FONDAMENTI DI AUTOMATICA

By Edoardo

$$\Phi(s) = (sI - A)^{-1}$$
$$H(s) = \Phi(s)B$$
$$\Psi(s) = C\,\Phi(s)$$
$$W(s) = C\,\Phi(s)B + D$$

TRASFORMATE LAPLACE

$$L(e^{\lambda t}) = \frac{1}{s-\lambda}$$
$$L(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$
$$L(\cos \omega t) = \frac{s}{s^2+\omega^2}$$
$$L(\delta_{-1}(t)) = 1/s$$
$$L(t\,\delta_{-1}(t)) = 1/s^2$$
$$L\left(\frac{t^k}{k!}\right) = \frac{1}{s^{k+1}}$$
$$L(e^{\alpha t}\sin \omega t) = \frac{\omega}{(s-\alpha)^2+\omega^2}$$
$$L(e^{\alpha t}\cos \omega t) = \frac{s-\alpha}{(s-\alpha)^2+\omega^2}$$

EVOLUZIONE LIBERA DELLO STATO

1) Trovo gli autovettori destri e sinistri
2) Calcolo $c_i = v_i^T x_0$

3a) Autovalori reali:

$$x_e(t) = \sum e^{\lambda t} u_i c_i$$

3b) Autovalori complessi:

$$x_e(t) = e^{\lambda_1 t} u_1 c_1 + e^{\alpha t} \cdot m \left(\sin(\omega t + \theta) u_a + \cos(\omega t + \theta) u_b\right)$$

con $m = \sqrt{c_a^2 + c_b^2}$ $\quad \theta = \operatorname{arctg}\left(\frac{c_a}{c_b}\right)$

EVOLUZIONE LIBERA DELL'USCITA

Autovalori reali:

$$\sum e^{\lambda t} C u_i c_i$$

Autovalori complessi

$$e^{\lambda t} C u_1 c_1 + e^{\alpha t} m \left[\sin(\omega t + \theta) C u_a + \cos(\omega t + \theta) C u_b\right]$$

RISPOSTA FORZATA

1) Trasformo $u(t)$ in $U(s)$ con le trasformate di Laplace
2) $Y_F(s) = W(s) \cdot u(s)$
3) Calcolo i residui
4) Ritrasformo in t

Residui

$\frac{R_1}{s} \longrightarrow \lim_{s \to 0} s\, Y_F(s)$

$\frac{N_1 s + N_2}{s^2 + as + b} \longrightarrow$ trovo N_1 e N_2 $\longrightarrow \frac{N_1 s + N_2}{s^2 + as + b} = \frac{L_1(s-\alpha) + L_2(\omega)}{(s-\alpha)^2 + \omega^2}$

$\frac{R_1}{s} + \frac{R_2}{s^2}$ per R_2 come sopra, per R_1 $\lim_{s \to 0} \frac{\partial f}{\partial s}\left(s^2\, Y_F(s)\right)$

$\frac{R_1}{s+\alpha} \longrightarrow \lim_{s \to -\alpha} (s+\alpha) \cdot Y_F(s)$

BODE

Termine Monomio S

Termine Binomio $S \pm$ termine noto

Termine Trinomio $S^2 \pm S \pm$ termine noto

Trasformazione in forma di Bode:

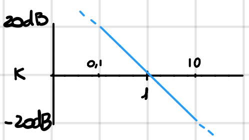
$(s+a) \longrightarrow a\left(\frac{s}{a}+1\right)$

Guadagno K = termine noto esplicitato totale

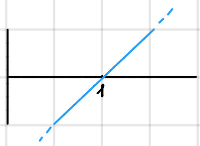
Grafico Modulo:

Tracciamo una retta a $20\log_{10}|K|$ su cui segnamo tutti i valori a_i della forma di Bode e le decadi successive e precedenti (es. $a = 10 \longrightarrow$ segno 1, 10, 100)

Termine Monomio:

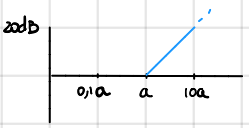


(se al denominatore)

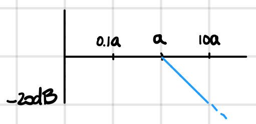


(se al numeratore)

Termine Binomio:



(se al numeratore)



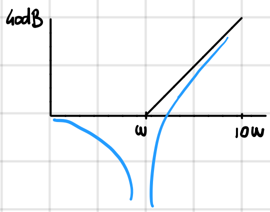
(se al denominatore)

Termine Trinomio: $(1 + aS + bS^2)$

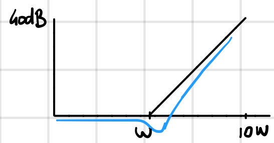
si confronta il termine trinomio con il polinomio caratteristico:

$aS = 2\varepsilon S/\omega \longrightarrow$ ricaviamo ε = smorzamento

$bS^2 = S^2/\omega \longrightarrow$ ricavo ω



(se al numeratore e $|\varepsilon| = 0$)



(se al numeratore e $|\varepsilon| \neq 0$)

I grafici sono ribaltati se il termine trinomio è al denominatore

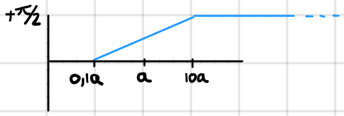
Grafico Fase

Fase "guadagno": 0 se $K>0$, $-\pi$ se $K<0$

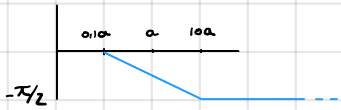
Fase termine monomio: $\pi/2$ se al numeratore, $-\pi/2$ se al denominatore

Tracciamo una retta a "fase guadagno + fase termine monomio" su cui segnamo le stesse decadi del grafico del modulo

Fase Termine Binomio

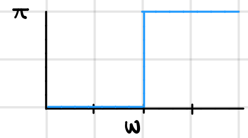


(se al numeratore $1/T>0$) inverso se $1/T<0$

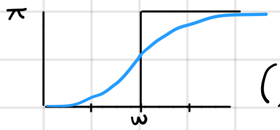


(se al denominatore e $1/T>0$) inverso se $1/T<0$

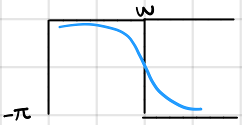
Fase Termine Trinomio



(se al numeratore e $\varepsilon=0$)



(se al numeratore e $\varepsilon>0$)



(se al numeratore e $\varepsilon<0$)

I grafici sono ribaltati se al denominatore

ANALISI MODALE

$\dot{x} = Ax + Bu$

$y = Cx + u$

1) Trovo gli autovalori di A calcolando $\det(A-\lambda I)$

2a) Autovalori reali

Calcolo $(A-\lambda I)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\forall \lambda$ e trovo gli autovettori

2b) Autovalori complessi coniugati $\alpha \pm j\omega$

Calcolo:

$$A \cdot \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}$$

Risolvo il sistema e trovo i due autovettori complessi

3) $T = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \longrightarrow T^{-1} = \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix}$

4) Calcolo $v_i^T B$ $\forall i$ $\longrightarrow$ se $\neq 0$ eccitabile

5) Calcolo $C u_i$ $\forall i$ $\longrightarrow$ se $\neq 0$ osservabile

6)

$Re > 0 \longrightarrow$ divergente

$Re < 0 \longrightarrow$ converge

Forma Canonica Osservabile

$$A=\left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & & -a_1 \\ 0 & 1 & 0 & \dots & \vdots \\ 0 & \dots & 1 & 0 & \\ 0 & \dots & & 1 & -a_n \end{array}\right) \quad B=\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} \quad C=(0 \dots 0\,1)$$

I termini a_i sono i coefficienti del denominatore
I termini b_i sono i coefficienti del numeratore
Il grado del denominatore da le dimensioni delle matrici

es. $\dfrac{s-10}{s(s^2-s+1)} \rightarrow \dfrac{0s^2+s-10}{s^3-s^2+s+0}$

$B_{3\times1}=\begin{pmatrix} -10 \\ 1 \\ 0 \end{pmatrix}$

NB. Segni cambiati

$A_{3\times3}=\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$

$C_{1\times3}=(0\,0\,1)$

Forma Canonica Raggiungibile

Si scambiano C e B e A viene riflessa lungo la diagonale

Stabilità

Prendo la parte reale degli autovalori
- Se tutte sono <0 asintoticamente stabile
- Se tutte sono ≤ 0 e m.g $=1$ di quelli $Re(\lambda)=0$ stabile semplice

Esistenza Risposta a Regime:
- stabilità
- $Re(\lambda)$ OSSERVABILI <0

Criterio di Routh

$\det(A-\lambda I) \longrightarrow$ polinomio caratteristico in λ : $a_n\lambda^n + a_{n-1}\lambda^{n-1} \dots a_0 = 0$

Si costruisce la tabella di routh:

n	a_n	a_{n-2}	a_{n-4}	...
$n-1$	a_{n-1}	a_{n-3}	a_{n-5}	...
$n-2$	b_{n-1}	b_{n-2}	b_{n-3}	...
$n-3$	c_{n-2}	c_{n-3}	...	

- $b_{n-1} = \dfrac{\det\begin{pmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{pmatrix}}{-a_{n-1}}$
- $b_{n-2} = \dfrac{\det\begin{pmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{pmatrix}}{-a_{n-1}}$
- $c_{n-2} = \dfrac{\det\begin{pmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-2} \end{pmatrix}}{-b_{n-1}}$

e così via fino a 0

Finita la tabella si controlla la prima colonna: per ogni cambio di segno corrisponde una radice a parte reale positiva, per ogni permanenza di segno una radice a parte reale negativa.

DECOMPOSIZIONE DI KALMAN

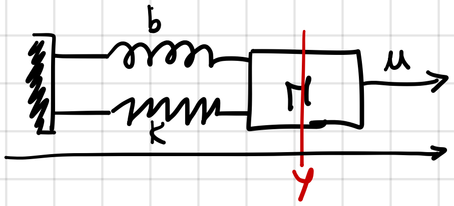
- Calcolo la matrice $\{B \ldots AB \ldots A^{n-1}B\}$ n = dimensione di A (K)
- Calcolo la matrice $\begin{Bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{Bmatrix}$ n = dimensione di A (K_o)
- Calcolo il rango di K $(\rho(K))$
- Prendo $\rho(K)$ colonne di K linearmente indipendenti e formo X_R
- Calcolo il rango di K_o
- Scrivo il sistema $K_o x = 0$ $\left(\text{es } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{matrix} x_1 + 2x = 0 \\ 3x_1 + 4x_2 = 0 \end{matrix}\right)$ e prendo $n - \rho(K)$ righe indipendenti
- Troviamo $n - \rho(k)$ vettori indipendenti con il sistema e formo X_{no}

- $X_1 = X_R \cap X_{NO}$
- $X_2 = X_1^{\perp} \cap X_R$
- $X_3 = X_1^{\perp} \cap X_{NO}$
- $X_4 = X_R^{\perp} \cap X_{NO}^{\perp}$

Andamento continuo
Andamento discreto

$$\begin{cases} \dot x = ax + bu & \text{(eq differenziali)} \\ x(k+1) = ax(k) + bu(k) & \text{(stato passato + sollecitazione)} \end{cases}$$

ES



$$Ma = u - b\dot y - ky \longrightarrow \underbrace{My^{(2)} + b\dot y + ky = u}_{\text{eq. diff. secondo ordine}}$$

$$\left.\begin{matrix} y = x_1 \\ \dot y = x_2 \end{matrix}\right\} \quad \left.\begin{matrix} \dot x_1 = x_2 \\ \dot x_2 = y^{(2)} = \frac{1}{M}(u - kx_1 - bx_2) \end{matrix}\right] \text{ sistema differenziale del primo ordine}$$

Forma Matriciale

$$\dot x = Ax + Bu$$
$$y = Cx + Du$$

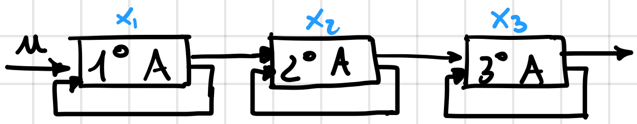
coeff.

$$A = \begin{array}{cc} x_1 & x_2 \\ \end{array} \atop \begin{bmatrix} 0 & 1 \\ -k/M & -b/M \end{bmatrix}$$

coeff.

$$B = \begin{array}{c} u \\ \end{array} \atop \begin{bmatrix} 0 \\ 1/M \end{bmatrix} \qquad D = 0 \qquad C = [1 \ \ 0]$$

ES



$$x_1(k+1) = a_1 x_1(k) + u(k)$$
$$x_2(k+1) = (1-a_1)x_1(k) + a_2 x_2(k)$$
$$x_3(k+1) = (1-a_2)x_2(k) + a_3 x_3(k)$$
$$y(k) = (1-a_3)x_3(k)$$

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 1-a_1 & a_2 & 0 \\ 0 & 1-a_2 & a_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad 1-a_3] \qquad D = 0$$

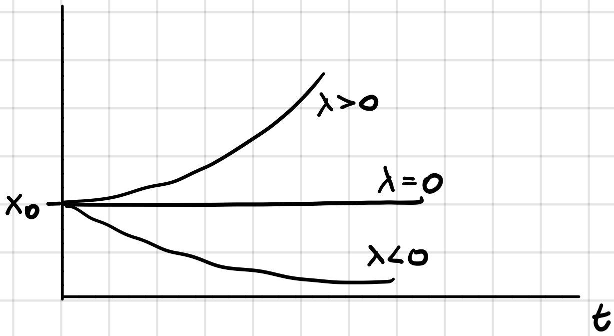
Sistema lineare, stazionario a dimensione finita.

Es

α - natalità β - morti $\qquad \dot{x} = (\alpha-\beta)x + bu$

$$x(t) = e^{(\alpha-\beta)(t-t_0)}x(t_0) \qquad \text{con } u=0$$

$$\alpha - \beta = \lambda \rightarrow x(t) = e^{\lambda t}x(t_0) \qquad (t_0=0)$$



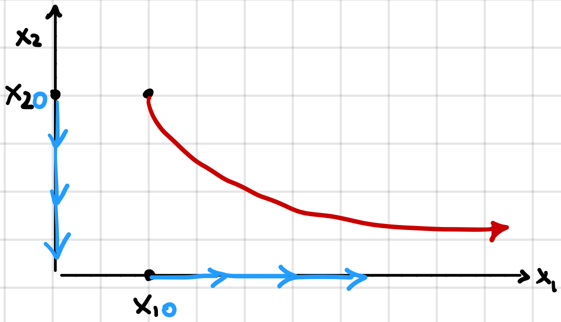
Es

$\dot{x}_1 = 3x_1 \qquad x_1(t) = e^{3t}x_{10} \qquad$ con $t_0=0$

$\dot{x}_2 = -2x_2 \qquad x_2(t) = e^{-2t}x_{20}$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \qquad \det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda)$$

$\hookrightarrow \lambda_1 = 3,\ \lambda_2 = -2$



Se non ho un sistema diagonale come $\dot{x} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{pmatrix}x + \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix}u$ posso ricondurlo ad uno diagonale

$x \longrightarrow z$

$A \longrightarrow \tilde{A}$

$\dot{x} = Ax + Bu$ $\xrightarrow{\text{cambio}}$ $z = Tx$ (T invertibile) $\longrightarrow \dot{z} = T\dot{x} = T[Ax+Bu]$

$y = Cx + Du$ coordinate

$$= \underbrace{TAT^{-1}}_{\tilde{A}}z + \underbrace{TB}_{\tilde{B}}u$$

$$y = \underbrace{CT^{-1}}_{\tilde{C}}z + \underbrace{D}_{\tilde{D}}u$$

Calcolo autovalori e autovettori

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u \qquad M=1 \quad b=4 \quad k=3$$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{vmatrix} = -\lambda(-4-\lambda)+3 = \lambda^2+4\lambda+3 \longrightarrow \lambda = -3, -1 \text{ Autovalori}$$

Gli autovettori rispettano la condizione $(A-\lambda I)u = 0$

$$\lambda_1 = -3 \rightarrow [A+3I]u_1 = 0 \rightarrow \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \longrightarrow 3a+b=0 \rightarrow b=-3a \qquad u_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$\lambda_2 = -1 \rightarrow [A+I]u_2 = 0 \rightarrow \begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightarrow a+b=0 \rightarrow a=-b \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

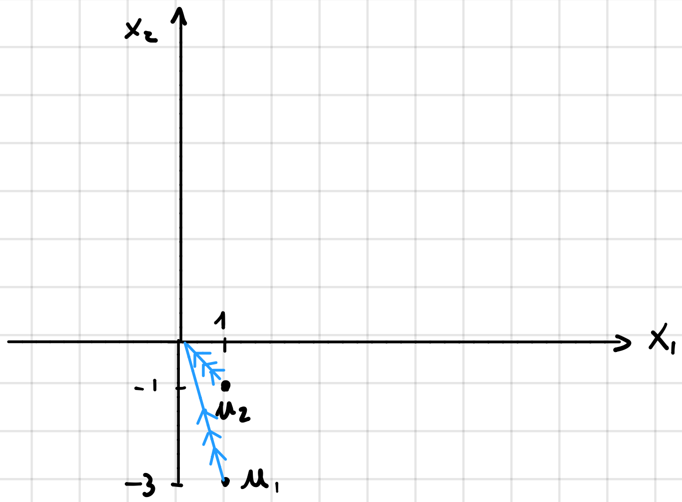
$$T^{-1} = (u_1, u_2) = \begin{bmatrix} 1 & 1 \\ -3 & -1 \end{bmatrix} \qquad T = \tfrac{1}{2}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix}$$

$$\tilde{A} = TAT^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -6 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{B} = TB = \tfrac{1}{2}\begin{pmatrix}-1 & -1\\ 3 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-1/2\\ 1/2\end{pmatrix}$$

$$\tilde{C} = CT^{-1} = (1 \;\; 0)\begin{pmatrix}1 & 1\\ -3 & -1\end{pmatrix} = (1 \;\; 1)$$

$$\begin{cases}\dot{z}_1 = -3z_1 + \tfrac{1}{2}u\\ \dot{z}_2 = -z_2 + \tfrac{1}{2}u\end{cases} \longrightarrow \begin{cases} z_1(t) = e^{-3t} z_{10}\\ z_2(t) = e^{-t} z_{20}\end{cases}$$



ES.

$$\dot{x} = \begin{pmatrix}-3 & 2 & 1\\ 0 & 1 & 2\\ 0 & 0 & -1\end{pmatrix}x + \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}u \qquad y = (1 \;\; 0 \;\; 0)\,x + u$$

• Calcolo autovalori e autovettori

det $A \Rightarrow \lambda_1 = -3 \quad \lambda_2 = 1 \quad \lambda_3 = -1$ gli autovalori sono sulla diagonale perché la matrice è triangolare

$\lambda_1 = -3 \qquad [A + 3I]\,u = 0$

$$\begin{pmatrix}0 & 2 & 1\\ 0 & 4 & 2\\ 0 & 0 & 2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = 0 \qquad \begin{matrix}2b + c = 0\\ 4b + 2c = 0\\ 2c = 0 \rightarrow c = 0\end{matrix} \quad b = 0 \longrightarrow \begin{pmatrix}a\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$\lambda_2 = 1$

$$\begin{pmatrix}-4 & 2 & 1\\ 0 & 0 & 2\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = 0 \qquad \begin{matrix}-4a + 2b + c = 0\\ 2c = 0 \longrightarrow c = 0\\ -2c = 0\end{matrix} \longrightarrow \begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$$

