Supplementary Material for Stagewise Newton Method for Dynamic Game Control with Imperfect State Observation

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1 Problem statement

Given a sequence of measurements $y_{1:t}$, a sequence of control inputs, $u_{0:t-1}$ and a prior on the initial state $\hat{x}_0$, we study the following minimization-maximization problem:

$$\min_{u_{t:T-1}} \max_{w_{0:T}} \max_{\gamma_{1:t}} \sum_{j=0}^{T-1} \ell_j(x_j, u_j) + \ell_T(x_T) \tag{1}$$
$$- \frac{1}{2\mu} \left(\omega_0^T P^{-1} \omega_0 + \sum_{j=1}^{t} \gamma_j^T R_j^{-1} \gamma_j + \sum_{j=1}^{T} w_j^T Q_j^{-1} w_j \right)$$

subject to the constraints:

$$x_0 = \hat{x}_0 + w_0, \tag{2a}$$
$$x_{j+1} = f_j(x_j, u_j) + w_{j+1}, \qquad 0 \leq j < T, \tag{2b}$$
$$y_j = h_j(x_j) + \gamma_j, \qquad 1 \leq j \leq t. \tag{2c}$$

where $\mu > 0$. x_j is the state, ω_j the process disturbance, γ_j the measurement disturbance, T the time horizon, t the current time. f_j, the transition model, h_j, measurement model and ℓ_j, the controller's cost are assumed to be $\mathcal{C}^2$. R_j the measurement uncertainty, Q_j the process uncertainty and P the initial state uncertainty are positive semi definite matrices. Instead of searching for a global solution, we search for a stationary point of the following unconstrained cost.

$$\begin{aligned} J(x_{0:T}, u_{t:T-1}) = \sum_{j=0}^{T-1} \ell_j(x_j, u_j) + \ell_T(x_T) &- \frac{1}{2\mu}(x_0 - \hat{x}_0)^T P^{-1}(x_0 - \hat{x}_0) \\ &- \frac{1}{2\mu} \sum_{j=1}^{t} (y_j - h_j(x_j))^T R_j^{-1} (y_j - h_j(x_j)) \\ &- \frac{1}{2\mu} \sum_{j=0}^{T-1} (x_{j+1} - f_j(x_j, u_j))^T Q_{j+1}^{-1} (x_{j+1} - f_j(x_j, u_j)) \end{aligned}$$

Notations: The gradient of a function f with respect to a vector v is denoted by f^v, similarly for second order derivatives w.r.t vectors u, v is denoted as f^{uv}. All functions are assumed to be $\mathcal{C}^2$. If $(v_i)_{i\in\mathbb{N}}$ is a sequence of vectors, then $v_{k:t}$ denotes the batch vector of all v_j for $k \leq j \leq t$. $\mathbf{1}_{x\in A}$ is the indicator function which equals 1 if $x \in A$ and 0 otherwise. I_n denotes the identity matrix of size n by n.

2 Characterization of the Newton step

Let's consider a step

$$p = \begin{bmatrix} p_{x_{0:T}} \\ p_{u_{t:T-1}} \end{bmatrix}, \tag{3}$$

we know that the Newton step p satisfies:

$$Hp = -\nabla J, \tag{4}$$

where

$$H = \begin{bmatrix} \frac{\partial^2 J}{\partial x_{0:T} \partial x_{0:T}} & \frac{\partial^2 J}{\partial x_{0:T} \partial u_{t:T-1}} \\ \frac{\partial^2 J}{\partial u_{t:T-1} \partial x_{0:T}} & \frac{\partial^2 J}{\partial u_{t:T-1} \partial u_{t:T-1}} \end{bmatrix} \tag{5}$$

and

$$\nabla J = \begin{bmatrix} \frac{\partial J}{\partial x_{0:T}} \\ \frac{\partial J}{\partial u_{t:T-1}} \end{bmatrix} \tag{6}$$

And we use the following notation for the Hessian:

$$\frac{\partial^2 J}{\partial x_{0:T} \partial x_{0:T}} = \begin{bmatrix} \frac{\partial^2 J}{\partial x_0^2} & \frac{\partial^2 J}{\partial x_0\, \partial x_1} & \cdots & \frac{\partial^2 J}{\partial x_0\, \partial x_T} \\ \frac{\partial^2 J}{\partial x_1\, \partial x_0} & \frac{\partial^2 T}{\partial x_1^2} & \cdots & \frac{\partial^2 J}{\partial x_1\, \partial x_T} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 J}{\partial x_T\, \partial x_0} & \frac{\partial^2 J}{\partial x_T\, \partial x_1} & \cdots & \frac{\partial^2 J}{\partial x_T^2} \end{bmatrix} \tag{7}$$

Note that: $\frac{\partial^2 J}{\partial x_i \partial x_j} \in \mathbb{R}^{n_x \times n_x}$ and $\frac{\partial^2 J}{\partial x_i \partial u_j} \in \mathbb{R}^{n_x \times n_u}$. To simplify the derivations, we define the gaps:

$$\gamma_k \coloneqq y_k - h_k(x_k) \tag{8}$$
$$w_{k+1} \coloneqq x_{k+1} - f_k(x_k, u_k) \tag{9}$$
$$w_0 \coloneqq x_0 - \hat{x}_0 \tag{10}$$

