Data Driven Integral Quadratic Constraints (IQCs)

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Abstract

This project presents a novel framework for designing a controller using a data-driven approach. The designed controller corresponds to the best characterization of the non-linearity part and guarantees the system's stability and safety constraints. The framework is formulated as a bi-level optimization problem. Within the upper level, the objective is to find sector-bound parameters such that these parameters satisfy all the observed input-output pairs of the non-linearity part with a less conservative bound. In the lower level, the goal is to design the controller parameter that ensures the stability and safety constraints of the system. Finally, The proposed approach is demonstrated numerically.

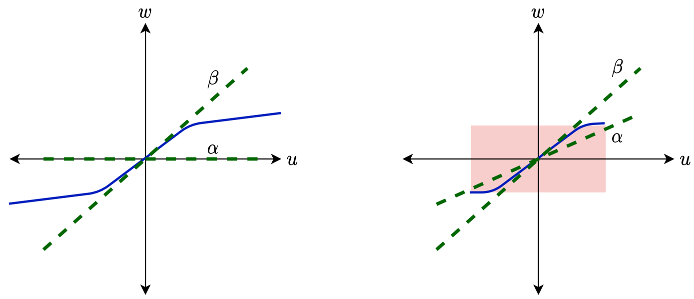
1 Introduction

Uncertain feedback systems with linear and nonlinear controller suffer from the lack of stability and safety certificates. The framework of stabilizing an uncertain feedback system and the stability analysis for systems with non-linear controllers is presented in [7]. In [5, 7], the uncertain plant is modeled as an interconnection of nominal plant and perturbation described by Integral Quadratic Constraints (IQCs).

(IQCs) have been widely used for modeling the uncertainty and bounding the non-linearity of dynamical systems [2]. Moreover, it gives a framework for robustness analysis and stability analysis [4]. One method is to use sector bound as quadratic constraints. The sector bound is given by the following

$$\begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \geq 0, \tag{1}$$

where $v, w \in \mathbb{R}$ are the input and the output of a system, and $\alpha, \beta \in \mathbb{R}$ are the lower and upper bounds, respectively. Fig 1-(a) illustrates a non-linear function bounded by the sector$[\alpha, \beta]$.



(a) (b)

Figure 1: Sector constraints of non-linear functions .

Typically, the values of α and β are set to be fixed to specific values before the interconnection of nominal and perturbation parts of the system. In this case, we assume all trajectories can be seen, and all of these trajectories are bounded by typical sector bounds. This project proposes the idea of data-driven (IQCs). It is described as the following; given data from the trajectories of the system under control, the goal is to design a controller using the observed data to ensure the stability and the safety of the system.

The motivating ideas behind proposing data-driven (IQCs) can be described twofold. First, we may not have access to all trajectories of the system. The second motivation is shown in Fig 1-(b). Suppose that the safe operating area of the system is the red region, where we will not see any input-output pairs outside this region, so that will lead us to a less conservative bound given by α and β.

In order to provide a general framework for solving the previously described problem. A bilevel optimization problem is proposed. In the upper level, the goal is to find the sector-bound parameters such that these parameters satisfy all the observed data. In the lower level, the goal is to find the controller parameters corresponding to α and β that guarantee the system's stability and safety with maximum Region Of Attraction (ROA).

2 Problem formulation

2.1 Modeling of the system

Consider a state space, discrete time, closed loop, non linear system G of order n, given by

$$x(k+1) = Ax(k) + Bu(k) + \phi(x(k), u(k)), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ models the linear time invariant system component, while ϕ captures the non-linearity and the uncertainty of the system G. Further, $B \in \mathbb{R}^{n\times n_u}$ represents the control input matrix. The state of the system and the system control input are $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^{n_u}$ respectively. Fig. 2 shows the system G, where $K \in \mathbb{R}^{n_u\times n}$ is the feedback gain matrix.

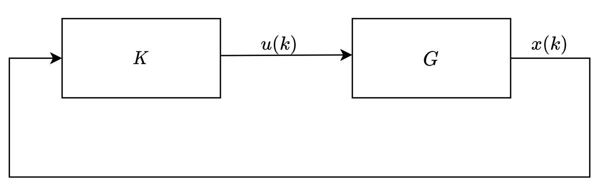


Figure 2: A closed loop, discrete time, nonlinear system G.