$\lambda_3 = -1$

$$\begin{pmatrix}-2 & 2 & 1\\ 0 & 2 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = 0 \qquad \begin{matrix}-2a + 2b + c = 0\\ 2b + 2c = 0\end{matrix} \quad \begin{matrix}-2a - c = 0\\ b = -c\end{matrix} \quad \begin{matrix}c = -2a\\ b = 2a\end{matrix} \longrightarrow \begin{pmatrix}1\\ 2\\ -2\end{pmatrix}$$

• Cambio coordinate che diagonalizzano

$$T^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix} = (u_1 \; u_2 \; u_3)$$

$$T = -\frac{1}{4} \begin{bmatrix} -4 & 2 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

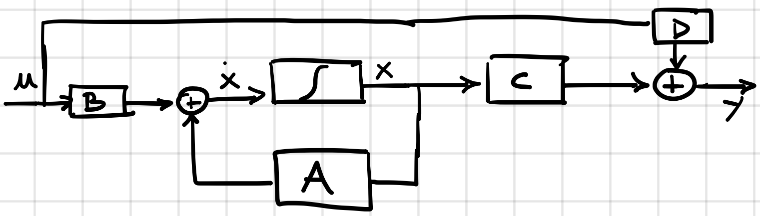
$$\tilde{A} = TAT^{-1} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\tilde{B} = \frac{1}{4} \begin{bmatrix} -4 & 2 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\tilde{C} = CT^{-1} = (1 \; 0 \; 0) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix} = (1 \; 1 \; 1)$$

$$\tilde{D} = D = 1$$

SCHEMA DI SIMULAZIONE



MODELLO ESPLICITO

$$\dot{x} = Ax$$
$$x(t) = e^{A(t-t_0)}x(t_0)$$

Con $t_0 = 0$ $\quad e^{At} = Id + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!}\dots$

$$x(t) = e^{At}x(0)$$
$$\dot{x}(t) = d/dt\,(e^{At})\,x(0) = d/dt\,(Id + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!}\dots)$$
$$= (A + A^2 t + A^3\frac{t^2}{2} + A^4\frac{t^3}{3!}\dots)\,x(0) = A[Id + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \dots]x(0)$$
$$= A\,e^{At}x(0)$$

$\Phi = e^{At}$ = matrice di transizione dello stato

Se $u \neq 0$:

$$\dot{x} = Ax + Bu$$
$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$
$$\dot{x}(t) = d/dt\,(e^{At})\,x(0) + d/dt\left[\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\right] =$$
$$= A\,e^{At}x(0) + \int_0^t d/dt\,[e^{A(t-\tau)}Bu(\tau)]\,d\tau + [e^{A(t-\tau)}Bu(\tau)]_{t=\tau} = Ax + Bu$$

$e^{At}B$ = matrice delle risposte impulsive dello stato $= H(t)$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$
$$y(t) = C\,e^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$\varphi(t) = Ce^{At}$ = matrice di trasformazione

$$A = T^{-1}\tilde{A}T \Rightarrow e^{At} = Id + T^{-1}\tilde{A}Tt + \underbrace{T^{-1}\tilde{A}TT^{-1}\tilde{A}T}_{T^{-1}\tilde{A}^2 T}\frac{t^2}{2} + \underbrace{T^{-1}\tilde{A}TT^{-1}\tilde{A}TT^{-1}\tilde{A}T}_{T^{-1}\tilde{A}^3 T}\frac{t^3}{3!}\dots$$
$$= T^{-1}\left(Id + \tilde{A}t + \tilde{A}^2\frac{t^2}{2} + \tilde{A}^3\frac{t^3}{3!}\dots\right)T$$

$$\Rightarrow \boxed{e^{At} = T^{-1}e^{\tilde{A}t}T}$$
$$= T^{-1}\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}T$$

es.

$$T^{-1} = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} \quad \lambda_1 = -3 \quad \lambda_2 = -1 \qquad T = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} e^{-3t} & e^{-t} \\ -3e^{-3t} & -e^{-t} \end{pmatrix} \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix} = \dots$$

$$T^{-1} = (u_1 \; u_2) \quad T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \Rightarrow \begin{matrix} [A - \lambda I]\, u = 0 \\ v\,[A - \lambda I] = 0 \end{matrix}$$

autovettori destri
autovettori sinistri

$$\boxed{e^{At} = (u_1 \; u_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}} = \left[e^{\lambda t} u_1 \,\middle|\, e^{\lambda_2 t} u_2 \right] \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = e^{\lambda t} u_1 v_1^T + e^{\lambda_2 t} u_2 v_2^T$$

$$x(t) = e^{At} x(0) = e^{\lambda_1 t} \underbrace{u_1 v_1^T x(0)}_{c_1} + e^{\lambda_2 t} \underbrace{u_2 v_2^T x(0)}_{c_2}$$

es

$$\dot{x} = \begin{pmatrix} -4 & 1 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \qquad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$y = (1 \quad 1)\, x$$

$$\lambda_1 = -4 \quad \lambda_2 = -2 \qquad u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \quad T = \frac{1}{2} \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

$$c_1 = v_1^T u_1 x_0 = \frac{1}{2} [2 \;\; -1] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$c_2 = \frac{1}{2} [0 \;\; 1] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$x(t) = e^{-4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e^{At} = \sum_1^n e^{\lambda_i t} u_i v_i$$

$$x_e(t) = e^{At} x(0) = \sum_1^n e^{\lambda_i t} u_i^o c_i^o$$

$A \Rightarrow \det(A - \lambda I) \Rightarrow \lambda = \alpha \pm j\omega$ complesso

$$u = u_a \pm j u_b$$

$[A - \lambda I] u = 0 \Rightarrow [A - (\alpha + j\omega) I](u_a + j u_b) = 0 \Rightarrow Re + j Im = 0$ $Re = Im = 0$

$$[A - \alpha I](u_a + j u_b) - j\omega I(u_a + j u_b) = 0$$
$$= [A - \alpha I) u_a + j(A - \alpha I) u_b - j\omega u_a + \omega u_b = 0$$

$\Rightarrow (A - \alpha I) u_a + \omega u_b = 0$ Re
$(A - \alpha I) u_b - \omega u_a = 0$ Im

$$A u_a = \alpha u_a - \omega u_b \qquad A(u_a \; u_b) = (u_a \; u_b)\begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix}$$
$$A u_b = \omega u_a + \alpha u_b$$

$$\Rightarrow \underbrace{(u_a \; u_b)^{-1} A \, (u_a \; u_b)}_{T \; A \; T^{-1}} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}$$

$A_{3\times3}$ $\lambda_1 =$ $\leftarrow u_1$
$\lambda_{2,3} = \alpha \pm j\omega$ $\leftarrow u = u_a \pm j\omega u_b$

$$T^{-1} = (u_1 \; u_a \; u_b) \longrightarrow T = \begin{pmatrix} v_1^T \\ v_a^T \\ v_b^T \end{pmatrix}$$

$$\Phi(t) = e^{\lambda_1 t} u_1 v_1^T + e^{\alpha t}\left(\cos(\omega t)(u_a v_a^T + u_b v_b^T) + \operatorname{sen}(\omega t)(u_a v_b^T - u_a v_b^T)\right)$$

$$\tilde{A} = T A T^{-1} = \left(\begin{array}{c|cc} \lambda_1 & 0 & 0 \\ \hline 0 & \alpha & \omega \\ 0 & -\omega & \alpha \end{array}\right) \qquad e^{\tilde{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\alpha t}\cos(\omega t) & e^{\alpha t}\operatorname{sen}(\omega t) \\ 0 & -e^{\alpha t}\operatorname{sen}(\omega t) & e^{\alpha t}\cos(\omega t) \end{pmatrix}$$

$$e^{At} = T^{-1} e^{\tilde{A}t} T = (u_1 \; u_a \; u_b)\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & \boxed{} \end{bmatrix}\begin{pmatrix} v_1^T \\ v_a^T \\ v_b^T \end{pmatrix} = \left[e^{\lambda_1 t} u_1 \,\middle|\, e^{\alpha t}(u_a \cos\omega t - u_b \operatorname{sen}\omega t) \,\middle|\, e^{\alpha t}(u_a \operatorname{sen}\omega t + u_b \cos\omega t) \right]\begin{pmatrix} v_1^T \\ v_a^T \\ v_b^T \end{pmatrix}$$

$$e^{\lambda_1 t} u_1 v_1^T + e^{\alpha t}(u_a v_a^T \cos\omega t - u_b v_a^T \operatorname{sen}\omega t) + e^{\alpha t}(u_a v_b^T \operatorname{sen}\omega t + u_b v_b^T \cos\omega t) =$$

$$e^{\lambda_1 t} u_1 v_1^T + e^{\alpha t}\left[\cos\omega t(u_a v_a^T + u_b v_b^T) + \operatorname{sen}\omega t(u_a v_b - u_b v_a^T)\right)$$

$$x(t) = e^{At}x(0) = e^{\lambda_1 t} u_1 v_1^T x(0) + e^{\alpha t}\left(\cos\omega t\left(u_a v_a^T x(0) + u_b v_b^T x(0)\right) + \operatorname{sen}\omega t\left(u_a v_b^T x(0) - u_b v_a^T x(0)\right)\right) =$$
$$= e^{\lambda_1 t} u_1 c_1 + e^{\alpha t}\left(\cos\omega t\left(u_a c_a + u_b c_b\right) + \operatorname{sen}\omega t\left(u_a c_b - u_b c_a\right)\right)$$

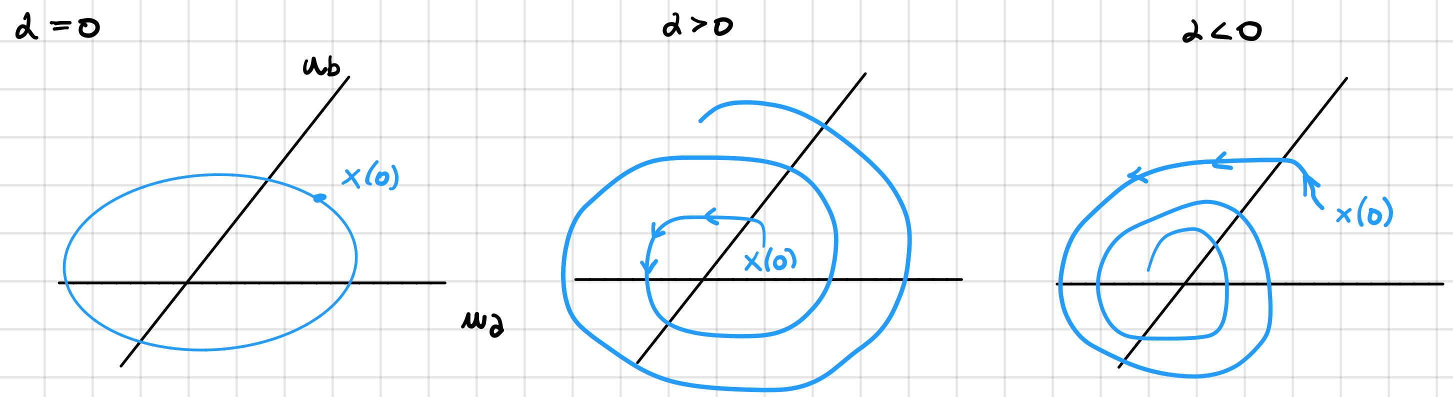
NB. $z = Tx \quad T^{-1} = (u_1\ u_a\ u_b) \quad z(0) = \begin{pmatrix} v_1^T \\ v_a^T \\ v_b^T \end{pmatrix} x(0) = \begin{pmatrix} c_1 \\ c_a \\ c_b \end{pmatrix}$

$$C = c_b + j c_a = m e^{j\phi} = m\cos\phi + j \operatorname{sen}\phi$$

$m = \sqrt{c_a^2 + c_b^2} \qquad c_b = m\cos\phi$

$\phi = \operatorname{arctg}\frac{c_a}{c_b} \qquad c_a = m \operatorname{sen}\phi$

$$\Rightarrow e^{\lambda_1 t} u_1 c_1 + m e^{\alpha t}\left(\left(\cos\omega t \frac{c_a}{m} + \operatorname{sen}\omega t \frac{c_b}{m}\right) u_a + \left(\cos\omega t \frac{c_b}{m} - \operatorname{sen}\omega t \frac{c_a}{m}\right) u_b\right)$$
$$= e^{\lambda_1 t} u_1 c_1 + m e^{\alpha t}\left(\left(\cos\omega t \operatorname{sen}\phi + \operatorname{sen}\omega t \cos\phi\right) u_a + \left(\cos\omega t\cos\phi - \operatorname{sen}\omega t \operatorname{sen}\phi\right) u_b\right)$$
$$= e^{\lambda_1 t} u_1 c_1 + m e^{\alpha t}\left(\operatorname{sen}(\omega t + \phi) u_a + \cos(\omega t + \phi) u_b\right)$$

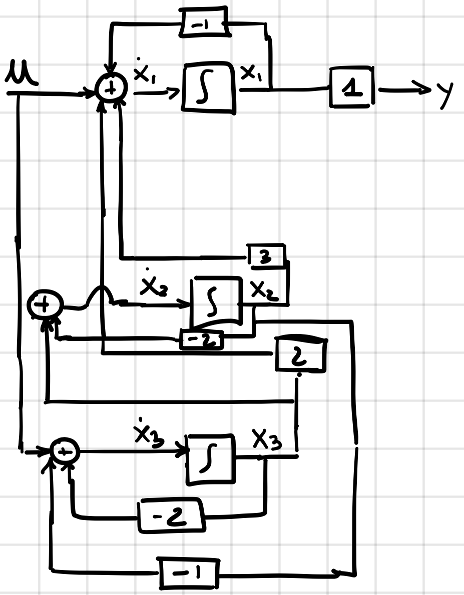


esercizio

$$\dot{x} = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x$$

1) Tracciare lo schema di simulazione
2) Calcolare $\Phi(t)$
3) $u=0$ calcolare $x(t)$ $\quad x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

1)



2) autovalori

$$\left(\begin{array}{c|cc} -1 & 3 & 2 \\ \hline 0 & -2 & 1 \\ 0 & -1 & -2 \end{array}\right) \quad \begin{pmatrix} \alpha & w \\ -w & \alpha \end{pmatrix} \Rightarrow \lambda_1 = -1 \qquad \lambda_{2,3} = -2 \pm j$$

$$u_1 : [A - (-1)I] \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad \begin{bmatrix} 0 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = 0 \Rightarrow b = c = 0 \Rightarrow u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[A - \underset{\alpha + jw}{\lambda} I] \underset{u_a + j u_b}{u} = 0 \qquad Re - Im = 0 \qquad A(\underset{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}{u_a} \; \underset{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}{u_b}) \begin{bmatrix} \alpha & w \\ -w & \alpha \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$$

$$\begin{array}{cc} 1 & 2 \end{array} \qquad\qquad \begin{array}{cc} 1 & 2 \end{array}$$

1

$$-a_1+3a_2+2a_3=-2a_1-b_1$$
$$-\cancel{2a_2}+a_3=-\cancel{2a_2}-b_2$$
$$-a_2-\cancel{2a_3}=-\cancel{2a_3}-b_3$$

$$a_3=-b_2$$
$$a_2=b_3$$

2

$$-b_1+3b_2+2b_3=a_1-2b_1$$
$$-\cancel{2b_2}+b_3=a_2-\cancel{2b_2}$$
$$-b_2-\cancel{2b_3}=a_3-\cancel{2b_3}$$

//

$$\left.\begin{aligned}&3b_3-2b_2=a_1-b_1\\&a_3=b_2\\&a_2=b_3\\&3b_2+2b_3=a_1-b_1\end{aligned}\right\}$$

$$b_1=-b_2/2-\frac{5b_2}{2}$$
$$a_1=5b_2/2-b_3/2$$

$$u_b=\begin{pmatrix}-1\\2\\0\end{pmatrix}\qquad u_a=\begin{pmatrix}5\\0\\-2\end{pmatrix}$$

$$T^{-1}=(u_1\ u_a\ u_b)=\begin{pmatrix}1&5&-1\\0&0&2\\0&-2&0\end{pmatrix}\qquad T=\begin{pmatrix}v_1^T\\v_a^T\\v_b^T\end{pmatrix}=1/4\begin{bmatrix}4&2&10\\0&0&-2\\0&2&0\end{bmatrix}=1/2\begin{bmatrix}2&1&5\\0&0&-1\\0&1&0\end{bmatrix}$$

$$\Phi(t)=e^{At}=$$

$$e^{-t}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1&1/2&5/2\end{pmatrix}+e^{-2t}\left(\cos t\left(\begin{pmatrix}5\\0\\-2\end{pmatrix}\begin{pmatrix}0&0&-1/2\end{pmatrix}+\begin{pmatrix}-1\\2\\0\end{pmatrix}\begin{pmatrix}0&1/2&0\end{pmatrix}\right)\right)+$$

$$e^{-2t}\left(\operatorname{sen} t\left(\begin{pmatrix}5\\0\\-2\end{pmatrix}\begin{pmatrix}0&1/2&0\end{pmatrix}-\begin{pmatrix}-1\\2\\0\end{pmatrix}\begin{pmatrix}0&0&-1/2\end{pmatrix}\right)\right)$$

$$=e^{-t}\begin{pmatrix}1&1/2&5/2\\0&0&0\\0&0&0\end{pmatrix}+e^{-2t}\left[\cos t\left(\begin{pmatrix}0&0&-5/2\\0&0&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&-1/2&0\\0&1&0\\0&0&8\end{pmatrix}\right)+\operatorname{sen} t\left(\begin{pmatrix}0&5/2&0\\0&0&0\\0&-1&0\end{pmatrix}-\begin{pmatrix}0&0&1/2\\0&0&-1\\0&0&0\end{pmatrix}\right)\right]$$

$$\begin{pmatrix}1&1/2&5/2\\0&0&-1/2\\0&1/2&0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}7/2\\-1/2\\0\end{pmatrix}\begin{matrix}\leftarrow c_1\\\leftarrow c_a\\\leftarrow c_b\end{matrix}$$

$$m=\sqrt{c_a^2+c_b^2}=1/2$$
$$\varphi=\operatorname{arctg}\frac{c_a}{c_b}-\pi/2$$

$$e^{-t}\begin{pmatrix}1\\0\\0\end{pmatrix}7/2+1/2\,e^{-2t}\left(\operatorname{sen}(t-\pi/2)\begin{pmatrix}5\\0\\-2\end{pmatrix}+\cos(t+\pi/2)\begin{pmatrix}-1\\2\\0\end{pmatrix}\right)$$