In the rest of this section, we show how to derive the Newton step sequentially, namely without inverting the Hessian matrix. To do so, let's retrieve the analytical expression of the gradient and Hessian of J. We assume that $t < T$, otherwise, there is no control input. However, we note that if $t = T$, the past stress recursion could be derived similarly. First, we derive the gradient with respect to every state variable and every control inputs:

$$\begin{aligned}
&\frac{\partial J}{\partial x_0}(x_{0:T}, u_{t:T-1}) = && \ell_0^x - \mu^{-1}P^{-1}w_0 + \mu^{-1}{f_0^x}^T Q_1^{-1} w_1 \\
\forall 1 \leq k \leq t-1,\ &\frac{\partial J}{\partial x_k}(x_{0:T}, u_{t:T-1}) = && \ell_k^x - \mu^{-1}Q_k^{-1}w_k + \mu^{-1}{f_k^x}^T Q_{k+1}^{-1} w_{k+1} + \mu^{-1}{h_k^x}^T R_k^{-1}\gamma_k \\
&\frac{\partial J}{\partial x_t}(x_{0:T}, u_{t:T-1}) = && \ell_t^x - \mu^{-1}Q_t^{-1}w_t + \mu^{-1}{f_t^x}^T Q_{t+1}^{-1} w_{t+1} + \mathbf{1}_{t\geq 1}\mu^{-1}{h_t^x}^T R_t^{-1}\gamma_t \\
\forall t+1 \leq k \leq T-1,\ &\frac{\partial J}{\partial x_k}(x_{0:T}, u_{t:T-1}) = && \ell_k^x - \mu^{-1}Q_k^{-1}w_k + \mu^{-1}{f_k^x}^T Q_{k+1}^{-1} w_{k+1} \\
&\frac{\partial J}{\partial x_T}(x_{0:T}, u_{t:T-1}) = && \ell_T^x - \mu^{-1}Q_T^{-1}w_T
\end{aligned}$$

$$\forall t \leq k \leq T-1,\ \frac{\partial J}{\partial u_k}(u_{t:T}, x_{0:T}) = \ell_k^u + \mu^{-1}{f_k^u}^T Q_{k+1}^{-1} w_{k+1}$$

Next, we derive each term of the Hessian. All the terms that are not equal to zero are of the form:

$$\frac{\partial^2 J}{\partial x_k \partial x_k} = \begin{cases} \ell_0^{xx} - \mu^{-1}P^{-1} - \mu^{-1}{f_0^x}^T Q_1^{-1} f_0^x + \mu^{-1}{f_0^{xx}}^T Q_1^{-1} w_1 & k = 0 \\ \ell_k^{xx} - \mu^{-1}Q_k^{-1} - \mu^{-1}{f_k^x}^T Q_{k+1}^{-1} f_k^x + \mu^{-1}{f_k^{xx}}^T Q_{k+1}^{-1} w_{k+1} - \mu^{-1}{h_k^x}^T R_k^{-1} h_k^x + \mu^{-1}{h_k^{xx}}^T R_k^{-1}\gamma_k & 1 \leq k \leq t \\ \ell_k^{xx} - \mu^{-1}Q_k^{-1} - \mu^{-1}{f_k^x}^T Q_{k+1}^{-1} f_k^x + \mu^{-1}{f_k^{xx}}^T Q_{k+1}^{-1} w_{k+1} & t < k < T \\ \ell_T^{xx} - \mu^{-1}Q_T^{-1} & k = T \end{cases}$$

$$\frac{\partial^2 J}{\partial x_{k+1}\partial x_k} = \mu^{-1}Q_{k+1}^{-1} f_k^x \tag{11}$$

$$\frac{\partial^2 J}{\partial x_{k-1}\partial x_k} = \mu^{-1}{f_{k-1}^x}^T Q_k^{-1} \tag{12}$$

$$\frac{\partial^2 J}{\partial u_{k-1}\partial x_k} = \mu^{-1}{f_{k-1}^u}^T Q_k^{-1} \tag{13}$$

$$\frac{\partial^2 J}{\partial u_k \partial x_k} = \ell_k^{ux} - \mu^{-1}{f_k^u}^T Q_{k+1}^{-1} f_k^x + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{ux} \tag{14}$$

$$\frac{\partial^2 J}{\partial x_k \partial u_k} = \ell_k^{xu} - \mu^{-1}{f_k^x}^T Q_{k+1}^{-1} f_k^u + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{xu} \tag{15}$$

$$\frac{\partial^2 J}{\partial x_{k+1}\partial u_k} = \mu^{-1}Q_{k+1}^{-1} f_k^u \tag{16}$$

$$\frac{\partial^2 J}{\partial u_k \partial u_k} = \ell_k^{uu} - \mu^{-1}{f_k^u}^T Q_{k+1}^{-1} f_k^u + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{uu} \tag{17}$$

where f_k^{ux} is a tensor and $w_{k+1}^T Q_{k+1}^{-1} f_k^{ux}$ is a matrix. More precisely,

$$(f_k^{ux})_{i,j,l} = \frac{\partial^2 (f_k)_l}{\partial u_i \partial x_k} \tag{18}$$

where $(f_k)_l$ is the l component of f_k. And, the product is defined as follows:

$$({w_{k+1}}^T Q_{k+1}^{-1} f_k^{ux})_{i,j} = \sum_{l=1}^{n} (f_k^{ux})_{i,j,l} (Q_{k+1}^{-1} w_{k+1}^i)_l \tag{19}$$