The dynamical system G can be represented as a combination of linear and non-linear part. The linear part of the system is given by

$$x(k+1) = Ax(k) + Bu(k) + \bar{B}w(k), \tag{3}$$

where $\bar{B} \in \mathbb{R}^{n\times 1}$ is the non-linear part input matrix.

$$v(k) = \bar{C}x(k) + \bar{D}u(k), \tag{4}$$

where $\bar{C} \in \mathbb{R}^{1\times n}$, and $\bar{D} \in \mathbb{R}^{1\times n_u}$. Also, a linear controller is consider, such that

$$u(k) = Kx(k). \tag{5}$$

The non-linear part of the system is given by

$$w(k) = \phi(v(k)). \tag{6}$$

Fig 3 shows this combination of linear and nonlinear part of the system G.

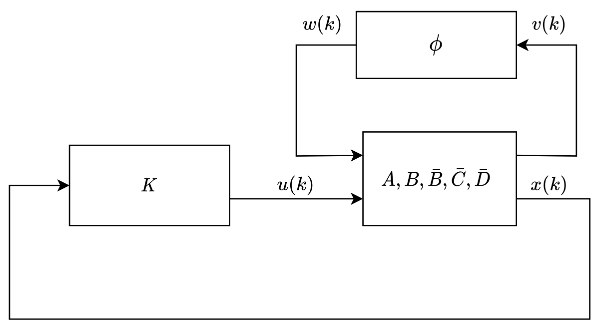


Figure 3: A closed loop, discrete time, nonlinear system G.

2.2 Quadratic Constraints for the non linear part ϕ

In order to bound the non linearity part, we need a quadratic constraints. A typical quadratic constraints is the sector bound. For single input, single output pair (v, w) of the nonlinear part ϕ, the quadratic constraints is given by the following lemma

Lemma 2.1 *Let $\alpha \leq \beta$, and $\phi : \mathbb{R} \to \mathbb{R}$ lies in the sector $[\alpha, \beta]$ if*

$$(\phi(v) - \alpha v)(\beta v - \phi(v)) \geq 0, \forall v \in \mathbb{R}.$$

$$\begin{bmatrix} v \\ w \end{bmatrix} \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \geq 0. \tag{7}$$

The significant of IQC is to provide a framework for robustness and stability analysis.

2.3 Loop Transformation

In order to derive a convex conditions with respect to the linear controller, a loop transformation is implemented as shown in Fig 4.

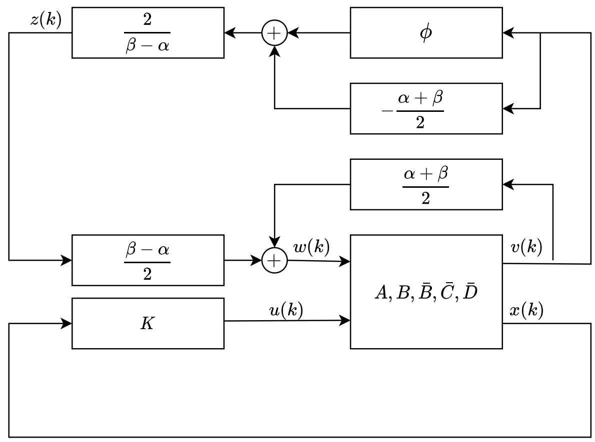


Figure 4: A closed loop, discrete time, nonlinear system G.