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\}$$ modello implicito

$$\left.\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-T)}Bu(T)\,dT \\ y(t) &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-T)}Bu(T)\,dT + Du(t)\end{aligned}\right\}$$ modello esplicito

$$H(t) = e^{At}B \qquad \Psi(t) = Ce^{At} \qquad W(t) = Ce^{At}B + D\delta(t)$$

$$\Rightarrow \begin{bmatrix} x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} H(t-T)u(T)\,dT \\ y(t) = \Psi(t-t_0)x(t_0) + \int_{t_0}^{t} W(t-T)u(T)\,dT \end{bmatrix}$$

$$x_{01}(t_0) \quad u_1[t_0, t) \rightarrow \begin{matrix}\bar{x}(t) \\ \bar{y}(t)\end{matrix}$$

$$\bar{x}(t) = \Phi(t-t_0)\,x_{01}(t_0) + \int_{t_0}^{t} H(t-T)u_1(T)\,dT$$

$$x_{02}(t_0) \quad u_2[t_0, t) \rightarrow \begin{matrix}\tilde{x}(t) \\ \tilde{y}(t)\end{matrix}$$

$$\tilde{x}(t_0) - \Phi(t-t_0)x_{02} + \int_{t_0}^{t} H(t-T)u_2(T)\,dT$$

$$x_0(t_0) = C_1 x_{01}(t_0) + C_2 x_{02}(t)$$
$$u[t_0, t) = C_1 u_1[t_0, t) + C_2 u_2[t_0, t)$$

$$x(t) = \Phi(t-t_0)(C_1 x_{01} + C_2 x_{02}) + \int_{t_0}^{t} H(t-T)(C_1 u_1(T) + C_2 u_2(T))\,dT$$
$$= \underbrace{\Phi(t-t_0)C_1 x_{01} + \int_{t_0}^{t} H(t-T)C_1 u_1(T)\,dT}_{C_1\bar{x}(t)} + \underbrace{\Phi(t-t_0)C_2 x_{02} + \int_{t_0}^{t} H(t-t_0)C_2 u_2(T)\,dT}_{C_2\tilde{x}(t)}$$

$\bar{x}(t)(u_1 - 0) =$ risposta al forzamento del sistema

$x(t) = x_l(t) + x_f(t) =$ composizione evoluzione totale (libera + forzata)

$$H(t) = e^{At}B = \left[e^{\lambda_1 t} u_1 v_1^T + e^{\alpha t}\left(\cos\omega t\,(u_a v_a^T + u_b v_b^T) + \operatorname{sen}\omega t\,(u_a v_b^T - u_b v_a^T)\right)\right] B$$

Se $v_1^T B = 0$ NON eccitabile con impulsi in ingresso

$\left.\begin{matrix} v_a^T B = 0 \\ v_b^T B = 0 \end{matrix}\right\}$ NON eccitabile nel modo pseudoperiodico

$$y(t) = \underbrace{\Psi(t-t_0)\,x(t_0)}_{y_e(t)} + \int_{t_0}^{t} w(t-\tau)\,u(\tau)\,d\tau$$

$$\Psi(t) = Ce^{At} = C\left(e^{\lambda_1 t} u_1 v_1^T + e^{\alpha t}\left(\cos\omega t\,(u_a v_a^T + u_b v_b^T) + \operatorname{sen}\omega t\,(v_a v_b^T - u_b v_b^T)\right)\right)$$

Se $Cu_1 = 0$ NON osservabile in uscita

$Cu_a = Cu_b = 0$ NON osservabile in uscita nel modo pseudoperiodico

$$W(t) = Ce^{At}B + D\delta(t) = \Psi(t)B + D\delta(t)$$

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 0$

Calcolare $\Phi(t)$, $H(t)$, $\Psi(t)$ e $W(t)$.

Studiare eccitabilità e osservabilità dei modi

$\lambda_1 = -3 \quad u_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad T^{-1} = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} \quad T = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 3 & 1 \end{bmatrix} \begin{matrix} = v_1^T \\ = v_2^T \end{matrix}$

$\lambda_2 = -1 \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\Phi(t) = e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

	Ecc	Oss.
λ_1	✓	✓
λ_2	✓	✓

$v_1^T B = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \neq 0$

$v_2^T B = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \neq 0$

$$H(t) = \Phi(t)B = e^{\lambda_1 t} u_1 v_1^T B + e^{\lambda_2 t} u_2 v_2^T B = e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}\left(-\frac{1}{2}\right) + e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\left(\frac{1}{2}\right)$$

$Cu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = 1 \neq 0$

$Cu_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \neq 0$

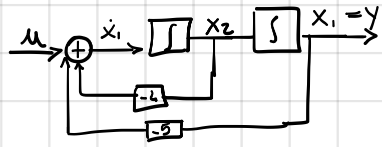
$$\Psi(t) = Ce^{At} = Ce^{\lambda_1 t} u_1 v_1^T + Ce^{\lambda_2 t} u_2 v_2^T = e^{-3t}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + e^{-t}\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

$$W(t) = C\Phi(t)B = Ce^{\lambda_1 t} u_1 v_1^T B + Ce^{\lambda_2 t} u_2 v_2^T B = -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}$$

$$\dot{x} = Ax + Bu \qquad A = \begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$y = Cx + Du \qquad D = 0$$

Tracciare schema simulazione, calcolare autovettori, $\Phi(t)$, $H(t)$, $\Psi(t)$ e $W(t)$. Studiare eccitabilità e osserv. modi

$$\begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \rightarrow \begin{aligned} \dot{x}_1 &= x_2 + v_2 \\ \dot{x}_2 &= -5x_1 - 5x_2 + u \end{aligned} \qquad y = x_1$$



autovalori: $(A - \lambda I) = 0 \rightarrow \lambda^2 + 4\lambda + 5 \rightarrow \lambda_{1,2} = -2 \pm j$

$$\begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \Longrightarrow$$

$$\left.\begin{aligned} a_2 &= -2a_1 - b_1 \\ -5a_1 - \cancel{4}^{2} a_2 &= -\cancel{2a_2} - b_2 \\ b_2 &= a_1 - 2b_1 \\ -5b_1 - \cancel{4}^{2} b_2 &= a_2 - \cancel{2b_2} \end{aligned}\right\} \quad \begin{aligned} a_2 &= -2a_1 - b_1 \\ -5a_1 - 2a_2 &= -b_1 \\ b_2 &= a_1 - 2b_1 \\ -5b_1 - 2b_2 &= a_2 \end{aligned} \Bigg\} \quad \dots \quad \Bigg\} \begin{aligned} a_1 &= 2b_1 + b_2 \\ a_2 &= -5b_1 - 2b_2 \end{aligned}$$

$$u_b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u_a = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad T^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{matrix} = v_a^T \\ = v_b^T \end{matrix}$$

$$\Phi(t) = e^{-2t}\left[\cos t\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix}\right) + \operatorname{sen} t\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right)\right]$$
$$= e^{-2t}\left[\cos t\left[\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\right] + \operatorname{sen} t\left[\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\right]$$
$$= e^{-2t}\left[\cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \operatorname{sen} t \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}\right]$$

Eccitabilità

$$\left.\begin{aligned} v_a^T B &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ v_b^T B &= \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \end{aligned}\right\} \text{Eccitabile}$$

Osservabilità

$$C u_a = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \qquad \text{Osservabile (basta uno solo)}$$
$$C u_b = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$H(t) = e^{\lambda t}\left(\cos \omega t\,(u_a \cdot 0 + u_b \cdot 1) + \operatorname{sen} \omega t\,(u_a \cdot 1 - u_b \cdot 0)\right)$$
$$= e^{-2t}\left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \operatorname{sen} t \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$$

$$\Psi(t) = e^{\alpha t}\left(\cos \omega t\,(1 \cdot v_a^T + 0 \cdot v_b^T) + \operatorname{sen} \omega t\,(1 \cdot v_b^T - 0 \cdot v_a^T)\right) = e^{-2t}\left(\cos t \begin{pmatrix} 1 & 0 \end{pmatrix} + \operatorname{sen} t \begin{pmatrix} 2 & 1 \end{pmatrix}\right)$$

$$W(t) = Ce^{At}B + D\delta(t) = C\, e^{\lambda t}\left(\cos \omega t\,(\cancel{u_a v_a^T} + \cancel{u_b v_b^T}) + \operatorname{sen} \omega t\,(u_a v_b^T - \cancel{u_b v_a^T})\right) B$$
$$= e^{-2t} \operatorname{sen} t \cdot 1$$

$$x(t) = \Phi(t-t_0)x(t_0) + \int_{t_0}^{t} H(t-T)u(T)\,dT \quad = x_L + x_F$$
$$y(t) = \Psi(t-t_0)x(t_0) + \int_{t_0}^{t} W(t-T)u(T)\,dT \quad = y_L + y_F$$

$$\left.\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx+Du \end{aligned}\right] \text{PROCESSO}$$

TRASFORMATA DI LAPLACE

$$f(t) \Rightarrow f(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\mathcal{L}(e^{\lambda t}) = \int_0^{\infty} e^{\lambda t} e^{-st} dt = \int_0^{\infty} e^{-(s-\lambda)t} dt = \left[-\frac{1}{s-\lambda} \cdot e^{-(s-\lambda)}\right]_0^{\infty} = \frac{1}{s-\lambda}$$

$$\mathcal{L}(\mathrm{sen}\,wt) = \mathcal{L}\left(\frac{e^{jwt}-e^{-jwt}}{2j}\right) = \frac{1}{2j}\left(\mathcal{L}(e^{jwt}) - \mathcal{L}(e^{-jwt})\right) = \frac{1}{2j}\left(\frac{1}{s-jw} - \frac{1}{s+jw}\right) = \frac{1}{2j}\left(\frac{w}{s^2+w^2}\right)$$

$$\mathcal{L}(\cos wt) = \frac{\mathcal{L}(e^{jwt}) + \mathcal{L}(e^{-jwt})}{2} = \frac{1}{2}\left(\frac{1}{s-jw} + \frac{1}{s+jw}\right) = \frac{s}{s^2+w^2}$$

$$\mathcal{L}(e^{\alpha t}\mathrm{sen}\,wt) = \frac{1}{2j} \cdot \mathcal{L}\left(e^{(\alpha+jw)t} - e^{(\alpha-jw)t}\right) = \frac{1}{2j}\left[\frac{1}{s-\alpha-jw} - \frac{1}{s-\alpha+jw}\right] = \frac{w}{(s-\alpha)^2+w^2}$$

$$\mathcal{L}(e^{\alpha t}\cos wt) = \frac{s-\alpha}{(s-\alpha)^2+w^2}$$

$$\mathcal{L}(\delta_{-1}(t)) - \text{funzione gradino} = 1/s$$

$$\mathcal{L}\left(\frac{t^k}{k!}\right) = \frac{1}{s^{k+1}}$$

Se $e^{\lambda t} \rightarrow \frac{1}{s-\lambda} = (s-\lambda)^{-1}$ allora $\mathcal{L}(e^{At}) = (sI - A)^{-1}$

$$\begin{aligned} \dot{x} &= Ax+Bu \\ y &= Cx+Du \end{aligned} \qquad \begin{aligned} x(t) &\rightarrow x(s) \\ u(t) &\rightarrow u(s) \\ y(t) &\rightarrow y(s) \end{aligned} \qquad \dot{x}(t) - sx(s) - x(s)$$

$$\mathcal{L}(x) = \mathcal{L}(Ax+Bu) = Ax(s) + Bu(s) = sx(s) - x(s) \Rightarrow (sI-A)x(s) = x(s) + Bu(s) \Rightarrow x(s) = (sI-A)^{-1}(x(s)+Bu(s))$$

$$\Rightarrow x(s) = (sI-A)^{-1}x(s) + (sI-A)^{-1}Bu(s)$$

$$\mathcal{L}(\Phi(t) = e^{At}) = (sI-A)^{-1} = \Phi(s)$$
$$\mathcal{L}(H(t) = e^{At}B) = (sI-A)^{-1}B = H(s)$$

$$x(s) = \Phi(s)x(0) + H(s)u(s)$$

$$\mathcal{L}\left(\int_0^t H(t-T)u(T)dT\right) = H(s)u(s)$$

$$y(s) = Cx(s) + Du(s) = C(sI - A)^{-1}x(0) + C(sI-A)^{-1}B + Du(s) = \Psi(s) + W(s)$$

$$y(t) = \Psi(t)x(0) + \int_0^t W(t-\tau)u(\tau)\,d\tau$$

es $\dot{x} = -3x + u$

$y = x + u \qquad u(s) = \delta_{-1} \to 1/s$

$$y_F(s) = W(s)u(s)$$

$$W(s) = C(sI-A)^{-1}B + D = 1(s+3)^{-1} \cdot 1 + 1 = \frac{s+4}{s+3}$$

$$\Rightarrow y_F(s) = 1/s \; \frac{s+4}{s+3} = \frac{R_1}{s} + \frac{R_2}{s+3}$$

$$s\,y_F(s) = \frac{s+4}{s+3} = R_1 + R_2 \cdot \frac{s}{s+3}$$

$$\lim_{s\to 0} \Rightarrow 4/3$$

$$\lim_{s\to -3} (s+3)y_F(s) = \frac{s+4}{s+3}\Big|_{s=-3} = \left[\frac{R_1(s+3)}{s} + R_2\right]_{s=3} = 1/3$$

$$\Rightarrow y_F(s) = 4/3 \cdot 1/3 = 4/9$$

es. $\dot{x} = Ax + Bu \qquad A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y = Cx + Du \qquad C = (-1 \;\; 1) \quad D = 0$

$$W(s) = C(sI-A)^{-1}B + D = (-1 \;\; 1)\begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (-1 \;\; 1)\frac{1}{(s+1)(s+2)}\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{[-(s+2)+s]}{(s+1)(s+2)}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{2}{(s+1)(s+2)}$$

$$y_F(s) = \frac{-2}{(s+1)(s+2)} \cdot 1/s =$$

$$= R_1/s + \frac{R_2}{s+2} + \frac{R_3}{s+1}$$

$$y_F(t) = (R_1 + R_2 e^{-2t} + R_3 e^{-t})\,\delta_{-1}(t)$$

$R_1 = \frac{-2}{(s+1)(s-2)} = -1$ con $s = 0$

$R_2 = \frac{-2}{s(s+1)} = -1$ con $s = -2$

$R_3 = \frac{-2}{s(s+2)} = 2$ con $s = -1$

Esercizio

$$A=\begin{pmatrix} -1 & 6 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad B=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad C=(1\ 1\ 0) \quad D=0$$

Calcolare $\Phi(s), H(s), W(s)$ e $\Psi(s)$.

Calcolare la risposta forzata in uscita $u(t)=\delta_{-1}(t)$, $u(t)=t\,\delta_{-1}(t)$

$$\Phi(s)=(sI-A)^{-1}=\frac{1}{\det(sI-A)}\begin{bmatrix} s+1 & -6 & -2 \\ 0 & s+3 & -4 \\ 0 & 0 & s-1 \end{bmatrix}^{-1}=\frac{1}{(s+1)(s+3)(s-1)}\begin{bmatrix} (s+3)(s-1) & 6(s-1) & [24+2(s+3)] \\ 0 & (s+1)(s-1) & 4(s+1) \\ 0 & 0 & (s+1)(s+3) \end{bmatrix}=\frac{R_1}{s+1}+\frac{R_2}{s+3}+\frac{R_3}{s-1}$$

$$R_1=\lim_{s\to-1}(s+1)\Phi(s)=\frac{1}{(s+3)(s-1)}\left[\dots\right]=-\frac{1}{4}\begin{bmatrix} -4 & -12 & 28 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{pmatrix} 1 & 3 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_2=\lim_{s\to-3}(s+3)\Phi(s)=\frac{1}{8}\begin{bmatrix} 0 & -24 & 24 \\ 0 & 8 & -8 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -3 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3=\lim_{s\to1}(s-1)\Phi(s)=\frac{1}{8}\begin{bmatrix} 0 & 0 & 32 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(t)=R_1e^{-t}+R_2e^{-3t}+R_3e^{t}=e^{\lambda_1 t}u_1v_1^T+e^{\lambda_2 t}u_2v_2^T+e^{\lambda_3 t}u_3v_3^T \to R_i=u_iv_i^T$$

$$H(s)=\Phi(s)B=\left(\frac{\overset{u_1v_1^T}{R_1}}{s+1}+\frac{\overset{u_2v_2^T}{R_2}}{s+3}+\frac{\overset{u_3v_3^T}{R_3}}{s-1}\right)B=\frac{1}{(s+1)(s+3)(s-1)}\begin{bmatrix} 30+2s \\ 4(s+1) \\ (s+1)(s+3) \end{bmatrix}=\frac{\begin{pmatrix} 28 \\ 0 \\ 0 \end{pmatrix}}{-4(s+1)}+\frac{\begin{pmatrix} 24 \\ -8 \\ 0 \end{pmatrix}}{(s+3)8}+\frac{\begin{pmatrix} 32 \\ 8 \\ 8 \end{pmatrix}}{(s-1)8}$$

$$\Psi(s)=C(sI-A)^{-1}=(1\ 1\ 0)\Phi(s)=\begin{bmatrix} (s+3)(s-1) & (s-1)(s+1+6) & 34+6s \end{bmatrix}=\frac{\hat R_1}{s+1}+\frac{\hat R_2}{s+3}+\frac{\hat R_3}{s-1}$$

$$=-\frac{1}{4}\frac{(-4\ -12\ 28)}{s+1}+\frac{(0\ -16\ 16)}{8(s+3)}+\frac{(0\ 0\ 40)}{(s-1)8}$$

$$W(s)=C\Phi(s)B+D=\frac{34+6s}{(s+1)(s+3)(s-1)}$$

$$u_1(t)=\delta_{-1}(t) \quad u_1(s)=1/s \quad Y_{F_1}=W(s)u(s)=\frac{34+6s}{(s+1)(s-1)(s+3)}\cdot 1/s$$

$$=\frac{R_1}{s+1}+\frac{R_2}{s+3}+\frac{R_3}{s-1}+\frac{R_4}{s}$$

$$Y_{F_1}(t)=(R_1e^{-t}+R_2e^{-3t}+R_3e^{t}+R_4)\delta_{-1}(t)$$

$$R_1=\lim_{s\to-1}(s+1)Y_F(s)=\frac{34+6s}{(s+3)(s-1)s}=\frac{28}{4}=7$$
$$R_2=\lim_{s\to-3}(s+3)Y_F(s)=\frac{34+6s}{(s+1)(s-1)s}=\frac{16}{-24}=-2/3$$
$$R_3=\lim_{s\to1}(s-1)Y_F(s)=\frac{34+6s}{(s+1)(s+3)s}=\frac{40}{8}=5$$
$$R_4=\lim_{s\to0}sY_F(s)=W(s)|_{s=0}=\frac{34}{-3}$$

$$u_2=t\,\delta_{-1}(t)=1/s^2 \quad Y_{F_2}(s)=W(s)\cdot 1/s^2=\frac{R_1}{s+1}+\frac{R_2}{s+3}+\frac{R_3}{s-1}+\frac{R_{4a}}{s^2}+\frac{R_{4b}}{s}$$

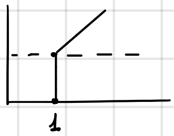
$$s^2Y_{F_2}(s)=\frac{R_1s^2}{s+1}+\frac{R_2s^2}{s+3}+\frac{R_3s^2}{s-1}+R_{4a}+R_{4b}s \to \lim_{s\to0}\left(s^2Y_{F_2}(s)\right)=R_{4a}$$

$$\lim_{s\to0}\left(d/ds\left(sY_{F_2}(s)=sR_1\left(\frac{s(s+1)-s}{(s+1)^2}\right)+sR_2(\ )+sR_3(\ )+R_{4b}\right)\right)=R_{4b}$$

$$R_1=\frac{34-6}{-4}=-7 \qquad R_{4a}=W(s)|_{s=0}=\frac{34}{-3} \qquad Y_{F_2}(t)=R_1e^{-t}+R_2e^{-3t}+R_3e^{t}+R_{4a}t+R_{4b}$$
$$R_2=\frac{34-18}{(-2)(-4)9}=2/9 \qquad R_{4b}=d/ds\,W(s)|_{s=0}=\frac{16}{9}$$
$$R_3=\frac{40}{8}$$

Es.