To simplify, let's denote for all $k < T$:

$$\begin{aligned}
\bar{\ell}_k^{xx} &= \ell_k^{xx} + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{xx} + \mu^{-1}\mathbf{1}_{k\leq t}{\gamma_k}^T R_k^{-1} h_k^{xx} \\
\bar{\ell}_k^{xu} &= {\bar{\ell}_k^{ux}}^T = \ell_k^{xu} + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{xu} \\
\bar{\ell}_k^{uu} &= \ell_k^{uu} + \mu^{-1}{w_{k+1}}^T Q_{k+1}^{-1} f_k^{uu}
\end{aligned} \tag{20}$$

Therefore, Equation (5) is equivalent to:

- If $k = 0$ and if $t \geq 1$:

$$\frac{\partial^2 J}{\partial x_0 \partial x_0} p_{x_0} + \frac{\partial^2 J}{\partial x_0 \partial x_1} p_{x_1} = -\frac{\partial J}{\partial x_0} \tag{21}$$

- $\forall\, k = 1, \ldots, t-1$:

$$\frac{\partial^2 J}{\partial x_k \partial x_{k-1}} p_{x_{k-1}} + \frac{\partial^2 J}{\partial x_k \partial x_k} p_{x_k} + \frac{\partial^2 J}{\partial x_k \partial x_{k+1}} p_{x_{k+1}} = -\frac{\partial J}{\partial x_k} \tag{22}$$

- $k = t$:

$$\frac{\partial^2 J}{\partial x_t \partial x_{t-1}} p_{x_{t-1}} \mathbf{1}_{t \geq 1} + \frac{\partial^2 J}{\partial x_t \partial x_t} p_{x_t} + \frac{\partial^2 J}{\partial x_t \partial x_{t+1}} p_{x_{t+1}} + \frac{\partial^2 J}{\partial x_t \partial u_t} p_{u_t} = -\frac{\partial J}{\partial x_t} \tag{23}$$

- $\forall\, k = t+1, \ldots, T-1$:

$$\frac{\partial^2 J}{\partial x_k \partial x_{k-1}} p_{x_{k-1}} + \frac{\partial^2 J}{\partial x_k \partial x_k} p_{x_k} + \frac{\partial^2 J}{\partial x_k \partial x_{k+1}} p_{x_{k+1}} + \frac{\partial^2 J}{\partial x_k \partial u_{k-1}} p_{u_{k-1}} + \frac{\partial^2 J}{\partial x_k \partial u_k} p_{u_k} = -\frac{\partial J}{\partial x_k} \tag{24}$$

- $k = T$:

$$\frac{\partial^2 J}{\partial x_T \partial x_{T-1}} p_{x_{T-1}} + \frac{\partial^2 J}{\partial x_T \partial x_T} p_{x_T} + \frac{\partial^2 J}{\partial x_T \partial u_{T-1}} p_{u_{T-1}} = -\frac{\partial J}{\partial x_T} \tag{25}$$

- $\forall\, k = t, \ldots, T-1$:

$$\frac{\partial^2 J}{\partial u_k \partial x_k} p_{x_k} + \frac{\partial^2 J}{\partial u_k \partial x_{k+1}} p_{x_{k+1}} + \frac{\partial^2 J}{\partial u_k \partial u_k} p_{u_k} = -\frac{\partial J}{\partial u_k} \tag{26}$$

3 Future stress

In this section, we derive the recursion for the future stress:

- From (24), $\forall\, k = t+1, \ldots, T-1$:

$$\begin{aligned}
&\mu^{-1} Q_k^{-1} f_{k-1}^x p_{x_{k-1}} \\
&+ \left(\bar{\ell}_k^{xx} - \mu^{-1} Q_k^{-1} - \mu^{-1} f_k^{x\,T} Q_{k+1}^{-1} f_k^x \right) p_{x_k} \\
&+ \mu^{-1} f_k^{x\,T} Q_{k+1}^{-1} p_{x_{k+1}} \\
&+ \mu^{-1} Q_k^{-1} f_{k-1}^u p_{u_{k-1}} \\
&+ \left(\bar{\ell}_k^{xu} - \mu^{-1} f_k^{x\,T} Q_{k+1}^{-1} f_k^u \right) p_{u_k} \\
&+ \ell_k^x - \mu^{-1} Q_k^{-1} w_k + \mu^{-1} f_k^{x\,T} Q_{k+1}^{-1} w_{k+1} = 0
\end{aligned} \tag{27}$$