The input to the linear dynamical system $w(k)$ is given by

$$w(k) = \frac{\alpha+\beta}{2}v(k) + \frac{\beta-\alpha}{2}z(k). \tag{8}$$

Substituting 8 and 4in the dynamical system represented by 3 and 7 yields

$$x(k+1) = (A + \bar{B}\frac{\alpha+\beta}{2}\bar{C})x(k) + (B + \bar{B}\frac{\alpha+\beta}{2}\bar{D})u(k) + (\bar{B}\frac{\beta-\alpha}{2})z(k), \tag{9}$$

$$v(k) = \bar{C}x(k) + \bar{D}u(k). \tag{10}$$

$$\begin{bmatrix} v(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \frac{(\beta-\alpha)^2}{2} & 0 \\ 0 & -\frac{(\beta-\alpha)^2}{2} \end{bmatrix} \begin{bmatrix} v(k) \\ z(k) \end{bmatrix} \geq 0, \tag{11}$$

Moreover, using 5, we can write 9, 10, and 11 as

$$x(k+1) = \underbrace{[(A + \bar{B}\frac{\alpha+\beta}{2}\bar{C}) + (B + \bar{B}\frac{\alpha+\beta}{2}\bar{D})(K)]}_{\mathcal{A}} x(k) + \underbrace{[\bar{B}\frac{\beta-\alpha}{2}]}_{\mathcal{B}} z(k), \tag{12}$$

$$v(k) = \underbrace{[\bar{C} + \bar{D}K]}_{\mathcal{C}} x(k). \tag{13}$$

$$\begin{bmatrix} v(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \frac{(\beta-\alpha)^2}{2} & 0 \\ 0 & -\frac{(\beta-\alpha)^2}{2} \end{bmatrix} \begin{bmatrix} v(k) \\ z(k) \end{bmatrix} \geq 0, \tag{14}$$

Dividing 15 by $\frac{(\beta-\alpha)^2}{2} > 0$, yields

$$\begin{bmatrix} v(k) \\ z(k) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(k) \\ z(k) \end{bmatrix} \geq 0, \tag{15}$$

Fig 5 shows the block diagram representation of 12, 13, and 15

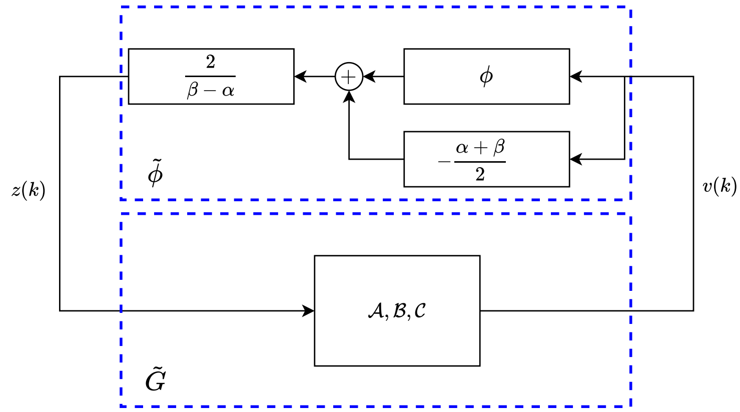


Figure 5: A closed loop, discrete time, nonlinear system G.

2.4 Stability and Safety conditions

2.4.1 Convex stability condition

The stability condition, from for close loop systems is given by

$$V(x(k+1)) - V(x(k)) < 0, \tag{16}$$

where,

$$V(x(k)) = x(k)^T P x(k). \tag{17}$$

where V is the Lyapunov function; a scalar value function, used to prove the stability of the system. Then, for the close loop system, given by 12, and 13, the stability condition is

$$\begin{bmatrix} x(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B} \\ \mathcal{B}^T P \mathcal{A} & \mathcal{B}^T P \mathcal{B} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \prec 0, \tag{18}$$

Moreover, adding the positive inequality given by 15, to 19 the stability condition is still satisfied as the following

$$\begin{bmatrix} x(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B} \\ \mathcal{B}^T P \mathcal{A} & \mathcal{B}^T P \mathcal{B} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} v(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(k) \\ z(k) \end{bmatrix} \prec 0, \tag{19}$$

Then, using 13, the stability condition is given by the following

$$\begin{bmatrix} x(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B} \\ \mathcal{B}^T P \mathcal{A} & \mathcal{B}^T P \mathcal{B} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} x(k) \\ z(k) \end{bmatrix}^T \begin{bmatrix} \mathcal{C} & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} \prec 0, \tag{20}$$

$$\begin{bmatrix} \mathcal{A}^T P \mathcal{A} - P & \mathcal{A}^T P \mathcal{B} \\ \mathcal{B}^T P \mathcal{A} & \mathcal{B}^T P \mathcal{B} \end{bmatrix} + \begin{bmatrix} \mathcal{C} & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & 1 \end{bmatrix} \prec 0, \tag{21}$$