$W(s) = \frac{1}{s+1}$

$u - t\,\delta_{-1}(t-1) \longrightarrow$ 

$t - 1 = \varepsilon \qquad t = \varepsilon + 1$

$\rightarrow u(\varepsilon) = (\varepsilon+1)\,\delta_{-1}(\varepsilon)$

$u(s) = 1/s^2 + 1/s$

$$Y_F(s) = \frac{1}{s+1}\left(1/s^2 + 1/s\right) = \frac{R_1}{s+1} + \frac{R_2}{s^2} + \frac{R_3}{s}$$

$$\left.\begin{array}{l} R_1 = \left(1/s^2 + 1/s\right)\Big|_{s=0} = 0 \\ R_2 = \left(1/(s+1) + 1/s\right)\Big|_{s=0} = 1 \\ R_3 = d/ds\, W(s)\Big|_{s=0} = \left(-\frac{1}{(s+1)^2}\right) = -1 \end{array}\right\} Y_F(s) = 1/s^2 - 1/s \longrightarrow y_F(\varepsilon) = (\varepsilon - 1)\,\delta_{-1}(\varepsilon) \longrightarrow y_F(t) = (t-1-1)\,\delta_{-1}(t-1)$$

$u(t) = \frac{t^2}{2}\,\delta_{-1}(t)$

$Y_F(s) = 1/(s+1) \cdot 1/s^3 = \frac{R_1}{s+1} + \frac{R_2}{s^3} + \frac{R_3}{s^2} + \frac{R_4}{s}$ perché $\frac{t^k}{k}\,\delta_{-1}(t) \longrightarrow 1/s^{k+1}$

$R_1 = \lim_{s \to -1} (s+1)\,Y_F(s)$

$R_2 = \lim_{s \to 0} s^3\,Y_F(s) = W(s)\big|_{s=0} = 1$

$R_3 = d/ds\, W(s)\big|_{s=0}$

$R_4 = 1/2\, d^2/ds^2\, W(s)\big|_{s=0}$

$\dot{x} = Ax + Bu$
$y = Cx + Du$

$u(t)$, $y(t)$ t sufficientemente grande
Per quali condizioni il sistema sia indipendente da x_0 e legato solo al tipo di ingresso.

$Y_R(t)$ risposta a regime permanente $= \lim_{t \to \infty} y(t)$

t_0 $x(t_0)$ $u[t_0, t]$ $\qquad u(t) = \dfrac{e^{jwt} - e^{-jwt}}{2j}$

$$y(t) = \underbrace{\Psi(t-t_0)x(t_0)}_{\text{Risposta libera}} + \int_{t_0}^{t} w(t-T)u(T)\,dT$$

$$\Rightarrow \lim_{t\to\infty} y(t) \to \lim_{t\to\infty} \int_{t_0}^{t} w(t-T)u(T)\,dT = \int_{-\infty}^{t} w(t-T)u(T)\,dT = \int_{-\infty}^{t} w(t-T)\left[\frac{e^{jwt} - e^{-jwt}}{2j}\right] dT$$

$$= \frac{1}{2j}\underbrace{\int_{-\infty}^{t} w(t-T)\, e^{jwt}\, dT}_{1} - \frac{1}{2j}\underbrace{\int_{-\infty}^{t} w(t-T)\, e^{-jwt}\, dT}_{2}$$

1 $\displaystyle\int_{-\infty}^{t} w(t-T)\, e^{jwt}\, dT$ con $u = e^{j\alpha t}$ $\to t - T = \varepsilon \to T = t - \varepsilon$
$dT = -d\varepsilon$

Per $T \to -\infty$ $\quad \varepsilon \to \infty$
$T \to t$ $\quad \varepsilon \to 0$

$$\Rightarrow -\int_{\infty}^{0} w(\varepsilon)\, e^{j\alpha(t-\varepsilon)}\, d\varepsilon \Rightarrow Y_R(t) = \int_{0}^{\infty} w(\varepsilon)\, e^{j\alpha t} e^{-j\alpha\varepsilon}\, d\varepsilon = e^{j\alpha t}\int_{0}^{\infty} w(\varepsilon)\, e^{-j\alpha\varepsilon}\, d\varepsilon = e^{j\alpha t} W(s)\big|_{s=j\alpha}$$

$W(jw) = |W(jw)| \cdot e^{j\phi(w)}$
$W(-jw) = |W(jw)| \cdot e^{-j\phi(w)}$

$$Y_R(t) = \frac{1}{2j}\left(e^{jwt} M(jw)\, e^{j\phi} - e^{-jwt} M(jw)\, e^{-j\phi}\right) = M(jw)\left[\frac{e^{j(wt+\phi)} - e^{-j(wt+\phi)}}{2j}\right]$$
Risposta a regime permanente

es. $W(s) = 1/_{s+1}$ $\qquad u(t) = \sin t \, \delta_{-1}(t)$

Trovo risposta regime permanente

$y_R(t) = M(j1) \sin(t + \phi) \, \delta_{-1}(t)$

$W(j1) = \frac{1}{1+j} = Re + j Im = \frac{1-j}{2}$ $\qquad M(j1) = \sqrt{Re^2 + Im^2} = \sqrt{1/2}$

$\phi = \operatorname{arctg} Im/Re = -\pi/4$

$\Rightarrow 1/\sqrt{2} \sin(t - \frac{\pi}{4}) \, \delta_{-1}(t)$

es $W(s) = \frac{s+1}{(s+10)(s+3)}$ $\qquad u(t) = \sin t \, \delta_{-1}(t)$

Trovare risposta forzata e regime permanente

$u(s) = 1/_{s^2+1}$

$Y_F(s) = \frac{s+1}{(s+10)(s+3)} \cdot \frac{1}{s^2+1} = \frac{R_1}{s+10} + \frac{R_2}{s+3} + \frac{N_1 s + N_2}{s^2+1}$

$Y_F(t) = (R_1 e^{-10t} + R_2 e^{-3t} + N_1 \cos t + N_2 \sin t) \, \delta_{-1}(t)$

$R_1 = \frac{s+1}{(s+3)(s^2+1)} \Big|_{s=-10} = \frac{-9}{7 \cdot 101}$

$R_2 = \frac{s+1}{(s+10)(s^2+1)} \Big|_{s=-3} = \frac{-2}{7 \cdot 10}$

$s=0 \Rightarrow 1/30 = \frac{9}{707 \cdot 10} + \frac{1}{35 \cdot 3} + N_2$

$s=-1 \Rightarrow 0 = \frac{R_1}{9} + \frac{R_2}{2} + \frac{N_1 - N_2}{2}$

$$W(s) = \frac{s+1}{s^2+s+1} \qquad u_1(t) = \delta_{-1}(t) \qquad u_2(t) = t\,\delta_{-1}(t)$$

Risposta forzata e regime

$$Y_F(s) = \frac{s+1}{s^2+s+1} \cdot 1/s$$

$$= \frac{R_1}{S} + \frac{N_1 s + N_2}{s^2+s+1} - \frac{R_1}{S} + \frac{N_1 s + N_2}{(s-\alpha)^2+w^2} = 1/s + \frac{N_1 S + N_2}{(s+1/2)^2+3/4}$$

$$s=-1 \Rightarrow 0 = -1 + N_2 - N_1 \Rightarrow N_2 - N_1 = 1$$

$$s=-1/2 \Rightarrow \frac{-1/2}{1/4-1/2+1} \cdot 2 = -2 + \frac{N_2 - 1/2 N_1}{3/4} \Rightarrow 1/2 = N_2 - 1/2 N_1 \Rightarrow 2N_2 - N_1 = 1$$

$$\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$Y_F(t) = (1 - e^{-1/2 t} \cos t + 1/\sqrt{3}\, e^{-1/2} \sin t)\, \delta_{-1}(t)$$

$$Y_R(t) = 1\, \delta_{-1}(t)$$

$$\hookrightarrow C_0 = W(s)\big|_{s=0}$$

$$2)\ Y_F(s) = \frac{s+1}{s^2+s+1}\, 1/s^2 = \frac{R_1}{s^2} + \frac{R_2}{s} + \frac{N_1 s + N_2}{s^2+s+1}$$

$$R_1 = W(s)\big|_{s=0} = 1$$
$$R_2 = \frac{d\,W(s)}{ds}\Big|_{s=0} = 0$$

$$(R_1 + R_2 + L_1 e^{-\alpha t} \cos wt + L_2 e^{\alpha t} \sin wt)\, \delta_{-1}(t)$$

$$0 = R_1 + N_2 - N_1 \Rightarrow N_2 - N_1 = -1 \qquad s=-1$$

$$2/3 = 1 + \frac{N_1 + N_2}{3} \Rightarrow N_1 + N_2 = -1 \qquad s=1$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{matrix} N_2 = -1 \\ N_1 = 0 \end{matrix}$$

$$L_1 = 0$$
$$L_2 = -2/\sqrt{3}$$

Se prendo un segnale $u(t) = \sin \omega t$ in uscita, a regime, la risposta è uguale $y_R(t) = M(\omega) \sin(\omega t + \phi(\omega))$

$$W(s)\big|_{s=j\omega} = W(j\omega) = \underbrace{M(\omega)}_{\text{modulo}} \underbrace{e^{j\phi(\omega)}}_{\text{fase}}$$

in base alla frequenza cambiano i risultati.

$W_1(j\omega) = M_1(\omega) e^{j\phi_1}$ $\qquad W_2(j\omega) = M_2(\omega) e^{j\phi_2}$

$m \rightarrow \boxed{W_2(s)} \xrightarrow{u} \boxed{W_1(s)} \xrightarrow{y}$ connessione in serie

$y(s) = W_1(s) u(s) = W_1(s) W_2(s) m(s)$ funzione di trasferimento

$u(s) = W_2(s) m(s)$

$$W(j\omega) = W_1(j\omega) \cdot W_2(j\omega) = M_1(j\omega) e^{j\phi_1} \cdot M_2(j\omega) e^{j\phi_2} = M_1(j\omega) M_2(j\omega) e^{j(\phi_1 + \phi_2)}$$

Se tracciamo il grafico di ϕ_1 e ϕ_2 e li sommiamo otteniamo il grafico $\phi_1 + \phi_2$ che rappresenta la fase

Per il modulo si usa una scala logaritmica:

$M_1 dB = 20 \log_{10} M_1(j\omega)$

$M_2 dB = 20 \log_{10} M_2(j\omega)$

$M dB = M_1 dB + M_2 dB$

La funzione di trasferimento $W(s)$ è data da un certo numero di zeri e poli:

$W(s) = \frac{K \pi_i (s + z_i)}{\pi_j (s + p_i) s^n}$ $\qquad z_i$ - poli $\quad p_i$ - poli $\quad K$ - costante $\quad \pi_i - \pi_j$ - produttoria

$$W(j\omega) = \frac{\pi_i M_i(j\omega) e^{j\phi_i}}{\pi_j M_j(j\omega) e^{j\phi_j}} s^n \longrightarrow \pi \hat{M}_k e^{j\hat{\phi}_k}$$

Forma di Bode

Normalizzo tutti i termini in modo da avere la parte reale $= 1$

$(s + z_i) \longrightarrow z_i(1 + s/z_i)$ e $1/z_i$ la chiamo T

con il suo coniugato: $(s + z_i)(s + z_i^*)$

Sostituisco e ottengo:

$$\frac{\pi_i (1 + T_i s) \pi_j \left(1 + 2\varepsilon_j \frac{s}{\omega_{nj}} + \frac{s^2}{\omega_{nj}^2}\right) K}{\pi_k (1 + T_k s) \pi_\ell \left(1 + 2\varepsilon_\ell \frac{s}{\omega_{n\ell}} + \frac{s^2}{\omega_{n\ell}^2}\right) s^r}$$ forma di bode

$-1 < \varepsilon < 1$

$\varepsilon =$ smorzamento $\qquad K =$ guadagno

s^r termine monomio $\qquad (1 + T_i s) =$ binomi

$1 + 2\varepsilon \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2} =$ trinomi

es.

$$\text{funzione trasferimento} = \frac{s + 5}{s^2 + s + 4} \implies \frac{5(1 + s/5)}{4\left[1 + s/4 + (s/2)^2\right]}$$

guadagno K: $\quad K \longrightarrow 20 \log_{10} |K|$

termine monomio (s): $\quad s = j\omega \qquad$ Modulo: $|s| = 20 \log_{10}(\omega)$

es. Voglio studiare $W(s) = 100/s$

$M dB = 20 \log_{10} \frac{|100|}{|j\omega|} = 20 \log_{10} |100| - 20 \log_{10} \omega$

Fase $- \varphi = \frac{e^{j\phi_1}}{e^{j\phi_2}} = e^{j\phi} = e^{j(\phi_1 - \phi_2)}$

termine binomio: al posto di s metto $j\omega$ e ottengo: $\underset{Re}{1} + \underset{Im}{j\omega T}$

Modulo $= \sqrt{Re^2 + Im^2} = \sqrt{1 + \omega^2 T^2}$ $\qquad M dB = 20 \log_{10} \sqrt{1 + \omega^2 T^2}$

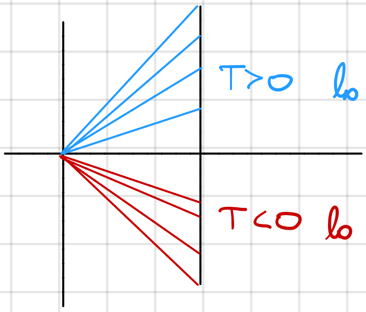
$|\omega T| \gg 1 \longrightarrow M dB \simeq 20 \log_{10} \sqrt{\omega^2 T^2} = 20 \log_{10} |\omega T|$

Se invece $|\omega T| \ll 1 \rightarrow M_{dB} \simeq 0 dB$

$\omega = 1/|T| \rightarrow M_{dB} = 20 \log_{10} \sqrt{2} = 3dB$

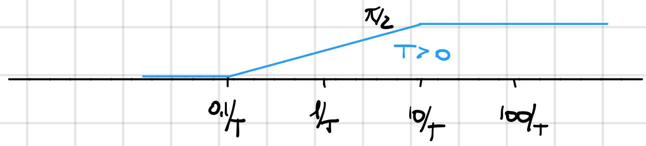
Fase di $1 + j\omega T$

$y = \arctan \omega t$



$T > 0$ la fase va da 0 a $\pi/2$

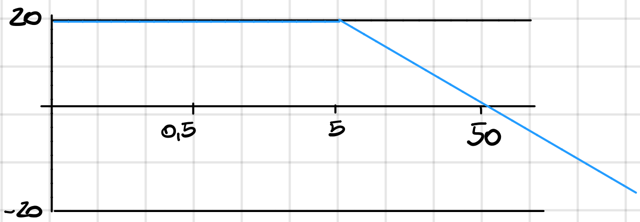
$T < 0$ la fase va da 0 a $-\pi/2$



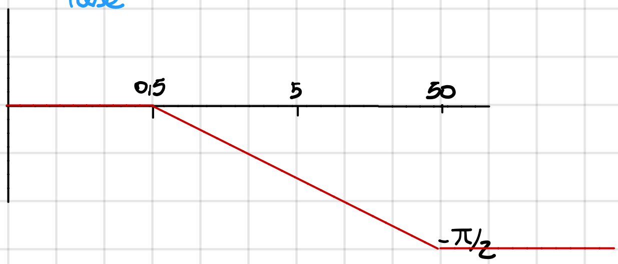
Con $1/t$ o $-T$ si ribalta il grafico

es. $\frac{50}{s+5} \xrightarrow{bode} \frac{\cancel{5}\cdot 10}{\cancel{5}(1+s/5)} \rightarrow s \cdot T \Rightarrow T = 1/5 \Rightarrow 1/T =$ punto di rottura $= 5$

$K_{dB} = 20 \log_{10} 10 = 20$



Fase



termine trinomio: $1 + 2\varepsilon \frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}$

Sostituisco $s = j\omega \rightarrow 1 + 2\varepsilon \frac{j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2} = \underbrace{1 - \frac{\omega^2}{\omega_n^2}}_{Re} + \underbrace{\frac{j 2\varepsilon\omega}{\omega_n}}_{Im}$

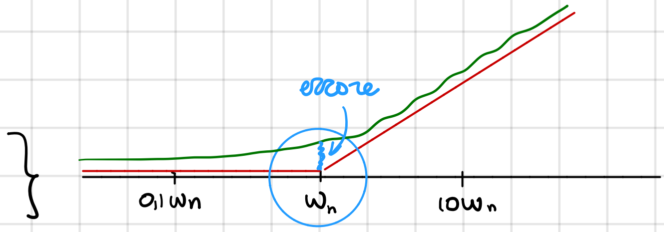
$M = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\varepsilon\omega}{\omega_n}\right)^2}$

$M_{dB} = 20 \log_{10} M$

caso 1 $\quad \omega \ll \omega_n \longrightarrow 20 \log_{10} 1 \simeq 0\,dB$

caso 2 $\quad \omega \gg \omega_n \longrightarrow 20 \log_{10} \left(\frac{\omega}{\omega_n}\right)^2 = 40 \log_{10} \omega/\omega_n$

caso 3 $\quad \omega = \omega_n \longrightarrow 20 \log_{10} 2|\varepsilon| = 40 \log_{10} \sqrt{2|\varepsilon|}$



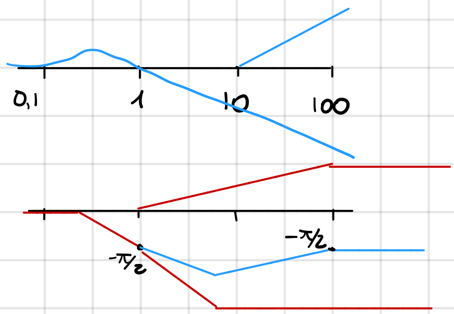
$\varphi = \text{arctg} \frac{2\varepsilon\,\omega/\omega_n}{1 - \omega^2/\omega_n^2}$

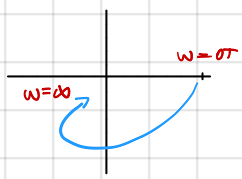
caso 1 $\quad \omega \ll \omega_n \longrightarrow \varphi \simeq 0$

caso 2 $\quad \omega \gg \omega_n \longrightarrow \varphi \simeq \pi$

es. $\frac{s+10}{s^2+s+1}$

$\Rightarrow \frac{10(1+s/10)}{1+2\varepsilon s/\omega_n + s^2/\omega_n^2} \Rightarrow \frac{10(1+s/10)}{1+2\cdot\frac{1}{2}\cdot s + s^2}$ perché $s^2 = s^2/\omega^2 \Rightarrow \omega_n^2 = 1$ e $2\varepsilon s/\omega_n = s \Rightarrow \frac{2\varepsilon}{\omega_n} = 1$ e $\varepsilon = 1/2$





es. $\frac{s+10}{s^2+3s+2}$

$\Rightarrow \frac{10(1+s/10)}{(s+2)(s+1)} = \frac{\cancel{10}^{\,5}(1+s/10)}{\cancel{2}(1+s/2)(s+1)}$