- From (25), for $k = T$:

$$\mu^{-1} Q_T^{-1} f_{T-1}^x p_{x_{T-1}} + \left(\ell_T^{xx} - \mu^{-1} Q_T^{-1} \right) p_{x_T} + \mu^{-1} Q_T^{-1} f_{T-1}^u p_{u_{T-1}} + \ell_T^x - \mu^{-1} Q_T^{-1} w_T = 0 \tag{28}$$

- From (26), $\forall\, k = t, \ldots, T-1$:

$$\begin{aligned}
&\left(\bar{\ell}_k^{ux} - \mu^{-1} f_k^{u\,T} Q_{k+1}^{-1} f_k^x \right) p_{x_k} + \mu^{-1} f_k^{u\,T} Q_{k+1}^{-1} p_{x_{k+1}} \\
&+ \left(\bar{\ell}_k^{uu} - \mu^{-1} f_k^{u\,T} Q_{k+1}^{-1} f_k^u \right) p_{u_k} + \ell_k^u + \mu^{-1} f_k^{u\,T} Q_{k+1}^{-1} w_{k+1} = 0
\end{aligned} \tag{29}$$

We define

$$\forall\, k = t, \dots, T-1 \quad \lambda_{k+1} \coloneqq \mu^{-1} Q_{k+1}^{-1} \left(p_{x_{k+1}} - f_k^x p_{x_k} - f_k^u p_{u_k} + w_{k+1} \right) \tag{30}$$

and find that (27), (29) and (28) can be written:

$$\forall\, k = t+1 \dots T-1 \;\; \bar{\ell}_k^{xx} p_{x_k} + \bar{\ell}_k^{xu} p_{u_k} + \ell_k^x + f_k^{x\,T} \lambda_{k+1} = \lambda_k \tag{31}$$

$$\forall\, k = t \dots T-1 \;\; \bar{\ell}_k^{ux} p_{x_k} + \bar{\ell}_k^{uu} p_{u_k} + \ell_k^u + f_k^{u\,T} \lambda_{k+1} = 0 \tag{32}$$

$$\ell_T^{xx} p_{x_T} + \ell_T^x = \lambda_T \tag{33}$$

Proposition 1.

$$\begin{aligned} k = t, \dots, T-1, \quad V_k p_{x_k} + v_k &= \bar{\ell}_k^{xx} p_{x_k} + \bar{\ell}_k^{xu} p_{u_k} + \ell_k^x + f_k^{x\,T} \lambda_{k+1} \\ V_T p_{x_T} + v_T &= \lambda_T \end{aligned} \tag{34}$$

where V_k and v_k are solution of the backward recursion:

$$\begin{aligned} \Gamma_{k+1} &= I - \mu V_{k+1} Q_{k+1} \\ Q_{uu} &= \bar{\ell}_k^{uu} + f_k^{u\,T} \Gamma_{k+1}^{-1} V_{k+1} f_t^u \\ Q_{ux} &= \bar{\ell}_k^{ux} + f_k^{u\,T} \Gamma_{k+1}^{-1} V_{k+1} f_k^x \\ Q_u &= \ell_k^u + f_k^{u\,T} \Gamma_{k+1}^{-1} \left(v_{k+1} - V_{k+1} w_{k+1} \right) \\ G_k &= -Q_{uu}^{-1} Q_{ux} \\ g_k &= -Q_{uu}^{-1} Q_u \\ V_k &= \bar{\ell}_k^{xx} + f_k^{x\,T} \Gamma_{k+1}^{-1} V_{k+1} f_k^x + Q_{ux}^T G_k \\ v_k &= \ell_k^x + f_k^{x\,T} \Gamma_{k+1}^{-1} \left(v_{k+1} - V_{k+1} w_{k+1} \right) + Q_{ux}^T g_k \end{aligned} \tag{35}$$

with the terminal condition:

$$\begin{aligned} V_T &= \ell_T^{xx} \\ v_T &= \ell_T^x \end{aligned} \tag{36}$$

Furthermore,

$$\begin{aligned} p_{x_{k+1}} &= \left(I - \mu Q_{k+1} V_{k+1} \right)^{-1} \left(f_k^x p_{x_k} + f_k^u p_{u_k} + \mu Q_{k+1} v_{k+1} - w_{k+1} \right) \\ p_{u_k} &= G_k p_{x_k} + g_k \end{aligned} \tag{37}$$

Proof. Clearly:

$$\begin{aligned} V_T &= \ell_T^{xx} \\ v_T &= \ell_T^x \end{aligned} \tag{38}$$

Let $t \leq k \leq T-1$. Assuming the property true at $k+1$, then, from (31) and (33), we must have $\lambda_{k+1} = V_{k+1} p_{x_{k+1}} + v_{k+1}$. Now, let's show that the property also holds for the index k. From (30), we have:

$$\begin{aligned} V_{k+1} p_{x_{k+1}} + v_{k+1} &= \mu^{-1} Q_{k+1}^{-1} \left(p_{x_{k+1}} - f_k^x p_{x_k} - f_k^u p_{u_k} + w_{k+1} \right) \\ \left(I - \mu Q_{k+1} V_{k+1} \right) p_{x_{k+1}} &= f_k^x p_{x_k} + f_k^u p_{u_k} + \mu Q_{k+1} v_{k+1} - w_{k+1} \end{aligned} \tag{39}$$