Rearrange 21 yields the following inequality

$$\begin{bmatrix} \mathcal{A}^T & \mathcal{C}^T \\ \mathcal{B}^T & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \prec 0. \tag{22}$$

Applying the Schur complement yields

$$\begin{bmatrix} P & 0 & \mathcal{A}^T & \mathcal{C}^T \\ 0 & 1 & \mathcal{B} & 0 \\ \mathcal{A} & \mathcal{B} & P^{-1} & 0 \\ \mathcal{C} & 0 & 0 & 1 \end{bmatrix} \succ 0. \tag{23}$$

In order to obtain a convex inequality, multiply 22 from the left and right by diag$[P^{-1}, I, I, I]$

$$\begin{bmatrix} P^{-1} & 0 & P^{-1}\mathcal{A}^T & P^{-1}\mathcal{C}^T \\ 0 & 1 & \mathcal{B} & 0 \\ \mathcal{A}P^{-1} & \mathcal{B} & P^{-1} & 0 \\ \mathcal{C}P^{-1} & 0 & 0 & 1 \end{bmatrix} \succ 0. \tag{24}$$

Let $P^{-1} = Q \succ 0$, then 24 is given by

$$\begin{bmatrix} Q & 0 & Q\mathcal{A}^T & Q\mathcal{C}^T \\ 0 & 1 & \mathcal{B} & 0 \\ \mathcal{A}Q & \mathcal{B} & Q & 0 \\ \mathcal{C}Q & 0 & 0 & 1 \end{bmatrix} \succ 0. \tag{25}$$

Also, assume and $KQ = L$, and recover the value of $\mathcal{A}$, and $\mathcal{C}$ from 12 and 13, respectively. Then 25 is given by

$$\begin{bmatrix} Q & 0 & Q\mathcal{A}_1^T + L^T\mathcal{A}_2^T & Q\bar{C}^T + L^T\bar{D}^T \\ 0 & 1 & \mathcal{B} & 0 \\ \mathcal{A}_1 Q + \mathcal{A}_2 L & \mathcal{B} & Q & 0 \\ \bar{C}Q + \bar{D}L & 0 & 0 & 1 \end{bmatrix} \succ 0, \tag{26}$$

where $\mathcal{A}_1 = (A + \bar{B}\frac{\alpha+\beta}{2}\bar{C})$, and $\mathcal{A}_2 = (B + \bar{B}\frac{\alpha+\beta}{2}\bar{D})$. Note that the stability condition in 26 is convex in the decision variables Q, and L. Variable K can be recovered using the computed Q, and L, through the following equation

$$K = LQ^{-1} \tag{27}$$

2.4.2 Safety condition

$$\mathcal{E}(Q^{-1}) = \left\{x \in \mathbb{R}^n : x^T Q^{-1} x \leq 1\right\} \tag{28}$$

We enforce a safety condition on the state x. We constrained the state x to a set $\mathcal{X}$. The set $\mathcal{X}$ is defined to be a polytope symmetric around the origin as the following

$$x \in \mathcal{X} = \{x \in \mathbb{R}^n : -h \leq Hx \leq h\}, \tag{29}$$

where $H \in \mathbb{R}^{n_{\mathcal{X}} \times n}$, and $h \in \mathbb{R}^{n_{\mathcal{X}}}$. Let H_i be the i^{th} row of the matrix H, then

$$x \in \mathcal{X} = \{x \in \mathbb{R}^n : -h_i \leq H_i x \leq h_i, i = 1, \ldots, n_{\mathcal{X}}\} \tag{30}$$

Using Lemma(1) in [3], the condition that the set $\mathcal{E}(Q^{-1})$ is contained in the space $\mathcal{X}$ is given by

$$H_i^T Q H_i \leq h_i^2, i = 1, \ldots, n_{\mathcal{X}}. \tag{31}$$

Moreover the volume of $\mathcal{E}(Q^{-1})$ is proportional to the $det(Q)$. While $det(Q)$ is a non convex function, $log\ det(Q)$ is a concave function.