$K = 5$ = guadagno $\longrightarrow K_{dB} = 20 \log_{10}(5) \simeq 14$

$T = 1/10 \rightarrow 1/T$ = Punto rottura $= 10$

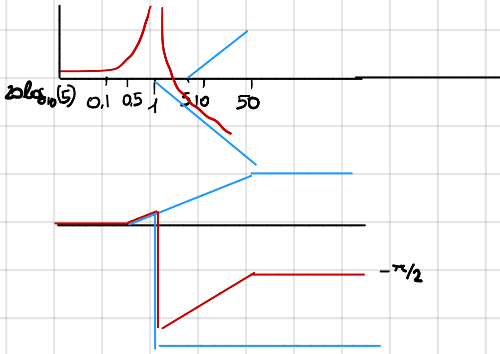
$T_2 = 1/2 \rightarrow 1/T_2 = 2$

$T_3 = 1$

es $\frac{s+5}{s^2+1}$

$\frac{5(1+s/5)}{1+s^2}$

$1/T_1 = 5 \quad 1/T_2 = 1$



modulo

fase

SISTEMI A TEMPO DISCRETO

$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}\right\} \text{modello implicito}$$

$$x(1) = Ax(0) + Bu(0)$$

$$x(2) = Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1)$$

evol. libera, evol. forzata

$$x(k) = A^k x(0) + A^{k-1}Bu(0) + A^{k-2}Bu(1) + \ldots + Bu(k-1) = A^k x(0) + \sum_0^{k-1} A^{k-1-\tau} Bu(\tau) = \underbrace{\Phi(k)x(0)}_{\text{evol. libera}} + \underbrace{\sum_0^{k-1} H(k-\tau)u(\tau)}_{\text{evol. forzata}}$$

$$H(k) = A^{k-1}B$$

$$\Phi(k) = A^k$$

caso autovalori reali e distinti:

$$A \to \tilde{A} = TAT^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad A = T^{-1}\tilde{A}T$$

$$A^k = T^{-1}\tilde{A}^k T$$

$$\Phi(k) = A^k = T^{-1}\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} T = (u_1 \ \ldots \ u_n)\begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}\begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} = \lambda_1^k u_1 v_1^T + \ldots + \lambda_n^k u_n v_n^T$$

$$x_0 = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n$$

Evoluzione libera: $x_\ell(k) = A^k x(0) = \lambda_1^k u_1 c_1 + \ldots + \lambda_n^k u_n c_n$

traiettoria:

- divergente se $|\lambda| > 1$
- costante se $\lambda = 1$
- convergente se $|\lambda| < 1$

Parliamo di modi naturali aperiodici nel caso di autovalori reali e positivi, modi naturali pseudoperiodici nel caso di autovalori complessi coniugati e modi naturali alternati nel caso di autovalori reali e negativi.

$$y(k) = Cx(k) + Du(k) = C\left(A^k x(0) + \sum_0^{k-1} A^{k-1-\tau}Bu(\tau)\right) + Du(k) = CA^k x(0) + \sum_0^{k} W(k-\tau)u(\tau)$$

$W(k) = CA^{k-1}B$ con $k > 0$ $\quad$ $(W(0) = D$ con $k = \tau)$

$$\Psi(k) = CA^k$$

Osservabilità

$\Psi(k) = CA^k = C\sum_0^n \lambda_i^k u_i v_i^T$

se $Cv_i^T = 0$ NON OSSERVABILITÀ

Eccitabilità

$x_F(k) = \sum_0^{k-1} A^{k-1-\tau}Bu(\tau)$ se per qualche valore di τ quel modo scompare allora non è eccitabile

$H(k) = A^{k-1}B = T^{-1}\tilde{A}^{k-1}T \cdot B = \sum_1^n \lambda_i^{k-1} u_i v_i^T B \Rightarrow v_i^T B = 0 \Rightarrow$ NON ECCITABILITÀ

$$W(k) = CA^{k-1}B = \sum_{i}^{n} \lambda_i^k (C u_i v_i^T B) \rightarrow \text{se } = 0 \text{ non eccitabile e non osservabile}$$

Modi Pseudoperiodici

$$\lambda = \alpha \pm j\omega = \sigma e^{\pm j\omega} = \sigma(\cos\phi \pm j \sin\phi)$$

$$\begin{pmatrix} \overset{\alpha}{\sigma\cos\phi} & \overset{\omega}{\sigma\sin\phi} \\ \underset{-\omega}{-\sigma\sin\phi} & \underset{\alpha}{\sigma\cos\phi} \end{pmatrix} \Rightarrow \text{per } A^k \Rightarrow \sigma^k \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^k = \sigma^k \begin{pmatrix} \cos\phi k & \sin\phi k \\ -\sin\phi k & \cos\phi k \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ -\sin 2\phi & \cos 2\phi \end{pmatrix}$$

$$\begin{pmatrix} \cos\phi(k-1) & \sin\phi(k-1) \\ -\sin\phi(k-1) & \cos\phi(k-1) \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \Rightarrow \begin{array}{l} \cos(\phi(k-1))\cos\phi - \sin(\phi(k-1))\sin\phi = \cos\phi k \\ \cos(\phi(k-1))\sin\phi + \sin(\phi(k-1))\cos\phi = \sin\phi k \end{array}$$

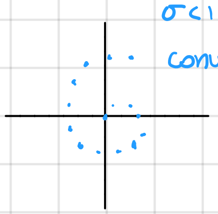
$$T^{-1}\tilde{A}^k T = (u_a, u_b) \left(\boxed{\dots} \right) \begin{pmatrix} v_a^T \\ v_b^T \end{pmatrix} = (u_a, u_b)\, g^k \begin{pmatrix} \cos\phi k & \sin\phi k \\ -\sin\phi k & \cos\phi k \end{pmatrix} \begin{pmatrix} v_a^T \\ v_b^T \end{pmatrix}$$

$$= g^k \left(\cos(\phi k) u_a - \sin(\phi k) u_b \,|\, \sin(\phi k) u_a + \cos(\phi k) u_b\right) \begin{pmatrix} v_a^T \\ v_b^T \end{pmatrix}$$

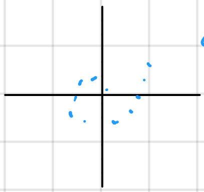
$$= g^k \left(\cos(\phi k)(u_a v_a^T + u_b v_b^T) + \sin(\phi k)(u_a v_b^T - u_b v_a^T)\right)$$

$$\sigma = \sqrt{Re^2 + Im^2}$$

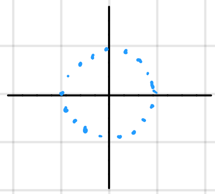
$\sigma < 1$ convergente

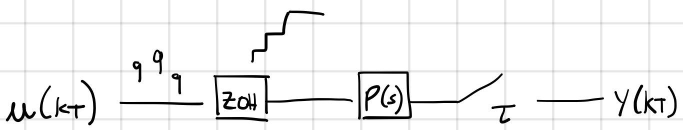


$\sigma > 1$ divergente



$\sigma = 1$ limitata





$$x(t) = \phi(t-t_0)x(t_0) + \int_{t_0}^{t} H(t-\tau)u(\tau)\,d\tau$$

$$y(t) = Cx(t) + Du(t)$$

$$y(kT) = \underbrace{C}_{C\text{discreta}} x(kT) + \underbrace{D}_{D\text{discreta}} u(kT)$$

$$x((k+1)T) = e^{A((k+1)T-kT)}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} Bu(\tau)\,d\tau$$

$$= \underbrace{e^{AT}}_{A\text{discreta}} x(kT) + \underbrace{\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,d\tau}_{B\text{discreta}}\, u(kT)$$

$$A_d = e^{At} \qquad C_d = C \qquad D_d = D$$

$$B_d = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B\,d\tau = \int_0^T e^{A\varepsilon} B\,d\varepsilon$$

sostituisco $(k+1)T - \tau = \varepsilon$

$d\varepsilon = -d\tau$

$\tau \rightarrow kT \rightarrow \varepsilon \rightarrow T$

$\tau \rightarrow (k+1)T \rightarrow \varepsilon \rightarrow 0$

$$\Rightarrow -\int_T^0 e^{A\varepsilon} B\,d\varepsilon = \int_0^T e^{A\varepsilon} B\,d\varepsilon$$

$$x(k+1)=Ax(k)+Bu(k)$$
$$y(k)=Cx(k)+Du(k)$$

Trasformata Zeta

$$Z(f(k))=\sum_0^n \frac{f(k)}{z^k}=f(0)+f(1)/z+f(2)/z^2\ldots$$

$$Z(f(k+1))=\sum_0^n \frac{f(k+1)}{z^k}=z\left(\frac{f(1)}{z}+f(2)/z^2+f(3)/z^3\ldots\right)=z\left(f(z)-f(0)\right)$$

$$Z(x(k+1))=zx(z)-zx(0)=Ax(z)+Bu(z)\rightarrow(zI-A)x(z)=zx(0)+Bu(z)$$
$$\rightarrow x(z)=(zI-A)^{-1}z\,x(0)+(zI-A)^{-1}Bu(z)$$
$$y(z)=Cx(z)+Du(z)=C(zI-A)^{-1}z\,x(0)+C(zI-A)^{-1}Bu(z)+Du(z)$$

$$x(k)=\Phi(k)x(0)+\sum_0^{k-1}H(k-T)u(T)$$
$$y(k)=\Psi(k)x(0)+\sum_0^{k}W(k-T)u(T)$$

$\phi(k)=A^k$ $\quad H(k)=A^{k-1}B$ $\quad \Psi(k)\,C A^k$ $\quad W(k)$

$\underline{u=0}$ $\quad x(z)=\phi(z)x(0)=(zI-A)^{-1}zx(0)$

$\phi(z)=(zI-A)^{-1}z$

	S	Z
ϕ	$(sI-A)^{-1}$	$(zI-A)^{-1}z$
H	$(sI-A)^{-1}B$	$(zI-A)^{-1}B$
Ψ	$C(sI-A)^{-1}$	$C(zI-A)^{-1}z$
W	$C(sI-A)^{-1}B+D$	$C(zI-A)^{-1}B+D$

$$H(z)=(zI-A)^{-1}B$$
$$W(z)=C(zI-A)^{-1}B+D$$
$$\Psi(z)=C(zI-A)^{-1}z$$

Trasformate

$$\lambda^k\rightarrow z/_{z-\lambda}$$

$$\sin\omega k\rightarrow\frac{(e^{j\omega})^k-(e^{-j\omega})^k}{2j}=1/_{2j}\left(\frac{z}{z-e^{j\omega}}-\frac{z}{z-e^{-j\omega}}\right)=1/_{2j}\,\frac{z(z-e^{-j\omega})-z(z-e^{j\omega})}{z^2-z(e^{j\omega}-e^{-j\omega})+1}=\frac{z\sin\omega}{z^2-2z\cos\omega+1}$$

$$\cos\omega k\rightarrow\frac{z^2-z\cos\omega}{z^2-2z\cos\omega+1}$$

$$\delta_{-1}(k)\rightarrow\frac{z}{z-1}$$

$$k^\ell/_{\ell!}\rightarrow z/_{(z-1)^{(\ell+1)}}$$

$$\sigma^k\sin\omega k\rightarrow\frac{z\sigma\sin\omega}{z^2-2z\sigma\cos\omega+\sigma^2}$$

N.B. $\frac{z}{z^2+z+2}$ $\rightarrow\sigma^2\rightarrow\sigma=\sqrt{2}$

$$\sigma^k\cos\omega k\rightarrow\frac{z^2-z\sigma\cos\omega}{z^2-2z\sigma\cos\omega+\sigma^2}$$

a) $W(z) = \frac{z}{(2z-1)(3z-1)}$

$Y_F(z)$ $u(z) = z/z-1$

$$\frac{z^2}{(2z-1)(3z-1)(z-1)} = \frac{R_1}{(2z-1)} + \frac{R_2}{(3z-1)} + \frac{R_3}{(z-1)}$$

$$Y_F(z)/z = \frac{z}{(2z-1)(3z-1)(z-1)} = \frac{R_1}{(z-1/2)} + \frac{R_2}{z-1/3} + \frac{R_3}{z-1}$$

$$R_1 = \lim_{z \to 1/2} (z-1/2) \frac{z}{2(z-1/2)(3z-1)(z-1)} = -1$$

$$R_2 = \lim_{z \to 1/3} (z-1/3) \frac{z}{3(z-1/3)(2z-1)(z-1)} = 1/2$$

$$R_3 = \lim_{z \to 1} (z-1) \frac{z}{(2z-1)(3z-1)(z-1)} = 1/2$$

$W(z)|_{z=1}$

$$Y_F(z) = -\frac{z}{z-1/2} + 1/2 \frac{z}{z-1/3} + 1/2 \frac{z}{z-1} = -(1/2)^k + 1/2 (1/3)^k + 1/2 (1)^k$$

RISPOSTA A REGIME

$x(k+1) = Ax(k) + Bu(k)$

$y(k) = Cx(k) + Du(k) = \Psi(k)x(0) + \sum_0^k w(k-\tau)u(\tau)$

modi osservabili convergenti $|\lambda^\circ| < 1$

$|\lambda| \le 1 \quad |\lambda| = 1 \quad mg = 1$

$y(k) = \Psi(k-t_0)x(t_0) + \sum_{t_0}^{k} w(k-\tau)u(\tau)$

$y_R(k) = \sum_{-\infty}^{k} w(k-\tau)u(\tau) = \sum_{-\infty}^{k} w(k-\tau)e^{\alpha\tau}$

$\varepsilon = k - \tau \qquad \tau \to -\infty \qquad \varepsilon \to \infty$

$\qquad\qquad\qquad \tau \to k \qquad \varepsilon \to 0$

$\Rightarrow y_R(k) = \left[\sum_0^\infty w(\varepsilon)e^{-\alpha\varepsilon}\right] e^{\alpha k}$

$e^{\alpha k} \mathcal{Z}(w(k)) = \sum_0^\infty \frac{w(k)}{z^k}$

$\Rightarrow z = e^\alpha \quad \varepsilon = k \Rightarrow y_R(k) = e^{\alpha k} W(z)\big|_{z=e^\alpha}$

es.

$u(k) = \frac{e^{jwk} - e^{-jwk}}{2j} = \sin wk$

$y_R(k) = 1/2j\,(W(z)|_{z=e^{jw}} \cdot e^{jwk} - W(z)|_{z=e^{-jw}} \cdot e^{-jwk})$

$\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$

$\qquad\qquad M(jw)\,e^{j\phi} \qquad\qquad M(jw)\,e^{-j\phi}$

$= M(jw)\left(\frac{e^{j(wk+\phi)} - e^{-j(wk+\phi)}}{2j}\right) \Rightarrow M(jw)\sin(wk+\phi)$

es. $u(k) = \frac{k^\ell}{\ell!}$

$y_R(k) = \sum_{-\infty}^{k} w(k-\tau)\,\tau^\ell/\ell! = \sum_0^\infty w(\varepsilon)\frac{(k-\varepsilon)^\ell}{} = (c_0 \frac{k^\ell}{\ell!} + \dots + c_{k-1}k + c_k)\,\delta_{-1}(k)$

$c_0 = W(z)|_{z=1}$

$c_1 = \frac{dW(z)}{dz}\Big|_{z=1}$

$c_2 = 1/2\,\frac{d^2W(z)}{dz^2}\Big|_{z=1}$

es.

$$W(z) = \frac{z}{z(2z-1)}$$

• Calcolare risposta forzata e se esiste a regime

$u_1(k) = \sin k \, \delta_{-1}(k)$

$$\bullet\ u_1(z) = \frac{z \sin 1}{z^2 - 2z\cos 1 + 1}$$

$$y_{F_1}(z) = \frac{2\cos 1}{(2z-1)(z^2-2z\cos 1+1)}$$

$$\frac{y_{F_1}(z)}{z} = \frac{2\sin 1}{z(2z-1)(z^2-2z\cos 1+1)} = \frac{R_1}{z} + \frac{R_2}{z-1/2} + \frac{R_3 z + R_4}{z^2-2z\cos 1+1}$$

$$R_1 = \lim_{z\to 0} z \frac{y_{F_1}(z)}{z} = -2\sin 1$$

$$R_2 = \lim_{z\to 1/2} (z-1/2)\frac{y_{F_1}(z)}{z} = \frac{4\sin 1}{(5/4 - \cos 1)}$$

$$\frac{2\sin 1}{2-2\cos 1} = R_1 + 2R_2 + \frac{R_3+R_4}{2-2\cos 1} \implies \frac{2\sin 1}{3(2+2\cos 1)} = -R_1 - \frac{2}{3}R_2 + \frac{R_4-R_3}{2+2\cos 1}$$

$$y_{F_1}(z) = R_1 + R_2 \frac{z}{z-1/2} + \frac{R_3 z^2 + R_4 z}{z^2-2z\cos 1+1}$$

$$N_1 \frac{z^2 - z\cos 1}{z^2-2z\cos 1+1} + N_2 \frac{z\sin 1}{z^2-2z\cos 1+1}$$

$$N_1 z^2 + (N_2 \sin 1 - N_1 \cos 1) z = R_3 z^2 + R_4 z$$

$N_1 = R_3$

$$N_2 = \frac{R_4 + R_3\cos 1}{\sin 1}$$

$N \sin(k+\phi)$

ANALISI → CONTROLLO

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$x(1) = A\cancel{x(0)} + Bu(0) \to B = \binom{1}{1}$

$x(2) = Ax(1) + Bu(1) = A^2\cancel{x}(0) + ABx(0) + Bu(1) \to$ Immagine $\{B\ AB\}$

$x(k) = A^k x(0) + A^{k-1}Bx(0) + \dots + Bu(k-1) \to$ Immagine $\{B\ AB \dots A^{k-1}B\}$ matrice di raggiungibilità

al passo n: $(B\ AB \dots A^{n-1}B)$

rango

$\rho(B\ AB \dots A^{n-1}B) = k \le n$

$$T^{-1} = \left(\begin{array}{c|c} \text{base } R & \text{compl.} \end{array} \right)$$

$$\tilde{B} = TB = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\tilde{A} = TAT^{-1} = \left(\begin{array}{c|c} \tilde{A}_{11} & \tilde{A}_{12} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array} \right)$$ divido $\tilde{A}$ in blocchi

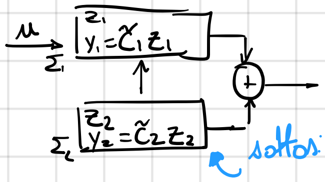
Se prendo uno stato raggiungibilità nella nuova base $\to v = \binom{*}{0}_{n-k}$

$$\tilde{A}v = \binom{*}{0}$$

$$\Rightarrow \begin{pmatrix}\tilde A_{11} & \tilde A_{12}\\ \tilde A_{21} & \tilde A_{22}\end{pmatrix}\begin{pmatrix}I\\0\end{pmatrix} = \begin{pmatrix}\tilde A_{11}\\ \tilde A_{21}\end{pmatrix} = \begin{pmatrix}*\\ \hline 0\end{pmatrix}$$

$$z_1(k+1) = \tilde A_{11} z_1(k) + \tilde A_{12} z_2(k) + \tilde B_1 u$$
$$z_2(k+1) = \tilde A_{22} z_2(k)$$