We define $\Gamma_{k+1} \coloneqq \left(I - \mu V_{k+1} Q_{k+1} \right)^{-1}$ and find that:

$$\begin{aligned} \lambda_{k+1} &= V_{k+1} \left(I - \mu Q_{k+1} V_{k+1} \right)^{-1} \left(f_k^x p_{x_k} + f_k^u p_{u_k} + \mu Q_{k+1} v_{k+1} - w_{k+1} \right) + v_{k+1} \\ &= \Gamma_{k+1}^{-1} V_{k+1} \left(f_k^x p_{x_k} + f_k^u p_{u_k} + \mu Q_{k+1} v_{k+1} - w_{k+1} \right) + v_{k+1} \\ &= \Gamma_{k+1}^{-1} V_{k+1} \left(f_k^x p_{x_k} + f_k^u p_{u_k} - w_{k+1} \right) + \Gamma_{k+1}^{-1} v_{k+1} \end{aligned} \tag{40}$$

as

$$V_{k+1}\left(I-\mu Q_{k+1}V_{k+1}\right)^{-1}=\left(V_{k+1}^{-1}-\mu Q_{k+1}\right)^{-1}=\left(I-\mu V_{k+1}Q_{k+1}\right)^{-1}V_{k+1} \tag{41}$$

Now from (32)

$$\begin{aligned}\bar{\ell}_k^{ux}p_{x_k}+\bar{\ell}_k^{uu}p_{u_k}+\ell_k^u+f_k^{uT}\left(\Gamma_{k+1}^{-1}V_{k+1}\left(f_k^x p_{x_k}+f_k^u p_{u_k}-w_{k+1}\right)+\Gamma_{k+1}^{-1}v_{k+1}\right)=0\\ \left(\bar{\ell}_k^{uu}+f_k^{uT}\Gamma_{k+1}^{-1}V_{k+1}f_k^u\right)p_{u_k}+\left(\bar{\ell}_k^{ux}+f_k^{uT}\Gamma_{k+1}^{-1}V_{k+1}f_k^x\right)p_{x_k}+\ell_k^u+f_k^{uT}\left(-\Gamma_{k+1}^{-1}V_{k+1}w_{k+1}+\Gamma_{k+1}^{-1}v_{k+1}\right)=0\end{aligned} \tag{42}$$

which can be written

$$p_{u_k}=G_k p_{x_k}+g_k \tag{43}$$

where

$$\begin{aligned}Q_{uu}&=\bar{\ell}_k^{uu}+f_k^{uT}\Gamma_{k+1}^{-1}V_{k+1}f_k^u\\ Q_{ux}&=\bar{\ell}_k^{ux}+f_k^{uT}\Gamma_{k+1}^{-1}V_{k+1}f_k^x\\ Q_u&=\ell_k^u+f_k^{uT}\Gamma_{k+1}^{-1}v_{k+1}-f_k^{uT}\Gamma_{k+1}^{-1}V_{k+1}w_{k+1}\\ G_k&=-Q_{uu}^{-1}Q_{ux}\\ g_k&=-Q_{uu}^{-1}Q_u\end{aligned} \tag{44}$$

Finally, we get:

$$\begin{aligned}\bar{\ell}_k^{xx}p_{x_k}+\bar{\ell}_k^{xu}p_{u_k}+\ell_k^x+f_k^{xT}\lambda_{k+1}&=\bar{\ell}_k^{xx}p_{x_k}+\bar{\ell}_k^{xu}p_{u_k}+\ell_k^x+f_k^{xT}\Gamma_{k+1}^{-1}\left(V_{k+1}\left(f_k^x p_{x_k}+f_k^u p_{u_k}-w_{k+1}\right)+v_{k+1}\right)\\ &=\bar{\ell}_k^{xx}p_{x_k}+\left(\bar{\ell}_k^{xu}+f_k^{xT}\Gamma_{k+1}^{-1}V_{k+1}f_k^u\right)p_{u_k}+\ell_k^x+f_k^{xT}\Gamma_{k+1}^{-1}\left(V_{k+1}\left(f_k^x p_{x_k}-w_{k+1}\right)+v_{k+1}\right)\\ &=\bar{\ell}_k^{xx}p_{x_k}+Q_{xu}p_{u_k}+\ell_k^x+f_k^{xT}\Gamma_{k+1}^{-1}\left(V_{k+1}\left(f_k^x p_{x_k}-w_{k+1}\right)+v_{k+1}\right)\\ &=V_k p_{x_k}+v_k\end{aligned} \tag{45}$$

with:

$$\begin{aligned}V_k&=\bar{\ell}_k^{xx}+f_k^{xT}\Gamma_{k+1}^{-1}V_{k+1}f_k^x-Q_{xu}Q_{uu}^{-1}Q_{ux}\\ v_k&=\ell_k^x+f_k^{xT}\Gamma_{k+1}^{-1}v_{k+1}-f_k^{xT}\Gamma_{k+1}^{-1}V_{k+1}w_{k+1}-Q_{xu}Q_{uu}^{-1}Q_u\end{aligned} \tag{46}$$