2.5 Formulate the problem as Bi-level optimization problem, "Hard Constraint"

The problem of finding less conservative bound using data driven (IQC) at the same time guarantee the stability and safety of the system is a bi-level problem.

2.5.1 Upper level

The upper level is developed to find the tightest sector bound represented by $[\alpha, \beta]$. Moreover, the (IQCs) should satisfied all the observed input and output pairs. Let the observed data set given by

$$\mathcal{D} = \{(v_i, w_i), \ldots, (v_k, w_k)\}. \tag{32}$$

Then, the upper level constrained optimization problem is given by

$$
\begin{aligned}
&\underset{\alpha,\beta}{\text{minimize}} \quad \|\beta-\alpha\|_2 \\
&\text{subject to} \quad \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 & -v_i^2 & w_i v_i \\ -v_i^2 & 0 & w_i v_i \\ v_i w_i & v_i w_i & -2w_i^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} \geq 0,\ \forall (v_i, w_i) \in \mathcal{D}.
\end{aligned} \tag{33}
$$

In order to simplify this optimization problem. The constraint of the optimization problem 33 can be seen as intersection between two plane with some modifications. The steps for this conversion are shown in Fig 6, 7, and 8, respectively. Then, the optimization problem 33 can be replace by

$$
\begin{aligned}
&\underset{\alpha,\beta}{\text{minimize}} \quad \|\beta-\alpha\|_2 \\
&\text{subject to} \quad \begin{bmatrix} -\alpha & 1 \\ \beta & -1 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} \geq 0,\ \forall (v_i, w_i) \in \mathcal{D}.
\end{aligned} \tag{34}
$$

It is good to note this modification is satisfying the original data. It is just a new representation to deal with simple and convex optimization problem.

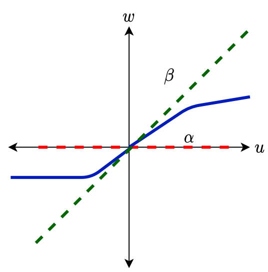


Figure 6: A non-decreasing monotone function bounded by sector bound $[\alpha, \beta]$

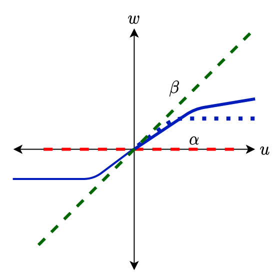


Figure 7: Multiplying the part of the data set that appears in the third quarter by $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

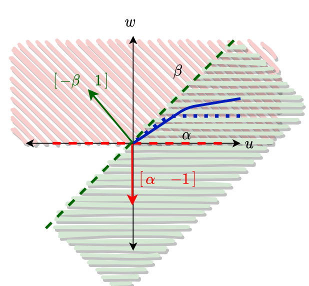


Figure 8: Obtaining the convex cone to be the upper level constrints set

2.5.2 Lower level

Given the bound parameters α and β, the goal of the lower level is to find the controller parameters that guarantee the stability and the safety of the system with maximum Region Of Attraction (ROA). The lower level is given by

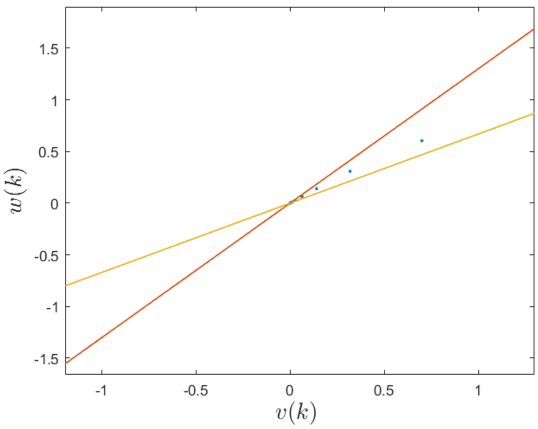


Figure 9: The tightest Sector bound represented by $[\alpha, \beta]$ that satisfied the observed data.