Ho due sistemi :



sottosistema non raggiungibile

$$\tilde A = \left(\begin{array}{c|c}\tilde A_{11} & \tilde A_{12}\\ \hline 0 & \tilde A_{22}\end{array}\right)$$

$$\mathcal{L}\int_0^t w(t-\tau)u(\tau)d\tau \longrightarrow W(s)u(s)$$
$$\mathcal{Z}\Big(\sum_0^k (k-\tau)u(\tau)\Big) \longrightarrow W(z)u(z)$$

$$\tilde A = \begin{pmatrix}\tilde A_{11} & \tilde A_{12}\\ 0 & \tilde A_{22}\end{pmatrix} \quad \tilde B = \begin{pmatrix}\tilde B_1\\ 0\end{pmatrix} \quad \tilde C = (\tilde C_1\,,\ \tilde C_2)$$

$v \in \text{span}\,(B\,AB \cdots A^{n-1}B)$

$Av \in \text{span}\,\{AB\ A^2B \,..\, A^nB\}$

$\quad \in \text{span}\,\{B\ \ AB\ \ A^n\}$

$$\tilde B = TB = \begin{pmatrix}*\\ \hline 0\end{pmatrix} \qquad \tilde A\begin{pmatrix}I\\0\end{pmatrix} = \begin{pmatrix}*\\ \hline 0\end{pmatrix}$$

es.

$$A = \begin{pmatrix}1 & 1\\ 3 & -1\end{pmatrix} \quad B = \begin{pmatrix}-1\\ 3\end{pmatrix} \quad C = (1 \ \ 1)$$

$$R = (B\ AB) = \begin{pmatrix}-1 & 2\\ 3 & -6\end{pmatrix} \longrightarrow \rho = \text{rango}(R) = 1$$

$$T^{-1} = \begin{pmatrix}-1 & 0\\ 3 & 1\end{pmatrix}$$ completamento

$$T = -1\begin{pmatrix}1 & 0\\ -3 & -1\end{pmatrix}$$

$$\tilde A = TAT^{-1} = \left(\begin{array}{c|c}-2 & -1\\ \hline 0 & 2\end{array}\right) \qquad \tilde B = TB = \begin{pmatrix}1\\ \hline 0\end{pmatrix}$$

$$\hat C = CT^{-1} = (2 \,|\, 1)$$

$x(k+1) = Ax(k) + Bu(k)$
$y(k) = Cx(k)$

$y(k+1) = CAx(k)$
$y(k+2) = CA^2x(k)$
$\vdots$

Supponiamo $x_0 \to y_a$, $x_b \to y_b$ t.c. $y_a = y_b$:

$Cx_a = Cx_b \longrightarrow C(x_a - x_b) = 0$
$CAx_a = CAx_b \longrightarrow CA(x_a - x_b) = 0$
$CA^2x_a = CA^2x_b \longrightarrow CA^2(x_a - x_b) = 0$
$\vdots$
$CA^\ell x_a = CA^\ell x_b \longrightarrow CA^\ell(x_a - x_b) = 0 \qquad \bar{x} = x_a - x_b$

$C\bar{x} = 0$
$CA\bar{x} = 0$
$\vdots$
$CA^\ell \bar{x} = 0 \quad \forall \ell \geq 0$ fino a $CA^{n-1}\bar{x} = 0$

$\begin{pmatrix} C \\ CA \\ | \\ CA^{n-1} \end{pmatrix} \bar{x} = 0$ matrice Osservabilità $(\mathcal{O})$

Se $\rho(\mathcal{O}) = k < n \longrightarrow n-k$ soluzioni = stati non osservabili

$T^{-1} = \left(\begin{array}{c|c} \text{compl.} & \text{base inoss.} \\ n-k & k \end{array} \right)$
$\qquad\qquad\qquad v_1 - v_k$

$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \quad \tilde{C} = (\tilde{C}_1 \;\; \tilde{C}_2) = CT^{-1} = C\left[n-k \,|\, \bar{x} = 0 \right] = (\tilde{C}_1 \;\; 0)$

$A\bar{x}$ = inosservabile $= \hat{x}$ (forma $\begin{pmatrix} 0 \\ * \end{pmatrix}$ nelle nuove coordinate)

$z = Tx \qquad \bar{z} = T\bar{x} = \begin{pmatrix} 0 \\ * \end{pmatrix}$

$CA^2 = (CA)A$

es. $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} = A \qquad B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad C = (0 \;\; 2 \;\; 0 \;\; 3)$

$\mathcal{O} = \begin{pmatrix} 0 & 2 & 0 & 3 \\ 0 & -2 & 0 & -8 \\ 0 & -2 & 0 & -36 \\ & \cdots & & \end{pmatrix} \qquad \rho(\mathcal{O}) = 2$

$\begin{pmatrix} 0 & 2 & 0 & 3 \\ 0 & -2 & 0 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \longrightarrow \left. \begin{array}{l} 2x_2 + 3x_4 = 0 \\ -2x_2 + 8x_4 = 0 \end{array} \right\} \; x_2 = x_4 = 0$

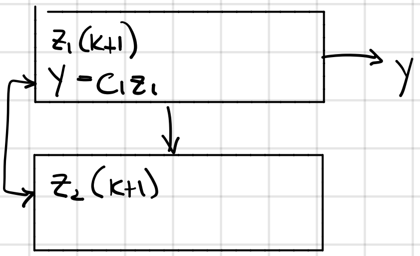
Kernel $\mathcal{O} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$T^{-1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right) \qquad \tilde{C} = \left(\begin{array}{cc|cc} 2 & 3 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = T \qquad \tilde{A} = TAT^{-1} = \left(\begin{array}{cc|cc} -1 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right)$$

$$z_1(k+1) = \tilde{A}_{11} z_1(k) + \tilde{B}_1 u(k)$$
$$z_2(k+1) = \tilde{A}_{21} z_1 + \tilde{A}_{21} z_2 + \tilde{B}_2 u$$

$$y = \tilde{C}_1 z_1$$



$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix}$$

$$W(z) = \tilde{C}(zI - \tilde{A})^{-1} B$$

$$(\tilde{C}_1 \ \tilde{C}_2) \begin{bmatrix} zI - \tilde{A}_{11} & -\hat{A}_{12} \\ 0 & zI - \tilde{A}_{22} \end{bmatrix}^{-1} \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} = (\tilde{C}_1 \ \tilde{C}_2) \begin{bmatrix} (zI - \tilde{A}_{11})^{-1} & * \\ 0 & (zI - \hat{A}_{22})^{-1} \end{bmatrix} \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} = (\tilde{C}_1 \ \tilde{C}_2) \begin{bmatrix} (zI - \tilde{A}_{11})^{-1} \tilde{B}_1 \\ 0 \end{bmatrix}$$
$$= \tilde{C}_1 (zI - \tilde{A}_{11}) \tilde{B}$$

$$(zI - \tilde{A}_{11})L - \hat{A}_{12}(zI - \hat{A}_{22}) = 0 \longrightarrow L = (zI - \tilde{A}_{11})^{-1} (\tilde{A}_{12}(zI - \tilde{A}_{22}))$$

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} \quad \tilde{C} = (\tilde{C}_1 \ 0)$$

$$(\tilde{C}_1 \ 0) \begin{bmatrix} (zI - \tilde{A}_{11}) & 0 \\ -\tilde{A}_{21} & zI - \tilde{A}_{41} \end{bmatrix}^{-1} \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} = \tilde{C}_1 (zI - \tilde{A}_{11})^{-1} \tilde{B}_1$$

$R =$

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad \ker \mathcal{O}$$

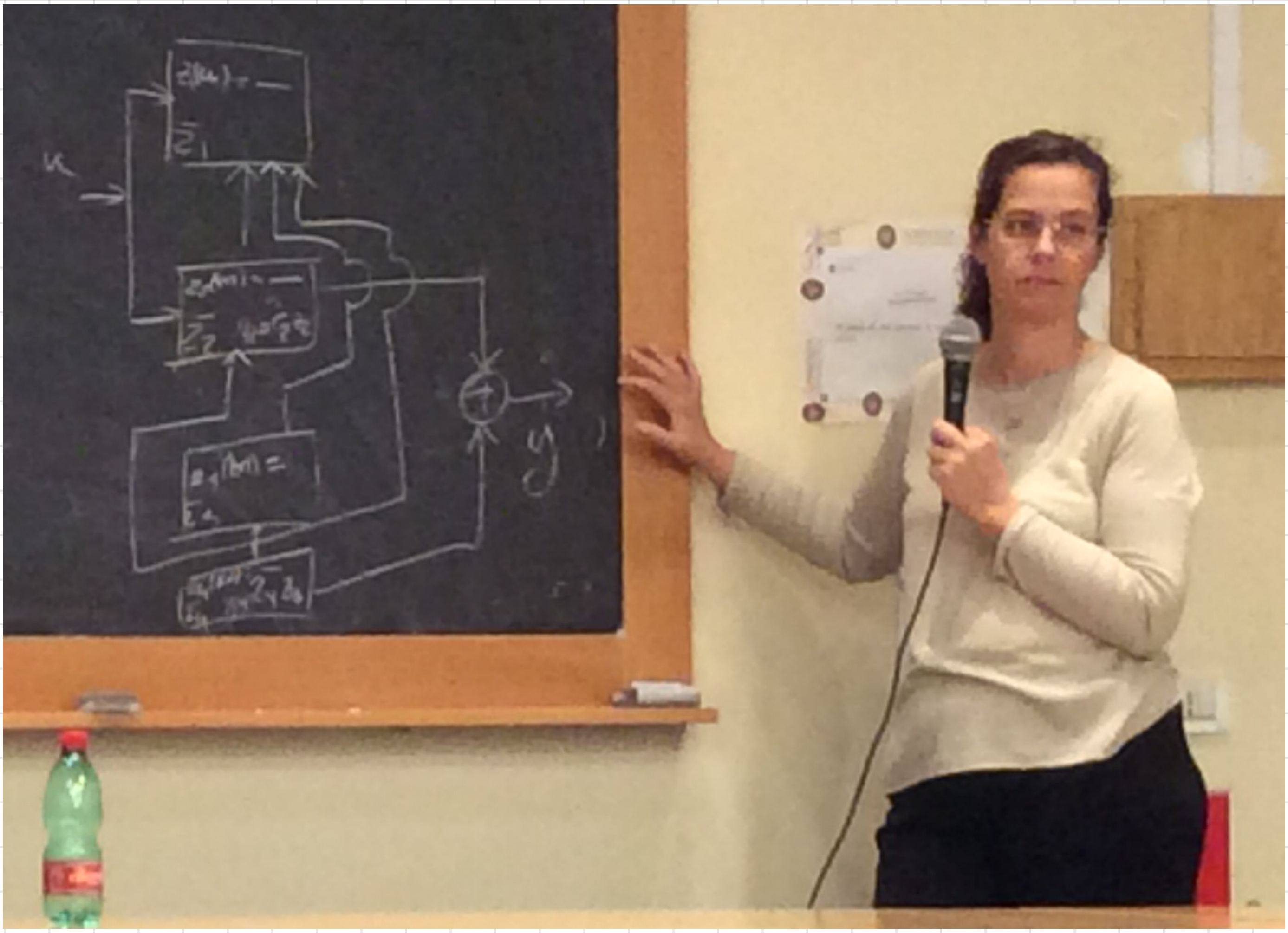
$X_1 = R \cap I$ — ragg. e inoss.
$X_2 \Rightarrow X_1 \oplus X_2 = R$ — ragg. e oss.
$X_3 \Rightarrow X_1 \oplus X_3 = I$ — inoss. e inragg.
$X_4 \rightarrow X_1 \oplus X_2 \oplus X_3 \oplus X_4 = R^n$ — oss. e inragg.

$$T^{-1} = \Big(\underbrace{\text{base } X_1 \ \Big| \ \text{base } X_2}_{\text{raggiungibili}} \ \Big| \ \text{base } X_3 \ \Big| \ \text{base } X_4 \Big)$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & & & \tilde{A}_{14} \\ \tilde{A}_{21} & & 0 & \tilde{A}_{24} \\ \tilde{A}_{31} & \cdots & & \tilde{A}_{34} \\ \tilde{A}_{41} & & 0 & \tilde{A}_{44} \end{bmatrix} \qquad \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 & \tilde{C}_4 \end{pmatrix} \qquad \tilde{C}\bar{x} = 0 \longrightarrow \begin{pmatrix} * \\ 0 \\ * \\ 0 \end{pmatrix}$$

$$z_1(k+1) = \tilde{A}_{11} z_1 + \tilde{A}_{12} z_2 + \tilde{A}_{13} z_3 + \tilde{A}_{14} z_4 + \tilde{B}_1 u$$
$$z_2(k+1) = \tilde{A}_{22} z_2 + \tilde{A}_{24} z_4 + \tilde{B}_1 u$$
$$z_3(k+1) = \tilde{A}_{33} z_3 + \tilde{A}_{34} z_4$$
$$z_4(k+1) = \tilde{A}_{44} z_4$$
$$y = \tilde{C}_2 z_2 + \tilde{C}_4 z_4$$



$K(s) = [\quad]$ trovare se esistono A, B, C, D t.c. $C(sI - A)^{-1}B + D = K(s)$

$W(s) = N(s)/D(s)$ $\quad m \le n$ condizione necessaria (m = grado D, n = grado N)

se $m = n$ $D \neq 0$ $\quad K(s) = K'(s) + D \Rightarrow N(s)/D(s) = N'(s)/D(s) + D$

es. $W(s) = \frac{s+1}{s+3} \Rightarrow N(s)/D(s) = -2/(s+3) + 1$ perché $C(sI-A)^{-1}B = -2/(s+3) = (-2)(s+3)^{-1}1$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & \\ -a_0 & -a_1 & \dots & & -a_n \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad C = (b_0\ b_1 \cdots b_{n-1})$$

$$K(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} \dots a_1 s + a_0}$$

es. $\frac{s+3}{3s^2+2s+1}$ $\quad D = 0$ perché $m > n$

$$\Rightarrow \frac{s/3 + 1}{s^2 + 2/3 s + 1/3} \longrightarrow A = \begin{pmatrix} 0 & 1 \\ -1/3 & -2/3 \end{pmatrix}_{2\times 2} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = (1\ \ 1/3)$$

es $\begin{pmatrix} \frac{s+3}{s^2+s+1} \\ \frac{s+2}{s+1} \end{pmatrix}$ due uscite $\Rightarrow \begin{pmatrix} \frac{s+3}{s^2+s+1} \\ 1 + 1/(s+1) \end{pmatrix} = \begin{pmatrix} \frac{s+3}{s^2+s+1} \\ \frac{1}{s+1} \end{pmatrix} + \overset{D}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$$\frac{\begin{bmatrix} (s+3)(s+1) \\ s^2+s+1 \end{bmatrix}}{(s^2+s+1)(s+1)} = \frac{\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} s + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s^2}{s^3 + 2s^2 + 2s + 1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad C = \left(\begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

Forma canonica osservabile

$$A_o = \left(\begin{array}{ccc|c} 0 & & & -a_0 \\ 1 & 1 & & -a_1 \\ 0 & & & \vdots \\ \dots & 0 & 1 & -a_{n-1} \end{array}\right) \quad B = \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} \quad C = (0 \dots 1)$$

$W(s) = -2/(s+3) \longrightarrow A_o = (-3)$ $\quad B_o = -2$ $\quad C_o = 1$

$w(s) = \frac{s+3}{3s^2+2s+1} \longrightarrow \frac{s/3 + 1}{s^2 + 2/3 s + 1/3} \longrightarrow A_o = \begin{pmatrix} 0 & -1/3 \\ 1 & -2/3 \end{pmatrix}$ $\quad B_o = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\quad C_o = (0\ 1)$

$$W(s) = \begin{pmatrix} \frac{s+3}{s^2+s+1} \\ \frac{1}{s+1} \end{pmatrix} = \frac{\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} s + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s^2}{s^3 + 2s^2 + 2s + 1}$$

$$A_o = \left(\begin{array}{ccc|c} 0 & & & -a_0 I \\ I & & & \vdots \\ 0 & \ddots & I & -a_{n-1} I \end{array}\right) \quad B_o = \begin{pmatrix} B_o \\ \vdots \\ B_{n-1} \end{pmatrix} \quad C = (0 \dots I)$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ \hline 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{array}\right) \left(\begin{array}{c} 3 \\ 1 \\ \hline 4 \\ 1 \\ \hline 1 \\ 1 \end{array}\right) \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

ingressi = 1 raggiungibilità min

uscite = 1 osservabilità min

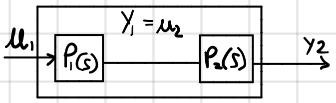
Metodo di Gilbert

$$W(s) = \begin{pmatrix} \frac{s+1}{s+3} & \frac{1}{s+2} \\ \frac{1}{s+3} & -1/s+2 \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{-2}{s+3} & \frac{1}{s+2} \\ \frac{1}{s+3} & -1/s+2 \end{pmatrix} + \overset{D}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\frac{R_0}{s+3} + \frac{R_1}{s+2} = \frac{\begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}^{C_1 B_1}_{P \times m}}{s+3} + \frac{\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}^{C_2 B_2}_{P \times m}}{s+2}$$

Vedi solo Diag

$$A_G = \left(\begin{array}{c|c} -3 & 0 \\ \hline 0 & -2 \end{array}\right) \quad B_G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C_G = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$



IN SERIE

$$y_2(s) = P_2(s) u_2(s)$$
$$y_1(s) = P_1(s) u_1(s)$$
$$u_2(s) = y_1(s)$$
$$y(s) = y_2(s) = P_2(s) P_1(s) u_1(s)$$

es. $P_2 = \frac{1}{s+3}$ $\quad P_1 = \frac{s+d}{s+2}$ $\Rightarrow \frac{s+a}{(s+3)(s+2)}$ se $a=3$ si eliminano $(s+3)$ quindi quell'autovalore non era osservabile o raggiungibile (ho una perdita)

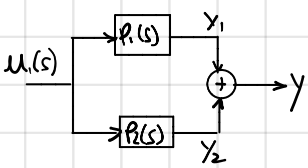
$\rightarrow \dot{x}_1 = A x_1 + B_1 u_1$

$\quad y_1 = C_1 x_1 + D_1 u_1$

$\rightarrow \dot{x}_2 = A_2 x_2 + B_2 \cancel{u_2} (C_1 x_1 + D_1 u_1)$

$\rightarrow y_2 = C_2 x_2 + D_2 \cancel{u_2} (C_1 x_1 + D_1 u_1)$

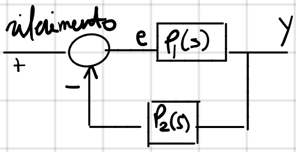
$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline B_2 C_1 & A_2 \end{array}\right) \quad B = \begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix} \quad C = \begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix} \quad D = D_2 D_1$$



IN PARALLELO

$$y = y_1 + y_2 = P_1 u(s) + P_2 u(s) = (P_1(s) + P_2(s)) u(s)$$
$$y = C_1 x_1 + D_1 u_1 + C_2 x_2 + D_2 u_1$$

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \quad D = D_1 + D_2$$



IN CONTROREAZIONE

$$y(s) = P_1(s)\,e(s) \longrightarrow P_1(s)\big(r(s) - y_2(s)\big) = P_1(s)r(s) - P_1(s)P_2(s)y(s)$$

$$e(s) = r(s) - y_2(s)$$

$$y_2(s) = P_2(s)y(s)$$

$$\Rightarrow y(s) + P_1(s)P_2(s)y(s) = P_1(s)r(s)$$

$$\Rightarrow [I + P_1P_2]\,y(s) = P_1(s)r(s)$$

$$y(s) = [I + P_1P_2]^{-1} P_1(s)\,r(s)$$

$$y(s) = \frac{P_1}{1 + P_1P_2}\,r(s)$$

$$P(s) = \frac{N_1(s)/D_1(s)}{1 + \dfrac{N_1(s)N_2(s)}{D_1(s)D_2(s)}} = \frac{N_1}{\cancel{D_1}} \cdot \frac{\cancel{D_1}D_2}{\underbrace{D_1D_2 + N_1N_2}_{D_1 + KN_1}}$$

$$N_2(s)/D_2(s) = K = \text{numero}$$

STABILITÀ E INSTABILITÀ

Fissato $\|x_0 - x_e\| < \delta \longrightarrow \|x(t) - x_e\| < \varepsilon$ stabilità semplice

$+ \quad \lim_{t\to\infty} \|x(t) - x_e\| = 0$ stabilità asintotica

Stabilità semplice $\Leftrightarrow$ autovalori $Re \leq 0$, $Re = 0 \Leftrightarrow m.g. = 1$

Stabilità asintotica $\Leftrightarrow$ autovalori $Re < 0$

LJAPUNOV

$V(x)$ simmetrica e definita positiva ≥ 0 : $V(x_e) = 0 \quad V(x) > 0 \quad \forall x \neq x_e$

$\dot{V}(x) < 0 \qquad V(x) = \frac{1}{2} x^T P x$

$V(x) = \frac{1}{2} x^T P x > 0$

$\dot{V} = \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x} = \frac{1}{2}(x^T A^T P x + x^T P A x) < 0$

$\Rightarrow A^T P + PA = -Q$ Q simmetrica e definita positiva

es.