□

4 Past stress

In this section, we derive the recursion for the past stress. Note that those derivations are valid only if $t \geq 1$. First, we define:

$$\forall\, k=0,\dots,t-1 \quad \bar{\lambda}_k \coloneqq \mu^{-1}f_k^{xT}Q_{k+1}^{-1}f_k^x p_{x_k}-\bar{\ell}_k^{xx}p_{x_k}-\mu^{-1}f_k^{xT}Q_{k+1}^{-1}(w_{k+1}+p_{x_{k+1}})-\ell_k^x \tag{47}$$

- From (21), for $k=0$:

$$\begin{aligned}\left(\bar{\ell}_0^{xx}-\mu^{-1}P^{-1}-\mu^{-1}f_0^{xT}Q_1^{-1}f_0^x\right)p_{x_0}+\mu^{-1}f_0^{xT}Q_1^{-1}p_{x_1}+\ell_0^x-\mu^{-1}P^{-1}w_0+\mu^{-1}f_0^{xT}Q_1^{-1}w_1=0\\ -\mu^{-1}P^{-1}(p_{x_0}+w_0)=\bar{\lambda}_0\end{aligned} \tag{48}$$

- From (22), $\forall\, k=1,\dots,t-1$:

$$\begin{aligned}&\mu^{-1}Q_k^{-1}f_{k-1}^x p_{x_{k-1}}+\left(\bar{\ell}_k^{xx}-\mu^{-1}Q_k^{-1}-\mu^{-1}f_k^{xT}Q_{k+1}^{-1}f_k^x-\mu^{-1}h_k^{xT}R_k^{-1}h_k^x\right)p_{x_k}+\mu^{-1}f_k^{xT}Q_{k+1}^{-1}p_{x_{k+1}}\\ &+\ell_k^x-\mu^{-1}Q_k^{-1}w_k+\mu^{-1}f_k^{xT}Q_{k+1}^{-1}w_{k+1}+\mu^{-1}h_k^{xT}R_k^{-1}\gamma_k=0\end{aligned}$$

$$Q_k^{-1}f_{k-1}^x p_{x_{k-1}}-\left(Q_k^{-1}+h_k^{xT}R_k^{-1}h_k^x\right)p_{x_k}-Q_k^{-1}w_k+h_k^{xT}R_k^{-1}\gamma_k=\mu\bar{\lambda}_k \tag{49}$$

Proposition 2.

$$Q_k^{-1} f_{k-1}^x p_{x_{k-1}} - \left(Q_k^{-1} + h_k^{xT} R_k^{-1} h_k^x\right) p_{x_k} - Q_k^{-1} w_k + h_k^{xT} R_k^{-1} \gamma_k = -P_k^{-1}(p_{x_k} - \hat{\mu}_k) \quad \forall k = 1, \dots, t$$
$$-P^{-1}(p_{x_0} + w_0) = -P_0^{-1}(p_{x_0} - \hat{\mu}_0) \tag{50}$$

where P_k and $\hat{\mu}_k$ are solution of the forward recursion:

$$\begin{aligned}
E_{k+1} &= P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx} \\
\bar{P}_{k+1} &= Q_{k+1} + f_k^x (P_k^{-1} - \mu \bar{\ell}_k^{xx})^{-1} f_k^{xT} \\
K_{k+1} &= \bar{P}_{k+1} h_{k+1}^{x^T} (R_{k+1} + h_{k+1}^x \bar{P}_{k+1} h_{k+1}^{x^T})^{-1} \\
P_{k+1} &= (I - K_{k+1} h_{k+1}^x) \bar{P}_{k+1} \\
\hat{\mu}_{k+1} &= (I - K_{k+1} h_{k+1}^x)(f_k^x \hat{\mu}_k - w_{k+1}) + K_{k+1} \gamma_{k+1} + \mu P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} (\bar{\ell}_k^{xx} \hat{\mu}_k + \ell_k^x)
\end{aligned} \tag{51}$$

with the initialization:

$$\begin{aligned}
P_0 &= P \\
\hat{\mu}_0 &= \hat{x}_0 - x_0^i
\end{aligned} \tag{52}$$

Furthermore:

$$\forall k = 0, \dots, t-1, \quad p_{x_k} = E_{k+1}^{-1} \left(f_k^{xT} Q_{k+1}^{-1} (w_{k+1} + p_{x_{k+1}}) + P_k^{-1} \hat{\mu}_k + \mu \ell_k^x \right) \tag{53}$$

Proof. The initialization is clear:

$$P_0 = P \tag{54}$$
$$\hat{\mu}_0 = -w_0 \tag{55}$$

If the prop is true at time k, from (48) and (49), we must have $\bar{\lambda}_k = -\mu^{-1} P_k^{-1}(p_{x_k} - \hat{\mu}_k)$. Now let's show that the property holds for $k+1$. From (47), we have:

$$-\mu^{-1} P_k^{-1}(p_{x_k} - \hat{\mu}_k) = \mu^{-1} f_k^{xT} Q_{k+1}^{-1} f_k^x p_{k_k} - \bar{\ell}_k^{xx} p_{k_k} - \mu^{-1} f_k^{xT} Q_{k+1}^{-1} (w_{k+1} + p_{x_{k+1}}) - \ell_k^x$$
$$\left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right) p_{x_k} = f_k^{xT} Q_{k+1}^{-1} (w_{k+1} + p_{x_{k+1}}) + P_k^{-1} \hat{\mu}_k + \mu \ell_k^x$$