$$
\begin{aligned}
\underset{Q,L}{\text{minimize}} \quad & -\log(\det(Q)) \\
\text{subject to} \quad & \begin{bmatrix} Q & 0 & Q\mathcal{A}_1^T + L^T\mathcal{A}_2^T & Q\bar{C}^T + L^T\bar{D}^T \\ 0 & 1 & \mathcal{B} & 0 \\ \mathcal{A}_1 Q + \mathcal{A}_2 L & \mathcal{B} & Q & 0 \\ \bar{C}Q + \bar{D}L & 0 & 0 & 1 \end{bmatrix} \succ 0, \\
& H_i^T Q H_i \leq h_i^2, i = 1, \ldots, n_{\mathcal{X}},
\end{aligned} \tag{35}
$$

where $\mathcal{A}_1 = (A + \bar{B}\frac{\alpha+\beta}{2}\bar{C})$, $\mathcal{A}_2 = (B + \bar{B}\frac{\alpha+\beta}{2}\bar{D})$, $\mathcal{B} = \bar{B}\frac{\beta-\alpha}{2}$, and $Q \succ 0$. Then, the control gain matrix K can be found by $K = LQ^{-1}$.

2.6 Solving the formulated Bilevel optimization problem

Bilevel optimization problems can be solved by transforming the bilevel optimization problem into a single-level optimization problem [1, 6]. The convex lower level problem 35 is replaced by its Karush–Kuhn–Tucker(KKT) conditions. Using the KKT conditions, the lower level appears as Lagrangian and complementarity constraints of the upper-level optimization problem.

3 Illustrative simple example

Consider the following discrete-time, nonlinear, unstable dynamical system

$$x(k+1) = 1.7x(k) + 0.3u(k) + 0.3w(k), \tag{36}$$

also, $\bar{C} = 1$ and $\bar{D} = 0$, so that $v(k) = x(k)$. The nonlinear part of the system is given by

$$w(k) = \tanh(v(k)). \tag{37}$$

The system is simulated for 20-time steps. The input and the output pairs (v_i, w_i) are observed. Then, using 34 and 35, the bilevel optimization problem is formulated. Next, A single-level optimization problem is formulated by replacing the lower-level optimization problem with its (KKT) condition. The single-level optimization problem is solved, and the optimization decision variables α, β, Q, L, and the Lagrangian parameters are obtained. The value of α, β that satisfied the observed data is shown in Fig 9. Using 31, the linear system controller is calculated. Next, the closed-loop system is simulated for 20-time steps. The system response is shown in Fig 10.

4 Conclusions

This project introduced a novel framework to design a controller that corresponds to the best characterization of the system's non-linearity part while ensuring the system's stability and safety constraints.

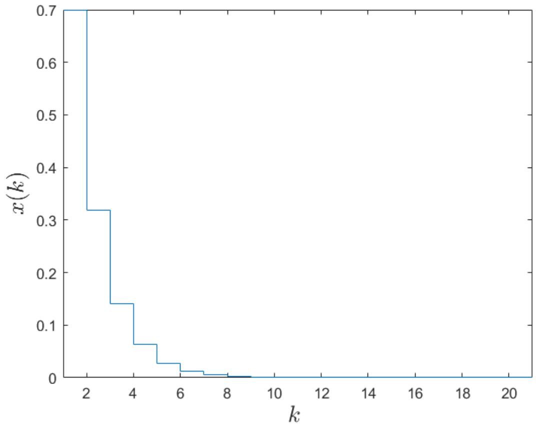


Figure 10: The closed loop system response, under the linear controller K, that satisfied the stability of the system.

The problem of finding a less conservative bound using data-driven (IQC) that guarantees the stability and safety of the system are formulated as a bi-level optimization problem. In the upper level, the goal is to find the sector-bound parameters such that these parameters satisfy all the observed data. The lower level is to find the controller parameters that guarantee the system's stability and safety with maximum Region Of Attraction (ROA). Finally, The proposed approach is demonstrated numerically.

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