① $W(s) = \begin{pmatrix} \frac{s-1}{s^2+3s+2} \\ \frac{s+10}{s+1} \end{pmatrix}$

a) Realizzazione forma canonica ragg. / oss. / gilbert

b) Risposta forzata e a regime : $u(t) = \sin(t)$

② Stabilità per $s^5 + (k-1)s^4 + s^3 + 2s^2 + s + 2$

a) $\frac{s+10}{s+1} \longrightarrow 1 + \frac{9}{s+1} \longrightarrow \begin{pmatrix} \frac{s-1}{(s+1)(s+2)} \\ \frac{9}{s+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_D$

$W'(s) = \frac{\begin{bmatrix} s-1 \\ 9(s+2) \end{bmatrix}}{(s+1)(s+2)} = \frac{\begin{pmatrix} -1 \\ 18 \end{pmatrix} + \begin{pmatrix} 1 \\ 9 \end{pmatrix} s}{s^2+3s+2}$

$A_{2\times 2} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \left(\begin{pmatrix} -1 \\ 18 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \end{pmatrix} \right)$ f.c. raggiungibile

$B_{oss} = \begin{pmatrix} -1 \\ 18 \\ \hline 1 \\ 9 \end{pmatrix} \qquad \left(\begin{array}{cc|cc} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ \hline 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \end{array} \right) = A$ } f.c. osservabile

$C = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$

Poli: $(s+1)$ e $(s+2)$

$R_1/s+1 + R_2/s+2 = \frac{\begin{bmatrix} -2 \\ 9 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}{s+2}$ $\longrightarrow \begin{pmatrix} -2 \\ 9 \end{pmatrix} 1 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} 3$

$C_1 B_1 \quad C_2 B_2$

$p \times p_1 \; p_1 \times q \quad p \times p_1 \; p_1 \times q$

$A_G = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad B_G = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad C_G = \begin{pmatrix} -2 & 1 \\ 9 & 0 \end{pmatrix}$

b) $u(s) = 1/s^2+1$

$Y_F(s) = W(s)u(s) = \frac{\begin{bmatrix} s-1 \\ (s+10)(s+2) \end{bmatrix}}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2} + \frac{R_3 s + R_4}{s^2+1}$

$R_1 = \lim_{s\to -1} \frac{\begin{bmatrix} s-1 \\ (s+10)(s+2) \end{bmatrix}}{(s+2)(s^2+1)} = \frac{\begin{pmatrix} -2 \\ 9 \end{pmatrix}}{2}$

$R_2 = \lim_{s\to -2} \frac{\begin{bmatrix} s-1 \\ (s+10)(s+2) \end{bmatrix}}{(s+1)(s^2+1)} = \frac{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}{5}$

$$\frac{\begin{bmatrix}-1\\20\end{bmatrix}}{2} = \frac{\begin{bmatrix}-2\\9\end{bmatrix}}{2} + \frac{\begin{bmatrix}3/5\\0\end{bmatrix}}{2} + R_4 \qquad s=0$$

$$\frac{\begin{bmatrix}0\\33\end{bmatrix}}{12} = \frac{\begin{bmatrix}-2\\9\end{bmatrix}}{4} + \frac{\begin{bmatrix}3/5\\0\end{bmatrix}}{3} + \frac{R_3+R_4}{2} \qquad s=1$$

$$R_4 = \begin{pmatrix}1/5\\11/2\end{pmatrix} \qquad R_3 = 2\begin{pmatrix}0 + 2/4 - 1/5\\ 11/4 - 9/4\end{pmatrix} - \begin{pmatrix}1/5\\11/2\end{pmatrix} = \begin{pmatrix}2/5\\-9/2\end{pmatrix}$$

$$y_F(t) = \begin{pmatrix}-1\\9/2\end{pmatrix}e^{-t} + \begin{pmatrix}3/5\\0\end{pmatrix}e^{-2t} + \underbrace{\begin{pmatrix}2/5\\-9/5\end{pmatrix}\cos t + \begin{pmatrix}1/5\\11/2\end{pmatrix}\sin t}_{M\sin(t+\varphi)}$$

$$\frac{s+1}{s^2+3s+2} \qquad W_{11}(j)\frac{1+j}{1+3j} = \frac{(1+j)(1-3j)}{10} = \frac{2-j}{5} \qquad M_1 = \frac{\sqrt{5}}{5} \qquad \varphi_1 = \operatorname{arctg} -1/2$$

$$W_{21}(j) = \frac{10+j}{1+j} = \frac{(10+j)(1-j)}{2} = \frac{11-9j}{2} \qquad M_2 = \frac{\sqrt{202}}{2} \qquad \varphi = \operatorname{arctg} -9/11$$

2. $K-1>0 \rightarrow K>1$

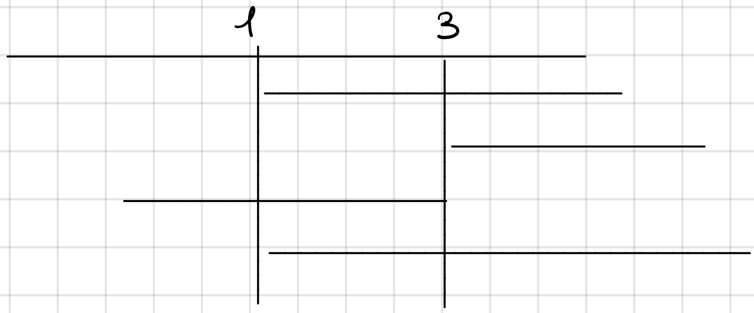
$$\begin{array}{c|ccc}
5 & 1 & 1 & 1\\
4 & K-1 & 2 & 2\\
3 & K-3 & K-3 & \\
2 & 3-K & 2 & \\
1 & \frac{(K-3)(3-K)-2(K-3)}{3-K} & & \\
0 & 2 & &
\end{array}$$

$K-1>0$

$K-3>0$

$3-K>0$

$K-3-2>0$



$$K=3 \quad \begin{array}{c|ccc}
5 & 1 & 1 & 1\\
4 & 2 & 2 & 2\\
\cancel{3} & \cancel{0} & \cancel{0} & \\
3 & 8 & 4 & \\
\cancel{2} & \cancel{8} & \cancel{16} & \\
2 & 1 & 2 & \\
1 & -12 & & \\
0 & 2 & &
\end{array}$$

$2\lambda^4 + 2\lambda^2 + 2$

$8\lambda^3 + 4\lambda$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$u = Fx + Gv \quad \Rightarrow \begin{cases} \dot{x} = Ax + BFx + BGv \\ y = Cx + DFx + DGv \end{cases} \quad \tilde{A} = A + BF$$

$$\tilde{A} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ -a_0 & \dots & & -a_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} F_1 & F_2 & \dots & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ -a_0 & \dots & & -a_{n-1} \end{pmatrix} + \begin{pmatrix} 0 & — & 0 \\ 0 & — & 0 \\ F_1 & — & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ -a_0+F_1 & — & & -a_{n-1}+F_n \end{pmatrix} \longrightarrow \lambda^n + \underbrace{(a_{n-1}-F_n)}_{b_{n-1}}\lambda^{n-1} + \dots + \underbrace{(a_1-F_2)}_{b_1}\lambda + \underbrace{(a_0-F_1)}_{b_0}$$

$$u = \tilde{F} z = \tilde{F} T x$$

$$R^{-1} = (B\ AB \dots A^{n-1}B)^{-1} \longrightarrow \gamma \text{ - ultima riga}$$

$$T \rightarrow \begin{bmatrix} \begin{pmatrix} \gamma \\ \gamma A \\ \vdots \\ \gamma A^{n-1} \end{pmatrix} \end{bmatrix} (B\ AB — A^{n-1}B) = \begin{pmatrix} 0 & 0 & — & 1 \\ 0 & 0 & — 1 & * \\ 1 & & & * \end{pmatrix} \Rightarrow T \text{ cambio coordinate}$$

$$F = \tilde{F} T = \begin{pmatrix} a_0 - b_0 & a_1 - b_1 & \dots & a_{n-1} - b_{n-1} \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma A \\ \gamma A^{n-1} \end{pmatrix} = (a_0 - b_0)\gamma + (a_1 - b_1)\gamma A + \dots + (a_{n-1} - b_{n-1})\gamma A^{n-1} =$$

$$= \gamma a_0 + \gamma a_1 A + \dots + \gamma a_{n-1} A^{n-1} - (\gamma b_0 + \gamma b_1 A + \dots + \gamma b_{n-1} A^{n-1}) = \underbrace{\gamma (a_0 + a_1 A + \dots + a_{n-1} A^{n-1})}_{-A^n} \underbrace{- \gamma (b_0 + \dots + b_{n-1} A^{n-1})}_{-\gamma \tilde{p}(A)}$$

$$F = -\gamma \tilde{p}(A)$$

es.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad u = Fx \quad \lambda_1 = -4 \quad \lambda_2 = -5$$

$$(B\ AB) = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \quad R^{-1} = \tfrac{1}{2} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad \gamma = \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} \end{pmatrix}$$

$$\tilde{p}(\lambda) = (\lambda+4)(\lambda+5) = \lambda^2 + 9\lambda + 20$$

$$F = -\gamma \tilde{p}(A) = -(\tfrac{1}{2}\ \tfrac{1}{2})[20 + 9A + A^2] = -(\tfrac{1}{2}\ \tfrac{1}{2}) \left[\begin{pmatrix} 20 & 0 \\ 0 & 20 \end{pmatrix} + \begin{pmatrix} 9 & 18 \\ 0 & -27 \end{pmatrix} + \begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix} \right]$$

$$= -(\tfrac{1}{2}\ \tfrac{1}{2}) \begin{pmatrix} 30 & 14 \\ 0 & 2 \end{pmatrix} = (-15\ -8)$$

$$A + BF : \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (-15\ -8) = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} -15 & -8 \\ 15 & 8 \end{pmatrix} = \begin{pmatrix} -14 & -6 \\ 15 & 5 \end{pmatrix} = \tilde{A}$$

$$(\tilde{A} - \lambda I) = \begin{pmatrix} -14-\lambda & -6 \\ 15 & 5-\lambda \end{pmatrix}$$

$$p(\lambda) = (-14-\lambda)(5-\lambda) + 6 \cdot 15 = -70 - 5\lambda + 14\lambda + \lambda^2 + 90 = \lambda^2 + 9\lambda + 20 \rightarrow \text{torno quindi giusto}$$

$\dot{x} = Ax + Bu \qquad R = (B \ AB - A^{n-1}B) \longrightarrow z = Tx \longrightarrow \dot{z}_1 = \tilde{A}_{11} z_1 + \tilde{A}_{12} z_2 + B_1 u$

$\dot{z}_2 = \tilde{A}_{22} z_2$

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix}$$

$u = \tilde{F} z = \tilde{F}_1 z_1 + \tilde{F}_2 z_2$

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ 0 \end{pmatrix} \begin{pmatrix} \tilde{F}_1 & \tilde{F}_2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix} + \begin{pmatrix} B_1\tilde{F}_1 & B_1\tilde{F}_2 \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \tilde{A}_{11} + B_1\tilde{F}_1 & \tilde{A}_{12} + B_1\tilde{F}_2 \\ \hline 0 & \tilde{A}_{22} \end{array}\right)$$

es $A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda = -3 \quad \lambda = -4$

$R = (B \ AB) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(F_1 \ \ 0)$ t.c $A_{11} + B_1\tilde{F}_1 = -4$ perché -3 già c'è $\longrightarrow 1 + F_1 = -4 \longrightarrow F_1 = 5$

$A + BF$:

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} -5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 0 & -3 \end{pmatrix}$$

es

$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \lambda_1 = -2 \quad \lambda_2 = -4 \quad \lambda_3 = -5$

$R = (B \ AB \ A^2B) = \begin{pmatrix} 1 & -1 & 5 \\ -1 & 3 & -9 \\ 0 & 0 & 0 \end{pmatrix} \quad T^{-1} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

$A_{11} + B_1\tilde{F}_1$: $\ p(\lambda) = (\lambda+4)(\lambda+5) \longrightarrow F_1 = (-15 \ \ -8) \longrightarrow \tilde{F} = (-15 \ \ -8 \mid 0)$

Se il sistema è tutto raggiungibile possiamo trovare $u = Fx + Gv$ tale che gli autovalori siano quelli che vogliamo noi.

$\dot{x} = Ax + Bu$

$y = Cx$

$x(t) = e^{A(t-t_0)} x(t) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$

NON CONOSCO LO STATO INIZIALE!

Ipotesi: A ha autovalori λ con parte reale < 0

$\dot{z} = Az + Bu$ errore $e = x - z \Rightarrow \dot{e} = \dot{x} - \dot{z} = Ax + Bu - Az - Bu = Ae$ $e \to 0$ per $t \to \infty$

Se $\dot{z} = Az + Bu + Ky - KCz$:

$e = x - z \Rightarrow \dot{e} = \dot{x} - \dot{z} = (A - KC)x - (A - KC)z = (A - KC)e$

Trovare K:

$(A - KC)^T = A^T - C^T K^T$

$\hat{A} + \hat{B}\,\hat{F}$

$\rho(\hat{B}\ \hat{A}\hat{B} \dots \hat{A}^{n-1}\hat{B}) = n = (C^T A^T C^T \dots (A^T)^{n-1} C^T) \Leftrightarrow \rho \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n \Leftrightarrow$ sistema osservabile

$\gamma = \eta^T$ = ultima colonna matrice inversa osservabilità

$\tilde{p}(x)$: $\hat{F} = -\gamma \tilde{p}(A^T)$

$-K^T = -\eta^T \tilde{p}(A^T) \longrightarrow K = \tilde{p}(A)\eta$

es. $A = \begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$

λ per dinamica errore $(-1, -2)$

$\tilde{p}(\lambda) = (\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$

$\Theta = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}$ $\Theta^{-1} = 1/6 \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ $\eta = \begin{pmatrix} -1/5 \\ 1/5 \end{pmatrix}$

$K = (A^2 + 3A - 2I)\begin{pmatrix} -1/5 \\ 1/5 \end{pmatrix} = \left[\begin{pmatrix} 9 & -2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -9 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} -1/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} -1/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 6/5 \end{pmatrix}$

$\dot{z} = \begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix} z + Bu + \begin{pmatrix} -1/5 \\ 6/5 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}(x - z)$

$\dot{e} = \left[\begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1/5 \\ 6/5 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}\right] e = \left[\begin{pmatrix} -3 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1/5 & -1/5 \\ 6/5 & 6/5 \end{pmatrix}\right] e = \begin{bmatrix} -14/5 & 6/5 \\ -6/5 & -1/5 \end{bmatrix} e$

$\begin{pmatrix} -14/5 - \lambda & 6/5 \\ -6/5 & -1/5 - \lambda \end{pmatrix} \longrightarrow (-14/5 - \lambda)(-1/5 - \lambda) + \frac{36}{25} = \lambda^2 + 3\lambda + 2$

• $u = Fz$ $\quad \dot{x} = Ax + Bu \quad y = Cx$

$\dot{x} = Ax + Bu$

$\dot{z} = Az + Bu + KC(x - z)$

$u = Fz + GV$

$$\begin{pmatrix} z \\ e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} z \\ x \end{pmatrix} \Rightarrow \dot{z} = Az + Bu + KCe \qquad \dot{e} = \dot{x} - \dot{z} = Ax + Bu - (Az + Bu + KCe) \quad u = Fz + GV$$

$$\Rightarrow \begin{cases} \dot{z} = (A+BF)z + KCe + BGV \\ \dot{e} = (A-KC)e \end{cases}$$

$$A_e = \left(\begin{array}{c|c} [A+BF] & KC \\ \hline & [A-KC] \end{array}\right) \Rightarrow \text{autovalori} = \text{autovalori}(A+BF) \text{ e } \text{autovalori}(A-KC)$$

es

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \quad C = (2 \ 2 \ 3)$$

Stabilizzare in maniera asintotica con reazione dall'uscita con autovalori con parte reale ≤ -4

$\dot{x} = Ax + Bu$

$y = Cx$

$u = -Ky$

$$C(sI - A)^{-1}B + D \qquad W(s) = \frac{F(s)}{1+KF(s)}$$

$$(sI-A)^{-1} = \begin{pmatrix} s+1 & -2 & -3 \\ 0 & s+3 & 1 \\ 0 & 0 & s-1 \end{pmatrix} = \frac{\begin{bmatrix} (s+3)(s-1) & 2(s-1) & -2+3(s+3) \\ 0 & (s+1)(s-1) & -(s+1) \\ 0 & 0 & (s+1)(s+3) \end{bmatrix}}{(s+1)(s+3)(s-1)} \Rightarrow C(sI-A)^{-1}B =$$

$$F(s) = \frac{2(3s+7) - 2(s+1) + 6(s+3)}{(s+3)(s-1)}$$

$$= \frac{10s + 30}{(s-1)(s+3)} = \frac{10}{s-1} \qquad D + KN \longrightarrow s-1 + K10 \longrightarrow 10K - 1 = 5 \longrightarrow K = 6/10$$

In questo caso modifico solo $\lambda = 1$ perchè gli altri non sono raggiungibili e/o osservabili

Supponiamo volessimo solo stabilizzare il sistema $\Rightarrow$ costruzione osservatore

$$(B \ AB \ A^2B) = \left(\begin{array}{c|c|c} 3 & 1 & 7 \\ -1 & 1 & -5 \\ 2 & 2 & 2 \end{array}\right) \longrightarrow T^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \rightarrow T = 1/4 \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ -4 & -4 & 4 \end{pmatrix}$$

$$\tilde{A} = TAT^{-1} = \left(\begin{array}{cc|c} 0 & 3 & 1 \\ 1 & -2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \qquad \tilde{B} = TB = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

devo avere due zeri

Sott. ragg.