Hence, we recover equation (53). Consequently,

$$\begin{aligned}
&- Q_{k+1}^{-1} f_k^x p_{x_k} + \left(Q_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x\right) p_{x_{k+1}} + Q_{k+1}^{-1} w_{k+1} - {h_{k+1}^x}^T R_{k+1}^{-1} \gamma_{k+1} \\
=&- Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} \left(f_k^{xT} Q_{k+1}^{-1} (w_{k+1} + p_{x_{k+1}}) + P_k^{-1} \hat{\mu}_k + \mu \ell_k^x\right) + \left(Q_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x\right) p_{x_{k+1}} \\
&+ Q_{k+1}^{-1} w_{k+1} - {h_{k+1}^x}^T R_{k+1}^{-1} \gamma_{k+1} \\
&= P_{k+1}^{-1} (p_{x_{k+1}} - \hat{\mu}_{k+1})
\end{aligned} \tag{56}$$

where:

$$\begin{aligned}
P_{k+1}^{-1} &= Q_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x - Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} f_k^{xT} Q_{k+1}^{-1} \\
&= {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x + \underbrace{\left(Q_{k+1} + f_k^x (P_k^{-1} - \mu \bar{\ell}_k^{xx})^{-1} f_k^{xT}\right)^{-1}}_{:= \bar{P}_{k+1}^{-1}}
\end{aligned} \tag{57}$$

and:

$$\begin{aligned}
P_{k+1}^{-1} \hat{\mu}_{t+1} &= -Q_{k+1}^{-1} w_{k+1} + {h_{k+1}^x}^T R_{k+1}^{-1} \gamma_{k+1} + Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} \left(f_k^{xT} Q_{k+1}^{-1} w_{k+1} + P_k^{-1} \hat{\mu}_k + \mu \ell_k^x\right) \\
&= -\left(Q_{k+1}^{-1} - Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} f_k^{xT} Q_{k+1}^{-1}\right) w_{k+1} + {h_{k+1}^x}^T R_{k+1}^{-1} \gamma_{k+1} \\
&+ Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} (P_k^{-1} \hat{\mu}_k + \mu \ell_k^x) \\
&= -\bar{P}_{k+1}^{-1} w_{k+1} + {h_{k+1}^x}^T R_{k+1}^{-1} \gamma_{k+1} + Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} (P_k^{-1} \hat{\mu}_k + \mu \ell_k^x)
\end{aligned} \tag{58}$$

Let's define:

$$K_{k+1} := (\bar{P}_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x)^{-1} {h_{k+1}^x}^T R_{k+1}^{-1} = \bar{P}_{k+1} {h_{k+1}^x}^T (R_{k+1} + h_{k+1}^x \bar{P}_{k+1} {h_{k+1}^x}^T)^{-1} \tag{59}$$

$$E_{k+1} := P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx} \tag{60}$$

We have:

$$\begin{aligned}(I - K_{k+1} h_{k+1}^x)\bar{P}_{k+1} &= (I - (\bar{P}_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x)^{-1} {h_{k+1}^x}^T R_{k+1} h_{k+1}^x)\bar{P}_{k+1} \\ &= (\bar{P}_{k+1}^{-1} + {h_{k+1}^x}^T R_{k+1}^{-1} h_{k+1}^x)^{-1} = P_{k+1}\end{aligned} \tag{61}$$

and get:

$$\hat{\mu}_{t+1} = -(I - K_{k+1} h_{k+1}^x) w_{k+1} + K_{k+1}\gamma_{k+1} + P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} (P_k^{-1}\hat{\mu}_k + \mu \ell_k^x) \tag{62}$$

but:

$$\begin{aligned}P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} P_k^{-1} &= P_{k+1} Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} P_k^{-1} \qquad (63) \\ &= P_{k+1} Q_{k+1}^{-1} f_k^x - P_{k+1} Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} \left(f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right) \\ &= P_{k+1}\left(Q_{k+1}^{-1} - Q_{k+1}^{-1} f_k^x \left(P_k^{-1} + f_k^{xT} Q_{k+1}^{-1} f_k^x - \mu \bar{\ell}_k^{xx}\right)^{-1} f_k^{xT} Q_{k+1}^{-1}\right) f_k^x + P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} \mu \bar{\ell}_k^{xx} \\ &= P_{k+1}\left(Q_{k+1} + f_k^x \left(P_k^{-1} - \mu \bar{\ell}_k^{xx}\right)^{-1} f_k^{xT}\right)^{-1} f_k^x + P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} \mu \bar{\ell}_k^{xx} \\ &= P_{k+1} \bar{P}_{k+1}^{-1} f_k^x + P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} \mu \bar{\ell}_k^{xx} \\ &= (I - K_{k+1} h_{k+1}^x) f_k^x + P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} \mu \bar{\ell}_k^{xx}\end{aligned}$$

