3}$.

4 Exercise 4

Suppose we have variables $X_i \in 0, 1$ describing if a student i deserves an award. X $= \sum_i^n X_i =$ number of students that deserve an award. A student i is worthy if:
$a_i \geq a_j \cup b_i \geq b_i \; \forall j \neq i \in \{1, .., n\}$
Ordering students according to a_i, "first" student gets the award for sure. Second one will win the award only if $b_2 \geq b_1$ and so on. $E[X_i] = \frac{1}{i}$. Linearity of expected value bring us to:

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{i} = H_n = 1 + \frac{1}{2} + \frac{1}{3} + .. + \frac{1}{n}$$

1)

$$H_n > \log(n+1)$$

2)

$$H_n \leq \log(n) + 1$$

Combined together we have $\log(n+1) < H_n \leq 1 + \log(n), n > 1$. So $H_n = O(\log(n))$.

5 Exercise 5

a) Suppose we have an edge e=(a,b) that connects node a with b and then pick node z to create the triple (a,b,z). A this point there are 3 cases:

1. (a,b,z) $\in T_1$ because there is only the edge connecting a to b
2. (a,b,z) $\in T_2$ because there is the edge connecting a to b + the edge connecting a to z OR b to z (2 edges total)
3. (a,b,z) $\in T_3$ because there is the edge connecting a to b + the edge connecting a to z AND b to z (3 edges total)

The triple can't be in T_0 because we will always have at least the edge connecting a to b.In conclusion the algorithm has $|T_1| + 2|T_2| + 3|T_3|$ choices. Knowing this and the fact that the number of triples that can be generated is $|E| \cdot (|V| - 2)$ because for every edge e in E we can pick a node in V different from e's endpoint, we can say that $|T_1| + 2|T_2| + 3|T_3| = |E| \cdot (|V| - 2)$.