$$(\tilde{A}_{11} \ \tilde{B}_1) \rightarrow F = -\gamma p^*(A_{11}) \rightarrow p^*(A_{11}) = (\lambda+2)(\lambda+3) = \lambda^2 + 5\lambda + 6$$

$$R^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Rightarrow F = -(0 \ 1)\left[\begin{pmatrix} 3 & -6 \\ -2 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 15 \\ 5 & -10 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right] = \begin{matrix} F \text{ su sistema ridotto} \\ (-3 \ \ 3) \end{matrix}$$

γ

$$\hat{F} = (-3\ -3\ \ 0) \Rightarrow F = \hat{F}T = (-3\ -3\ 0)T = \left(-\tfrac{3}{2}\ \ -\tfrac{3}{2}\ \ 0\right)$$

$$\Theta = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ -2 & -2 & 7 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 3 \\ -2 & -2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad \left\{\begin{aligned} 2a+2b+3c&=0 \\ -2a-2b+7c&=0 \end{aligned}\right\} \ c=0 \Rightarrow \mathrm{Ker} = \left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$$

$$\Rightarrow T^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{A} = TAT^{-1} = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ oss.} \qquad \tilde{C} = CT^{-1} = (0\ \ 2\ \ 3)$$

$$\lambda = -5, -6 \Rightarrow \lambda^2 + 11\lambda + 30$$

$$\tilde{A}_{11}\ \tilde{C}_1 \qquad \tilde{\Theta}^{-1} = \begin{pmatrix} 2 & 3 \\ -2 & 7 \end{pmatrix}^{-1} = \frac{1}{20}\begin{pmatrix} x & -3 \\ x & 2 \end{pmatrix} \qquad \eta = \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix} \qquad \hat{K} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -11 & 22 \\ 0 & 11 \end{pmatrix} + \begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix}\right]\begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix} = \begin{pmatrix} -8/10 \\ \frac{42}{10} \end{pmatrix}$$

$$\Rightarrow \tilde{K} = \begin{pmatrix} 0 \\ -4/5 \\ 21/5 \end{pmatrix}$$

$$\dot{\varepsilon}_1 = \tilde{A}_{11}\varepsilon_1 + \tilde{A}_{12}\varepsilon_2 + \tilde{B}_1 u \quad + (Ky - KC\varepsilon)$$

$$\hookrightarrow \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}(0\ \ \tilde{C}_2)\varepsilon$$

$$\dot{\varepsilon}_2 = \tilde{A}_{22}\varepsilon_2 + \tilde{B}_2 u$$

$$y = \tilde{C}_2 \varepsilon_2$$

$$\hat{A} = T^{-1}(\tilde{A} - \tilde{K}\tilde{C})T = \overbrace{T^{-1}AT}^{A} - T^{-1}\tilde{K}\overbrace{\tilde{C}T}^{C}$$

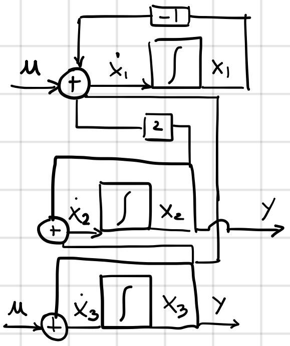
$$K = \begin{pmatrix} -4/5 \\ 0 \\ 21/5 \end{pmatrix} \qquad A - KC = \begin{pmatrix} 3/5 & 18/5 & 27/5 \\ 0 & -3 & -1 \\ -42/5 & -42/5 & -58/5 \end{pmatrix} - \lambda I = (3/5 - \lambda)\left((-3-\lambda)(-18/5 - \lambda) - 42/5\right) = -42/5\left(-18/5 + (\lambda+3)\,27/5\right)$$

2) $\dot{x} = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} x$

a) Analisi Modali
b) Risposta forzata $u = \sin t$ e a regime
c) Schema simulazione

c)

$$\begin{cases} -x_1 + 2x_2 + x_3 + u \\ x_2 - x_3 \\ x_2 + x_3 + u \end{cases}$$



b) $\det(A - \lambda I)$:

$$\begin{pmatrix} -1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{pmatrix} \longrightarrow (-1-\lambda)\left[(1-\lambda)(1-\lambda)+1\right] \longrightarrow (-1-\lambda)\left[1+\lambda^2-\lambda+1\right]$$

$\lambda_1 = -1 \qquad \lambda_{2,3} = 1 \pm J$

$$\lambda = -1 \longrightarrow \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{cases} 2b+c \\ 2b-c \\ b-2c \end{cases} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 1 \pm J$

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{cases} -a_1 + 2a_2 + a_3 = a_1 + b_1 \\ \cancel{a_2} - a_3 = \cancel{a_2} + b_2 \\ a_2 + \cancel{a_3} = \cancel{a_3} + b_3 \end{cases} \cup \begin{cases} -b_1 + 2b_2 + b_3 = -a_1 + b_1 \\ \cancel{b_2} - b_3 = -a_2 + \cancel{b_2} \\ b_2 + \cancel{b_3} = -a_3 + \cancel{b_3} \end{cases} \Rightarrow \begin{cases} b_1 = -2a_1 + 2a_2 + a_3 \\ -b_2 = a_3 \\ a_2 = b_3 \\ a_1 = 2b_1 - 2b_2 - b_3 \end{cases}$$

$b_1 = -4b_1 + 4b_2 + 2b_3 + 2b_3 - b_2 \Rightarrow b_1 = \frac{3b_2}{5} + \frac{4}{5}b_3 \qquad b = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix} \; a = \begin{pmatrix} -1 \\ 5 \\ -5 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 7 \\ 0 & 5 & 5 \\ 0 & -5 & 5 \end{pmatrix} \Rightarrow \frac{1}{50}\begin{pmatrix} 50 & 0 & 0 \\ -30 & 5 & 5 \\ -40 & -5 & 5 \end{pmatrix}^{-1} = \frac{1}{50}\begin{pmatrix} 50 & -30 & -40 \\ 0 & 5 & -5 \\ 0 & 5 & 5 \end{pmatrix} - \begin{pmatrix} 1 & -3/5 & -4/5 \\ 0 & 1/10 & -1/10 \\ 0 & 1/10 & 1/10 \end{pmatrix}$$

$$\left.\begin{aligned} &\left(1 \;\; -3/5 \;\; -4/5\right)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 - 4/5 = 1/5 \neq 0 \\ &\left(0 \;\; 1/10 \;\; -1/10\right) B = -1/10 \neq 0 \\ &\left(0 \;\; 1/10 \;\; 1/10\right) B = 1/10 \neq 0 \end{aligned}\right\} \text{eccitabili}$$

$(0 \;\; 1 \;\; 1)\, u_a = 0$

$(0 \;\; 1 \;\; 1)\, u_1 = 0$

$(0 \;\; 1 \;\; 1)\, u_2 = 10$

	ECC	OSS
λ_1	V	X
$\lambda_{2,3}$	V	V

$$W(s) = \frac{F(s)}{1+\underbrace{KF(s)}_{\bar F(s)}} \cdot \frac{K}{K} = \frac{1}{K} \cdot \frac{\bar F(s)}{1+\bar F(s)} = \frac{1}{K} \cdot \frac{\bar N / \bar D}{1+\bar N/\bar D} = \frac{1}{K} \cdot \frac{\bar N / \bar D}{\frac{\bar D + \bar N}{\bar D}} = \frac{1}{K} \cdot \frac{\bar N}{\bar D + \bar K} = \frac{N}{D+KN}$$

$1 + \bar F(s)$:

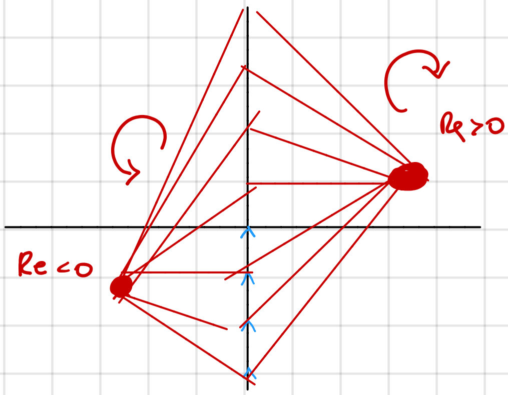
$$1 + \frac{\bar N(s)}{\bar D(s)} = \frac{\bar D(s) + \bar N(s)}{\bar D(s)} = \frac{d_{CH}(s)}{d_{AP}(s)}$$

$1 + \bar F(j\omega)$: studio ω che va da $-\infty$ a ∞

d_{AP} : $n - z_p$ radici reali negative

z_p radici reali positive

d_{CH} : n radici reali



$\Delta\varphi = \Delta\varphi(num) - \Delta\varphi(den.) = \Delta\varphi d_{CH} - \Delta\varphi d_{AP}$

ogni radice $Re < 0$: $\overset{\curvearrowleft}{\pi}$

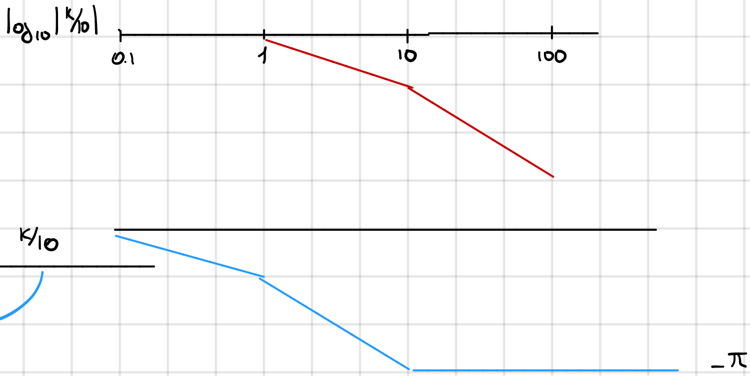
ogni radice $Re > 0$: $\overset{\curvearrowright}{\pi} = -\overset{\curvearrowleft}{\pi}$

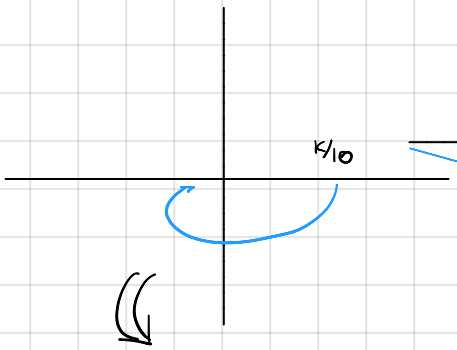
$$\Delta\varphi = n\overset{\curvearrowleft}{\pi} - \left[z_p \overset{\curvearrowright}{\pi} + (n - z_p)\overset{\curvearrowleft}{\pi}\right] = n\overset{\curvearrowleft}{\pi} - \left[(n - z_p)\overset{\curvearrowleft}{\pi}\right] - z_p 2 \overset{\curvearrowleft}{\pi}$$

$1 + \bar F(j\omega)$ intorno l'origine $\Leftrightarrow \bar F(j\omega)$ intorno $(-1, j_0) = KF(j\omega) = \frac{KN}{D}$

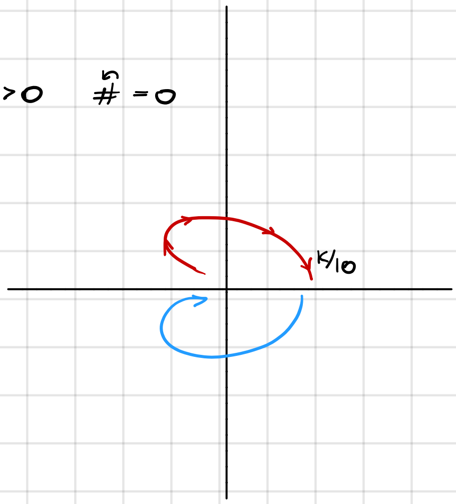
es. $\frac{K}{(s+1)(s+10)}$

$= \frac{K/10}{(1+s)(1+s/10)}$





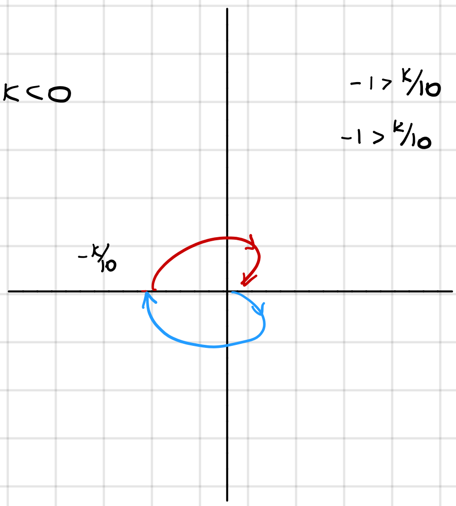
K > 0 $\quad \overset{\curvearrowleft}{\#} = 0$

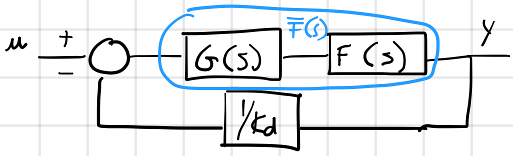


K < 0

$-1 > K/10 \quad \overset{\curvearrowleft}{\#} = -1$

$-1 > K/10 \quad \overset{\curvearrowleft}{\#} = 0$





$$G(s) = \frac{K_G}{s^r}\bar{G}(s)$$

$$\bar{G}(s) = \frac{1+\tau_1 s}{1+\tau_2 s} \qquad m>1 \qquad \frac{1+\tau s}{1+\frac{\tau}{m}s} \qquad \frac{1+\frac{\tau}{m}s}{1+\tau s}$$

Banda passante: modulo della risonanza

$Y(s) = W(s)u(s)$ = uscita reale

$Y_d(s) = K_d u(s)$ = uscita desiderata

funzione errore

$$Y_d - Y = K_d u(s) - W(s)u(s) = (K_d - W(s))u(s) = W_e(s)u(s)$$

$$W_e(s) = K_d - \frac{\bar{F}(s)}{1+\bar{F}(s)/K_d} = K_d - \frac{K_d\bar{F}(s)}{K_d+\bar{F}(s)} = \frac{K_d^2 + \cancel{K_d\bar{F}(s)} - \cancel{K_d\bar{F}(s)}}{K_d+\bar{F}(s)}$$

Se $u(s) = 1/s$

$u(t) = \delta_{-1}(t) \rightarrow Y_R(t) = C_0\delta_{-1}(t) = W(0)\delta_{-1}(t)$

$\Rightarrow u(t) = \delta_{-1}(t) \rightarrow e_R(t) = W_e(0)\delta_{-1}(t)$ $\quad$ $\bar{F}(s)$ ha poli $= \frac{\hat{F}(s)}{s^\ell}$

$$\frac{K_d^2}{K_d + \hat{F}/s^\ell} = \frac{K_d^2 s^\ell}{K_d s^\ell + \hat{F}} \qquad e_R(t) = 0$$

$$K_G R(s) = \bar{F}(s)$$

$$\frac{K_d^2}{K_d + G(s)} = \frac{K_d^2}{K_d + K_G\frac{\bar{G}}{s^r}}$$

$$\dot{x} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 4 & -5 \end{pmatrix} x + \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} x + u$$

1) Analisi Modale

2) Scomposizione di Kalman

$\lambda = 3, -1, -5$

$$\lambda = 3 \quad \begin{pmatrix} 0 & 0 & 0 \\ 2 & -4 & 0 \\ 1 & 4 & -8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{cases} 2a - 4b = 0 \\ a + 4b - 8c = 0 \end{cases} \quad \begin{cases} a = 2b \\ 6b = 8c \end{cases} \quad \begin{cases} a = 2b \\ b = \frac{4}{3} c \end{cases} \quad \begin{array}{c} u_1 \\ \begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix} \end{array}$$

$$\lambda = -1 \quad \begin{pmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{cases} 4a = 0 \rightarrow a = 0 \\ 2a = 0 \\ a + 4b - 4c = 0 \end{cases} \quad \begin{cases} a = 0 \end{cases} \quad \begin{array}{c} u_2 \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

$$\lambda_3 = \begin{pmatrix} 8 & 0 & 0 \\ 2 & 7 & 0 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{cases} a = 0 \\ b = 0 \\ c = 1 \end{cases} \quad \begin{array}{c} u_3 \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$$T^{-1} = \begin{pmatrix} 8 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \qquad T = \frac{1}{8} \begin{pmatrix} 1 & -4 & 1 \\ 0 & 8 & -8 \\ 0 & 0 & 8 \end{pmatrix}^T = \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 \\ -4 & 8 & 0 \\ 1 & -8 & 8 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 1/8 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/8 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{c} V_1^T \\ V_2^T \\ V_3^T \end{array}$$

$$V_1^T B = \begin{pmatrix} 1/8 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 1/2$$

$$V_2^T B = -1/2 \cdot 4 + 2 = 0$$

$$V_3^T B = 1/2 - 2 + 1 = 1/2 - 1 = -1/2$$

$$C_1 u_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ 3 \end{pmatrix} = 8 - 4 = 4$$

$$C_2 u_2 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -1$$

$$C_3 u_3 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

	ECC	OSS
λ_1	✓	✓
λ_2	✗	✓
λ_3	✓	✗

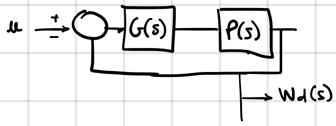
x_1 $R \cap I$

x_2 $R \cap O$

x_3 $IR \cap I$

x_4 $IR \cap O$

SINTESI DIRETTA



$$W_{desiderata}(s) = \frac{G(s)P(s)}{1+G(s)P(s)}$$

$$G(s)P(s) = [1+G(s)P(s)]\,W_d(s)$$

$$G(s)[P(s)-P(s)W_d(s)] - W_d(s)$$

$$G(s) - \frac{W_d(s)}{P(s)[1+W_d(s)]}$$

Il grado di $W_d(s)$ deve essere minore o uguale del denominatore

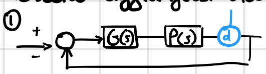
$$G(s) = \frac{\frac{N_d}{D_d}}{\frac{N_p}{D_p}\left[1-\frac{N_d}{D_d}\right]} = \frac{N_d}{D_d}\cdot\frac{D_p}{N_p}\cdot\frac{D_d}{D_d-N_d}$$

grado N_d + grado D_p $\leq$ grado D_d + grado N_p

grado D_p − grado N_p $\leq$ grado D_d − grado N_d

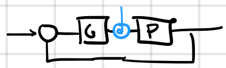
Possono aggiungersi dei disturbi

①



$$\frac{GP}{1+GP} \qquad \frac{1}{1+GP} = \frac{1}{1+\frac{N_g}{D_g}\cdot\frac{N_p}{D_p}} = \frac{D_gD_p}{D_gD_p+N_gN_p} =$$ funzione di disturbo

② $G(s)$ o $P(s)$ devono avere almeno un polo: se $P(s)$ non lo ho lo inserisco in $G(s)$



$$\frac{P}{1+GP} = \frac{N_p/D_p}{\frac{D_gD_p+N_gN_p}{D_gD_p}} = \frac{N_pD_g}{D_gD_p+N_pN_g}$$

③