Therefore,

$$\hat{\mu}_{k+1} = (I - K_{k+1} h_{k+1}^x)(f_k^x \hat{\mu}_k - w_{k+1}) + K_{k+1}\gamma_{k+1} + \mu P_{k+1} Q_{k+1}^{-1} f_k^x E_{k+1}^{-1} (\bar{\ell}_k^{xx}\hat{\mu}_k + \ell_k^x) \tag{64}$$

□

5 Coupling

In this section, we study (23) and show that it allows to couple the past and future stress:

- Case 1. From (23), if $t = 0$:

$$\begin{aligned}&\left(\bar{\ell}_0^{xx} - \mu^{-1} P_0^{-1} - \mu^{-1} f_0^{xT} Q_1^{-1} f_0^x\right) p_{x_0} + \mu^{-1} f_0^{xT} Q_1^{-1} p_{x_1} \\ &+ \left(\bar{\ell}_0^{xu} - \mu^{-1} f_0^{xT} Q_1^{-1} f_0^u\right) p_{u_0} + \ell_0^x - \mu^{-1} P_0^{-1} w_0 + \mu^{-1} f_0^{xT} Q_1^{-1} w_1 = 0\end{aligned} \tag{65}$$

  From (50), we have:

$$\left(\bar{\ell}_0^{xx} - \mu^{-1} f_0^{xT} Q_1^{-1} f_0^x\right) p_{x_0} + \mu^{-1} f_0^{xT} Q_1^{-1} p_{x_1} + \left(\bar{\ell}_0^{xu} - \mu^{-1} f_0^{xT} Q_1^{-1} f_0^u\right) p_{u_0} + \ell_0^x + \mu^{-1} f_0^{xT} Q_1^{-1} w_1 = \mu^{-1} P_0^{-1}(p_{x_0} - \hat{\mu}_0)$$

- Case 2. From (23), if $t \geq 1$:

$$\begin{aligned}&\mu^{-1} Q_t^{-1} {f_{t-1}^x}^T p_{x_{t-1}} + \left(\bar{\ell}_t^{xx} - \mu^{-1} Q_t^{-1} - \mu^{-1} f_t^{xT} Q_{t+1}^{-1} f_t^x - \mu^{-1} h_t^{xT} R_t^{-1} h_t^x\right) p_{x_t} + \mu^{-1} f_t^{xT} Q_{t+1}^{-1} p_{x_{t+1}} \\ &+ \left(\bar{\ell}_t^{xu} - \mu^{-1} f_t^{xT} Q_{t+1}^{-1} f_t^u\right) p_{u_t} + \ell_t^x - \mu^{-1} Q_t^{-1} w_t + \mu^{-1} f_t^{xT} Q_{t+1}^{-1} w_{t+1} + \mu^{-1} h_t^{xT} R_t^{-1} \gamma_t = 0\end{aligned} \tag{66}$$

  and from (50), we have:

$$-Q_t^{-1} f_{t-1}^x p_{x_{t-1}} + \left(Q_t^{-1} + h_t^{xT} R_t^{-1} h_t^x\right) p_{x_t} + Q_t^{-1} w_t - h_t^{xT} R_t^{-1} \gamma_t = P_t^{-1}(p_{x_t} - \hat{\mu}_t)$$

Therefore, in both cases, we find that:

$$\left(\bar{\ell}_t^{xx} - \mu^{-1} f_t^{x\,T} Q_{t+1}^{-1} f_t^x\right) p_{x_t} + \mu^{-1} f_t^{x\,T} Q_{t+1}^{-1} p_{x_{t+1}} + \left(\bar{\ell}_t^{xu} - \mu^{-1} f_t^{x\,T} Q_{t+1}^{-1} f_t^u\right) p_{u_t} \\ + \ell_t^x + \mu^{-1} f_t^{x\,T} Q_{t+1}^{-1} w_{t+1} = \mu^{-1} P_t^{-1} (p_{x_t} - \hat{\mu}_t) \tag{67}$$

And from (34)

$$\begin{aligned} V_t p_{x_t} + v_t &= \bar{\ell}_t^{xx} p_{x_t} + \bar{\ell}_t^{xu} p_{u_k} + \ell_t^x + f_t^{x\,T} \lambda_{t+1} \\ &= \bar{\ell}_t^{xx} p_{x_t} + \bar{\ell}_t^{xu} p_{u_k} + \ell_t^x + f_t^{x\,T} \left(\mu^{-1} Q_{t+1}^{-1} \left(p_{x_{t+1}} - f_t^x p_{x_t} - f_t^u p_{u_t} + w_{t+1}\right)\right) \end{aligned}$$

Hence, Equation (67) becomes:

$$\begin{aligned} V_t p_{x_t} + v_t &= \mu^{-1} P_t^{-1} (p_{x_t} - \hat{\mu}_t) \\ \mu P_t V_t p_{x_t} + \mu P_t v_t &= p_{x_t} - \hat{\mu}_t \end{aligned}$$

Hence:

$$p_{x_t} = \left(P_t^{-1} - \mu V_t\right)^{-1} \left(P_t^{-1} \hat{\mu}_t + \mu v_t\right) \tag{68}$$